A TANGENT CATEGORY APPROACH TO OPERADIC GEOMETRY

by

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"If there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither 'number' nor 'size,' but always form."

A. Grothendieck

"Ma non ti rendi conto di quant'è bello? Che non ti porti il peso del mondo sulle spalle, che sei soltanto un filo d'erba in un prato? Non ti senti più leggero?"

Sarah - Strappare lungo i bordi. Zerocalcare

Contents

Abstract . . . . **v**

Acknowledgements . . . . **vi**

Chapter 1 Introduction **1**

1.1 Outline 6

1.2 Contribution statement 8

1.3 Notation and naming conventions 10

1.4 Style of the thesis 12

1.5 Concept map of the thesis 12

Chapter 2 Towards a universal language of differentiation in geometry **16**

2.1 An introduction to tangent category theory 17

2.2 Tangent category concepts 32
2.2.1 Vector fields 33
2.2.2 Differential objects and linearity 33
2.2.3 Tangent display maps 35
2.2.4 The slice tangent category 49
2.2.5 Differential bundles 55
2.2.6 The adjoint tangent category 58

2.3 Tangent objects: a formal approach to tangent categories 63

2.4 Formal tangent monads 82

Chapter 3 The geometric theories of an operad **89**

3.1 Motivation 90

3.2 Operads: the factories of algebraic objects 92

3.3 Algebras of operads 100

3.4 The coCartesian differential monad of an operad 104

3.5 The algebraic tangent category of an operad 113
3.5.1 Vector fields in the algebraic tangent category of an operad . . 116

3.5.2 The functoriality of the algebraic construction 120

3.6 The geometric tangent category of an operad 134
3.6.1 Vector fields in the geometric tangent category of an operad . 152
3.6.2 The functoriality of the geometric construction 154

Chapter 4 The differential bundles of the geometric tangent category of an operad 162

4.1 The slice of the geometric tangent category of an operad 164

4.2 The slice tangent categories of the affine schemes over an operad . . . 175
4.2.1 The geometric tangent category of the enveloping operad . . . 177

4.3 The classification of differential bundles 184
4.3.1 The classification of differential objects 185
4.3.2 Differential bundles are modules 201

Chapter 5 A tangent category approach to deformation theory 203

5.1 An introduction to algebraic deformation theory 205

5.2 The category of operads is itself a tangent category 212
5.2.1 The tangent category of tangent monads 229
5.2.2 The adjoint tangent category of operads 241

5.3 Towards a local approach to deformations 256

Chapter 6 Conclusions 266

6.1 What this thesis is about 266

6.2 Future work 268
6.2.1 Operadic constructions for tangent categories 268
6.2.2 Tangent constructions for operads 270
6.2.3 Towards a formal theory of tangent objects 271
6.2.4 Deformation theory 272

Abstract

In his *Lectures on Noncommutative Geometry*, Ginzburg proposes a theory of algebraic noncommutative (affine) geometry. One of the fundamental insights of noncommutative geometry is to regard associative, not necessarily commutative, algebras as geometric spaces. In the last section of the aforementioned lectures, Ginzburg suggests an ambitious generalization of his work: he observes that most of the constructions he characterized in the noncommutative case, carry over into the realm of operadic algebras and he proposes a theory of *operadic geometry*. From a philosophical viewpoint one wonders if the similarities captured by Ginzburg could hide a deeper phenomenon: a common language which captures some important features of these examples. In this thesis, tangent category theory is applied for the first time to describe the patterns and similarities observed by Ginzburg. This work largely extends Cruttwell and Lemay's attempt to employ tangent category theory to capture significant features of commutative algebraic geometry. From the perspective of operad theory, this thesis translates in the context of tangent categories some important operadic constructions, such as derivations, enveloping operads, and modules. From the perspective of tangent category theory, it provides new examples of noncommutative non-pointwise models of geometry described with tangent categories. First, we show that each operad is canonically associated with two tangent categories: the algebraic and the geometric tangent categories. Once we have established the functorial correspondence between operads and tangent categories, we describe two important constructions. First, we show an equivalence between slice tangent categories and enveloping operads; second, we employ this result to classify differential bundles as modules over the operadic algebras. In the last chapter, we apply the established relationship between operads and tangent categories to the theory of algebraic deformation. First, we prove that the category of operads itself and its opposite carry two tangent structures, which are closely related to deformations. Finally, we explore some ideas, inspired by tangent category theory, to classify all infinitesimal deformations of an operadic algebra.

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The author would like to express his gratitude towards his friends at the Department of Mathematics & Statistics of Dalhousie University, for creating a wonderful and safe environment in which to share ideas. Special thanks to Deni Salja, Geoffrey Vooys, Sadie Richards, and Adrien Delazzer Meunier for the numerous discussions and the support they showed to the author during his Ph.D.

Last but not least, the author would like to thank his family and friends, who helped him handle the stress and emotions of living abroad and made him feel at home regardless of the language barrier, or the spacial distance.

Chapter 1

Introduction

One of the fundamental insights of differential calculus is that polynomial functions can locally approximate any sufficiently smooth real-valued function. The simplest case consists of approximating a real-valued function near a certain point by a first-order polynomial. The function's first derivative at the given point fully determines such a polynomial.

Similarly, differential geometry allows one to locally approximate some sufficiently smooth space near a certain point via a linear space at the given point. However, in the same way that not every function can be differentiated, not every geometric space admits a local linear approximation at each point. To solve this technicality, differential geometry restricts its attention to a certain class of geometric spaces, known as smooth manifolds. These spaces exhibit a local linear behaviour. Informally, this means that, at any given point, there is a vector space, known as the *tangent space* at that point, which provides a good approximation of the space in a neighbourhood of the point.

A philosophical question arises: is differential geometry the ultimate theory of geometric spaces or are smooth manifolds just an example of such an object? Is there a more general theory of geometry for which differential geometry is only one of many models? A natural mathematical language to formulate and answer this question is category theory.

One of the aims of category theory consists of studying mathematical objects in a model-independent fashion: instead of characterizing a specific class of objects by a direct description, the categorical approach is interested in the operations and structures required to define and study these objects.

Starting from Lawvere's ideas, Kock developed a categorical approach to differential geometry, known as *synthetic differential geometry* (cf. [37]) which aims to give a precise interpretation of the concept of infinitesimal quantity. Partially inspired by this attempt to categorify differential geometry, Rosický proposed a more general approach in [53], in which he introduced the concept of *tangent category*. He also showed that representable tangent categories capture synthetic differential geometry, proving this approach is more general. The simplicity and generality of this theory perfectly fit our purposes.

Informally, a tangent category is a collection of objects, interpreted as geometric spaces, a collection of morphisms, representing the transformations between these spaces, and a particular structure which allows one to axiomatize the idea that these spaces are locally linear. Cockett and Cruttwell in [12] revisited and generalized this notion and showed how tangent category theory generalizes categorical theories of differentiation such as Cartesian differential categories.

One of the interesting aspects of tangent category theory is encapsulated within the categorical framework in which this language is developed. Instead of selecting a specific class of spaces which manifest a local linear behaviour, the approach of tangent category theory is to provide the fundamental structures and operations which allow for a geometric interpretation of a collection of abstract objects. Within this perspective, differential geometry becomes a model of a more general theory of geometry: tangent category theory.

Assuming this interpretation, it is natural to wonder which other models of geometry can be described by tangent category theory. Recently, Cruttwell and Lemay in [18] employed tangent category theory to describe certain aspects of algebraic geometry. In particular, they proved that the opposite of the category of commutative and unital algebras, which is equivalent to the category of affine schemes, comes equipped with a tangent structure. They also extended this result to the category of schemes and they classified differential bundles, which can be interpreted as vector bundles in a tangent category, in terms of modules over the algebras. This striking connection between commutative algebra, algebraic geometry, and tangent category theory suggests a deeper phenomenon: other algebraic objects, like associative, Lie, or Poisson algebras, might also carry a tangent structure. The first goal of this thesis is to explore this possibility, linking operad theory with tangent categories.

The relationship between algebra and geometry is one of the most fascinating well-studied mathematical phenomena: from algebraic topology to algebraic geometry, from noncommutative geometry to Lie group theory, geometry and algebra appear as manifestations of two sides of the same coin. An important example of this relationship is represented by the celebrated theorem of Gelfand and Naimark [49, Theorem 8.33] which proves that a locally compact Hausdorff topological space is fully described by the associated unital and commutative $\mathbf{C}^*$-algebra of its continuous complex-valued functions. This result is the starting point of noncommutative geometry which aims to interpret associative, not necessarily commutative, algebras as geometric spaces. For an introduction to this subject, we advise the reader to consult [36] and [16].

In theoretical physics, noncommutative geometry is related to the quantization of a classical theory.

The Gelfand-Naimark-Segal theorem [49, Theorem 14.3] classifies the representations of the $\mathbf{C}^*$-algebra of observables of a system. In particular, it shows that the observables in a commutative algebra are represented as continuous functions over a suitable topological space while observables in a noncommutative algebra are represented as linear operators over a suitable Hilbert space. From the point of view of Physics, this means that commutative algebras represent classical systems, while noncommutative algebras encode quantum systems.

Furthermore, noncommutative geometry finds application in quantum gravity, suggesting that spacetime itself could be regarded as a noncommutative space (see for example, [2]).

One striking feature of noncommutative geometry is the absence of a classical notion of points: noncommutative spaces cannot be interpreted as sets of points equipped with extra structure. On the other hand, the categorical approach rejects a pointwise interpretation and embraces the idea that mathematical objects can be fully described, up to isomorphism, from their transformations, that is from the study of the morphisms among those objects. This encourages us to wonder whether or not non-commutative geometry can be interpreted as a model of tangent category theory.

However, so far, the main examples of tangent categories in the literature are only *classical* examples; they describe the geometry of pointwise spaces, like differential geometry or algebraic geometry. One of the goals of this thesis is to present a nontrivial example of a tangent category which can be employed to study noncommutative geometry. The main inspiration comes from Ginzburg's work on noncommutative algebraic geometry, presented in [25].

Ginzburg, in the introduction of these notes, describes two kinds of noncommutative geometries: noncommutative geometry *in the small* and noncommutative geometry *in the large*. In his words: [1]

> *"The former is a* ***generalization*** *of the conventional 'commutative' algebraic geometry to the noncommutative world. The objects that one studies here should be thought of as noncommutative deformations, sometimes referred to as* ***quantizations****, of their commutative counterparts. A typical example of this approach is the way of thinking about the universal enveloping algebra of a finite dimensional Lie algebra* $\mathfrak{g}$ *as a deformation of the symmetric algebra* $\mathsf{S}(\mathfrak{g})$*, which is isomorphic to the polynomial algebra.*
> *As opposed to the noncommutative geometry 'in the small', noncommutative geometry 'in the large' is* ***not*** *a generalization of commutative theory. The world of noncommutative geometry 'in the large' does not contain [the] commutative world as a special case, but is only similar, parallel, to it. The concepts and results that one develops here, do* ***not*** *specialize to their commutative analogues. Consider for instance the notion of* ***smoothness*** *that exists both in commutative algebraic geometry and in noncommutative algebraic geometry 'in the large'. A commutative algebra A may be smooth in the sense of commutative algebraic geometry, and at the same time be non-smooth from the point of view of noncommutative geometry 'in the large'."*

In Example 3.81, we discuss this distinction from the point of view of tangent category theory; we show that the tangent morphism which compares the tangent category of commutative algebraic geometry with the tangent category of noncommutative algebraic geometry does not preserve the tangent structure strongly, i.e. it is not a strong tangent morphism.

[1] Paragraph in the introduction of [25]. The original text is in normal font. The words here in bold are in italics in the original text

Ginzburg also suggests the existence of other kinds of geometries associated with other algebraic objects. He dedicates a final section of his notes [25] to extend his results to a new plethora of algebraic theories, parametrized by operads.

An operad is a mathematical machinery which encodes the n-ary operations and axioms of an algebraic theory. Consider for instance associative algebras. These objects are vector spaces equipped with a binary associative operation. Similarly, Lie algebras are vector spaces, equipped with a binary operation, satisfying two conditions: anticommutativity and the Jacobi identity. Both associative and Lie algebras are examples of algebraic objects generated by a suitable operad.

Concretely, an operad is a mathematical object which can be presented as a list of operations of an algebraic theory, grouped by their arity, subject to the relations established by the theory. For example, the operad which generates associative algebras contains all n-ary operations generated by composing a binary operation μ which satisfies the associativity condition (Example 3.7).

The representations of an operad form the algebraic theory associated with the operad. In particular, the theory of associative algebras is the algebraic theory of the operad $\mathscr{Ass}$, known as the *associative operad* and the theory of Lie algebras is the algebraic theory of the operad $\mathscr{Lie}$, known as the *Lie operad*.

Although the main motivation of this thesis is to investigate the relationship between noncommutative geometry and tangent category theory, our approach extends to a larger family of geometries: we present a canonical construction that associates to every operad a corresponding geometric theory, axiomatized by a tangent structure on the opposite of the category of the representations of the operad, also known as *algebras* of the operad.

We also show that this construction is functorial: morphisms of operads correspond to morphisms of tangent categories.

We also show that this functorial correspondence reflects an intimate relationship between operads and tangent categories by discussing how some constructions of operad theory translate to the associated tangent categories. In particular, we focus our attention on the geometric theory of the enveloping operad associated with a $\mathscr{P}$-algebra, on the compatibility between this functorial relationship and we employ this relationship to classify differential bundles.

One important source of noncommutative spaces is provided by deformation theory. The main idea of deformation theory consists of slightly deforming an algebraic or geometric object in a compatible way with the operations and axioms that describe such an object. The deformation of associative algebras, first studied by Gerstenhaber (see for example [23]), allows one to *deform* a commutative algebra to a noncommutative one, by twisting the original commutative multiplication map with some extra noncommutative terms. This idea extends to the realm of algebraic theories generated by operads, so one can deform operadic algebras to obtain new ones.

The idea of deforming an algebraic object can be seen as defining a path in the space of algebraic objects of a certain type. However, there is not such a thing as a differential structure over the collection of algebras of an operad. We believe that tangent category theory can provide a minimal geometric setting in which deformations can be interpreted as paths of a suitable geometric space.

In the last chapter we introduce some ideas to explore this intuition. In particular, we show that the category Operad of operads is itself a tangent category whose vector fields are strictly related to infinitesimal deformations of operadic algebras. We show that the opposite category, $\mathsf{Operad}^{\mathsf{op}}$, carries a tangent structure which is also related to infinitesimal deformations. We dedicate a final section to identifying some of the issues of this approach. To solve these issues, we propose two new approaches to classifying infinitesimal deformations as sections of a suitable differential bundle in the geometric tangent category of the operad, and as sections of the unit of a tangent comonad.

1.1 Outline

This thesis is organized into six distinct chapters. In Chapter 1, we establish the structure of the thesis and we set up the adopted notation and naming conventions. Chapter 2 is dedicated to tangent category theory. In particular, in Section 2.1, we motivate and introduce the concept of a tangent category. Section 2.2 is dedicated to exploring some of the main constructions of tangent category theory.

In particular, in Section 2.2.1, we recall the notion of vector fields over an object of a tangent category, in Section 2.2.2, we recall the notion of differential objects and explain why a tangent structure provides a notion of local linearity for its objects. Section 2.2.3 is dedicated to a new concept in tangent category theory, tangent display maps.

This new notion plays a crucial role in Section 2.2.4 in which we revisit the construction of the slice tangent category and in Section 2.2.5, dedicated to differential bundles. We conclude Section 2.2, by recalling, in Section 2.2.6 an important construction: the adjoint of a tangent category.

We conclude Chapter 2 with an introduction to a formal approach to tangent category theory, introduced by the author in [42]. Section 2.3 is dedicated to exploring this new approach, by introducing the notion of tangent objects. Finally, in Section 2.4 we review the concept of tangent monad and prove that every tangent monad admits the construction of algebras (Theorem 2.73).

In Chapter 3 we present the two main constructions of this thesis: the algebraic and the geometric tangent categories of an operad. First, in Section 3.2 we recall the definition of an operad over a symmetric monoidal category. In Section 3.3, we recall the definition of algebras of an operad. Section 3.4 is dedicated to the first main result of this thesis: we prove that the monad associated with an algebraic operad is a coCartesian differential monad.

In Section 3.5 we construct the algebraic tangent category of an algebraic operad harnessing the fact that the associated monad is a coCartesian differential monad (Theorem 3.26). We dedicate Section 2.2.1 and 3.5.2 to classify vector fields and to prove the functoriality of the operation which sends an operad to the corresponding algebraic tangent category, respectively.

Section 3.6 presents the construction of the geometric tangent category associated with an operad. We also classify the corresponding vector fields, in Section 3.6.1, and discuss functoriality in Section 3.6.2.

In Chapter 4, we examine the relationship between some constructions of operad theory and tangent category theory. We start, in Section 4.2, by reconsidering the construction of the slice tangent category in terms of a right adjoint functor of the functor $\mathsf{Term}\colon \mathsf{cTngCat} \to \mathsf{TngPair}$ (Theorem 4.12).

This allows us to compare in Section 4.2.1 the tangent category of the enveloping operad of a $\mathscr{P}$-algebra A with the slice tangent category over A of the geometric tangent category of the operad $\mathscr{P}$. In particular, we show they are equivalent (Theorem 4.17).

In Section 4.3, we harness this equivalence to classify differential bundles. First, in Section 4.3.1, we classify differential objects in the geometric tangent category of a given operad and then, in Section 4.3.2, we show that differential bundles are equivalent to modules over operadic algebras.

In Chapter 5, we explore some connections between operad theory, tangent category theory, and algebraic deformation theory. In Section 5.1, we recall the main ideas and definitions of algebraic deformation theory and in Section 5.2 we link infinitesimal deformations of operadic algebras with vector fields of a new tangent category: the tangent category of operads (Theorem 5.22).

In Section 5.2.1, we show that the category of tangent monads over a given tangent category also constitutes a tangent category (Theorem 5.30) and we show the relationship between the tangent category of operads and the one of tangent monads (Theorem 5.33). We also show in Section 5.2.2 that the tangent category of operads is corepresentable (Theorem 5.38) and consequently that the opposite of the category of operads is also a tangent category (Corollary 5.39).

Finally, in Section 5.3 we discuss two different approaches to classify all infinitesimal deformations of an operadic algebra.

Finally, Chapter 6 is dedicated to the conclusions. First, in Section 6.1 we briefly recall the story of this thesis highlighting the main results. Second, in Section 6.2 we discuss a few directions for future work on this subject.

1.2 Contribution statement

Part of the work of this thesis has been written up as papers which are currently in the process of publication:

- *The Rosický Tangent Categories of Algebras over an Operad*, in collaboration with Sacha Ikonicoff and Jean-Simon Lemay ([29]).
  Chapter 3 contains most of the work done in this paper. This paper was the natural confluence of three different and independent research projects of the three authors.

  In particular, Ikonicoff proved that every operad is associated with a coCartesian differential monad and consequently, the opposite of the Kleisli category of the operad is a Cartesian differential category. The author of this thesis, independently showed that the monad associated with an operad is a tangent monad and consequently, he showed that the category of algebras of an operad and its opposite carry each a tangent structure, the one adjoint to the other.

  Finally, Lemay investigated the relationship between coCartesian differential monads and tangent monads, providing a natural language to compare the approaches explored by Ikonicoff and the author. The paper [29] is the natural result of these three efforts. Functoriality was not explored in much detail in the paper [29] and only in the paper [41] it was properly addressed. The classification of differential objects presented in the first paper and inspired by the work of Cruttwell and Lemay in [18], was proved in the operadic case by the author.

- *The differential bundles of the geometric tangent category of an operad* ([41]).
  Chapter 4 presents the work done in this paper, with the exception of Section 4.3.1 which is still part of the first paper.

- *The Grothendieck construction in the context of tangent categories* ([42]).
  The notion of tangent objects presented in Section 2.3 was first presented in this paper.

In addition, the research done for Section 2.2.3, dedicated to tangent display maps, is a joint effort between Geoffrey Cruttwell and the author and it will appear in a future paper.

In Section 2.3, Definition 2.52 was proposed by Rory Lucyshyn-Wright during an informal discussion with the author. As reported in Remark 2.53, Lucyshyn-Wright correctly pointed out that the limits in the definition of a tangent object must be pointwise for this definition to agree with the usual definition of a tangent category when the base 2-category is the 2-category Cat of categories.

1.3 Notation and naming conventions

A generic category is denoted by $\mathbb{X}$; we adopt the font Cat to denote a specific category, e.g. the category of categories. Identity morphisms are denoted by $\mathsf{id}_A\colon A \to A$, for an object A, or simply by id, when the object is clear from the context. To denote the composition of two composable morphisms $f\colon A \to B$ and $g\colon B \to C$ we adopt both the following conventions:

- *diagrammatic composition*, simply denoted by juxtaposition, i.e. $fg\colon A \xrightarrow{f} B \xrightarrow{g} C$;

- *functional composition*, denoted by the usual $g \circ f := fg$.

We adopt diagrammatic composition when we interpret two composable morphisms as abstract morphisms of an ambient category, while we adopt functional composition when we interpret two composable morphisms as functors between two categories, or as pointwise-defined functions between concrete objects.

To show the commutativity of a given diagram we often decompose the diagram into smaller ones, each commutative. For instance, consider the following diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\mathsf{Nat}"] & \bullet \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "(g,g';\beta,\gamma)"] & \bullet \arrow[d, "\gamma"] \\
\bullet \arrow[r, "f'"] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "(1.2)"] & \bullet \arrow[r, "g'"] \arrow[d, "\beta'"] & \bullet \arrow[d, "\gamma'"] \\
\bullet \arrow[r, "f''"'] & \bullet \arrow[r, "g''"'] & \bullet
\end{tikzcd}$$

In the left top square, we used *naturality*; in the right top square we used the compatibility between the morphisms g and g' with β and γ; in the bottom left square we employed Equation (1.2); finally, in the bottom right square, we used that $g'\gamma' = \beta' h'$ on the nose.

For a pullback diagram:

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

we denote the projections by $\pi_1\colon A \times_C B \to A$ and by $\pi_2\colon A \times_C B \to B$ and when the diagram is an n-fold pullback then the k-th projection is denoted by π_k. Given two morphisms $\alpha\colon D \to A$ and $\beta\colon D \to B$ such that $\alpha f = \beta g$, the unique morphism $D \to A \times_C B$ defined by the universality of the pullback is denoted by $\langle \alpha, \beta \rangle$. For a pushout diagram:

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \\ A \arrow[r] & A +_C B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

the injections $A \to A +_C B$ and $B \to A +_C B$ are respectively denoted by ι_1 and ι_2 and for n-fold pushouts the k-th injection is denoted by ι_k. Given two morphisms $\alpha\colon A \to D$ and $\beta\colon B \to D$ such that $f\alpha = g\beta$, the unique morphism induced by the universality of the pushout is denoted by $[\alpha, \beta]$.

Given two pullback diagrams:

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] & A' \times_{C'} B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, "g'"] \\ A \arrow[r, "f"'] & C & A' \arrow[r, "f'"'] & C' \end{tikzcd}$$

and two morphisms $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ such that $\pi_1 \alpha f' = \pi_2 \beta g'$, we denote the unique morphism $\langle \pi_1 \alpha, \pi_2 \beta \rangle$ simply by $\alpha \times \beta$. Similarly, for pushouts, we denote the unique morphism $[\alpha \iota_1, \beta \iota_2]\colon A +_C B \to A +_{C'} B'$ (whenever well-defined) by $\alpha + \beta$.

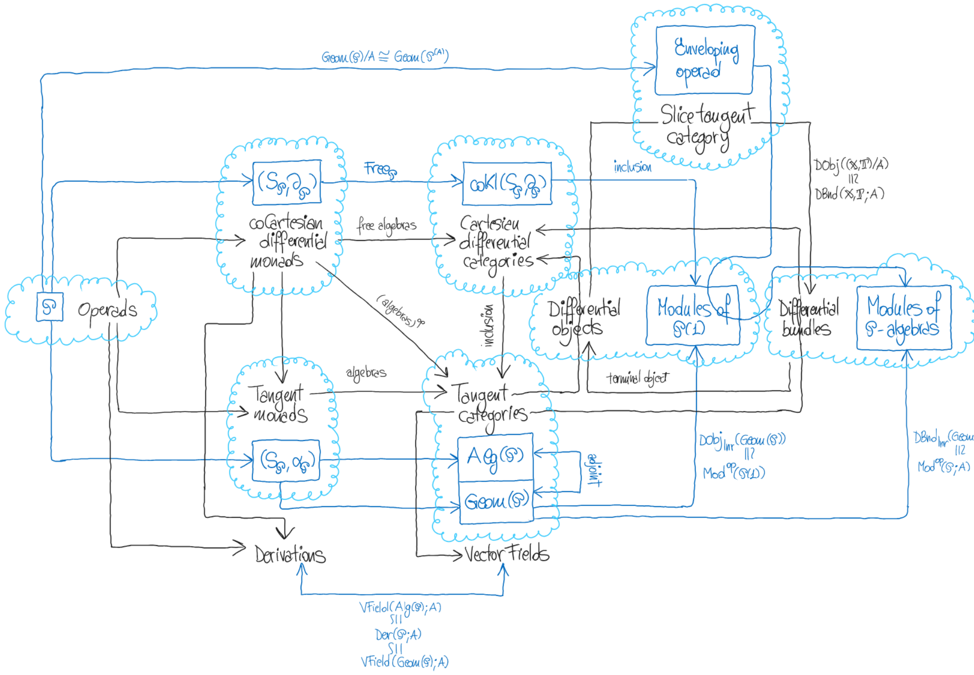

Figure 1.1: Concept map of Chapters 2, 3, and 4

1.4 Style of the thesis

For a smoother narration, we decided not to separate the background from the new contents presented in this thesis. In particular, we adopted the convention to dedicate background sections only to the concepts, like for tangent categories and operads, which appear at least twice in the thesis and introduce the rest of the known definitions and results only where required. Consequently, every chapter is a mix of known and new results. To avoid confusion and to stress the difference we decorate with an asterisk * any definition or result that is already present in the literature not including those by the author. In the text preceding each such definition or result, we also report the reference to the original source. On the other hand, the new definitions and results are not decorated with the asterisk.

1.5 Concept map of the thesis

The diagram in Figure 1.1 represents the structure of Chapters 2, 3, and 4:

from operads to cCDMs Theorem 3.26: the monad associated with an algebraic operad is a coCartesian differential monad;

from cCDMs to tangent monads Propositions 3.25 and 3.24: coCartesian differential monads are tangent monads over a tangent category induced by biproducts;

from operads to tangent monads Corollary 3.27: the monad associated with an algebraic operad is a tangent monad;

from tangent monads to tangent categories Theorem 2.73 and Proposition 3.40 (cf. [15]): the category of algebras of a tangent monad has a canonical tangent structure and the resulting tangent category is precisely the construction of algebras of a tangent monad.
Definition 3.28: algebraic tangent category of an operad;

cCDMs to tangent categories Corollary 3.54: the opposite of the category of algebras of a coCartesian differential monad is a tangent category (provided the existence of reflexive coequalizers).
Definition 3.56 and Theorem 3.68: geometric tangent category of an operad;

from cCDMs to CDCs Proposition 3.23 (cf. [30]): the coKleisli category of a coCartesian differential monad is a Cartesian differential category.
Theorem 4.20; the free algebras of an algebraic operad are differential objects in the geometric tangent category of the operad;

from CDCs to tangent categories Proposition [12, Proposition 4.7]: every Cartesian differential category is a Cartesian tangent category;

from tangent categories to CDCs via diff. objects [12, Theorem 4.11]: the category of differential objects of a Cartesian tangent category is a Cartesian differential category.
Corollary 4.31: the differential objects in the geometric tangent category of an operad $\mathscr{P}$ are equivalent to left modules over $\mathscr{P}(1)$;

from diff. objects to diff. bundles [11, Proposition 5.12]: differential objects are differential bundles over the terminal object;

from diff. bundles to diff. objects via the slice tangent category [11, Proposition 5.12]: differential bundles are differential objects in the slice tangent category.

Theorem 4.17: the slice tangent category over a $\mathscr{P}$-affine scheme A in the geometric tangent category of an operad $\mathscr{P}$ is equivalent to the geometric tangent category of the enveloping operad of $(\mathscr{P}; A)$;

from tangent categories to CDCs via diff. bundles Consequence of Theorem [11, Theorem 5.14]: the category of differential bundles of a Cartesian tangent category is a Cartesian differential category.

Theorem 4.35; differential bundles over a $\mathscr{P}$-affine scheme A in the geometric tangent category of an operad $\mathscr{P}$ are equivalent to modules over A in the operadic sense;

from tangent categories to vector fields Definition 2.15 (cf. [12]): vector fields are sections of the projection;

from cCDMs to derivations Definition 3.33 (cf. [30]): for an algebra of a coCartesian differential monad there is a well-defined notion of a derivation.

Lemma 3.34: derivations in the operadic sense are precisely derivations w.r.t. the associated coCartesian differential monad;

from operads to derivations Definition 5.12 (cf. [46]): for an algebra of an algebraic operad there is a well-defined notion of derivation.

Theorems 3.36 and 3.74: vector fields over a $\mathscr{P}$-affine scheme in the algebraic and the geometric tangent categories of an operad are equivalent to derivations over A.

The diagram in Figure 1.2 represents the structure of Chapter 5:

from operads to tangent monads Theorem 5.33: the functor which sends an algebraic operad to the corresponding tangent monad extends to a strong tangent morphism between the tangent category of operads (Theorem 5.22) and the tangent category of tangent monads over the base tangent category of R-modules (Theorem 5.30);

operad$^{\mathrm{op}}$ Theorem 5.38: the opposite of the category of algebraic operads is a representable tangent category;

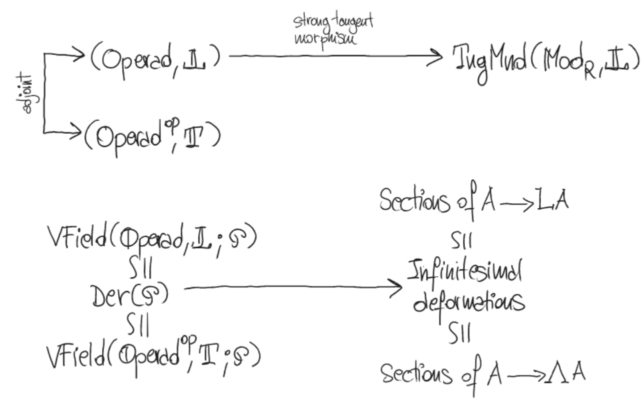

Figure 1.2: Concept map of Chapter 5

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relationship between vector fields and derivations Theorem 5.50: vector fields over an operad in the tangent categories of operads and its opposite are equivalent to derivations over the operad;

from derivations to deformations Example 5.11: derivations of an operad generate (only trivial, see Section 5.3) infinitesimal deformations of the corresponding algebras;

classification infinitesimal deformations Theorems 5.58 and 5.60: for every $\mathscr{P}$-algebra A, there is a differential bundle $q\colon A \to \mathrm{L}A$ in the geometric tangent category of the operad $\mathscr{P}$ whose sections classify all infinitesimal deformations of A. Moreover, there is a tangent comonad Λ over the geometric tangent category of $\mathscr{P}$ whose counit classifies all infinitesimal deformations of each A.

Chapter 2

Towards a universal language of differentiation in geometry

Tangent category theory aims to axiomatize the fundamental structures of differential geometry, such as the tangent bundle functor, in a categorical approach. One of the main goals of this thesis is to explore what other models of geometry can be described employing tangent category theory, with particular interests in applications to non-commutative geometry. This chapter is dedicated to reviewing the main definitions and results of the theory which play a role in our story. We also introduce two new concepts: tangent display maps, which will be employed in the construction of the slice tangent category, and tangent objects, which will be used to prove that every tangent monad admits the construction of algebras.

Section 2.1 is dedicated to motivating and reviewing the definition and the axioms of a tangent category. In the second section (Section 2.2), we review some constructions of tangent category theory. First (Section 2.2.1), we introduce vector fields and we discuss the definition of the Lie bracket between vector fields; then, in Section 2.2.2, we review the notion of differential object. Section 2.2.3 is dedicated to introducing the new notion of tangent display map and in Section 2.2.4 we employ it to discuss the construction of the slice tangent category. We then recall (Section 2.2.5) the notion of differential bundle and we show (Section 2.2.6) under which conditions the opposite of a tangent category is still a tangent category. Section 2.3 is dedicated to introducing the concept of tangent object, which formalizes the notion of tangent category in a generic strict 2-category. Therefore, in Section 5.2.1 we first review the concept of tangent monad and then we employ the notion of tangent object to prove that tangent monads over a given tangent category admit the construction of algebras, which defines the same tangent category as the one discussed by Cockett, Lemay, and Lucyshyn-Wright in [15].

Figure 2.1 displays the concept map of this chapter.

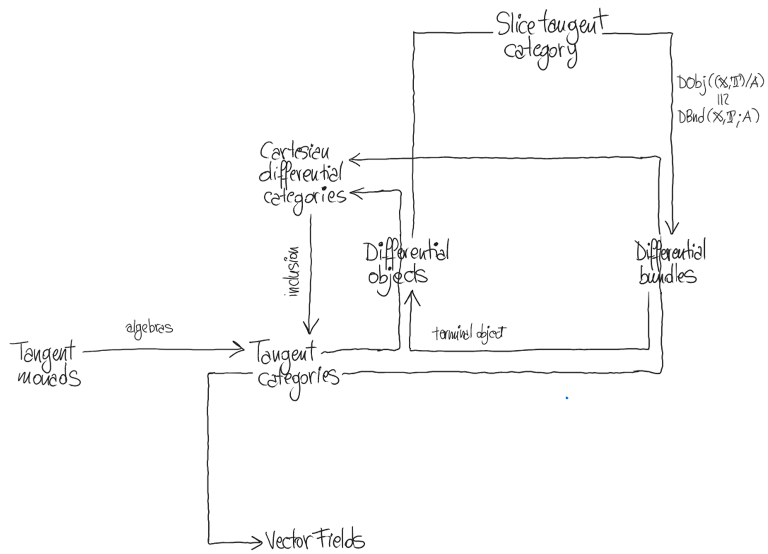


Figure 2.1: The concept map of the chapter

2.1 An introduction to tangent category theory

One of the most crucial insights of differential geometry is the idea of replicating mathematically the geometric experience we have of our planet: even if the global geometry of planet Earth is approximately that of a sphere, our daily experience suggests that we can locally approximate it by a plane. Mathematically, this intuition can be formalized by associating to each point of a given space M, e.g. a sphere, a linear space $\mathrm{T}_x M$, known as the *tangent space* of M at x, which, within a local chart, is approximately equal to the original space M. The collection of all these linear spaces $\mathrm{T}_x M$, parametrized by the points x of M, forms a new geometric space $\mathrm{T}M$ known as the *tangent bundle* of M, which locally can be described via a pair of coordinates (x, v) formed by a point x of the original space M with a tangent vector v belonging to the tangent space $\mathrm{T}_x M$ of M at x.

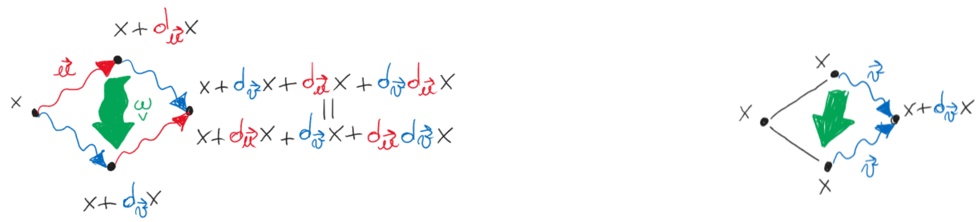
The tangent bundle of a space is the main protagonist in tangent category theory: the idea is to give a geometric description of an abstract object, assuming that such an object has an associated tangent bundle. But what is a tangent bundle of an abstract object? From differential geometry, one knows that the tangent bundle of a geometric space is, first of all, a vector bundle. Informally, a vector bundle E is a collection of disjoint but isomorphic linear spaces E_x, known as the *fibres*, indexed by the points x of another geometric space M, called the *base space*.

Mathematically, this can be modelled by a surjection $q\colon E \to M$, called the **projection**, which "forgets" about the fibres, together with an injection $z_q\colon M \to E$, called the **zero-morphism** which associates to each point of the base space M, the zero vector in the linear space $E_x = q^{-1}(x)$, and with a collection of binary operations $(s_q)_x\colon E_x \times E_x \to E_x$ on each fibre, which allows one to sum vectors of the same fibre. Instead of considering collections of binary operations, one can consider the vector bundle E_2 whose fibre $(E_2)_x$ is the product $E_x \times E_x$ and then introduce a function $s_q\colon E_2 \to E$. Categorically, E_2 is the pullback of the projection with itself:

$$\begin{tikzcd} E_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\ E \arrow[r, "q"'] & M \end{tikzcd}$$

One also notices that the sum s_q is commutative, associative, and unital with the unit given by qz_q. So far, we described the algebraic structures of a vector bundle. So, the tangent bundle must be equipped with a projection $p\colon \mathrm{T}M \to M$, a zero-morphism $z\colon M \to \mathrm{T}M$, and a sum morphism $s\colon \mathrm{T}_2M \to \mathrm{T}M$, where T_2M denotes the pullback of p along itself. However, in order to give a fully geometric interpretation of an abstract object M, one also needs $\mathrm{T}M$ to be locally trivial. This means that the tangent bundle must be locally isomorphic to a product of a local chart of the original space and of a linear space.

Surprisingly, this property can be axiomatized with the introduction of the **vertical lift**, that is a map $l\colon \mathrm{T}M \to \mathrm{T}^2M$ whose codomain is the tangent bundle of the tangent bundle of M, i.e. $\mathrm{T}^2M := \mathrm{T}(\mathrm{T}M)$. To understand the role of the vertical lift, consider first the *vertical bundle*, which is the vector bundle $VM \to M$ whose fibres are the kernels of the differential of the projection. Categorically, VM is the



(a) Representation of a double tangent vector

(b) Representation of $(x, 0, 0, v) \in \mathrm{T}^2M$

Figure 2.2: Heuristic representations of the elements of T^2M.

---

pullback of $\mathrm{T}p$ along the zero morphism:

$$\begin{tikzcd} VM \arrow[r, "\xi"] \arrow[d, "p^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}^2M \arrow[d, "\mathrm{T}p"] \\ M \arrow[r, "z"'] & \mathrm{T}M \end{tikzcd}$$

where, for a map $f\colon M \to N$ of geometric spaces, locally $\mathrm{T}f\colon \mathrm{T}M \to \mathrm{T}N$ sends a pair (x, v) formed by a point x of M and a tangent vector $v \in \mathrm{T}_xM$ to $(f(x), \mathsf{d}f_x(v))$, where $\mathsf{d}f_x$ is the differential of f at x. With this definition, one can immediately see that VM is a subbundle of T^2M, via the inclusion $\xi\colon VM \to \mathrm{T}^2M$.

Thanks to the local triviality of the tangent bundle, one can locally represent a point of $\mathrm{T}M$ as a pair (x, v) formed by a point $x \in M$ and by a tangent vector $v \in \mathrm{T}_xM$. Similarly, the *double tangent bundle* T^2M can be locally represented by a tuple (x, u, v, ω) formed by a point $x \in M$, two tangent vectors $u, v \in \mathrm{T}_xM$, and a double tangent vector ω, which is a tangent vector of the space $\mathrm{T}M$ at (x, u). Informally, a tangent vector can be regarded as an infinitesimal path. Similarly, a double tangent vector can be regarded as an infinitesimal homotopy between infinitesimal paths. In Figure 2.2a we represented this intuition.

Thanks to the local triviality of $\mathrm{T}M$ and T^2M we can then introduce a function $l\colon \mathrm{T}M \to \mathrm{T}^2M$ which sends (x, v) to $(x, 0, 0, v)$. Pictorially, the element $(x, 0, 0, v)$ can be represented as in Figure 2.2b. We can also define the map $\xi\colon \mathrm{T}_2M \to \mathrm{T}^2M$, which sends a triple $(x; u, v)$ formed by a point $x \in M$ and two tangent vectors

$u, v \in \mathrm{T}_x M$ to $(x, u, 0, v)$. Notice that:

$$\xi = (z_{\mathrm{T}} \times l)\mathrm{T}s$$

It is straightforward to prove that, by definition, $\xi \mathrm{T}p = \xi p_{\mathrm{T}} p z$, which implies that the image of ξ is a subbundle of the vertical bundle. Categorically, from the universality of the pullback, we define a unique dashed morphism:

$$\begin{tikzcd}
\mathrm{T}_2 M \arrow[rr, "\xi"] \arrow[dr, dashed] \arrow[dd, "\pi_1"'] & & \mathrm{T}^2 M \arrow[d, equal] \\
& VM \arrow[r, "\xi"] \arrow[d, "p^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}^2 M \arrow[d, "\mathrm{T}p"] \\
\mathrm{T}M \arrow[r, "p"'] & M \arrow[r, "z"'] & \mathrm{T}M
\end{tikzcd}$$

However, thanks to the local representation of $\mathrm{T}^2 M$ we conclude that if (x, u, v, ω) belongs in the image of ξ, i.e. is a point of the vertical bundle, then $v = 0$, since $\mathrm{T}p(x, u, v, \omega) = (x, v)$. This implies that the unique morphism $\mathrm{T}_2 M \to VM$ is an isomorphism. This is precisely the universality axiom formulated by Cockett and Cruttwell to axiomatize the local triviality of the tangent bundle in a tangent category.

In Figure 2.2a we schematically represented the idea of a double tangent vector, that is a point in $\mathrm{T}^2 M$, as an infinitesimal homotopy between infinitesimal paths. In this pictorial interpretation, a tangent vector u is an infinitesimal path which "moves" a point x of M of an infinitesimal distance in the direction and orientation of the arrow u. In the picture, we represented the endpoint of this infinitesimal path as $x + \mathsf{d}_u x$, where $\mathsf{d}_u x$ represents an infinitesimal quantity in the direction and orientation of the arrow u. Similarly, the vector v can be regarded as an infinitesimal path from x to $x + \mathsf{d}_v x$. So, an infinitesimal homotopy ω can be regarded as an infinitesimal path of infinitesimal paths. Notice that since the homotopy is infinitesimal, it can be regarded as an infinitesimal parallelogram, that is the left and right sides and the top and bottom sides must be parallel. Using the notation introduced, one finds that the right bottom corner in the picture can be represented in two different ways: $x + \mathsf{d}_u x + \mathsf{d}_v x + \mathsf{d}_v \mathsf{d}_u x$ and $x + \mathsf{d}_v x + \mathsf{d}_u x + \mathsf{d}_u \mathsf{d}_v x$. Since these are two representations of the same point one wants $\mathsf{d}_u\mathsf{d}_v x = \mathsf{d}_v\mathsf{d}_u x$. This heuristic explanation in differential geometry is equivalent to the symmetry of the Hessian matrix, that is the commutativity of the partial derivatives $\partial_i\partial_j f = \partial_j\partial_i f$. To axiomatize this symmetry one introduces an isomorphism $c\colon \mathrm{T}^2M \to \mathrm{T}^2M$, called the **canonical flip**. This is precisely the axiomatization of a tangent category proposed by Cockett and Cruttwell in [12]. Let's recall this concept formally. Let's start recalling the notion of additive bundle.

Definition* **2.1.** *In a category $\mathbb{X}$, an additive bundle consists of a morphism $q\colon E \to M$ which, in the slice category $\mathbb{X}/M$ of $\mathbb{X}$ over M is a commutative monoid, with respect to the Cartesian product of $\mathbb{X}/M$, i.e pullbacks of q along itself in $\mathbb{X}$. More concretely, an additive bundle is a morphism $q\colon E \to M$ equipped with a section $z\colon M \to E$, for which the n-fold pullbacks $q_n\colon E_n \to M$ of q along itself exist, and together with a morphism $s\colon E_2 \to E$ for which $sq = q_2$ which satisfies associativity and unitality.*

Definition* **2.2.** *A **tangent structure** $\mathbb{T}$ over a category $\mathbb{X}$ consists of the following data:*

1. *An endofunctor $\mathrm{T}\colon \mathbb{X} \to \mathbb{X}$, called the **tangent bundle functor**;*

2. *A natural transformation $p\colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathbb{X}}$, called the **projection**, for which the n-fold pullbacks of p along itself exist. Such pullbacks are denoted by T_n and the corresponding projections by $\pi_k\colon \mathrm{T}_n \Rightarrow \mathrm{T}$;*

3. *A natural transformation $z\colon \mathsf{id}_{\mathbb{X}} \Rightarrow \mathrm{T}$, called the **zero morphism**;*

4. *A natural transformation $s\colon \mathrm{T}_2 \Rightarrow \mathrm{T}$, called the **sum morphism**;*

such that, for every object M of $\mathbb{X}$, the triple $(p\colon \mathrm{T}M \to M, z\colon M \to \mathrm{T}M, s\colon \mathrm{T}_2M \to \mathrm{T}M)$ constitutes an additive bundle;

5. *A natural transformation $l\colon \mathrm{T} \Rightarrow \mathrm{T}^2$, called the **vertical lift**, for which (l, z) is a morphism of additive bundles;*

where, given two additive bundles $(q\colon E \to M, z_q\colon M \to E, s_q\colon E_2 \to E)$ and $(q'\colon E' \to M', z'_q\colon M' \to E', s'_q\colon E'_2 \to E')$, we call a pair $(f\colon E \to E', g\colon M \to M')$ a morphism *of additive bundles if the following diagrams commute:*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\ M \arrow[r, "g"'] & M' \end{tikzcd} \quad \begin{tikzcd} E \arrow[r, "f"] & E' \\ M \arrow[u, "z_q"] \arrow[r, "g"'] & M' \arrow[u, "z'_q"'] \end{tikzcd} \quad \begin{tikzcd} E_2 \arrow[r, "f\times_M f"] \arrow[d, "s_q"'] & E'_2 \arrow[d, "s'_q"] \\ E \arrow[r, "f"'] & E' \end{tikzcd}$$

6. *A natural transformation $c\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$, called the **canonical flip**, for which $(c, \mathsf{id}_{\mathrm{T}})$ is a morphism of additive bundles.*

Moreover, we have the following compatibilities:

$$\begin{tikzcd} \mathrm{T}^2M \arrow[r, "c"] \arrow[dr, equal] & \mathrm{T}^2M \arrow[d, "c"] \\ & \mathrm{T}^2M \end{tikzcd} \quad \begin{tikzcd} \mathrm{T}M \arrow[r, "l"] \arrow[dr, "l"'] & \mathrm{T}^2M \arrow[d, "c"] \\ & \mathrm{T}^2M \end{tikzcd} \quad \begin{tikzcd} \mathrm{T}^2M \arrow[rr, "c"] \arrow[d, "l_{\mathrm{T}}"'] & & \mathrm{T}^2M \arrow[d, "\mathrm{T}l"] \\ \mathrm{T}^3M \arrow[r, "\mathrm{T}c"'] & \mathrm{T}^3M \arrow[r, "c_{\mathrm{T}}"'] & \mathrm{T}^3M \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{T}M \arrow[r, "l"] \arrow[d, "l"'] & \mathrm{T}^2M \arrow[d, "l_{\mathrm{T}}"] \\ \mathrm{T}^2M \arrow[r, "\mathrm{T}l"'] & \mathrm{T}^3M \end{tikzcd} \quad \begin{tikzcd} \mathrm{T}^3M \arrow[r, "\mathrm{T}c"] \arrow[d, "c_{\mathrm{T}}"'] & \mathrm{T}^3M \arrow[r, "c_{\mathrm{T}}"] & \mathrm{T}^3M \arrow[d, "\mathrm{T}c"] \\ \mathrm{T}^3M \arrow[r, "\mathrm{T}c"'] & \mathrm{T}^3M \arrow[r, "c_{\mathrm{T}}"'] & \mathrm{T}^3M \end{tikzcd}$$

Finally, the vertical lift is universal, that is the following diagram:

$$\begin{tikzcd} \mathrm{T}_2M \arrow[r, "\xi"] \arrow[d, "\pi_1 p"'] & \mathrm{T}^2M \arrow[d, "\mathrm{T}p"] \\ M \arrow[r, "z"'] & \mathrm{T}M \end{tikzcd}$$

is a pullback diagram, where:

$$\xi := (l \times_M z_{\mathrm{T}})\mathrm{T}s\colon \mathrm{T}_2 \Rightarrow \mathrm{T}^2$$

*A category $\mathbb{X}$ with a tangent structure $\mathbb{T}$ forms a **tangent category**. A tangent category **has negatives** if there is also an extra:*

7. *A natural transformation $n\colon \mathrm{T} \Rightarrow \mathrm{T}$, called the **negation**, for which the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{T}M \arrow[r, "\langle pz,n\rangle"] \arrow[dr, equal] & \mathrm{T}_2M \arrow[d, "s"] \\
& \mathrm{T}M
\end{tikzcd}$$

Remark 2.3. The original definition of a tangent category due to Rosický was equivalent to a tangent category with negatives, in the sense we consider here.

Notation 2.4. A generic tangent category is denoted by $(\mathbb{X}, \mathbb{T})$ where $\mathbb{X}$ represents the underlying category and $\mathbb{T}$ is the tangent structure. In particular, the tangent bundle functor is denoted by the same letter as the one used for the tangent structure but in a different font, i.e. if $\mathbb{T}$ denotes the tangent structure then T denotes the associated tangent bundle functor; the projection, the zero morphism, the sum morphism, the vertical lift, and the canonical flip are denoted respectively by the letters p, z, s, l, and c.

When the tangent structure has negatives, the negation is denoted by n. Moreover, we adopt the following convention: when the symbol used to indicate the tangent structure is decorated with a superscript or a subscript, the same superscript or subscript is applied to the tangent bundle functor, the projection, the zero morphism, the sum morphism, the vertical lift, and the canonical flip. The same convention extends to the negation if the tangent structure has negatives.

Sometimes, for the sake of clarity, we add the superscript $^{(\mathrm{T})}$ to the natural transformations p, z, s, l, c and n to stress the fact that these are part of the tangent structure $\mathbb{T}$, where T denotes the tangent bundle functor.

In the original definition of [12], the map $\xi\colon \mathrm{T}_2M \to \mathrm{T}^2M$, defined by $\xi\colon = (z_\mathrm{T} \times l)\mathrm{T}s$, was denoted by the letter v. Since v is already adopted in other contexts in this thesis, for the sake of clarity, we decided to employ the Greek letter ξ, instead.

Morphisms of tangent categories come in different flavours and, for the purpose of our discussion, we need to distinguish them.

Definition* 2.5. *Given two tangent categories $(\mathbb{X}, \mathbb{T})$ and $(\mathbb{X}', \mathbb{T}')$, a **lax tangent morphism** $(F, \alpha)\colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}', \mathbb{T}')$ consists of a functor $F\colon \mathbb{X} \to \mathbb{X}'$ together with a natural* *transformation* $\alpha\colon F\circ\mathrm{T}\Rightarrow\mathrm{T}'\circ F$, *called the* ***lax distributive law*** *of the morphism, compatible with the two tangent structures as follows:*

$$\begin{tikzcd} F\circ\mathrm{T} \arrow[r, "\alpha"] \arrow[d, "Fp"'] & \mathrm{T}'\circ F \arrow[d, "p'F"] \\ F \arrow[r, equal] & F \end{tikzcd} \quad \begin{tikzcd} F\circ\mathrm{T} \arrow[r, "\alpha"] & \mathrm{T}'\circ F \\ F \arrow[u, "Fz"] \arrow[r, equal] & F \arrow[u, "z'F"'] \end{tikzcd} \quad \begin{tikzcd} F\circ\mathrm{T}_2 \arrow[r, "\langle F\pi_1\alpha, F\pi_2\alpha\rangle"] \arrow[d, "Fs"'] & \mathrm{T}'_2\circ F \arrow[d, "s'F"] \\ F\circ\mathrm{T} \arrow[r, "\alpha"'] & \mathrm{T}'\circ F \end{tikzcd}$$

$$\begin{tikzcd} F\circ\mathrm{T} \arrow[rr, "\alpha"] \arrow[d, "Fl"'] & & \mathrm{T}'\circ F \arrow[d, "l'F"] \\ F\circ\mathrm{T}^2 \arrow[r, "\alpha\mathrm{T}"'] & \mathrm{T}'\circ F\circ\mathrm{T} \arrow[r, "\mathrm{T}'\alpha"'] & \mathrm{T}'^2\circ F \end{tikzcd}$$

$$\begin{tikzcd} F\circ\mathrm{T}^2 \arrow[r, "\alpha\mathrm{T}"] \arrow[d, "Fc"'] & \mathrm{T}'\circ F\circ\mathrm{T} \arrow[r, "\mathrm{T}'\alpha"] & \mathrm{T}'\circ F \arrow[d, "c'F"] \\ F\circ\mathrm{T}^2 \arrow[r, "\alpha\mathrm{T}"'] & \mathrm{T}'\circ F\circ\mathrm{T} \arrow[r, "\mathrm{T}'\alpha"'] & \mathrm{T}'^2\circ F \end{tikzcd}$$

Similarly, a ***colax tangent morphism*** $(G,\beta)\colon(\mathbb{X},\mathbb{T})\nrightarrow(\mathbb{X}',\mathbb{T}')$ *consists of a functor* $G\colon\mathbb{X}\to\mathbb{X}'$ *and a natural transformation* $\beta\colon\mathrm{T}'\circ G\Rightarrow G\circ\mathrm{T}$*, called the* ***colax distributive law*** *of the morphism, which is compatible with the tangent structures precisely as for a lax tangent morphism morphism but with the direction of the distributive law reversed.*

A ***strong tangent morphism*** *is lax tangent morphism whose distributive law is invertible. Finally, a strong tangent morphism is* ***strict*** *if the distributive law is the identity morphism.*

Notation 2.6. When (F,α) is a strong tangent morphism, (F,α^{-1}) is a colax tangent morphism, so in the future, we call both (F,α) and (F,α^{-1}) strong tangent morphisms. Since, by definition, the distributive law of a strict tangent morphism is trivial, we refer to a functor as a strict tangent morphism, omitting explicitly the distributive law.

To stress the difference between lax and colax tangent morphisms, we denote the former one by $(F,\alpha)\colon(\mathbb{X},\mathbb{T})\to(\mathbb{X}',\mathbb{T}')$ and the latter by $(G,\beta)\colon(\mathbb{X},\mathbb{T})\nrightarrow(\mathbb{X}',\mathbb{T}')$. We interchangeably adopt the two notations for strong and strict tangent morphisms according to the context.

Definition* 2.7. *Given two lax tangent morphisms $(F,\alpha),(F',\alpha')\colon(\mathbb{X},\mathbb{T})\to(\mathbb{X}',\mathbb{T}')$, a* ***lax tangent natural transformation*** *$\varphi\colon(F,\alpha)\Rightarrow(F',\alpha')$ consists of a natural transformation $\varphi\colon F\Rightarrow F'$ satisfying the following compatibility condition:*

$$\begin{tikzcd}
F\mathrm{T}A \arrow[r, "\alpha"] \arrow[d, "\varphi\mathrm{T}"'] & \mathrm{T}'FA \arrow[d, "\mathrm{T}'\varphi"] \\
F'\mathrm{T}A \arrow[r, "\alpha'"'] & \mathrm{T}'F'A
\end{tikzcd}$$

Similarly, given two colax tangent morphisms $(G,\beta),(G',\beta')\colon(\mathbb{X},\mathbb{T})\nrightarrow(\mathbb{X}',\mathbb{T}')$, a ***colax tangent natural transformation*** *$\psi\colon(G,\beta)\Rightarrow(G',\beta')$ consists of a natural transformation $\psi\colon G\Rightarrow G'$ satisfying the dual of the compatibility condition of a lax tangent natural transformation, i.e.:*

$$\begin{tikzcd}
\mathrm{T}'GA \arrow[r, "\beta"] \arrow[d, "\mathrm{T}'\psi"'] & G\mathrm{T}A \arrow[d, "\psi\mathrm{T}"] \\
\mathrm{T}'G'A \arrow[r, "\beta'"'] & G'\mathrm{T}A
\end{tikzcd}$$

Finally, a ***double tangent cell****:*

$$\begin{tikzcd}
(\mathbb{X}_\circ,\mathbb{T}_\circ) \arrow[r, "{(F,\alpha)}"] \arrow[d, "{(G_\circ,\beta_\circ)}"', "\shortmid" marking] & (\mathbb{X}_\bullet,\mathbb{T}_\bullet) \arrow[d, "{(G_\bullet,\beta_\bullet)}", "\shortmid" marking] \arrow[dl, Rightarrow, "\theta"] \\
(\mathbb{X}'_\circ,\mathbb{T}'_\circ) \arrow[r, "{(F',\alpha')}"'] & (\mathbb{X}'_\bullet,\mathbb{T}'_\bullet)
\end{tikzcd}$$

where the horizontal morphisms are lax tangent morphisms while the vertical ones are colax tangent morphisms, is a natural transformation $\theta\colon F'\circ G_\circ\Rightarrow G_\bullet\circ F$ satisfying the following compatibility condition:

$$\begin{tikzcd}
F'\mathrm{T}'_\circ G_\circ A \arrow[r, "\alpha' G_\circ"] \arrow[d, "F'\beta_\circ"'] & \mathrm{T}'_\bullet F' G_\circ A \arrow[r, "\mathrm{T}'_\bullet\theta"] & \mathrm{T}'_\bullet G_\bullet FA \arrow[d, "\beta_\bullet F"] \\
F' G_\circ \mathrm{T}_\circ A \arrow[r, "\theta\mathrm{T}_\circ"] & G_\bullet F\mathrm{T}_\circ A \arrow[r, "G_\bullet\alpha"] & G_\bullet\mathrm{T}_\bullet FA
\end{tikzcd}$$

Notation 2.8. We denote by TngCat the 2-category of tangent categories, whose 1-morphisms are lax tangent morphisms, and 2-morphisms are lax tangent natural transformations. Similarly, the 2-category of tangent categories, whose 1-morphisms are colax tangent morphisms, and 2-morphisms are colax tangent natural transformations is denoted by $\mathsf{TngCat}_{\mathsf{co}}$. The 2-subcategory of TngCat whose 1-morphisms are strong is denoted by $\mathsf{TngCat}_{\cong}$, and the 2-subcategory of TngCat whose 1-morphisms are strict is denoted by $\mathsf{TngCat}_{=}$. Finally, we abuse notation and we also denote by TngCat, $\mathsf{TngCat}_{\mathsf{co}}$, $\mathsf{TngCat}_{\cong}$, and $\mathsf{TngCat}_{=}$ the corresponding underlying 1-categories.

Example 2.9. Every category $\mathbb{X}$ comes equipped with a trivial tangent structure whose tangent bundle functor, projection, zero, sum, vertical lift, and canonical flip are all identities.

Example 2.10. The archetypical example of a tangent category is the category of finite-dimensional smooth manifolds. The tangent bundle functor is the functor which sends a smooth manifold M to its tangent bundle $\mathrm{T}M$, in the classical sense, and a smooth function $f\colon M \to N$ to the function $\mathrm{T}f\colon \mathrm{T}M \to \mathrm{T}N$ which in local coordinates sends a pair (x, v) formed by a point x of M and a tangent vector v over x of M to $(f(x), \mathsf{d}_x f(v))$, where $\mathsf{d}_x f$ is the differential of f at x. The projection, the zero morphism, the sum morphism, the vertical lift, and the canonical flip are precisely the homonymous smooth maps of the category of smooth manifolds (cf. [12, Section 6] for a construction of this tangent structure employing manifold completion).

Example 2.11. Recall that a category **with biproducts**, or sometimes called a **semi-additive** category, is a category with finite products, denoted by $\times$, finite coproducts, denoted by $+$, and for which the unique morphisms $X_1 + \ldots + X_n \to X_1 \times \cdots \times X_n$, induced by universality, are isomorphisms, for every non-negative integer n. Semi-additive categories are enriched over the monoidal category of commutative monoids. To see this, let $\mathbb{X}$ be a category with biproducts and let $f, g\colon X \to Y$ be two morphisms. Let's define $f + g\colon X \to Y$ as follows:

$$f + g\colon X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \cong Y + Y \xrightarrow{+} Y$$

where $\Delta\colon = \langle \mathsf{id}, \mathsf{id}\rangle$ and $+\colon = [\mathsf{id}, \mathsf{id}]$ are the unique morphisms induced by the universality of the binary product and the binary coproduct, respectively. Finally, define $0\colon X \to Y$ to be the morphism:

$$0\colon X \xrightarrow{!} \mathbb{1} \cong 0 \xrightarrow{!} Y$$

where $\mathbb{1}$ denotes the terminal object and 0 the initial object of $\mathbb{X}$. Since products and coproducts are isomorphic, we usually use the word **biproducts** for both and the notation $\oplus$.

Every category $\mathbb{X}$ with biproducts has a canonical tangent structure, denoted by $\mathbb{L}$, defined as follows:

tangent bundle functor The tangent bundle functor $\mathrm{L}\colon \mathbb{X} \to \mathbb{X}$ is the diagonal functor, that is the functor which sends an object X to $X \oplus X$ and a morphism $f\colon X \to Y$ to $f \oplus f\colon X \oplus X \to Y \oplus Y$;

projection The projection $p^{(\mathbb{L})}\colon \mathrm{L} \Rightarrow \mathsf{id}_{\mathbb{X}}$ is the projection along the first component:

$$p := \pi_1 = [\mathsf{id}_X, !]\colon X \oplus X \to X \oplus \mathbb{1} \cong X$$

zero morphism The zero morphism $z^{(\mathbb{L})}\colon \mathsf{id}_{\mathbb{X}} \Rightarrow \mathrm{L}$ is the inclusion in the first component:

$$z^{(\mathbb{L})} := \iota_1 = \langle \mathsf{id}_X, !\rangle\colon X \to X \oplus X$$

n-fold pullback The n-fold pullback $\mathrm{L}_n\colon \mathbb{X} \to \mathbb{X}$ sends an object X to $\underbrace{X \oplus \cdots \oplus X}_{n+1 \text{ times}}$ and a morphism $f\colon X \to Y$ to $\underbrace{f \oplus \cdots \oplus f}_{n+1 \text{ times}}$. The projections $\pi_k^{(\mathbb{L})}\colon \mathrm{L}_n \Rightarrow \mathrm{L}$ send the first component to the first, and the $k+1$-th one to the second one:

$$\pi_k^{(\mathbb{L})} = \langle \pi_1 \iota_1, \pi_{k+1} \iota_2\rangle\colon X \oplus \cdots \oplus X \to X \oplus X$$

sum morphism The sum morphism $s^{(\mathbb{L})}\colon \mathrm{L}_2 \Rightarrow \mathrm{L}$ sends the first component to the first component and sums the second and the third one together and sends them to the second component:

$$s^{(\mathbb{L})} = \mathsf{id}_X \oplus +\colon X \oplus X \oplus X \to X \oplus X$$

vertical lift The vertical lift $l^{(\mathbb{L})} \colon \mathbb{L} \Rightarrow \mathbb{L}^2$ sends the first component to the first and the second one to the fourth one:

$$l^{(\mathbb{L})} = \langle \iota_1\pi_1, \iota_2\pi_4 \rangle \colon X \oplus X \to X \oplus X \oplus X \oplus X$$

canonical flip The canonical flip $c^{(\mathbb{L})} \colon \mathbb{L}^2 \Rightarrow \mathbb{L}^2$ flips the order of the internal components:

$$c^{(\mathbb{L})} = \mathsf{id}_X \oplus \tau \oplus \mathsf{id}_X \colon X \oplus X \oplus X \oplus X \to X \oplus X \oplus X \oplus X$$

where $\tau = \langle \iota_2\pi_1, \iota_1\pi_2 \rangle \colon X \oplus X \to X \oplus X$.

We want to point out that even though the category $\mathbb{X}$ might not have all pullbacks, the existence of the n-fold pullbacks of the projection along itself and the universality of the vertical lift are guaranteed by the existence of biproducts.

If $\mathbb{X}$ is enriched over the monoidal category of Abelian groups, so it is sometimes called **additive**, then we can also introduce negatives as follows:

negation The negation $n^{(\mathbb{L})} \colon \mathbb{L} \Rightarrow \mathbb{L}$ sends the first component to the first one and sends the second one to the second one with a negative sign:

$$n^{(\mathbb{L})} = \mathsf{id}_X \oplus - \colon X \oplus X \to X \oplus X$$

An example of a category with biproducts is the category of R-modules over a commutative and unital ring R.

Remark 2.12. Given a category $\mathbb{X}$ with biproducts, also the opposite category $\mathbb{X}^{\mathsf{op}}$ has biproducts. Therefore, from Example 2.11, we conclude that also $\mathbb{X}^{\mathsf{op}}$ has a tangent structure, denoted by $\mathbb{T}$. In Section 2.2.6, we show that, under some conditions, the opposite of a tangent category is also a tangent category. A category with biproducts satisfies those conditions, therefore, the opposite of the category $\mathbb{X}$ comes equipped with another tangent structure. By direct inspection, one can see that these two tangent structures over $\mathbb{X}^{\mathsf{op}}$ are isomorphic.

Example 2.13. The category cRing of commutative and unital rings has a tangent structure, denoted by $\mathbb{L}$, so defined:

tangent bundle functor The tangent bundle functor $\mathbb{T}\colon \mathsf{cRing} \to \mathsf{cRing}$ sends a ring R to the ring of dual numbers $R\langle\varepsilon\rangle = R[x]/(x^2)$, that is the ring obtained by quotienting the ring $R[x]$ of polynomials in 1 variable and coefficients in R, by the ideal generated by x^2. Moreover, $\mathbb{T}$ sends a morphism $: R \to R'$ of rings to the morphism $\mathbb{T}f\colon R\langle\varepsilon\rangle \to R'\langle\varepsilon\rangle$, which sends the generator ε to itself;

The elements of $R\langle\varepsilon\rangle$ are terms of the form $a + b\varepsilon$ with $a, b \in R$ and $\varepsilon^2 = 0$. So, $\mathbb{T}f$ sends $a + b\varepsilon$ to $f(a) + f(b)\varepsilon$. An equivalent description of $R\langle\varepsilon\rangle$ is given by the so-called semi-direct product. Consider the ring formed by pairs (a, b) with $a, b \in R$. The multiplication is defined as follows:

$$(a, b)(a', b') := (aa', ab' + ba')$$

and the unit is $(1, 0)$. This is a commutative and unital ring, denoted by $R \ltimes R$, isomorphic to $R\langle\varepsilon\rangle$ via the morphism $(a, b) \mapsto a + b\varepsilon$.

projection The projection $p^{(\mathbb{T})}\colon \mathbb{T} \Rightarrow \mathsf{id}_{\mathsf{cRing}}$ is the map which sends the generator ε to 0:

$$p\colon R\langle\varepsilon\rangle \to R$$
$$p^{(\mathbb{T})}(a + b\varepsilon) := a$$

zero morphism The zero morphism $z^{(\mathbb{T})}\colon \mathsf{id}_{\mathsf{cRing}} \Rightarrow \mathbb{T}$ is the inclusion of R into $R\langle\varepsilon\rangle$:

$$z^{(\mathbb{T})}\colon R \to R\langle\varepsilon\rangle$$
$$z^{(\mathbb{T})}(a) := a = a + 0\varepsilon$$

n-fold pullback The n-fold pullback $\mathbb{T}_n\colon \mathsf{cRing} \to \mathsf{cRing}$ sends a ring R to $R\langle\varepsilon_1, \dots, \varepsilon_n\rangle$, which is the ring obtained by quotienting $R[x_1, \dots, x_n]$ by the ideals generated by all $x_i x_j$, for $i, j = 1, \dots, n$. Moreover, it sends a morphism $f\colon R \to R'$ to $\mathbb{T}_n f$ which sends each generators ε_k to ε_k. The projections $\pi_k^{(\mathbb{T})}\colon \mathbb{T}_n \Rightarrow \mathbb{T}$ send ε_i to 0 for every $i \neq k$ and ε_k to ε:

$$\pi_k^{(\mathbb{T})}\colon R\langle\varepsilon_1, \dots, \varepsilon_n\rangle \to R\langle\varepsilon\rangle$$
$$\pi_k^{(\mathbb{T})}(a + b_1\varepsilon_1 + \dots + b_n\varepsilon_n) := a + b_k\varepsilon$$

sum morphism The sum morphism $s^{(\mathbb{L})}\colon \mathbb{L}_2 \Rightarrow \mathbb{L}$ sends both ε_1 and ε_2 to ε:

$$s^{(\mathbb{L})}\colon R\langle\varepsilon_1, \varepsilon_2\rangle \to R\langle\varepsilon\rangle$$
$$s^{(\mathbb{L})}(a + b_1\varepsilon + b_2\varepsilon_2) := a + (b_1 + b_2)\varepsilon$$

vertical lift The vertical lift $l^{(\mathbb{L})}\colon \mathbb{L} \Rightarrow \mathbb{L}^2$ sends ε to $\varepsilon'\varepsilon$:

$$l^{(\mathbb{L})}\colon R\langle\varepsilon\rangle \to R\langle\varepsilon\rangle\langle\varepsilon'\rangle$$
$$l^{(\mathbb{L})}(a + b\varepsilon) := a + b\varepsilon'\varepsilon$$

canonical flip The canonical flip $c^{(\mathbb{L})}\colon \mathbb{L}^2 \Rightarrow \mathbb{L}^2$ sends ε to ε' and ε' to ε:

$$c^{(\mathbb{L})}\colon R\langle\varepsilon\rangle\langle\varepsilon'\rangle \to R\langle\varepsilon\rangle\langle\varepsilon'\rangle$$
$$c^{(\mathbb{L})}(a + b\varepsilon + c\varepsilon' + d\varepsilon'\varepsilon) := a + c\varepsilon + b\varepsilon' + d\varepsilon'\varepsilon$$

Finally, since rings have also negatives, we can define the negation as follows:

negation The negation $n^{(\mathbb{L})}\colon \mathbb{L} \Rightarrow \mathbb{L}$ sends ε to $-\varepsilon$:

$$n^{(\mathbb{L})}\colon R\langle\varepsilon\rangle \to R\langle\varepsilon\rangle$$
$$n^{(\mathbb{L})}(a + b\varepsilon) := a - b\varepsilon$$

Example 2.14. Interestingly, also the opposite of the category of commutative and unital rings $\mathsf{cRing}^{\mathsf{op}}$ has a tangent structure that can be described as follows:

tangent bundle functor The tangent bundle functor $\mathrm{T}\colon \mathsf{cRing}^{\mathsf{op}} \to \mathsf{cRing}^{\mathsf{op}}$ sends a ring R to the ring $R^{\mathbb{Z}\langle\varepsilon\rangle}$, which is an exponential object in the category cRing and that corresponds to the symmetric algebra of the module of Kähler differentials of R (cf. [12, Section 5.4]). Moreover, it sends a morphism $f\colon R \to S$ of rings to $f^{\mathbb{Z}\langle\varepsilon\rangle}\colon R^{\mathbb{Z}\langle\varepsilon\rangle} \to S^{\mathbb{Z}\langle\varepsilon\rangle}$;

projection The projection $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathsf{cRing}} \Rightarrow \mathrm{T}$ is induced by the augmentation map of $\mathbb{Z}\langle\varepsilon\rangle$, which sends $a + b\varepsilon$ to a;

zero morphism The zero morphism $z^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathsf{cRing}}$ is induced by the inclusion of $\mathbb{Z}$ into $\mathbb{Z}\langle\varepsilon\rangle$, i.e. $\mathbb{Z} \ni a \mapsto a \in \mathbb{Z}\langle\varepsilon\rangle$;

n-fold pullback The n-fold pushout (in cRing) $\mathrm{T}_n \colon \mathsf{cRing} \to \mathsf{cRing}$ sends a ring R to $R^{\mathbb{Z}\langle\varepsilon_1,\dots,\varepsilon_n\rangle}$. Moreover, T_n sends a morphism $f \colon R \to S$ of rings to $f^{\mathbb{Z}\langle\varepsilon_1,\dots,\varepsilon_n\rangle} \colon R^{\mathbb{Z}\langle\varepsilon_1,\dots,\varepsilon_n\rangle} \to S^{\mathbb{Z}\langle\varepsilon_1,\dots,\varepsilon_n\rangle}$;

sum morphism The sum morphism $s^{(\mathrm{T})} \colon \mathrm{T} \Rightarrow \mathrm{T}_2$ is induced by the sum morphism $\mathbb{Z}\langle\varepsilon_1,\varepsilon_2\rangle \ni \varepsilon_1,\varepsilon_2 \mapsto \varepsilon \in \mathbb{Z}\langle\varepsilon\rangle$;

vertical lift The vertical lift $l^{(\mathrm{T})} \colon \mathrm{T}^2 \Rightarrow \mathrm{T}$ is induced by the vertical lift $\mathbb{Z}\langle\varepsilon\rangle \ni \varepsilon \mapsto \varepsilon'\varepsilon \in \mathbb{Z}\langle\varepsilon\rangle\langle\varepsilon'\rangle$;

canonical flip The canonical flip $c^{(\mathrm{T})} \colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$is induced by the flip $\mathbb{Z}\langle\varepsilon'\rangle\varepsilon\rangle \ni \varepsilon,\varepsilon' \mapsto \varepsilon',\varepsilon \in \mathbb{Z}\langle\varepsilon\rangle\langle\varepsilon'\rangle$.

Finally, since rings have also negatives, we can define the negation as follows:

negation The negation $n^{(\mathrm{T})} \colon \mathrm{T} \Rightarrow \mathrm{T}$ is induced by the negation $\mathbb{Z}\langle\varepsilon\rangle \ni \varepsilon \mapsto -\varepsilon \in \mathbb{Z}\langle\varepsilon\rangle$.

An equivalent characterization of this tangent structure will be given in Example 3.16.

At this point, one could notice that in the axioms of a tangent category two important properties of the tangent bundle in the context of differential geometry are missing. The first is the absence of any reference to the ring of real numbers. The local approximation of a manifold with an open subset of $\mathbb{R}^n$ is probably one of the most memorable aspects of differential geometry. In particular, the continuity of real numbers allows one to interpret tangent vectors as tiny paths. Surprisingly, Cockett and Cruttwell in [14], suggested a way to reconstruct the ring of real numbers within a tangent category. Their intuition was to define a *curve object* in a tangent category as an object together with a fixed point and a vector field representing an infinitesimal displacement to add to the fixed point recursively. This shows that the axioms of a tangent category are sufficient to have the geometric intuition of infinitesimal paths and to reconstruct a "ri(n)g of real numbers". At this point, it is important to mention that not every tangent category admits a curve object.

The second missing characteristic in the axiomatization of tangent categories is the idea that the tangent space at a point should *approximate* the geometric space within a local neighbourhood of the point. The notion of locality, surprisingly, is not a natural notion in tangent category theory due to the lack of a topology. In [12, Chapter 5], Cockett and Cruttwell proposed the notion of **restriction** tangent category, aiming to axiomatize partial maps, that are functions whose domain is a subset of a given set.

Finally, in differential geometry, the ring of real numbers plays another important role: it provides an action over the fibres of the tangent bundle, i.e. the scalar multiplication, so that each fibre becomes a real vector space. Surprisingly, when a *differential* curve object exists in a tangent category, such an object acts over each fibre of the tangent bundle of any object (see [14, Section 5.4]). In particular, the ring of real numbers is a differential curve object for the tangent category of smooth manifolds. This shows that, even without an explicit reference to the ring of real numbers, tangent category theory is capable of capturing geometric aspects comparable to the ones studied by differential geometry.

According to the philosophy of tangent category theory, a tangent structure is the context which establishes both a notion of linearity and what it means for a space to be locally linear. Different tangent structures give different notions of linearity and of local linear behaviour. For example, in the *trivial tangent structure* described in Example 2.9 every object is linear. To put this in a slogan: *linearity and local linear behaviour are contextual notions and such a context is established by a tangent structure*. Crucially, for defining the notion of a curve object, encoding the local triviality condition of the tangent bundle, and finally, establishing the notion of linearity and local linear behaviour, the universality of the vertical lift is the key ingredient (cf.'[11] and [14]).

2.2 Tangent category concepts

So far, we introduced the main ideas that led to the axiomatization of a tangent category. We also briefly discussed the main philosophy underpinning this theory, encapsulated in the slogan which sees linearity and local linear behaviour of a geometric space as contextual notions. To motivate this idea, we dedicate this section to reviewing some key concepts of tangent category theory.

2.2.1 Vector fields

Probably one of the most important concepts in differential geometry is the notion of vector field. Informally, a vector field is a collection of vectors spread along the entire space which vary smoothly. Mathematically, a vector field is a section of the projection of the tangent bundle of a space. This definition has an immediate generalization in the context of tangent category theory. Let's recall this definition, introduced in [53].

Definition* 2.15. *A **vector field** on an object M of a tangent category $(\mathbb{X}, \mathbb{T})$ is a morphism $v\colon M \to \mathrm{T}M$, which is a section of the projection, that is $vp = \mathsf{id}_M$.*

Crucially, the vector fields of a given space in differential geometry exhibit an important algebraic structure: they form a Lie algebra. As initially shown by Rosický and subsequently revisited by Cockett and Cruttwell in [13], a tangent category with negatives exhibits such a structure on the set of vector fields over a fixed object. In particular, the Lie bracket of two vector fields $u, v\colon M \to \mathrm{T}M$ is defined as follows:

$$[u, v] := \{u\mathrm{T}v - v\mathrm{T}uc\}$$

where c denotes the canonical flip, for two maps $f, g\colon N \to \mathrm{T}^2M$, $f - g$ denotes $\langle f, gn_\mathrm{T}\rangle s_\mathrm{T}$ and where, for a morphism $f\colon N \to \mathrm{T}^2M$ for which $f\mathrm{T}p = fp_\mathrm{T}pz$, $\{f\}\colon N \to \mathrm{T}M$ is $\tilde{f}\pi_2$ where $\tilde{f}$ denotes the unique morphism defined by the universality of the vertical lift:

$$\begin{tikzcd}
N \arrow[rr, "f"] \arrow[d, "f"'] \arrow[dr, dashed, "\tilde{f}"] & & \mathrm{T}^2M \arrow[d, equal] \\
\mathrm{T}^2M \arrow[d, "p_\mathrm{T}"'] & \mathrm{T}_2M \arrow[r, "\xi"] \arrow[d, "\pi_1 p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}^2M \arrow[d, "\mathrm{T}p"] \\
\mathrm{T}M \arrow[r, "p"'] & M \arrow[r, "z"'] & \mathrm{T}M
\end{tikzcd}$$

2.2.2 Differential objects and linearity

We want to motivate our belief that a tangent structure provides the context to introduce a notion of linearity by recalling the notion of a differential object. To do so, first, recall that a **Cartesian** tangent category $(\mathbb{X}, \mathbb{T})$ is a category $\mathbb{X}$ with finite products, equipped with a tangent structure $\mathbb{T}$ whose tangent bundle functor preserves finite products (see the original [12, Definition 2.8]).

Definition* 2.16. *A **differential object** in a Cartesian tangent category $(\mathbb{X}, \mathbb{T})$ consists of an object A of $\mathbb{X}$ together with a morphism $\zeta \colon \mathbb{1} \to A$, $\mathbb{1}$ being the terminal object of $\mathbb{X}$, and a binary operation $\sigma \colon A \times A \to A$ so that the triple (A, ζ, σ) forms a commutative monoid object with respect to the monoidal structure induced by the Cartesian product. Moreover, a differential object comes equipped with a **differential projection**, which is a morphism $\hat{p} \colon \mathrm{T}A \to A$, compatible with the monoidal structure (A, ζ, σ), and satisfying the following universality property. The diagram:*

$$A \xleftarrow{p} \mathrm{T}A \xrightarrow{\hat{p}} A$$

*is a product diagram. Finally, the differential projection is **linear**, that is, it is compatible with the vertical lift:*

$$\begin{tikzcd}
\mathrm{T}^2A \arrow[d, "\mathrm{T}\hat{p}"'] & \mathrm{T}A \arrow[l, "l"'] \arrow[d, "\hat{p}"] \\
\mathrm{T}A \arrow[r, "\hat{p}"'] & A
\end{tikzcd}$$

Differential objects, first introduced in [12, Definition 4.8] and then revisited in [11, Definition 3.1] where the compatibility with the vertical lift was added, form a generalization of the linear spaces $\mathbb{R}^n$ in a tangent category. One of the key results of Cockett and Cruttwell (cf. [12, Theorems 4.11 and 4.12]) proves that the category of differential objects of a (Cartesian) tangent category forms a Cartesian differential category, that is a Cartesian left-additive category (cf. [7]), equipped with an operation D, known as a **derivation operator**, which sends a morphism $f \colon A \to B$ to a morphism $\mathsf{D}[f] \colon A \times A \to B$. Informally, as explained in [7], by currying the map $\mathsf{D}[f] \colon A \times A \to B$ one obtains a map $\mathsf{J}[f] \colon A \to \mathsf{Lnr}(A, B)$ that can be interpreted as the Jacobian matrix of the smooth map f. In particular, the category $\mathsf{DObj}(\mathsf{Smooth}, \mathbb{T})$ of the Cartesian tangent category of smooth manifolds is the Cartesian differential category whose objects are the linear spaces $\mathbb{R}^n$ and whose morphisms $f \colon \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions between linear spaces. The map $\mathsf{J}[f]$ is then precisely the Jacobian matrix of f.

2.2.3 Tangent display maps

In differential geometry, there are a few important classes of morphisms, which we generically refer to as bundles. A bundle consists of a pair of smooth manifolds E and M respectively known as the total and the base space, together with a smooth morphism $q \colon E \to M$. Usually, one wants E to satisfy some nice properties, like local triviality, e.g. being a fibre bundle, being equipped with some extra structure, like being a vector bundle or a principal bundle, or one wants q to be regular in some sense, e.g. being a submersion. The name bundle, which refers to a generic morphism $q \colon E \to M$, evokes a specific interpretation of E as a collection of fibres $E_x := q^{-1}(x)$, indexed by the elements x of M. The regularity conditions on E or q usually address the question of how to make use of this interpretation: for example, one would like to have extra structures on the fibres or have a notion of transport of elements from one fibre to the adjacent ones.

Because of the importance of the notion of a bundle in differential geometry, a large part of research in tangent category theory is devoted to axiomatizing such classes in the general context of a tangent category. In Section 2.2.5, we recall the notion of a differential bundle and we discuss how it axiomatizes a vector bundle in differential geometry.

One technical difficulty encountered by Cockett and Cruttwelll in the attempt to define the concept of a differential bundle was the requirement for such a bundle to have all pullbacks along any morphism. Cockett and Cruttwell in [11] proved that, whenever the pullback exists, the pulled-back morphism of a differential bundle is still a differential bundle. However, this does not guarantee the existence of such pullbacks.

The solution presented by Cockett and Cruttwell and also adopted by MacAdam in [47] was to introduce the notion of a tangent display system, which consists of a family of morphisms stable under pullbacks and the tangent bundle functor. Before recalling this definition, we would like to introduce the following jargon distinction.

Jargon 2.17. In a category $\mathbb{X}$, a morphism $q \colon E \to M$, that is any morphism of $\mathbb{X}$,

admits all pullbacks if for any morphism $f\colon N \to M$ the pullback diagram:

$$\begin{tikzcd}
N \times_M E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\
N \arrow[r, "f"'] & M
\end{tikzcd}$$

exists. We also say that a family of morphisms $\mathscr{F}$ of $\mathbb{X}$ is **closed under pullbacks** if, whenever the pullback of a morphism $q\colon E \to M$ of $\mathscr{F}$ along a generic morphism $f\colon N \to M$ of $\mathbb{X}$ exists, then the pulled-back morphism $N \times_M E \to N$ is also a morphism of $\mathscr{F}$. Finally, we say that a family $\mathscr{F}$ of morphisms of $\mathbb{X}$ is **stable under pullbacks** if each morphism q of $\mathscr{F}$ admits all pullbacks and $\mathscr{F}$ is closed under pullbacks.

In a tangent category, one usually prefers to work with tangent pullbacks. Let's briefly recall this notion, introduced by MacAdam in [47].

Definition* 2.18. *In a tangent category $(\mathbb{X}, \mathbb{T})$, a* ***tangent limit*** *is a limit diagram which is preserved by all functors T^n, for every positive integer n. In particular, a* ***tangent pullback*** *is a pullback whose universality is preserved by all T^n. We are going to extend the same convention to all other limits, so for example a tangent equalizer is a tangent limit which is an equalizer and so on.*

We now extend the jargon we introduced before to tangent pullbacks.

Jargon 2.19. A morphism $q\colon E \to M$ of a tangent category $(\mathbb{X}, \mathbb{T})$ **admits all tangent pullbacks** if it admits all pullbacks and each of these is a tangent pullback. We also say that a family of morphisms $\mathscr{F}$ of $(\mathbb{X}, \mathbb{T})$ is **closed under tangent pullbacks** if, whenever a tangent pullback of a morphism $q\colon E \to M$ of $\mathscr{F}$ along a morphism $f\colon N \to M$ of $\mathbb{X}$ exists, then the pulled-back morphism $N \times_M E \to N$ is also a morphism of $\mathscr{F}$. Finally, a family of morphism $\mathscr{F}$ is **stable under tangent pullbacks** if it is closed under tangent pullbacks and each morphism of $\mathscr{F}$ admits all tangent pullbacks.

We can now recall the definition of a tangent display system.

Definition* 2.20. *In a tangent category $(\mathbb{X}, \mathbb{T})$ a* ***tangent display system*** *consists of a family $\mathscr{F}$ of morphisms of $\mathbb{X}$ which is stable under tangent pullbacks and stable under T.*

For a family of morphisms in a generic category $\mathbb{X}$, being stable under an endofunctor T means if q is a morphism of this family, so is $\mathrm{T}q$.

Remark 2.21. Cockett and Cruttwell in [11] required a tangent display system of a tangent category to also contain all the tangent bundles, that are all morphisms $p\colon \mathrm{T}M \to M$, for each M. Then, they defined a display tangent category as a tangent category equipped with such a tangent display system. Here we decided to adopt MacAdam's convention (cf. [47]) for which a tangent display system is not required to contain the tangent bundles.

A tangent display system is a technical answer to the technical issue of ensuring the families of bundles we are interested in are stable under tangent pullbacks. However, from a philosophical point of view, this axiomatization choice is, in our opinion, awkwardly constrained to treat the interesting class of bundles as a structure of the tangent category instead of being an intrinsic property of these bundles. In particular, this approach goes against the general philosophy of differential geometry of requiring *local* properties of bundles, instead of asking for *global* properties of families of bundles.

Finding this approach unsatisfactory for these reasons, we investigated under which minimal assumptions a bundle is an element of a tangent display system. We want to dedicate this section to exploring our results, believing that in future work one will not require the notion of a tangent display system anymore. The first insight in the right direction comes from the following proposition. First, let's recall a classic result of pullbacks.

Lemma* 2.22. *Consider the following diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C'
\end{tikzcd}$$

If the outer square and the right square are pullback diagrams, so is the left square.

Proof. This is a well-known property of pullbacks, but for completeness, let's give a sketch of the proof. Consider the following diagram:

$$\begin{tikzcd}
D \arrow[rr, "t"] \arrow[dd, "h"'] & & B \arrow[d, equal] & \\
& A \arrow[r, "g'"] \arrow[d, "s"'] & B \arrow[r, "f'"] \arrow[d, "p"] & C \arrow[d, "q"] \\
A' \arrow[r, equal] & A' \arrow[r, "g"'] & B' \arrow[r, "f"'] & C'
\end{tikzcd}$$

By employing the universality of the outer diagram, one finds a unique morphism $\varphi \colon D \dashrightarrow A$ such that $\varphi g' f' = t f'$ and $\varphi s = h$:

$$\begin{tikzcd}
D \arrow[rr, "t"] \arrow[dd, "h"'] \arrow[dr, dashed, "\varphi"] & & B \arrow[d, equal] & \\
& A \arrow[r, "g'"] \arrow[d, "s"'] & B \arrow[r, "f'"] \arrow[d, "p"] & C \arrow[d, "q"] \\
A' \arrow[r, equal] & A' \arrow[r, "g"'] & B' \arrow[r, "f"'] & C'
\end{tikzcd}$$

We want to show that $\varphi g' = t$. However, this is a consequence of the right square being a pullback. It is also straightforward to check that if $\varphi' \colon D \to A$ is a second morphism for which $\varphi' g' = t$ and $\varphi' s = h$, then $\varphi' = \varphi$. ☐

Lemma 2.23. *Consider the family $\mathscr{F}(\mathbb{X})$ of morphisms $q \in \mathbb{X}$ which admit all pullbacks. Then $\mathscr{F}(\mathbb{X})$ is stable under pullbacks.*

Proof. The proof makes use of Lemma 2.22. Consider first a morphism $q \colon E \to M$ of $\mathscr{F}(\mathbb{X})$ and suppose $f \colon N \to M$ is any morphism of $\mathbb{X}$. Since q admits all pullbacks the pullback of q along f is well-defined. As a shorthand, let $q' \colon E' \to N$ denote the pullback of q along f. We want to show that q' also admits all pullbacks. So, let $g\colon P \to N$ another morphism of $\mathbb{X}$ and consider the following diagram:

$$\begin{tikzcd}
E'' \arrow[rr, bend left] \arrow[d, "q''"'] & E' \arrow[r] \arrow[d, "q'"] & E \arrow[d, "q"] \\
P \arrow[r, "g"'] \arrow[rr, bend right, "gf"'] & N \arrow[r, "f"'] & M
\end{tikzcd}$$

where $q''\colon E'' \to P$ denotes the pullback of q along the composition gf, which exists because q admits all pullbacks. However, this implies the existence of a unique dash morphism:

$$\begin{tikzcd}
E'' \arrow[rr, bend left] \arrow[r, dashed] \arrow[d, "q''"'] & E' \arrow[r] \arrow[d, "q'"] & E \arrow[d, "q"] \\
P \arrow[r, "g"'] \arrow[rr, bend right, "gf"'] & N \arrow[r, "f"'] & M
\end{tikzcd}$$

It is clear that the outer and the right squares are pullback diagrams, so thanks to the Lemma 2.22, also the left square is a pullback. So, we conclude that also q' admits all pullbacks, so in particular q' belongs to $\mathscr{F}(\mathbb{X})$. So, $\mathscr{F}(\mathbb{X})$ is closed under pullbacks and, by definition, each morphism of $\mathscr{F}(\mathbb{X})$ admits all pullbacks, so $\mathscr{F}(\mathbb{X})$ is stable under pullbacks as expected. □

Let $\mathscr{F}$ be a family of morphisms of a category $\mathbb{X}$ and suppose $\mathscr{F}$ is stable under pullbacks. So, in particular, every morphism of $\mathscr{F}$ admits all pullbacks. This implies that $\mathscr{F}$ is included in $\mathscr{F}(\mathbb{X})$, or in other words, $\mathscr{F}(\mathbb{X})$ is a maximal family of morphisms of $\mathbb{X}$ which is stable under pullbacks, where the maximality is with respect to the partial order defined by the inclusion of families of morphisms. So, without loss of generality, if one needs a class of morphisms in a category $\mathbb{X}$ to be stable under pullbacks, instead of choosing one family of morphisms with this property, one can simply take any morphism of $\mathscr{F}(\mathbb{X})$. This is precisely what we want for tangent display systems: instead of choosing a specific family of morphisms satisfying the property of being stable under tangent pullbacks and stable under T, we want to find the right property of morphisms which allows us to have such stabilities for free. To understand how to define such a morphism, notice that for a morphism $q\colon E \to M$ in a tangent display system, not just q admits all tangent pullbacks, but so does $\mathrm{T}^n q$. This is due to the fact that a tangent display system is stable under T, so $\mathrm{T}^n q$ must be in the tangent display system and moreover, the tangent display system must be stable under tangent pullbacks. So, the intuition is to require for a morphism $q\colon E \to M$ to have that each $\mathrm{T}^n q$ admits tangent pullbacks. Let's officially introduce this new concept.

Definition 2.24. *A **tangent display map** $q\colon E \to M$ in a tangent category $(\mathbb{X}, \mathbb{T})$ is a morphism for which, for every non-negative integer n, $\mathrm{T}^n q$ (when $n = 0$, $\mathrm{T}^0 q = q$ and for $n = 1$, $\mathrm{T}^1 q = \mathrm{T}q$) admits all tangent pullbacks.*

The following theorem proves that tangent display maps are precisely the desired notion of morphism to work with.

Theorem 2.25. *Let $\mathscr{D}(\mathbb{X}, \mathbb{T})$ denote the family of tangent display maps of a tangent category $(\mathbb{X}, \mathbb{T})$. Then, $\mathscr{D}(\mathbb{X}, \mathbb{T})$ is the (unique) maximal tangent display system of $(\mathbb{X}, \mathbb{T})$ under the partial order of inclusions of families of morphisms of $\mathbb{X}$.*

Proof. It is straightforward to see that if q is a tangent display map, so is $\mathrm{T}q$. Let's then prove that $\mathscr{D}(\mathbb{X}, \mathbb{T})$ is stable under tangent pullbacks. We are going to make repeated use of Lemmas 2.22 and 2.23. Consider a tangent display map $q\colon E \to M$ and let $f\colon N \to M$ be a morphism of $\mathbb{X}$. By definition, q admits all pullbacks so let $q'\colon E' \to M$ denote the pullback of q along f. We want to show that q' is still a tangent display map. Consider a morphism $g\colon P \to N$. Since q admits all pullbacks, by Lemma 2.23, also q' admits all pullbacks. So, the pullback q'' of q' along g is well-defined. We want to show that this pullback is a tangent pullback. Consider a non-negative integer n and the following diagram:

$$\begin{tikzcd}
\mathrm{T}^n E'' \arrow[r] \arrow[d, "\mathrm{T}^n q''"'] & \mathrm{T}^n E' \arrow[r] \arrow[d, "\mathrm{T}^n q'"] & \mathrm{T}^n E \arrow[d, "\mathrm{T}^n q"] \\
\mathrm{T}^n P \arrow[r, "\mathrm{T}^n g"'] & \mathrm{T}^n N \arrow[r, "\mathrm{T}^n f"'] & \mathrm{T}^n M
\end{tikzcd}$$

The outer and the right square diagrams are both pullbacks since q admits all tangent pullbacks. So, by Lemma 2.22, also the left square diagram is a pullback.

This shows that also q' admits all tangent pullbacks. We now need to show that so does each $\mathrm{T}^m q$, for each non-negative integer m. However, since the pullback of q along f must be a tangent pullback, $\mathrm{T}^m q'$ is the pullback of $\mathrm{T}^m q$ along $\mathrm{T}^m f$. However, we already discussed that $\mathrm{T}q$ is also a tangent display map, and by induction, so is $\mathrm{T}^m q$. We also proved that if q admits all tangent pullbacks so does the pullback of q along any morphism f. So, we conclude that also $\mathrm{T}^m q'$ admits all tangent pullbacks. This proves that $\mathscr{D}(\mathbb{X}, \mathbb{T})$ is indeed a tangent display system.

Finally, we previously discussed that if $\mathscr{F}$ is any tangent display system and q is a morphism of $\mathscr{F}$, then for every non-negative integer n, $\mathrm{T}^n q$ admits all tangent pullbacks, which is precisely the definition of a tangent display map. So, each tangent display system is a subfamily of $\mathscr{D}(\mathbb{X}, \mathbb{T})$. This implies $\mathscr{D}(\mathbb{X}, \mathbb{T})$ to be the maximal tangent display system, with respect to the partial order of the inclusion between families of morphisms of $\mathbb{X}$. $\square$

In our convention, we intentionally did not include tangent bundles as part of a tangent display system (see Remark 2.21). We decide to call a tangent category a display tangent category when this is the case.

Definition 2.26. *A* ***display tangent category*** *consists of a tangent category in which the tangent bundles $p\colon \mathrm{T}M \to M$ are all tangent display maps.*

Remark 2.27. As discussed in Remark 2.21, in Cockett and Cruttwell's original definition a display tangent category consists of a tangent category equipped with a given tangent display system for which every tangent bundle $p\colon \mathrm{T}M \to M$ is a morphism of the tangent display system. The entire purpose of this section is to eliminate the necessity of choosing a specific tangent display system and instead take the maximal one, which is characterized as the family of tangent display maps. So, in some sense, our definition of a display tangent category refines Cockett and Cruttwell's original version, in the sense that every display tangent category in their sense is also a display tangent category in our sense.

Another important operation employed often in the context of classes of morphisms in differential geometry is the notion of retraction.

Jargon 2.28. We say that a morphism $q'\colon E' \to M$ is a **retraction of a morphism** $q\colon E \to M$ if there is a pair of morphisms $r\colon E \leftrightarrows E'\colon s$, such that $rq' = q$ and

$sr = \mathsf{id}_{E'}$. r is known as a **retraction** and s as a **section**. We also say that a family of morphisms $\mathscr{F}$ is **closed under retraction** if, for each morphism $q\colon E \to M$ and for any section-retraction pair $r\colon E \leftrightarrows E'\colon s$, q' is also a morphism in $\mathscr{F}$.

Notice that, if q' is a retract of a map q along the section-retraction pair $r\colon E \leftrightarrows E'\colon s$, then $sq = srq' = q'$, so, since r commutes with q and q', and s being a section of r, also s commutes with q and q'. The converse is also true: assuming s commutes with q and q' implies that so does r, i.e. $rq' = rsq = q$.

One would like tangent display maps to be closed under retraction. This desired condition plays an important role in the theory of differential bundles, as we clarify in the following sections. Luckily, we have an interesting criterion under which tangent display maps in a given tangent category are closed under retraction. To understand the origin of this result, let's start by recalling that an idempotent in a category $\mathbb{X}$ consists of an endomorphism $e\colon E \to E$ over a given object E, for which the composition of e with itself, i.e. e^2, is equal to itself, that is $e^2 = e$. In some sense, an idempotent can be interpreted as an operation which projects elements of E into a subspace A of E, so that over A it acts as the identity. In particular, this analogy is particularly striking in the context of linear operators over vector spaces equipped with an internal product, in which projectors $e\colon E \to E$ consist of orthogonal idempotents, that, in particular, are in bijective correspondence with subspaces of E.

This correspondence between subspaces and idempotents is precisely formulated in terms of split idempotents. We say that an idempotent $e\colon E \to E$ splits if it is the composition of a section-retraction pair $r\colon E \leftrightarrows A\colon s$. Indeed, if $e := rs$, using that $sr = \mathsf{id}_A$, we immediately conclude $e^2 = rsrs = rs = e$. In general, an idempotent does not split; to make every idempotent split one can Cauchy complete the category. More precisely, the *Cauchy completion* of a category $\mathbb{X}$, sometimes called the Karoubi envelope of $\mathbb{X}$, is a category $\bar{\mathbb{X}}$ together with a fully faithful functor $\iota\colon \mathbb{X} \to \bar{\mathbb{X}}$ for which every idempotent of $\bar{\mathbb{X}}$ splits and every object of $\bar{\mathbb{X}}$ is a retract of an object in the image of ι. In particular, this last condition means that for every object A of $\bar{\mathbb{X}}$ there exists a retraction $r\colon \iota(E) \to A$ for some object E of $\mathbb{X}$. We invite the interested reader to consult [9] for a discussion of this topic.

One of the crucial examples of Cauchy completion is given by the category of finite-dimensional smooth manifolds, which is precisely the Cauchy completion of the category Open of open subsets of $\mathbb{R}^n$ and smooth maps between them. In particular, this implies that in differential geometry every idempotents split. The result we are going to prove is that assuming idempotents split, then tangent display maps are closed under retraction. Let's start by proving an important property of the retract of a pullback. We suspect this is a known result in the literature, however, instead of giving a reference, we decide to give proof.

Lemma 2.29. *Consider a pullback diagram:*

$$\begin{tikzcd}
E' \arrow[r, "f'"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\
N \arrow[r, "f"'] & M
\end{tikzcd}$$

Suppose also that A is a retract of E and A' a retract of E', which means that there are section-retraction pairs $r\colon E \leftrightarrows A\colon s$ and $r'\colon E' \to A'\colon s'$, moreover $rsq = q$ and $r's'q' = q'$. Consider now the diagram:

$$\begin{tikzcd}
E' \arrow[rr, "r'"] \arrow[dr, "f'"] \arrow[dd, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & & A' \arrow[rr, "s'"] \arrow[dr, "g'"] \arrow[dd, "u'"] & & E' \arrow[dr, "f'"] \arrow[dd, "q'"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
& E \arrow[rr, "r"] \arrow[dd, "q"] & & A \arrow[rr, "s"] \arrow[dd, "u"] & & E \arrow[dd, "q"] \\
N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[dr, "f"] & \\
& M \arrow[rr, equal] & & M \arrow[rr, equal] & & M
\end{tikzcd}$$

where $u := sq$, $u' := s'q'$, anf $g := s'f'r$. Then, the diagram:

$$\begin{tikzcd}
A' \arrow[r, "g'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\
N \arrow[r, "f"'] & M
\end{tikzcd}$$

is also a pullback.

Proof. Let's consider two morphisms $\alpha\colon B \to A$ and $\beta\colon B \to N$, such that $\alpha u = \beta f'$. Then, by universality, we obtain a morphism $\gamma\colon B \to E'$ as follows:

$$
\begin{tikzcd}
 & B \arrow[drrr, dashed, bend left, "\gamma"] \arrow[ddrr, bend left, "\alpha"] \arrow[dddr, bend right, "\beta"'] & & & & \\
E' \arrow[rr, "r'"] \arrow[dr, "f'"] \arrow[dd, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & & A' \arrow[rr, "s'"] \arrow[dr, "g'"] \arrow[dd, "u'"] & & E' \arrow[dr, "f'"] \arrow[dd, "q'"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
 & E \arrow[rr, "r"] \arrow[dd, "q"] & & A \arrow[rr, "s"] \arrow[dd, "u"] & & E \arrow[dd, "q"] \\
N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[dr, "f"] & \\
 & M \arrow[rr, equal] & & M \arrow[rr, equal] & & M
\end{tikzcd}
$$

which is the unique morphism $B \to E'$ satisfying the following two equations:

$$
\gamma f' = \alpha s
$$
$$
\gamma q' = \beta
$$

Let's define $\gamma' := \gamma r'\colon B \to A'$. We have:

$$
\begin{aligned}
& \gamma' u' \\
= \; & \gamma r' u' \\
= \; & \gamma r' s' q' \\
= \; & \gamma q' \\
= \; & \beta
\end{aligned}
$$

Moreover:

$$
\begin{aligned}
& \gamma' g' \\
= \; & \gamma' g' s r
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma' s' f' r\\
&= \gamma r' s' f' r\\
&= \gamma f' r s r\\
&= \gamma f' r\\
&= \alpha s r\\
&= \alpha
\end{aligned}
$$

So, γ' satisfies $\gamma' u' = \beta$ and $\gamma' g' = \alpha$. Let now $\gamma'' \colon B \to A'$ be a second morphism satisfying the same equations of γ', so let $\delta := \gamma'' s'$, so we have:

$$
\begin{aligned}
&\delta f'\\
= \;&\gamma'' s' f'\\
= \;&\gamma'' g' s\\
= \;&\alpha s
\end{aligned}
$$

$$
\begin{aligned}
&\delta q'\\
= \;&\gamma'' s' q'\\
= \;&\gamma'' u'\\
= \;&\beta
\end{aligned}
$$

From the universality of the pullback, we conclude that $\delta = \gamma$, therefore:

$$\gamma'' = \gamma'' s' r' = \delta r' = \gamma r' = \gamma'$$

This proves the desired result. □

Adopting MacAdam's naming convention (cf. [47]), we recall the following definition.

Definition⋆ 2.30. *In a tangent category $(\mathbb{X}, \mathbb{T})$ a **retractive tangent display system** is a tangent display system which is closed under retracts.*

We also introduce the following definition.

Definition 2.31. *A **retractive tangent category** is a tangent category whose tangent display maps are closed under retraction.*

In particular, a **retractive display** tangent category is a display tangent category which is also retractive.

Theorem 2.32. *Suppose the class of split idempotents of a tangent category* $(\mathbb{X}, \mathbb{T})$ *is closed under pullbacks, then* $(\mathbb{X}, \mathbb{T})$ *is a retractive tangent category. In particular, every tangent display system of* $(\mathbb{X}, \mathbb{T})$ *is a retractive tangent display system.*

Proof. To prove this result we need to show that if $u\colon A \to M$ is a retract of a tangent display map $q\colon E \to M$ via a section-retraction pair $r\colon E \leftrightarrows A\colon s$ for which $rp = q$ (which implies $sq = p$), then also u is a tangent display map. Let's start by considering a morphism $f\colon N \to M$ and let's show that u admits the pullback along f. In order to see this, let's take a look at the following diagram:

$$\begin{tikzcd}
E' \arrow[rrrr, "e'"] \arrow[dr, "f'"] \arrow[dd, "q'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & & & E' \arrow[dr, "f'"] \arrow[dd, "q'"] \arrow[dr, phantom, very near start, "\lrcorner"] & \\
& E \arrow[rr, "r"] \arrow[dd, "q"] & & A \arrow[rr, "s"] \arrow[dd, "u"] & & E \arrow[dd, "q"] \\
N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[dr, "f"] & \\
& M \arrow[rr, equal] & & M \arrow[rr, equal] & & M
\end{tikzcd}$$

where e' is induced by the universality of the pullback. Notice the pullback of q along f exists, since q is a tangent display map. Notice also that e' is an idempotent. To see this,let $e = rs$ be the split idempotent defined by r and s and consider the following:

$$\begin{aligned}
& e'e'f' \\
= \; & e'f'e \\
= \; & f'ee \\
= \; & f'e \\
= \; & e'f'
\end{aligned}$$

$$
\begin{aligned}
& e'e'q' \\
= \;& e'q' \\
= \;& q'
\end{aligned}
$$

So, e'^2 satisfies the same equations which uniquely define e'. So, e' is an idempotent. In particular, e' is the pullback of a split idempotent, so by hypothesis, it also splits. Let $r' : E' \leftrightarrows A' : s'$ be the corresponding section-retraction pair, so now we have the following diagram:

$$
\begin{tikzcd}
E' \arrow[rr, "r'"] \arrow[dr, "f'"] \arrow[dd, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & & A' \arrow[rr, "s'"] \arrow[dr, "g'"] \arrow[dd, "u'"] & & E' \arrow[dr, "f'"] \arrow[dd, "q'"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
& E \arrow[rr, "r"] \arrow[dd, "q"] & & A \arrow[rr, "s"] \arrow[dd, "u"] & & E \arrow[dd, "q"] \\
N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[rr, equal] \arrow[dr, "f"] & & N \arrow[dr, "f"] & \\
& M \arrow[rr, equal] & & M \arrow[rr, equal] & & M
\end{tikzcd}
$$

By Lemma 2.29, the diagram:

$$
\begin{tikzcd}
A' \arrow[r, "g'"] \arrow[d, "u"'] & A \arrow[d, "q'"] \\
N \arrow[r, "f"'] & M
\end{tikzcd}
\tag{2.2.1}
$$

is a pullback diagram. This shows that if $\mathscr{F}(\mathbb{X})$ is the family of morphisms which admit all pullbacks, then under the condition that split idempotents are closed under pullbacks, $\mathscr{F}(\mathbb{X})$ is closed under retraction. By definition, a tangent display map $q : E \to M$ is precisely a morphism for each, for every n, $\mathrm{T}^n q$ belongs to $\mathscr{F}(\mathbb{X})$ and each pullback of $\mathrm{T}^n q$ along any morphism is preserved by each functor T^m. Let's show that the pullback diagram (2.2.1) is a tangent pullback. To do so, consider a non-negative integer m, then:

$$
\begin{tikzcd}
\mathrm{T}^m E' \arrow[rr, "\mathrm{T}^m r'"] \arrow[dr, "\mathrm{T}^m f'"] \arrow[dd, "\mathrm{T}^m q'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathrm{T}^m A' \arrow[rr, "\mathrm{T}^m s'"] \arrow[dr, "\mathrm{T}^m g'"] \arrow[dd, "\mathrm{T}^m u'" near start] & & \mathrm{T}^m E' \arrow[dr, "\mathrm{T}^m f'"] \arrow[dd, "\mathrm{T}^m q'" near start] \arrow[dr, phantom, very near start, "\lrcorner"] & \\
& \mathrm{T}^m E \arrow[rr, "\mathrm{T}^m r" near end] \arrow[dd, "\mathrm{T}^m q" near start] & & \mathrm{T}^m A \arrow[rr, "\mathrm{T}^m s" near end] \arrow[dd, "\mathrm{T}^m u" near start] & & \mathrm{T}^m E \arrow[dd, "\mathrm{T}^m q"] \\
\mathrm{T}^m N \arrow[rr, equal] \arrow[dr, "\mathrm{T}^m f"] & & \mathrm{T}^m N \arrow[rr, equal] \arrow[dr, "\mathrm{T}^m f"] & & \mathrm{T}^m N \arrow[dr, "\mathrm{T}^m f"] & \\
& \mathrm{T}^m M \arrow[rr, equal] & & \mathrm{T}^m M \arrow[rr, equal] & & \mathrm{T}^m M
\end{tikzcd}
$$

This in particular means that T^m of the diagram of Equation (2.2.1) is the retract of the pullback of $\mathrm{T}^m q$ along $\mathrm{T}^m f$, which, by again Lemma 2.29, is also a pullback. So, $u\colon A \to M$ admits all tangent pullbacks. Finally, for every non-negative integer n, $\mathrm{T}^n u$ is again the retract of $\mathrm{T}^n q$, so, since $\mathrm{T}^n q$ admits every tangent pullbacks, so does $\mathrm{T}^n u$, proving once for all that u is a tangent display map. □

The condition of Theorem 2.32 which requires the family of split idempotents to be closed under pullbacks is in particular satisfied by any Cauchy complete category $\mathbb{X}$, where every idempotent splits. This is because the family of idempotents is always closed (not necessarily stable) under pullbacks since the pullback of an idempotent is always an idempotent. As a consequence, we have the following important corollary.

Corollary 2.33. *A tangent structure $\mathbb{T}$ over a Cauchy complete category $\mathbb{X}$ forms a retractive tangent category $(\mathbb{X}, \mathbb{T})$.*

Every category $\mathbb{X}$ is canonically embedded in a Cauchy complete category, called the Cauchy completion of $\mathbb{X}$ (The notion of a Cauchy complete category was introduced by Lawvere in [43]. For more details on Cauchy completion, we refer to [9]). Concretely the Cauchy completion of $\mathbb{X}$, known also as the Karoubi envelope of $\mathbb{X}$, is the category $\mathsf{Split}(\mathbb{X})$ whose objects are pairs (M, e) formed by an object M of $\mathbb{X}$ and an idempotent $e\colon M \to M$ of M, and whose morphisms $f\colon (M, e) \to (M', e')$ are morphisms $f\colon M \to M'$ of $\mathbb{X}$ for which $fe' = ef$.

Corollary 2.33 establishes an interesting relationship between Cauchy categories and tangent structures, so it is natural to wonder if the Cauchy completion of a tangent category is still a tangent category. Let's start by observing a simple fact.

Lemma 2.34. *Given a category $\mathbb{X}$, the functor $(\bar{-})\colon \mathsf{End}(\mathbb{X}) \to \mathsf{End}(\mathsf{Split}(\mathbb{X}))$ which sends an endofunctor $F\colon \mathbb{X} \to \mathbb{X}$ to the endofunctor $\bar{F}\colon \mathsf{Split}(\mathbb{X}) \to \mathsf{Split}(\mathbb{X})$ so defined:*

$$\begin{aligned}
&\bar{F}(M,e) := (FM, Fe)\\
&\bar{F}(f\colon (M,e) \to (M',e')) := Ff\colon (FM,Fe) \to (FM',Fe')
\end{aligned}$$

is a strong monoidal functor with respect to the monoidal structure defined by the composition of endofunctors.

Using Leung's definition of a tangent category $(\mathbb{X},\mathbb{T})$ as a strong monoidal functor $\mathsf{Weil}_1 \to \mathsf{End}(\mathbb{X})$ which preserves some pullback diagrams (see Section 2.3 for details), one can post-compose this strong monoidal functor with $(\bar{-})$ and obtain a strong monoidal functor $\mathsf{Weil}_1 \to \mathsf{End}(\mathsf{Split}(\mathbb{X}))$. It is not hard to show that this strong monoidal functor preserves the required pullbacks to define a tangent structure on $\mathsf{Split}(\mathbb{X})$.

Theorem 2.35. *The Cauchy completion $\mathsf{Split}(\mathbb{X})$ of a tangent category $(\mathbb{X},\mathbb{T})$ (with negatives) is still a tangent category (with negatives) denoted by $\mathsf{Split}(\mathbb{X},\mathbb{T})$ and the fully faithful functor $\iota\colon \mathbb{X} \to \mathsf{Split}(\mathbb{X})$ strictly preserves the tangent structures. Moreover, ι preserves tangent display maps and in particular, if $(\mathbb{X},\mathbb{T})$ is a display tangent category (with negatives), then so is $\mathsf{Split}(\mathbb{X},\mathbb{T})$.*

Corollary 2.36. *Every (display) tangent category $(\mathbb{X},\mathbb{T})$ (with negatives) is strictly embedded in a retractive (display) tangent category (with negatives).*

2.2.4 The slice tangent category

The second construction we want to recall in this brief introduction to tangent category theory is the slice tangent category of a tangent category $(\mathbb{X},\mathbb{T})$ over a given object M. First, recall that the slice category $\mathbb{X}/M$ of a category $\mathbb{X}$ over an object M of $\mathbb{X}$ is the category whose objects q_M^E are morphisms of $\mathbb{X}$ of the form $q\colon E \to M$ and whose morphisms $f\colon q_M^E \to {q'}_M^{E'}$ are morphisms $f\colon E \to E'$ of $\mathbb{X}$ for which $fq' = q$. The idea is to lift the tangent structure $\mathbb{T}$ of $\mathbb{X}$ to the slice category $\mathbb{X}/M$.

In order to lift $\mathbb{T}$ to the slice category one can employ two different contructions. On the one hand, one can define the tangent bundle functor $\mathrm{T}/M\colon \mathbb{X}/M \to \mathbb{X}/M$

which sends a bundle q^E_M to $\mathrm{T}q/M$, which is the bundle $\mathrm{T}E \xrightarrow{\mathrm{T}q} \mathrm{T}M \xrightarrow{p} M$. The natural transformations of this tangent structure are precisely the same natural transformations of the tangent structure $\mathbb{T}$ over the base category $\mathbb{X}$. We are going to refer to this tangent structure as the "**trivial slice tangent structure**".

A second, and more interesting construction, is the "non-trivial" slice tangent category. For the goals of this thesis, we are going to consider only this last construction and therefore, we are going to omit the adjective "non-trivial". The main idea is to define the tangent bundle of a bundle $q\colon E \to M$ as the pullback of $\mathrm{T}q$ along the zero morphism, that is the bundle $q^{(M)}\colon \mathrm{T}^{(M)}E \to M$:

$$\begin{tikzcd}
\mathrm{T}^{(M)}E \arrow[r, "\iota"] \arrow[d, "q^{(M)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\
M \arrow[r, "z"'] & \mathrm{T}M
\end{tikzcd} \tag{2.2.2}$$

Notice that the notation $\mathrm{T}^{(M)}E$ could be a little misleading, since, as we see in a moment, $\mathrm{T}^{(-)}$ is a functor on the slice category but not on the base category.

The main issue with this definition is that, in a generic tangent category, this pullback diagram is not defined for every bundle $q\colon E \to M$. Moreover, one needs this pullback diagram to be a tangent pullback to define the corresponding tangent structure and one also needs the bundle $\mathrm{T}q^{(M)}$ to admit the tangent pullack along the zero morphism. To solve these issues, there are two main approaches. The first approach is to focus the attention only on objects M of the category $\mathbb{X}$ for which every bundle $q \in \mathbb{X}/M$ admits all tangent pullbacks. In the original paper [41] of the author of this thesis, we employed this approach. Here we focus on a different approach: consider tangent display maps only. This has the advantage of furnishing a global picture of the classification of differential bundles, by employing the technology of fibrations, as we discuss in the next section. The main insight is given by the following lemma.

Lemma 2.37. *The bundles $q\colon E \to M$ of a tangent display system $\mathscr{F}$ in a tangent category $(\mathbb{X}, \mathbb{T})$ admit the $\mathbb{T}$-pullback of Equation (2.2.2). Moreover, for each q in the tangent display system, $q^{(M)}$ is automatically a bundle of the same tangent display system.*

Proof. Since a tangent display system, by definition, is stable under the tangent bundle functor, for every bundle q of the tangent display system, $\mathrm{T}q$ is also part of the tangent display system. Moreover, since a tangent display system is also stable under pullbacks, then the pullback of $\mathrm{T}q$ along the zero morphism is well-defined and it defines a new bundle of the tangent display system. Thus, also $\mathrm{T}^{(M)}q = q^{(M)}$ is part of the tangent display system. Finally, since every bundle of a tangent display system is necessarily a tangent display map, the pullback diagram is a tangent pullback. $\square$

In the following, we denote by $\mathscr{D}(\mathbb{X}, \mathbb{T}; M)$ the category whose objects are tangent display maps of $(\mathbb{X}, \mathbb{T})$ whose target is the fixed object M of $\mathbb{X}$, and whose morphisms are morphisms between bundles. Then, the **slice tangent category** $\mathsf{Slice}(\mathbb{X}, \mathbb{T}; M)$ is the tangent category so defined:

tangent bundle functor The tangent bundle functor $\mathrm{T}^{(M)} \colon \mathscr{D}(\mathbb{X}, \mathbb{T}; M) \to \mathscr{D}(\mathbb{X}, \mathbb{T}; M)$ is the functor which sends a tangent display map $q^E_M = q \colon E \to M$ to $q^{(M)}{}^{\mathrm{T}^{(M)}E}_{M}$, defined by the pullback diagram (2.2.2). Moreover, $\mathrm{T}^{(M)}$ sends a morphism $f \colon q^E_M \to q'^{E'}_M$ to the unique morphism $f^{(M)} \colon q^{(M)}{}^{\mathrm{T}^{(M)}E}_{M} \to q'^{(M)}{}^{E'^{(M)}}_{M}$, induced by the universality of the pullback diagram (2.2.2):

$$\begin{tikzcd}
 & & & E'^{*} \arrow[d, "q'^{(M)}"'] \arrow[rr, "\iota"] \arrow[dr, phantom, "\lrcorner"] & & \mathrm{T}E' \arrow[d, "\mathrm{T}q'"] \\
\mathrm{T}^{(M)}E \arrow[urrr, dashed, "f^{(M)}"] \arrow[d, "q^{(M)}"'] \arrow[rr, "\iota"] \arrow[dr, phantom, "\lrcorner"] & & \mathrm{T}E \arrow[urrr, "\mathrm{T}f"] \arrow[d, "\mathrm{T}q"] & M \arrow[dlll, equal] \arrow[rr, "z"] & & \mathrm{T}M \\
M \arrow[rr, "z"'] & & \mathrm{T}M \arrow[urrr, equal] & & &
\end{tikzcd}$$

projection The projection $p^{(M)} \colon \mathrm{T}^{(M)} \Rightarrow \mathsf{id}_{\mathbb{X}/M}$ is induced by the natural transformation:

$$\mathrm{T}^{(M)}E \xrightarrow{\iota} \mathrm{T}E \xrightarrow{p} E$$

for any object q^E_M;

zero morphism The zero morphism $z^{(M)}\colon \mathsf{id}_{\mathbb{X}/M} \Rightarrow \mathrm{T}^{(M)}$ is induced by the natural transformation defined by the universality of the pullback diagram (2.2.2):

$$
\begin{tikzcd}
E \arrow[rr, "z"] \arrow[dr, dashed, "z^{(M)}"] \arrow[dd, "q"'] & & \mathrm{T}E \arrow[d, equal] \\
& \mathrm{T}^{(M)}E \arrow[r, "\iota"] \arrow[d, "q^{(M)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\
M \arrow[r, equal] & M \arrow[r, "z"'] & \mathrm{T}M
\end{tikzcd}
$$

n**-fold pullback** The n-fold pullback $\mathrm{T}^{(M)}_n$ of the projection $p^{(M)}$ along itself:

$$
\begin{tikzcd}
\mathrm{T}^{(M)}_n q^E_M \arrow[rr, "\pi^{(M)}_n"] \arrow[dr, "\pi^{(M)}_k"] \arrow[dd, "\pi^{(M)}_1"'] & & \mathrm{T}^{(M)} q^E_M \arrow[dd, "p^{(M)}"] \\
& \mathrm{T}^{(M)} q^E_M \arrow[dr, "p^{(M)}"] & \\
\mathrm{T}^{(M)} q^E_M \arrow[rr, "p^{(M)}"'] & & q^E_M
\end{tikzcd}
$$

is given by the pullback diagram:

$$
\begin{tikzcd}
E^*_n \arrow[r, "\langle \iota, \ldots, \iota \rangle"] \arrow[d, "q^*_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}_n E \arrow[d, "\mathrm{T}_n q"] \\
M \arrow[r, "\langle z, \ldots, z \rangle"'] & \mathrm{T}_n M
\end{tikzcd}
$$

that is $\mathrm{T}^{(M)}_n q^E_M = q^{(M)}_n$, and the k-th projection $\pi^{(M)}_k\colon \mathrm{T}^{(M)}_n \Rightarrow \mathrm{T}^{(M)}$ is given by the natural transformation induced by the universality of the following diagram:

$$
\begin{tikzcd}
 & & \mathrm{T}^{(M)}E \arrow[d, "q^{(M)}"'] \arrow[dr, "\iota"] \arrow[d, phantom, shift left=3, "\lrcorner"] & \\
E_n^{(M)} \arrow[urr, dashed, "\pi_k^{(M)}"] \arrow[d, "q_n^{(M)}"'] \arrow[dr, "\langle\iota,\ldots,\iota\rangle"] \arrow[dr, phantom, shift right=3, "\lrcorner"] & & M \arrow[dr, "z"'] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\
M \arrow[urr, equal] \arrow[dr, "\langle z,\ldots,z\rangle"'] & \mathrm{T}_nE \arrow[urr, "\pi_k"'] \arrow[d, "\mathrm{T}_nq"] & & \mathrm{T}M \\
 & \mathrm{T}_nM \arrow[urr, "\pi_k"] & &
\end{tikzcd}
$$

sum morphism The sum morphism $s^{(M)}\colon \mathrm{T}_2^{(M)} \Rightarrow \mathrm{T}^{(M)}$ is induced by the natural transformation defined by the universality of the pullback diagram (2.2.2):

$$
\begin{tikzcd}
\mathrm{T}^{(M)}E_2 \arrow[rr, "\langle\iota,\iota\rangle"] \arrow[dr, dashed, "s^{(M)}"] \arrow[dd, "q_2^{(M)}"'] & & \mathrm{T}_2E \arrow[d, "s"] \\
 & \mathrm{T}^{(M)}E \arrow[r, "\iota"] \arrow[d, "q^{(M)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\
M \arrow[r, equal] & M \arrow[r, "z"'] & \mathrm{T}M
\end{tikzcd}
$$

vertical lift The vertical lift $l^{(M)}\colon \mathrm{T}^{(M)} \Rightarrow \mathrm{T}^{(M)^2}$ is induced by the natural transformation defined by the universality of the pullback diagram (2.2.2):

$$
\begin{tikzcd}
\mathrm{T}^{(M)}E \arrow[rrr, "\iota"] \arrow[dr, dashed, "l^{(M)}"] \arrow[dd, "q^{(M)}"'] & & & \mathrm{T}E \arrow[d, "l"] \\
 & E^{(M)^2} \arrow[r, "\iota"] \arrow[d, "q^{(M)^2}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{T}E)^{*} \arrow[r, "\iota"] \arrow[d, "(\mathrm{T}q)^{(\mathrm{T}M)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}^2E \arrow[d, "\mathrm{T}^2q"] \\
M \arrow[r, equal] & M \arrow[r, "z"'] & \mathrm{T}M \arrow[r, "z_{\mathrm{T}}"'] & \mathrm{T}^2M
\end{tikzcd}
$$

where ${q^{(M)^2}}^{E^{**}}_{M} := \mathrm{T}^{(M)^2} q^E_M$;

canonical flip The canonical flip $c^{(M)}\colon \mathrm{T}^{(M)^2} \Rightarrow \mathrm{T}^{(M)^2}$ is induced by the natural transformation defined by the universality of the pullback diagram (2.2.2):

$$\begin{tikzcd}
E^{(M)^2} \arrow[rr, "\iota"] \arrow[dr, dashed, "c^{(M)}"] \arrow[dd, "q^{(M)^2}"'] & & (\mathrm{T}E)^{(\mathrm{T}M)} \arrow[r, "\iota"] & \mathrm{T}^2E \arrow[d, "c"] \\
 & E^{(M)^2} \arrow[r, "\iota"] \arrow[d, "q^{(M)^2}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{T}E)^{(\mathrm{T}M)} \arrow[r, "\iota"] \arrow[d, "(\mathrm{T}q)^{(\mathrm{T}M)}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}^2E \arrow[d, "\mathrm{T}^2q"] \\
M \arrow[r, equal] & M \arrow[r, "z"'] & \mathrm{T}M \arrow[r, "z_{\mathrm{T}}"'] & \mathrm{T}^2M
\end{tikzcd}$$

Moreover, if $(\mathbb{X}, \mathbb{T})$ has negatives with negation $n\colon \mathrm{T} \Rightarrow \mathrm{T}$, then so does $\mathsf{Slice}(\mathbb{X}, \mathbb{T}; M)$:

negation The negation $n^{(M)}\colon \mathrm{T}^{(M)} \Rightarrow \mathrm{T}^{(M)}$ is induced by the natural transformation defined by the universality of the pullback diagram (2.2.2):

$$\begin{tikzcd}
\mathrm{T}^{(M)}E \arrow[rr, "\iota"] \arrow[dr, dashed, "n^{(M)}"] \arrow[dd, "q^{(M)}"'] & & \mathrm{T}_2E \arrow[d, "n"] \\
 & \mathrm{T}^{(M)}E \arrow[r, "\iota"] \arrow[d, "q^{(M)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\
M \arrow[r, equal] & M \arrow[r, "z"'] & \mathrm{T}M
\end{tikzcd}$$

This construction was first introduced by Rosický in his seminal article [53]. More recently in [11, Section 5], Cockett and Crutwell showed how this construction is naturally contextualized within the theory of tangent fibrations. In particular, they proved that the fibres of the functor $\mathscr{D}(\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$ from the tangent category of morphisms of $\mathbb{X}$, whose tangent bundle functor is T acting on morphisms, to a fixed tangent category $(\mathbb{X}, \mathbb{T})$, are precisely the slice tangent categories $\mathsf{Slice}(\mathbb{X}, \mathbb{T}; M)$, parametrized by the objects M of $\mathbb{X}$. This result inspired the author to investigate the relationship between tangent fibrations and the celebrated Grothendieck construction. We suggest the interested reader to consult [42].

Remark 2.38. In Rosickỳ's original version for the construction of the slice tangent category the pullback diagram (2.2.2) was replaced by the equalizer diagram:

$$\begin{tikzcd} \mathrm{T}^{(M)}E \arrow[r, dashed, "\iota"] & \mathrm{T}E \arrow[r, shift left, "\mathrm{T}q"] \arrow[r, shift right, "\mathrm{T}qpz"'] & \mathrm{T}M \end{tikzcd}$$

So, in Rosický's version, the tangent bundle functor $\mathrm{T}^{(M)}$ sends q^E_M to $\mathrm{T}^{(M)}E \xrightarrow{\iota} \mathrm{T}E \xrightarrow{\mathrm{T}q} \mathrm{T}M \xrightarrow{p} M$. It is straightforward to show that these two definitions are equivalent.

Example 2.39. In [18, Section 4.1], Cruttwell and Lemay showed that the tangent category $(\mathsf{cAlg}^{\mathsf{op}}_R, \mathbb{T})$ can also be characterized as the slice tangent category of $(\mathsf{cRing}^{\mathsf{op}}, \mathbb{T})$ over the ring R. Indeed, unital and commutative algebras over a ring R are equivalently characterized as morphisms $R \to A$ of rings.

2.2.5 Differential bundles

The universality of the vertical lift establishes that the vertical bundle $VM \to M$, which is the pullback of $\mathrm{T}p$ along the zero morphism, is trivial; that is, $VM \to M$ is isomorphic to the pullback $\mathrm{T}_2M \to M$. In fact, the vertical bundle is precisely the slice tangent bundle $\mathrm{T}^{(M)}$ applied to the projection $p^{\mathrm{T}M}_M$. Putting these two facts together, one concludes that the tangent bundle $p\colon \mathrm{T}M \to M$ is a differential object of the slice tangent category $\mathsf{Slice}(\mathbb{X}, \mathbb{T}; M)$, when the projection $p^{\mathrm{T}M}_M$ is regarded as an object of such a category.

This suggests interpreting differential objects of the slice tangent category over an object M of $(\mathbb{X}, \mathbb{T})$ as vector bundles over M. This is precisely the intuition underpinning the definition of **differential bundles**, first introduced by Cockett and Cruttwell in [11]. Here, we recall the original definition.

Definition* 2.40. *A **differential bundle** in a tangent category $(\mathbb{X}, \mathbb{T})$ consists of an additive bundle $(q\colon E \to M, z_q\colon M \to E, s_q\colon E_2 \to E)$ together with a morphism $l_q\colon E \to \mathrm{T}E$, called the **vertical lift**, satisfying the following conditions:*

1. *$(l_q, z)\colon (q, z_q, s_q) \to (\mathrm{T}q, \mathrm{T}z_q, \mathrm{T}s_q)$ is an additive bundle morphism;*
2. *$(l_q, z_q)\colon (q, z_q, s_q) \to (p, z, s)$ is an additive bundle morphism;*

3. *The vertical lift is universal, that is, the following diagram:*

$$\begin{tikzcd} E_2 \arrow[r, "\xi_q"] \arrow[d, "\pi_1 q"'] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\ M \arrow[r, "z_q"'] & \mathrm{T}M \end{tikzcd}$$

is a pullback diagram, and it is preserved by all functors T^n*, where:*

$$\xi_q := (l_q \times_M z)\mathrm{T}s_q$$

4. *The vertical lifts* l *and* l_q *are compatible:*

$$\begin{tikzcd} E \arrow[r, "l_q"] \arrow[d, "l_q"'] & \mathrm{T}E \arrow[d, "Tl_q"] \\ \mathrm{T}E \arrow[r, "l"'] & \mathrm{T}^2E \end{tikzcd}$$

The interpretation of differential bundles as vector bundles in a tangent category acquires solidity in light of MacAdam's result, presented in [47], which shows that differential bundles in the tangent category of (connected) smooth manifolds are precisely vector bundles.

Cockett and Cruttwell also showed that (cf. [11, Corollary 3.5]), for a chosen point $x\colon \mathbb{1} \to M$, i.e. a morphism from the terminal object $\mathbb{1}$ of $\mathbb{X}$ to the object M, the local fibre E_x of a differential bundle $q\colon E \to M$ over x, obtained by pulling back $q\colon E \to M$ along $x\colon \mathbb{1} \to M$ as follows:

$$\begin{tikzcd} E_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "q"] \\ \mathbb{1} \arrow[r, "x"'] & M \end{tikzcd}$$

is a differential object. The relationship between differential bundles and differential objects is deeper. Under some conditions, differential bundles are precisely the differential objects in the slice tangent category. The main issue for this to be true is that the slice tangent category should contain the differential bundle itself as an object. Tangent display systems were initially introduced specifically to solve this issue. Now, with the new notion of tangent display map, we can simply take the largest tangent display system: the family of tangent display maps. For this scope, let's introduce this concept.

Definition 2.41. *A* ***display differential bundle*** *in a tangent category* $(\mathbb{X}, \mathbb{T})$ *consists of a differential bundle whose underlying bundle is a tangent display map.*

In a display tangent category every tangent bundle $p \colon \mathrm{T}M \to M$ is a tangent display map and therefore a display differential bundle. However, it is not obvious that a generic differential bundle would be a tangent display map as well. Let's introduce formally this concept.

Definition 2.42. *A* ***fully display tangent category*** *is a tangent category in which every differential bundle is a display differential bundle.*

In particular, every fully display tangent category is a display tangent category. In [47, Corollary 3.1.4], MacAdam proved an important characterization of differential bundles: when the tangent category has negatives, every differential bundle is the retract of the pullback of a tangent bundle. We can employ this characterization to show that when the tangent bundles are tangent display maps, the tangent category has negatives, and the tangent display maps are closed under retraction, then every differential bundle becomes automatically a tangent display map.

Theorem 2.43. *A retractive display tangent category with negatives is a fully display tangent category.*

Corollary 2.33, proves that every Cauchy complete tangent category is retractive, so we have the following result.

Corollary 2.44. *A Cauchy complete display tangent category with negatives is a fully display tangent category with negatives.*

Putting together that the Cauchy completion of a display tangent category with negatives is still a display tangent category with negatives (Theorem 2.35 and Corollary 2.36) and Corollary 2.44 we obtain the following result.

Corollary 2.45. *Every display tangent category* $(\mathbb{X}, \mathbb{T})$ *with negatives is strictly embedded in a fully display tangent category with negatives* $\mathsf{Split}(\mathbb{X}, \mathbb{T})$*.*

2.2.6 The adjoint tangent category

In Example 2.13, we showed that the category of commutative and unital rings comes equipped with a tangent structure $\mathbb{L}$ whose tangent bundle functor L is the functor which sends a ring R to the associated ring of dual numbers $R\langle\varepsilon\rangle = R[x]/(x^2)$. Notice that $R\langle\varepsilon\rangle$ is isomorphic to $R \otimes D$ where $D := \mathbb{Z}\langle\varepsilon\rangle$ and the tensor product is over the ring $\mathbb{Z}$. On the other hand, also the opposite of the category of commutative and unital rings has a tangent structure $\mathbb{T}$, whose tangent bundle functor T sends a ring R to R^D.

It is not a lucky coincidence that in both $\mathbb{L}$ and $\mathbb{T}$ the ring D appears in the definition of the tangent bundle functor. One can also notice that the functors $-\otimes D$ and $(-)^D$ form an adjunction. This is indeed the reason why these two tangent structures share D in the definition of the tangent bundle functor. In this section, we recall an important construction due to Cockett and Cruttwell (see [12, Section 5]) in tangent category theory which explains the relationship between the two tangent structure on the category of commutative and unital rings and its opposite. Let's first recall the notion of an adjunctable tangent category.

Definition* 2.46. *A tangent category $(\mathbb{X}, \mathbb{L})$ is **adjunctable** if the functors $\mathrm{L}_n : \mathbb{X} \to \mathbb{X}$ obtained by pulling back the tangent bundle $p : \mathrm{L} \Rightarrow \mathsf{id}_{\mathbb{X}}$ along itself n times, admit left adjoints $\mathrm{T}_n \dashv \mathrm{L}_n$, for every positive integer n (when $n = 1$, $\mathrm{T} := \mathrm{T}_1$ and $\mathrm{L}_1 = \mathrm{L}$).*

Remark 2.47. The terminology *adjunctable* is new. In the original paper [12], Cockett and Cruttwell did not introduce a dedicated definition for this concept but they added the expression *dual tangent category*. We decided to not use the word *dual* since it can be confused with a *cotangent structure*, which is not related to this construction. Notice also that in [29], Ikonicoff, the author of this thesis, and Lemay used the expression "*a tangent structure with adjoint tangent structure*" for adjunctable tangent structure.

Cockett and Cruttwell's result proves that, if a tangent category is adjunctable, then also the opposite of the underlying category has a tangent structure. Since this result plays a crucial role in our story, we report here the statement.

Theorem* 2.48. *If a tangent category $(\mathbb{X}, \mathbb{L})$ is adjunctable, then the opposite of the category $\mathbb{X}$, i.e. $\mathbb{X}^{\mathsf{op}}$, has also a tangent structure $\mathbb{T}$ whose tangent bundle functor is the* *left adjoint T of $\mathbb{L}$ and whose natural transformations are the mates of the corresponding natural transformations of $\mathbb{L}$ via the adjunctions $\mathrm{T}_n \dashv \mathbb{L}_n$.*

Let's unwrap the construction of the **adjoint tangent structure**, i.e. the tangent structure defined by Theorem 2.48. We denote by $(\eta_n, \varepsilon_n)\colon \mathrm{T}_n \dashv \mathbb{L}_n$ the adjunctions of Definition 2.46.

tangent bundle functor The tangent bundle functor $\mathrm{T}\colon \mathbb{X}^{\mathsf{op}} \to \mathbb{X}^{\mathsf{op}}$ is the left adjoint T of $\mathbb{L}$;

projection The projection $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathbb{X}} \Rightarrow \mathrm{T}$, regarded as an $\mathbb{X}$-morphism, is defined as follows:

$$p^{(\mathrm{T})}\colon \mathsf{id}_{\mathbb{X}} \xrightarrow{\eta} \mathbb{L} \circ \mathrm{T} \xrightarrow{p^{(\mathbb{L})}\mathrm{T}} \mathrm{T}$$

zero morphism The zero morphism $z^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathbb{X}}$, regarded as an $\mathbb{X}$-morphism, is defined as follows:

$$z^{(\mathrm{T})}\colon \mathrm{T} \xrightarrow{\mathrm{T}z^{(\mathbb{L})}} \mathrm{T} \circ \mathbb{L} \xrightarrow{\varepsilon} \mathsf{id}_{\mathbb{X}}$$

***n*-fold pullback** The n-fold pushout (in $\mathbb{X}$) of the projection along itself is given by the left adjoint T_n of $\mathbb{L}_n$. Moreover, the k-th injection $\pi_k^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}_n$, regarded as an $\mathbb{X}$-morphism, is defined as follows:

$$\pi_k^{(\mathrm{T})}\colon \mathrm{T} \xrightarrow{\mathrm{T}\eta_n} \mathrm{T} \circ \mathbb{L}_n \circ \mathrm{T}_n \xrightarrow{\mathrm{T}\pi_k^{(\mathbb{L})}\mathrm{T}_n} \mathrm{T} \circ \mathbb{L} \circ \mathrm{T}_n \xrightarrow{\varepsilon\mathrm{T}_n} \mathrm{T}_n$$

sum morphism The sum morphism $s^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}_2$, regarded as an $\mathbb{X}$-morphism, is defined as follows:

$$s^{(\mathrm{T})}\colon \mathrm{T} \xrightarrow{\mathrm{T}\eta_2} \mathrm{T} \circ \mathbb{L}_2 \circ \mathrm{T}_2 \xrightarrow{\mathrm{T}s^{(\mathbb{L})}\mathrm{T}_2} \mathrm{T} \circ \mathbb{L} \circ \mathrm{T}_2 \xrightarrow{\varepsilon\mathrm{T}_2} \mathrm{T}_2$$

vertical lift The vertical lift $l^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}$, regarded as an $\mathbb{X}$-morphism, is defined as follows:

$$l^{(\mathrm{T})}\colon \mathrm{T}^2 \xrightarrow{\mathrm{T}^2\eta} \mathrm{T}^2 \circ \mathbb{L} \circ \mathrm{T} \xrightarrow{\mathrm{T}^2 l^{(\mathbb{L})}\mathrm{T}} \mathrm{T}^2 \circ \mathbb{L}^2 \circ \mathrm{T} \xrightarrow{\mathrm{T}\varepsilon\mathbb{L}\mathrm{T}} \mathrm{T} \circ \mathbb{L} \circ \mathrm{T} \xrightarrow{\varepsilon\mathrm{T}} \mathrm{T}$$

canonical flip The canonical flip $c^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$, regarded as an $\mathbb{X}$-morphism, is defined as follows:

$$c^{(\mathrm{T})}\colon \mathrm{T}^2 \xrightarrow{\mathrm{T}^2\eta} \mathrm{T}^2 \circ \mathbb{L} \circ \mathrm{T} \xrightarrow{\mathrm{T}^2\mathbb{L}\eta\mathrm{T}} \mathrm{T}^2 \circ \mathbb{L}^2 \circ \mathrm{T}^2 \xrightarrow{\mathrm{T}^2 c^{(\mathbb{L})}\mathrm{T}^2}$$
$$\to \mathrm{T}^2 \circ \mathbb{L}^2 \circ \mathrm{T}^2 \xrightarrow{\mathrm{T}\varepsilon\mathbb{L}\mathrm{T}^2} \mathrm{T} \circ \mathbb{L} \circ \mathrm{T}^2 \xrightarrow{\varepsilon\mathrm{T}^2} \mathrm{T}^2$$

Moreover, if $(\mathbb{X}, \mathbb{L})$ has negatives, then also the adjoint tangent category has negatives:

negation The negation $n^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}$, regarded as an $\mathbb{X}$-morphism, is defined as follows:

$$n^{(\mathrm{T})}\colon \mathrm{T} \xrightarrow{\mathrm{T}\eta} \mathrm{T} \circ \mathbb{L} \circ \mathrm{T} \xrightarrow{\mathrm{T}n^{(\mathbb{L})}\mathrm{T}} \mathrm{T} \circ \mathbb{L} \circ \mathrm{T} \xrightarrow{\varepsilon\mathrm{T}} \mathrm{T}$$

Checking that all functors $\mathbb{L}_n$ have left adjoints can be a painful exercise; however, such a procedure can be simplified under mild conditions. The key observation is that $\mathbb{L}_n$ is the n-fold pullback of the tangent bundle functor T along the projection. So, when the tangent bundle functor has a left adjoint and there are enough pushouts, it comes for free that also the $\mathbb{L}_n$ have left adjoints. This observation came as a result of a discussion between the author and Martin Frankland, during the annual Foundational Method of Computer Science conference of 2022. We would like to thank Frankland for the suggestions and to propose a full proof of this statement. Here we propose our proof.

Lemma 2.49. *Let $\mathbb{L}\colon \mathbb{X} \to \mathbb{X}$ be an endofunctor over a category $\mathbb{X}$ and let $p^{(\mathbb{L})}\colon \mathbb{L} \Rightarrow \mathsf{id}_{\mathbb{X}}$ be a natural transformation for which the n-fold pullback of $p^{(\mathbb{L})}$ along itself exists, for every non-negative integer n. Suppose that $\mathbb{L}$ admits a left adjoint T so that $(\eta, \varepsilon)\colon \mathrm{T} \dashv \mathbb{L}$ is an adjunction. Define the mate of $p^{(\mathbb{L})}$ as follows:*

$$p^{(\mathrm{T})}\colon \mathsf{id}_{\mathbb{X}} \xrightarrow{\eta} \mathbb{L} \circ \mathrm{T} \xrightarrow{p^{(\mathbb{L})}\mathrm{T}} \mathrm{T}$$

If for a non-negative integer n, $p^{(\mathrm{T})}$ admits the n-fold pushout T_n along itself, then T_n is a left adjoint of $\mathbb{L}_n$.

Proof. The goal is to define the unit $\eta_n : \mathsf{id}_{\mathbb{X}} \Rightarrow \mathbb{L}_n \circ \mathrm{T}_n$ and the counit $\varepsilon_n : \mathrm{T}_n \circ \mathbb{L}_n \Rightarrow \mathsf{id}_{\mathbb{X}}$. Let's start with η_n and consider the following diagram:

$$\begin{tikzcd}
A \arrow[rrr, "\eta"] \arrow[dr, dashed, "\eta_n"] \arrow[ddd, "\eta"'] & & & \mathbb{L}\mathrm{T}A \arrow[dll, "\mathbb{L}\iota_n"] \arrow[dd, "p^{(\mathbb{L})}\mathrm{T}"] \\
& \mathbb{L}_n\mathrm{T}_nA \arrow[r, "\pi_n^{(\mathbb{L})}\mathrm{T}_n"] \arrow[d, "\pi_1^{(\mathbb{L})}\mathrm{T}_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{L}\mathrm{T}_nA \arrow[d, "p^{(\mathbb{L})}\mathrm{T}_n"] & \\
& \mathbb{L}\mathrm{T}_nA \arrow[r, "p^{(\mathbb{L})}\mathrm{T}_n"'] & \mathrm{T}_nA & \mathrm{T}A \arrow[l, "\iota_n"'] \\
\mathbb{L}\mathrm{T}A \arrow[ur, "\mathbb{L}\iota_1"'] \arrow[rr, "p^{(\mathbb{L})}\mathrm{T}"'] & & \mathrm{T}A \arrow[u, "\iota_1"'] &
\end{tikzcd}$$

where $\iota_k : \mathrm{T} \Rightarrow \mathrm{T}_n$ denotes the kth injection, so in particular $\eta p^{(\mathbb{L})}_{\mathrm{T}} \iota_i = p^{(\mathrm{T})}\iota_i = p^{(\mathrm{T})}\iota_j = \eta p^{(\mathbb{L})}_{\mathrm{T}} \iota_i$, for $i, j = 1, \ldots, n$. So, the morphism η_n is well-defined as the unique morphism for which:

$$\eta_n \pi_k^{(\mathbb{L})}\mathrm{T}_n = \eta \mathbb{L}\iota_k$$

for $k = 1, \ldots, n$. Similarly, for ε_n consider the following diagram:

$$\begin{tikzcd}
\mathbb{L}_nA \arrow[r, "p^{(\mathrm{T})}"] \arrow[d, "p^{(\mathrm{T})}"'] & \mathrm{T}\mathbb{L}_nA \arrow[r, "\mathrm{T}\pi_1^{(\mathbb{L})}"] \arrow[d, "\iota_n\mathbb{L}_n"] & \mathrm{T}\mathbb{L}A \arrow[dd, "\varepsilon"] \\
\mathrm{T}\mathbb{L}_nA \arrow[r, "\iota_1\mathbb{L}_n"'] \arrow[d, "\mathrm{T}\pi_1^{(\mathbb{L})}"'] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{T}_n\mathbb{L}_nA & \\
\mathrm{T}\mathbb{L}A \arrow[rr, "\varepsilon"'] & & A
\end{tikzcd}$$

Notice that:

$$
\begin{tikzcd}
\perp_n A \arrow[rr, "p^{(\mathrm{T})}"] \arrow[d, "\pi_i^{(\perp)}"'] \arrow[drr, phantom, "p^{(\mathrm{T})}=\eta p^{(\perp)}\mathrm{T}"] & & \mathrm{T}\perp_n A \arrow[d, "\mathrm{T}\pi_i^{(\perp)}"] \\
\perp A \arrow[r, "\eta\perp"'] \arrow[dr, equal] \arrow[dr, phantom, "\Delta"] & \perp\mathrm{T}\perp A \arrow[r, "p^{(\perp)}\mathrm{T}\perp"'] \arrow[d, "\perp\varepsilon"'] \arrow[dr, phantom, "\mathsf{Nat}"] & \mathrm{T}\perp A \arrow[d, "\varepsilon"] \\
& \perp A \arrow[r, "p^{(\perp)}"'] & A
\end{tikzcd}
$$

and moreover that $\pi_i^{(\perp)} p^{(\perp)} = \pi_j^{(\perp)} p^{(\perp)}$, for $i, j = 1, \ldots, n$. So, we have a unique morphism $\varepsilon_n \colon \mathrm{T}_n \perp_n \Rightarrow \mathsf{id}_{\mathbb{X}}$ such that:

$$
(\iota_k)_{\perp_n} \varepsilon_n = \mathrm{T}\pi_k^{(\perp)} \varepsilon
$$

for $k = 1, \ldots, n$. The naturality of η_n and ε_n is a direct consequence of the naturality of all morphisms involved in the diagrams in which they are constructed. Let's prove the triangle equalities for η_n and ε_n. In particular, we need to show that $\mathrm{T}_n \eta_n (\varepsilon_n)_{\mathrm{T}_n} = \mathsf{id}_{\mathrm{T}_n}$ and that $(\eta_n)_{\perp_n} \perp_n \varepsilon_n = \mathsf{id}_{\perp_n}$. Notice first that:

$$
\begin{tikzcd}
\mathrm{T}A \arrow[rr, "\iota_k"] \arrow[dr, "\mathrm{T}\eta_n"] \arrow[ddd, "\mathrm{T}\eta"'] & & \mathrm{T}_n A \arrow[d, "\mathrm{T}_n\eta_n"] \\
& \mathrm{T}\perp_n \mathrm{T}_n A \arrow[r, "\iota_k \perp_n \mathrm{T}_n"] \arrow[d, "\mathrm{T}\pi_k^{(\perp)}"] \arrow[ur, phantom, "\mathsf{Nat}"] \arrow[dr, phantom, "(\varepsilon_n{,}\iota_k)"] & \mathrm{T}_n \perp_n \mathrm{T}_n A \arrow[d, "\varepsilon_n \mathrm{T}_n"] \\
\arrow[r, phantom, "(\eta_n{,}\pi_k^{(\perp)})"] & \mathrm{T}\perp\mathrm{T}_n A \arrow[r, "\varepsilon\mathrm{T}_n"'] \arrow[dr, phantom, "\mathsf{Nat}"] & \mathrm{T}_n A \\
\mathrm{T}\perp\mathrm{T}A \arrow[ur, "\mathrm{T}\perp\iota_k"] \arrow[rr, "\varepsilon\mathrm{T}"'] & & \mathrm{T}A \arrow[u, "\iota_k"']
\end{tikzcd}
$$

Finally, by employing the triangle equality $\mathrm{T}\eta\varepsilon_{\mathrm{T}} = \mathsf{id}_{\mathrm{T}}$ one finds out that $\iota_k \mathrm{T}_n \eta_n (\varepsilon_n)_{\mathrm{T}_n} = \iota_k$, for $k = 1, \ldots, n$. So, from the universality of the n-fold pushout of $p^{(\mathrm{T})}$ along itself, we conclude that $\mathrm{T}_n\eta_n(\varepsilon_n)_{\mathrm{T}_n} = \mathsf{id}_{\mathrm{T}_n}$. Conversely, consider th following diagram:

$$\begin{tikzcd}
\perp_n A \arrow[r, "\eta_n\perp_n"] \arrow[dd, "\eta\perp_n"'] & \perp_n \mathrm{T}_n \perp_n A \arrow[r, "\perp_n\varepsilon_n"] \arrow[d, "\pi_k^{(\perp)}\mathrm{T}_n\perp_n"] & \perp_n A \arrow[dd, "\pi_k^{(\perp)}"] \\
& \perp\mathrm{T}_n\perp_n A \arrow[dr, "\perp\varepsilon_n"] & \\
\perp\mathrm{T}\perp_n A \arrow[ur, "\perp\iota_k\perp_n"] \arrow[r, "\varepsilon\perp_n"'] & \perp_n A \arrow[r, "\pi_k^{(\perp)}"'] & \perp A
\end{tikzcd}$$

So, $(\eta_n)_{\perp_n}\perp_n\varepsilon_n\pi_k^{(\perp)}\eta_{\perp}\perp\varepsilon\pi_k^{(\perp)} = \pi_k^{(\perp)}$, for $k = 1, \dots, n$. Thus, by employing the universality of the n-fold pullback of $p^{(\perp)}$ along itself, we conclude that $(\eta_n)_{\perp_n}\perp_n\varepsilon_n = \mathsf{id}_{\perp_n}$. ◻

Proposition 2.50. *Let $(\mathbb{X}, \mathbb{\perp})$ be a tangent category whose tangent bundle functor $\perp$ admits a left adjoint T so that $(\eta, \varepsilon)\colon \mathrm{T} \dashv \perp$ is an adjunction. If the mate $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathbb{X}} \Rightarrow \mathrm{T}$ of the projection $p^{(\perp)}\colon \perp \Rightarrow \mathsf{id}_{\mathbb{X}}$ admits n-fold pushouts T_n along itself, then $(\mathbb{X}, \mathbb{\perp})$ is adjunctable.*

Corollary 2.51. *If $\mathbb{X}$ is finitely cocomplete, then a tangent structure $\mathbb{\perp}$ over $\mathbb{X}$ forms an adjunctable tangent category $(\mathbb{X}, \mathbb{\perp})$ if and only if its tangent bundle functor $\perp$ admits a left adjoint T.*

2.3 Tangent objects: a formal approach to tangent categories

This section is dedicated to exploring a formal approach to tangent category theory. The author first introduced this approach, in the context of a Grothendieck construction for tangent fibrations (see [42]). The core concept of this discussion is the concept of tangent objects.

In this thesis, this formal approach plays an important role in contextualizing the theory of tangent monads. Tangent monads are 2-monads in the 2-category TngCat of tangent categories. In Section 2.4, we show that tangent monads are also tangent objects on the 2-category of monads and we employ this characterization to show an important result: the tangent category of algebras of a tangent monad described in [15] is precisely the algebra construction introduced by Street in [56] for a 2-monad.

Leung in their Ph.D. thesis [45] proposed a simple and effective classification of the tangent structures for a given category $\mathbb{X}$. In particular, they show that tangent structures $\mathbb{T}$ for $\mathbb{X}$ are in one-to-one correspondence with strong monoidal functors:

$$F_{\mathbb{T}} \colon \mathsf{Weil}_1 \to \mathsf{End}(\mathbb{X})$$

from the monoidal category of Weil algebras to the monoidal category of endofunctors over the category $\mathbb{X}$, satisfying extra conditions. A Weil algebra is a commutative and unital $\mathbb{N}$-algebra, obtained by quotienting the $\mathbb{N}$-algebra $\mathbb{N}[x_1, \dots, x_n]$ of $\mathbb{N}$-polynomials in n variables by an ideal generated by monomials of order 2. In particular, Weil_1 is the monoidal category generated by the Weil algebras $W^n := \mathbb{N}[x_1, \dots, x_n]/(x_i x_j, i \leq j)$, for positive integers n. As shown by Leung, in the category Weil_1 one can define the following morphisms:

projection The projection $p \colon W \to \mathbb{N}$, which sends the generator x of $W := W^1$ to 0;

zero morphism The zero morphism $z \colon \mathbb{N} \to W$, which sends an integer to itself;

sum morphism The sum morphism $s \colon W^2 \to W$, which sends the two generators x_1 and x_2 to the unique generator x;

vertical lift The vertical lift $l \colon W \to W \otimes W$, which sends the generator x to $x \otimes y$;

canonical flip The canonical flip $c \colon W \otimes W \to W \otimes W$, which sends the generator x of the left W to the generator y of the right W, and viceversa, i.e. y to x.

Leung's classification results state that a tangent structure $\mathbb{T}$ over a category $\mathbb{X}$ is precisely given by a strong monoidal functor $F_{\mathbb{T}}$ which sends the generators W^n of Weil_1 to the functors T_n, and the morphisms listed above to the synonymous natural transformations of $\mathbb{T}$. In particular, the tangent bundle functor is $\mathrm{T} = F_{\mathbb{T}}(W)$, the double tangent bundle functor is $\mathrm{T}^2 = F_{\mathbb{T}}(W \otimes W)$, the projection is $p = F_{\mathbb{T}}(p) \colon \mathrm{T} = F_{\mathbb{T}}(W) \to F_{\mathbb{T}}(\mathbb{N}) = \mathsf{id}_{\mathbb{X}}$, etcetera.

In this section we explore a generalization of this classification which leads to a simple but important tool for our discussion: the concept of tangent object. The idea of tangent objects was first introduced by the author of this thesis in [42] to extend the classical equivalence between fibrations and indexed categories known as the Grothendieck construction to the realm of tangent categories. The idea is to propose a formal theory of tangent structures for objects in a strict 2-category. This is similar in spirit to the formal theory of monads proposed by Street to generalize the notion of a monad in the context of 2-category theory. For our goal, let $\mathbf{C}$ be a fixed strict 2-category, that is a category enriched over Cat. In future work, we would like to explore weaker versions of this concept, but for now, let's focus on the strict case.

Before defining a tangent object, we first need to introduce an important technical definition, suggested by Lucyshyn-Wright in an informal discussion with the author.

Definition 2.52. *Given a strict 2-category $\mathbf{C}$ and two objects $\mathbb{X}$ and $\mathbb{Y}$ of $\mathbf{C}$, a limit in the category $\mathbf{C}(\mathbb{X},\mathbb{Y})$ is **pointwise** when it is preserved by all functors $\mathbf{C}(f,\mathbb{Y})\colon \mathbf{C}(\mathbb{X},\mathbb{Y}) \to \mathbf{C}(\mathbb{X}',\mathbb{Y})$ for each 1-morphism $f\colon \mathbb{X}' \to \mathbb{X}$ in $\mathbf{C}$.*

Remark 2.53. We would like to thank Rory Lucyshyn-Wright for pointing out the importance of this assumption for tangent objects. This aspect was missing in the original definition provided by the author.

When the 2-category $\mathbf{C}$ is the 2-category Cat of categories, pointwise limits of $\mathbf{C}(\mathbb{X},\mathbb{Y})$ are those limit diagrams in the category of functors of type $F\colon \mathbb{X} \to \mathbb{Y}$ that are preserved by the evaluation functor. Concretely, this means that, for an object X of $\mathbb{X}$, and a diagram $D\colon \mathbb{X}_0 \to \mathbf{C}(\mathbb{X},\mathbb{Y})$, the functor $\lim D\colon \mathbb{X} \to \mathbb{Y}$ evaluated at X is isomorphic to the object $\lim D(X)$ of $\mathbb{Y}$, where $D(X)$ represents the diagram $\mathbb{X}_0 \to \mathbb{Y}$ obtained by evaluating each functor $D_A\colon \mathbb{X} \to \mathbb{Y}$, corresponding to each A of $\mathbb{X}_0$, at X.

When the target category $\mathbb{Y}$ has all finite limits, then also the category of functors $\mathbf{C}(\mathbb{X},\mathbb{Y})$ has all finite limits and each limit is pointwise. However, when the target category is not known to be finitely complete, there is no guarantee that the limits of $\mathbf{C}(\mathbb{X},\mathbb{Y})$ will be pointwise. A counterexample can be found in [34, Section 3.3].

Unfortunately, in tangent category theory, the requirement of the existence of limits is a subtle matter since in differential geometry not every pair of morphisms admits a pullback. In particular, a tangent category cannot be required to be finitely complete since this would rule out one of the main examples of a tangent category. Consequently, in order to make our definition of tangent objects compatible with the usual notion of a tangent category when $\mathbf{C}$ is assumed to be the 2-category of categories, we need to require the limits involved in the definition of a tangent object to be pointwise.

Definition 2.54. *A* ***tangent object*** *in a* 2*-category* $\mathbf{C}$ *is an object* $\mathbb{X}$ *of* $\mathbf{C}$ *equipped with a* ***tangent structure*** $\mathbb{T}$*, which consists of a strong monoidal functor* $F_{\mathbb{T}} \colon \mathsf{Weil}_1 \to \mathsf{End}(\mathbb{X})$ *from the monoidal category of Weil algebras to the monoidal category of endomorphisms over* $\mathbb{X}$ *in* $\mathbf{C}$*, satisfying the following two universal conditions:*

1. $F_{\mathbb{T}}$ *preserves the* ***foundational pullbacks****, which are pullbacks of the form:*

$$\begin{tikzcd}
A \otimes (B \times C) \arrow[r, "A\otimes\pi_2"] \arrow[d, "A\otimes\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \otimes C \arrow[d, "A\otimes p"] \\
A \otimes B \arrow[r, "A\otimes p"'] & A
\end{tikzcd}$$

   for all $A, B, C \in \mathsf{Weil}_1$ *(cf. [45, Definition 3.17]). Moreover, these pullbacks are pointwise limits;*

2. $F_{\mathbb{T}}$ *preserves the universality of the vertical lift, i.e. the pullback diagram:*

$$\begin{tikzcd}
W^2 \arrow[r, "\xi"] \arrow[d, "\pi_1 p"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \otimes W \arrow[d, "W\otimes p"] \\
\mathbb{N} \arrow[r, "z"'] & W
\end{tikzcd}$$

   where $\xi \colon= \langle z \otimes W, l\rangle(W \otimes s)$ *and* $\pi_1 \colon W^2 \to W$ *sends* x_1 *to* x *and* x_2 *to zero. Moreover, this pullback is a pointwise limit.*

Remark 2.55. In Leung's original result, the universality of the vertical lift of Definition 2.54, is replaced with the universality of an equalizer. However, Cockett and Cruttwell proved in [12, Lemma 2.12] that the universality of the pullback diagram of Definition 2.54 is equivalent to the universality of the equalizer diagram proposed by Leung. To stay consistent with the rest of the thesis, we adopted the pullback version.

Remark 2.56. To classify tangent structures with negatives one can replace the rig $\mathbb{N}$ with the ring $\mathbb{Z}$ in the definition of a Weil algebra and then introduce the negation as follows:

negation The negation $n\colon W \to W$ sends the generator x to $-x$.

Thus, Leung's classification extends as follows: tangent structures with negatives $\mathbb{T}$ over a category $\mathbb{X}$ are in correspondence with strong monoidal functors $F_{\mathbb{T}}\colon \mathsf{Weil}_1^- \to \mathsf{End}(\mathbb{X})$ preserving foundational pullbacks and the universality of the vertical lift, where Weil_1^- is the category of Weil algebras over the ring $\mathbb{Z}$.

Thanks to Remark 2.56, we can also define a tangent object with negatives as follows.

Definition 2.57. *A tangent object **with negatives** in a 2-category $\mathbf{C}$ is an object $\mathbb{X}$ of $\mathbf{C}$ together with a tangent structure with negatives $\mathbb{T}$, which consists of a strong monoidal functor $F_{\mathbb{T}}\colon \mathsf{Weil}_1^- \to \mathsf{End}(\mathbb{X})$, preserving foundational pullbacks and the universality of the vertical lift as in Definition 2.54.*

Using a similar strategy to the one used by Leung to classify tangent structure on a given category, we can unwrap Definitions 2.54 and 2.57 to have a more concrete understanding of these key notions. Let's start by introducing a useful concept.

Definition 2.58. *An **additive bundle object** in a 2-category $\mathbf{C}$ is an additive bundle in the category $\mathsf{End}(\mathbb{X})$ of endomorphisms of $\mathbb{X}$. Concretely, it consists of an object $\mathbb{X} \in \mathbf{C}$, two 1-endomorphisms $B\colon \mathbb{X} \to \mathbb{X}$ and $E\colon \mathbb{X} \to \mathbb{X}$, together with a 2-morphism $q\colon E \Rightarrow \mathsf{id}_{\mathbb{X}}$, called the **projection**, a 2-morphism $z_q\colon \mathsf{id}_{\mathbb{X}} \Rightarrow E$ called the **zero morphism**, and a 2-morphism $s_q\colon E_2 \Rightarrow E$, called the **sum morphism**, satisfying the following properties:*

1. *z_q is a section of q:*

$$\begin{tikzcd} E \arrow[dr, "q"] & \\ B \arrow[u, "z_q"] \arrow[r, equal] & B \end{tikzcd}$$

2. *n-fold pullbacks: for any positive integer n, the n-fold pullback of the projection q along itself exists in the category $\mathsf{End}(\mathbb{X})$ of endomorphisms of $\mathbb{X}$, is a pointwise limit,*

and its preserved by each $E^m := E \circ \ldots \circ E$*, for every positive integer* m*. The k-th projection* $\pi_k \colon E_n \Rightarrow E$ *is denoted by* π_k*;*

3. s_q *is a bundle morphism:*

$$\begin{tikzcd} E_2 \arrow[r, "s_q"] \arrow[d, "\pi_1"'] & E \arrow[d, "q"] \\ E \arrow[r, "q"'] & B \end{tikzcd} \quad \begin{tikzcd} E_2 \arrow[r, "s_q"] \arrow[d, "\pi_2"'] & E \arrow[d, "q"] \\ E \arrow[r, "q"'] & B \end{tikzcd}$$

   $\pi_k \colon E_n \to E$ *being the k-th projection of the pointwise n-fold pullback;*

4. *Associativity:*

$$\begin{tikzcd} E_3 \arrow[r, "\mathsf{id}_E \times_A s_q"] \arrow[d, "s_q \times_A \mathsf{id}_E"'] & E_2 \arrow[d, "s_q"] \\ E_2 \arrow[r, "s_q"'] & E \end{tikzcd}$$

5. *Unitality:*

$$\begin{tikzcd} E_2 \arrow[r, "s_q"] & E \\ E \arrow[u, "\langle q z_q, \mathsf{id}_E \rangle"] \arrow[ur, equal] & \end{tikzcd}$$

6. *Commutativity:*

$$\begin{tikzcd} E_2 \arrow[r, "s_q"] \arrow[d, "\tau"'] & E \arrow[d, equal] \\ E_2 \arrow[r, "s_q"'] & E \end{tikzcd}$$

   where $\tau \colon E_2 \to E_2$ *denotes the flip* $\langle \pi_2, \pi_1 \rangle$*.*

A 1-***morphism of additive bundle objects*** $(\psi, \varphi) \colon (B, E, q, z_q, s_q) \to (B', E', q', z'_q, s'_q)$ *over* $\mathbb{X}$ *consists of two* 2*-morphisms* $\varphi \colon E \Rightarrow E'$ *and* $\psi \colon B \Rightarrow B'$*, satisfying the following properties:*

1. *Compatibility with the projections:*

$$
\begin{tikzcd}
E \arrow[r, "g"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\
B \arrow[r, "f"'] & B'
\end{tikzcd}
$$

2. *Compatibility with the zero morphisms:*

$$
\begin{tikzcd}
E \arrow[r, "g"] & E' \\
B \arrow[r, "f"'] \arrow[u, "z_q"] & B' \arrow[u, "z'_q"']
\end{tikzcd}
$$

3. *Additivity:*

$$
\begin{tikzcd}
E_2 \arrow[r, "g\times_B g"] \arrow[d, "s_q"'] & E'_2 \arrow[d, "s'_q"] \\
E \arrow[r, "g"'] & E'
\end{tikzcd}
$$

Notation 2.59. In the following, we adopt the following convention: given two 1-morphisms $\mathrm{T}\colon \mathbb{X} \to \mathbb{X}$, $\mathrm{T}'\colon \mathbb{X}' \to \mathbb{X}'$ and two 2-morphisms:

$$
\begin{tikzcd}
\mathbb{X} \arrow[r] \arrow[d] & \mathbb{X}' \arrow[d, equal] \arrow[dl, Rightarrow, "f"'] \\
\mathbb{X}'' \arrow[r] & \mathbb{X}'
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{X} \arrow[r] \arrow[d, equal] & \mathbb{X}' \arrow[d] \arrow[dl, Rightarrow, "g"'] \\
\mathbb{X} \arrow[r] & \mathbb{X}''
\end{tikzcd}
$$

we write $\mathrm{T}'f$ for:

$$
\begin{tikzcd}
\mathbb{X} \arrow[r] \arrow[d] & \mathbb{X}' \arrow[r, "\mathrm{T}'"] \arrow[d, equal] \arrow[dl, Rightarrow, "f"'] & \mathbb{X}' \arrow[d, equal] \\
\mathbb{X}'' \arrow[r] & \mathbb{X}' \arrow[r, "\mathrm{T}'"'] & \mathbb{X}'
\end{tikzcd}
$$

and g_{T} for:

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r] \arrow[d, equal] & \mathbb{X}' \arrow[d] \arrow[dl, Rightarrow, "g"'] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r] & \mathbb{X}''
\end{tikzcd}
$$

A tangent object $(\mathbb{X}, \mathbb{T})$ in $\mathbf{C}$ is an object $\mathbb{X}$ of $\mathbf{C}$ equipped with the following data:

tangent 1-morphism A 1-morphism $\mathrm{T} \colon \mathbb{X} \to \mathbb{X}$;

projection A 2-morphism p:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[d, equal] \arrow[dl, Rightarrow, "p"] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd}$$

zero 2-morphism A 2-morphism z:

$$\begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, equal] & \mathbb{X} \arrow[d, equal] \arrow[dl, Rightarrow, "z"] \\ \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \end{tikzcd}$$

sum 2-morphism A 2-morphism s:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\mathrm{T}_2"] \arrow[d, equal] & \mathbb{X} \arrow[d, equal] \arrow[dl, Rightarrow, "s"] \\ \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \end{tikzcd}$$

such that (T, p, z, s) is an additive bundle object of $\mathbf{C}$;

vertical lift A 2-morphism l:

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "\mathrm{T}"] \arrow[d, equal] & & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "l"] \\ \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \end{tikzcd}$$

so that $(z, l) \colon (\mathsf{id}_{\mathbb{X}}, \mathrm{T}, p, z, s) \to (\mathrm{T}, \mathrm{T}^2, \mathrm{T}p, \mathrm{T}z, \mathrm{T}s)$ is a morphism of additive bundle objects;

canonical flip A 2-morphism c:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

> so that $(\mathsf{id}_T, c)\colon (T, T^2, Tp, Tz, Ts) \to (T, T^2, p_T, z_T, s_T)$ is a morphism of additive bundle objects.

Moreover, the following conditions are satisfied:

$$\begin{tikzcd}
\mathbb{X} \arrow[rr, "T"] \arrow[d, equal] & & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "l"] \\
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[rr, "T"] \arrow[d, equal] & & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "l"] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{X} \arrow[rrr, "T"] \arrow[d, equal] & & & \mathbb{X} \arrow[d, equal] \arrow[dlll, Rightarrow, "l"] \\
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[rr, "T"] \arrow[d, equal] & & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "l"] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[rrr, "T"] \arrow[d, equal] & & & \mathbb{X} \arrow[d, equal] \arrow[dlll, Rightarrow, "l"] \\
\mathbb{X} \arrow[rr, "T"] \arrow[d, equal] & & \mathbb{X} \arrow[r, "T"] \arrow[d, equal] \arrow[dll, Rightarrow, "l"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[rr, "\mathrm{T}"] & & \mathbb{X} \arrow[d, equal] \arrow[dlll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[rr, "\mathrm{T}"] \arrow[d, equal] & & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "l"] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[rr, "\mathrm{T}"] \arrow[d, equal] & & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] \arrow[dll, Rightarrow, "l"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] \arrow[dll, Rightarrow, "c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] \arrow[dll, Rightarrow, "c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] \arrow[dll, Rightarrow, "c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] \arrow[dll, Rightarrow, "c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
$$

Finally, the vertical lift is universal in the following sense. The diagram:

$$
\begin{tikzcd}
\mathrm{T}_2 \arrow[r, Rightarrow, "\xi"] \arrow[d, Rightarrow, "\pi_1 p"'] & \mathrm{T}^2 \arrow[d, Rightarrow, "\mathrm{T}p"] \\
\mathsf{id}_{\mathbb{X}} \arrow[r, Rightarrow, "z"'] & \mathrm{T}
\end{tikzcd}
$$

is a pointwise pullback in $\mathbf{C}(\mathbb{X}, \mathbb{X})$, where $\xi \colon = (l \times z_{\mathrm{T}})\mathrm{T}s$. We refer to the tuple $\mathbb{T} \colon = (\mathrm{T}, p, z, s, l, c)$ as a **tangent structure** over $\mathbb{X}$. Finally, a tangent object with negatives is a tangent object equipped with an extra structure:

negation A 2-morphism:

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[d, equal] \arrow[dl, Rightarrow, "n"] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
$$

satisfying the following property:

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow\langle n,\mathsf{id}\rangle"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}_2"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow s"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
$$

We introduce the following naming convention.

Convention 2.60. Given a 2-category $\mathbf{C}$ whose objects are called with a name x, we refer to a tangent object of $\mathbf{C}$ as a **tangent** $\boldsymbol{x}$.

The next example shows that our naming convention is consistent with the notion of tangent category, that is, tangent categories are tangent objects in the 2-category of categories.

Example 2.61. The obvious example of tangent objects is given by tangent categories. Thanks to Leung's classification theorem, a tangent category is a category $\mathbb{X}$ equipped with a strong monoidal functor $F_{\mathbb{T}}\colon \mathsf{Weil}_1 \to \mathsf{End}(\mathbb{X})$ satisfying some universality conditions. So, by taking the 2-category Cat of (small) categories, functors and natural transformations, we see that a tangent object of Cat is precisely a tangent category.

Notice that, as pointed out by Lucyshyn-Wright (see Remark 2.53), for tangent objects of Cat to be tangent categories, it is important that the limit diagrams involved in Definition 2.54 are pointwise.

Example 2.62. Let $\mathbf{C}$ be a 2-category and consider the 2-category $\mathsf{Mnd}(\mathbf{C})$ whose objects are pairs $(\mathbb{X}, S)$ formed by an object $\mathbb{X}$ of $\mathbf{C}$ and a formal monad S of $\mathbb{X}$. Recall that a formal monad in a 2-category over an object $\mathbb{X}$ consists of a monoid in the monoidal category $\mathsf{End}(\mathbb{X})$ of endomorphisms of $\mathbb{X}$. Concretely, a formal monad consists of an endomorphism $S\colon \mathbb{X} \to \mathbb{X}$ together with two 2-morphisms $\eta\colon \mathsf{id}_{\mathbb{X}} \Rightarrow S$ and $\gamma\colon S^2 \Rightarrow S$, where $S^2 := S \circ S$, satisfying associativity and unitality conditions. A morphism of formal monads $(F,\alpha)\colon(\mathbb{X},S)\to(\mathbb{X}',S')$ consists of a 1-morphism $F\colon\mathbb{X}\to\mathbb{X}'$ together with a 2-morphism:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "F"] \arrow[d, "S"'] & \mathbb{X}' \arrow[d, "S'"] \arrow[dl, phantom, "\swarrow\alpha"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{X}'
\end{tikzcd}$$

satisfying the following compatibilities with the units η and η' and the multiplications γ and γ' of the monads S and S', respectively:

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "F"] \arrow[d, "S"'] & \mathbb{X}' \arrow[r, equal] \arrow[d, "S'"] \arrow[dl, phantom, "\swarrow\alpha"] & \mathbb{X}' \arrow[d, equal] \arrow[dl, phantom, "\swarrow\eta'"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{X}' \arrow[r, equal] & \mathbb{X}'
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[r, equal] \arrow[d, "S"'] & \mathbb{X} \arrow[r, "F"] \arrow[d, equal] \arrow[dl, phantom, "\swarrow\eta"] & \mathbb{X}' \arrow[d, equal] \\
\mathbb{X} \arrow[r, equal] & \mathbb{X}' \arrow[r, "F"'] & \mathbb{X}'
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "F"] \arrow[d, equal] & \mathbb{X}' \arrow[r, "S'"] & \mathbb{X}' \arrow[d, equal] \arrow[dll, phantom, "\swarrow\alpha"] \\
\mathbb{X} \arrow[r, "S"] \arrow[d, equal] & \mathbb{X} \arrow[r, "F"] \arrow[d, "S"] \arrow[dl, phantom, "\swarrow\gamma"] & \mathbb{X}' \arrow[d, "S'"] \arrow[dl, phantom, "\swarrow\alpha"] \\
\mathbb{X} \arrow[r, "S"'] & \mathbb{X} \arrow[r, "F"'] & \mathbb{X}'
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[r, "F"] \arrow[d, equal] & \mathbb{X}' \arrow[r, "S'"] \arrow[d, equal] & \mathbb{X}' \arrow[d, "S'"] \arrow[dl, phantom, "\swarrow\gamma'"] \\
\mathbb{X} \arrow[r, "F"] \arrow[d, equal] & \mathbb{X}' \arrow[r, "S'"] & \mathbb{X}' \arrow[d, equal] \arrow[dll, phantom, "\swarrow\alpha"] \\
\mathbb{X} \arrow[r, "S"'] & \mathbb{X} \arrow[r, "F"'] & \mathbb{X}'
\end{tikzcd}$$

Finally, given two 1-morphisms $(F,\alpha),(F',\alpha')\colon(\mathbb{X},S)\to(\mathbb{X}',S')$, a 2-morphism $\theta\colon(F,\alpha)\to(F',\alpha')$ of $\mathsf{Mnd}(\mathbf{C})$ is a natural transformation $\theta\colon F\Rightarrow F'$ satisfying the following condition:

$$\begin{tikzcd}
F\circ S \arrow[r, "\alpha"] \arrow[d, "\theta S"'] & S'\circ F \arrow[d, "S'\theta"] \\
F'\circ S \arrow[r, "\alpha'"'] & S'\circ F'
\end{tikzcd}$$

By spelling out the details one finds out that a tangent object of $\mathsf{Mnd}(\mathbf{C})$ consists of a tangent object $(\mathbb{X},\mathbb{T})$ of $\mathbf{C}$ together with a formal monad S of $\mathbb{X}$ with a 2-morphism $\alpha\colon S\circ\mathrm{T}\Rightarrow\mathrm{T}\circ S$ compatible with the tangent structure $\mathbb{T}$ of $\mathbb{X}$. We refer to $(\mathbb{X},\mathbb{T};S,\alpha)$ as a **formal tangent monad**. When $\mathbf{C}$ is the 2-category Cat, formal tangent monads are precisely tangent monads, as introduced by Cockett, Lemay, and Lucyshyn-Wright in [15]. We return to this example in a moment, since tangent monads play a crucial role in our discussion.

Example 2.63. Let's consider the 2-category MonCat whose objects are monoidal categories $(\mathbb{X},\otimes,\mathbb{1},\alpha,\lambda,\rho)$ with associator α and left and right unitors λ and ρ, respectively, 1-morphisms are strong monoidal functors $(F,\varepsilon,\mu)\colon(\mathbb{X},\otimes,\mathbb{1},\alpha,\lambda,\rho)\to(\mathbb{X}',\otimes',\mathbb{1}',\alpha',\lambda',\rho')$, i.e. functors $F\colon\mathbb{X}\to\mathbb{X}'$ together with an isomorphism $\varepsilon\colon\mathbb{1}'\to F\mathbb{1}$ and a natural isomorphism $\mu_{X,Y}\colon F(X)\otimes' F(Y)\to F(X\otimes Y)$, compatible with the associators and the unitors, and 2-morphisms are natural transformations compatible with the morphisms ε and μ of the strong monoidal functors.

Then a tangent object of MonCat consists of a monoidal category $(\mathbb{X},\otimes,\mathbb{1},\alpha,\lambda,\rho)$ equipped with a tangent structure, so that $(\mathbb{X},\mathbb{T})$ is also a tangent category and with an isomorphism $\mathbb{1}\to\mathrm{T}(\mathbb{1})$, that we call **tangent unitor**, and a natural isomorphism $\mathrm{T}M\otimes\mathrm{T}N\to\mathrm{T}(M\otimes N)$ that we call **tangent distributor**, compatible with the associator and the unitors. Employing the Convention 2.60, we call the tangent objects of MonCat, **tangent monoidal categories**.

Notice that the 2-category TngCat of tangent categories admits products. This allows one to wonder what are pseudomonoids of TngCat. Recall that a pseudomonoid in a 2-category $\mathbf{C}$ with products consists of an object $\mathbb{X}$, together with two 1-morphisms $\otimes\colon\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ and $\eta\colon\mathbb{1}\to\mathbb{X}$, $\mathbb{1}$ being terminal in $\mathbf{C}$, and three 2-isomorphisms:

$$\begin{tikzcd}
\mathbb{X}\times\mathbb{X}\times\mathbb{X} \arrow[r, "\otimes\times\mathsf{id}_{\mathbb{X}}"] \arrow[d, "\mathsf{id}_{\mathbb{X}}\times\otimes"'] & \mathbb{X}\times\mathbb{X} \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathbb{X}\times\mathbb{X} \arrow[r, "\otimes"'] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "\langle !,\mathsf{id}_{\mathbb{X}}\rangle"] \arrow[d, equal] & \mathbb{1}\times\mathbb{X} \arrow[r, "\eta\times\mathsf{id}_{\mathbb{X}}"] & \mathbb{X}\times\mathbb{X} \arrow[d, "\otimes"] \arrow[dll, Rightarrow, "\lambda"'] \\
\mathbb{X} \arrow[rr, equal] & & \mathbb{X}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\langle\mathsf{id}_{\mathbb{X}},!\rangle"] \arrow[d, equal] & X\times\mathbb{1} \arrow[r, "\mathsf{id}_{\mathbb{X}}\times\eta"] & \mathbb{X}\times\mathbb{X} \arrow[d, "\otimes"] \arrow[dll, Rightarrow, "\rho"'] \\
\mathbb{X} \arrow[rr, equal] & & \mathbb{X}
\end{tikzcd}$$

satisfying the same pentagonal and hexagonal diagrams of the associator and unitors in the definition of a monoidal category.

In a similar way tangent monads can be equivalently described as formal monads in the 2-category TngCat of tangent categories (see [15]) or as tangent objects in the 2-category $\mathsf{Mnd}(\mathsf{Cat})$ of monads, it turns out that tangent monoidal categories can also be equivalently described as pseudo-monoids in the 2-category TngCat of tangent categories. We refer to this second description as **monoidal tangent categories** and in the future, we use tangent monoidal categories and monoidal tangent categories, interchangeably.

Proposition 2.64. *There is an equivalence between the category of tangent monoidal categories and the category of monoidal tangent categories, defined as pseudo-monoids in the category of tangent categories.*

Example 2.63 can be extended to other classes of monoidal categories. For example, one can consider braided monoidal categories, symmetric monoidal categories, or closed symmetric monoidal categories. The corresponding tangent objects are then called **tangent braided monoidal categories**, **tangent symmetric monoidal categories**, and **tangent closed symmetric monoidal categories**.

Example 2.65. An enriched category $\mathbb{Y}$ over a monoidal category $(\mathbb{X},\otimes,\mathbb{1},\alpha,\lambda,\rho)$ consists of a collection of objects together, for each pair M,N of objects, an object $\mathbb{Y}(M,N)$ of $\mathbb{X}$, which plays the role of the Hom-Set functor (cf. [34]). Moreover, an enriched category comes equipped with a collection $\circ\colon \mathbb{Y}(N,P)\otimes\mathbb{Y}(M,N)\to\mathbb{Y}(M,P)$ of morphisms of $\mathbb{X}$, which plays the role of the composition of morphisms, and a collection $\mathsf{id}\colon \mathbb{1}\to\mathbb{Y}(M,M)$ of morphisms of $\mathbb{X}$, which plays the role of the identity morphisms. One can define a 2-category Enrch whose objects are pairs $(\mathbb{X},\mathbb{Y})$ formed by a monoidal category $\mathbb{X}\colon=(\mathbb{X},\otimes,\mathbb{1},\alpha,\lambda,\rho)$ together with an enriched category $\mathbb{Y}$ over $\mathbb{X}$. A 1-morphism $(F,G,\beta)\colon(\mathbb{X},\mathbb{Y})\to(\mathbb{X}',\mathbb{Y}')$ of Enrch consists of a strong monoidal functor $F\colon=(F,\varepsilon,\eta)\colon\mathbb{X}\to\mathbb{X}'$ together with a pair $(G,\beta)\colon\mathbb{Y}\to\mathbb{Y}'$, formed by an operation G which sends an object M of $\mathbb{Y}$ to an object GM of $\mathbb{Y}'$, together with a collection of isomorphisms of $\mathbb{X}$:

$$\beta\colon F(\mathbb{Y}(M,N))\to\mathbb{Y}'(GM,GN)$$

compatible with the morphisms $\circ$ and id. Finally, 2-morphisms $(\varphi,\psi)\colon(F,G,\beta)\to(F',G',\beta')$ between two 1-morphisms $(F,G,\beta),(F',G',\beta')\colon(\mathbb{X},\mathbb{Y})\to(\mathbb{X}',\mathbb{Y}')$ consist of a natural transformation of strong monoidal functors $\varphi\colon F \to F'$ together with a collection of morphisms:

$$\psi\colon \mathbb{Y}'(GM, GN) \to \mathbb{Y}'(G'M, G'N)$$

satisfying the following condition:

$$\begin{tikzcd}
F\mathbb{Y}(M,N) \arrow[r, "\varphi"] \arrow[d, "\beta"'] & F'\mathbb{Y}(M,N) \arrow[d, "\beta'"] \\
\mathbb{Y}'(GM,GN) \arrow[r, "\psi"'] & \mathbb{Y}'(G'M,G'N)
\end{tikzcd}$$

Spelling out the details one finds that a tangent object of Enrch consists of tangent monoidal category $(\mathbb{X}, \mathbb{T})$ together with an $\mathbb{X}$-enriched category $\mathbb{Y}$ equipped with an operation T' which sends an object M to another object $\mathrm{T}'M$ of $\mathbb{Y}$, with a collection of isomorphisms $\beta\colon \mathrm{T}(\mathbb{Y}(M,N)) \to \mathbb{Y}(\mathrm{T}'M, \mathrm{T}'N)$, compatible with $\circ$ and id. Moreover, $\mathbb{Y}$ comes equipped with a list of collections of morphisms of $\mathbb{X}$, $p'\colon \mathbb{Y}(\mathrm{T}'M, \mathrm{T}'N) \to \mathbb{Y}(M,N)$, $z'\colon \mathbb{Y}(M,N) \to (\mathrm{T}'M, \mathrm{T}'N)$, $s'\colon \mathbb{Y}_2(\mathrm{T}'M, \mathrm{T}'N) \to \mathbb{Y}(\mathrm{T}'M, \mathrm{T}'N)$, $\mathbb{Y}_2(\mathrm{T}'M, \mathrm{T}'N)$ denoting the pullback of p' along itself, $l'\colon \mathbb{Y}(\mathrm{T}'M, \mathrm{T}'N) \to \mathbb{Y}(\mathrm{T}'^2M, \mathrm{T}'^2N)$, and $c\colon \mathbb{Y}(\mathrm{T}'^2M, \mathrm{T}'^2N) \to \mathbb{Y}(\mathrm{T}'^2M, \mathrm{T}'^2N)$, satisfying some compatibility conditions with the tangent structure of $\mathbb{X}$.

These are only some of the infinitely many examples of tangent objects of a given 2-category. In the future, we intend to explore notions like tangent model categories, tangent internal categories, double tangent categories (that are tangent objects in the 2-category of tangent categories), tangent double categories (that are tangent objects in the 2-category of double categories), tangent topoi, tangent sheaves and many more.

The next step is to introduce 1-morphisms of tangent objects.

Definition 2.66. *A* ***lax*** $\mathbf{1}$***-morphism of tangent objects*** $(F, \alpha)\colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}', \mathbb{T}')$ *between two tangent objects* $(\mathbb{X}, \mathbb{T})$ *and* $(\mathbb{X}', \mathbb{T}')$ *in a* 2*-category* $\mathbf{C}$ *consists of a* 1*-morphism* $F\colon \mathbb{X} \to \mathbb{X}'$ *of* $\mathbf{C}$ *together with a* 2*-morphism:*

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, "F"'] & \mathbb{X} \arrow[d, "F"] \arrow[dl, phantom, "\swarrow\alpha"] \\
\mathbb{X}' \arrow[r, "\mathrm{T}'"'] & \mathbb{X}'
\end{tikzcd}$$

so that $(F, \alpha, \mathsf{id}_{\mathbb{X}})$ is a morphism of additive bundle objects and the following conditions are satisfied:

$$
\begin{tikzcd}
\mathbb{X} \arrow[rr, "\mathrm{T}"] \arrow[d, "F"'] & & \mathbb{X} \arrow[d, "F"] \arrow[dll, Rightarrow, "\alpha"'] \\
\mathbb{X}' \arrow[rr, "\mathrm{T}'"] \arrow[d, equal] & & \mathbb{X}' \arrow[d, equal] \arrow[dll, Rightarrow, "l'"'] \\
\mathbb{X}' \arrow[r, "\mathrm{T}'"'] & \mathbb{X} \arrow[r, "\mathrm{T}'"'] & \mathbb{X}'
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[rr, "\mathrm{T}"] \arrow[d, equal] & & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "l"'] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, "F"'] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, "F"] \arrow[dl, Rightarrow, "\alpha"'] & \mathbb{X} \arrow[d, "F"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathbb{X}' \arrow[r, "\mathrm{T}'"'] & \mathbb{X}' \arrow[r, "\mathrm{T}'"'] & \mathbb{X}'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] & \mathbb{X} \arrow[d, equal] \arrow[dll, Rightarrow, "c"'] \\
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, "F"'] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, "F"] \arrow[dl, Rightarrow, "\alpha"'] & \mathbb{X} \arrow[d, "F"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathbb{X}' \arrow[r, "\mathrm{T}'"'] & \mathbb{X}' \arrow[r, "\mathrm{T}'"'] & \mathbb{X}'
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, "F"'] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, "F"] \arrow[dl, Rightarrow, "\alpha"'] & \mathbb{X} \arrow[d, "F"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathbb{X}' \arrow[r, "\mathrm{T}'"] \arrow[d, equal] & \mathbb{X}' \arrow[r, "\mathrm{T}'"] & \mathbb{X}' \arrow[d, equal] \arrow[dll, Rightarrow, "c"'] \\
\mathbb{X} \arrow[r, "\mathrm{T}'"'] & \mathbb{X} \arrow[r, "\mathrm{T}'"'] & \mathbb{X}
\end{tikzcd}
$$

*Similarly, a **colax 1-morphism of tangent objects** $(G, \beta)\colon (\mathbb{X}, \mathbb{T}) \nrightarrow (\mathbb{X}', \mathbb{T}')$ consists of a 1-morphism $G\colon \mathbb{X} \to \mathbb{X}'$ together with a 2-morphism:*

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "F"] \arrow[d, "\mathrm{T}"'] & \mathbb{X}' \arrow[d, "\mathrm{T}'"] \arrow[dl, Rightarrow, "\beta"'] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{X}'
\end{tikzcd}
$$

*satisfying similar conditions as α above. A lax morphism (F, α) of tangent objects is **strong** if α is invertible and **strict** if α is the identity.*

We are also interested in defining 2-morphisms of tangent objects.

Definition 2.67. *Given two lax 1-morphisms $(F, \alpha), (F', \alpha')\colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}', \mathbb{T}')$ of tangent objects, a **lax 2-morphism of tangent ojects** $\varphi\colon (F, \alpha) \Rightarrow (F', \alpha')$ consists of a*

2-morphism $\varphi\colon F \Rightarrow F'$ satisfying the following compatibility condition:

$$\begin{tikzcd}
F \circ \mathrm{T} \arrow[r, "\alpha"] \arrow[d, "\varphi\mathrm{T}"'] & \mathrm{T}' \circ F \arrow[d, "\mathrm{T}'\varphi"] \\
F' \circ \mathrm{T} \arrow[r, "\alpha'"'] & \mathrm{T}' \circ F'
\end{tikzcd}$$

Similarly, given two colax 1-morphisms $(G, \beta), (G', \beta')\colon (\mathbb{X}, \mathbb{T}) \nrightarrow (\mathbb{X}', \mathbb{T}')$ of tangent objects, a ***colax 2-morphism of tangent objects*** *$\psi\colon (G, \beta) \Rightarrow (G', \beta')$ consists of a 2-morphism $\psi\colon G \Rightarrow G'$ satisfying the dual of the compatibility condition of a lax 2-morphism, i.e.:*

$$\begin{tikzcd}
\mathrm{T}' \circ G \arrow[r, "\beta"] \arrow[d, "\mathrm{T}'\psi"'] & G \circ \mathrm{T} \arrow[d, "\psi\mathrm{T}"] \\
\mathrm{T}' \circ G' \arrow[r, "\beta'"'] & G' \circ \mathrm{T}
\end{tikzcd}$$

Finally, a ***double morphism of tangent objects****:*

$$\begin{tikzcd}
(\mathbb{X}_\circ, \mathbb{T}_\circ) \arrow[r, "{(G,\beta)}"] \arrow[d, "{(F_\circ,\alpha_\circ)}"', "\shortmid" marking] & (\mathbb{X}_\bullet, \mathbb{T}_\bullet) \arrow[d, "{(F_\bullet,\alpha_\bullet)}", "\shortmid" marking] \arrow[dl, Rightarrow, "\theta"] \\
(\mathbb{X}'_\circ, \mathbb{T}'_\circ) \arrow[r, "{(G',\beta')}"'] & (\mathbb{X}'_\bullet, \mathbb{T}'_\bullet)
\end{tikzcd}$$

for the lax 1-morphisms (G, β) and (G', β') and the colax 1-morphisms $(F_\circ, \alpha_\circ), (F_\bullet, \alpha_\bullet)$ is a 2-morphism:

$$\begin{tikzcd}
\mathbb{X}_\circ \arrow[r, "F_\circ"] \arrow[d, "G"'] & \mathbb{X}'_\circ \arrow[d, "G'"] \arrow[dl, Rightarrow, "\theta"] \\
\mathbb{X}_\bullet \arrow[r, "F_\bullet"'] & \mathbb{X}'_\bullet
\end{tikzcd}$$

satisfying the following properties:

$$\begin{tikzcd}
\mathbb{X}_\circ \arrow[r, "\mathrm{T}_\circ"] \arrow[d, equal] & \mathbb{X}_\circ \arrow[r, "F_\circ"] \arrow[dl, phantom, "\Swarrow\alpha_\circ"] & \mathbb{X}'_\circ \arrow[d, equal] \\
\mathbb{X}_\circ \arrow[r, "F_\circ"] \arrow[d, "G"'] & \mathbb{X}'_\circ \arrow[r, "\mathrm{T}'_\circ"] \arrow[d, "G'"] \arrow[dl, phantom, "\Swarrow\theta"] & \mathbb{X}'_\circ \arrow[d, "G'"] \arrow[dl, phantom, "\Swarrow\beta'"] \\
\mathbb{X}_\bullet \arrow[r, "F_\bullet"'] & \mathbb{X}'_\bullet \arrow[r, "\mathrm{T}'_\bullet"'] & \mathbb{X}'_\bullet
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{X}_\circ \arrow[r, "\mathrm{T}_\circ"] \arrow[d, "G"'] & \mathbb{X}_\circ \arrow[r, "F_\circ"] \arrow[d, "G"] \arrow[dl, phantom, "\Swarrow\beta"] & \mathbb{X}'_\circ \arrow[d, "G'"] \arrow[dl, phantom, "\Swarrow\theta"] \\
\mathbb{X}_\bullet \arrow[r, "\mathrm{T}_\bullet"] \arrow[d, equal] & \mathbb{X}_\bullet \arrow[r, "F_\bullet"] \arrow[dl, phantom, "\Swarrow\alpha_\bullet"] & \mathbb{X}'_\bullet \arrow[d, equal] \\
\mathbb{X}_\bullet \arrow[r, "F_\bullet"'] & \mathbb{X}'_\bullet \arrow[r, "\mathrm{T}'_\bullet"'] & \mathbb{X}'_\bullet
\end{tikzcd}$$

Tangent objects of a 2-category $\mathbf{C}$ together with lax tangent 1-morphisms and lax 2-morphisms form a 2-category $\mathsf{Tng}(\mathbf{C})$. Similarly, tangent objects of $\mathbf{C}$ together with colax tangent 1-morphisms and colax 2-morphisms form a 2-category $\mathsf{Tng}_{\mathsf{co}}(\mathbf{C})$. The 2-subcategory of $\mathsf{Tng}(\mathbf{C})$ whose 1-morphisms are strong, that is the distributive law is invertible, is denoted by $\mathsf{Tng}_{\cong}(\mathbf{C})$ and the 2-subcategory of $\mathsf{Tng}(\mathbf{C})$ whose 1-morphisms are strict, i.e. the distributive law is the identity, is denoted by $\mathsf{Tng}_{=}(\mathbf{C})$. Finally, tangent objects together with lax tangent 1-morphisms as horizontal morphisms, colax tangent 1-morphisms as vertical morphisms, and double tangent cells for double cells form also a double category denoted by $\mathbf{Tng}(\mathbf{C})$. When $\mathbf{C}$ is the 2-category Cat of categories, the double category $\mathbf{Tng}(\mathsf{Cat})$ is precisely the double category $\mathbb{T}\mathsf{ngCat}$ of tangent categories, first described in [41, Proposition 2.2].

We conclude this section, by showing that the operation Tng which sends a 2-category $\mathbf{C}$ to the 2-category $\mathsf{Tng}(\mathbf{C})$ of tangent objects of $\mathbf{C}$ extends to a 2-functor. For this purpose, we first need to select the correct class of morphisms between 2-categories. Indeed, if $F\colon \mathbf{C} \to \mathbf{C}'$ is an arbitrary 2-functor and $(\mathbb{X}, \mathbb{T})$ a tangent object of $\mathbf{C}$, there is no reason, in general, that $F\mathbb{X}$ is also a tangent object of $\mathbf{C}'$. The main issue is that, to make $F\mathbb{X}$ into a tangent object, the 2-functor F must preserve the n-fold pullbacks of the projection with itself and the universality of the vertical lift. Recall that a 2-functor $F\colon \mathbf{C} \to \mathbf{C}'$ (notice that here we work with strict 2-functors) is an operation which sends objects M, 1-morphisms $f\colon M \to N$, and 2-morphisms $\theta\colon f \Rightarrow g$ of $\mathbf{C}$ to objects F_0M, 1-morphisms $F_1f\colon F_0M \to F_0N$, and 2-morphisms $F_2\theta\colon F_1f \Rightarrow F_2g$, respectively, in a compatible way with the composition and the identities.

Definition 2.68. *A* 2*-functor* $F\colon \mathbf{C} \to \mathbf{C}'$ *is* 2***-pullback preserving*** *if it preserves pullbacks of the form:*

$$\begin{tikzcd}
\bullet \arrow[r, Rightarrow, "\pi_2"] \arrow[d, Rightarrow, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, Rightarrow, "p"] \\
\bullet \arrow[r, Rightarrow, "q"'] & \bullet
\end{tikzcd}$$

where p, q, π_1*, and* π_2 *are* 2*-morphisms.*

If F is a 2-pullback preserving 2-functor and $(\mathbb{X}, \mathbb{T})$ a tangent object of $\mathbf{C}$, it is not hard to see that, $F\mathbb{X} := F_0\mathbb{X}$ comes equipped with a tangent structure so defined:

tangent bundle morphism The tangent bundle morphism $F\mathrm{T}\colon F\mathbb{X} \to F\mathbb{X}$ is given by:

$$F_1\mathrm{T}\colon F_0\mathbb{X} \to F_0\mathbb{X}$$

projection The projection $Fp\colon F\mathrm{T} \Rightarrow \mathsf{id}_{F\mathbb{X}}$ is given by:

$$F_2p\colon F_1\mathrm{T} \Rightarrow F_1\mathsf{id}_{\mathbb{X}} = \mathsf{id}_{F_0\mathbb{X}}$$

zero morphism The zero morphism $Fz\colon \mathsf{id}_{F\mathbb{X}} \Rightarrow F\mathrm{T}$ is given by:

$$F_2z\colon \mathsf{id}_{F_0\mathbb{X}} = F_1\mathsf{id}_{\mathbb{X}} \Rightarrow F_1\mathrm{T}$$

sum morphism The sum morphism $Fs\colon (F\mathrm{T})_2 \Rightarrow F\mathrm{T}$ is given by::

$$(F_1\mathrm{T})_2 \cong F_1(\mathrm{T}_2) \xRightarrow{F_2s} F_1\mathrm{T}$$

vertical lift The vertical lift $Fl\colon F\mathrm{T} \Rightarrow (F\mathrm{T})^2$ is given by:

$$F_2l\colon F_1\mathrm{T} \Rightarrow F_1(\mathrm{T}^2) = (F_1\mathrm{T})^2$$

canonical flip The canonical flip $Fc\colon (F\mathrm{T})^2 \Rightarrow (F\mathrm{T})^2$ is given by:

$$F_2c\colon (F_1\mathrm{T})^2 = F_1(\mathrm{T}^2) \Rightarrow F_1(\mathrm{T}^2) = (F_1\mathrm{T})^2$$

Moreover, if $(\mathbb{X}, \mathbb{T})$ has negatives with negation n, then:

negation The negation $Fn\colon F\mathrm{T} \Rightarrow F\mathrm{T}$ is given by:

$$F_2 n\colon F_1\mathrm{T} \Rightarrow F_1\mathrm{T}$$

Remark 2.69. Note that tangential continuity is only a sufficient condition for a 2-functor to preserve tangent objects. Indeed, one can ask for stricter conditions on 2-functors. For the sake of simplicity, we decided to adopt the weaker condition expressed by Definition 2.68.

(Strict) 2-categories, 2-functors, and 2-natural transformations form a 2-category denoted by $2\mathsf{Cat}$. Moreover, the composition of two 2-pullback preserving 2-functors is still a 2-pullback preserving 2-functor. Thus, also 2-categories, 2-pullback preserving 2-functors and 2-natural transformations form a 2-category that will be denoted by $2\mathsf{Cat}_{\mathbb{T}\text{-}\mathsf{cts}}$.

Proposition 2.70. *The operation* Tng *which sends a* 2*-category* $\mathbf{C}$ *to the* 2*-category* $\mathsf{Tng}(\mathbf{C})$ *of tangent objects of* $\mathbf{C}$ *extends to a* 2*-functor* $\mathsf{Tng}\colon 2\mathsf{Cat}_{\mathbb{T}\text{-}\mathsf{cts}} \to 2\mathsf{Cat}$*. Similarly, also* $\mathsf{Tng}_{\mathsf{co}}$*,* $\mathsf{Tng}_{\cong}$*, and* $\mathsf{Tng}_{=}$ *extend to* 2*-functors* $2\mathsf{Cat}_{\mathbb{T}\text{-}\mathsf{cts}} \to 2\mathsf{Cat}$.

2.4 Formal tangent monads

In Example 2.62, we introduced tangent monads as tangent objects of the 2-category $\mathsf{Mnd}(\mathsf{Cat})$ of monads. This notion plays a crucial role in the whole story of this thesis. Therefore, we dedicate this section to recalling the fundamental results of the theory of tangent monads.

Recall first that a monad over a category $\mathbb{X}$ consists of an endofunctor $S\colon \mathbb{X} \to \mathbb{X}$ together with a multiplication $\gamma\colon S^2 \to S$ and a unit $\eta\colon \mathsf{id}_{\mathbb{X}} \to S$, satisfying associativity and unitality. The simplest example of a monad is given by a monoid M of a monoidal category $\mathbb{X}$. Indeed, if M is such a monoid, then the functor $M \otimes -\colon \mathbb{X} \to \mathbb{X}$ acting on $\mathbb{X}$ by tensoring by M on the left, inherits a monad structure from the multiplication and the unit of M. Informally, a monad can be interpreted as an *algebraic theory*. This interpretation is particularly striking when the monad is the monad of an operad, as we will discuss later.

The *models* of such a theory are called **algebras** of the monad and the corresponding category $\mathsf{Alg}(S)$ is known as the Eilenberg-Moore category of S. Concretely, an algebra of S consists of an object A of the category $\mathbb{X}$ where the monad is defined upon, together with a morphism $SA \to A$, called the **structure map** of the algebra, compatible with the multiplication and the unit of the monad S. When the monad is the functor $M \otimes -$ associated with a monoid M, then the structure map $M \otimes A \to A$ can be interpreted as an action of the monoid M on the object A. So, in this sense, the structure map can be phrased as a representation of the monad over the object A. This interpretation will be more clear in the case of operads, as we will discuss later. In this section, we show that this interpretation, when applied to tangent monads, allows one to think of them as *geometric theories* and their algebras as geometric spaces.

Tangent monads were first introduced in [15] as monads S over a given tangent category $(\mathbb{X}, \mathbb{T})$ together with a distributive law $\alpha \colon S \circ \mathrm{T} \Rightarrow \mathrm{T} \circ S$, which is a natural transformation compatible with the composition $\gamma \colon S^2 \Rightarrow S$ and the unit $\eta \colon \mathsf{id}_{\mathbb{X}} \Rightarrow S$ of the monad, compatible with the tangent structure. In particular, this latter assumption is equivalent to stating that $(S, \alpha) \colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$ is a lax tangent morphism. As mentioned in the original paper, a tangent monad can be equivalently characterized as a formal monad in the 2-category TngCat of tangent categories.

The first main result (cf. [15, Proposition 20]) establishes that the Eilenberg-Moore category of a tangent monad comes equipped with a tangent structure which is strictly preserved by the forgetful functor. In this section, we use the new technology of tangent objects to extend this result to formal monads of an arbitrary 2-category.

For this purpose, recall the notion from Street's formal monad theory [56] of a 2-category $\mathbf{C}$ which *admits the construction of algebras*. This happens when the 2-functor $\mathbf{C} \to \mathsf{Mnd}(\mathbf{C})$ which sends an object $\mathbb{X}$ of $\mathbf{C}$ to $(\mathbb{X}, \mathbb{1})$, $\mathbb{1}$ being the trivial monad, i.e. the monad whose underlying endofunctor is the identity functor and so are its multiplication and unit, admits a 2-right adjoint $\mathsf{Alg}_{\mathbf{C}} \colon \mathsf{Mnd}(\mathbf{C}) \to \mathbf{C}$. The 2-category Cat admits the construction of algebras and in particular, the 2-functor $\mathsf{Alg} := \mathsf{Alg}_{\mathsf{Cat}}$ sends a monad S over a given category $\mathbb{X}$ to the corresponding Eilenberg-Moore category.

Street proved in [56, Theorem 2] that if $\mathbf{C}$ is a 2-category which admits the construction of algebras and S is a formal monad of $\mathbf{C}$, then S is generated by an adjunction of $\mathbf{C}$, that is, there are two 1-morphisms $F\colon \mathbb{X} \leftrightarrows \mathsf{Alg}(S)\colon U$ together with two 2-morphisms $\eta\colon \mathsf{id}_{\mathbb{X}} \Rightarrow U \circ F$ and $\varepsilon\colon F \circ U \Rightarrow \mathsf{id}_{\mathbb{X}}$ satisfying the triangle identities and respectively called the *unit* and the *counit* of the adjunction. This result, which is a classic result of monad theory (see [3, Proposition 10.3]), extends to formal monads.

Lemma 2.71. *The* 2*-category* $\mathsf{Tng}(\mathsf{Mnd}(\mathbf{C}))$ *of tangent objects of the* 2*-category of formal monads of a* 2*-category* $\mathbf{C}$ *is isomorphic to the* 2*-category* $\mathsf{Mnd}(\mathsf{Tng}(\mathbf{C}))$ *of formal monads of the* 2*-category of tangent objects of* $\mathbf{C}$*.*

Proof. First, consider a formal monad S over an object $\mathbb{X}$ of $\mathbf{C}$. Then, given a tangent structure on S, the tangent bundle functor consists of a morphism $(\mathrm{T},\alpha)\colon(\mathbb{X},S)\to(\mathbb{X},S)$ of formal monads, that is a 1-morphism $\mathrm{T}\colon\mathbb{X}\to\mathbb{X}$ of $\mathbf{C}$ together with a 2-morphism $\alpha\colon S\circ\mathrm{T}\Rightarrow S\circ\mathrm{T}$ of $\mathbf{C}$, compatible with the multiplication and the unit of S. Moreover, the projection, the zero morphism, the sum morphism, the vertical lift, and the canonical flip of such a tangent structure correspond to 2-morphisms p,z,s,l, and c, respectively, of $\mathbf{C}$, compatible with the distributive law α, and satisfying the axioms of a tangent structure. It is easy to see that $(\mathbb{X},\mathbb{T})$, where $\mathbb{T}$ is precisely given by (T,p,z,s,l,c) is a tangent object of $\mathbf{C}$, and that (S,α) constitutes a formal monad over $(\mathbb{X},\mathbb{T})$ in the 2-category $\mathsf{Tng}(\mathbf{C})$ of tangent objects over $\mathbf{C}$. So, every object $(\mathbb{X},S;\mathbb{T},\alpha)$ of $\mathsf{Tng}(\mathsf{Mnd}(\mathbf{C}))$ defines an object $(\mathbb{X},\mathbb{T};S,\alpha)$ of $\mathsf{Mnd}(\mathsf{Tng}(\mathbf{C}))$. Conversely, it is clear that a formal monad $(\mathbb{X},\mathbb{T};S,\alpha)$ of $\mathsf{Tng}(\mathbf{C})$ defines a tangent object $(\mathbb{X},S;\mathbb{T},\alpha)$ of $\mathsf{Mnd}(\mathbf{C})$.

Let $(F,\varphi;\beta)\colon(\mathbb{X},S;\mathbb{T},\alpha)\to(\mathbb{X}',S';\mathbb{T}',\alpha')$ be a lax tangent morphism of $\mathsf{Tng}(\mathsf{Mnd}(\mathbf{C}))$. This is a 1-morphism $F\colon\mathbb{X}\to\mathbb{X}'$ of $\mathbf{C}$ together with a distributive law $\varphi\colon S'\circ F\Rightarrow F\circ S$, compatible with the monad structures, and a lax distributive law $\beta\colon F\circ\mathrm{T}\Rightarrow\mathrm{T}'\circ F$, compatible with φ,α, and α'. This corresponds to a morphism $(F,\beta;\varphi)\colon(\mathbb{X},\mathbb{T};S,\alpha)\to(\mathbb{X}',\mathbb{T}';S',\alpha')$ of formal monads of $\mathsf{Tng}(\mathbf{C})$. The converse is straightforward. Finally, take into consideration a lax 2-morphism $\theta\colon(F,\varphi;\beta)\Rightarrow(G,\psi;\gamma)$ between two lax tangent 1-morphisms $(F,\varphi;\beta),(G,\psi;\gamma)\colon(\mathbb{X},S;\mathbb{T},\alpha)\to(\mathbb{X}',S';\mathbb{T}',\alpha')$. This consists of a 2-morphism $\theta\colon F\Rightarrow G$, compatible with φ and ψ, and with α and α'. However, this is also a 2-morphism $\theta\colon(F,\beta;\varphi)\Rightarrow(G,\gamma;\psi)$

between the corresponding morphisms of formal monads of $\mathsf{Tng}(\mathbf{C})$. The converse is also true. □

Remark 2.72. In light of Lemma 2.71, it is natural to wonder whether or not $\mathsf{Tng}_{\mathsf{co}}(\mathsf{Mnd}(\mathbf{C}))$ and $\mathsf{Mnd}(\mathsf{Tng}_{\mathsf{co}}(\mathbf{C}))$ are also isomorphic 2-categories, for a 2-category $\mathbf{C}$. However, this is not the case. To see this, notice that objects of $\mathsf{Tng}_{\mathsf{co}}(\mathsf{Mnd}(\mathbf{C}))$ are tangent objects over the 2-category of formal monads over $\mathbf{C}$. Thus, the objects of $\mathsf{Tng}_{\mathsf{co}}(\mathsf{Mnd}(\mathbf{C}))$ are the same objects as those of $\mathsf{Tng}(\mathsf{Mnd}(\mathbf{C}))$. On the contrary, the objects of $\mathsf{Mnd}(\mathsf{Tng}_{\mathsf{co}}(\mathbf{C}))$ are formal monads over the 2-category $\mathsf{Tng}_{\mathsf{co}}(\mathbf{C})$, that are tuples $(\mathbb{X},\mathbb{T};S,\beta)$, where $\beta\colon \mathrm{T}\circ S \Rightarrow S\circ \mathrm{T}$, since $(S,\beta)\colon(\mathbb{X},\mathbb{T}) \nrightarrow (\mathbb{X},\mathbb{T})$ is a colax tangent 1-morphism. To fix this discrepancy, one can consider the 2-category $\mathsf{Mnd}_{\mathsf{co}}(\mathbf{C})$ of formal monads, colax 1-morphisms of monads $(F,\beta)\colon(\mathbb{X},S) \nrightarrow (\mathbb{X}',S')$ which are 1-morphisms $F\colon\mathbb{X}\to\mathbb{X}'$ together with a 2-morphism:

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "S"] \arrow[d, "F"'] & \mathbb{X} \arrow[d, "F"] \arrow[dl, phantom, "\Swarrow\beta"] \\
\mathbb{X}' \arrow[r, "S'"'] & \mathbb{X}'
\end{tikzcd}
$$

compatible with the multiplication and the unit of the monads, and 2-morphisms $\theta\colon(F,\beta)\Rightarrow(G,\gamma)$ which consists of natural transformations $\theta\colon F\Rightarrow G$, compatible with β and γ. Then, it is not hard to see that $\mathsf{Tng}_{\mathsf{co}}(\mathsf{Mnd}_{\mathsf{co}}(\mathbf{C}))=\mathsf{Mnd}_{\mathsf{co}}(\mathsf{Tng}_{\mathsf{co}}(\mathbf{C}))$.

Theorem 2.73. *If $\mathbf{C}$ is a 2-category which admits the construction of algebras, so does the 2-category $\mathsf{Tng}(\mathbf{C})$ of tangent objects of $\mathbf{C}$. Moreover, if $(\mathbb{X},\mathbb{T})$ is a tangent object of $\mathbf{C}$ and (S,α) is a formal tangent monad of $(\mathbb{X},\mathbb{T})$, then the object $\mathsf{Alg}(S)\in\mathbf{C}$ is also a tangent object of $\mathbf{C}$ and the right adjoint morphism $U\colon(\mathsf{Alg}(S),\mathbb{T}(S))\to(\mathbb{X},\mathbb{T})$ is a strict tangent morphism.*

Proof. First, notice that by definition the 2-functor $\mathsf{Alg}\colon\mathsf{Mnd}(\mathbf{C})\to\mathbf{C}$ is a right adjoint, thus it preserves limits. Similarly, the 2-functor $\mathsf{Inc}\colon\mathbf{C}\to\mathsf{Mnd}(\mathbf{C})$ which sends $\mathbb{X}\in\mathbf{C}$ to $(\mathbb{X},\mathbb{1})$ is also a right adjoint, as proved in [56, Theorem 1]. In particular, it is the right adjoint of the 2-functor $\mathsf{Mnd}(\mathbf{C})\to\mathbf{C}$ which sends a pair $(\mathbb{X},S)$ to $\mathbb{X}$. So, also Inc is 2-pullback preserving and therefore, $\mathsf{Inc}\dashv\mathsf{Alg}$ forms an adjunction in $2\mathsf{Cat}_{\mathbb{T}\text{-cts}}$. As a 2-functor, Tng preserves adjunctions, thus

$\mathsf{Tng}(\mathsf{Inc})\colon \mathsf{Tng}(\mathbf{C}) \leftrightarrows \mathsf{TngMnd}(\mathbf{C})\colon \mathsf{Tng}(\mathsf{Alg})$ is an adjunction. Finally, the 2-category $\mathsf{TngMnd}(\mathbf{C})$ of tangent monads is equivalent to the 2-category $\mathsf{MndTng}(\mathbf{C})$ of formal monads in the 2-category of tangent objects of $\mathbf{C}$. This proves that $\mathsf{Tng}(\mathbf{C})$ admits the construction of algebras.

To prove that the right adjoint $U\colon (\mathsf{Alg}(S), \mathbb{T}(S)) \to (\mathbb{X}, \mathbb{T})$ is a strict tangent morphism, first let's recall the construction of the adjunction $F\colon \mathbb{X} \leftrightarrows \mathsf{Alg}(S)\colon U$ associated to a generic formal monad, as illustrated by Street's original paper. By definition, the 2-functors Alg and Inc form an adjunction $\mathsf{Inc} \dashv \mathsf{Alg}$, whose counit, for every formal monad S over an object $\mathbb{X}$, is a morphism of monads $(\mathsf{Alg}(S), \mathbb{1}_{\mathsf{Alg}(S)}) \to (\mathbb{X}, S)$, where $\mathbb{1}_{\mathsf{Alg}(S)}$ denotes the trivial formal monad over $\mathsf{Alg}(S)$. In particular, the underlying 1-morphism of the counit is the morphism $U\colon \mathsf{Alg}(S) \to \mathbb{X}$, which represents the right adjoint in the adjunction $F \dashv U$, associated with the formal monad S. Keeping in mind the origin of this right adjoint U, notice that the counit of the induced adjunction $\mathsf{Alg}\colon \mathsf{Tng}(\mathbf{C}) \leftrightarrows \mathsf{Mnd}(\mathsf{Tng}(\mathbf{C}))\colon \mathsf{Inc}$ is precisely given by $\mathsf{Tng}(\varepsilon)$. By definition, given a natural transformation $\theta\colon F \Rightarrow G$ from a (2-pullback preserving) 2-functor $F\colon \mathbf{C} \to \mathbf{C}'$ to another 2-functor $G\colon \mathbf{C} \to \mathbf{C}'$, the corresponding natural transformation $\mathsf{Tng}(\theta)\colon \mathsf{Tng}(F) \Rightarrow \mathsf{Tng}(G)$ is defined, for every tangent object $(\mathbb{X}, \mathbb{T})$ of $\mathbf{C}$, as the tangent morphism $(F\mathbb{X}, F\mathbb{T}) \to (G\mathbb{X}, G\mathbb{T})$, whose underlying 1-morphism is $\theta\colon F\mathbb{X} \to G\mathbb{X}$. Interestingly, the distributive law is just the identity, since $\theta(F\mathrm{T}) = (G\mathrm{T})\theta$ on the nose. In short, every natural transformation θ is sent by Tng to a strict tangent natural transformation. In particular, this applies to the counit of the adjunction $\mathsf{Inc} \dashv \mathsf{Alg}$ and therefore, the right adjoint $U\colon (\mathsf{Alg}(S), \mathbb{T}(S)) \to (\mathbb{X}, \mathbb{T})$ is a strict tangent morphism. $\square$

Remark 2.74. One of the key facts employed in the proof of Theorem 2.73 is that $\mathsf{TngMnd}(\mathbf{C}) \cong \mathsf{Mnd}(\mathsf{Tng}(\mathbf{C}))$. One could wonder if, assuming that $\mathbf{C}$ admits the construction of algebras, then also $\mathsf{Tng}_{\mathsf{co}}(\mathbf{C})$ will admit this construction. However, as pointed out in Remark 2.72, $\mathsf{Tng}_{\mathsf{co}}(\mathsf{Mnd}(\mathbf{C}))$ is not isomorphic to $\mathsf{Mnd}(\mathsf{Tng}_{\mathsf{co}}(\mathbf{C}))$.

When $\mathbf{C}$ is the 2-category Cat of categories, Cockett, Lemay, and Lucyshyn-Wright already showed (cf. [12, Proposition 20]) that the category of algebras $\mathsf{Alg}(S)$ of a tangent monad (S, α) over a tangent category $(\mathbb{X}, \mathbb{T})$ is also a tangent category for which the forgetful functor $U\colon \mathsf{Alg}(S) \to \mathbb{X}$ preserves the tangent structure strictly. Concretely, the tangent structure $\mathbb{T}(S)$ over the category of algebras $\mathsf{Alg}(S)$

of a tangent monad (S, α) is so defined:

tangent bundle morphism The tangent bundle morphism $\mathrm{T}^{(S)} \colon \mathsf{Alg}(S) \to \mathsf{Alg}(S)$ sends and algebra A of S with structure map $\theta \colon SA \to A$ to the algebra $\mathrm{T}A$ with structure map:

$$S\mathrm{T}A \xrightarrow{\alpha} \mathrm{T}SA \xrightarrow{\mathrm{T}\theta} \mathrm{T}A$$

Moreover, it sends a morphism $f \colon A \to B$ of algebras of S to $\mathrm{T}f$;

while the projection, the zero morphism, the sum morphism, the vertical lift, and the canonical flip are defined by the corresponding natural transformations of $\mathbb{T}$. When $\mathbb{T}$ has negatives, so does $\mathbb{T}(S)$, with negation n as in $\mathbb{T}$.

In the next proposition, we show that this construction is precisely the one obtained by Theorem 2.73 in the special case of $\mathbf{C} = \mathsf{Cat}$, that is the 2-functor $\mathsf{Alg} \colon \mathsf{TngMnd} \to \mathsf{TngCat}$ which sends a tangent monad (S, α) to the tangent category $(\mathsf{Alg}(S), \mathbb{T}(S))$ is precisely the algebra 2-functor, right adjoint to the inclusion functor Inc.

Proposition 2.75. *The 2-category* TngCat *of tangent categories admits the construction of algebras. Moreover, the Eilenberg-Moore object* $(\mathsf{Alg}(S), \mathbb{T}(S))$ *associated with a tangent monad* (S, α) *over a given tangent category* $(\mathbb{X}, \mathbb{T})$ *is precisely the tangent category described by Cockett, Lemay, and Lucyshyn-Wright in [15].*

Proof. First, as proved by Street in [56, Theorem 7], notice that the 2-category Cat of categories admits the construction of algebras and that the 2-functor Alg sends a monad to the corresponding Eilenberg-Moore category. Thanks to Theorem 2.73, $\mathsf{Tng}(\mathsf{Cat})$ also admits the construction of algebras. However, as noticed in Example 2.61, $\mathsf{Tng}(\mathsf{Cat})$ is the 2-category TngCat of tangent categories.

In order to prove the second part of this result, recall the definition of the 2-functor $\mathsf{Alg} \colon \mathsf{Mnd} \to \mathsf{Cat}$. Alg sends a morphism of monads $(F, \alpha) \colon (\mathbb{X}, S) \to (\mathbb{X}, S)$, formed by a functor $F \colon \mathbb{X} \to \mathbb{X}'$ together with a natural transformation $\alpha \colon S' \circ F \Rightarrow F \circ S$, to the functor $\mathsf{Alg}(F, \alpha)$ which sends a algebra A of S with structure map $\theta \colon SA \to A$, to the algebra FA of S' with structure map:

$$S'FA \xrightarrow{\alpha} FSA \xrightarrow{F\theta} FA$$

Moreover, Alg sends a natural transformation $\varphi\colon (F,\alpha) \Rightarrow (G,\beta)$ between two morphisms of monads $(F,\alpha),(G,\beta)\colon (\mathbb{X},S) \to (\mathbb{X}',S')$, to the natural transformation $\mathsf{Alg}(F,\alpha) \Rightarrow \mathsf{Alg}(G,\beta)$, defined by φ. Now, recall that the 2-functor:

$$\mathsf{Alg}\colon \mathsf{Mnd}(\mathsf{TngCat}) \to \mathsf{TngCat}$$

defined in Theorem 2.73, under the identification $\mathsf{Mnd}(\mathsf{TngCat}) = \mathsf{TngMnd}$, sends a tangent object $(\mathbb{X},\mathbb{T};S,\alpha)$ of Mnd, whose tangent bundle morphism $(\mathrm{T},\alpha)\colon (\mathbb{X},S) \to (\mathbb{X},S)$ is given by the tangent bundle functor $\mathrm{T}\colon \mathbb{X} \to \mathbb{X}$ together with the distributive law $\alpha\colon S\circ\mathrm{T} \Rightarrow \mathrm{T}\circ S$, to the tangent object $(\mathsf{Alg}(\mathbb{X},S),\mathsf{Alg}(\mathbb{T}))$. The tangent bundle morphism is given by $\mathsf{Alg}(\mathrm{T},\alpha)$ which sends an algebra A of S with structure map $\theta\colon SA \to A$ to the algebra $\mathrm{T}A$ with structure map:

$$S\mathrm{T}A \xrightarrow{\alpha} \mathrm{T}SA \xrightarrow{\mathrm{T}\theta} \mathrm{T}A$$

Moreover, it sends a morphism $f\colon A \to B$ of algebras of S to $\mathrm{T}f$. Finally, all the natural transformations p,z,s,l, and c are precisely given by the corresponding natural transformations of $\mathbb{T}$. □

Remark 2.76. Proposition 2.75 explains why $\mathsf{Tng}_{\mathsf{co}}(\mathbf{C})$ does not admit the construction of algebras. Take $\mathbf{C}$ to be the 2-category Cat of categories and a formal monad (S,β) over $\mathsf{Tng}_{\mathsf{co}}(\mathsf{Cat})$, which consists of a formal monad S over a category $\mathbb{X}$, together with a colax distributive law $\beta\colon \mathrm{T}\circ S \Rightarrow S\circ\mathrm{T}$. Now, let A be an algebra with structure map $SA \to A$ of S. In Proposition 2.75, in order to lift the tangent structure to $\mathsf{Alg}(S)$, we employed the distributive law $\alpha\colon S\circ\mathrm{T} \Rightarrow \mathrm{T}\circ S$ and defined the tangent bundle of A as the algebra $\mathrm{T}A$ with structure map $S\mathrm{T}A \xrightarrow{\alpha} \mathrm{T}SA \xrightarrow{\mathrm{T}\theta} \mathrm{T}A$. However, the colax distributive law β is pointing in the wrong direction.

Chapter 3

The geometric theories of an operad

In the previous chapter, we explored some important examples (Examples 2.10, 2.13, and 2.14) of models of geometry described using tangent category theory. These examples have in common that they represent the geometry of commutative and point-wise spaces, like affine schemes or smooth manifolds. The first main goal of this thesis is to show that tangent category theory applies to a larger family of geometries, including algebraic non-commutative geometry. To construct these examples, we employ the concept of operad and we show that every algebraic operad generates two tangent categories: the algebraic and the geometric tangent categories of an operad.

We start in Section 3.1 by recalling the tangent category of affine schemes and by discussing the motivation which brought us to employ operad theory to explore new kinds of geometries. In Section 3.2, we review the definition of an operad and in Section 3.3 we recall the notion of an algebra over an operad. Section 3.4 is dedicated to proving that the monad associated with an algebraic operad carries the structure of a coCartesian differential monad. Section 3.5 is entirely dedicated to the algebraic tangent category of an operad and to classifying its vector fields (Section 3.5.1) and to prove the functoriality of the operation which sends an operad to its algebraic tangent category (Section 3.5.2). Similarly, in Section 3.6, we introduce the geometric tangent category of an operad and we classify its vector fields (Section 3.6.1) and discuss the functoriality of the operation which sends an operad to its geometric tangent category (Section 3.6.2).

Figure 3.1 displays the concept map of this chapter.

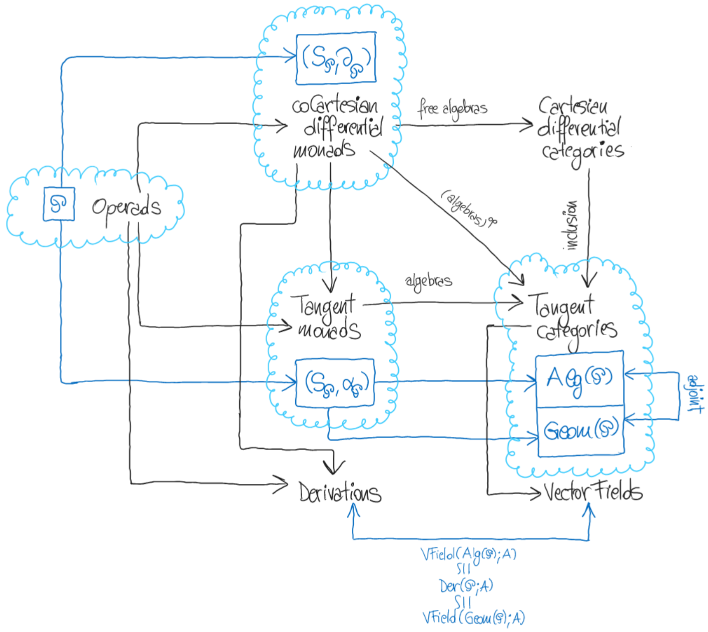


Figure 3.1: The concept map of the chapter

3.1 Motivation

One of the main goals of this thesis is to employ tangent category theory in the study of noncommutative geometry. The original idea of noncommutative geometry is to treat associative, not necessarily commutative, algebras as geometric spaces. Morally this resembles the approach of algebraic geometry, which looks at commutative and unital rings as *affine schemes*. The starting point of our work is the paper [18] in which Cruttwell and Lemay show how to construct a tangent structure to capture some key geometric features of affine schemes. In particular, they prove that the opposite of the category of unital and commutative algebras $\mathsf{cAlg}^{\mathsf{op}}$ over a commutative and unital ring R comes equipped with the following tangent structure:

tangent bundle functor The tangent bundle functor $\mathrm{T}\colon \mathsf{cAlg}^{\mathsf{op}} \to \mathsf{cAlg}^{\mathsf{op}}$ sends an algebra A to the symmetric algebra of the module of Kähler differentials, that is $\mathrm{T}A := \mathsf{S}_A\Omega A$. Concretely, such an algebra is generated by all the elements a of A together with the symbols $\mathrm{d}a$, for every $a \in A$, satisfying the following relations:

$$
\begin{aligned}
a \cdot b &= ab \\
\mathsf{d}(ra + sb) &= r\mathsf{d}a + s\mathsf{d}b \\
\mathsf{d}(ab) &= a\mathsf{d}b + b\mathsf{d}a
\end{aligned}
$$

where $\cdot$ represents the multiplication of $\mathrm{T}A$ and the juxtaposition the one of A; for every $a, b \in A$ and every $r, s \in R$. Moreover, T sends a morphism of algebras $f \colon A \to B$ to the morphism $\mathrm{T}f$ of algebras which sends each generator $a \in A$ to $f(a)$ and each $\mathsf{d}a$ to $\mathsf{d}f(a)$;

projection The projection $p \colon \mathsf{id}_{\mathsf{cAlg}^{\mathrm{op}}} \Rightarrow \mathrm{T}$, regarded as a morphism of algebras, sends each $a \in A$ to itself;

zero morphism The zero morphism $z \colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathsf{cAlg}^{\mathrm{op}}}$, regarded as a morphism of algebras, sends each generator $a \in A$ to itself, and each $\mathsf{d}a$ to 0;

n-fold pullback The n-fold pullback of the projection along itself, in the category cAlg corresponds to the pushout T_n. Concretely, $\mathrm{T}_n A$ is the tensor product of $\mathrm{T}A$ over A n-times, where the A-module structure of $\mathrm{T}A$ is induced by the projection. Alternatively, $\mathrm{T}_n A$ can be described as the commutative and unital algebra generated by all elements a of A and symbols $\mathsf{d}_1 a, \dots, \mathsf{d}_n a$, satisfying the following relations:

$$
\begin{aligned}
a \cdot b &= ab \\
\mathsf{d}_k(ra + sb) &= r\mathsf{d}_k a + s\mathsf{d}_k b \\
\mathsf{d}_k(ab) &= a\mathsf{d}_k b + b\mathsf{d}_k a
\end{aligned}
$$

for every $a, b \in A$, $r, s \in R$, and $k = 1, \dots, n$;

sum morphism The sum morphism $s \colon \mathrm{T} \Rightarrow \mathrm{T}_2$, regarded as a morphism of algebras, sends each generator $a \in A$ to itself and each $\mathsf{d}_1 a$ to $\mathsf{d}_1 a + \mathsf{d}_2 a$. In the equivalent description of $\mathrm{T}_2 A$ as the tensor product $\mathrm{T}A \otimes_A \mathrm{T}A$, $\mathsf{d}_1 a + \mathsf{d}_2 a$ is represented by $\mathsf{d}a \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}a$, $\mathbb{1}$ being the unit of A;

vertical lift The vertical lift $l \colon \mathrm{T}^2 \Rightarrow \mathrm{T}$, regarded as a morphism of algebras, sends each generator $a \in A$ to itself, each $\mathsf{d}a$ and each $\mathsf{d}'a$ to 0, and each $\mathsf{d}'\mathsf{d}a$ to $\mathsf{d}a$, where d' denotes the Kähler differentials associated to the second T in the composition $\mathrm{T}^2 = \mathrm{T} \circ \mathrm{T}$;

canonical flip The canonical flip $c\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$, regarded as a morphism of algebras, sends each generator $a \in A$ to itself, each $\mathsf{d}a$ to $\mathsf{d}'a$, each $\mathsf{d}'a$ to $\mathsf{d}a$, and each $\mathsf{d}'\mathsf{d}a$ to $\mathsf{d}'\mathsf{d}a$.

Moreover, since R has negatives, this tangent structure has also negatives given by:

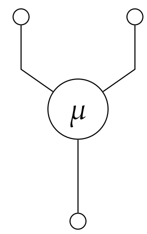
negation The negation $n\colon \mathrm{T} \Rightarrow \mathrm{T}$, regarded as a morphisms of algebras, sends each generator $a \in A$ to itself and each $\mathsf{d}a$ to $-\mathsf{d}a$.

We refer to the tangent category just described as the **geometric tangent category** of affine schemes.

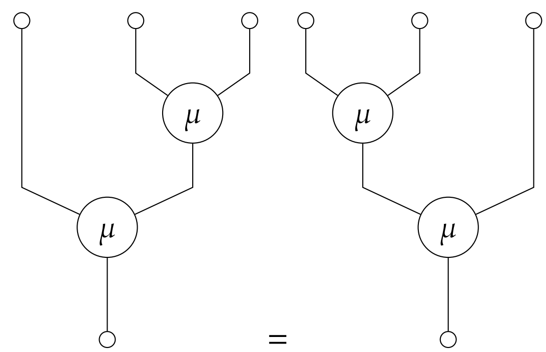
3.2 Operads: the factories of algebraic objects

Our first goal is to generalize this construction to other kinds of algebraic objects, like associative algebras. This will produce the first example of a tangent category which captures some key geometric features of algebraic noncommutative geometry. This is of course far from being an exhaustive description of noncommutative geometry with tangent category theory. However, it opens the doors to a new exploration and builds the basis for future work in this direction.

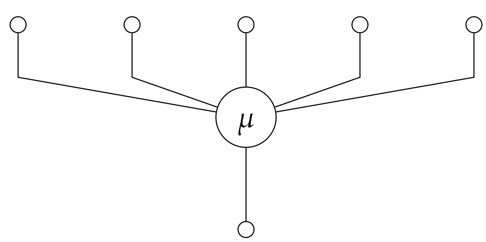
In this section we extend Cruttwell and Lemay's construction to a large family of algebraic objects: the algebras of an operad. To put it in a slogan: *operads are mathematical factories of algebraic objects*. In this sense, operads are related to monads, as we will soon discuss. Informally, an algebraic object A is given by an object, e.g. a set, a space, or a module, together with a list of operations μ, which take n inputs from A and return a single output in A. These operations satisfy some relations. For example, an associative algebra is an R-module together with an operation μ which takes two inputs of A and returns an output of A and such that, for every a, b, and c of A, $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$. One can start thinking of such a μ as a tree:

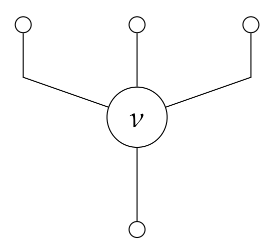


Then, the associativity relation can be expressed as an equation between trees:

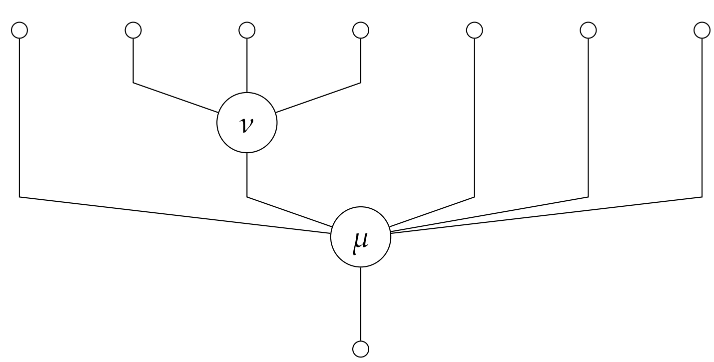


Notice that in order to express such a relation one needs a composition of trees, that is, given two trees:





the first one with m inputs and the second one with n inputs, we want to compose ν with μ along the k-th input of μ:



More generally, if $\mu_1, \ldots, \mu_m$ are trees with $k_1, \ldots, k_m$ inputs, respectively, and μ is a tree with m inputs, we want to be able to compose each μ_i with μ along the i-th input of μ. We will represent the resulting tree with $k_1 + \ldots + k_m$ inputs by:

$$\mu(\mu_1, \ldots, \mu_m)$$

or by:

$$\gamma(\mu; \mu_1, \ldots, \mu_m)$$

Notice that such an operation γ is associative in the following sense:

$$\begin{aligned} \mu\left(\mu_1(\nu_1^{(1)}, \ldots, \nu_{k_1}^{(1)}), \ldots, \mu_m(\nu_1^{(m)}, \ldots, \nu_{k_m}^{(m)})\right) = \\ \mu(\mu_1, \ldots, \mu_m)\left(\nu_1^{(1)}, \ldots, \nu_{k_1}^{(1)}, \ldots, \nu_1^{(m)}, \ldots, \nu_{k_k}^{(m)}\right) \end{aligned}$$

where the left-hand side of this equation is the tree formed by composing each μ_i with the trees $\nu_1^{(i)}, \ldots, \nu_{k_i}^{(i)}$ and then composing the resulting trees $\mu_i\left(\nu_1^{(i)}, \ldots, \nu_{k_i}^{(i)}\right)$ with μ along the corresponding inputs, while the right-hand side represents the tree formed by first composing each $\mu_1, \ldots, \mu_m$ with μ and then composing the $\nu_{j_i}^{(i)}$ to the corresponding input of $\mu(\mu_1, \ldots, \mu_m)$.

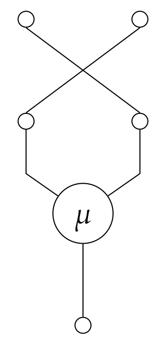
Secondly, to be able to abstractly represent the equation that expresses associativity for the binary tree μ, we also need to have a special tree $\mathbb{1}$ with 1 input and 1 output which plays the role of the identity, that is such that:

$$\mathbb{1}(\mu) = \mu = \mu(\mathbb{1}, \ldots, \mathbb{1})$$

for any other tree μ with m inputs and 1 output. Thanks to the operation γ and the unit $\mathbb{1}$, we can represent the associativity condition by the equation:

$$\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu) \tag{3.2.1}$$

Finally, to express symmetries we also need an action of the symmetric groups over the spaces of trees. To understand this, consider commutative algebras. These are algebraic objects equipped with a binary operation μ which satisfies the associativity condition of Equation (3.2.1), together with the condition $\mu(a, b) = \mu(b, a)$. The idea is to change the tree μ into a new tree μ^{op}:



where the two inputs have been shuffled by the permutation $\tau = (1 \quad 2)$. In general, given a tree μ with m inputs and a permutation $\sigma \in \mathbb{S}_m$ over m elements, we denote by $\mu \cdot \sigma$ the tree with m inputs obtained by shuffling the m inputs of μ with the permutation σ. So, in our example, we write $\mu^{\mathsf{op}} = \mu \cdot \tau$. So, the commutation condition reads as:

$$\mu \cdot \tau = \mu$$

In a nutshell, this shows that every algebraic object can be axiomatized by a sequence $\{\mathscr{P}(n)\}$ where $\mathscr{P}(n)$ is an object which collects all trees with n inputs, with an associative operation $\gamma\colon \mathscr{P}(m) \otimes \mathscr{P}(k_1) \otimes \ldots \otimes \mathscr{P}(k_m) \longrightarrow \mathscr{P}(k_1 + \ldots + k_m)$ which composes trees, with a unit $\mathbb{1}_{\mathscr{P}} \in \mathscr{P}(1)$ which plays the role of the identity, and with a right action $\mathscr{P}(n) \times \mathbb{S}_n \longrightarrow \mathscr{P}(n)$ of the symmetric group $\mathbb{S}_n$ which shuffles the n inputs of the trees in $\mathscr{P}(n)$.

This is precisely the definition of an operad. For a complete introduction to the theory of operads, we advise the reader to consult [46]. Let's recall the definition, beginning with establishing some useful notation.

Notation 3.1. Symmetric monoidal categories are denoted by the letter $\mathbb{E}$ and, for the sake of simplicity, in the computations, we treat them as strict monoidal categories equipped with a symmetric braiding.

A sequence of objects in a symmetric monoidal category $\mathbb{E}$ is denoted by $\{E(n)\}$. The symmetric group over n distinct elements is denoted by $\mathbb{S}_n$ and the right action over the entries of a symmetric sequence by ρ. The n-fold tensor product $A \otimes \ldots \otimes A$ of A with itself is denoted by $A^{\otimes n}$, and the left action of $\mathbb{S}_n$ on $A^{\otimes n}$ via the symmetric braiding is denoted by λ.

Definition⋆ 3.2. *An **operad** over a symmetric monoidal category $\mathbb{E}$ consists of a sequence $\{\mathscr{P}(n)\}$ of objects of $\mathbb{E}$ together with a collection of morphisms $\gamma_{m;k_1,\ldots,k_m} : \mathscr{P}(m)\otimes\mathscr{P}(k_1)\otimes\ldots\otimes\mathscr{P}(k_m) \to \mathscr{P}(k_1+\ldots+k_m)$ for every tuple of non-negative integers $m; k_1, \ldots, k_m \in \mathbb{N}$, called the **multiplication** maps of $\mathscr{P}$, and a morphism $\mathbb{1} \to \mathscr{P}(1)$, called the **unit** of $\mathscr{P}$, satisfying the following properties:*

$$\begin{tikzcd}
\mathscr{P}(m)\otimes\mathscr{P}(k_1)\otimes\mathscr{P}(i_1^{(1)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_1}}^{(1)})\otimes\ldots\otimes\mathscr{P}(k_m)\otimes\mathscr{P}(i_1^{(m)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_m}}^{(m)}) \arrow[r, "\mathscr{P}(m)\otimes\gamma\otimes\ldots\otimes\gamma"] \arrow[d, "\cong"'] & \mathscr{P}(m)\otimes\mathscr{P}(i_1^{(1)}+\ldots+i_{j_{k_1}}^{(1)})\otimes\ldots\otimes\mathscr{P}(i_1^{(m)}+\ldots+i_{j_{k_m}}^{(m)}) \arrow[dd, "\gamma"] \\
\mathscr{P}(m)\otimes\mathscr{P}(k_1)\otimes\ldots\otimes\mathscr{P}(k_m)\otimes\mathscr{P}(i_1^{(1)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_1}}^{(1)})\otimes\ldots\otimes\mathscr{P}(i_1^{(m)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_m}}^{(m)}) \arrow[d, "\gamma\otimes\mathscr{P}(i_1^{(1)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_1}}^{(1)})\otimes\ldots\otimes\mathscr{P}(i_1^{(m)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_m}}^{(m)})"'] & \\
\mathscr{P}(k_1+\ldots+k_m)\otimes\mathscr{P}(i_1^{(1)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_1}}^{(1)})\otimes\ldots\otimes\mathscr{P}(i_1^{(m)})\otimes\ldots\otimes\mathscr{P}(i_{j_{k_m}}^{(m)}) \arrow[r, "\gamma"'] & \mathscr{P}(i_1^{(1)}+\ldots+i_{j_{k_1}}^{(1)}+\ldots+i_1^{(m)}+\ldots+i_{j_{k_m}}^{(m)})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{P}(1)\otimes\mathscr{P}(n) \arrow[r, "\gamma"] & \mathscr{P}(n) \\
\mathbb{1}\otimes\mathscr{P}(n) \arrow[u, "\eta\otimes\mathscr{P}(n)a"] \arrow[ur, "\cong"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{P}(n)\otimes\mathscr{P}(1) \arrow[r, "\gamma"] & \mathscr{P}(n) \\
\mathscr{P}(n)\otimes\mathbb{1} \arrow[u, "\mathscr{P}(n)\otimes\eta"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Moreover, for each $n \in \mathbb{N}$, there is a right action of the symmetric group $\mathbb{S}_n$ on $\mathscr{P}(n)$. More precisely, if $\mathsf{Aut}(\mathscr{P}(n))$ denotes the group of automorphisms of $\mathscr{P}(n)$, then there is a group homomorphism $\rho_n : \mathbb{S}_n^{\mathsf{op}} \to \mathsf{Aut}(\mathscr{P}(n))$, where $\mathbb{S}_n^{\mathsf{op}}$ denotes the group of permutations over n elements and composition, the opposite of the composition of $\mathbb{S}_n$. The multiplication maps are equivariant with respect to ρ, that is:

$$\begin{tikzcd}
\mathscr{P}(m)\otimes\mathscr{P}(k_1)\otimes\ldots\otimes\mathscr{P}(k_m) \arrow[r, "\rho(\sigma)\otimes\varepsilon(\sigma)"] \arrow[d, "\gamma"'] & \mathscr{P}(m)\otimes\mathscr{P}(k_{\sigma(1)})\otimes\ldots\otimes\mathscr{P}(k_{\sigma(m)}) \arrow[d, "\gamma"] \\
\mathscr{P}(k_1+\ldots+k_m) \arrow[r, "\rho(\sigma_{k_1,\ldots,k_m})"'] & \mathscr{P}(k_1+\ldots+k_m)
\end{tikzcd}$$

where $\sigma \in \mathbb{S}_m$ and $\sigma_{k_1,\ldots,k_m} \in \mathbb{S}_{k_1+\ldots+k_m}$ denotes the permutation which shuffles each block of $k_1, k_2, \ldots, k_m$ elements via σ as each block was a single element. Moreover, $\varepsilon(\sigma)$ represents the action of the symmetric group via the symmetric braiding. Finally:

$$\begin{tikzcd}
\mathscr{P}(m)\otimes\mathscr{P}(k_1)\otimes\ldots\otimes\mathscr{P}(k_m) \arrow[r, "\mathscr{P}(m)\otimes\rho(\sigma_1)\otimes\ldots\otimes\rho(\sigma_m)"] \arrow[d, "\gamma"'] & \mathscr{P}(m)\otimes\mathscr{P}(k_{\sigma(1)})\otimes\ldots\otimes\mathscr{P}(k_{\sigma(m)}) \arrow[d, "\gamma"] \\
\mathscr{P}(k_1+\ldots+k_m) \arrow[r, "\rho(\sigma_1\oplus\sigma_2\oplus\cdots\oplus\sigma_m)"'] & \mathscr{P}(k_1+\ldots+k_m)
\end{tikzcd}$$

where $\sigma_1 \oplus \cdots \oplus \sigma_m$ denotes the permutation over $k_1 + \ldots + k_m$ elements which acts on each block of $k_1, k_2, \ldots, k_m$ elements as $\sigma_1, \sigma_2, \ldots, \sigma_m$, respectively.

Remark 3.3. It is important to point out that there are many alternative but equivalent characterizations of (symmetric) operads, among which operads are defined as monoids in the symmetric monoidal category of symmetric sequences, or as algebras of a monad. We invite the interested reader to consult [46, Section 5], which explores different approaches.

Notation 3.4. For operads in a symmetric monoidal category $\mathbb{E}$ we adopt the font $\mathscr{P}$. We refer to operads over the symmetric monoidal category of R-modules, for a commutative and unital ring R, as **algebraic operads**. For an algebraic operad $\mathscr{P}$, we denote the generic element of $\mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}$ by $(\mu; a_1, \ldots, a_n)$ or, sometimes by $(\mu; \vec{a})$. In particular, this denotes an orbit of the right action of the symmetric group of the representative of $\mu \otimes a_1 \otimes \ldots \otimes a_n$.

The unit of an operad $\mathscr{P}$ is denoted by $\mathbb{1}_{\mathscr{P}}$ or simply by $\mathbb{1}$, the multiplication by $\gamma_{\mathscr{P}}$ or just γ, and the associated monad by $\mathsf{S}_{\mathscr{P}}$. When the operad is algebraic, we denote the multiplication map by:

$$\mu(\mu_1, \ldots, \mu_n) := \gamma(\mu; \mu_1, \ldots, \mu_n)$$

Usually, one requires the symmetric monoidal category $\mathbb{E}$ to have colimits. This assumption will be clear in a moment. For now, let's clarify the assumptions we need for our construction in the following convention:

Convention 3.5. In the following, we denote by $\mathbb{E}$ a strict symmetric monoidal category whose tensor product is denoted by $\otimes$ and unit by $\mathbb{1}$ with countable colimits and for which the tensor product $\otimes$ commutes with countable colimits in each variable.

As proved in [48, Proposition 4.7], under Convention 3.5, an operad $\mathscr{P}$ of $\mathbb{E}$ generates a monad defined as follows:

endofunctor The endofunctor $\mathsf{S}_{\mathscr{P}} : \mathbb{E} \to \mathbb{E}$, called the **Shur functor** of $\mathscr{P}$, is defined as follows:

$$\mathsf{S}_{\mathscr{P}}(V) := \bigoplus_{n \in \mathbb{N}} \mathscr{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$$

where $\otimes_{\mathbb{S}_n}$ denotes the coequalizer $\mathscr{P}(n) \otimes V^{\otimes n}/\mathbb{S}_n$ between the right action of $\mathbb{S}_n$ acting over $\mathscr{P}(n)$ and the left action of $\mathbb{S}_n$ over $V^{\otimes n}$ by shuffling via the symmetric braiding;

multiplication The multiplication $\gamma\colon \mathsf{S}^2_{\mathscr{P}} \Rightarrow \mathsf{S}_{\mathscr{P}}$ is induced by the maps $\gamma\colon \mathscr{P}(n) \otimes \mathscr{P}(k_1) \otimes \ldots \otimes \mathscr{P}(k_n) \to \mathscr{P}(k_1 + \ldots + k_n)$, which lift to the coequalizer being equivariant under the action of the symmetric groups;

unit The unit $\eta\colon \mathsf{id}_{\mathbb{E}} \Rightarrow \mathsf{S}_{\mathscr{P}}$ is induced by the unit $\eta\colon \mathbb{1} \to \mathscr{P}(1)$.

Our discussion will mostly focus on *algebraic operads*, which are operads over the symmetric monoidal category Mod_R of modules over a commutative and unital ring R. In this case, we can interpret $\mathsf{S}_{\mathscr{P}}V$ in a pointwise fashion as the R-module generated by elements of the form:

$$(\mu; x_1, \ldots, x_m)$$

formed by trees with m inputs μ together with m elements $x_1, \ldots, x_m$ of V, fufilling the following equivariant relation:

$$(\mu \cdot \sigma; x_1, \ldots, x_m) = (\mu; x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

Similarly, $\mathsf{S}^2_{\mathscr{P}}V$ is the R-module generated by elements:

$$\left(\mu; (\mu_1; x^{(1)}_1, \ldots, x^{(1)}_{k_1}), \ldots, (\mu_m; x^{(m)}_1, \ldots, x^{(m)}_{k_m})\right)$$

satisfying a similar equivariant relation. So, the multiplication map is defined by:

$$\begin{aligned} &\gamma\left(\mu; (\mu_1; x^{(1)}_1, \ldots, x^{(1)}_{k_1}), \ldots, (\mu_m; x^{(m)}_1, \ldots, x^{(m)}_{k_m})\right) := \\ &= \left(\mu(\mu_1, \ldots, \mu_m); x^{(1)}_1, \ldots, x^{(1)}_{k_1}, \ldots, x^{(m)}_1, \ldots, x^{(m)}_{k_m}\right) \end{aligned}$$

The unit map η is defined as follows:

$$\eta(x) = (\mathbb{1}_{\mathscr{P}}; x)$$

for any $x \in V$. For now, we are going to keep the discussion more general and $\mathbb{E}$ will be considered a generic symmetric monoidal category satisfying Convention 3.5.

Example 3.6. Consider an associative and unital algebra A over a commutative and unital ring R. Then, we can define the operad $A^{\bullet}$ which is an operad over the monoidal category Mod_R, and whose sequence $A^{\bullet}(n)$ is the trivial R-module 0 for every $n \neq 1$ and $A^{\bullet}(1) = A$. The multiplication and the unit of $A^{\bullet}$ is induced by the multiplication and the unit of A. So, the corresponding monad $\mathsf{S}_{A^{\bullet}}$ is the monad which sends an R-module V to $A \otimes V$.

More generally, given a monoid object A of the symmetric monoidal category $\mathbb{E}$, $A^{\bullet}$ is the operad over $\mathbb{E}$ whose only non-trivial entry (0 represents here the initial object of $\mathbb{E}$) is $A^{\bullet}(1) = A$ and whose multiplication and unit are induced by the ones of A. So, $\mathsf{S}_{A^{\bullet}} V = A \otimes V$.

Example 3.7. The associative operad $\mathscr{Ass}$ is the algebraic operad generated by the binary tree μ, satisfying Relation (3.2.1). Concretely, $\mathscr{Ass}(n)$ is the group ring $R[\mathbb{S}_n]$, the multiplication sends $(\sigma; \sigma_1, \ldots, \sigma_m) \in R[\mathbb{S}_m] \otimes R[\mathbb{S}_{k_1}] \otimes \ldots \otimes R[\mathbb{S}_{k_m}]$ to the permutation which acts on $M := k_1 + \ldots + k_m$ elements as follows: the first $k_{\sigma(1)}$ elements are shuffled by $\sigma_{\sigma(1)}$, the second $k_{\sigma(2)}$ elements are shuffled by $\sigma_{\sigma(2)}$ and so on up to the last $k_{\sigma(m)}$ elements that are shuffled by $\sigma_{\sigma(m)}$. Finally, the unit is the unique generator of $\mathscr{Ass}(1) = R[\mathbb{S}_1] = R[\mathbb{1}_{\mathscr{Ass}}]$.

To justify this presentation of $\mathscr{Ass}$, notice that $\mathscr{Ass}(2)$ is generated by the two binary trees μ and μ^{op}. Similarly, $\mathscr{Ass}(3)$ is generated by the ternary trees $\mu(\mu, \mathbb{1})$ and $\mu(\mathbb{1}, \mu)$, together with all the possible permutations of the inputs of these trees. However, because of the associativity relation, we have that $\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu)$, so the remaining trees after the imposition of this relation must be in bijection with $\mathbb{S}_3$. Similarly, the generators of $\mathscr{Ass}(n)$ are in bijection with the elements of $\mathbb{S}_n$.

The operad $u\mathscr{Ass}$ is also generated by $\mu \in u\mathscr{Ass}(2)$ satisfying Relation (3.2.1), and by a 0-ary operation $\eta \in u\mathscr{Ass}(0)$, for which, $\mu(\eta, \mathbb{1}) = \mathbb{1} = \mu(\mathbb{1}, \eta)$. Concretely, $u\mathscr{Ass}(n) = \mathscr{Ass}(n)$ for every $n > 0$ and $u\mathscr{Ass}(0) = R$.

Example 3.8. The commutative operad $\mathscr{Com}$ is the algebraic operad generated by the binary tree μ which satisfies the associativity condition of Equation (3.2.1) together with $\mu^{\mathsf{op}} = \mu \cdot \tau = \mu$. To construct $\mathscr{Com}$, one can start with $\mathscr{Ass}$ and quotient by the relation $\mu^{\mathsf{op}} = \mu$. So, $\mathscr{Com}(2) = R$. $\mathscr{Com}(3)$ is generated by $\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu)$. Moreover, for any permutation $\sigma \in \mathbb{S}_3$, $\mu(\mu, \mathbb{1}) \cdot \sigma$ is equivalent to one of the following expressions: $\mu(\mu, \mathbb{1}), \mu^{\mathsf{op}}(\mu, \mathbb{1}), \mu(\mu^{\mathsf{op}}, \mathbb{1}), \mu^{\mathsf{op}}(\mu^{\mathsf{op}}, \mathbb{1})$. However, because of the identification $\mu^{\mathrm{op}} = \mu$, these are equal to $\mu(\mu, \mathbb{1})$. So, also $\mathscr{Com}(3) = R$. Similarly, for any $n > 0$, $\mathscr{Com}(n) = R$. For $n = 0$, we have $\mathscr{Com}(0) = 0$. If we take $\mathscr{Com}(1)$, then we obtain the operad $u\mathscr{Com}$, which is the unitary and commutative operad.

Example 3.9. The Lie operad $\mathscr{Lie}$ is the algebraic operad generated by the binary tree ν satisfying the following relations:

$$\begin{aligned}
&\nu + \nu \cdot \tau = 0\\
&\nu(\nu, \mathbb{1}) + \nu(\nu, \mathbb{1}) \cdot \sigma + \nu(\nu, \mathbb{1}) \cdot \sigma^2 = 0
\end{aligned}$$

where $\tau = (1 \quad 2) \in \mathbb{S}_2$ and $\sigma := (1 \quad 2 \quad 3) \in \mathbb{S}_3$. For a more explicit presentation of $\mathscr{Lie}$ we refer the reader to [46, Section 13.2.3].

Example 3.10. The Poisson operad $\mathscr{Pois}$ is the algebraic operad generated by two binary trees μ and ν, μ satisfying the same relations as the binary tree that generates $\mathscr{Com}$ and ν satisfying the same relations as the binary tree that generates $\mathscr{Lie}$. Moreover, they satisfy the following compatibility:

$$\nu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \nu) + \mu(\nu, \mathbb{1}) \cdot \sigma$$

where $\sigma := (2 \quad 3) \in \mathbb{S}_3$. Concretely, this relation reads as $[ab, c] = a[b, c] + [a, c]b$, for any a, b, c, where $[,] := \nu$ and juxtaposition represents μ.

Example 3.11. Suppose that $\mathbb{E}$ is also closed, that is it admits an internal Hom-functor $[,]$. Then, we can introduce the $\mathscr{End}(V)$ operad, for $V \in \mathbb{E}$, defined by $\mathscr{End}(V)(n) := [V^{\otimes n}, V]$. The multiplication is induced by the internal composition:

$$\gamma \colon [V^{\otimes m}, V] \otimes [V^{\otimes k_1}, V] \otimes \ldots \otimes [V^{\otimes k_m}, V] \to [V^{\otimes (k_1 + \ldots + k_m)}, V]$$

and the unit is given by the internal identity morphism $\mathbb{1} \in [V, V]$.

3.3 Algebras of operads

So far we recalled and motivated the definition of an operad. At this point, we want to point out that there are numerous generalizations of this concept, among which there are coloured operads (which are related to multicategories), non-symmetric operads, planar operads, props, pros, and many others. We invite the reader to consult [46, Chapter 5] for a list of some of these extensions. The next step is to show that an operad generates algebraic objects. The main idea is to refer to the *algebraic theory* of an operad as the category of representations of the operad. Let's briefly recall this notion (see [46, Chapter 5]).

Definition⋆ 3.12. *An* ***algebra*** *of an operad $\mathscr{P}$ consists of an object A of $\mathbb{E}$ together with a collection of morphisms:*

$$\theta_n \colon \mathscr{P}(n) \otimes A^n \to A$$

called ***structure maps*** *of the algebra, compatible with the multiplication, the unit of the operad, as follows:*

$$\begin{tikzcd}
\mathscr{P}(m) \otimes \mathscr{P}(k_1) \otimes \ldots \otimes \mathscr{P}(k_m) \otimes A^{\otimes(k_1+\ldots+k_m)} \arrow[r, "\gamma\otimes A^{\otimes(k_1+\ldots+k_m)}"] \arrow[d, "\cong"'] & \mathscr{P}(k_+\ldots+k_m) \otimes A^{\otimes(k_1+\ldots+k_m)} \arrow[dd, "\theta"] \\
\mathscr{P}(m) \otimes \mathscr{P}(k_1) \otimes A^{\otimes k_1} \otimes \ldots \otimes \mathscr{P}(k_m) \otimes A^{\otimes k_m} \arrow[d, "\mathscr{P}(m)\otimes\theta\otimes\ldots\otimes\theta"'] & \\
\mathscr{P}(m) \otimes A^{\otimes m} \arrow[r, "\theta"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{P}(1) \otimes A \arrow[r, "\theta"] & A \\
\mathbb{1} \otimes A \arrow[u, "\eta\otimes A"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

and satisfying the following equivariant condition with respect to the symmetric actions:

$$\begin{tikzcd}
\mathscr{P}(n) \otimes A^{\otimes n} \arrow[r, "\rho(\sigma)\otimes\varepsilon(\sigma)"] \arrow[d, "\theta"'] & \mathscr{P}(n) \otimes A^{\otimes n} \arrow[d, "\theta"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

Thanks to the compatibility between θ and the operad structure, the relations on the trees μ are reflected in the properties of the operations $\mu \colon A \otimes \ldots \otimes A \to A$.

Notation 3.13. For an algebra A of an operad $\mathscr{P}$, we denote by $\theta\colon \mathsf{S}_{\mathscr{P}}A \to A$ the corresponding structure map and, when $\mathscr{P}$ is algebraic, we adopt the convention to write:

$$\mu_A(a_1,\ldots,a_n) := \mu(a_1,\ldots,a_n) := \theta(\mu; a_1,\ldots,a_n)$$

Example 3.14. Consider an associative and unital algebra A, then an algebra M of the operad $A^{\bullet}$ has a unique non-trivial map $\theta\colon A\otimes M \to M$, which is compatible with the multiplication and the unit of A, that is:

$$(ab)(x) = a(b(x))$$
$$\mathbb{1}(x) = x$$

So, M is a left-module of A.

Example 3.15. Consider the operad $\mathscr{Ass}$. An algebra A of $\mathscr{Ass}$ consists of an R-module together with linear morphisms $\theta\colon R[\mathbb{S}_n]\otimes A^{\otimes n} \to A$. Since $\mathscr{Ass}$ is generated by the binary tree μ, the algebraic structure of A is fully specified by $\theta_2\colon R[\mathbb{S}_2]\otimes A\otimes A \to A$, which sends $(\mu; a, b)$ to $\mu(a,b)$ and $(\mu^{\mathsf{op}}; a, b) = (\mu\cdot\tau; a, b) = (\mu; b, a)$ to $\mu^{\mathsf{op}}(a,b) = \mu(b,a)$. Moreover, $\theta_3\colon R[\mathbb{S}_3]\otimes A\otimes A\otimes A \to A$ sends $(\mu(\mu,\mathbb{1}); a,b,c)$ to $\mu(\mu(a,b),c)$ and sends $(\mu(\mathbb{1},\mu); a,b,c)$ to $\mu(a,\mu(b,c))$. However, since $\mu(\mu,\mathbb{1}) = \mu(\mathbb{1},\mu)$, we have that:

$$\mu(\mu(a,b),c) = \mu(a,\mu(b,c))$$

So, A is precisely an associative algebra.

Example 3.16. Consider the operad $\mathscr{Com}$. An algebra A of $\mathscr{Com}$ consists of an R-module together with linear morphisms $\theta\colon A^{\otimes n} = R\otimes A^{\otimes n} \to A$. Since $\mathscr{Com}$ is generated by the binary tree μ, the algebraic structure of A is fully specified by $\theta_2\colon A\otimes A \to A$, which sends $(\mu; a, b)$ to $\mu(a,b)$ and $(\mu^{\mathsf{op}}; a, b)$ to $\mu^{\mathsf{op}}(a,b)$. However, since $\mu = \mu^{\mathsf{op}}$ we have that:

$$\mu(a,b) = \mu^{\mathsf{op}}(a,b) = \mu(b,a)$$

Using a similar argument of Example 3.15, we also conclude that $\mu(\mu(a,b),c) = \mu(a,\mu(b,c))$. So, A is precisely a commutative algebra. When we consider the operad $u\mathscr{Com}$, then an algebra A of $u\mathscr{Com}$ is a unital and commutative algebra.

Example 3.17. Consider the operad $\mathscr{L}ie$. An algebra A of $\mathscr{L}ie$ consists of an R-module together with linear morphisms $\theta\colon A^{\otimes n} = \mathscr{L}ie(n) \otimes A^{\otimes n} \longrightarrow A$. Since $\mathscr{L}ie$ is generated by the binary tree ν, the algebraic structure of A is fully specified by $\theta_2\colon \mathscr{L}ie(2) \otimes A \otimes A \longrightarrow A$, which sends $(\nu; a, b)$ to $\nu(a, b)$. Moreover, since $\nu + \nu^{\mathsf{op}} = 0$, we have that $\theta_2(\nu + \nu^{\mathsf{op}}; a, b)$ is sent to $\nu(a, b) + \nu^{\mathsf{op}}(a, b) = 0$, so we have that:

$$\nu(b, a) = \nu^{\mathsf{op}}(a, b) = -\nu(a, b)$$

Moreover, $\theta_3\colon \mathscr{L}ie(3) \otimes A \otimes A \otimes A \longrightarrow A$ sends $(\nu(\nu, \mathbb{1}) + \nu(\nu, \mathbb{1}) \cdot \sigma + \nu(\nu, \mathbb{1}) \cdot \sigma^2; a, b, c)$ to:

$$\nu(\nu(a, b), c) + \nu(\nu(c, a), b) + \nu(\nu(b, c), a) = 0$$

which is the Jacobi identity. So, A is precisely a Lie algebra.

Example 3.18. Adopting a similar argument as in Examples 3.16 and 3.17, one can easily see that an algebra A of the operad $\mathscr{P}ois$ is an R-module equipped with two binary operations μ and ν, such that (A, μ) is a commutative algebra and (A, ν) is a Lie algebra. Moreover, the structure map $\theta_3\colon \mathscr{P}ois(3) \otimes A \otimes A \otimes A \longrightarrow A$ sends $(\nu(\mu, \mathbb{1}); a, b, c)$ to $\nu(\mu(a, b), c)$ and $(\mu(\mathbb{1}, \nu) + \mu(\nu, \mathbb{1}) \cdot \sigma; a, b, c)$ to $\mu(a, \nu(b, c)) + \mu(\nu(a, c), b)$ and thanks to the relation $\nu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \nu) + \mu(\nu, \mathbb{1}) \cdot \sigma$ we have that:

$$\nu(\mu(a, b), c) = \mu(a, \nu(b, c)) + \mu(\nu(a, c), b)$$

This implies that A is a Poisson algebra.

Let's introduce now morphisms of $\mathscr{P}$-algebras.

Definition⋆ 3.19. *Given two algebras A and B of an operad $\mathscr{P}$ with structure maps $\theta^{(A)}$ and $\theta^{(B)}$, respectively, a morphism of algebras $f\colon A \longrightarrow B$ consists of a morphism $f\colon A \longrightarrow B$ of $\mathbb{E}$, compatible with the structure maps of A and B, that is:*

$$\begin{tikzcd}
\mathscr{P}(n) \otimes A^{\otimes n} \arrow[r, "\theta^{(A)}"] \arrow[d, "\mathscr{P}(n) \otimes f^{\otimes n}"'] & A \arrow[d, "f"] \\
\mathscr{P}(n) \otimes B^{\otimes n} \arrow[r, "\theta^{(B)}"'] & B
\end{tikzcd}$$

Algebras of an operad $\mathscr{P}$ together with their morphisms form a category denoted by $\mathsf{Alg}_{\mathscr{P}}$.

In Section 2.4 we recalled that a monad is a machine which produces algebraic objects and we also mentioned that every operad $\mathscr{P}$ is associated with a monad $\mathsf{S}_{\mathscr{P}}$. So, it is natural to wonder if the algebraic objects produced by the monad $\mathsf{S}_{\mathscr{P}}$ are the same as the algebras of $\mathscr{P}$. Indeed, an algebra of the monad $\mathsf{S}_{\mathscr{P}}$ consists of an object A of $\mathbb{E}$ together with a map $\mathsf{S}_{\mathscr{P}}A \to A$, that is:

$$\bigoplus_{n\in\mathbb{N}} \mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \to A$$

By definition of coproduct, this is equivalent to a collection of morphisms $\theta_n \colon \mathscr{P}(n)\otimes A^{\otimes n} \to A$, compatible with the multiplication and the unit of the monad and equivariant with respect to the symmetric actions. This last property comes directly from the definition of $\otimes_{\mathbb{S}_n}$. Finally, recall that the multiplication and the unit of $\mathsf{S}_{\mathscr{P}}$ are induced by the multiplication and the unit of $\mathscr{P}$, so it is easy to see that an algebra of $\mathsf{S}_{\mathscr{P}}$ is precisely an algebra of $\mathscr{P}$ and vice versa. Similarly, morphisms of algebras of operads correspond to morphisms of algebras of the corresponding monads, so we have that $\mathsf{Alg}_{\mathscr{P}} \cong \mathsf{Alg}_{\mathsf{S}_{\mathscr{P}}}$.

3.4 The coCartesian differential monad of an operad

The main goal of this chapter is to associate each algebraic operad with a tangent category which can be interpreted as a legitimate *geometric theory* of the operad. The key step of our argument is to show that the monad associated with an operad is a tangent monad. However, in order to obtain a tangent monad, we first need a tangent structure on the base category $\mathbb{E}$. For this purpose, in this section, we focus on a special class of symmetric monoidal categories which carry a canonical tangent structure. Let's introduce this via the following convention.

Convention 3.20. We denote by $\mathbb{E}$ a symmetric monoidal category satisfying Convention 3.5. We also assume that $\mathbb{E}$ has finite biproducts (see Example 2.11) and that finite biproducts commute with the tensor product in each variable.

The archetypical example of such a monoidal category is the category of R-modules over a commutative and unital ring R. When $\mathbb{E} = \mathsf{Mod}_R$ we use the convention to call operads over such an $\mathbb{E}$, *algebraic operads*. This convention is the one adopted by Loday and Vallette in [46].

The key ingredient is the presence of biproducts, which means that finite products and finite coproducts exist and that the unique morphism between them induced by universality is an isomorphism. As already discussed in Example 2.11, a category with biproducts has a canonical tangent structure, called the **tangent structure induced by biproducts**, denoted by $\mathbb{L}$. Our goal consists of proving that the monad associated with an operad is a tangent monad over the tangent category induced by biproducts. Interestingly, when the base tangent category is induced by biproduct, tangent monads are equivalent to a simpler concept, investigated by Ikonicoff and Lemay in [30]: coCartesian differential monads. Let's recall here this notion.

Definition* 3.21. *Let $\mathbb{X}$ be a category with biproducts. Then, a* ***coCartesian differential monad*** *over $\mathbb{X}$ consists of a monad S over $\mathbb{X}$ together with a natural transformation $\partial\colon S \Rightarrow S \circ \mathbb{L}$, called the* ***differential combinator****, where $\mathbb{L}$ is the functor which sends each object X of $\mathbb{X}$ to $X \oplus X$ and each morphism $f\colon X \to Y$ to $f \oplus f$, satisfying the following conditions:*

zero rule

$$\begin{tikzcd}
SA \arrow[r, "\partial"] \arrow[dr, "0"'] & S(A \oplus A) \arrow[d, "S\pi_1"] \\
& SA
\end{tikzcd}$$

additive rule

$$\begin{tikzcd}
SA \arrow[rr, "\partial"] \arrow[d, "\partial"'] & & S(A \oplus A) \arrow[dd, "S(\mathsf{id}_A \oplus \Delta)"] \\
S(A \oplus A) \arrow[d, "\langle S(\mathsf{id}_A \oplus \iota_1), S(\mathsf{id}_A \oplus \iota_2)\rangle"'] & & \\
S(A \oplus A \oplus A) \oplus S(A \oplus A \oplus A) \arrow[rr, "+"'] & & S(A \oplus A \oplus A)
\end{tikzcd}$$

linear rule

$$\begin{tikzcd} SA \arrow[r, "\partial"] & S(A \oplus A) \\ A \arrow[u, "\eta"] \arrow[r, "\iota_2"'] & A \oplus A \arrow[u, "\eta"'] \end{tikzcd}$$

chain rule

$$\begin{tikzcd} S^2A \arrow[r, "\partial S"] \arrow[d, "\gamma"'] & S(SA \oplus SA) \arrow[r, "S(\langle S\iota_1, \partial \rangle)"] & S^2(A \oplus A) \arrow[d, "\gamma"] \\ SA \arrow[rr, "\partial"'] & & S(A \oplus A) \end{tikzcd}$$

lift rule

$$\begin{tikzcd} S(A \oplus A) \arrow[r, "\partial"] & S(A \oplus A \oplus A \oplus A) \arrow[d, "S(\pi_1 \oplus \pi_4)"] \\ SA \arrow[u, "\partial"] \arrow[r, "\partial"'] & S(A \oplus A) \end{tikzcd}$$

symmetry rule

$$\begin{tikzcd} SA \arrow[r, "\partial"] \arrow[d, "\partial"'] & S(A \oplus A) \arrow[dd, "\partial"] \\ S(A \oplus A) \arrow[d, "\partial"'] & \\ S(A \oplus A \oplus A \oplus A) \arrow[r, "S(\mathsf{id}_A \oplus \tau \oplus \mathsf{id}_A)"'] & S(A \oplus A \oplus A \oplus A) \end{tikzcd}$$

where $\tau \colon A \oplus A \to A \oplus A$ *is the canonical flip* $\langle \iota_1 \pi_2, \iota_2 \pi_1 \rangle$.

Remark 3.22. In the original [30, Definition 3.1], instead of a coCartesian differential monad, the authors introduced the dual notion, that is a Cartesian differential comonad. Since operads are usually associated with monads, not comonads, here we prefer to employ the version introduced in Definition 3.21. The name was already introduced in [29].

Ikonicoff and Lemay's original motivation for introducing coCartesian differential monads was to construct a monad which could *generate* a Cartesian differential category (see [7]). They showed that the opposite of the Kleisli category of a coCartesian differential monad is, in fact, a Cartesian differential category. Since this result plays an important role in our story we want to recall this result here. First, recall that the coKleisli category $\mathsf{Kl}^{\mathsf{op}}(S)$ of a monad S over a category $\mathbb{E}$ is the category whose objects are the objects of $\mathbb{X}$ and morphisms $f\colon A \to B$ corresponds to morphisms of $\mathbb{X}$ of the form $[\![f]\!]\colon B \to SA$, and composition of $f\colon A \to B$ and $g\colon B \to C$ is given by:

$$[\![fg]\!]\colon C \xrightarrow{[\![g]\!]} SB \xrightarrow{S[\![f]\!]} S^2A \xrightarrow{\gamma} SA$$

where $\gamma\colon S^2 \Rightarrow S$ is the monad multiplication. Finally, the identity morphisms are given by $[\![\mathsf{id}_A]\!] := \eta\colon A \to SA$, where η is the unit of the monad.

Proposition⋆ 3.23. *The coKleisli category* $\mathsf{Kl}^{\mathsf{op}}(S)$ *of a coCartesian differential monad* (S, ∂) *is a Cartesian differential category, whose product is the same as the one on* $\mathbb{E}$ *and whose differential combinator* D *sends a morphism* $f\colon A \to B$ *to:*

$$[\![\mathsf{D}f]\!]\colon B \xrightarrow{[\![f]\!]} SA \xrightarrow{\partial} S(A \oplus A)$$

The Kleisli category of a monad, i.e. the dual of $\mathsf{Kl}^{\mathsf{op}}(S)$, is the category of free algebras of the monad. For an algebra A to be free means that $A = SA'$ for some object A' and the structure map of A is given by $SA = S^2A' \xrightarrow{\gamma} SA' = A$. So, a morphism $f\colon A \to B$ between two free algebras is a morphism $f\colon SA' \to SB'$, for some A' and B'. By precomposing with the unit η of the monad, we obtain a morphism $[\![f]\!]\colon A' \xrightarrow{\eta} SA' \xrightarrow{f} B$. Let $\mathsf{Free}(S)$ be the category of free algebras of S and all morphisms of algebras between them. Then, we have a functor:

$$\mathsf{Free}(S) \to \mathsf{Kl}(S)$$

which sends each free algebra SA' to A' and each morphism $f\colon SA' \to SB'$ to $[\![f]\!]\colon A' \xrightarrow{\eta} SA' \xrightarrow{f} SB'$. Similarly, each object A' of $\mathsf{Kl}(S)$ can be sent to $SA' \in \mathsf{Free}(S)$ and each morphism $f\colon A' \to B'$ is sent to $SA' \xrightarrow{S[\![f]\!]} S^2B' \xrightarrow{\gamma} SB'$. This induces an equivalence between $\mathsf{Kl}(S)$ and $\mathsf{Free}(S)$. So, in particular, dualizing, we have that $\mathsf{Kl}^{\mathsf{op}}(S) \cong \mathsf{Free}^{\mathsf{op}}(S)$.

This, in particular, implies that the subcategory of free algebras of a coCartesian differential monad is a Cartesian differential category. So, it is natural to wonder if this Cartesian differential category could be associated with the category of differential objects of a larger tangent category (see Section 2.2.2). The trick consists of showing that every coCartesian differential monad is also a tangent monad with respect to the base tangent category induced by biproducts.

Proposition 3.24. *Consider a category with biproducts $\mathbb{X}$. If (S, ∂) is a coCartesian differential monad over $\mathbb{X}$ then, S equipped with the natural transformation:*

$$\alpha\colon S(A \oplus A) \xrightarrow{\langle S(\pi_1), \nabla\rangle} SA \oplus SA$$

is a tangent monad over the tangent category $(\mathbb{X}, \mathbb{L})$ induced by biproducts, where $\nabla\colon S(A \oplus A) \to SA$ is the natural transformation defined as follows:

$$\nabla\colon S(A \oplus A) \xrightarrow{\partial} S(A \oplus A \oplus A \oplus A) \xrightarrow{S(\pi_1 + \pi_4)} SA$$

Proof. The proof is a long but straightforward computation. We invite the interested reader to consult the original paper [29]. □

Surprisingly, the converse of Proposition 3.24 also holds: every tangent monad over a tangent category induced by biproducts is a coCartesian differential monad.

Proposition 3.25. *If $\mathbb{X}$ is a category with biproducts and $\mathbb{L}$ is the tangent structure on $\mathbb{X}$ induced by biproducts, then every tangent monad (S, α) over $(\mathbb{X}, \mathbb{L})$ defines a coCartesian differential monad (S, ∂) whose differential combinator is defined as follows:*

$$\partial\colon SA \xrightarrow{S\langle \mathsf{id}_A, 0, 0, \mathsf{id}_A\rangle} S(A \oplus A \oplus A \oplus A) \xrightarrow{\alpha_{A\oplus A}} S(A \oplus A) \oplus S(A \oplus A) \xrightarrow{\pi_2} S(A \oplus A)$$

Proof. The proof is a long but straightforward computation. We invite the interested reader to consult the original paper [29]. □

Propositions 3.24 and 3.25 show that the notion of coCartesian differential monad over a category with biproducts and the one of a tangent monad over a tangent category induced by biproducts are equivalent. We now employ this equivalence to prove the main result of this section. For this purpose, consider an object

A of the base monoidal category $\mathbb{E}$ and, for each integer n, consider the maps so defined:

$$\delta_k := \iota_1 \otimes \ldots \otimes \underbrace{\iota_2}_{k\text{-th position}} \otimes \ldots \otimes \iota_1 \colon A^{\otimes n} \longrightarrow (A \oplus A)^{\otimes n}$$

where the index k runs from 1 to n and ι_1 and ι_2 denote the inclusions $A \to A \oplus A$ in the first and the second component, respectively. For $\mathbb{E} = \mathsf{Mod}_R$, δ_k corresponds to the following map:

$$\delta_k(x_1, \ldots, x_n) = ((x_1, 0), \ldots, (0, x_k), \ldots, (x_n, 0))$$

This family of maps induces a morphism, for each n:

$$\partial_n := \sum_{k=1}^{n} \mathsf{id}_{\mathscr{P}(n)} \otimes \delta_k \colon \mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \longrightarrow \mathscr{P}(n) \otimes_{\mathbb{S}_n} (A \oplus A)^{\otimes n}$$

In particular, one can employ such morphisms to define:

$$\partial_{\mathscr{P}} \colon \bigoplus_{n \in \mathbb{N}} \mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \xrightarrow{\bigoplus_{n \in \mathbb{N}} \partial_n} \bigoplus_{n \in \mathbb{N}} \mathscr{P}(n) \otimes_{\mathbb{S}_n} (A \oplus A)^{\otimes n} \tag{3.4.1}$$

Theorem 3.26. *The monad $\mathsf{S}_{\mathscr{P}}$ associated with an operad $\mathscr{P}$ over a monoidal category $\mathbb{E}$ satisfying Convention 3.20 is a coCartesian differential monad whose differential combinator $\partial_{\mathscr{P}}$ is defined in Equation (3.4.1).*

Proof. We decide here to give the proof in the algebraic case, i.e. when $\mathbb{E} = \mathsf{Mod}_R$. This is for two reasons: first, the proof in the algebraic case really clarifies each aspect of the theorem, while the notation for the general case obscures the meaning of it. Second, when one understands the proof in the algebraic case, one can easily see how the proof generalizes to the general case. This is because all the steps in the proof involve only morphisms which can be defined in any monoidal category with biproducts. In the algebraic case, we do not use any specific aspect of the category Mod_R.

Let's start with the zero rule, which reads $\mathsf{S}_{\mathscr{P}}(\pi_1) \circ \partial = 0$:

$$\begin{aligned} & (\mathsf{S}_{\mathscr{P}}(\pi_1)) \circ \partial)(\mu; a_1, \ldots, a_n) \\ = \ & \mathsf{S}_{\mathscr{P}}(\pi_1) \left(\sum_{k=1}^{n} (\mu; (a_1, 0), \ldots, (0, a_k), \ldots, (a_n, 0)) \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} (\mu; \pi_1(a_1, 0), \ldots, \pi_1(0, a_k), \ldots, \pi_1(a_n, 0)) \\
&= \sum_{k=1}^{n} (\mu; a_1, \ldots, 0, \ldots, a_n) \\
&= 0
\end{aligned}
$$

The additive rule, which reads as $\mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \Delta) \circ \partial = + \oplus \langle \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \iota_1), \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \iota_2) \rangle \circ \partial$, can be proven as follows:

$$
\begin{aligned}
& (\mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \Delta) \circ \partial)(\mu; a_1, \ldots, a_n) \\
&= \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \Delta) \left(\sum_{k=1}^{n} (\mu; (a_1, 0), \ldots, (0, a_k), \ldots, (a_n, 0)) \right) \\
&= \sum_{k=1}^{n} (\mu; (a_1, 0, 0), \ldots, (0, a_k, a_k), \ldots, (a_n, 0, 0))
\end{aligned}
$$

Let's now consider the right-hand side of the equation:

$$
\begin{aligned}
& (+ \oplus \langle \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \iota_1), \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \iota_2) \rangle \circ \partial)(\mu; a_1, \ldots, a_n) \\
&= (+ \oplus \langle \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \iota_1), \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \iota_2) \rangle) \left(\sum_{k=1}^{n} (\mu; (a_1, 0), \ldots, (0, a_k), \ldots, (a_n, 0)) \right) \\
&= \sum_{k=1}^{n} (\mu; (a_1, 0, 0), \ldots, (0, a_k, 0), \ldots, (a_n, 0, 0)) + \\
&\qquad + (\mu; (a_1, 0, 0), \ldots, (0, 0, a_k), \ldots, (a_n, 0, 0)) \\
&= \sum_{k=1}^{n} (\mu; (a_1, 0, 0), \ldots, (0, a_k, a_k), \ldots, (a_n, 0, 0))
\end{aligned}
$$

The next step is to show the linear rule $\partial \circ \eta = \eta \circ \iota_2$:

$$
\begin{aligned}
& (\partial \circ \eta)(a) \\
&= \partial(\mathbb{1}_{\mathscr{P}}; a) \\
&= (\mathbb{1}_{\mathscr{P}}; (0, a)) \\
&= (\mathbb{1}_{\mathscr{P}}; \iota_2(a)) \\
&= \eta(\iota_2(a))
\end{aligned}
$$

The chain rule requires that $\gamma \circ \mathsf{S}_{\mathscr{P}}(\langle \mathsf{S}_{\mathscr{P}}(\iota_1), \partial \rangle) \circ \partial_{\mathsf{S}_{\mathscr{P}}} = \partial \circ \gamma$. To represent the generic element of $\mathsf{S}^2_{\mathscr{P}} A$, we adopt the convention of denoting by $\vec{a}_k$ a tuple $a_1^{(k)}, \ldots, a_n^{(k)}$ of elements of A:

$$\begin{aligned}
& (\gamma \circ \mathsf{S}_{\mathscr{P}}(\langle \mathsf{S}_{\mathscr{P}}(\iota_1), \partial \rangle) \circ \partial_{\mathsf{S}_{\mathscr{P}}})\,(\mu; (\mu_1; \vec{a}_1), \ldots, (\mu_n; \vec{a}_n)) \\
= \;& (\gamma \circ \mathsf{S}_{\mathscr{P}}(\langle \mathsf{S}_{\mathscr{P}}(\iota_1), \partial \rangle)) \left(\sum_{j=1}^{n} (\mu; ((\mu_1; \vec{a}_1), 0), \ldots, (0, (\mu_j; \vec{a}_j)), \ldots, ((\mu_n; \vec{a}_n), 0)) \right) \\
= \;& \gamma \left(\sum_{j=1}^{n} \Big(\mu; (\mu_1; ((a_1^{(1)}, 0), \ldots, (a_{k_1}^{(1)}, 0)), \ldots \right. \\
& \qquad \ldots, \sum_{i_j=1}^{k_j} (\mu_k; (a_1^{(j)}, 0), \ldots, (0, a_{i_j}^{(j)}), \ldots, (a_{k_j}^{(j)}, 0)), \ldots \\
& \qquad \left. \ldots, (\mu_n; ((a_1^{(n)}, 0), \ldots, (a_{k_n}^{(n)}, 0)))) \Big) \right) \\
= \;& \sum_{j=1}^{n} \sum_{i_j=1}^{k_j} (\mu(\mu_1, \ldots, \mu_n); (a_1^{(1)}, 0), \ldots, (0, a_{i_j}^{(j)}), \ldots, (a_{k_n}^{(n)}, 0)) \\
= \;& \partial(\mu(\mu_1, \ldots, \mu_n); \vec{a}_1, \ldots, \vec{a}_n) \\
= \;& (\partial \circ \gamma)(\mu; (\mu_1; \vec{a}_1), \ldots, (\mu_n; \vec{a}_n))
\end{aligned}$$

To prove the lift rule is to show that $\mathsf{S}_{\mathscr{P}}(\pi_1 \oplus \pi_4) \circ \partial \circ \partial = \partial$:

$$\begin{aligned}
& (\mathsf{S}_{\mathscr{P}}(\pi_1 \oplus \pi_4) \circ \partial \circ \partial)(\mu; a_1, \ldots, a_n) \\
= \;& (\mathsf{S}_{\mathscr{P}}(\pi_1 \oplus \pi_4) \circ \partial) \left(\sum_{k=1}^{n} (\mu; (a_1, 0), \ldots, (0, a_k), \ldots, (a_n, 0)) \right) \\
= \;& \mathsf{S}_{\mathscr{P}}(\pi_1 \oplus \pi_4) \left(\sum_{k=1}^{n} \sum_{j \neq k} (\mu; (a_1, 0, 0, 0), \ldots, (0, 0, a_j, 0), \ldots \right. \\
& \qquad \ldots, (0, a_k, 0, 0), \ldots, (a_n, 0, 0, 0)) + \\
& \qquad \left. + \sum_{k=1}^{n} (\mu; (a_1, 0, 0, 0), \ldots, (0, 0, 0, a_k), \ldots, (a_n, 0, 0, 0)) \right) \\
= \;& \sum_{k=1}^{n} \sum_{j \neq k} (\mu; (a_1, 0), \ldots, (0, 0), \ldots, (0, 0), \ldots, (a_n, 0)) + \\
& \qquad + \sum_{k=1}^{n} (\mu; (a_1, 0), \ldots, (0, a_k), \ldots, (a_n, 0)) \\
= \;& \sum_{k=1}^{n} (\mu; (a_1, 0), \ldots, (0, a_k), \ldots, (a_n, 0))
\end{aligned}$$

$$= \quad \partial(\mu; a_1, \ldots, a_n)$$

Finally, the symmetry rule $\partial \circ \partial = \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \tau \oplus \mathsf{id}) \circ \partial \circ \partial$ reads as follows:

$$\begin{aligned}
& (\mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \tau \oplus \mathsf{id}) \circ \partial \circ \partial)(\mu; a_1, \ldots, a_n) \\
= \quad & \mathsf{S}_{\mathscr{P}}(\mathsf{id} \oplus \tau \oplus \mathsf{id}) \Bigg(\sum_{k=1}^{n} \sum_{j \neq k} (\mu; (a_1, 0, 0, 0), \ldots, (0, 0, a_j, 0), \ldots \\
& \quad , \ldots (0, a_k, 0, 0), \ldots, (a_n, 0, 0, 0)) + \\
& \quad + \sum_{k=1}^{n} (\mu; (a_1, 0, 0, 0), \ldots, (0, 0, 0, a_k), \ldots, (a_n, 0, 0, 0)) \Bigg) \\
= \quad & \sum_{k=1}^{n} \sum_{j \neq k} (\mu; (a_1, 0, 0, 0), \ldots, (0, a_j, 0, 0), \ldots, (0, 0, a_k, 0), \ldots, (a_n, 0, 0, 0)) + \\
& \quad + \sum_{k=1}^{n} (\mu; (a_1, 0, 0, 0), \ldots, (0, 0, 0, a_k), \ldots, (a_n, 0, 0, 0) \\
= \quad & (\partial \circ \partial)(\mu; a_1, \ldots, a_n)
\end{aligned}$$

This concludes the proof. □

In the algebraic case, i.e. when $\mathbb{E} = \mathsf{Mod}_R$, the differential combinator $\partial_{\mathscr{P}}$ is defined over generators as follows:

$$\partial_{\mathscr{P}}(\mu; x_1, \ldots, x_n) = \sum_{k=1}^{n} (\mu; (x_1, 0), \ldots, (0, x_k), \ldots, (x_n, 0))$$

Since coCartesian differential monads are tangent monads, we obtain that the monad associated with an operad is a tangent monad. Consider now the following morphisms:

$$\rho_k \colon \pi_1 \otimes \ldots \otimes \underbrace{\pi_2}_{k\text{-th position}} \otimes \ldots \otimes \pi_1 \colon (A \oplus A)^{\otimes n} \to A^{\otimes n}$$

In the algebraic case, ρ_k is defined as follows:

$$\rho_k((x_1, y_1), \ldots, (x_n, y_n)) = (x_1, \ldots, y_k, \ldots, x_n)$$

From this, let's introduce the following:

$$\alpha_n := \sum_{k=1}^{n} \langle \mathsf{id}_{\mathscr{P}}(n) \otimes \pi_1^{\otimes n}, \mathsf{id}_{\mathscr{P}} \otimes \rho_k \rangle \colon \mathscr{P}(n) \otimes_{\mathbb{S}_n} (A \oplus A)^{\otimes n} \to$$

$$\to \left(\mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}\right) \oplus \left(\mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}\right)$$

In the algebraic case, we have:

$$\alpha_n(\mu; (x_1, \dots, y_1), \dots, (x_n, y_n)) = \left((\mu; x_1, \dots, x_n), \sum_{k=1}^{n} (\mu; x_1, \dots, y_k, \dots, x_n)\right)$$

Finally, let α be so defined:

$$\alpha_{\mathscr{P}} \colon \bigoplus_{n \in \mathbb{N}} \mathscr{P}(n) \otimes_{\mathbb{S}_n} (A \oplus A)^{\otimes n} \xrightarrow{\oplus_{n\in\mathbb{N}} \alpha_n} \bigoplus_{n \in \mathbb{N}} \mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \oplus \mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \tag{3.4.2}$$

Corollary 3.27. *The monad associated with an operad $\mathscr{P}$ is a tangent monad whose distributive law is defined in Equation (3.4.2).*

3.5 The algebraic tangent category of an operad

Theorem 2.73 shows that the category of algebras of a tangent monad is a tangent category. On the other hand, Corollary 3.27 establishes that the monad associated with an operad over a symmetric monoidal category $\mathbb{E}$ with biproducts is a tangent monad. Putting together these two facts, we conclude that the category of algebras of an operad comes equipped with a canonical tangent structure.

Definition 3.28. *The **algebraic tangent category** $\mathsf{Alg}(\mathscr{P})$ of an operad $\mathscr{P}$ over a symmetric monoidal category $\mathbb{E}$ satisfying Convention 3.20, is the Eilenberg-Moore object of the tangent monad associated with the operad $\mathscr{P}$, described in Corollary 3.27.*

This section is dedicated to providing a complete description of this tangent category. The first step is to understand the tangent bundle functor. It turns out that the tangent bundle functor $\mathbb{T}$ sends a $\mathscr{P}$-algebra A with structure map θ to the $\mathscr{P}$-algebra $A \ltimes A$, known as the semidirect product of A with itself (see [46, Section 12.3.2]). Concretely, $A \ltimes A$ is the $\mathscr{P}$-algebra over the object $A \oplus A$ whose structure map is defined as follows:

$$\mathscr{P}(n) \otimes (A \oplus A)^{\otimes n} \xrightarrow{\alpha_n} \mathscr{P}(n) \otimes A^{\otimes n} \oplus \mathscr{P}(n) \otimes A^{\otimes n} \xrightarrow{\theta \oplus \theta} A \oplus A$$

When $\mathbb{E} = \mathsf{Mod}_R$, the structure map is defined by:

$$\mu((a_1, b_1), \dots, (a_m, b_m)) := \left(\mu(a_1, \dots, a_m), \sum_{k=1}^{m} (a_1, \dots, b_k, \dots, a_m)\right)$$

Lemma 3.29. *The tangent bundle functor* $\mathbb{T}\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}_{\mathscr{P}}$ *of the algebraic tangent category of an operad* $\mathscr{P}$ *sends an algebra* A *to the semi-direct product of* A *with itself, i.e.* $A \ltimes A$*, and a morphism of algebras* $f\colon A \to B$ *to* $f \ltimes f\colon A \ltimes A \to B \ltimes B$*, which, as a* $\mathbb{E}$*-morphism is just* $f \oplus f$*.*

Then, the algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ of an operad $\mathscr{P}$ is defined as follows:

category The objects of $\mathsf{Alg}(\mathscr{P})$ are $\mathscr{P}$-algebras and morphisms are morphisms of $\mathscr{P}$-algebras;

tangent bundle functor The tangent bundle functor $\mathbb{T}\colon \mathsf{Alg}(\mathscr{P}) \to \mathsf{Alg}(\mathscr{P})$ sends A to $A \ltimes A$ and $f\colon A \to B$ to $f \ltimes f\colon A \ltimes A \to B \ltimes B$;

projection The projection $p^{(\mathbb{T})}\colon \mathbb{T} \Rightarrow \mathsf{id}_{\mathsf{Alg}(\mathscr{P})}$ is the natural transformation whose underlying $\mathbb{E}$-morphism is the projection from the first component $\pi_1\colon A \oplus A \to A$;

zero morphism The zero morphism $z^{(\mathbb{T})}\colon \mathsf{id}_{\mathsf{Alg}(\mathscr{P})} \Rightarrow \mathbb{T}$ is the natural transformation whose underlying $\mathbb{E}$-morphism is the injection into the first component $\iota_1\colon A \to A \oplus A$;

n**-fold pullback** The n-fold pullback $\mathbb{T}_n\colon \mathsf{Alg}(\mathscr{P}) \to \mathsf{Alg}(\mathscr{P})$ of the projection along itself is the functor which sends an algebra A to $A \ltimes A^n$, with $A^n = A \oplus \cdots \oplus A$. Moreover, the k-th projection, denoted by $\pi_k^{(\mathbb{T})}\colon \mathbb{T}_n \Rightarrow \mathbb{T}$ is the natural transformation whose underlying $\mathbb{E}$-morphism projects the first component in the first component and the $(k+1)$-th component to the second component, i.e. $A \oplus A^n \xrightarrow{\mathsf{id}_A \oplus \iota_k \pi_2} A \oplus A$;

sum morphism The sum morphism $s^{(\mathbb{T})}\colon \mathbb{T}_2 \Rightarrow \mathbb{T}$ is the natural transformation whose underlying $\mathbb{E}$-morphism acts as the identity in the first component and sums the other two, that is $A \oplus A^2 \xrightarrow{\mathsf{id}_A \oplus +} A \oplus A$;

vertical lift The vertical lift $l^{(\mathbb{T})}\colon \mathbb{T} \Rightarrow \mathbb{T}^2$, where $\mathbb{T}^2$ denotes $\mathbb{T} \circ \mathbb{T}$, is the natural transformation whose underlying $\mathbb{E}$-morphism sends the first component to the first one and the second component to the fourth one, that is $A \oplus A \xrightarrow{\mathsf{id}_A \oplus \pi_4} A \oplus A \oplus A \oplus A$;

canonical flip The canonical flip $c^{(\mathbb{L})}\colon \mathbb{L}^2 \Rightarrow \mathbb{L}^2$ is the natural transformation whose underlying $\mathbb{E}$-morphism switches the internal components, that is $A \oplus A \oplus A \oplus A \xrightarrow{\mathsf{id}_A \oplus \tau \oplus \mathsf{id}_A} A \oplus A \oplus A \oplus A$, where $\tau\colon A \oplus B \to B \oplus A$ is the canonical braiding.

Finally, if $\mathbb{E}$ is additive, which means that it is also Ab-enriched, then $\mathsf{Alg}(\mathscr{P})$ has also negatives:

negation The negation $n^{(\mathbb{L})}\colon \mathbb{L} \Rightarrow \mathbb{L}$ is the natural transformation whose underlying $\mathbb{E}$-morphism acts as the identity on the first component and negates the second one, that is $A \oplus A \xrightarrow{\mathsf{id}_A \oplus -} A \oplus A$.

When $\mathscr{P}$ is algebraic, that is $\mathbb{E} = \mathsf{Mod}_R$, then we have a more concrete description of the algebraic tangent structure of $\mathscr{P}$. We already described the semi-direct product of a $\mathscr{P}$-algebra A with itself, then the natural transformations are defined as follows:

$$\begin{aligned}
&p^{(\mathbb{L})}(a,u) := a \\
&z^{(\mathbb{L})}(a) := (a,0) \\
&s^{(\mathbb{L})}(a;u_1,u_2) := (a,u_1+u_2) \\
&l^{(\mathbb{L})}(a,u) := (a,0,0,u) \\
&c^{(\mathbb{L})}(a,u,v,w) := (a,v,u,w) \\
&n^{(\mathbb{L})}(a,u) := (a,-u)
\end{aligned} \tag{3.5.1}$$

Example 3.30. Consider the symmetric monoidal category of $\mathbb{Z}$-modules, i.e. the category of abelian groups. This category is additive. Consider the operad $u\mathscr{C}om$ described in Example 3.8 over $\mathsf{Mod}_{\mathbb{Z}}$. Then, the category of algebras of $u\mathscr{C}om$ is the category cRing of commutative and unital rings. Spelling out the details about the tangent bundle functor $\mathbb{L}$, one realizes that the semi-direct product $R \ltimes R$ of a ring R with itself, is isomorphic to the ring $R\langle\varepsilon\rangle$ of dual numbers, which we recall is the ring of terms $r + s\varepsilon$ with $\varepsilon^2 = 0$. In particular, the isomorphism sends $R \ltimes R \ni (r,s)$ to $r + s\varepsilon$. Thus the algebraic tangent category $\mathsf{Alg}(u\mathscr{C}om)$ is isomorphic to the tangent category $(\mathsf{cRing}, \mathbb{L})$ described in Example 2.13.

Example 3.31. Consider the symmetric monoidal category of $\mathbb{Z}$-modules and the operad $u\mathscr{A}ss$ described in Example 3.7 over $\mathsf{Mod}_{\mathbb{Z}}$. The category of algebras of $u\mathscr{A}ss$

is the category Ring of unital and associative, but not necessarily commutative, rings. Again, one can represent the tangent bundle functor $\mathbb{T}R$ as $R \ltimes R \cong R\langle\varepsilon\rangle$. Although this tangent structure is quite similar to the one of cRing, this is the first example of a non-trivial tangent structure on the category of non-commutative rings.

Example 3.32. The algebraic tangent category $\mathsf{Alg}(\mathscr{Lie})$ of the operad $\mathscr{Lie}$, described in Example 3.9, over Mod_R is the tangent category over the category of Lie algebras, whose tangent bundle functor $\mathbb{T}$ sends a Lie algebra $\mathfrak{g}$ to the Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}$, which is the the Lie algebra over $\mathfrak{g} \oplus \mathfrak{g}$ whose Lie bracket is so defined:

$$[(g,h),(g',h')] = ([g,g'],[g,h'] + [g',h])$$

The projection, the zero morphism, the sum morphism, the lift, and the canonical flip are defined as in Equation (3.5.1).

3.5.1 Vector fields in the algebraic tangent category of an operad

Remember that a vector field in a tangent category is a section of the projection. So, it is natural to wonder what vector fields in the algebraic tangent category of an operad represent. Since the algebraic tangent category of an operad is the Eilenberg-Moore object in the category of tangent categories of a tangent monad, one could also ask the same question for the tangent category of algebras of a generic tangent monad. In general, we do not have a good characterization of vector fields in this abstract context, unless the tangent monad is in fact a coCartesian differential monad. Let's start by recalling the definition of a derivation over an algebra of a coCartesian differential monad (see [44]).

Definition* 3.33. *Let (S,∂) be a coCartesian differential monad over a category $\mathbb{X}$ with biproducts. An S-**derivation** over an S-algebra $(A,\theta\colon SA \to A)$ is an $\mathbb{X}$-morphism $\delta\colon A \to A$ making the following diagram commutative:*

$$\begin{tikzcd}
SA \arrow[rr, "\theta"] \arrow[d, "\partial"'] & & A \arrow[d, "\delta"] \\
S(A \oplus A) \arrow[r, "S(\pi_1+\pi_2\delta)"'] & SA \arrow[r, "\theta"'] & A
\end{tikzcd}$$

Consider now an algebraic operad $\mathscr{P}$ and a $\mathscr{P}$-algebra A. A $\mathscr{P}$-derivation of A over A consists of a morphism of R-modules $\delta\colon A \to A$ such that, for every $a_1, \ldots, a_m$ and every $\mu \in \mathscr{P}(m)$:

$$\delta(\mu(a_1, \ldots, a_m)) = \sum_{k=1}^{m} \mu(a_1, \ldots, \delta(a_k), \ldots, a_m) \tag{3.5.2}$$

It is natural to wonder if this notion of derivation coincides with the derivation associated with the coCartesian differential monad of the operad.

Lemma 3.34. *For an algebraic operad $\mathscr{P}$, $\mathscr{P}$-derivations of a $\mathscr{P}$-algebra A are precisely $\mathsf{S}_{\mathscr{P}}$-derivations of A, when the monad $\mathsf{S}_{\mathscr{P}}$ is equipped with the differential combinator defined in Theorem 3.26.*

Proof. Spelling out the definition of an $\mathsf{S}_{\mathscr{P}}$-derivation, one finds out that such a derivation consists of a morphism of R-modules $\delta\colon A \to A$ such that:

$$\begin{aligned}
& \delta(\mu(a_1, \ldots, a_m)) \\
= \;& \theta(\mathsf{S}_{\mathscr{P}}(\pi_1 + \pi_2\delta)(\partial(\mu; a_1, \ldots, a_m))) \\
= \;& \theta\left(\mathsf{S}_{\mathscr{P}}(\pi_1 + \pi_2\delta)\left(\sum_{k=1}^{m}(\mu; (a_1, 0), \ldots, (0, a_k), \ldots, (a_m, 0))\right)\right) \\
= \;& \theta\left(\sum_{k=1}^{n}(\mu; a_1, \ldots, \delta(a_k), \ldots, a_m)\right) \\
= \;& \sum_{k=1}^{n} \mu(a_1, \ldots, \delta(a_k), \ldots, a_m)
\end{aligned}$$

which is precisely the Leibniz rule of Equation 3.5.2. □

The main result of this section is that, for a coCartesian differential monad, the vector fields in the corresponding algebraic tangent category are equivalent to derivations.

Proposition 3.35. *Let $\mathsf{Alg}(S, \partial)$ denote the tangent category associated with a coCartesian differential monad (S, ∂). The vector fields $v\colon A \to \mathbb{L}A$ over an S-algebra A in the tangent category $\mathsf{Alg}(S, \partial)$ are equivalent to S-derivations over A. Moreover, if the base category is additive, then this equivalence lifts to an isomorphism of Lie algebras between the Lie algebra of vector fields, whose Lie bracket is defined as in Section 2.2.1, and the Lie algebra of S-derivations, whose Lie bracket is given by the commutator.*

Proof. Let's start by proving that vector fields are equivalent to S-derivations. Consider a vector field $v\colon A \to \mathbb{L}A$. Since $\mathbb{L}A$, as an $\mathbb{E}$-object, is $A \oplus A$, one can consider the $\mathbb{E}$-morphism $\delta_v\colon A \xrightarrow{v} \mathbb{L}A \xrightarrow{\pi_2} A$. We want to show that δ_v is indeed an S-derivation. First of all, since v is an S-algebra morphism, the following diagram commutes:

$$\begin{tikzcd}
SA \arrow[r, "Sv"] \arrow[dd, "\theta"'] & S\mathbb{L}A \arrow[d, "\alpha"] \\
 & \mathbb{L}SA \arrow[d, "\mathbb{L}\theta"] \\
A \arrow[r, "v"'] & \mathbb{L}A
\end{tikzcd}$$

where $\alpha\colon S \circ \mathbb{L} \Rightarrow \mathbb{L} \circ S$ is the natural transformation:

$$\alpha\colon S(A \oplus A) \xrightarrow{\langle S(\pi_1), \partial S(\pi_1+\pi_4)\rangle} SA \oplus SA$$

Since, by definition, $\delta_v = v\pi_2$ and $v\pi_1 = vp = \mathsf{id}_A$, we have that:

$$(v \oplus v)(\pi_1 + \pi_4) = \pi_1 + \pi_2\delta_v$$

Thus:

$$\begin{aligned}
& \partial_A S(\pi_1 + \pi_2\delta_v)\theta \\
= \;& \partial_A S((v \oplus v)(\pi_1 + \pi_4))\theta \\
= \;& \partial_A S(v \oplus v)S(\pi_1 + \pi_4)\theta \\
= \;& S(v)\partial_{A\oplus A} S(\pi_1 + \pi_4)\theta \\
= \;& S(v)\mathbb{L}\theta\alpha\pi_2 \\
= \;& \theta v\pi_2 \\
= \;& \theta\delta_v
\end{aligned}$$

where we computed $\partial_{A\oplus A}S(\pi_1+\pi_4)\theta = \mathbb{L}\theta\alpha\pi_2$. Let's now consider an S-derivation $\delta\colon A \to A$ and let's consider $v_\delta\colon A \xrightarrow{\langle \mathsf{id}_A, \delta\rangle} \mathbb{L}A$. By construction, $v_\delta p = v_\delta\pi_1 = \mathsf{id}_A$. We need to show that v is a morphism of S-algebras. Notice first:

$$(v_\delta \oplus v_\delta)(\pi_1 + \pi_4) = \pi_1 + \pi_2\delta$$

Then:

$$
\begin{aligned}
& S(v_\delta)\langle S(\pi_1), \partial S(\pi_1+\pi_4)\rangle\theta \\
= \quad & \langle S(v_\delta\pi_1)\theta, S(v_\delta)\partial S(\pi_1+\pi_4)\theta\rangle \\
= \quad & \langle \theta, \partial S(v_\delta \oplus v_\delta)S(\pi_1+\pi_4)\theta\rangle \\
= \quad & \langle \theta, \partial S((v_\delta \oplus v_\delta)(\pi_1+\pi_4))\theta\rangle \\
= \quad & \langle \theta, \partial S(\pi_1+\pi_2\delta)\theta\rangle \\
= \quad & \langle \theta, \theta\delta\rangle \\
= \quad & \theta\langle \mathsf{id}_A, \delta\rangle \\
= \quad & \theta v_\delta
\end{aligned}
$$

Clearly, the functions $v \mapsto \delta_v$ and $\delta \mapsto v_\delta$ are inverse to each other. Now, suppose that the base category is additive and let's show that the Lie bracket is preserved by $v \mapsto \delta_v$. Let's start by noticing that on the base category, where biproducts induce the tangent structure, vector fields are precisely morphisms $v\colon A \to A \oplus A$ for which $v\pi_1 = \mathsf{id}_A$. So, vector fields are in bijective correspondence with morphisms $\delta_v\colon A \to A$. Let's now consider two vector fields $u, v\colon A \to A \oplus A$; then:

$$[u, v] = \{u \mathbb{L} v - v \mathbb{L} u c\}$$

In particular:

$$
\begin{aligned}
& u\mathbb{L}v\colon A \xrightarrow{\langle \mathsf{id}_A, \delta_v, \delta_u, \delta_v\circ\delta_u\rangle} A \oplus A \oplus A \oplus A \\
& v\mathbb{L}uc\colon A \xrightarrow{\langle \mathsf{id}_A, \delta_v, \delta_u, \delta_u\circ\delta_v\rangle} A \oplus A \oplus A \oplus A \\
& u\mathbb{L}v - v\mathbb{L}uc\colon A \xrightarrow{\langle \mathsf{id}_A, 0, 0, \delta_v\circ\delta_u - \delta_u\circ\delta_v\rangle} A \oplus A \oplus A \oplus A
\end{aligned}
$$

Since the base tangent structure is induced by biproducts, every object has the structure of a differential object, so, in particular, the bracket operation $\{-\}$ consists of precomposing a morphism $f\colon B \to \mathbb{L}^2 A = A \oplus A \oplus A \oplus A$ for which $f\pi_3 = 0$ with π_4. So:

$$[u, v] = \{u\mathbb{L}v - v\mathbb{L}uc\} = (u\mathbb{L}v - v\mathbb{L}uc)\pi_4\colon A \xrightarrow{\langle \mathsf{id}_A, \delta_v\circ\delta_u - \delta_u\circ\delta_v\rangle} A \oplus A$$

Therefore, under the bijective correspondence $v \mapsto \langle \mathsf{id}_A, \delta_v\rangle$ between vector fields of the base tangent category and morphisms $\delta_v\colon A \to A$, the Lie bracket induced by the tangent structure corresponds to the Lie bracket defined by the commutator. To conclude, notice that the forgetful functor $\mathsf{Alg}(S,\partial) \to (\mathbb{X},\mathbb{L})$ is a strict tangent morphism. Thus, the Lie bracket lifts along the forgetful functor and therefore, when $\mathbb{X}$ is additive, the bijective correspondence between vector fields in $\mathsf{Alg}(S,\partial)$ and S-derivations becomes an isomorphism of Lie algebras. □

By putting together that the algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ of an operad $\mathscr{P}$ is the algebraic tangent category of a coCartesian differential monad $(\mathsf{S}_{\mathscr{P}},\partial_{\mathscr{P}})$ and that, for an algebraic operad $\mathscr{P}$ $\mathsf{S}_{\mathscr{P}}$-derivations are $\mathscr{P}$-derivations, one finds a complete classification of vector fields in the algebraic tangent category of an operad.

Theorem 3.36. *For an algebraic operad $\mathscr{P}$, there is an isomorphism of Lie algebras between the Lie algebra of vector fields $v\colon A \to \mathbb{L}A$ of the algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ whose Lie bracket is induced by the algebraic tangent structure, and derivations $\delta\colon A \to A$ in the operadic sense, whose Lie bracket is defined by the commutator.*

3.5.2 The functoriality of the algebraic construction

So far, we have shown that the category of algebras $\mathsf{Alg}_{\mathscr{P}}$ of an operad $\mathscr{P}$ over a symmetric monoidal category with biproducts carries a canonical tangent structure $\mathbb{L}_{\mathscr{P}}$. In particular, we call this tangent category the algebraic tangent category of $\mathscr{P}$ and, when $\mathscr{P}$ is algebraic, we have a complete characterization of the vector fields in terms of derivations over operadic algebras.

It is natural to wonder if the operation which sends an operad $\mathscr{P}$ to its corresponding algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ is functorial. In particular, one is interested in seeing if morphisms of operads are canonically sent to morphisms between the corresponding tangent categories in a compatible way with the composition and the identities. In this section, we explore this question. The first step is to recall the definition of a morphism of operads. For the purpose of this discussion, we keep the base symmetric category $\mathbb{E}$ fixed and we only consider morphisms of operads over this base category (see [46, Chapter 5]).

Definition⋆ 3.37. *A **morphism of operads** $\varphi\colon \mathscr{P} \to \mathscr{P}'$ between two operads $\mathscr{P}$ and $\mathscr{P}'$ over the base category $\mathbb{E}$ consists of a sequence $\{\varphi_n\colon \mathscr{P}(n) \to \mathscr{P}'(n)\}$ of morphisms of*

$\mathbb{E}$, compatible with the unit and the symmetric actions of the operad $\mathscr{P}$. Concretely, these compatibilities are expressed via the commutativity of the following diagrams:

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & \mathscr{P}(1) \arrow[d, "\gamma"] \\ & \mathscr{P}'(1) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{P}(n) \otimes \mathscr{P}(k_1) \otimes \ldots \otimes \mathscr{P}(k_n) \arrow[r, "\gamma"] \arrow[d, "\varphi_n \otimes \varphi_{k_1} \otimes \ldots \otimes \varphi_{k_n}"'] & \mathscr{P}(k_1 + \ldots + k_n) \arrow[d, "\varphi_{k_1+\ldots+k_n}"] \\ \mathscr{P}'(n) \otimes \mathscr{P}'(k_1) \otimes \ldots \otimes \mathscr{P}'(k_n) \arrow[r, "\gamma'"'] & \mathscr{P}'(k_1 + \ldots + k_n) \end{tikzcd}$$

Furthermore, φ is equivariant with respect to the symmetric actions of the operads.

In the following, when there is no confusion, we simplify notation by omitting the subscript n in the morphisms φ_n. For algebraic operads $\mathscr{P}$ and $\mathscr{P}'$, a morphism of operads is a sequence $\{\varphi_n \colon \mathscr{P}(n) \to \mathscr{P}'(n)\}$ of morphisms of R-modules satisfying the following conditions:

$$\begin{aligned} &\varphi(\mathbb{1}_{\mathscr{P}}) = \mathbb{1}_{\mathscr{P}'} \\ &\varphi(\mu(\mu_1, \ldots, \mu_m)) = \varphi(\mu)(\varphi(\mu_1), \ldots, \varphi(\mu_n)) \\ &\varphi(\mu \cdot \sigma) = \varphi(\mu) \cdot \sigma \end{aligned}$$

Example 3.38. Consider the algebraic operads $\mathscr{Ass}$ and $\mathscr{Com}$, respectively described in Examples 3.7 and 3.8. Both $\mathscr{Ass}$ and $\mathscr{Com}$ are generated by a binary tree. To distinguish, we denote by μ and ν the two binary trees of $\mathscr{Ass}$ and $\mathscr{Com}$, respectively. Moreover, both μ and ν satisfy the same relation:

$$\begin{aligned} \mu(\mathbb{1}, \mu) &= \mu(\mu, \mathbb{1}) \\ \nu(\mathbb{1}, \nu) &= \nu(\nu, \mathbb{1}) \end{aligned}$$

which encodes associativity. However, ν satisfies an extra relation:

$$\nu \cdot \tau = \nu$$

which encodes commutativity. Despite this difference, thanks to the associativity relation, there is a morphism of operads $\mathscr{Ass} \to \mathscr{Com}$ which sends μ to ν. In fact, one can see $\mathscr{Com}$ as the coequalizer in the category of operads of the maps $\mathrm{id}, \tau \colon \mathscr{Ass} \to \mathscr{Ass}$, where τ sends μ to $\mu \cdot \tau$, with $\tau = (1 \quad 2)$. Then, the coequalizer map $\mathscr{Ass} \dashrightarrow \mathscr{Com}$ is precisely the morphism which sends μ to ν.

Equivalently, from the concrete description of the operads $\mathscr{A}ss$ and $\mathscr{C}om$, one can describe the morphism $\varphi\colon \mathscr{A}ss \to \mathscr{C}om$ as the sequence of morphisms $R[\mathbb{S}_n] \to R$ which send each generator $\sigma \in \mathbb{S}_n$ to 1, for every $n > 0$, and the trivial morphism $0 \to 0$ for $n = 0$.

This morphism lifts to the unital versions of these two operads. In particular, the operads $u\mathscr{A}ss$ and $u\mathscr{C}om$ are generated by a binary tree and by a unitary tree e, for which:

$$\mu(e, \mathbb{1}) = \mathbb{1} = \mu(\mathbb{1}, e)$$
$$\nu(e, \mathbb{1}) = \mathbb{1} = \nu(\mathbb{1}, e)$$

Thus, there is a morphism of operads $\varphi\colon u\mathscr{A}ss \to u\mathscr{C}om$ which sends e to itself and μ to ν. Concretely, this morphism is defined as $\varphi\colon \mathscr{A}ss \to \mathscr{C}om$ for each $n > 0$ and for $n = 0$ it is just the identity $R \to R$.

Example 3.39. It is well-known that an associative algebra A equipped with the commutator $[,]$ defines a Lie algebra. This extends to a functor $\mathsf{Alg}_{\mathscr{A}ss} \to \mathsf{Alg}_{\mathscr{L}ie}$ from the category of associative algebras to the one of Lie algebras. It turns out this functor is induced by a morphism of operads $\mathscr{L}ie \to \mathscr{A}ss$. For a description of this morphism we refer to [46, Section 13.2.5].

Operads over $\mathbb{E}$ together with their morphisms form a category denoted by $\mathsf{Operad}(\mathbb{E})$. When $\mathbb{E}$ is clear from the context we omit it from the notation. Moreover, for algebraic operads, we simplify notation and denote by Operad_R the category $\mathsf{Operad}(\mathsf{Mod}_R)$.

A morphism of operads $\varphi\colon \mathscr{P} \to \mathscr{P}'$ induces a morphism of monads $\mathsf{S}_\varphi\colon \mathsf{S}_{\mathscr{P}} \to \mathsf{S}_{\mathscr{P}'}$ between the corresponding monads. When the operads are algebraic, this morphism is defined as follows:

$$(\mu; x_1, \ldots, x_m) \mapsto (\varphi(\mu); x_1, \ldots, x_m)$$

where $(\mu; x_1, \ldots, x_m)$ is the generic element of $\mathsf{S}_{\mathscr{P}}V$, for an R-module V. On the other hand, a morphism of monads $\varphi\colon S \to S'$ between two monads over a category $\mathbb{X}$, induces a functor $\varphi^*\colon \mathsf{Alg}_{S'} \to \mathsf{Alg}_S$, which sends an S'-algebra A with structure map $\theta\colon S'A \to A$ to the S-algebra A with structure map $SA \xrightarrow{\varphi} S'A \xrightarrow{\theta} A$. Moreover,

φ^* sends a morphism of S'-algebras $f\colon A \to B$ to the morphism of S-algebra $\varphi^* f\colon \varphi^* A \to \varphi^* B$ whose underlying $\mathbb{X}$-morphism is just f. So, by putting together these two functorial operations, $\mathsf{Operad}(\mathbb{E}) \to \mathsf{Mnd}(\mathbb{E})$ and $\mathsf{Alg}\colon \mathsf{Mnd}^{\mathsf{op}}(\mathbb{E}) \to \mathsf{Cat}$, one finds a functor $\mathsf{Alg}\colon \mathsf{Operad}^{\mathsf{op}}(\mathbb{E}) \to \mathsf{Cat}$ which sends an operad to the corresponding category $\mathsf{Alg}_{\mathscr{P}}$ of algebras and a morphism of operads $\varphi\colon \mathscr{P} \to \mathscr{P}'$ to the corresponding functor $\varphi^*\colon \mathsf{Alg}_{\mathscr{P}'} \to \mathsf{Alg}_{\mathscr{P}}$.

The first question is to see whether or not, for a morphism of operads $\varphi\colon \mathscr{P} \to \mathscr{P}'$, φ^* lifts to a tangent morphism between the algebraic tangent categories of the two operads. To begin with, notice that, for any tangent category $(\mathbb{X}, \mathbb{T})$, there is a functor $\mathsf{TngMnd}(\mathbb{X}, \mathbb{T})^{\mathsf{op}} \to \mathsf{TngCat}$ which sends a tangent monad (S, α) over $(\mathbb{X}, \mathbb{T})$ to the corresponding algebraic tangent category $\mathsf{Alg}(S, \alpha)$ and a morphism $\varphi\colon (S, \alpha) \to (S', \alpha)$ of tangent monads, which consists of a morphism of monads $\varphi\colon S \to S'$ which is compatible with the distributive laws, to the strict tangent morphism $\varphi^*\colon \mathsf{Alg}(S, \alpha) \to \mathsf{Alg}(S', \alpha')$.

To construct this functor, recall that, for Theorem 2.73, if a 2-category $\mathbf{C}$ admits the construction of algebras, so does $\mathsf{Tng}(\mathbf{C})$, so in particular there is a 2-functor $\mathsf{Alg}\colon \mathsf{TngMnd}(\mathbf{C}) \cong \mathsf{Mnd}(\mathsf{Tng}(\mathbf{C})) \to \mathbf{C}$. Since we are only interested in tangent monads over a given tangent category, we can precompose this functor with the inclusion functor $\mathsf{TngMnd}^{\mathsf{op}}(\mathbf{C}; \mathbb{X}, \mathbb{T}) \to \mathsf{TngMnd}(\mathbf{C})$ of tangent monads over a fixed tangent object $(\mathbb{X}, \mathbb{T})$.

Notice that, by convention, morphisms $\varphi\colon S' \leftarrow S$ in the category $\mathsf{Mnd}^{\mathsf{op}}(\mathbf{C}; \mathbb{X})$ of monads over a fixed $\mathbb{X}$ correspond to morphisms $(\mathsf{id}_{\mathbb{X}}, \varphi)\colon (\mathbb{X}, S) \to (\mathbb{X}, S')$. Similarly, morphisms of $\mathsf{TngMnd}^{\mathsf{op}}(\mathbf{C}; \mathbb{X}, \mathbb{T})$ $\varphi(S', \alpha') \leftarrow (S, \alpha)$ correspond to morphisms $(\mathsf{id}_{(\mathbb{X},\mathbb{T})}, \varphi)\colon \mathsf{Alg}(S, \alpha) \to \mathsf{Alg}(S', \alpha')$. This is the reason why one needs to take the opposite of the category $\mathsf{TngMnd}(\mathbf{C}; \mathbb{X}, \mathbb{T})$.

When we apply this construction to the 2-category Cat of algebras, which admits the construction of algebras, we obtain precisely the functor $\mathsf{TngMnd}(\mathbb{X}, \mathbb{T}) \to \mathsf{TngCat}$. Moreover, by unpacking the construction of algebras $\mathsf{Alg}\colon \mathsf{Mnd}(\mathsf{Cat}) \to \mathsf{Cat}$ it is easy to see that a morphism $\varphi\colon (S, \alpha) \to (S', \alpha)$ of tangent monads, which consists of a morphism of monads $\varphi\colon S \to S'$ which is compatible with the distributive laws, is sent by Alg to a strict tangent morphism. Let's summarise this discussion as a result.

Proposition 3.40. *The functor* $\mathsf{Alg}\colon \mathsf{Mnd}^{\mathsf{op}}(\mathbb{X}) \to \mathsf{Cat}$ *which sends a monad S over a fixed category $\mathbb{X}$ to the corresponding category of algebras* Alg_S *and a morphism $\varphi\colon S' \leftarrow S$ of monads (opposite) to $\varphi^*\colon \mathsf{Alg}_{S'} \to \mathsf{Alg}_S$, extends to a functor* $\mathsf{Alg}\colon \mathsf{TngMnd}^{\mathsf{op}}(\mathbb{X};\mathbb{T}) \to \mathsf{TngCat}_{=}$.

The next step consists of showing that a morphism $\varphi\colon \mathscr{P} \to \mathscr{P}'$ of operads is compatible with the differential combinators of the corresponding coCartesian differential monads. In particular, we want to show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{S}_{\mathscr{P}}A \arrow[r, "\partial_{\mathscr{P}}"] \arrow[d, "\mathsf{S}_{\varphi}A"'] & \mathsf{S}_{\mathscr{P}}(A \oplus A) \arrow[d, "\mathsf{S}_{\varphi}(A\oplus A)"] \\
\mathsf{S}_{\mathscr{P}'}A \arrow[r, "\partial_{\mathscr{P}'}"'] & \mathsf{S}_{\mathscr{P}'}(A \oplus A)
\end{tikzcd}$$

Proposition 3.41. *The morphism $\mathsf{S}_{\varphi}\colon \mathsf{S}_{\mathscr{P}} \to \mathsf{S}_{\mathscr{P}'}$ of monads induced by a morphism $\varphi\colon \mathscr{P} \to \mathscr{P}'$ is compatible with the differential combinators. In particular, the functor* $\mathsf{S}\colon \mathsf{Operad}(\mathbb{E}) \to \mathsf{Mnd}(\mathbb{E})$ *extends to a functor* $\mathsf{S}\colon \mathsf{Operad}(\mathbb{E}) \to \mathsf{TngMnd}(\mathbb{E},\mathbb{L})$*, which sends an operad $\mathscr{P}$ to the corresponding tangent monad $(\mathsf{S}_{\mathscr{P}},\alpha_{\mathscr{P}})$ and a morphism $\varphi\colon \mathscr{P} \to \mathscr{P}'$ of operads to a morphism $\mathsf{S}_{\varphi}\colon (\mathsf{S}_{\mathscr{P}},\alpha_{\mathscr{P}}) \to (\mathsf{S}_{\mathscr{P}'},\alpha_{\mathscr{P}'})$ of tangent monads.*

Proof. Recalling Equation (3.4.1), it is easy to see that the diagram:

$$\begin{tikzcd}
\mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \arrow[r, "\sum_{k=1}^{n} \mathsf{id}_{\mathscr{P}}\otimes\delta_k"] \arrow[d, "\varphi\otimes A^{\otimes n}"'] & \mathscr{P}(n) \otimes_{\mathbb{S}_n} (A \oplus A)^{\otimes n} \arrow[d, "\varphi\otimes(A\oplus A)^{\otimes n}"] \\
\mathscr{P}'(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \arrow[r, "\sum_{k=1}^{n} \mathsf{id}_{\mathscr{P}'}\otimes\delta_k"'] & \mathscr{P}'(n) \otimes_{\mathbb{S}_n} (A \oplus A)^{\otimes n}
\end{tikzcd}$$

commutes and so, in particular, S_{φ} is compatible with the differential combinators. Moreover, the correspondence between coCartesian differential monads and tangent monads over the tangent category induced by biproducts extends to a functor. In particular, it is not hard to see that a morphism of coCartesian differential monads, which consists of a morphism of monads compatible with the differential combinators, is sent to a morphism of tangent monads. Therefore, the operation which sends an operad to its corresponding tangent monad extends to a functor $\mathsf{Operad}(\mathbb{E}) \to \mathsf{TngMnd}(\mathbb{E},\mathbb{L})$. $\square$

By putting together the two functors $\mathsf{S}\colon \mathsf{Operad}(\mathbb{E}) \to \mathsf{TngMnd}(\mathbb{E},\mathbb{L})$ and $\mathsf{Alg}\colon \mathsf{TngMnd}^{\mathsf{op}}(\mathbb{E},\mathbb{L}) \to \mathsf{TngCat}_=$ we obtain a functor $\mathsf{Alg}^*\colon \mathsf{Operad}^{\mathsf{op}}(\mathbb{E}) \to \mathsf{TngCat}_=$.

Proposition 3.42. *The operation which sends an operad $\mathscr{P}$ to the corresponding algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ extends to a functor $\mathsf{Alg}^*\colon \mathsf{Operad}^{\mathsf{op}}(\mathbb{E}) \to \mathsf{TngCat}_=$.*

Interestingly, the functor $\varphi^*\colon \mathsf{Alg}_{\mathscr{P}'} \to \mathsf{Alg}_{\mathscr{P}}$ induced by a morphism $\varphi\colon \mathscr{P} \to \mathscr{P}'$ of operads admits a left adjoint $\varphi_!\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}_{\mathscr{P}'}$. Concretely, $\varphi_!$ sends a $\mathscr{P}$-algebra A to the $\mathscr{P}'$-algebra $\varphi_!A$ defined by identifying the two algebra structures over the free algebra $\mathsf{S}_{\mathscr{P}'}A$, induced by the multiplication of the operad $\mathscr{P}'$, that is the free algebra structure, and the one induced by the structure map of A as a $\mathscr{P}$-algebra. In a nutshell, $\varphi_!$ is the coequalizer in the category of $\mathscr{P}'$-algebras between the morphisms $\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}A \xrightarrow{\mathsf{S}_{\mathscr{P}'}\mathsf{S}_\varphi} \mathsf{S}^2_{\mathscr{P}'}A \xrightarrow{\gamma'} \mathsf{S}_{\mathscr{P}'}A$ and $\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}A \xrightarrow{\mathsf{S}_{\mathscr{P}'}\theta} \mathsf{S}_{\mathscr{P}'}A$:

$$\begin{tikzcd}
\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}A \arrow[r, "\mathsf{S}_{\mathscr{P}'}\mathsf{S}_\varphi"] \arrow[rr, bend right, "\mathsf{S}_{\mathscr{P}'}\theta"'] & \mathsf{S}^2_{\mathscr{P}'}A \arrow[r, "\gamma'"] & \mathsf{S}_{\mathscr{P}'}A \arrow[r, dashed] & \varphi_!A
\end{tikzcd}$$

In particular, the structure map of $\varphi_!A$ is induced by γ' as follows:

$$\begin{tikzcd}
\mathsf{S}^2_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}A \arrow[r, "\mathsf{S}^2_{\mathscr{P}'}\mathsf{S}_\varphi"] \arrow[rr, bend left, "\mathsf{S}^2_{\mathscr{P}'}\theta"] \arrow[d, "\gamma'\mathsf{S}_{\mathscr{P}}"'] & \mathsf{S}^3_{\mathscr{P}'}A \arrow[r, "\mathsf{S}_{\mathscr{P}'}\gamma'"] \arrow[d, "\gamma'\mathsf{S}_{\mathscr{P}'}"] & \mathsf{S}^2_{\mathscr{P}'}A \arrow[r] \arrow[d, "\gamma'"] & \mathsf{S}_{\mathscr{P}'}\varphi_!A \arrow[d, dashed, "\theta_!"] \\
\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}A \arrow[r, "\mathsf{S}_{\mathscr{P}'}\mathsf{S}_\varphi"] \arrow[rr, bend right, "\mathsf{S}_{\mathscr{P}'}\theta"'] & \mathsf{S}^2_{\mathscr{P}'}A \arrow[r, "\gamma'"] & \mathsf{S}_{\mathscr{P}'}A \arrow[r] & \varphi_!A
\end{tikzcd}$$

Moreover, a morphism $f\colon A \to B$ between two $\mathscr{P}$-algebras is sent to the unique morphism $\varphi_!f\colon \varphi_!A \to \varphi_!B$:

$$\begin{tikzcd}
\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}A \arrow[r, "\mathsf{S}_{\mathscr{P}'}\mathsf{S}_\varphi"] \arrow[rr, bend left, "\mathsf{S}_{\mathscr{P}'}\theta"] \arrow[d, "\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}f"'] & \mathsf{S}^2_{\mathscr{P}'}A \arrow[r, "\gamma'"] \arrow[d, "\mathsf{S}^2_{\mathscr{P}'}f"] & \mathsf{S}_{\mathscr{P}'}A \arrow[r] \arrow[d, "\mathsf{S}_{\mathscr{P}'}f"] & \varphi_!A \arrow[d, dashed, "\varphi_!f"] \\
\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}B \arrow[r, "\mathsf{S}_{\mathscr{P}'}\mathsf{S}_\varphi"] \arrow[rr, bend right, "\mathsf{S}_{\mathscr{P}'}\theta"'] & \mathsf{S}^2_{\mathscr{P}'}B \arrow[r, "\gamma'"] & \mathsf{S}_{\mathscr{P}'}B \arrow[r] & \varphi_!B
\end{tikzcd}$$

Example 3.43. As discussed in Example 3.38, there is a canonical morphism of operads $\varphi\colon \mathscr{Ass} \to \mathscr{Com}$, which sends the generator μ of $\mathscr{Ass}$ to the generator ν of $\mathscr{Com}$. This induces an a pair of adjoint functors $\varphi_!\colon \mathsf{Alg}_{\mathscr{Ass}} \leftrightarrows \mathsf{Alg}_{\mathscr{Com}} \colon \varphi^*$. The right adjoint φ^* sends a commutative algebra to its underlying associative algebra and a morphism of commutative algebras to the underlying morphism of associative algebras. More interesting is the left adjoint $\varphi_!$ which consists of the abelianization functor, which sends an associative algebra A to its abelianization $A/[A,A]$. Concretely, $A/[A,A]$ is the quotient between the algebra A and the ideal known as the commutator of A, generated by all the terms of the form $[a,b]\colon = ab - ba$. A morphism $f\colon A \to B$ of associative algebras lifts to the quotient, since $f[a,b] = f(ab - ba) = f(a)f(b) - f(b)f(a) = [f(a), f(b)]$. The lift $\bar{f}\colon A/[A,A] \to B/[B,B]$ is precisely $\varphi_! f$.

Example 3.44. As discussed in Example 3.39, there is a canonical morphism of operads $\varphi\colon \mathscr{Lie} \to u\mathscr{Ass}$. This induces an a pair of adjoint functors $\varphi_!\colon \mathsf{Alg}_{\mathscr{Lie}} \leftrightarrows \mathsf{Alg}_{u\mathscr{Ass}} \colon \varphi^*$. The right adjoint φ^* sends an associative algebra A to the Lie algebra over A whose Lie bracket is defined by the commutator $[,]\colon A \otimes A \to A, [,]\colon (a,b) \mapsto [a,b]\colon = ab - ba$ of A. The left adjoint $\varphi_!$ sends a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $\mathsf{Env}(\mathfrak{g})$. Concretely, $\mathsf{Env}(\mathfrak{g})$ is obtained by quotienting the tensor algebra $\mathsf{Tens}(\mathfrak{g})$, i.e. the free associative algebra generated by elements of $\mathfrak{g}$, by the quotient generated by $a \otimes b - b \otimes a - [a,b]$, where $[,]$ denotes the Lie bracket of $\mathfrak{g}$.

A morphism $f\colon \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras lifts to the quotient, since $f(a \otimes b - b \otimes a - [a,b]) = f(a) \otimes f(b) - f(b) \otimes f(a) - [f(a), f(b)]$, and the lifted morphism $\bar{f}\colon \mathsf{Env}(\mathfrak{g}) \to \mathsf{Env}(\mathfrak{g}')$ is precisely $\varphi_! f$.

At this point, it is natural to wonder if the functor $\mathsf{Operad}(\mathbb{E}) \to \mathsf{Cat}$ which sends an operad $\mathscr{P}$ to $\mathsf{Alg}_{\mathscr{P}}$ and a morphism $\varphi\colon \mathscr{P} \to \mathscr{P}'$ of operads to the left adjoint $\varphi_!\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}_{\mathscr{P}'}$ also extends to a tangent morphism. This question opens a more general question: if the underlying functor $G\colon \mathbb{X} \to \mathbb{X}'$ of a tangent morphism $(G,\beta)\colon (\mathbb{X},\mathbb{T}) \to (\mathbb{X}',\mathbb{T}')$ between two tangent categories admits a left adjoint $F\colon \mathbb{X}' \to \mathbb{X}$, does the left adjoint F extend to a tangent morphism $(F,\alpha)\colon (\mathbb{X}',\mathbb{T}') \to (\mathbb{X},\mathbb{T})$?

The answer is positive whenever (G,β) is a *colax* tangent morphism. In particular, the left adjoint extends to a *lax* tangent morphism (F,α). This interesting relationship between colax and lax tangent morphisms has a natural contextualization in the double category $\mathbb{T}\mathsf{ngCat}$ of tangent categories. Recall, that tangent objects can be organized in a double category whose horizontal morphisms are lax tangent morphisms and vertical morphisms are colax tangent morphisms (cf. Section 2.3). Recall also that a conjunction is a generalization in the context of double categories of an adjunction (see [52, Definition 7]).

Definition⋆ 3.45. *In a double category, a* ***conjunction*** $(\eta, \varepsilon)\colon F \dashv G$ *consists of a horizontal morphism* $F\colon \mathbb{X} \to \mathbb{X}'$ *and vertical morphism* $G\colon \mathbb{X}' \nrightarrow \mathbb{X}$ *together with two double cells:*

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, "F"] \arrow[d, equal, "\shortmid" marking] \arrow[dr, phantom, "\Nearrow\eta"] & \mathbb{X}' \arrow[d, "G", "\shortmid" marking] \\
\mathbb{X} \arrow[r, equal] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X}' \arrow[r, equal] \arrow[d, "G"', "\shortmid" marking] \arrow[dr, phantom, "\Nearrow\varepsilon"] & \mathbb{X}' \arrow[d, equal, "\shortmid" marking] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{X}'
\end{tikzcd}
$$

satisfying triangle equalities.

Proposition 3.46. *A conjunction* $(F, \alpha)\colon (\mathbb{X}, \mathrm{T}) \leftrightarrows (\mathbb{X}', \mathrm{T}')\colon (G, \beta)$ *in the double category* ***Tng***$(\mathbf{C})$ *of tangent objects of a* 2*-category* $\mathbf{C}$ *consists of an adjunction* $(\eta, \varepsilon)\colon F \dashv G$ *in* $\mathbf{C}$ *together with a colax distributive law* $\beta\colon \mathrm{T} \circ G \Rightarrow G \circ \mathrm{T}'$ *and its mate* $\alpha\colon F \circ \mathrm{T} \Rightarrow \mathrm{T}' \circ F$ *along the adjunction* $(\eta, \varepsilon)\colon F \dashv G$*. In particular,* α *is defined by:*

$$\alpha\colon F \circ \mathrm{T} \xrightarrow{F\mathrm{T}\eta} F \circ \mathrm{T} \circ G \circ F \xrightarrow{F\beta F} F \circ G \circ \mathrm{T}' \circ F \xrightarrow{\varepsilon \mathrm{T}' F} \mathrm{T}' \circ F$$

Proof. Let's start by proving that (F, α) is a lax tangent morphism. The first step is to show that α is compatible with the projections, i.e. $\alpha p'_F = Fp$, where p' denotes the projection of the tangent structure $\mathbb{T}'$. We will adopt a similar notation for the other 2-morphisms of the tangent structures. This amounts to showing the commutativity of the following diagram:

$$
\begin{tikzcd}
F \circ \mathrm{T} \arrow[r, "(F\circ\mathrm{T})\eta"] \arrow[dd, "Fp"'] \arrow[dr, phantom, "\mathsf{Nat}"] & F \circ \mathrm{T} \circ G \circ F \arrow[r, "F\beta_F"] \arrow[d, "Fp_{G\circ F}"'] \arrow[dr, phantom, "(\beta;p,p')"] & F \circ G \circ \mathrm{T}' \circ F \arrow[r, "\varepsilon_{\mathrm{T}'\circ F}"] \arrow[d, "(F\circ G)p'_F"] \arrow[dr, phantom, "\mathsf{Nat}"] & \mathrm{T}' \circ F \arrow[dd, "p'_F"] \\
 & F \circ G \circ F \arrow[r, equal] & F \circ G \circ F \arrow[dr, "\varepsilon_F"] \arrow[d, phantom, "\Delta"] & \\
F \arrow[ur, "F\eta"] \arrow[rrr, equal] & & & F
\end{tikzcd}
$$

To express the commutativity of the diagrams that compose the whole diagram we adopted the following convention: with Nat we denote commutativity by naturality; by $(\beta; p, p')$ we denote the compatibility between β and the projections; and Δ indicates the triangle identities between the unit and the counit of the adjunction. In the following, we adopt a similar notation.

The second step is to prove the compatibility with the zero morphisms. This amounts to showing that $Fz\alpha = z'_F$, i.e.:

$$\begin{tikzcd}
F \circ \mathrm{T} \arrow[r, "(F\circ\mathrm{T})\eta"] \arrow[dr, phantom, "\mathsf{Nat}"] & F \circ \mathrm{T} \circ G \circ F \arrow[r, "F\beta_F"] \arrow[dr, phantom, "(\beta;z,z')"] & F \circ G \circ \mathrm{T}' \circ F \arrow[r, "\varepsilon_{\mathrm{T}'\circ F}"] \arrow[dr, phantom, "\mathsf{Nat}"] & \mathrm{T}' \circ F \\
 & F \circ G \circ F \arrow[u, "Fz_{G\circ F}"] \arrow[r, equal] \arrow[dr, phantom, "\Delta"] & F \circ G \circ F \arrow[u, "(F\circ G)z'_F"'] \arrow[dr, "\varepsilon_F"] & \\
F \arrow[uu, "Fz"] \arrow[ur, "F\eta"'] \arrow[rrr, equal] & & & F \arrow[uu, "z'_F"']
\end{tikzcd}$$

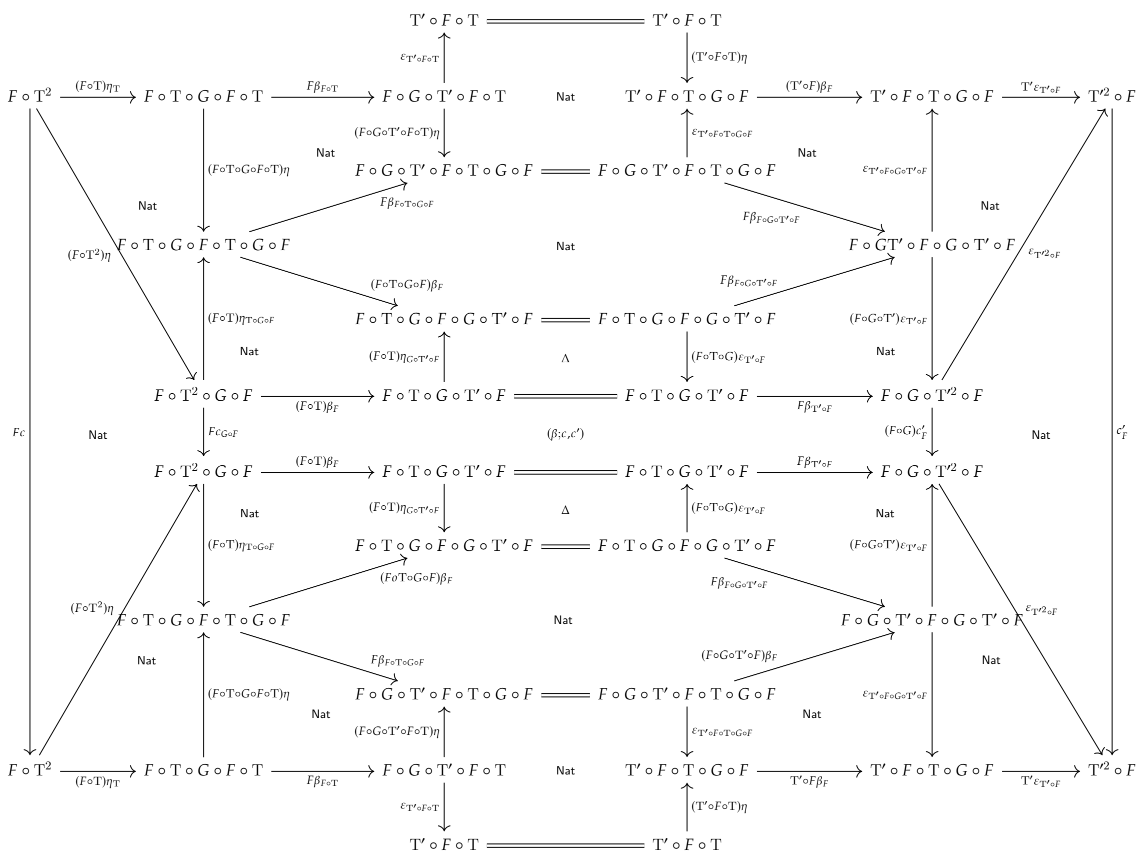
Let's show the compatibility with the sum morphism, which is $(\alpha)_2 s'_F = Fs\alpha$:

$$\begin{tikzcd}
F \circ \mathrm{T}_2 \arrow[r, "(F\circ\mathrm{T}_2)\eta_2"] \arrow[d, "Fs"'] \arrow[dr, phantom, "\mathsf{Nat}"] & F \circ \mathrm{T}_2 \circ G_2 \circ F_2 \arrow[r, "F(\beta_2)_{F_2}"] \arrow[d, "Fs_{G\circ F}"'] \arrow[dr, phantom, "(\beta;s,s')"] & F \circ G_2 \circ \mathrm{T}'_2 \circ F_2 \arrow[r, "(\varepsilon_2)_{\mathrm{T}'_2\circ F_2}"] \arrow[d, "(F\circ G)s'_F"] \arrow[dr, phantom, "\mathsf{Nat}"] & \mathrm{T}'_2 \circ F_2 \arrow[d, "s'_F"] \\
F \circ \mathrm{T} \arrow[r, "(F\circ\mathrm{T})\eta"'] & F \circ \mathrm{T} \circ G \circ F \arrow[r, "F\beta_F"'] & F \circ G \circ \mathrm{T}' \circ F \arrow[r, "\varepsilon_{\mathrm{T}'\circ F}"'] & F
\end{tikzcd}$$

Let's now show the compatibility with the vertical lifts, i.e. $\alpha l'_F = Fl\alpha_{\mathrm{T}}\mathrm{T}'\alpha$:

$$\begin{tikzcd}
F \circ \mathrm{T} \arrow[r, "(F\circ\mathrm{T})\eta"] \arrow[ddddd, "Fl"'] & F \circ \mathrm{T} \circ G \circ F \arrow[rrr, "F\beta_F"] \arrow[d, "Fl_{G\circ F}"'] & \arrow[r, phantom, "(\beta;l,l')"] & & F \circ G \circ \mathrm{T}' \circ F \arrow[r, "\varepsilon_{\mathrm{T}'\circ F}"] \arrow[d, "(F\circ G)l'_F"] & F \arrow[ddddd, "l'_F"] \\
\arrow[r, phantom, "\mathsf{Nat}"] & F \circ \mathrm{T}^2 \circ G \circ F \arrow[r, "(F\circ\mathrm{T})\beta_F"] \arrow[dd, "(F\circ\mathrm{T})\eta_{\mathrm{T}\circ G\circ F}"] \arrow[dr, phantom, "\mathsf{Nat}"] & F \circ \mathrm{T} \circ G \circ \mathrm{T}' \circ F \arrow[r, equal] \arrow[d, "(F\circ\mathrm{T})\eta_{G\circ\mathrm{T}'\circ F}"'] \arrow[dr, phantom, "\Delta"] & F \circ \mathrm{T} \circ G \circ \mathrm{T}' \circ F \arrow[r, "F\beta_{\mathrm{T}'\circ F}"] & F \circ G \circ \mathrm{T}'^2 \circ F \arrow[ddddr, "\varepsilon_{\mathrm{T}'^2\circ F}"] \arrow[r, phantom, "\mathsf{Nat}"] & \\
 & & F \circ \mathrm{T} \circ G \circ F \circ G \circ \mathrm{T}' \circ F \arrow[r, equal] & F \circ \mathrm{T} \circ G \circ F \circ G \circ \mathrm{T}' \circ F \arrow[u, "(F\circ\mathrm{T}\circ G)\varepsilon_{\mathrm{T}'\circ F}"'] \arrow[dr, "F\beta_{F\circ G\circ\mathrm{T}'\circ F}"] \arrow[r, phantom, "\mathsf{Nat}"] & & \\
 & F \circ \mathrm{T} \circ G \circ F \circ \mathrm{T} \circ G \circ F \arrow[ur, "(F\circ\mathrm{T}\circ G\circ F)\beta_F"'] \arrow[dr, "F\beta_{F\circ\mathrm{T}\circ G\circ F}"] \arrow[rrr, phantom, "\mathsf{Nat}"] & & & F \circ G \circ \mathrm{T}' \circ F \circ G \circ \mathrm{T}' \circ F \arrow[uu, "(F\circ G\circ\mathrm{T}')\varepsilon_{\mathrm{T}'\circ F}"'] \arrow[dd, "\varepsilon_{\mathrm{T}'\circ F\circ G\circ\mathrm{T}'\circ F}"'] & \\
\arrow[r, phantom, "\mathsf{Nat}"] & \arrow[r, phantom, "\mathsf{Nat}"] & F \circ G \circ \mathrm{T}' \circ F \circ \mathrm{T} \circ G \circ F \arrow[r, equal] & F \circ G \circ \mathrm{T}' \circ F \circ \mathrm{T} \circ G \circ F \arrow[ur, "(F\circ G\circ\mathrm{T}'\circ F)\beta_F"] \arrow[d, "\varepsilon_{\mathrm{T}'\circ F\circ\mathrm{T}\circ G\circ F}"] \arrow[r, phantom, "\mathsf{Nat}"] & \arrow[r, phantom, "\mathsf{Nat}"] & \\
F \circ \mathrm{T}^2 \arrow[uuuur, "(F\circ\mathrm{T}^2)\eta"] \arrow[r, "(F\circ\mathrm{T})\eta_{\mathrm{T}}"'] & F \circ \mathrm{T} \circ G \circ F \circ \mathrm{T} \arrow[uu, "(F\circ\mathrm{T}\circ G\circ F\circ\mathrm{T})\eta"'] \arrow[r, "F\beta_{F\circ\mathrm{T}}"'] & F \circ G \circ \mathrm{T}' \circ F \circ \mathrm{T} \arrow[u, "(F\circ G\circ\mathrm{T}'\circ F\circ\mathrm{T})\eta"] \arrow[d, "\varepsilon_{\mathrm{T}'\circ F\circ\mathrm{T}}"'] \arrow[r, phantom, "\mathsf{Nat}"] & \mathrm{T}' \circ F \circ \mathrm{T} \circ G \circ F \arrow[r, "(\mathrm{T}'\circ F)\beta_F"'] & \mathrm{T}' \circ F \circ \mathrm{T} \circ G \circ F \arrow[r, "\mathrm{T}'\varepsilon_{\mathrm{T}'\circ F}"'] & \mathrm{T}'^2 \circ F \\
 & & \mathrm{T}' \circ F \circ \mathrm{T} \arrow[r, equal] & \mathrm{T}' \circ F \circ \mathrm{T} \arrow[u, "(\mathrm{T}'\circ F\circ\mathrm{T})\eta"'] & &
\end{tikzcd}$$

Finally, the compatibility with the canonical flips, i.e. $\alpha_{\mathrm{T}}\mathrm{T}'\alpha c'_F = Fc\alpha_{\mathrm{T}}\mathrm{T}'\alpha$:



So far, we proved that (F, α) is a lax tangent morphism. The next step is to prove that:

$$
\begin{tikzcd}
(\mathbb{X},\mathbb{T}) \arrow[r, "(F,\alpha)"] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & (\mathbb{X}',\mathbb{T}') \arrow[d, "(G,\beta)"] & (\mathbb{X}',\mathbb{T}') \arrow[r, equal] \arrow[d, "(G,\beta)"'] \arrow[dr, phantom, "\varepsilon"] & (\mathbb{X}',\mathbb{T}') \arrow[d, equal] \\
(\mathbb{X},\mathbb{T}) \arrow[r, equal] & (\mathbb{X},\mathbb{T}) & (\mathbb{X},\mathbb{T}) \arrow[r, "(F,\alpha)"'] & (\mathbb{X}',\mathbb{T}')
\end{tikzcd}
$$

are tangent double cells. This amounts to showing the commutativity of the following diagrams:

$$
\begin{tikzcd}
\mathrm{T} \arrow[rr, "\eta_{\mathrm{T}}"] \arrow[d, "\mathrm{T}\eta"'] \arrow[drr, phantom, "\mathsf{Nat}"] & & G \circ F \circ \mathrm{T} \arrow[d, "(G\circ F\circ \mathrm{T})\eta"] \\
\mathrm{T} \circ G \circ F \arrow[rr, "\eta_{\mathrm{T}\circ G\circ F}"'] \arrow[d, "\beta_F"'] \arrow[drr, phantom, "\mathsf{Nat}"] & & G \circ F \circ \mathrm{T} \circ G \circ F \arrow[d, "(G\circ F)\beta_F"] \\
G \circ \mathrm{T}' \circ F \arrow[rr, "\eta_{G\circ \mathrm{T}'\circ F}"] \arrow[d, equal] \arrow[drr, phantom, "\Delta"] & & G \circ F \circ G \circ \mathrm{T}' \circ F \arrow[d, "G\varepsilon_{Q\circ F}"] \\
G \circ \mathrm{T}' \circ F \arrow[rr, equal] & & G \circ \mathrm{T}' \circ F
\end{tikzcd}
\qquad
\begin{tikzcd}
F \circ \mathrm{T} \circ G \arrow[rr, equal] \arrow[d, "(F\circ \mathrm{T})\eta_G"'] \arrow[drr, phantom, "\Delta"] & & F \circ \mathrm{T} \circ G \arrow[d, equal] \\
F \circ \mathrm{T} \circ G \circ F \circ G \arrow[rr, "(F\circ \mathrm{T}\circ G)\varepsilon"'] \arrow[d, "F\beta_{F\circ G}"'] \arrow[drr, phantom, "\mathsf{Nat}"] & & F \circ \mathrm{T} \circ G \arrow[d, "F\beta"] \\
F \circ G \circ \mathrm{T}' \circ F \circ G \arrow[rr, "(F\circ G\circ \mathrm{T}')\varepsilon"'] \arrow[d, "\varepsilon_{\mathrm{T}'\circ F\circ G}"'] \arrow[drr, phantom, "\mathsf{Nat}"] & & F \circ G \circ \mathrm{T}' \arrow[d, "\varepsilon'_{\mathrm{T}}"] \\
\mathrm{T}' \circ F \circ G \arrow[rr, "\mathrm{T}'\varepsilon"'] & & \mathrm{T}'
\end{tikzcd}
$$

The converse is a straightforward computation we leave for the reader to spell out. □

Remark 3.47. Consider a strong tangent morphism $(G, \beta)\colon (\mathbb{X}', \mathbb{T}') \to (\mathbb{X}, \mathbb{T})$ whose underlying functor G admits a left adjoint $F\colon \mathbb{X} \to \mathbb{X}'$. Since a strong tangent morphism is, in particular, a colax tangent morphism, by Proposition 3.46, we conclude that F extends to a lax tangent morphism (F, α) whose distributive law is induced by β via mates along the adjunction between the two underlying functors. Since (G, β) is strong, a natural question is whether or not also (F, α) is strong, i.e. if also α is an isomorphism. The answer is no, as explained in Remark 3.52. The reason why the operation of taking the conjoint of a colax tangent morphism does not preserve strength comes from the fact that the mate of the inverse of the distributive law β is not, in general, well-defined.

The key observation is that the distributive law α that makes the left adjoint F of the underlying functor of a colax tangent morphism (G, β) into a lax tangent morphism (F, α) is fully determined by the colax distributive law β. On the other hand, the tangent morphism $\mathsf{Alg}^*(\varphi)$ associated with a morphism φ of operads is a strict tangent morphism whose underlying functor φ^* has a left adjoint $\varphi_!$. Since strong, and then, in particular, strict, tangent morphisms are also colax tangent morphisms, Proposition 3.46 implies that $\varphi_!$ comes equipped with a lax distributive law which makes it into a lax tangent morphism. From this, we argue that the operation which sends an operad $\mathscr{P}$ to its algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ extends to a covariant functor which sends a morphism of operads φ to a lax tangent morphism whose underlying functor is $\varphi_!$.

In order to fully characterize the associated distributive law $\beta_!\colon \varphi_! \circ \Perp \Rightarrow \Perp \circ \varphi_!$, we first need to fully understand the unit and the counit of the adjunction $(\eta, \varepsilon)\colon \varphi_! \dashv \varphi^*$. The unit $\eta\colon A \to \varphi^*\varphi_! A$ is defined by the $\mathbb{E}$-morphism:

$$\eta\colon A \xrightarrow{\mathbb{1}_{\mathscr{P}'}} \mathsf{S}_{\mathscr{P}'}A \to \varphi_! A$$

where $\mathbb{1}_{\mathscr{P}'}$ denotes the unit of the monad associated to $\mathscr{P}'$ and the second morphism is the coequalizer morphism. To define the counit, consider a $\mathscr{P}'$-algebra B with structure map $\theta\colon \mathsf{S}_{\mathscr{P}'}B \to B$, then:

$$\begin{tikzcd}
\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\mathscr{P}}B \arrow[r, "\mathsf{S}_{\mathscr{P}'}\mathsf{S}_{\varphi}"] \arrow[dr, "\mathsf{S}_{\mathscr{P}'}(\varphi^*\theta)"'] & \mathsf{S}^2_{\mathscr{P}'}B \arrow[r, "\gamma'"] \arrow[d, "\mathsf{S}_{\mathscr{P}'}\theta"] & \mathsf{S}_{\mathscr{P}'}B \arrow[r] \arrow[dr, "\theta"] & \varphi_!\varphi^*B \arrow[d, dashed, "\varepsilon"] \\
 & \mathsf{S}_{\mathscr{P}'}B \arrow[rr, "\theta"'] & & B
\end{tikzcd}$$

where $\varphi^*\theta$ denotes the structure map of φ^*B.

Proposition 3.48. *The operation which sends an operad $\mathscr{P}$, over a symmetric monoidal category $\mathbb{E}$ which satisfies Convention 3.20, to its algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ extends to a pseudofunctor $\mathsf{Operad}(\mathbb{E}) \to \mathsf{TngCat}$ which sends a morphism $\varphi_!\colon \mathscr{P} \to \mathscr{P}'$ to the lax tangent morphism $(\varphi_!, \beta_!)\colon \mathsf{Alg}(\mathscr{P}) \to \mathsf{Alg}(\mathscr{P}')$, where $\beta_!$ is the distributive law defined as follows:*

$$\beta_!\colon \varphi_! \circ \Perp \xrightarrow{\varphi_!\Perp\eta} \varphi_! \circ \Perp \circ \varphi^* \circ \varphi_! = \varphi_! \circ \varphi^* \circ \Perp \circ \varphi_! \xrightarrow{\varepsilon\Perp\varphi_!} \Perp \circ \varphi_!$$

Remark 3.49. Notice that, since the left adjoint of a given functor is only determined up to a unique isomorphism, we need to choose for each morphism φ of operads of a representative of the class of isomorphism of left adjoints of the functor φ^*. This implies that $\mathsf{Alg}_!$ cannot be a strict functor, but rather a pseudofunctor whose associator and unitors are induced by these unique isomorphisms between left adjoints.

For algebraic operads $\mathscr{P}$ and $\mathscr{P}'$ and a morphism $\varphi\colon \mathscr{P} \to \mathscr{P}'$ of operads the distributive law $\beta_!$ is defined as follows. Given a $\mathscr{P}$-algebra A, $\varphi_!(\mathbb{L}^{(\mathscr{P})}A)$ is the $\mathscr{P}'$-algebra generated by pairs (a,b) for $a,b \in A$, satisfying some suitable relations defined by the coequalizer that defines $\varphi_!$. Similarly, also $\mathbb{L}^{(\mathscr{P}')}(\varphi_!A)$ is generated by pairs (a,b) for $a,b \in A$. So, $\beta_!$ sends each generator (a,b) to the corresponding generator (a,b).

Example 3.50. Consider the operads $\mathscr{Ass}$ and $\mathscr{Com}$ of Examples 3.7 and 3.8, respectively. Since the generator ν of $\mathscr{Com}$ satisfies the same relation as the generator μ of $\mathscr{Ass}$, there is a quotient morphism $\varphi\colon \mathscr{Ass} \to \mathscr{Com}$ of operads (see Example 3.43), which sends μ to ν, and that induces an adjunction:

$$\varphi_!\colon \mathsf{Alg}_{\mathscr{Ass}} \leftrightarrows \mathsf{Alg}_{\mathscr{Com}} : \varphi^*$$

φ^* sends a commutative algebra B to the underlying associative algebra φ^*B, while $\varphi_!$ sends an associative algebra A to its abelianization $A/[A,A]$, where $[A,A]$ denotes the commutator, i.e. the ideal generated by symbols $ab - ba$, for any $a,b \in A$.

The functor Alg^* maps the morphism of operads φ to the strict tangent morphism over the pullback functor φ^*, which makes $\mathsf{Alg}(\mathscr{Com})$ a tangent subcategory of $\mathsf{Alg}(\mathscr{Ass})$.

The functor $\mathsf{Alg}_!$ maps the morphism of operads φ to the lax tangent morphism whose underlying functor is the abelianization functor $\varphi_!$. To understand what is the corresponding distributive law $\varphi_! \circ \mathbb{L}^{(\mathscr{Ass})} \to \mathbb{L}^{(\mathscr{Com})} \circ \varphi_!$, first notice that, for an associative algebra A, $\varphi_!(\mathbb{L}^{(\mathscr{Ass})}(A))$ is the abelianization of $A \ltimes A$. It is not hard to see that this is isomorphic to $\varphi_!(A) \ltimes \varphi_!(A)$ which is precisely $\mathbb{L}^{(\mathscr{Com})}(\varphi_!(A))$. On the other hand, the distributive law sends the generator $(a,b) \in \varphi_!(A \ltimes A)$ to $(a,b) \in \varphi_!(A) \ltimes \varphi_!(A)$. Thus, the distributive law is precisely the isomorphism between the abelianization of $A \ltimes A$ and the semi-direct product of the abelianization of A with itself.

Example 3.51. Consider the operad $\mathscr{Lie}$ described in Example 3.9. There is a canonical morphism of operads $\varphi\colon \mathscr{Lie} \to u\mathscr{Ass}$ (see 3.39). Consider the induced adjunction:

$$\varphi_!\colon \mathsf{Alg}_{\mathscr{Lie}} \leftrightarrows \mathsf{Alg}_{u\mathscr{Ass}} : \varphi^*$$

The pullback functor φ^* sends an associative algebra A to the underlying Lie algebra with Lie brackets defined by the commutator $[a,b] := ab - ba$. On the other hand, the left adjoint $\varphi_!$ sends a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $\mathsf{Env}\mathfrak{g}$.

The functor Alg^* sends φ to the strict tangent morphism whose underlying functor is the pullback functor φ^*. The functor $\mathsf{Alg}_!$ sends φ to the lax tangent morphism whose underlying functor is the universal enveloping algebra functor $\varphi_!$. To understand the distributive law $\varphi_! \circ \mathbb{T}^{(\mathscr{Lie})} \to \mathbb{T}^{(u\mathscr{Ass})} \circ \varphi_!$, we first take a closer look at $\varphi_!(\mathbb{T}^{(\mathscr{Lie})}(\mathfrak{g}))$ and $\mathbb{T}^{(u\mathscr{Ass})}(\varphi_!(\mathfrak{g}))$, for a Lie algebra $\mathfrak{g}$. The former is the universal enveloping algebra of the semi-direct product $\mathfrak{g} \ltimes \mathfrak{g}$. Concretely, this is the associative algebra generated by pairs (g,h) for each $g,h \in \mathfrak{g}$, satisfying the relation:

$$(g,h)(g',h') - (g',h')(g,h) = ([g,g'],[g,h'] + [h,g']) \tag{3.5.3}$$

The second one is the semi-direct product of the universal enveloping algebra with itself. Concretely, this is the associative algebra of pairs (g,h) for $g,h \in \mathsf{Env}\mathfrak{g}$, satisfying the relations:

$$\begin{aligned} &(g,h)(g',h') = (gg', gh' + hg') \\ &gh - hg = [g,h] \end{aligned} \tag{3.5.4}$$

It is straightforward to see that the latter relations imply the former ones, thus there is a canonical morphism of Lie algebras $\varphi_!(\mathbb{T}^{(\mathscr{Lie})}(\mathfrak{g})) \to \mathbb{T}^{(u\mathscr{Ass})}(\varphi_!(\mathfrak{g}))$, which corresponds to the distributive law.

Remark 3.52. Proposition 3.42 shows that the functor Alg^* sends a morphism of operads to a strict tangent morphism. One can imagine that $\mathsf{Alg}_!$ should also send a morphism of operads to a strict, or maybe strong, tangent morphism. However, as discussed in Remark 3.47, the operation of taking conjoints does not preserve strength. In particular, Example 3.51 furnishes a counterexample of this conjecture, indeed the relations of Equation (3.5.3) do not imply relations of Equation (3.5.4).

3.6 The geometric tangent category of an operad

A well-known fact of algebraic geometry (cf. [19]) establishes that the category of affine schemes over R and the opposite of the category of commutative and unital R-algebras are equivalent. Starting from this observation, one would like to interpret the opposite of the category of algebras over a given operad $\mathscr{P}$ as operadic affine schemes of type $\mathscr{P}$. Inspired by this insight, Ginzburg in [25] and [26] suggested to think of operadic algebras in a geometric sense. Crucially, not the category of operadic algebras, but its opposite is the correct category to establish a geometric interpretation of these objects.

To understand why this is the case, let us consider the example of commutative and unital algebras. From a categorical point of view, a point of an object A in a category consists of a morphism from the terminal object to A. In the category cAlg of commutative and unital algebras, the terminal object is the zero algebra, so the only point of an algebra A is just the zero $0\colon 0 \to A$. On the contrary, in the opposite of the category of commutative and unital algebras $\mathsf{cAlg}^{\mathsf{op}}$, the terminal object is the ring R, so a point of an affine scheme A, i.e. an algebra seen as an object in $\mathsf{cAlg}^{\mathsf{op}}$, consists of an algebra morphism $\omega\colon A \to R$.

To understand why this is a good notion of point, consider the coordinate ring $A\colon = R[x,y]/(p(x,y))$ of a polynomial $p(x,y)$ in two variables. Then, a point of A consists of a morphism of algebras $\omega\colon R[x,y]/(p(x,y)) \to R$, which is fully determined by the values $x_0\colon = \omega(x)$ and $y_0\colon = \omega(y)$. However, from the relations which define the algebra A, $p(x_0,y_0) = 0$. So, a point $\omega\colon A \to R$ in the categorical sense is equivalent to a point on the affine scheme represented by the locus of the polynomial $p(x,y)$.

Another important reason why one should consider the opposite of the category of algebras as the category of true geometric objects, is the Gelfand-Naimark functor. Such a functor sends a finite-dimensional Hausdorff locally compact topological space M to the commutative C^*-algebra of complex-valued continuous functions of M. This functor, which is contravariant, establishes an equivalence between the categories of Hausdorff, locally compact, topological spaces and the opposite of the category of commutative C^*-algebras. In particular, this equivalence suggests treating the opposite of the category of C^*-algebras as a category of geometric objects.

From the perspective of tangent category theory, one would like to define a tangent structure on the opposite of the category of operadic algebras capable of capturing the geometry of operadic affine schemes. This has already been explored by Cruttwell and Lemay for commutative and unital algebras ([18]). In particular, as shown in Example 2.14, the opposite of commutative and unital rings $\mathsf{cRing}^{\mathsf{op}}$ comes equipped with a tangent structure. This tangent structure, first introduced by Cockett and Cruttwell in [12], was then extensively studied by Cruttwell and Lemay as the correct tangent structure capable of capturing some important geometric features of affine schemes.

As explained in Section 2.2.6, the existence of a tangent structure on $\mathsf{cRing}^{\mathsf{op}}$ is a consequence of the adjunctability of the tangent structure on cRing. With this in mind, it is natural to wonder a more general question: when is the tangent category of algebras of a tangent monad adjunctable (see Section 2.2.6)? Cockett, Lemay, and Lucyshyn-Wright in [15, Theorem 26] answered this question in the context of differential categories with finite biproducts. Here, we want to extend the same idea for a general tangent monad. The main idea is to employ Johnstone's adjoint lifting theorem [33, Theorem 2] which establishes that, for a monad S over a category $\mathbb{X}$, given an endofunctor $\mathrm{L}\colon \mathbb{X} \to \mathbb{X}$ of $\mathbb{X}$ and a distributive law $\alpha\colon S \circ \mathrm{L} \Rightarrow \mathrm{L} \circ S$ so that L can be lifted to the algebras $\overline{\mathrm{L}}\colon \mathsf{Alg}_S \to \mathsf{Alg}_S$, if the category Alg_S of algebras of S admits coequalizers of reflexive pairs, then whenever L has a left adjoint T, so does the left $\overline{\mathrm{L}}$. In particular, this implies that when Alg_S is finitely cocomplete, then adjunctions of endofunctors $\mathrm{T} \dashv \mathrm{L}$ over the base category $\mathbb{X}$ can be lifted along the forgetful functor $\mathsf{Alg}_S \to \mathbb{X}$.

Proposition 3.53. *Suppose the category of algebras* Alg_S *of a tangent monad* (S, α) *over a tangent category* $(\mathbb{X}, \mathbb{L})$ *admits coequalizers of reflexive pairs. If* $(\mathbb{X}, \mathbb{L})$ *is adjunctable, so is the tangent category* $\mathsf{Alg}(S, \alpha) = (\mathsf{Alg}_S, \mathbb{L}^{(S)})$. *In particular,* $\mathsf{Geom}(S, \alpha) := (\mathsf{Alg}_S^{\mathsf{op}}, \mathbb{T}^{(S)})$ *is a tangent category. Finally, if* $\mathbb{X}$ *has negatives, then so does* $\mathsf{Geom}(S, \alpha)$.

Proof. This result follows directly from the adjoint lifting theorem: since the category Alg_S admits coequalizers of reflexive pairs, the adjunctions $\mathrm{T}_n \dashv \mathrm{L}_n$ defined by the adjunctability of $(\mathbb{X}, \mathbb{L})$ can be lifted to Alg_S, so in particular, the lifted tangent bundle functor $\mathrm{L}^{(S)}$ admits a left adjoint $\mathrm{T}^{(S)}$. Now, consider the n-fold pullback

L_n of the projection $p^{(\mathrm{L})}$ along itself. The lift $\mathrm{L}_n^{(S)}$ is the functors which map an S-algebra A with structure map $\theta\colon SA \to A$, to the S-algebra L_nA with structure map $S\mathrm{L}_nA \xrightarrow{\alpha_n} \mathrm{L}_nSA \xrightarrow{\mathrm{L}_n\theta} \mathrm{L}_nA$, where α_n is the unique morphism defined by the universality of the diagram:

$$
\begin{tikzcd}
S\mathrm{L}_nA \arrow[rrr, "S\pi_n^{(\mathrm{L})}"] \arrow[dr, dashed, "\alpha_n"] \arrow[ddd, "S\pi_1^{(\mathrm{L})}"'] & & & S\mathrm{L}A \arrow[dl, "\alpha"'] \arrow[ddd, "Sp^{(\mathrm{L})}"] \\
& \mathrm{L}_nSA \arrow[r, "\pi_n^{(\mathrm{L})}S"] \arrow[d, "\pi_1^{(\mathrm{L})}S"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{L}SA \arrow[d, "p^{(\mathrm{L})}S"] & \\
& \mathrm{L}SA \arrow[r, "p^{(\mathrm{L})}S"'] & SA \arrow[dr, equal] & \\
S\mathrm{L}A \arrow[ur, "\alpha"] \arrow[rrr, "Sp^{(\mathrm{L})}"'] & & & SA
\end{tikzcd}
$$

On the other hand, this is also the n-fold pullback of the projection $p^{(S)}\colon \mathrm{L}^{(S)} \Rightarrow \mathsf{id}_{\mathsf{Alg}_S}$ along itself. Thus, the left adjoint T_n of L_n is lifted to the left adjoint $\mathrm{T}_n^{(S)}$ of $\mathrm{L}_n^{(S)}$. Thus, $\mathsf{Alg}(S,\alpha)$ is adjunctable. $\square$

Corollary 3.54. *Suppose $\mathbb{X}$ is a category with biproducts and (S,∂) a coCartesian differential monad over $\mathbb{X}$. Then, if the category Alg_S of algebras of S admits coequalizers of reflexive pairs, the algebraic tangent category $\mathsf{Alg}(S,\partial) = (\mathsf{Alg}_S, \mathbb{L}^{(S)})$ is adjunctable and in particular, $\mathsf{Geom}(S,\partial) = (\mathsf{Alg}_S^{\mathsf{op}}, \mathbb{T}^{(S)})$ is a tangent category. Finally, if $\mathbb{X}$ is also additive, then $\mathsf{Geom}(S,\partial)$ has negatives.*

Proof. The algebraic tangent category $\mathsf{Alg}(S,\partial)$ of a coCartesian differential monad is the algebraic tangent category of the associated tangent monad over the tangent category $(\mathbb{X},\mathbb{L})$ induced by biproducts. However, each functor L_n, which sends an object A of $\mathbb{X}$ to $A \oplus A \oplus \cdots \oplus A$ ($(n+1)$-times) is self-adjoint. So, by Proposition 3.53 we conclude that the algebraic tangent category $\mathsf{Alg}(S,\partial)$ of (S,∂) is adjunctable. $\square$

We are finally in the position to define the tangent category of operadic affine schemes over a given operad. The idea is to apply Corollary 3.54 to the coCartesian differential monad associated with an operad $\mathscr{P}$ and obtain a tangent structure $\mathbb{T}^{(\mathscr{P})}$ over the category $\mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}}$ of operadic affine schemes over $\mathscr{P}$.

Theorem 3.55. *The algebraic tangent category* $\mathsf{Alg}(\mathscr{P})$ *of an operad* $\mathscr{P}$ *defined over a symmetric monoidal category* $\mathbb{E}$ *with biproducts satisfying Convention 3.20 is adjunctable and in particular, the category of operadic affine schemes* $\mathsf{Geom}(\mathscr{P}) := (\mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}}, \mathbb{T}^{(\mathscr{P})})$ *is a tangent category. Finally, if* $\mathbb{E}$ *is also additive, then* $\mathsf{Geom}(\mathscr{P})$ *has negatives.*

Proof. The first step is to realize that the category $\mathsf{Alg}_{\mathscr{P}}$ of operadic algebras has finite colimits, so in particular, has coequalizers of reflexive pairs. This is proved in [48, Proposition 6.4]. Notice in particular, that $\mathbb{E}$, by Convention 3.20, satisfies the hypothesis of this result. Then, for Corollary 3.54, the algebraic tangent category of the operad $\mathscr{P}$, which is, by definition, the algebraic tangent category of the associated coCartesiann differential monad, is adjunctable. $\square$

Definition 3.56. *The **geometric tangent category** of an operad* $\mathscr{P}$ *is the tangent category* $\mathsf{Geom}(\mathscr{P})$ *over the category of operadic affine schemes defined in Theorem 3.55.*

Theorem 3.55 furnishes an abstract definition of the tangent category of operadic affine schemes over a given operad. In the rest of this section, we want to give a concrete description of this tangent category. The key is to characterize the left adjoint $\mathrm{T}^{(\mathscr{P})}$ of the algebraic tangent bundle functor $\mathbb{L}^{(\mathscr{P})}$. A suggestion to construct $\mathrm{T}^{(\mathscr{P})}$ comes from the definition of the tangent bundle functor T over the opposite category of commutative and unital algebras, as described by Cruttwell and Lemay in [18]. Concretely, the tangent bundle functor sends a commutative and unital algebra A to the symmetric algebra over A of the module of Kähler differentials ΩA of A, i.e. $\mathrm{T}A = \mathsf{S}_A\Omega A$. One can extend the notion of the module of Kähler differentials to operadic algebras. For this purpose, let's first recall the notion of a module over an algebra over an operad. The interested reader can find this notion in [27] or [46, Section 12.3.1].

Definition⋆ 3.57. *A **module** over a* $\mathscr{P}$*-algebra* A *consists of an object* M *of the base monoidal category* $\mathbb{E}$ *equipped with a collection of morphisms of* $\mathbb{E}$*, called the **structure map** of* M*:*

$$\psi_{n+1} \colon \mathscr{P}(n+1) \otimes A^{\otimes n} \otimes M \to M$$

compatible with the algebra structure map as follows:

$$\begin{tikzcd}
\mathscr{P}(n)\otimes\mathscr{P}(k_1)\otimes A^{\otimes k_1}\otimes\ldots\otimes\mathscr{P}(k_n+1)\otimes A^{\otimes k_n}\otimes M \arrow[r, "\mathscr{P}(n+1)\otimes\theta\otimes\ldots\otimes\psi"] \arrow[d, "\cong"'] & \mathscr{P}(n+1)\otimes A^{\otimes n}\otimes M \arrow[dd, "\psi"] \\
\mathscr{P}(n)\otimes\mathscr{P}(k_1)\otimes\ldots\otimes\mathscr{P}(k_n+1)\otimes A^{\otimes(k_1+\ldots+k_n)}\otimes M \arrow[d, "\gamma\otimes A^{\otimes k_1}\otimes\ldots\otimes A^{\otimes k_n}\otimes M"'] & \\
\mathscr{P}(k_1+\ldots+k_n+1)\otimes A^{\otimes(k_1+\ldots+k_n)}\otimes M \arrow[r, "\psi"'] & M
\end{tikzcd}$$

and satisfying an equivariant condition with respect to the symmetric actions. Given two A-modules M and M' a morphism of A-modules $g\colon M\to M'$ consists of an $\mathbb{E}$-morphism $g\colon M\to M'$ satisfying the following compatibility with the module structures:

$$\begin{tikzcd}
\mathscr{P}(n+1)\otimes A^{\otimes n}\otimes M \arrow[r, "\psi"] \arrow[d, "\mathscr{P}(n+1)\otimes A^{\otimes n}\otimes g"'] & M \arrow[d, "g"] \\
\mathscr{P}(n+1)\otimes A^{\otimes n}\otimes M' \arrow[r, "\psi'"'] & M'
\end{tikzcd}$$

Notation 3.58. For an A-module M of a $\mathscr{P}$-algebra A, we denote the structure map by $\psi\colon \mathscr{P}(n+1)\otimes A^{\otimes n}\otimes M\to M$. When $\mathscr{P}$ is algebraic, we adopt the convention to write:

$$\mu_M(a_1,\ldots,a_n,x) := \mu(a_1,\ldots,a_n,x) := \psi(\mu;a_1,\ldots,a_n,x)$$

Moreover, we also write $\mu_M(a_1,\ldots,x,\ldots,a_n) := \mu(a_1,\ldots,a_{k-1},x,a_{k+1},\ldots,a_n)$ for $\mu\in\mathscr{P}(n+1)$, $a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n\in A$ and $x\in M$ to denote $(\mu\cdot\sigma)(a_1,\ldots,a_n,x)$, for $\sigma=(k\quad k+1\quad\ldots\quad n+1)$.

For algebraic operads, the compatibility condition of Definition 3.57 reads as follows:

$$\mu\left({\mu_1}_A(a_1^{(1)},\ldots,a_{k_1}^{(1)}),\ldots,{\mu_n}_M(a_1^{(n)},\ldots,a_{k_n}^{(n)},x)\right) = \mu(\mu_1,\ldots,\mu_n)_M(a_1^{(1)},\ldots,a_{k_n}^{(n)},x)$$

The equivariant condition, in this context, reads as follows:

$$(\mu\cdot\sigma)(a_1,\ldots,a_n,x) = \mu(a_{\sigma(1)},\ldots,a_{\sigma(n)},x)$$

for every $\sigma \in \mathbb{S}_n$ where $\mathbb{S}_n$ is regarded as subgroup of $\mathbb{S}_{n+1}$. In the following, in the context of algebraic operads, we adopt the convention of writing:

$$\mu(a_1, \ldots, x, \ldots, a_n)$$

where $x \in M$ is in the k-th position, for:

$$(\mu \cdot \sigma)(a_1, \ldots, a_n, x)$$

where $\sigma = (k \quad \ldots \quad n+1)$ represents the cyclic permutation (going counterclockwise) which shifts the last $n+1-k$ terms. The next step is to recall the notion of derivation for operadic algebras (see [46, Section 12.3.7]).

Definition⋆ 3.59. *A **derivation** of a $\mathscr{P}$-algebra A over an A-module M consists of an $\mathbb{E}$-morphism $\delta \colon A \longrightarrow M$ satisfying the following condition:*

$$\begin{tikzcd}
\mathscr{P}(n+1) \otimes A^{\otimes(n+1)} \arrow[r, "\mathscr{P}(n+1)\otimes A^{\otimes n}\otimes\delta"] \arrow[d, "\sum_{k=1}^{n+1} \delta_k"'] & \mathscr{P}(n+1) \otimes A^{\otimes n} \otimes M \arrow[d, "\psi"] \\
\mathscr{P}(n+1) \otimes A^{\otimes n} \otimes M \arrow[r, "\psi"'] & M
\end{tikzcd}$$

where δ_k is defined as follows:

$$\delta_k \colon \mathscr{P}(n+1) \otimes A^{\otimes(n+1)} \xrightarrow{\mathscr{P}(n+1)\otimes A^{\otimes(k-1)}\otimes\delta\otimes A^{\otimes(n-k)}} \mathscr{P}(n+1) \otimes A^{\otimes(k-1)} \otimes M \otimes A^{\otimes(n-k)} \longrightarrow$$
$$\xrightarrow{\rho_\sigma \otimes \varepsilon} \mathscr{P}(n+1) \otimes A^{\otimes n} \otimes M$$

with ρ_σ the right action of the symmetric group $\mathbb{S}_{n+1}$ over $\mathscr{P}(n+1)$, $\sigma = (k \quad \ldots \quad n+1)$, and ε the shuffling induced by the braiding.

For algebraic operads, a derivation of a $\mathscr{P}$-algebra A over an A-module M is an R-linear morphism $\delta \colon A \longrightarrow M$ satisfying the following condition:

$$\delta(\mu(a_1, \ldots, a_{n+1})) = \sum_{k=1}^{n+1} \mu(a_1, \ldots, \delta(a_k), \ldots, a_{n+1})$$

In particular, for the operad $\mathscr{A}ss$ of Example 3.7, since it is generated by a binary tree $\mu \in \mathscr{A}ss(2)$, the condition for a R-linear morphism $\delta \colon A \longrightarrow M$ to be a derivation in the operadic sense is precisely the usual Leibniz rule:

$$\delta(\mu(a, b)) = \mu(\delta(a), b) + \mu(a, \delta(b))$$

It is straightforward to see that for a morphism $f\colon A \to B$ of $\mathscr{P}$-algebras, and a derivation $\delta\colon B \to N$ of B over a B-module N, the composition $A \xrightarrow{f} B \xrightarrow{\delta} N$ defines a derivation $f^*\delta\colon A \to f^*N$ of A over the A-module f^*N, whose module structure is the restriction of scalars of N via f. Moreover, for an A-module morphism $g\colon M \to N$ and a derivation $\delta\colon A \to M$, it is also straightforward to see that the composition $A \xrightarrow{\delta} M \xrightarrow{g} N$ is also a derivation of A. This induces a functor:

$$\mathsf{Der}(A,-)\colon \mathsf{Mod}_A \to \mathsf{Mod}_R$$

which sends an A-module M to the R-module of derivations $\delta\colon A \to M$ of A over M and a morphism $g\colon M \to N$ of A-modules to the R-linear morphism which sends a derivation $\delta A \to M$ to the composition $\delta g\colon A \to N$. When the operad is algebraic this functor is representable by an A-module ΩA. This is equivalent to saying that there is a natural isomorphism:

$$\mathsf{Der}(A,M) \cong \mathsf{Mod}_A(\Omega A, M)$$

Concretely, ΩA is the A-module generated by symbols $\mathsf{d}a$, for every $a \in A$ satisfying the following relations:

$$\begin{aligned}\mathsf{d}(ra+sb) &= r\mathsf{d}a + s\mathsf{d}b\\ \mathsf{d}(\mu(a_1,\dots,a_n)) &= \sum_{k=1}^{n} \mu(a_1,\dots,\mathsf{d}a_k,\dots,a_n)\end{aligned}$$

where on the right side of the last equation we employed the A-module structure. In particular, ΩA is the A-module equipped with a derivation $\mathsf{d}\colon A \to \Omega A$, which is the R-linear morphism which sends $a \in A$ to $\mathsf{d}a \in \Omega A$, uniquely defined by the following universality condition. If $\delta\colon A \to M$ is a derivation of A over an A-module M, δ factors uniquely through d, i.e. there is a unique morphism of A-modules $\bar{\delta}\colon \Omega A \to M$ such that the diagram:

$$\begin{tikzcd} A \arrow[r, "\mathsf{d}"] \arrow[dr, "\delta"'] & \Omega A \arrow[d, dashed, "\bar{\delta}"] \\ & M \end{tikzcd}$$

commutes.

Definition* 3.60. *For an algebraic operad $\mathscr{P}$ the **module of Kähler differentials** of a $\mathscr{P}$-algebra A is the A-module ΩA equipped by the universal derivation $\mathsf{d}\colon A \to \Omega A$ as defined above.*

For the operad $u\mathscr{C}om$ of Example 3.8, it is not hard to see that the module of Kähler differentials ΩA of a unital and commutative algebra A is precisely the usual notion of this module. In particular, $\Omega A \cong I/I^2$, where I is the kernel of the multiplication map $\nu\colon A \otimes A \to A$.

Since a morphism $f\colon A \to B$ sends derivations $\delta\colon B \to N$ of B over a B-module N to derivations $f^*\delta\colon A \to f^*N$, this, together with the representability of $\mathsf{Der}(A, -)$, implies that the operation which sends an $\mathscr{P}$-algebra A to the module of Kähler differentials ΩA, extends to a functor $\Omega\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Mod}$ which sends a $\mathscr{P}$-algebra A to the pair $(A, \Omega A)$ and a morphism $f\colon A \to B$ to $f\colon A \to B$ equipped with the morphism $\Omega A \to f^*\Omega B$ of A-modules which send each $\mathsf{d}a \in \Omega A$ to $\mathsf{d}f(a) \in f^*\Omega B$.

In order to define the tangent bundle functor $\mathrm{T}\colon \mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}} \to \mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}}$ we need to send the A-module ΩA back to a $\mathscr{P}$-algebra without losing the information about the Kähler differentials. For the commutative and unital case, this was done by sending ΩA to the symmetric algebra over A of ΩA, $\mathsf{S}_A\Omega A$. We want to define a similar construction for operadic algebras. To characterize more generally the functor S_A, consider the functor:

$$\mathsf{Restr}_A\colon A/\mathsf{Alg}_{\mathscr{P}} \to \mathsf{Mod}_A$$

which sends a morphism of $\mathscr{P}$-algebras $q\colon A \to B$ to the A-module q^*B over B induced by q and a morphism $f\colon B \to B'$, such that $qf = q'$ for $q\colon A \to B$ and $q'\colon A \to B'$, to the morphism of A-modules $q^*B \to q^*B'$, whose underlying R-linear morphism is f. This functor admits a left adjoint $\mathsf{Free}_A\colon \mathsf{Mod}_A \to A/\mathsf{Alg}_{\mathscr{P}}$, which sends an A-module M to the free $\mathscr{P}$-algebra under A, $A \to \mathsf{Free}_A M$ of M.

Lemma 3.61. *For an algebraic operad $\mathscr{P}$ and a $\mathscr{P}$-algebra A, the functor $\mathsf{Restr}_A\colon A/\mathsf{Alg}_{\mathscr{P}} \to \mathsf{Mod}_A$ which sends a morphism of $\mathscr{P}$-algebras $A \to B$ to the corresponding A-module, has a left adjoint $\mathsf{Free}_A\colon \mathsf{Mod}_A \to A/\mathsf{Alg}_{\mathscr{P}}$, which sends an A-module M to the $\mathscr{P}$-algebra under A, $A \to \mathsf{Free}_A M$. In particular, $\mathsf{Free}_A M$, is the $\mathscr{P}$-algebra obtained by quotienting the free $\mathscr{P}$-algebra $\mathsf{S}_{\mathscr{P}}(A \oplus M)$ by the ideal generated by the relations:*

$$(\mu; (a_1, 0), \ldots, (a_k, x), \ldots, (a_n, 0)) = (\mu_A(a_1, \ldots, a_n), \mu_M(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n))$$

for every $a_1, \dots, a_n \in A$, $x \in M$, $\mu \in \mathscr{P}(n)$, and any positive integer n.

Proof. First, notice that the R-linear morphism $\iota_A\colon A \to \mathsf{Free}_A M$ which sends $a \in A$ to $(a, 0) \in \mathsf{Free}_A M$ is a well-defined $\mathscr{P}$-algebra morphism, since the relations imply that:

$$\begin{aligned}
& \mu_{\mathsf{Free}_A M}(\iota_A(a_1), \dots, \iota_A(a_n)) \\
= \; & (\mu; (a_1, 0), \dots, (a_n, 0)) \\
= \; & (\mu_A(a_1, \dots, a_n), \mu_M(a_1, \dots, 0, \dots, a_n)) \\
= \; & (\mu_a(a_1, \dots, a_n), 0) \\
= \; & \iota_A(\mu_A(a_1, \dots, a_n))
\end{aligned}$$

In order to show that Free_A is a left adjoint of Restr_A, let's define the unit $\eta\colon M \to \mathsf{Restr}_A \mathsf{Free}_A M$ and the counit $\varepsilon\colon \mathsf{Free}_A \mathsf{Restr}_A(q\colon A \to B) \to q$. First, notice that $\mathsf{Restr}_A \mathsf{Free}_A M$ is the A-module induced by the $\mathscr{P}$-algebra morphism $\iota_A\colon A \to \mathsf{Free}_A M$. Let's show that the R-linear morphism $\iota_M\colon M \to \mathsf{Free}_A M$ which sends $x \in M$ to $(0, x) \in \mathsf{Free}_A M$ is a well-defined morphism of A-modules:

$$\begin{aligned}
& \mu_{\mathsf{Free}_A M}(a_1, \dots, a_n, \iota_M(x)) \\
= \; & (\mu; (a_1, 0), \dots, (a_n, 0), (0, x)) \\
= \; & (\mu_A(a_1, \dots, a_n, 0), \mu_M(a_1, \dots, a_n, x)) \\
= \; & (0, \mu_M(a_1, \dots, a_n, x)) \\
= \; & \iota_M(\mu_M(a_1, \dots, a_n, x))
\end{aligned}$$

So, $\eta_M \colon = \iota_M$. Let's now focus on the counit. First, realize that, for a morphism $q\colon A \to B$ of $\mathscr{P}$-algebras, $\mathsf{Free}_A \mathsf{Restr}_A(q)$ is the morphism $\iota_A\colon A \to \mathsf{Free}_A B$ of $\mathscr{P}$-algebras which sends $a \in A$ to $(a, 0) \in \mathsf{Free}_A B$, where B is the A-module with module structure induced by q. Let's define on generators $(a, b) \in A \oplus B$ the morphism $\pi_B\colon \mathsf{Free}_A B \to B$ which sends (a, b) to b. As an A-module morphism, this extends to a morphism which sends $(\mu; (a_1, b_1), \dots, (a_n, b_n))$ to $\sum_{k=1}^{n} \mu_B(q(a_1), \dots, q(a_{k-1}), b_k, q(a_{k+1}), \dots, q(a_n))$. Let's show this lifts to the quotient:

$$\pi_B(\mu; (a_1, 0), \dots, (a_k, b), \dots, (a_n, 0)) = \mu_B(a_1, \dots, a_{k-1}, b, a_{k+1}, \dots, a_n)$$

$$\pi_B(\mu_A(a_1, \dots, a_n), \mu_{q^*B}(a_1, \dots, b, \dots, a_n)) =$$
$$= \mu_B(q(a_1), \dots, q(a_{k-1}), b, q(a_{k+1}), \dots, q(a_n))$$

where we denoted by q^*B the A-module B induced by q. Let ε be π_B. We now need to prove the triangle identities $\varepsilon_{\mathsf{Free}_A} \circ \mathsf{Free}_A\eta = \mathrm{id}_{\mathsf{Free}_A}$ and $\mathsf{Restr}_A\varepsilon \circ \eta_{\mathsf{Restr}_A} = \mathrm{id}_{\mathsf{Restr}_A}$. Consider $(a, x) \in A \oplus M$. Then, we have:

$$\begin{aligned} & \varepsilon_{\mathsf{Free}_A}(\mathsf{Free}_A\eta(a, x)) \\ = \ & \varepsilon_{\mathsf{Free}_A}(\mathsf{Free}_A\eta((0, 0), (a, x))) \\ = \ & (a, x) \end{aligned}$$

Let's now consider $q \colon A \to B$ and $b \in B$:

$$\begin{aligned} & \mathsf{Restr}_A\varepsilon(\eta_{\mathsf{Restr}_A}(b)) \\ = \ & \mathsf{Restr}_A\varepsilon(0, b) \\ = \ & b \end{aligned}$$

This proves that Free_A is a left adjoint of Restr_A. □

Remark 3.62. In the following, for the sake of simplicity, we adopt the same notation for the functor $\mathsf{Free}_A \colon \mathsf{Mod}_A \to A/\mathsf{Alg}_{\mathscr{P}}$ which sends an A-module M to the $\mathscr{P}$-algebra morphism $\iota_A \colon A \to \mathsf{Free}_A M$ and the $\mathscr{P}$-algebra $\mathsf{Free}_A M$, codomain of ι_A.

Remark 3.63. Free_A sends an A-module morphism $g \colon M \to M'$ to the morphism of $\mathscr{P}$-algebras under A, $\mathsf{Free}_A g \colon \mathsf{Free}_A M \to \mathsf{Free}_A M'$ which sends the generator (a, x) to $(a, f(x))$.

The next step is to compose the two functors $\Omega \colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Mod}$ and $\mathsf{Free}_A \colon \mathsf{Mod}_A \to A/\mathsf{Alg}_{\mathscr{P}}$ together.

Lemma 3.64. *For a $\mathscr{P}$-algebra A, the $\mathscr{P}$-algebra $\mathsf{Free}_A\Omega A$ can be represented as the $\mathscr{P}$-algebra generated by all elements a of A and symbols $\mathsf{d}^{(\mathscr{P})}a$, for each a of A, satisfying the following relations:*

$$\mu_{\mathrm{T}A}(a_1, \dots, a_n) = \mu_A(a_1, \dots, a_n)$$
$$\mathsf{d}^{(\mathscr{P})}(ra + sb) = r\mathsf{d}^{(\mathscr{P})}a + s\mathsf{d}^{(\mathscr{P})}b$$

$$\mathsf{d}^{(\mathscr{P})}(\mu_{\mathrm{T}A}(a_1,\dots,a_n)) = \sum_{k=1}^{n} \mu_{\mathrm{T}A}(a_1,\dots,\mathsf{d}^{(\mathscr{P})}a_k,\dots,a_n)$$

for every $a_1,\dots,a_n \in A$.

Proof. By construction, $\mathsf{Free}_A\Omega A$ is the $\mathscr{P}$-algebra defined as the quotient of the free $\mathscr{P}$-algebra over $A \oplus \Omega A$ by the relations:

$$(\mu;(a_1,0),\dots,(a_k,\mathsf{d}^{(\mathscr{P})}b),\dots,(a_n,0)) = (\mu_A(a_1,\dots,a_n),\mu_{\Omega A}(a_1,\dots,\mathsf{d}^{(\mathscr{P})}b,\dots,a_n))$$

for every $a_1,\dots,a_n,b \in A$. The image of the $\mathscr{P}$-algebra $\iota_A\colon A \to \mathsf{Free}_A\Omega A$ which sends $a \in A$ to $(a,0)$ defines a copy of A inside of $\mathsf{Free}_A\Omega A$. Denote by $\mathrm{T}A$ the $\mathscr{P}$-algebra generated by all a and all $\mathsf{d}^{(\mathscr{P})}A$ as above and define the morphism $\varphi\colon \mathsf{Free}_A\Omega A \to \mathrm{T}A$ which sends $(a,\mathsf{d}^{(\mathscr{P})}b)$ to $a + \mathsf{d}^{(\mathscr{P})}b$. Let's prove this is a well-defined morphism of $\mathscr{P}$-algebras:

$$\begin{aligned}
&\varphi(\mu;(a_1,0),\dots,(a_k,\mathsf{d}^{(\mathscr{P})}b),\dots,(a_n,0)\\
=\;&\mu_{\mathrm{T}A}(a_1+0,\dots,a_k+\mathsf{d}^{(\mathscr{P})}b,\dots,a_n+0)\\
=\;&\mu_{\mathrm{T}A}(a_1,\dots,a_k,\dots,a_n)+\mu_{\mathrm{T}A}(a_1,\dots,\mathsf{d}^{(\mathscr{P})}b,\dots,a_n)\\
=\;&\mu_A(a_1,\dots,a_n)+\mu_{\Omega A}(a_1,\dots,\mathsf{d}^{(\mathscr{P})}b,\dots,a_n)
\end{aligned}$$

$$\begin{aligned}
&\varphi(\mu_A(a_1,\dots,a_n),\mu_{\Omega A}(a_1,\dots,\mathsf{d}^{(\mathscr{P})}b,\dots,a_n))\\
=\;&\mu_A(a_1,\dots,a_n)+\mu_{\Omega A}(a_1,\dots,\mathsf{d}^{(\mathscr{P})}b,\dots,a_n)
\end{aligned}$$

Let's now consider the morphism $\psi\colon \mathrm{T}A \to \mathsf{Free}_A\Omega A$ which sends the generator a to $(a,0)$ and $\mathsf{d}^{(\mathscr{P})}a$ to $(0,\mathsf{d}^{(\mathscr{P})}a)$. Let's first show this is a well-defined morphism of $\mathscr{P}$-algebras:

$$\begin{aligned}
&\psi(\mu_{\mathrm{T}A}(a_1,\dots,a_n))\\
=\;&(\mu;(a_1,0),\dots,(a_n,0))\\
=\;&(\mu_A(a_1,\dots,a_n),0)\\
=\;&\psi(\mu_A(a_1,\dots,a_n))
\end{aligned}$$

$$\psi(\mathsf{d}^{(\mathscr{P})}(ra+sb))$$

$$
\begin{aligned}
&= (0, \mathsf{d}^{(\mathscr{P})}(ra + sb)) \\
&= r(0, \mathsf{d}^{(\mathscr{P})}a) + s(0, \mathsf{d}^{(\mathscr{P})}b) \\
&= r\psi(\mathsf{d}^{(\mathscr{P})}a) + s\psi(\mathsf{d}^{(\mathscr{P})}b)
\end{aligned}
$$

$$
\begin{aligned}
&\psi(\mathsf{d}^{(\mathscr{P})}(\mu_{\mathrm{T}A}(a_1, \dots, a_n))) \\
= \;& (0, \mathsf{d}^{(\mathscr{P})}(\mu_A(a_1, \dots, a_n))) \\
= \;& \left(0, \sum_{k=1}^{n} \mu_{\Omega A}(a_1, \dots, \mathsf{d}^{(\mathscr{P})}a_k, \dots, a_n)\right) \\
= \;& \psi(\mu_{\mathrm{T}A}(a_1, \dots, \mathsf{d}^{(\mathscr{P})}a_k, \dots, a_n))
\end{aligned}
$$

Finally, notice that:

$$
\begin{aligned}
&\varphi(\psi(a)) = \varphi(a, 0) = a + 0 = a \\
&\varphi(\psi(\mathsf{d}^{(\mathscr{P})}a)) = \varphi(0, \mathsf{d}^{(\mathscr{P})}a) = 0 + \mathsf{d}^{(\mathscr{P})}a = \mathsf{d}^{(\mathscr{P})}a
\end{aligned}
$$

$$
\psi(\varphi(a, \mathsf{d}^{(\mathscr{P})}b)) = \psi(a + \mathsf{d}^{(\mathscr{P})}b)) = (a, 0) + (0, \mathsf{d}^{(\mathscr{P})}b) = (a, \mathsf{d}^{(\mathscr{P})}b)
$$

So, ψ and φ are inverses to each other. $\square$

Remark 3.65. In the following, for the sake of simplicity, we simplify notation and omit the superscript $^{(\mathscr{P})}$ when the operad $\mathscr{P}$ is clear from the context. Moreover, we will not distinguish between $\mathsf{Free}_A \Omega A$ and $\mathrm{T}A$.

Remark 3.66. Given a morphism $f \colon A \to B$ of $\mathscr{P}$-algebras, T sends f to the $\mathscr{P}$-algebra morphism $\mathrm{T}f \colon \mathrm{T}A \to \mathrm{T}B$ which sends the generators a and $\mathsf{d}^{(\mathscr{P})}a$ to $f(a)$ and $\mathsf{d}^{(\mathscr{P})}f(a)$, respectively.

Lemma 3.67. *The functor $\mathrm{T}^{(\mathscr{P})} \colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}_{\mathscr{P}}$, which sends a $\mathscr{P}$-algebra A to $\mathrm{T}^{(\mathscr{P})}A := \mathrm{T}A$, is a left adjoint of the tangent bundle functor $\perp^{(\mathscr{P})} \colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}_{\mathscr{P}}$ of the algebraic tangent structure of $\mathscr{P}$.*

Proof. To prove that $\mathrm{T} := \mathrm{T}^{(\mathscr{P})}$ is a left adjoint of $\perp := \perp^{(\mathscr{P})}$, let's define the unit $\eta \colon A \to \perp\mathrm{T}A$ and the counit $\varepsilon \mathrm{T}\perp A \to A$. Let η be the morphism which sends $a \in A$ to $(a, \mathsf{d}a) \in \perp\mathrm{T}A$ and let ε be the morphism which sends the generators $(a, b) \in \mathrm{T}\perp A$ to a and $\mathsf{d}(a, b) \in \mathrm{T}\perp A$ to b. Let's start by showing they are well-defined:

$$
\eta(\mu_A(a_1, \dots, a_n))
$$

$$
\begin{aligned}
&= (\mu_{\mathrm{T}A}(a_1, \ldots, a_n), \mathsf{d}(\mu_{\mathrm{T}A}(a_1, \ldots, a_n)) \\
&= (\mu_A(a_1, \ldots, a_n), \sum_{k=1}^{n} \mu_{\mathrm{T}A}(a_1, \ldots, \mathsf{d}a_k, \ldots, a_n)) \\
&= \mu_{\perp\mathrm{T}A}((a_1, \mathsf{d}a_1), \ldots, (a_n, \mathsf{d}a_n)) \\
&= \mu_{\perp\mathrm{T}A}(\eta(a_1), \ldots, \eta(a_n))
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(\mu_{\mathrm{T}\perp A}((a_1, b_1), \ldots, (a_n, b_n))) \\
= \;& \mu_A(a_1, \ldots, a_n) \\
= \;& \mu_A(\varepsilon(a_1, b_1), \ldots, \varepsilon(a_n, b_n)) \\
= \;& \mu_{\mathrm{T}\perp A}(\varepsilon(a_1, b_1), \ldots, \varepsilon(a_n, b_n))
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(\mathsf{d}(r(a, b) + s(a', b'))) \\
= \;& (0, \mathsf{d}(rb + sb')) \\
= \;& r(0, \mathsf{d}b) + s(0, \mathsf{d}b') \\
= \;& r\varepsilon(\mathsf{d}(a, b)) + s\varepsilon(\mathsf{d}(a', b'))
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(\mathsf{d}(\mu((a_1, b_1), \ldots, (a_n, b_n)))) \\
= \;& \varepsilon(\mathsf{d}(\mu(a_1, \ldots, a_n), \sum_{k=1}^{n} \mu(a_1, \ldots, b_k, \ldots, a_n))) \\
= \;& \sum_{k=1}^{n} \mu(a_1, \ldots, b_k, \ldots, a_n) \\
= \;& \sum_{k=1}^{n} \mu(\varepsilon(a_1, b_1), \ldots, \varepsilon(\mathsf{d}(a_k, b_k)), \ldots, \varepsilon(a_n, b_n)) \\
= \;& \varepsilon(\sum_{k=1}^{n} \mu((a_1, b_1), \ldots, \mathsf{d}(a_k, b_k), \ldots, (a_n, b_n))
\end{aligned}
$$

This shows that η and ε are well-defined $\mathscr{P}$-algebra morphisms. Let's prove the triangle identities:

$$
\begin{aligned}
&\perp\varepsilon(\eta_{\perp)}(a, b)) \\
= \;& \perp\varepsilon((a, b), \mathsf{d}(a, b)) \\
= \;& (\varepsilon(a, b), \varepsilon(\mathsf{d}(a, b)))
\end{aligned}
$$

$$= \quad (a, b)$$

$$\begin{aligned} & \varepsilon_{\mathrm{T}}(\mathrm{T}\eta(a)) \\ = \quad & \varepsilon_{\mathrm{T}}(a, \mathsf{d}a) \\ = \quad & a \end{aligned}$$

$$\begin{aligned} & \varepsilon_{\mathrm{T}}(\mathrm{T}\eta(\mathsf{d}a)) \\ = \quad & \varepsilon_{\mathrm{T}}(\mathsf{d}(a, \mathsf{d}a)) \\ = \quad & \mathsf{d}a \end{aligned}$$

This proves that T is a left adjoint of $\mathbb{L}$. □

Theorem 3.68. *For an algebraic operad $\mathscr{P}$, the geometric tangent category $\mathsf{Geom}(\mathscr{P})$ defined in Theorem 3.55 is given as follows. For the sake of clarity, all morphisms are regarded as morphisms of $\mathscr{P}$-algebras:*

tangent bundle functor *The tangent bundle functor $\mathrm{T}^{(\mathscr{P})}\colon \mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}} \to \mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}}$, regarded as an endofunctor of $\mathsf{Alg}_{\mathscr{P}}$, is the left adjoint of the algebraic tangent bundle functor $\mathbb{L}^{(\mathscr{P})}$ described in Lemma 3.64;*

projection *The projection $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathsf{Geom}(\mathscr{P})} \Rightarrow \mathrm{T}^{(\mathscr{P})}$ is the natural transformation $p^{(\mathrm{T})}\colon A \to \mathrm{T}^{(\mathscr{P})}A$ which sends $a \in A$ to $a \in \mathrm{T}^{(\mathscr{P})}A$;*

zero morphism *The zero morphism $z^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})} \Rightarrow \mathsf{id}_{\mathsf{Geom}(\mathscr{P})}$ is the natural transformation $z^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})}A \to A$ which sends $a \in \mathrm{T}^{(\mathscr{P})}A$ to $a \in A$ and $\mathsf{d}^{(\mathscr{P})}a$ to 0;*

n-fold pullback *The n-fold pushout (in $\mathsf{Alg}_{\mathscr{P}}$) of the projection along itself is the functor $\mathrm{T}_n^{(\mathscr{P})}\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}_{\mathscr{P}}$ which sends an algebra A to the algebra $\mathrm{T}_n^{(\mathscr{P})}A$ generated by all a of A and symbols $\mathsf{d}_1^{(\mathscr{P})}a, \dots, \mathsf{d}_n^{(\mathscr{P})}a$, for each $a \in A$, satisfying the following relations:*

$$\begin{aligned} & \mu_{\mathrm{T}_n^{(\mathscr{P})}A}(a_1, \dots, a_n) = \mu_A(a_1, \dots, a_n) \\ & \mathsf{d}_i^{(\mathscr{P})}(ra + sb) = r\mathsf{d}_i^{(\mathscr{P})}a + s\mathsf{d}_i^{(\mathscr{P})}b \\ & \mathsf{d}_i^{(\mathscr{P})}(\mu_{\mathrm{T}_n^{(\mathscr{P})}A}(a_1, \dots, a_n)) = \sum_{k=1}^{n} \mu_{\mathrm{T}_n^{(\mathscr{P})}A}(a_1, \dots, \mathsf{d}_i^{(\mathscr{P})}a_k, \dots, a_n) \end{aligned}$$

for every $a_1, \dots, a_n \in A$ and every $i = 1, \dots, n$;

sum morphism *The sum morphism* $s^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})} \Rightarrow \mathrm{T}_2^{(\mathscr{P})}$ *is the natural transformation* $s^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})}A \to \mathrm{T}_2^{(\mathscr{P})}A$ *which sends* $a \in \mathrm{T}^{(\mathscr{P})}A$ *to* $a \in \mathrm{T}_2^{(\mathscr{P})}A$ *and* $\mathsf{d}^{(\mathscr{P})}a$ *to* $\mathsf{d}_1^{(\mathscr{P})}a + \mathsf{d}_2^{(\mathscr{P})}a$*;*

vertical lift *The vertical lift* $l^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})^2} \Rightarrow \mathrm{T}^{(\mathscr{P})}$ *is the natural transformation* $l^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})^2}A \to \mathrm{T}^{(\mathscr{P})}A$ *which sends* $a \in \mathrm{T}^{(\mathscr{P})^2}A$ *to* $a \in \mathrm{T}^{(\mathscr{P})}A$*,* $\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})^2}A$ *and* $\mathsf{d}^{(\mathscr{P})\prime}a \in \mathrm{T}^{(\mathscr{P})^2}A$ *to* $0 \in \mathrm{T}^{(\mathscr{P})}A$*, and* $\mathsf{d}^{(\mathscr{P})\prime}\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})^2}A$ *to* $\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})}A$*;*

canonical flip *The canonical flip* $c^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})^2} \Rightarrow \mathrm{T}^{(\mathscr{P})^2}$ *is the natural transformation* $c^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})^2}A \to \mathrm{T}^{(\mathscr{P})^2}A$ *which sends* $a \in \mathrm{T}^{(\mathscr{P})^2}A$ *to* $a \in \mathrm{T}^{(\mathscr{P})^2}A$*,* $\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})^2}A$ *to* $\mathsf{d}^{(\mathscr{P})\prime}a \in \mathrm{T}^{(\mathscr{P})^2}A$*,* $\mathsf{d}^{(\mathscr{P})\prime}a \in \mathrm{T}^{(\mathscr{P})^2}A$ *to* $\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})^2}A$*, and* $\mathsf{d}^{(\mathscr{P})\prime}\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})^2}A$ *to* $\mathsf{d}^{(\mathscr{P})\prime}\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})^2}A$*;*

negation *The negation* $n^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})} \Rightarrow \mathrm{T}^{(\mathscr{P})}$ *is the natural transformation* $n^{(\mathrm{T})}\colon \mathrm{T}^{(\mathscr{P})}A \to \mathrm{T}^{(\mathscr{P})}A$ *which sends* $a \in \mathrm{T}^{(\mathscr{P})}A$ *to* $a \in \mathrm{T}^{(\mathscr{P})}A$ *and* $\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})}A$ *to* $-\mathsf{d}^{(\mathscr{P})}a \in \mathrm{T}^{(\mathscr{P})}A$*.*

Example 3.69. Consider the operad $u\mathscr{C}om$ described in Example 3.8. The module of Kähler differentials ΩA of a commutative and unital algebra A is precisely the usual notion of the module of Kähler differentials of A. Concretely, ΩA is the quotient of the kernel I of the multiplication map $\nu\colon A \otimes A \to A$ by I^2, i.e. $\Omega A = I/I^2$. Equivalently, ΩA can be described as the A-module of symbols $\mathsf{d}a$, for each $a \in A$, such that:

$$\mathsf{d}(ab) = a\mathsf{d}b + b\mathsf{d}a$$

for every $a, b \in A$. The functor Free_A sends an A-module, which is precisely a left A-module, M to the symmetric algebra over A, i.e. $\mathsf{S}_A M$. Concretely, if M is a free module of A generated by a set of generators Γ, then $\mathsf{S}_A\Gamma$ is the algebra of polynomials in the variables Γ and coefficients in A.

Putting together these two pieces of information we obtain a description of the geometric tangent structure associated with $u\mathscr{C}om$:

tangent bundle functor The tangent bundle functor $\mathrm{T}\colon \mathsf{cAlg}^{\mathsf{op}} \to \mathsf{cAlg}^{\mathsf{op}}$, regarded as an endofunctor of cAlg, sends a commutative and unital algebra A to $\mathsf{S}_A\Omega A$ and a morphism $f\colon A \to B$ to the morphism $\mathsf{S}_f\Omega f$ which sends each $a \in A$ and each $\mathsf{d}a \in \Omega A$ to $f(a)$ and $\mathsf{d}f(a)$, respectively;

projection The projection $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathsf{Geom}(u\mathscr{C}om)} \Rightarrow \mathrm{T}$, regarded as a morphism of cAlg, is the natural transformation $p^{(\mathrm{T})}\colon A \to \mathrm{T}A$ which sends $a \in A$ to $a \in \mathrm{T}A$;

zero morphism The zero morphism $z^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathsf{Geom}(u\mathscr{C}om)}$, regarded as a morphism of cAlg, is the natural transformation $z^{(\mathrm{T})}\colon \mathrm{T}A \to A$ which sends $a \in \mathrm{T}A$ to $a \in A$ and $\mathsf{d}a$ to 0;

n-fold pullback The n-fold pushout (in cAlg) of the projection along itself is the functor $\mathrm{T}_n\colon \mathsf{cAlg} \to \mathsf{cAlg}$ which sends an algebra A to the algebra T_nA which is the n-fold tensor product $\mathrm{T}A \otimes_A \ldots \otimes_A \mathrm{T}A$ of $\mathrm{T}A$ over A n-times;

sum morphism The sum morphism $s^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}_2$, regarded as a morphism of cAlg, is the natural transformation $s^{(\mathrm{T})}\colon \mathrm{T}A \to \mathrm{T}_2A$ which sends $a \in \mathrm{T}A$ to $a \in \mathrm{T}_2A$ and $\mathsf{d}a$ to $\mathsf{d}a \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}a$;

vertical lift The vertical lift $l^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}$, regarded as a morphism of cAlg, is the natural transformation $l^{(\mathrm{T})}\colon \mathrm{T}^2A \to \mathrm{T}A$ which sends $a \in \mathrm{T}^2A$ to $a \in \mathrm{T}A$, $\mathsf{d}a \in \mathrm{T}^2A$ and $\mathsf{d}'a \in \mathrm{T}^2A$ to $0 \in \mathrm{T}A$, and $\mathsf{d}'\mathsf{d}a \in \mathrm{T}^2A$ to $\mathsf{d}a \in \mathrm{T}A$;

canonical flip The canonical flip $c^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$, regarded as a morphism of cAlg, is the natural transformation $c^{(\mathrm{T})}\colon \mathrm{T}^2A \to \mathrm{T}^2A$ which sends $a \in \mathrm{T}^2A$ to $a \in \mathrm{T}^2A$, $\mathsf{d}a \in \mathrm{T}^2A$ to $\mathsf{d}'a \in \mathrm{T}^2A$, $\mathsf{d}'a \in \mathrm{T}^2A$ to $\mathsf{d}a \in \mathrm{T}^2A$, and $\mathsf{d}'\mathsf{d}a \in \mathrm{T}^2A$ to $\mathsf{d}'\mathsf{d}a \in \mathrm{T}^2A$;

negation The negation $n^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}$, regarded as a morphism of cAlg, is the natural transformation $n^{(\mathrm{T})}\colon \mathrm{T}A \to \mathrm{T}A$ which sends $a \in \mathrm{T}A$ to $a \in \mathrm{T}A$ and $\mathsf{d}a \in \mathrm{T}A$ to $-\mathsf{d}a \in \mathrm{T}A$.

This tangent category is precisely the tangent structure originally described by Cockett and Cruttwell in [12] and recently re-analyzed by Cruttwell and Lemay in [18]. In particular, when the base ring is $R = \mathbb{Z}$, this is also precisely the tangent category described in Example 2.14.

Example 3.70. Consider the operad $u\mathscr{A}ss$ described in Example 3.7. The module of Kähler differentials ΩA of an associative and unital algebra A is the noncommutative version of the usual notion of the module of Kähler differentials. Concretely, ΩA is precisely the kernel I of the multiplication map $\nu\colon A \otimes A \to A$ by I^2, i.e.

$\Omega A = I$, as described by Ginzburg in [25]. Equivalently, ΩA can be described as the A-bimodule of symbols $\mathsf{d}a$, for each $a \in A$, such that:

$$\mathsf{d}(ab) = a\mathsf{d}b + \mathsf{d}a \cdot b$$

for every $a, b \in A$. The functor Free_A sends an A-module, which is precisely a A-bimodule, M to the tensor algebra over A, i.e. $\mathsf{Tens}_A M$. Concretely, if M is a free module of A generated by a set of generators Γ, then $\mathsf{Tens}_A\Gamma$ is the algebra of noncommutative polynomials in the variables Γ and coefficients in A.

Putting together these two pieces of information we obtain a description of the geometric tangent structure associated with $u\mathscr{A}ss$:

tangent bundle functor The tangent bundle functor $\mathrm{T}\colon \mathsf{Alg}^{\mathsf{op}} \to \mathsf{Alg}^{\mathsf{op}}$, regarded as an endofunctor of Alg, sends a commutative and unital algebra A to $\mathsf{Tens}_A\Omega A$ and a morphism $f\colon A \to B$ to the morphism $\mathsf{Tens}_f\Omega f$ which sends each $a \in A$ and each $\mathsf{d}a \in \Omega A$ to $f(a)$ and $\mathsf{d}f(a)$, respectively;

projection The projection $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathsf{Geom}(u\mathscr{A}ss)} \Rightarrow \mathrm{T}$, regarded as a morphism of Alg, is the natural transformation $p^{(\mathrm{T})}\colon A \to \mathrm{T}A$ which sends $a \in A$ to $a \in \mathrm{T}A$;

zero morphism The zero morphism $z^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathsf{Geom}(u\mathscr{A}ss)}$, regarded as a morphism of Alg, is the natural transformation $z^{(\mathrm{T})}\colon \mathrm{T}A \to A$ which sends $a \in \mathrm{T}A$ to $a \in A$ and $\mathsf{d}a$ to 0;

n-fold pullback The n-fold pushout (in Alg) of the projection along itself is the functor $\mathrm{T}_n\colon \mathsf{Alg} \to \mathsf{Alg}$ which sends an algebra A to the algebra $\mathrm{T}_n A$ which is the n-fold free product $\mathrm{T}A *_A \cdots *_A \mathrm{T}A$ of $\mathrm{T}A$ over A n-times;

sum morphism The sum morphism $s^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}_2$, regarded as a morphism of Alg, is the natural transformation $s^{(\mathrm{T})}\colon \mathrm{T}A \to \mathrm{T}_2 A$ which sends $a \in \mathrm{T}A$ to $a \in \mathrm{T}_2 A$ and $\mathsf{d}a$ to $\mathsf{d}a * \mathbb{1} + \mathbb{1} * \mathsf{d}a$;

vertical lift The vertical lift $l^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}$, regarded as a morphism of Alg, is the natural transformation $l^{(\mathrm{T})}\colon \mathrm{T}^2 A \to \mathrm{T}A$ which sends $a \in \mathrm{T}^2 A$ to $a \in \mathrm{T}A$, $\mathsf{d}a \in \mathrm{T}^2 A$ and $\mathsf{d}'a \in \mathrm{T}^2 A$ to $0 \in \mathrm{T}A$, and $\mathsf{d}'\mathsf{d}a \in \mathrm{T}^2 A$ to $\mathsf{d}a \in \mathrm{T}A$;

canonical flip The canonical flip $c^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$, regarded as a morphism of Alg, is the natural transformation $c^{(\mathrm{T})}\colon \mathrm{T}^2A \to \mathrm{T}^2A$ which sends $a \in \mathrm{T}^2A$ to $a \in \mathrm{T}^2A$, $\mathsf{d}a \in \mathrm{T}^2A$ to $\mathsf{d}'a \in \mathrm{T}^2A$, $\mathsf{d}'a \in \mathrm{T}^2A$ to $\mathsf{d}a \in \mathrm{T}^2A$, and $\mathsf{d}'\mathsf{d}a \in \mathrm{T}^2A$ to $\mathsf{d}'\mathsf{d}a \in \mathrm{T}^2A$;

negation The negation $n^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}$, regarded as a morphism of Alg, is the natural transformation $n^{(\mathrm{T})}\colon \mathrm{T}A \to \mathrm{T}A$ which sends $a \in \mathrm{T}A$ to $a \in \mathrm{T}A$ and $\mathsf{d}a \in \mathrm{T}A$ to $-\mathsf{d}a \in \mathrm{T}A$.

This represents a completely new example of a tangent category and the first instance of non-commutative geometry, this being the category of non-commutative affine schemes, described using tangent category theory.

Example 3.71. Consider the operad $\mathscr{Lie}$ described in Example 3.9.

tangent bundle functor The tangent bundle functor $\mathrm{T}\colon \mathsf{LieAlg}^{\mathsf{op}} \to \mathsf{LieAlg}^{\mathsf{op}}$, regarded as an endofunctor of LieAlg, sends a Lie algebra $\mathfrak{g}$ to the Lie algebra generated by all $g \in \mathfrak{g}$ and all $\mathsf{d}g$, for each $g \in \mathfrak{g}$, such that:

$$[g,h]_{\mathrm{T}\mathfrak{g}} = [g,h]_{\mathfrak{g}}$$
$$\mathsf{d}(rg + sh) = r\mathsf{d}g + s\mathsf{d}h$$
$$\mathsf{d}[g,h] = [\mathsf{d}g,h] + [g,\mathsf{d}h]$$

projection The projection $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathsf{Geom}(\mathscr{Lie})} \Rightarrow \mathrm{T}$, regarded as a morphism of LieAlg, is the natural transformation $p^{(\mathrm{T})}\colon \mathfrak{g} \to \mathrm{T}\mathfrak{g}$ which sends $g \in \mathfrak{g}$ to $g \in \mathrm{T}\mathfrak{g}$;

zero morphism The zero morphism $z^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathsf{Geom}(\mathscr{Lie})}$, regarded as a morphism of LieAlg, is the natural transformation $z^{(\mathrm{T})}\colon \mathrm{T}\mathfrak{g} \to \mathfrak{g}$ which sends $g \in \mathrm{T}\mathfrak{g}$ to $g \in \mathfrak{g}$ and $\mathsf{d}g$ to 0;

n-fold pullback The n-fold pushout (in LieAlg) of the projection along itself is the functor $\mathrm{T}_n\colon \mathsf{LieAlg} \to \mathsf{LieAlg}$ which sends an algebra $\mathfrak{g}$ to the algebra $\mathrm{T}_n\mathfrak{g}$ which is generated by all $g \in \mathfrak{g}$ and all $\mathsf{d}_1g, \dots, \mathsf{d}_ng$, for each $g \in \mathfrak{g}$, such that:

$$[g,h]_{\mathrm{T}\mathfrak{g}} = [g,h]_{\mathfrak{g}}$$
$$\mathsf{d}_k(rg + sh) = r\mathsf{d}_kg + s\mathsf{d}_kh$$
$$\mathsf{d}_k[g,h] = [\mathsf{d}_kg,h] + [g,\mathsf{d}_kh]$$

for every $k = 1, \dots, n$;

sum morphism The sum morphism $s^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}_2$, regarded as a morphism of LieAlg, is the natural transformation $s^{(\mathrm{T})}\colon \mathrm{T}\mathfrak{g} \to \mathrm{T}_2\mathfrak{g}$ which sends $g \in \mathrm{T}\mathfrak{g}$ to $g \in \mathrm{T}_2\mathfrak{g}$ and $\mathsf{d}g$ to $\mathsf{d}_1 g + \mathsf{d}_2 g$;

vertical lift The vertical lift $l^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}$, regarded as a morphism of LieAlg, is the natural transformation $l^{(\mathrm{T})}\colon \mathrm{T}^2\mathfrak{g} \to \mathrm{T}\mathfrak{g}$ which sends $g \in \mathrm{T}^2\mathfrak{g}$ to $g \in \mathrm{T}\mathfrak{g}$, $\mathsf{d}g \in \mathrm{T}^2\mathfrak{g}$ and $\mathsf{d}'g \in \mathrm{T}^2\mathfrak{g}$ to $0 \in \mathrm{T}\mathfrak{g}$, and $\mathsf{d}'\mathsf{d}g \in \mathrm{T}^2\mathfrak{g}$ to $\mathsf{d}g \in \mathrm{T}\mathfrak{g}$;

canonical flip The canonical flip $c^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$, regarded as a morphism of LieAlg, is the natural transformation $c^{(\mathrm{T})}\colon \mathrm{T}^2\mathfrak{g} \to \mathrm{T}^2\mathfrak{g}$ which sends $g \in \mathrm{T}^2\mathfrak{g}$ to $g \in \mathrm{T}^2\mathfrak{g}$, $\mathsf{d}g \in \mathrm{T}^2\mathfrak{g}$ to $\mathsf{d}'g \in \mathrm{T}^2\mathfrak{g}$, $\mathsf{d}'g \in \mathrm{T}^2\mathfrak{g}$ to $\mathsf{d}g \in \mathrm{T}^2\mathfrak{g}$, and $\mathsf{d}'\mathsf{d}g \in \mathrm{T}^2\mathfrak{g}$ to $\mathsf{d}'\mathsf{d}g \in \mathrm{T}^2\mathfrak{g}$;

negation The negation $n^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}$, regarded as a morphism of LieAlg, is the natural transformation $n^{(\mathrm{T})}\colon \mathrm{T}\mathfrak{g} \to \mathrm{T}\mathfrak{g}$ which sends $g \in \mathrm{T}\mathfrak{g}$ to $g \in \mathrm{T}\mathfrak{g}$ and $\mathsf{d}g \in \mathrm{T}\mathfrak{g}$ to $-\mathsf{d}g \in \mathrm{T}\mathfrak{g}$.

3.6.1 Vector fields in the geometric tangent category of an operad

For an adjunctable tangent category $(\mathbb{X}, \mathbb{L})$ vector fields over an object M are in bijection with the vector fields of the adjoint tangent category $(\mathbb{X}^{\mathsf{op}}, \mathbb{T})$ over M. This comes directly from the adjunction $\mathrm{T} \dashv \mathrm{L}$ between the tangent bundle functors. Indeed, each morphism $v\colon M \to \mathrm{L}M$ corresponds to a morphism $\bar{v}\colon \mathrm{T}M \to M$ and vice versa. Less obviously, in the presence of negatives, the Lie bracket of vector fields (see Section 2.2.1 for this construction) is also preserved. Recall, that, the Lie bracket between two vector fields $v_1, v_2\colon M \to \mathrm{L}M$ is defined as follows:

$$[v_1, v_2]^{(\mathrm{L})} = \{v_1 \mathrm{L} v_2 - v_2 \mathrm{L}_1 c^{(\mathrm{L})}\}^{(\mathrm{L})}$$

Recall that, given a morphism $f\colon N \to \mathrm{L}^2 M$ for which $f\mathrm{L}p^{(\mathrm{L})} = fp^{(\mathrm{L})}p^{(\mathrm{L})}z^{(\mathrm{L})}$, $\{f\}^{(\mathrm{L})}\colon N \to \mathrm{L}M$ is the unique morphism such that $\left\langle \{f\}^{(\mathrm{L})}, fp^{(\mathrm{L})}\right\rangle \nu^{(\mathrm{L})} = f$ (see [12, Section 2.5]). To simplify notation, let's write $\{\{f\}\}^{(\mathrm{L})}$ for the morphism $\left\langle \{f\}^{(\mathrm{L})}, fp\right\rangle^{(\mathrm{L})}\colon N \to \mathrm{L}_2 M$.

Lemma 3.72. *Let $(\mathbb{X}, \mathbb{L})$ be an adjunctable tangent category with adjoint tangent category $(\mathbb{X}^{\mathsf{op}}, \mathbb{T})$ and let $f\colon N \to \mathrm{T}\mathrm{L}^2 M$ be a morphism, where M and N are objects of $\mathbb{X}$, for which $f\mathrm{L}p^{(\mathrm{L})} = fp^{(\mathrm{L})}p^{(\mathrm{L})}z^{(\mathrm{L})}$. The following formula holds:*

$$\{f^\sharp\}^{(\mathrm{T})} = \{f\}^{(\mathrm{L})\sharp}$$

where $f^\sharp\colon \mathrm{T}^2N \to M$ *is the mate of* f *along the adjunction* $(\eta, e\varepsilon)\colon \mathrm{T} \dashv \mathbb{L}$*, that is:*

$$f^\sharp\colon \mathrm{T}^2N \xrightarrow{\mathrm{T}^2 f} \mathrm{T}^2\mathbb{L}^2M \xrightarrow{\mathrm{T}\varepsilon\mathbb{L}} \mathrm{T}\mathbb{L}M \xrightarrow{\varepsilon} M$$

Proof. To prove this result, we need to show that the mate $\{\{f\}\}^{(\mathbb{L})^\sharp}\colon \mathrm{T}_2N \to M$ of the unique morphism $\{\{f\}\}^{(\mathbb{L})}: N \to \mathbb{L}_2M$ satisfies the same universality property as $\left\{\{f^\sharp\}\right\}^{(\mathrm{T})}$, that is:

$$\nu^{(\mathrm{T})}\{\{f\}\}^{(\mathbb{L})^\sharp} = f^\sharp$$

Recalling that $\nu^{(\mathrm{T})}$ is the mate of $\nu^{(\mathbb{L})}$ and using that mates preserve pasting diagrams (cf. [1, Proposition 2.2]), one concludes that $\nu^{(\mathrm{T})}\{\{f\}\}^{(\mathbb{L})^\sharp}$ is the mate of $\{\{f\}\}^{(\mathbb{L})}\,\nu^{(\mathbb{L})} = f$. Finally, from the definition of $\{\{f\}\}$ we conclude that $\{f^\sharp\}^{(\mathrm{T})} = \{f\}^{(\mathbb{L})^\sharp}$. ◻

Proposition 3.73. *Suppose* $(\mathbb{X}, \mathbb{L})$ *is an adjunctable tangent category. There is a bijective correspondence between vector fields of* $(\mathbb{X}, \mathbb{L})$ *over an object* $M \in \mathbb{X}$ *and vector fields of the adjoint tangent category* $(\mathbb{X}^{\mathsf{op}}, \mathbb{T})$ *over* M*. Moreover, if* $(\mathbb{X}, \mathbb{L})$ *has negatives (and thus so does* $(\mathbb{X}^{\mathsf{op}}, \mathbb{T})$*) this bijective correspondence preserves the Lie bracket.*

Proof. Let's start by defining the correspondence. Consider a vector field $v\colon M \to \mathbb{L}M$ over an object M in $(\mathbb{X}\mathbb{L})$. Since T is left adjoint of $\mathbb{L}$, v corresponds to a morphism $v^\sharp\colon \mathrm{T}M \to M$. Concretely, $v^\sharp$ is defined as follows:

$$v^\sharp\colon \mathrm{T}M \xrightarrow{\mathrm{T}v} \mathrm{T}\mathbb{L}M \xrightarrow{\varepsilon} M$$

where ε represents the counit of the adjunction $\mathrm{T} \dashv \mathbb{L}$. We need to show that $v^\sharp$, regarded as a morphism $v^\sharp\colon M \leftarrow \mathrm{T}M$ of $\mathbb{X}^{\mathsf{op}}$, is a section of the projection $p^{(\mathrm{T})}\colon \mathrm{T}M \leftarrow M$. Recall that $p^{(\mathrm{T})}$ (as an $\mathbb{X}$-morphism) is the mate of $p^{(\mathbb{L})}$. Using that mates preserve pasting diagrams whenever the mate of each diagram is well-defined (cf. [1, Proposition 2.2]), one concludes that $p^{(\mathrm{T})}v^{(\sharp)}$ (in $\mathbb{X}$) is the mate of $vp^{(\mathbb{L})} = \mathsf{id}_M$. So, we conclude that $v^{(\sharp)}$ is a vector field in $(\mathbb{X}^{\mathsf{op}}, \mathbb{T})$. Similarly, given a vector field $u\colon M \leftarrow \mathrm{T}M$ over M in $(\mathbb{X}^{\mathsf{op}}, \mathbb{T})$, employing the adjunction $\mathrm{T} \dashv \mathbb{L}$, we find a morphism $u^\flat\colon M \to \mathbb{L}M$, defined as follows:

$$u^\flat\colon M \xrightarrow{\eta} \mathbb{L}\mathrm{T}M \xrightarrow{\mathbb{L}u} \mathbb{L}M$$

Using a similar argument as the one employed before, it is straightforward to see that $u^\flat$ is a section of $p^{(\mathbb{L})}$. Moreover, it is also easy to see that $(v^\sharp)^\flat = v$ and that $(u^\flat)^\sharp = u$. Let's now focus on the Lie brackets. First, recall that the Lie brackets between two vector fields $u_1, u_2 \colon M \leftarrow \mathrm{T}M$ of $(\mathbb{X}^{\mathsf{op}}, \mathbb{T})$ are defined as follows:

$$[u_1, u_2] = \{\mathrm{T}u_2 u_1 - c^{(\mathrm{T})}\mathrm{T}u_1 u_2\}^{(\mathrm{T})}$$

where the composition is in $\mathbb{X}$. Consider now, two vector fields $v_1, v_2 \colon M \to \mathrm{L}M$ of $(\mathbb{X}, \mathbb{L})$ and consider $[v_1^\sharp, v_2^\sharp]$. Since the negation $n^{(\mathrm{T})}$, the sum morphism $s^{(\mathrm{T})}$, and canonical flip $c^{(\mathrm{T})}$ are mates along the adjunction $(\eta, \varepsilon) \colon \mathrm{T} \dashv \mathrm{L}$, employing that mates of pasting diagrams is the pasting of the mates, whenever the mate of each diagram is well-defined we conclude that $\varphi^\sharp \colon= \mathrm{T}v_2^\sharp v_1^\sharp - c^{\mathrm{T}}\mathrm{T}v_1^\sharp v_2^\sharp$ is the mate of $\varphi \colon= v_1 \mathrm{L} v_2 - v_2 \mathrm{L} v_1 c^{(\mathrm{L})}$. Finally, employing Lemma 3.72, we conclude:

$$[v_1^\sharp, v_2^\sharp]^{(\mathrm{T})} = \{\varphi^\sharp\}^{(\mathrm{T})} = \{\varphi\}^{(\mathrm{L})^\sharp} = [v_1, v_2]^{(\mathrm{L})^\sharp}$$

This concludes the proof. $\square$

Theorem 3.74. *For an operad $\mathscr{P}$ the Lie algebra $\mathsf{VField}(\mathsf{Geom}(\mathscr{P}); A)$ of vector fields in the geometric tangent category $\mathsf{Geom}(\mathscr{P})$ over an operadic affine scheme A is isomorphic to the Lie algebra $\mathsf{Der}_{\mathsf{S}_{\mathscr{P}}}(A)$ of $\mathsf{S}_{\mathscr{P}}$-derivations over A. Moreover, when $\mathscr{P}$ is algebraic, these two Lie algebras are also isomorphic to the Lie algebra $\mathsf{Der}_{\mathscr{P}}(A)$ of $\mathscr{P}$-derivations.*

3.6.2 The functoriality of the geometric construction

In this section, we want to address the question of how the adjunctions induced by morphisms of operads between the corresponding categories of algebras interact with the geometric tangent structures. In Section 3.5.2 we fully characterized how these adjunctions interact with the algebraic tangent structures. We also characterized the geometric tangent category of an operad as the adjoint tangent category of the algebraic one. So, it is natural to pose a more general question: does the operation $(-)^{\mathsf{op}}$ which sends an adjunctable tangent category to its adjoint extend to a functor which sends a tangent morphism to a tangent morphism between the adjoint tangent categories? Solving this problem will answer the first question.

The crucial observation is the following. Suppose $(\mathbb{X}, \mathbb{L})$ and $(\mathbb{X}', \mathbb{L}')$ are two adjunctable tangent categories whose adjoint tangent categories are respectively

$(\mathbb{X}^{\mathsf{op}},\mathbb{T})$ and $(\mathbb{X}'^{\mathsf{op}},\mathbb{T}')$. Consider a lax tangent morphism $(F,\alpha)\colon(\mathbb{X},\mathbb{L})\to(\mathbb{X}',\mathbb{L}')$, whose distributive law is $\alpha\colon F\circ\mathrm{L}\Rightarrow\mathrm{L}'\circ F$. Then we can define α^{op} as follows:

$$\alpha^{\mathsf{op}}\colon \mathrm{T}'\circ F\xrightarrow{\mathrm{T}'F\theta}\mathrm{T}'\circ F\circ\mathrm{L}\circ\mathrm{T}\xrightarrow{\mathrm{T}'\alpha\mathrm{T}}\mathrm{T}'\circ\mathrm{L}'\circ F\circ\mathrm{T}\xrightarrow{\tau'F\mathrm{T}}F\circ\mathrm{T}$$

where $(\theta,\tau)\colon\mathrm{T}\dashv\mathrm{L}$ and $(\theta',\tau')\colon\mathrm{T}'\dashv\mathrm{L}'$. We claim that $(F^{\mathsf{op}},\alpha^{\mathsf{op}})\colon(\mathbb{X}^{\mathsf{op}},\mathbb{T})\to(\mathbb{X}'^{\mathsf{op}},\mathbb{T}')$ is also a lax tangent morphism.

In the following, let's denote by $\mathsf{adjTngCat}$ the 2-category of adjunctable tangent categories, lax tangent morphisms, and natural transformations compatible with the distributive laws of the tangent morphisms.

Proposition 3.75. *The operation which takes an adjunctable tangent category* $(\mathbb{X},\mathbb{L})$ *to its associated adjoint tangent category* $(\mathbb{X}^{\mathsf{op}},\mathbb{T})$ *extends to a pseudofunctor* $(-)^{\mathsf{op}}\colon\mathsf{adjTngCat}\to\mathsf{adjTngCat}$*, which equips the 2-category* $\mathsf{adjTngCat}$ *with an endofunctor together with a natural isomorphism* $(-)^{\mathsf{op}}\circ(-)^{\mathsf{op}}\cong\mathsf{id}_{\mathsf{adjTngCat}}$.

Proof. By definition, the natural transformations (i.e. projection etcetera) of the adjoint tangent structure $\mathbb{T}$ of a tangent structure $\mathbb{L}$ are mates along the adjunction $(\theta,\tau)\colon\mathrm{T}\dashv\mathrm{L}$ between the tangent bundle functors of the corresponding natural transformations of $\mathbb{L}$. Thanks to [1, Proposition 2.2], the mate of a pasting diagram is the pasting diagram of the mates, as long as the mate of each morphism of the diagram is well-defined. Therefore, given a lax tangent morphism (F,α) the distributive law α^{op} is compatible with the tangent structures and thus $(F^{\mathsf{op}},\alpha^{\mathsf{op}})$ is a lax tangent morphism between the corresponding adjoint tangent categories. To prove that $(-)^{\mathsf{op}}$ is a pseudofunctor notice first that, given three adjunctable tangent categories $(\mathbb{X},\mathbb{L}),(\mathbb{X}',\mathbb{L}')$, and $(\mathbb{X}'',\mathbb{L}'')$ with adjoint tangent categories $(\mathbb{X}^{\mathsf{op}},\mathbb{T}),(\mathbb{X}'^{\mathsf{op}},\mathbb{T}')$ and $(\mathbb{X}''^{\mathsf{op}},\mathbb{T}'')$, respectively, and two lax tangent morphisms $(F,\alpha)\colon(\mathbb{X},\mathbb{L})\to(\mathbb{X}',\mathbb{L}')$ and $(G,\beta)\colon(\mathbb{X}',\mathbb{L}')\to(\mathbb{X}'',\mathbb{L}'')$, the composition of $(F^{\mathsf{op}},\alpha^{\mathsf{op}})$ with $(G^{\mathsf{op}},\beta^{\mathsf{op}})$ is $(G^{\mathsf{op}}\circ F^{\mathsf{op}},G\alpha^{\mathsf{op}}\circ\beta^{\mathsf{op}}_F)$. This must be compared with the opposite of the composition $(G\circ F,\beta_F\circ G\alpha)$. However, for the pasting diagram property of mates, these are the same lax tangent morphism. Similarly, we can argue that $(\mathsf{id}^{\mathsf{op}}_{\mathbb{X}},\mathsf{id}^{\mathsf{op}}_{\mathrm{L}})$ corresponds precisely to $(\mathsf{id}_{\mathbb{X}^{\mathsf{op}}},\mathsf{id}_{\mathrm{T}})$. Finally, notice that if $(\mathbb{X},\mathbb{L})$ is adjunctable, then so is its adjoint tangent category $(\mathbb{X}^{\mathsf{op}},\mathbb{T})$ and its adjoint is (isomorphic to) $(\mathbb{X},\mathbb{L})$. $\square$

Remark 3.76. We point out that $(-)^{\mathsf{op}}$ defined by Proposition 3.75 is only a pseudofunctor and not a strict functor because the choice of a left adjoint for the tangent bundle functor $\mathbb{L}$ is only unique up to a unique isomorphism. This implies that associativity and unitality are only defined up to a unique isomorphism, which defines the associator and the left and the right unitors of $(-)^{\mathsf{op}}$.

Remark 3.77. One could hope that a similar endofunctor $(-)^{\mathsf{op}}$ could also occur in the 2-category $\mathsf{adjTngCat}_{\mathsf{co}}$ of adjunctable tangent categories, colax tangent morphisms, and corresponding tangent natural transformations. However, this is not the case. The reason is that mates of the colax distributive laws along the adjunctions of the tangent bundle functors are simply not well-defined. This breaking of symmetry plays a crucial role in understanding the differences between noncommutative algebraic geometry and the geometry of affine schemes. We will come back to this point later in Example 3.81.

Before proving the functoriality of the operation which takes an operad to its geometric tangent category, we notice an interesting fact.

Lemma 3.78. *Consider a strong tangent morphism $(G, \alpha)\colon (\mathbb{X}', \mathbb{L}') \to (\mathbb{X}, \mathbb{L})$ between two adjunctable tangent categories. Suppose also that the functor G has a left adjoint $F \dashv G$ and write $\beta := \alpha^{-1}\colon \mathbb{L} \circ G \Rightarrow G \circ \mathbb{L}'$ for the inverse of $\alpha\colon G \circ \mathbb{L}' \Rightarrow \mathbb{L} \circ G$. Then the corresponding tangent morphism $(F^{\mathsf{op}}, (\beta_!)^{\mathsf{op}})\colon (\mathbb{X}^{\mathsf{op}}, \mathbb{T}) \to (\mathbb{X}'^{\mathsf{op}}, \mathbb{T}')$ over the left adjoint F and between the adjoint tangent categories is also strong.*

Proof. By Proposition 3.46, the mate of β along the adjunction $F \dashv G$ defines a lax tangent morphism $(F, \beta_!)\colon (\mathbb{X}, \mathbb{L}) \to (\mathbb{X}', \mathbb{L}')$, where $\beta_!\colon F \circ \mathbb{L} \Rightarrow \mathbb{L}' \circ F$.

By Proposition 3.75, the mate of the distributive law α along the adjunctions between the tangent bundle functors and their left adjoint defines a lax tangent morphism $(G^{\mathsf{op}}, \alpha^{\mathsf{op}})\colon (\mathbb{X}'^{\mathsf{op}}, \mathbb{T}') \to (\mathbb{X}^{\mathsf{op}}, \mathbb{T})$, so that, as an $\mathbb{X}$-morphism, $\alpha^{\mathsf{op}}\colon \mathrm{T} \circ G \Rightarrow G \circ \mathrm{T}'$. Similarly, $\beta_!$ defines, again by mating, a lax tangent morphism $(F^{\mathsf{op}}, (\beta_!)^{\mathsf{op}})\colon (\mathbb{X}^{\mathsf{op}}, \mathbb{T}) \to (\mathbb{X}'^{\mathsf{op}}, \mathbb{T}')$, so that, as a $\mathbb{X}'$-morphism, $(\beta_!)^{\mathsf{op}}\colon \mathrm{T}' \circ F \Rightarrow F \circ \mathrm{T}$. Interestingly, α^{op} admits a second mate along the adjunction $(\eta, \varepsilon)\colon F \dashv G$:

$$(\alpha^{\mathsf{op}})_!\colon F \circ \mathrm{T} \xrightarrow{F\mathrm{T}\eta} F \circ \mathrm{T} \circ G \circ F \xrightarrow{F\alpha^{\mathsf{op}}F} F \circ G \circ \mathrm{T}' \circ F \xrightarrow{\varepsilon \mathrm{T}' F} \mathrm{T}' \circ F$$

regarded as a morphism in $\mathbb{X}'$. Thus, we also obtain a colax tangent morphism $(F^{\mathsf{op}}, (\alpha^{\mathsf{op}})_!)\colon (\mathbb{X}^{\mathsf{op}}, \mathbb{T}) \nrightarrow (\mathbb{X}'^{\mathsf{op}}, \mathbb{T}')$. To prove that $(\alpha^{\mathsf{op}})_!$ is the inverse of $(\beta_!)^{\mathsf{op}}$, consider the following diagram:

$$
\begin{tikzcd}
F\circ\mathrm{T} \arrow[r, "F\mathrm{T}\eta"] \arrow[d, "F\mathrm{T}\theta"'] & F\circ\mathrm{T}\circ G\circ F \arrow[r, "F\mathrm{T}G\theta'F"] \arrow[d, "F\mathrm{T}GF\theta"] & F\circ\mathrm{T}\circ G\circ\bot'\circ\mathrm{T}'\circ F \arrow[r, "F\mathrm{T}\alpha\mathrm{T}'F"] \arrow[d, "F\mathrm{T}G\bot'\mathrm{T}'F\theta"] & F\circ\mathrm{T}\circ\bot\circ G\circ\mathrm{T}'\circ F \arrow[r, "F\tau G\mathrm{T}'F"] & F\circ G\circ\mathrm{T}'\circ F \arrow[r, "\varepsilon\mathrm{T}'F"] & \mathrm{T}'\circ F \arrow[d, "\mathrm{T}'F\theta"] \\
F\circ\mathrm{T}\circ\bot\circ\mathrm{T} \arrow[d, "F\mathrm{T}\bot\eta\mathrm{T}"'] \arrow[r, phantom, "\mathsf{Nat}"] & F\circ\mathrm{T}\circ G\circ F\circ\bot\circ\mathrm{T} \arrow[d, "F\mathrm{T}GF\bot\eta\mathrm{T}"] & F\circ\mathrm{T}\circ G\circ\bot'\circ\mathrm{T}'\circ F\circ\bot\circ\mathrm{T} \arrow[d, "F\mathrm{T}G\bot'\mathrm{T}'F\bot\eta\mathrm{T}"] & & & \mathrm{T}'\circ F\circ\bot\circ\mathrm{T} \arrow[d, "\mathrm{T}'F\bot\eta\mathrm{T}"] \\
F\circ\mathrm{T}\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[d, "F\mathrm{T}\beta F\mathrm{T}"'] & F\circ\mathrm{T}\circ G\circ F\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[d, "F\mathrm{T}GF\beta F\mathrm{T}"] \arrow[r, phantom, "\mathsf{Nat}"] & F\circ\mathrm{T}\circ G\circ\bot'\circ\mathrm{T}'\circ F\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[d, "F\mathrm{T}G\bot'\mathrm{T}'F\beta F\mathrm{T}"] \arrow[rrr, phantom, "\mathsf{Nat}"] & & & \mathrm{T}'\circ F\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[d, "\mathrm{T}'F\beta F\mathrm{T}"] \\
F\circ\mathrm{T}\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[r, "F\mathrm{T}\eta G\bot'F\mathrm{T}"] \arrow[dr, equal] & F\circ\mathrm{T}\circ G\circ F\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[d, "F\mathrm{T}G\varepsilon\bot'F\mathrm{T}"] \arrow[dl, phantom, "\Delta"] & F\circ\mathrm{T}\circ G\circ\bot'\circ\mathrm{T}'\circ F\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[d, "F\mathrm{T}G\bot'\mathrm{T}'\varepsilon\bot'F\mathrm{T}"] & & & \mathrm{T}'\circ F\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[d, "\mathrm{T}'\varepsilon\bot'F\mathrm{T}"] \\
& F\circ\mathrm{T}\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[r, "F\mathrm{T}G\theta'\bot'F\mathrm{T}"] \arrow[dr, equal] & F\circ\mathrm{T}\circ G\circ\bot'\circ\mathrm{T}'\circ\bot'\circ F\circ\mathrm{T} \arrow[d, "F\mathrm{T}G\bot'\tau'F\mathrm{T}"] \arrow[dl, phantom, "\Delta"] & & & \mathrm{T}'\circ\bot'\circ F\circ\mathrm{T} \arrow[d, "\tau'F\mathrm{T}"] \\
& & F\circ\mathrm{T}\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[r, "F\mathrm{T}\alpha F\mathrm{T}"'] & F\circ\mathrm{T}\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[r, "F\tau GF\mathrm{T}"'] & F\circ G\circ F\circ\mathrm{T} \arrow[r, "\varepsilon F\mathrm{T}"'] & F\circ T
\end{tikzcd}
$$

where $(\eta, \varepsilon)\colon F \dashv G$, $(\theta, \tau)\colon \mathrm{T} \dashv \bot$ and $(\theta', \tau')\colon \mathrm{T}' \dashv \bot'$. This shows that the following diagram commutes:

$$
\begin{tikzcd}
F\circ\mathrm{T} \arrow[rrr, "(\alpha^{\mathsf{op}})_!"] \arrow[d, "F\mathrm{T}\theta"'] & & & \mathrm{T}'\circ F \arrow[ddd, "(\beta_!)^{\mathsf{op}}"] \\
F\circ\mathrm{T}\circ\bot\circ\mathrm{T} \arrow[d, "F\mathrm{T}\bot\eta\mathrm{T}"'] & & & \\
F\circ\mathrm{T}\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[d, "F\mathrm{T}\beta F\mathrm{T}"'] & & & \\
F\circ\mathrm{T}\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[r, "F\mathrm{T}\alpha F\mathrm{T}"'] & F\circ\mathrm{T}\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[r, "F\tau GF\mathrm{T}"'] & F\circ G\circ F\circ\mathrm{T} \arrow[r, "\varepsilon F\mathrm{T}"'] & F\circ\mathrm{T}
\end{tikzcd}
$$

However:

$$
\begin{tikzcd}
F\circ\mathrm{T} \arrow[rrrr, equal] \arrow[d, "F\mathrm{T}\theta"'] \arrow[drrr, phantom, "\Delta"] & & & & F\circ\mathrm{T} \arrow[ddd, equal] \arrow[dl, equal] \\
F\circ\mathrm{T}\circ\bot\circ\mathrm{T} \arrow[rrr, "F\tau\mathrm{T}"] \arrow[d, "F\mathrm{T}\bot\eta\mathrm{T}"'] \arrow[drrr, phantom, "\mathsf{Nat}"] & & & F\circ\mathrm{T} \arrow[dd, "F\eta\mathrm{T}"] \arrow[dr, phantom, "\Delta"] & \\
F\circ\mathrm{T}\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[r, equal] \arrow[d, "F\mathrm{T}\beta F\mathrm{T}"'] \arrow[dr, phantom, "\beta=\alpha^{-1}"] & F\circ\mathrm{T}\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[d, equal] & & & \\
F\circ\mathrm{T}\circ G\circ\bot'\circ F\circ\mathrm{T} \arrow[r, "F\mathrm{T}\alpha F\mathrm{T}"'] & F\circ\mathrm{T}\circ\bot\circ G\circ F\circ\mathrm{T} \arrow[rr, "F\tau GF\mathrm{T}"'] & & F\circ G\circ F\circ\mathrm{T} \arrow[r, "\varepsilon F\mathrm{T}"'] & F\circ\mathrm{T}
\end{tikzcd}
$$

We just proved that $(\beta_!)^{\mathsf{op}} \circ (\alpha^{\mathsf{op}})_! = \mathsf{id}_{F\mathrm{T}}$. Similarly, one can prove the converse and conclude that $(\alpha^{\mathsf{op}})_!$ is the inverse of $(\beta_!)^{\mathsf{op}}$, as expected. $\square$

Remark 3.79. Given a pair of conjoints $(F, \beta_!) \dashv (G, \alpha)$ in the double category of tangent categories where (G, α) is a strong tangent morphism, Lemma 3.78 establishes that the pseudofunctor $(-)^{\mathsf{op}}$ maps $(F, \beta_!) \dashv (G, \alpha)$ to another pair of conjoints $(G^{\mathsf{op}}, \alpha^{\mathsf{op}}) \dashv (F^{\mathsf{op}}, (\beta_!)^{\mathsf{op}})$ and that $(F^{\mathsf{op}}, (\beta_!)^{\mathsf{op}})$ is also a strong tangent morphism. However, if (G, α) is strict this does not imply that $(F^{\mathsf{op}}, (\beta_!)^{\mathsf{op}})$ is strict as well.

In the following diagram, we represent the proof of Lemma 3.78.

$$\begin{tikzcd}
& \beta \arrow[r, "F\dashv G"] \arrow[dl, leftrightarrow, "\text{inverses}"'] & \beta_! \arrow[d, "(-)^{\mathsf{op}}"] \\
\alpha \arrow[d, "(-)^{\mathsf{op}}"'] & & (\beta_!)^{\mathsf{op}} \\
\alpha^{\mathsf{op}} \arrow[r, "F\dashv G"'] & (\alpha^{\mathsf{op}})_! \arrow[ur, leftrightarrow, dashed, "\text{inverses}"'] &
\end{tikzcd}$$

Starting from β, which is the inverse of the strong distributive law α, by moving to the right, i.e. by mating along the adjunction $F \dashv G$, we obtain a lax distributive $\beta_!$, which, as noticed in Remark 3.52, in general, is not invertible. By moving down from $\beta_!$, by applying the pseudofunctor $(-)^{\mathsf{op}}$, we obtain a lax distributive law $(\beta_!)^{\mathsf{op}}$. Similarly, by starting from α and moving down, i.e. applying $(-)^{\mathsf{op}}$, we obtain a lax distributive law α^{op}, which, as mentioned in Remark 3.76, in general, is not invertible. Finally, by moving from α^{op} to the right, i.e. by mating along the adjunction $F \dashv G$, we obtain a colax distributive law $(\alpha^{\mathsf{op}})_!$ which turns out to be the inverse of $(\beta_!)^{\mathsf{op}}$.

We can now prove the functoriality of the operation which takes an operad to its associated geometric tangent category. Similarly, as for the algebraic counterpart of this construction, this operation extends to two functors, one mapping operad morphisms φ to a lax tangent morphism whose underlying functor is $(\varphi^*)^{\mathsf{op}}$ and the second to a strong tangent morphism whose underlying functor is $\varphi_!^{\mathsf{op}}$.

Proposition 3.80. *The operation which takes an operad $\mathscr{P}$ to its associated geometric tangent category* $\mathsf{Geom}(\mathscr{P})$ *extends to a contravariant pseudofunctor* $\mathsf{Geom}^*\colon \mathsf{Operad}^{\mathsf{op}} \to \mathsf{TngCat}$ *which sends a morphism of operads $\varphi\colon \mathscr{P} \to \mathscr{P}'$ to the lax tangent morphism*

$(\varphi^*, \alpha^*)\colon \mathsf{Geom}(\mathscr{P}') \to \mathsf{Geom}(\mathscr{P})$*, where* α^* *is defined as follows:*

$$\alpha^*\colon \mathrm{T}\circ\varphi^* \xrightarrow{\mathrm{T}\varphi^*\theta'} \mathrm{T}\circ\varphi^*\circ\mathbb{L}'\circ\mathrm{T}' = \mathrm{T}\circ\mathbb{L}\circ\varphi^*\circ\mathrm{T}' \xrightarrow{\tau\varphi^*\mathrm{T}'} \varphi^*\circ\mathrm{T}'$$

where $(\theta,\tau)\colon \mathrm{T} \dashv \mathbb{L}$ *and* $(\theta',\tau')\colon \mathrm{T}' \dashv \mathbb{L}'$ *and we adopted the notation* $\mathrm{T} := \mathrm{T}^{(\mathscr{P})}$*,* $\mathbb{L} := \mathbb{L}^{(\mathscr{P})}$*,* $\mathrm{T}' := \mathrm{T}^{(\mathscr{P}')}$*, and* $\mathbb{L}' := \mathbb{L}^{(\mathscr{P}')}$*.*

Moreover, the same operation extends also to a covariant pseudofunctor $\mathsf{Geom}_!\colon \mathsf{Operad} \to \mathsf{TngCat}_{\cong}$ *which sends a morphism of operads* $\varphi\colon \mathscr{P} \to \mathscr{P}'$ *to the strong tangent morphism* $(\varphi_!, \alpha_!)\colon \mathsf{Geom}(\mathscr{P}) \to \mathsf{Geom}(\mathscr{P}')$*, where* $\alpha_!$ *is defined as follows:*

$$\alpha_!\colon \mathrm{T}'\circ\varphi_! \xrightarrow{\mathrm{T}'\varphi_!\theta} \mathrm{T}'\circ\varphi_!\circ\mathbb{L}\circ\mathrm{T} \xrightarrow{\mathrm{T}'\beta_!\mathrm{T}} \mathrm{T}'\circ\mathbb{L}'\circ\varphi_!\circ\mathrm{T} \xrightarrow{\tau'\varphi_!\mathrm{T}} \varphi_!\circ\mathrm{T}$$

where $\beta_!$ *is defined as in Proposition 3.48.*

Concretely, given a morphism $\varphi\colon \mathscr{P} \to \mathscr{P}'$ and a $\mathscr{P}'$-algebra B, $\varphi^*(\mathrm{T}'B)$ is a $\mathscr{P}$-algebra generated by all $b \in B$ and by symbols $\mathrm{d}'b$, for $b \in B$, satisfying suitable relations. On the other hand, $\mathrm{T}(\varphi^*B)$ is generated by all $b \in B$ and by symbols $\mathrm{d}b$, for $b \in B$, satisfying suitable relations. Thus, the distributive law $\alpha^*\colon \mathrm{T}(\varphi^*B) \to \varphi^*(\mathrm{T}'B)$ associated with φ^* sends each b to b and each $\mathrm{d}b$ to $\mathrm{d}'b$.

Similarly, given a $\mathscr{P}$-algebra A, $\varphi_!(\mathrm{T}A)$ is generated by all $a \in A$ and by $\mathrm{d}a$ for $a \in A$, satisfying suitable relations. On the other hand, $\mathrm{T}'(\varphi_!A)$ is generated by all $a \in A$ and by $\mathrm{d}'a$, for $a \in A$, satisfying suitable relations. Thus, the distributive law $\alpha_!\colon \varphi_!(\mathrm{T}A) \to \mathrm{T}'(\varphi_!A)$ sends each a to a and each $\mathrm{d}a$ to $\mathrm{d}'a$.

Example 3.81. In Example 3.51 we showed how the canonical morphism of operads $\varphi\colon u\mathscr{A}ss \to u\mathscr{C}om$ is mapped by the functors Alg^* and $\mathsf{Alg}_!$. The functor Geom^* maps φ to the lax tangent morphism defined over the pullback functor φ^*. Interestingly, this lax tangent morphism is not strong, i.e. the distributive law $\mathrm{T}^{(u\mathscr{A}ss)}\circ\varphi^* \to \varphi^*\circ\mathrm{T}^{(u\mathscr{C}om)}$ (as a $u\mathscr{A}ss$-algebra morphism) is not an isomorphism.

To prove that, notice that the module of Kähler differentials $\Omega^{(u\mathscr{C}om)}A$ of a commutative algebra A is given by quotienting the ideal $I := \ker(\nu\colon A\otimes_R A \to A)$, where ν represents the multiplication of A, by I^2, i.e. $\Omega A = I/I^2$. If B is an associative algebra, the corresponding module of Kähler differentials $\Omega^{(u\mathscr{A}ss)}B$ is simply given by the ideal $I := \ker(\mu\colon B\otimes_R B \to B)$, where μ represents the multiplication of B (see [25] for a detailed description of both the modules of Kähler differentials in the commutative and in the associative case).

Thus, for a commutative algebra A, there is a natural quotient map $\Omega^{(u\mathscr{A}ss)}\pi^*(A) = I \to I/I^2 = \Omega^{(u\mathscr{C}om)}A$. The distributive law is induced precisely by this quotient map since it maps the symbols $\mathsf{d}^{(u\mathscr{A}ss)}a$ to $\mathsf{d}^{(u\mathscr{C}om)}a$. If the distributive law was an isomorphism such a comparison map between the modules of Kähler differentials would be invertible, which it clearly is not. We note that a similar argument was used by Ginzburg in [25] to distinguish between "noncommutative geometry *in the small*, and noncommutative geometry *in the large*", meaning that the former "is a generalization of the conventional 'commutative' algebraic geometry to the non-commutative world". The latter instead "is not a generalization of commutative theory. The world of noncommutative geometry 'in the large' does not contain commutative world as a special case, but is only similar, parallel, to it." ([25, Introduction]).

Finally, the functor $\mathbb{T}_!$ maps the morphism of operads φ to the strong tangent morphism whose underlying functor is the (opposite of the) abelianization functor $\varphi_!$. The corresponding distributive law $\mathrm{T}^{(u\mathscr{C}om)} \circ \varphi_! \Rightarrow \varphi_! \circ \mathrm{T}^{(u\mathscr{A}ss)}$ (as a commutative algebra morphism) is the commutative algebra morphism:

$$\mathrm{T}^{(u\mathscr{C}om)}\,(A/[A,A]) \to \mathrm{T}^{(u\mathscr{A}ss)}A/\left[\mathrm{T}^{(u\mathscr{A}ss)}A, \mathrm{T}^{(u\mathscr{A}ss)}A\right]$$

which sends the generators $[a]$ and $\mathsf{d}^{(u\mathscr{C}om)}[a]$ to $[a]$ and $[\mathsf{d}^{(u\mathscr{A}ss)}a]$, respectively, where we used the square brackets to indicate the left coset given by the commutator and an element of the associative algebra A. It is not hard to see that the algebra morphism $\mathrm{T}^{(u\mathscr{A}ss)}A \to \mathrm{T}^{(u\mathscr{C}om)}(A/[A,A])$ which sends each a to $[a]$ and $\mathsf{d}^{(u\mathscr{A}ss)}a$ to $\mathsf{d}^{(u\mathscr{C}om)}[a]$ is well-defined and provides an inverse for the distributive law.

Example 3.82. In Example 3.51 we showed how the canonical morphism of operads $\varphi\colon \mathscr{L}ie \to u\mathscr{A}ss$ is mapped by the functors Alg^* and $\mathsf{Alg}_!$. The functor Geom^* maps φ to the lax tangent morphism whose underlying functor is (the opposite of) φ^*. In order to understand the distributive law $\mathrm{T}^{(\mathscr{L}ie)} \circ \varphi^* \Rightarrow \varphi^* \circ \mathrm{T}^{(u\mathscr{A}ss)}$ (as an associative algebra morphism), let's first take a closer look at $\mathrm{T}^{(\mathscr{L}ie)}(\varphi^*(A))$ and $\varphi^*(\mathrm{T}^{(u\mathscr{A}ss)}(A))$ for an associative algebra A. The former one is the Lie algebra generated by $a \in A$ and by symbols $\mathsf{d}^{(\mathscr{L}ie)}a$ for each $a \in A$, satisfying the following relations:

$$[a,b] = ab - ba$$

$$\mathsf{d}^{(\mathscr{L}ie)}([a,b]) = [\mathsf{d}^{(\mathscr{L}ie)}a, b] + [a, \mathsf{d}^{(\mathscr{L}ie)}b]$$

The second algebra is generated by $a \in A$ and by symbols $\mathsf{d}^{(u\mathscr{A}ss)}a$ for each $a \in A$, satisfying the following relations:

$$\begin{aligned}
&[a,b] = ab - ba\\
&\mathsf{d}^{(u\mathscr{A}ss)}(ab) = \mathsf{d}^{(u\mathscr{A}ss)}a \cdot b + a\mathsf{d}^{(u\mathscr{A}ss)}b\\
&[a, \mathsf{d}^{(u\mathscr{A}ss)}b] = a\mathsf{d}^{(u\mathscr{A}ss)}b - \mathsf{d}^{(u\mathscr{A}ss)}b \cdot a\\
&[\mathsf{d}^{(u\mathscr{A}ss)}a, \mathsf{d}^{(u\mathscr{A}ss)}b] = \mathsf{d}^{(u\mathscr{A}ss)}a\mathsf{d}^{(u\mathscr{A}ss)}b - \mathsf{d}^{(u\mathscr{A}ss)}b\mathsf{d}^{(u\mathscr{A}ss)}a
\end{aligned}$$

Note that the relations of the former one are implied by the relations of the latter. The canonical quotient map $\mathrm{T}^{(\mathscr{L}ie)}(\varphi^*(A)) \to \varphi^*(\mathrm{T}^{(u\mathscr{A}ss)}(A))$ corresponds to the distributive law. Note that such a map is not an isomorphism.

Finally, the functor $\mathsf{Geom}_!$ maps φ to the lax tangent morphism whose underlying functor is the (opposite of the) universal enveloping algebra functor $\varphi_!$. To understand the distributive law $\mathrm{T}^{(u\mathscr{A}ss)} \circ \varphi_! \Rightarrow \varphi_! \circ \mathrm{T}^{(\mathscr{L}ie)}$ (as an associative algebra morphism), we first take a closer look at $\mathrm{T}^{(u\mathscr{A}ss)}(\varphi_!(\mathfrak{g}))$ and $\varphi_!(\mathrm{T}^{(\mathscr{L}ie)}(\mathfrak{g}))$ for a Lie algebra $\mathfrak{g}$. The former is the associative algebra generated by all $g \in \mathfrak{g}$ and by symbols $\mathsf{d}^{(u\mathscr{A}ss)}g$ for each $g \in \mathfrak{g}$ and satisfying the relations:

$$\begin{aligned}
&gh - hg = [g,h]\\
&\mathsf{d}^{(u\mathscr{A}ss)}(gh) = \mathsf{d}^{(u\mathscr{A}ss)}g \cdot h + g\mathsf{d}^{(u\mathscr{A}ss)}h
\end{aligned}$$

The latter is the associative algebra generated by $g \in \mathfrak{g}$ and by symbols $\mathsf{d}^{(\mathscr{L}ie)}g$ for each $g \in \mathfrak{g}$, satisfying the relations:

$$\begin{aligned}
&gh - hg = [g,h]\\
&\mathsf{d}^{(\mathscr{L}ie)}g \cdot h - h\mathsf{d}^{(\mathscr{L}ie)}g = [\mathsf{d}^{(\mathscr{L}ie)}g, h]\\
&g\mathsf{d}^{(\mathscr{L}ie)}h - \mathsf{d}^{(\mathscr{L}ie)}h \cdot g = [g, \mathsf{d}^{(\mathscr{L}ie)}h]\\
&\mathsf{d}^{(\mathscr{L}ie)}g \cdot \mathsf{d}^{(\mathscr{L}ie)}h - \mathsf{d}^{(\mathscr{L}ie)}h \cdot \mathsf{d}^{(\mathscr{L}ie)}g = [\mathsf{d}^{(\mathscr{L}ie)}g, \mathsf{d}^{(\mathscr{L}ie)}h]\\
&\mathsf{d}^{(\mathscr{L}ie)}[g,h] = [\mathsf{d}^{(\mathscr{L}ie)}g, h] + [h, \mathsf{d}^{(\mathscr{L}ie)}g]
\end{aligned}$$

Because the first set of relations implies the latter, this allows us to define a morphism of associative algebras $\varphi_!(\mathrm{T}^{(\mathscr{L}ie)}(\mathfrak{g})) \to \mathrm{T}^{(u\mathscr{A}ss)}(\varphi_!(\mathfrak{g}))$, which corresponds to the (inverse of the) distributive law. Thanks to Lemma 3.78, this morphism is an isomorphism.

Chapter 4

The differential bundles of the geometric tangent category of an operad

In Section 2.2.5 we recalled an important construction of tangent category theory: differential bundles. As mentioned earlier, differential bundles represent the analogs of smooth vector bundles in the context of a tangent category. In the original definition of Cockett and Cruttwell (cf. [10, Definition 2.2]), the underlying bundle of a differential bundle was not required to be a tangent display map (indeed this is only a more recent concept). However, in [11, Section 4.21] the same authors introduced the concept of display differential bundle as a differential bundle which is also a map of a given tangent display system. Then, using this notion, they proved that display differential bundles over a given object M in a tangent category $(\mathbb{X}, \mathbb{T})$ are precisely differential objects in the slice tangent category $(\mathbb{X}, \mathbb{T})/M$ over M.

Thanks to the introduction of tangent display maps as in Definition 2.24, we can now forget about the choice of a tangent display system and work directly with tangent display maps. So, as in Definition 2.41, for a display differential bundle we mean a differential bundle which is also a tangent display map.

As mentioned, the construction of the slice category $(\mathbb{X}, \mathbb{T})/M$ plays a crucial role in the story of differential bundles. The goal of this chapter is twofold: understanding what the slice category of the geometric category $\mathsf{Geom}(\mathscr{P})$ is for a given algebraic operad $\mathscr{P}$ over an operadic affine scheme A, and classifying differential bundles in $\mathsf{Geom}(\mathscr{P})$. In particular, in operad theory, the enveloping operad $\mathscr{P}^{(A)}$ of a $\mathscr{P}$-algebra A is the operad whose algebras are equivalent to morphisms of $\mathscr{P}$-algebras of type $A \to E$, i.e. $\mathsf{Alg}_{\mathscr{P}^{(A)}} \cong A/\mathsf{Alg}_{\mathscr{P}}$. When we consider the opposite of the category of $\mathscr{P}$-algebras we obtain:

$$\mathsf{Alg}_{\mathscr{P}^{(A)}}^{\mathsf{op}} \cong \mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}}/A$$

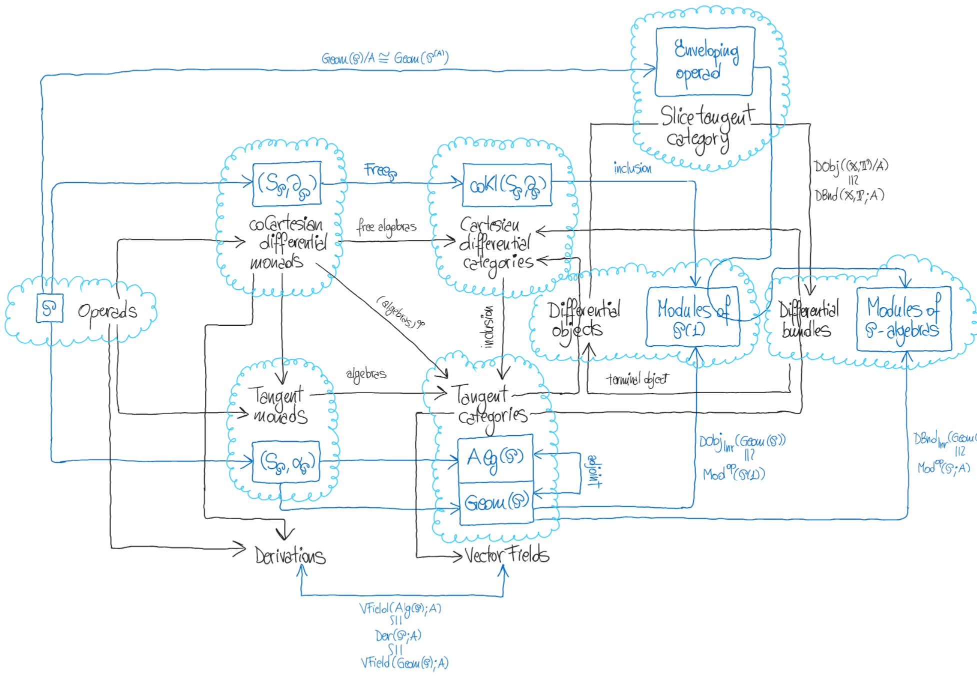


Figure 4.1: The concept map of the chapter

since the coslice of a category is the slice of the opposite category. So, it is natural to wonder what is the relationship between the geometric tangent category $\mathsf{Geom}(\mathscr{P}^{(A)})$ of the enveloping operad of a $\mathscr{P}$-algebra A and the slice tangent category $\mathsf{Geom}(\mathscr{P})/A$ of the geometric tangent category of $\mathscr{P}$ over A. In Section 4.2.1 we prove that these two tangent categories are equivalent to each other, proving that slicing behaves well with respect to the operation which sends an operad to its geometric tangent category. In particular, this shows that the slice tangent category of the geometric tangent category of an operad is still a geometric tangent category of an operad.

This fact can be harnessed to simplify the classification of differential bundles in $\mathsf{Geom}(\mathscr{P})$. The idea is to classify differential objects in the geometric tangent category of an arbitrary operad and then look at differential objects in $\mathsf{Geom}(\mathscr{P}^{(A)})$. Employing that $\mathsf{Geom}(\mathscr{P}^{(A)}) \cong \mathsf{Geom}(\mathscr{P})/A$ and that differential bundles in $\mathsf{Geom}(\mathscr{P})$ over A are precisely differential objects in $\mathsf{Geom}(\mathscr{P})/A$ we conclude the classification of differential bundles.

This proof strategy for the classification of differential bundles fully relies on the equivalence between the slice of $\mathsf{Geom}(\mathscr{P})$ and $\mathsf{Geom}(\mathscr{P}^{(A)})$. In order to prove this equivalence, we first need to reconceptualize the operation of taking the slice of a tangent category in terms of a right adjoint. In Section 4.1 we discuss this point, in Section 4.2.1 we recall the construction of the enveloping operad, and we prove the first the equivalence $\mathsf{Geom}(\mathscr{P}^{(A)}) \cong \mathsf{Geom}(\mathscr{P})/A$. In Section 4.3.1 we classify differential objects in the geometric tangent category of any given operad, and finally, we classify differential bundles, in Section 4.3.2.

Figure 4.1 displays the concept map of this chapter.

4.1 The slice of the geometric tangent category of an operad

This section is dedicated to understanding the slice tangent category of the geometric tangent category of a given operad over an operadic affine scheme. The first step is to revisit the construction of the slice of a given tangent category. We want to show that the operation which sends a pair formed by a tangent category $(\mathbb{X}, \mathbb{T})$ together with an object A of $\mathbb{X}$, to the corresponding slice tangent category $(\mathbb{X}, \mathbb{T})/A$ extends to a right adjoint functor of the functor Term which singles out the terminal object $*$ of a Cartesian tangent category, i.e. $\mathsf{Term}(\mathbb{X}, \mathbb{T}) := ((\mathbb{X}, \mathbb{T}); *)$. Let's start by recalling the notion of a Cartesian tangent category.

Definition* 4.1. *A **Cartesian** tangent category is a tangent category $(\mathbb{X}, \mathbb{T})$ whose underlying category $\mathbb{X}$ has finite products (including the terminal object) and for which the tangent bundle functor preserves these products. Moreover, given two Cartesian tangent categories $(\mathbb{X}, \mathbb{T})$ and $(\mathbb{X}', \mathbb{T}')$, a **Cartesian** (lax/colax/strong/strict) tangent morphism $(F, \alpha)\colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}', \mathbb{T}')$ is a (lax/colax/strong/strict) tangent morphism (F, α) whose underlying functor $F\colon \mathbb{X} \to \mathbb{X}'$ preserves Cartesian products (including the terminal object).*

Cartesian tangent categories together with Cartesian lax tangent morphisms form a category denoted by $\mathsf{cTngCat}$.

Example 4.2. For a given operad $\mathscr{P}$ the algebraic tangent category $\mathsf{Alg}(\mathscr{P})$ of $\mathscr{P}$ is Cartesian. This is a direct consequence of the fact that $\mathsf{Alg}_{\mathscr{P}}$ is complete and that the algebraic tangent bundle functor $\mathbb{L}^{(\mathscr{P})}$ is a right adjoint and thus it preserves all limits.

Example 4.3. For a given operad $\mathscr{P}$ the geometric tangent category $\mathsf{Geom}(\mathscr{P})$ of $\mathscr{P}$ is Cartesian. This is a direct consequence of the fact that $\mathsf{Alg}_{\mathscr{P}}$ is cocomplete and that the geometric tangent bundle functor $\mathbb{L}^{(\mathscr{P})}$, regarded as an endofunctor of $\mathsf{Alg}_{\mathscr{P}}$, is a left adjoint and thus it preserves all colimits.

One can see that the slice tangent category $(\mathbb{X}, \mathbb{T})/A$ of any (non-necessarily Cartesian) tangent category over a given object is always a Cartesian tangent category. Indeed, since the objects of $(\mathbb{X}, \mathbb{T})/A$ are the tangent display maps $q\colon E \to A$ of $(\mathbb{X}, \mathbb{T})$ with A for codomain, given two such tangent display maps $q\colon E \to A$ and $q'\colon E' \to A$, the pullback $q \times_A q'\colon E \times_A E' \to A$ in $\mathbb{X}$ of q along q' always exists. However, regarded as an object of $(\mathbb{X}, \mathbb{T})/A$, $q \times_A q'$ is the Cartesian product of q with q'. By induction, all finite products (the terminal objects will just be the tangent display map $\mathsf{id}_A\colon A \to A$) exist in $(\mathbb{X}, \mathbb{T})/A$. Finally, since T in $\mathbb{X}$ preserves pullbacks between tangent display maps, the tangent bundle functor $\mathrm{T}^{(A)}$ of $(\mathbb{X}, \mathbb{T})/A$ preserves products in $(\mathbb{X}, \mathbb{T})/A$.

Lemma 4.4. *The slice tangent category $(\mathbb{X}, \mathbb{T})/A$ of a tangent category $(\mathbb{X}, \mathbb{T})$ over a given object A of $\mathbb{X}$ is a Cartesian tangent category.*

Let's introduce the category of tangent pairs.

Definition 4.5. *A **tangent pair** consists of a pair $((\mathbb{X}, \mathbb{T}); A)$ formed by a tangent category $(\mathbb{X}, \mathbb{T})$ and an object A of $\mathbb{X}$. Moreover, a morphism $((\mathbb{X}, \mathbb{T}); A) \to ((\mathbb{X}', \mathbb{T}'); A')$ of tangent pairs consists of a pair $((F, \alpha); \varphi)$ formed by a lax tangent morphism $(F, \alpha)\colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}', \mathbb{T}')$ which preserves tangent display maps over A i.e. for which every tangent display map $q\colon E \to A$ is sent to a tangent display map $Fq\colon FE \to FA$, together with an isomorphism $\varphi\colon FA \to A'$ of $\mathbb{X}'$.*

The composition of two morphisms $((F, \alpha); \varphi)\colon ((\mathbb{X}, \mathbb{T}); A) \to ((\mathbb{X}', \mathbb{T}'); A')$, $((G, \beta); \psi)\colon ((\mathbb{X}', \mathbb{T}'); A') \to ((\mathbb{X}'', \mathbb{T}''); A'')$ of tangent pairs is defined as the tangent morphism $(G, \beta) \circ (F, \alpha) = (G \circ F, G\alpha\beta_F)$ together with the isomorphism $GFA \xrightarrow{G\varphi} GA' \xrightarrow{\psi} A''$. Notice, in particular, that since G preserves tangent display maps and tangent display maps are closed under composition, this defines a morphism of tangent pairs. Therefore, tangent pairs together with their morphisms form a category denoted by $\mathsf{TngPair}$.

We are now in the position to introduce the pseudofunctor $\mathsf{Term}\colon \mathsf{cTngCat} \to \mathsf{TngPair}$ which sends a Cartesian tangent category $(\mathbb{X}, \mathbb{T})$ to the tangent pair $((\mathbb{X}, \mathbb{T}); *)$, $*$ being the terminal object of $\mathbb{X}$. The terminal object $*$ is uniquely defined only up to unique isomorphisms; therefore, to define Term we are making a choice of one of the terminal objects of each $\mathbb{X}$. In the following, we abuse notation and refer to $*$ as *the* terminal object.

Notice that, since pullbacks over the terminal objects are precisely Cartesian products, tangent display maps over the terminal object consist of those objects E of $\mathbb{X}$ for which the Cartesian product $E' \times E$ exists for any other object E' and for which this product is preserved by T, i.e. $\mathrm{T}(E' \times E) \cong \mathrm{T}E' \times \mathrm{T}E$. Therefore, for a Cartesian tangent category $(\mathbb{X}, \mathbb{T})$ tangent display maps over the terminal object are all the objects. This observation implies that Term is a well-defined functor. Indeed, given a Cartesian tangent morphism $(F, \alpha)\colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}', \mathbb{T}')$ let $\mathsf{Term}(F, \alpha)$ be the pair $((F, \alpha); !)$ formed by (F, α) (notice that Cartesianity implies that F sends tangent display maps of $(\mathbb{X}, \mathbb{T})$ over the terminal object $*$ of $\mathbb{X}$ to tangent display maps of $\mathbb{X}'$ over $F* \cong *'$) and by the isomorphism $!\colon F* \to *'$ (notice that this is trivially a tangent display map since it is an isomorphism).

On the other hand, we can also define the pseudofunctor $\mathsf{Slice}\colon \mathsf{TngPair} \to \mathsf{cTngCat}$ which sends a tangent pair $((\mathbb{X}, \mathbb{T}); A)$ to the slice tangent category $\mathsf{Slice}((\mathbb{X}, \mathbb{T}); A)\colon = (\mathbb{X}, \mathbb{T})/A$. To understand how Slice acts on morphisms of tangent pairs, we first need to show that we can lift such a morphism to the slice tangent categories.

In the following, we adopt the following notation. For a tangent display map $q\colon E \to A$ we denote by $q^*\colon \mathrm{T}^{(A)}E \to A$ the pullback of $\mathrm{T}q$ along z (which exists since q is a tangent display map). Moreover, we denote by $\xi_q\colon \mathrm{T}^{(A)}E \to \mathrm{T}E$ the morphism in the pullback diagram commute:

$$
\begin{tikzcd}
\mathrm{T}^{(A)}E \arrow[r, "\xi_q"] \arrow[d, "q^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}E \arrow[d, "\mathrm{T}q"] \\
A \arrow[r, "z"'] & \mathrm{T}A
\end{tikzcd}
$$

When q is clear from the context, we omit it from the notation.

Remark 4.6. Term and Slice are not strict functors but rather pseudofunctors. This comes from the fact that terminal objects and slice tangent structures are only defined up to unique isomorphisms. Thus, the associators and unitors are defined by these unique isomorphisms.

Proposition 4.7. *Consider two tangent pairs $((\mathbb{X},\mathbb{T});A)$ and $((\mathbb{X}',\mathbb{T}');A')$ and a morphism of tangent pairs $((F,\alpha);\varphi)\colon((\mathbb{X},\mathbb{T});A)\to((\mathbb{X}',\mathbb{T}');A')$. Let $q\colon E\to A$ be a tangent display map in $(\mathbb{X},\mathbb{T})$ over A. Finally, consider the morphism $\theta_q\colon F\mathrm{T}^{(A)}E\to\mathrm{T}'^{(A')}FE$, as the unique morphism which makes commutative the following diagram:*

$$
\begin{tikzcd}
F\mathrm{T}^{(A)}E \arrow[rrr, "F\xi_q"] \arrow[dr, dashed, "\theta_q"] \arrow[dddd, "Fq^*"'] & & & F\mathrm{T}E \arrow[dl, "\alpha"'] \arrow[dddd, "F\mathrm{T}q"] \\
& \mathrm{T}'^{(A')}FE \arrow[r, "\xi_{Fq}"'] \arrow[dd, "(Fq\varphi)^*"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}'FE \arrow[d, "\mathrm{T}'Fq"'] & \\
& & \mathrm{T}'FA \arrow[d, "\mathrm{T}'\varphi"'] \arrow[r, phantom, "\mathsf{Nat}"] & {} \\
& A' \arrow[r, "z"] & \mathrm{T}'A' & \\
FA \arrow[ur, "\varphi"] \arrow[rrr, "Fz"'] & & & F\mathrm{T}A \arrow[uul, "\alpha"']
\end{tikzcd}
$$

Therefore, the functor:

$$
\begin{aligned}
&F\colon \mathbb{X}/A\to\mathbb{X}'/A'\\
&F(q\colon E\to A)\mapsto (FE\xrightarrow{Fq}FA\xrightarrow{\varphi}A')\\
&F(g\colon(q\colon E\to A)\to(q'\colon E'\to A))\mapsto(Fg\colon(Fq\varphi)\to(Fq'\varphi))
\end{aligned}
$$

extends to a lax tangent morphism:

$$(F,\alpha)/\varphi\colon(\mathbb{X},\mathbb{T})/A\to(\mathbb{X}',\mathbb{T}')/A'$$

whose distributive law is defined by the natural transformation $\theta_q\colon F\mathrm{T}^{(A)}q\to\mathrm{T}'^{(A')}Fq$.

Proof. For starters, let's prove the compatibility between θ and the projections:

$$\begin{tikzcd} FT^{(A)}q \arrow[r, "\theta"] \arrow[d, "Fp^{(A)}"'] & T'^{(A')}Fq \arrow[dl, "p_F^{(A')}"] \\ Fq & \end{tikzcd}$$

which corresponds to the diagram:

$$\begin{tikzcd} FT^{(A)}E \arrow[r, "\theta"] \arrow[d, "F\xi"'] \arrow[dr, phantom, "(\theta{,}\alpha;\xi)"] & T'^{(A')}FE \arrow[d, "\xi_F"] \\ FTE \arrow[r, "\alpha"] \arrow[d, "Fp"'] \arrow[dr, phantom, "(\alpha;p)"] & T'FE \arrow[d, "p_F"] \\ FE \arrow[r, equal] & FE \end{tikzcd}$$

Let's take into consideration the compatibility diagram between θ and the zero morphisms:

$$\begin{tikzcd} FT^{(A)}q \arrow[r, "\theta"] & T'^{(A')}Fq \\ Fq \arrow[u, "Fz^{(A)}"] \arrow[ur, "z_F^{(A')}"'] & \end{tikzcd}$$

To show that, first, consider the diagram:

$$\begin{tikzcd} FT^{(A)}E \arrow[rrr, "\theta"] \arrow[dr, "F\xi"] & & & T'^{(A')}FE \arrow[dl, "\xi_F"'] \\ & FTE \arrow[r, "\alpha"'] \arrow[r, phantom, "(\alpha{,}\theta;\xi)", shift left=6ex] \arrow[r, phantom, "(\alpha;z)"', shift right=6ex] & T'FE & \\ FE \arrow[uu, "Fz^{(A)}"] \arrow[ur, "Fz"'] \arrow[rrr, equal] & & & FE \arrow[uu, "z_F^{(A')}"'] \arrow[ul, "z_F"] \end{tikzcd}$$

with $(z;\xi)$ in the left and right regions.

Thus $Fz^{(A)}\theta\xi_F = z_F^{(A')}\xi_F$ and from the universality of ξ_F we conclude that $Fz^{(A)}\theta =$

$z_F^{(A')}$, as expected. The next step is to prove the compatibility with the sum morphism:

$$\begin{tikzcd}
F\mathrm{T}_2^{(A)}q \arrow[r, "\theta\times\theta"] \arrow[d, "Fs^{(A)}"'] & \mathrm{T'}_2^{(A')}Fq \arrow[d, "s_F^{(A)}"] \\
F\mathrm{T}^{(A)}q \arrow[r, "\theta"'] & \mathrm{T'}^{(A')}Fq
\end{tikzcd}$$

Thus, consider the following diagram:

$$\begin{tikzcd}
F\mathrm{T}_2^{(A)}E \arrow[rrr, "\theta\times\theta"] \arrow[dr, "F\xi\times F\xi"] \arrow[ddd, "Fs^{(A)}"'] & {} \arrow[r, phantom, "(\alpha{,}\theta;\xi)"] & {} & \mathrm{T'}_2^{(A')}FE \arrow[dl, "\xi_F\times\xi_F"'] \arrow[ddd, "s_F^{(A')}"] \\
{} \arrow[d, phantom, "(s;\xi)"] & F\mathrm{T}_2E \arrow[r, "\alpha\times\alpha"] \arrow[d, "Fs"'] \arrow[dr, phantom, "(\alpha;s)"] & \mathrm{T}'_2FE \arrow[d, "s_F"] & {} \arrow[d, phantom, "(s;\xi)"] \\
{} & F\mathrm{T}E \arrow[r, "\alpha"] & \mathrm{T}'FE & {} \\
F\mathrm{T}^{(A)}E \arrow[rrr, "\theta"'] \arrow[ur, "F\xi"'] & {} \arrow[r, phantom, "(\alpha{,}\theta;\xi)"] & {} & \mathrm{T'}^{(A')}FE \arrow[ul, "\xi_F"]
\end{tikzcd}$$

Thus, $Fs^{(A)}\theta\xi_F = (\theta\times\theta)s_F^{(A')}\xi_F$ and from the universality of ξ_F we conclude that $Fs^{(A)}\theta = (\theta\times\theta)s^{(A')}$, as expected. Let's prove the compatibility with the lift:

$$\begin{tikzcd}
F\mathrm{T}^{(A)}q \arrow[rr, "\theta"] \arrow[d, "Fl^{(A)}"'] & & \mathrm{T'}^{(A')}Fq \arrow[d, "l_F^{(A')}"] \\
F(\mathrm{T}^{(A)})^2q \arrow[r, "\theta_{\mathrm{T}^{(A)}}"'] & \mathrm{T'}^{(A')}F\mathrm{T}^{(A)}q \arrow[r, "\mathrm{T'}^{(A')}\theta"'] & (\mathrm{T'}^{(A')})^2Fq
\end{tikzcd}$$

As before, consider the following diagram:

$$
\begin{tikzcd}
F\mathrm{T}^{(A)}E \arrow[rrrr, "\theta"] \arrow[dr, "F\xi"'] \arrow[dddd, "Fl^{(A)}"'] \arrow[ddrr, phantom, "(\alpha,\theta;\xi)"] & & & & \mathrm{T}'^{(A')}FE \arrow[dl, "\xi_F"'] \arrow[dddd, "l_F^{(A')}"] \\
& FTE \arrow[rr, "\alpha"] \arrow[d, "Fl"'] \arrow[drr, phantom, "(\alpha;l)"] & & \mathrm{T}'FE \arrow[d, "l_F"] & \\
{} \arrow[r, phantom, "(l;\xi)"] & F\mathrm{T}^2E \arrow[r, "\alpha_\mathrm{T}"] & \mathrm{T}'F\mathrm{T}E \arrow[r, "\mathrm{T}'\alpha"] & \mathrm{T}'^2FE & {} \arrow[l, phantom, "(l;\xi)"'] \\
& F\mathrm{T}^{(A)}\mathrm{T}E \arrow[u, "F\xi_\mathrm{T}"] \arrow[r, "\theta_\mathrm{T}"] \arrow[ur, phantom, "(\alpha,\theta;\xi)"] & \mathrm{T}'^{(A')}F\mathrm{T}E \arrow[u, "\xi_{F\mathrm{T}}"] \arrow[r, "\mathrm{T}'^{(A')}\alpha"] \arrow[ur, phantom, "\mathsf{Nat}"] & \mathrm{T}'^{(A')}\mathrm{T}'FE \arrow[u, "\xi_{\mathrm{T}'F}"'] & \\
F(\mathrm{T}^{(A)})^2E \arrow[ur, "F\mathrm{T}^{(A)}\xi"] \arrow[rr, "\theta_{\mathrm{T}^{(A)}}"'] \arrow[urr, phantom, "\mathsf{Nat}"] & & \mathrm{T}'^{(A')}F\mathrm{T}^{(A)}E \arrow[u, "\mathrm{T}'^{(A')}F\xi"] \arrow[rr, "\mathrm{T}'^{(A')}\theta"'] \arrow[ur, phantom, "(\alpha,\theta;\xi)"] & & (\mathrm{T}'^{(A')})^2FE \arrow[ul, "\mathrm{T}'^{(A')}\xi"']
\end{tikzcd}
$$

Therefore, $\theta l_F^{(A')}\mathrm{T}'^{(A')}\xi\xi_{\mathrm{T}'F} = Fl^{(A)}\theta_{\mathrm{T}^{(A)}}\mathrm{T}'^{(A')}\theta\mathrm{T}'^{(A')}\xi\xi_{\mathrm{T}'F}$. By the universality of $\mathrm{T}'^{(A')}\xi\xi_{\mathrm{T}'F}$ we conclude that $\theta l_F^{(A')} = Fl^{(A)}\theta_{\mathrm{T}^{(A)}}\mathrm{T}'^{(A')}\theta$, as expected. Finally, let's prove the compatibility with the canonical flip:

$$
\begin{tikzcd}
F(\mathrm{T}^{(A)})^2q \arrow[r, "\theta_{\mathrm{T}^{(A)}}"] \arrow[d, "Fc^{(A)}"'] & \mathrm{T}'^{(A')}F\mathrm{T}^{(A)}q \arrow[r, "\mathrm{T}'^{(A')}\theta"] & (\mathrm{T}'^{(A')})^2Fq \arrow[d, "c_F^{(A')}"] \\
F(\mathrm{T}^{(A)})^2q \arrow[r, "\theta_{\mathrm{T}^{(A)}}"'] & \mathrm{T}'^{(A')}F\mathrm{T}^{(A)}q \arrow[r, "\mathrm{T}'^{(A')}\theta"'] & (\mathrm{T}'^{(A')})^2Fq
\end{tikzcd}
$$

Thus:

$$\begin{tikzcd}
F(\mathrm{T}^{(A)})^2E \arrow[rr, "\theta_{\mathrm{T}^{(A)}}"] \arrow[dr, "F\mathrm{T}^{(A)}\xi"'] \arrow[ddddd, "Fc^{(A)}"'] & & \mathrm{T}'^{(A')}F\mathrm{T}^{(A)}E \arrow[rr, "\mathrm{T}'^{(A')}\theta"] \arrow[d, "\mathrm{T}'^{(A')}F\xi"] \arrow[dl, phantom, "\mathsf{Nat}"] \arrow[dr, phantom, "(\alpha{,}\theta;\xi)"] & & (\mathrm{T}'^{(A')})^2FE \arrow[dl, "\mathrm{T}'^{(A')}\xi"'] \arrow[ddddd, "c_F^{(A')}"] \\
& F\mathrm{T}^{(A)}\mathrm{T}E \arrow[r, "\theta_\mathrm{T}"'] \arrow[d, "F\xi_\mathrm{T}"'] \arrow[dr, phantom, "(\alpha{,}\theta;\xi)"] & \mathrm{T}'^{(A')}F\mathrm{T}E \arrow[r, "\mathrm{T}'^{(A')}\alpha"'] \arrow[d, "\xi_{F\mathrm{T}}"] \arrow[dr, phantom, "\mathsf{Nat}"] & \mathrm{T}'^{(A')}\mathrm{T}'FE \arrow[d, "\xi_{\mathrm{T}'F}"] & \\
\arrow[dr, phantom, "(c;\xi)"] & F\mathrm{T}E \arrow[r, "\alpha_\mathrm{T}"'] \arrow[d, "Fc"'] & \mathrm{T}'F\mathrm{T}E \arrow[r, "\mathrm{T}'\alpha"'] \arrow[d, phantom, "(\alpha;c)"] & \mathrm{T}'^2FE \arrow[d, "c_F"] & \arrow[dl, phantom, "(c;\xi)"] \\
& F\mathrm{T}^2E \arrow[r, "\alpha_\mathrm{T}"] \arrow[dr, phantom, "(\alpha{,}\theta;\xi)"] & \mathrm{T}'F\mathrm{T}E \arrow[r, "\mathrm{T}'\alpha"] \arrow[dr, phantom, "\mathsf{Nat}"] & \mathrm{T}'^2FE & \\
& F\mathrm{T}^{(A)}\mathrm{T}E \arrow[u, "F\xi_\mathrm{T}"] \arrow[r, "\theta_\mathrm{T}"] & \mathrm{T}'^{(A')}F\mathrm{T}E \arrow[u, "\xi_{F\mathrm{T}}"'] \arrow[r, "\mathrm{T}'^{(A')}\alpha"] & \mathrm{T}'^{(A')}\mathrm{T}'FE \arrow[u, "\xi_{\mathrm{T}'F}"'] & \\
F(\mathrm{T}^{(A)})^2E \arrow[ur, "F\mathrm{T}^{(A)}\xi"] \arrow[rr, "\theta_{\mathrm{T}^{(A)}}"'] & & \mathrm{T}'^{(A')}F\mathrm{T}^{(A)}E \arrow[u, "\mathrm{T}'^{(A')}F\xi"] \arrow[rr, "\mathrm{T}'^{(A')}\theta"'] \arrow[ul, phantom, "\mathsf{Nat}"] \arrow[ur, phantom, "(\alpha{,}\theta;\xi)"] & & (\mathrm{T}'^{(A')})^2FE \arrow[ul, "\mathrm{T}'^{(A')}\xi"']
\end{tikzcd}$$

This proves that $\theta_{\mathrm{T}^{(A)}}\mathrm{T}'^{(A')}\theta c_F^{(A')}\mathrm{T}'^{(A')}\xi\xi_{\mathrm{T}'F} = Fc^{(A)}\theta_{\mathrm{T}^{(A)}}\mathrm{T}'^{(A')}\theta\mathrm{T}'^{(A')}\xi\xi_{\mathrm{T}'F}$. Finally, using the universality of $\mathrm{T}'^{(A')}\xi\xi_{\mathrm{T}'F}$ we conclude that $\theta_{\mathrm{T}^{(A)}}\mathrm{T}'^{(A')}\theta c_F^{(A')} = Fc^{(A)}\theta_{\mathrm{T}^{(A)}}\mathrm{T}'^{(A')}\theta$, as expected. □

Proposition 4.7 allows us to lift morphisms of tangent pairs to the corresponding slice tangent categories. Thanks to this, we can define Slice to be the pseudofunctor which sends a morphism $((F,\alpha);\varphi)\colon((\mathbb{X},\mathbb{T});A)\to((\mathbb{X}',\mathbb{T}');A')$ to the lax tangent morphism $(F,\alpha)/\varphi$. Notice also that since F preserves tangent display maps over A, it also preserves the Cartesian products between tangent display maps over A.

Remark 4.8. Notice that, to define morphisms of tangent pairs one could have asked φ to simply be a tangent display map. However, in order for $(F,\alpha)/\varphi$ to preserve Cartesian products we needed φ to be an isomorphism.

The next step is to find sufficient conditions on a morphism of tangent pairs for the corresponding tangent morphism over the slice categories to be strong. This will play a key role in our story. Let's introduce a definition.

Definition 4.9. *Given two tangent pairs* $((\mathbb{X},\mathbb{T});A)$ *and* $((\mathbb{X}',\mathbb{T}');A')$*, a morphism* $((F,\alpha);\varphi)\colon((\mathbb{X},\mathbb{T});A)\to((\mathbb{X}',\mathbb{T}');A')$ *of tangent pairs is* ***Cartesian*** *if the following diagrams:*

$$\begin{tikzcd}
F\mathrm{T}E \arrow[r, "\alpha"] \arrow[d, "F\mathrm{T}q"'] & \mathrm{T}'FE \arrow[d, "\mathrm{T}'Fq"] \\
F\mathrm{T}A \arrow[r, "\alpha"'] & \mathrm{T}'FA
\end{tikzcd}
\qquad
\begin{tikzcd}
F\mathrm{T}^{(A)}E \arrow[r, "F\xi_q"] \arrow[d, "Fq^*"'] & F\mathrm{T}E \arrow[d, "F\mathrm{T}q"] \\
FA \arrow[r, "Fz"'] & F\mathrm{T}A
\end{tikzcd}$$

are pullback diagrams, for every tangent display map $q\colon E\to A$ *of* $(\mathbb{X},\mathbb{T})$ *over* A*.*

Remark 4.10. Notice that, even if the functor F underlying a morphism of tangent pairs preserves tangent display maps over the given object A of the pair, it is not guaranteed that F preserves also tangent display maps over $\mathrm{T}A$. This is the reason why in Definition 4.9 we required the right diagram to be a pullback.

Lemma 4.11. *A Cartesian morphism of tangent pairs* $((F,\alpha);\varphi)\colon((\mathbb{X},\mathbb{T});A)\to((\mathbb{X}',\mathbb{T}');A')$ *lifts as a strong tangent morphism to the slice tangent categories. Concretely, this means that the natural transformation* $\theta_q\colon F\mathrm{T}^{(A)}q\to\mathrm{T}'^{(A')}Fq$ *defined in Proposition 4.7 is invertible.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
F\mathrm{T}^{(A)}E \arrow[r, "Fv"] \arrow[dd, "F\mathrm{T}^{(A)}q"'] \arrow[dr, phantom, "\lrcorner", very near start] & F\mathrm{T}E \arrow[r, "\alpha"] \arrow[d, "F\mathrm{T}q"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{T}'FE \arrow[dd, "\mathrm{T}'Ff"] \\
 & F\mathrm{T}A \arrow[dr, "\alpha"] & \\
FA \arrow[ur, "Fz"] \arrow[rr, "z_F"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{T}'FA \arrow[d, "\mathrm{T}'\varphi"] \\
A' \arrow[rr, "z"'] & & \mathrm{T}'A'
\end{tikzcd}$$

where we used that $Fz\alpha=z_F$. Thanks to the Cartesianity of $((F,\alpha);\varphi)$, this is a pullback diagram since it is formed by pullback diagrams. In particular, the bottom square diagram is a pullback because φ is an isomorphism. On the other hand, by definition, θ is defined by the diagram:

$$
\begin{tikzcd}
\mathrm{T}'^{(A')}FE \arrow[drrr, "\xi_F"] \arrow[dd, "\mathrm{T}'^{(A')}(Fq\varphi)"'] & & & \\
 & & & \mathrm{T}'FE \arrow[d, "\mathrm{T}'Fq"] \\
A' \arrow[drrr, "z"] \arrow[dddr, equal] & & F\mathrm{T}E \arrow[ur, "\alpha"] & \mathrm{T}'FA \arrow[d, "\mathrm{T}'\varphi"] \\
 & F\mathrm{T}^{(A)}E \arrow[uuul, dashed, "\theta"] \arrow[ur, "Fv"] \arrow[d, "F\mathrm{T}^{(A)}q"] & & \mathrm{T}'A' \\
 & FA \arrow[d, "\varphi"] & & \\
 & A' \arrow[uurr, "z"] & &
\end{tikzcd}
$$

However, the top and the right rectangular sides of this triangular diagram are pullbacks, so θ must be an isomorphism. $\square$

We can finally characterize the operation which takes a tangent pair to its slice tangent category as an adjunction between pseudofunctors.

Theorem 4.12. *The pseudofunctors* $\mathsf{Slice}\colon \mathsf{TngPair} \leftrightarrows \mathsf{cTngCat}\colon \mathsf{Term}$ *form an adjunction whose left adjoint is* Term*, the right adjoint is* Slice*, the unit* $(U,\eta)\colon (\mathbb{X},\mathbb{T}) \to \mathsf{Slice}(\mathsf{Term}(\mathbb{X},\mathbb{T})) = (\mathbb{X},\mathbb{T})/*$*, as a Cartesian tangent morphism between Cartesian tangent categories, is the isomorphism:*

$$
\begin{aligned}
&U\colon \mathbb{X} \to \mathbb{X}/* \\
&U(A) \mapsto (!\colon A \to *) \\
&U(f\colon A \to B) \mapsto (f(!\colon A \to *) \to (!\colon B \to *)) \\
&\eta\colon (U(\mathrm{T}A)) = (!\colon \mathrm{T}A \to *) \xrightarrow{\mathsf{id}_{\mathrm{T}A}} (!\colon \mathrm{T}A \to *) = \mathrm{T}(U(A))
\end{aligned}
$$

and the counit $((C,\varepsilon);\varphi)\colon \mathsf{Term}(\mathsf{Slice}((\mathbb{X},\mathbb{T});A)) = ((\mathbb{X},\mathbb{T})/A;\mathsf{id}_A) \to ((\mathbb{X},\mathbb{T});A)$ *is the morphism of tangent pairs:*

$$
\begin{aligned}
&C\colon (\mathbb{X},\mathbb{T})/A \mapsto (\mathbb{X},\mathbb{T}) \\
&C(q\colon E \to A) \mapsto E \\
&C(g\colon (q\colon E \to A) \to (q'\colon E' \to A)) \mapsto (g\colon E \to E')
\end{aligned}
$$

$$\varepsilon\colon C(\mathrm{T}^{(A)}(q\colon E\to A))=\mathrm{T}^{(A)}E\xrightarrow{\xi_q}\mathrm{T}E=\mathrm{T}(C(q\colon E\to A))$$
$$\varphi\colon C(\mathsf{id}_A\colon A\to A)=A\xrightarrow{\mathsf{id}_A}A$$

Proof. Let's start by noticing that the unit and the counit are well-defined morphisms. The underlying functor U of the unit is clearly Cartesian, so (U,η) is well-defined. Let's focus on the counit. A tangent display map in $(\mathbb{X},\mathbb{T})/A$ over $\mathsf{id}_A\colon A\to A$ consists of an object $q\colon E\to A$ of $(\mathbb{X},\mathbb{T})/A$, i.e. a tangent display map of $(\mathbb{X},\mathbb{T})$ over A, together with a morphism $q'\colon E'\to A$ for which $q'\mathsf{id}_A=q$. This implies that tangent display maps of $(\mathbb{X},\mathbb{T})/A$ over id_A are also tangent display maps of $(\mathbb{X},\mathbb{T})$ over A. So, the underlying functor C of the counit sends tangent display maps to tangent display maps.

The next step is to show that the unit (U,η) and the counit $((C,\varepsilon);\varphi)$ satisfy the triangle identities. Let's start by considering the following diagram:

$$\begin{tikzcd}
\mathsf{Term}(\mathbb{X},\mathbb{T}) \arrow[r, "\mathsf{Term}(U,\eta)"] \arrow[dr, equal] & \mathsf{Term}(\mathsf{Slice}(\mathsf{Term}(\mathbb{X},\mathbb{T}))) \arrow[d, "((C,\varepsilon);\varphi)_{\mathsf{Term}}"] \\
& \mathsf{Term}(\mathbb{X},\mathbb{T})
\end{tikzcd}$$

for a tangent category $(\mathbb{X},\mathbb{T})$ with terminal object. However, it is straightforward to realize that the underlying tangent morphisms (C,ε) and (U,η) of $((C,\varepsilon);\varphi)_{\mathsf{Term}}$ and $\mathsf{Term}(U,\eta)$ define the equivalence between $(\mathbb{X},\mathbb{T})$ and $(\mathbb{X},\mathbb{T})/*$ and that, by the universality of the terminal object, the composition of the comparison morphisms $\varphi=\mathsf{id}_*$ and $!\colon U*\to *$ is the identity over the terminal object. Similarly, by considering the diagram:

$$\begin{tikzcd}
\mathsf{Slice}((\mathbb{X},\mathbb{T});A) \arrow[r, "(U,\eta)_{\mathsf{Slice}}"] \arrow[dr, equal] & \mathsf{Slice}(\mathsf{Term}(\mathsf{Slice}((\mathbb{X},\mathbb{T});A))) \arrow[d, "\mathsf{Slice}((C,\varepsilon);\varphi)"] \\
& \mathsf{Slice}((\mathbb{X},\mathbb{T});A)
\end{tikzcd}$$

for a tangent pair $((\mathbb{X},\mathbb{T});A)$, it is straightforward to show the underlying tangent morphisms of $\mathsf{Slice}((C,\varepsilon);\varphi)$ and $(U,\eta)_{\mathsf{Slice}}$ define the equivalence between $(\mathbb{X},\mathbb{T})/A$ and $((\mathbb{X},\mathbb{T})/A)/\mathsf{id}_A$ and that the composition of the comparison morphisms gives the identity. Finally, notice that the unit is always an isomorphism. $\square$

Remark 4.13. As mentioned in the introduction of Section 2.2.4, in the original paper [41], the author of this thesis employed a different approach to define the slice tangent category. Instead of considering only tangent display maps as objects of the slice tangent category, all morphisms with a fixed codomain were considered. However, since the existence of tangent pullbacks along these morphisms is required in order to define the slice tangent structure, only so-called *sliceable objects* were considered. We suggest the interested reader to consult the original paper for details. This discrepancy in the definition of the slice tangent category in the original paper results in a different adjunction. Instead of having an adjunction $\mathsf{Term} \dashv \mathsf{Slice}$ between tangent pairs and Cartesian tangent categories, in the original paper, we proved the existence of an adjunction between tangent pairs and tangent categories with a terminal object. We also need to point out that the morphisms of $\mathsf{TngPair}$ in the original paper were not required to preserve tangent display maps.

4.2 The slice tangent categories of the affine schemes over an operad

The previous section was dedicated to characterizing the slicing of tangent categories via the adjunction between two pseudofunctors. A similar phenomenon happens in the realm of operads: given an operad $\mathscr{P}$ and a $\mathscr{P}$-algebra A the enveloping operad $\mathscr{P}^{(A)}$ of $\mathscr{P}$ over A is the operad whose category of algebras is equivalent to the coslice category of $\mathsf{Alg}_{\mathscr{P}}$ under A.

The goal of this section is to prove that these phenomena are two faces of the same coin: the geometric tangent category of the enveloping operad of $\mathscr{P}$ over A is equivalent to the slice tangent category of the geometric tangent category of $\mathscr{P}$ over A.

Let's start by recalling the definition of the enveloping operad of a pair $(\mathscr{P}; A)$. We advise the interested reader to consult [6], [46], or [21]. For this purpose, recall that since the category of algebras of an operad $\mathscr{P}$ is cocomplete, each operad has an initial algebra, which corresponds to the R-module $\mathscr{P}(0)$ together with the structure map $\mathscr{P}(m) \otimes \mathscr{P}(0)^{\otimes m} \to \mathscr{P}(0)$ defined by the operadic composition. This allows us to introduce an operation $\mathscr{P} \mapsto (\mathscr{P}; \mathscr{P}(0))$ between operads and operadic pairs. Notice that by a **operadic pair** we mean a pair $(\mathscr{P}; A)$ formed by an operad $\mathscr{P}$ and a

$\mathscr{P}$-algebra A. Moreover, given two operadic pairs $(\mathscr{P}; A)$ and $(\mathscr{P}'; A')$ a **morphism of operadic pairs** $(f; \varphi)\colon (\mathscr{P}; A) \to (\mathscr{P}'; A')$ is a morphism of operads $f : \mathscr{P} \to \mathscr{P}'$ together with a morphism of $\mathscr{P}$-algebras $\varphi\colon A \to f^*A'$, $f^*\colon \mathsf{Alg}_{\mathscr{P}'} \to \mathsf{Alg}_{\mathscr{P}}$ being the pullback functor induced by f. Operadic pairs together with their morphisms form a category that we denote by $\mathsf{OprPair}$. So, we have:

$$\begin{aligned}
&\mathsf{Init}\colon \mathsf{Operad} \to \mathsf{OprPair}\\
&\mathsf{Init}(\mathscr{P}) := (\mathscr{P}; \mathscr{P}(0))\\
&\mathsf{Init}(f : \mathscr{P} \mapsto \mathscr{P}') := (f; !\colon \mathscr{P}(0) \to f^*\mathscr{P}'(0))
\end{aligned}$$

$!$ being the unique morphism of $\mathscr{P}$-algebras induced by the universality of the initial algebra $\mathscr{P}(0)$. Concretely, $!$ sends an element $u \in \mathscr{P}(0)$ to $f_0(u)$.

Init admits a left adjoint Env (cf. [6]), which sends an operadic pair $(\mathscr{P}; A)$ to the corresponding enveloping operad $\mathsf{Env}(\mathscr{P}; A) := \mathscr{P}^{(A)}$. Following the description provided by [21, Section 4.1.3], $\mathscr{P}^{(A)}$ is generated by the symbols $(\mu; a_1, \dots, a_k|$, for every $\mu \in \mathscr{P}(m+k)$, $a_1, \dots, a_k \in A$ and every non-negative integer k (when $k = 0$, $(\mu|$ are the only terms) which satisfy the following relations:

$$(\mu; a_1, \dots, \nu(a_i, \dots, a_{i+n}), \dots, a_{k+n}| = (\mu \circ_i \nu; a_1, \dots, a_{k+n}| \tag{4.2.1}$$

for $\mu \in \mathscr{P}(m+k)$, $\nu \in \mathscr{P}(n)$ and $a_1, \dots, a_{k+n} \in A$, where we used the notation $\mu \circ_i \nu$ for $\mu(1_{\mathscr{P}}, \dots, \nu, \dots, 1_{\mathscr{P}})$. In particular, it is not hard to see that $\mathscr{P}^{(A)}(0) \cong A$. So, the functor Env sends a morphism of operadic pairs $(f, \varphi)\colon (\mathscr{P}; A) \to (\mathscr{P}'; A')$ to the morphism of operads $\mathsf{Env}(f; \varphi)\colon \mathscr{P}^{(A)} \to \mathscr{P}'^{(A')}$ defined on generators as follows:

$$(\mu; a_1, \dots, a_k| \mapsto (f(\mu); \varphi(a_1), \dots, \varphi(a_k)|$$

From this description of the enveloping operad, it is not hard to see that an algebra A' of the enveloping operad $\mathscr{P}^{(A)}$ is precisely given by a $\mathscr{P}$-algebra C^*A', $C : \mathscr{P} \to \mathscr{P}^{(A)}$ being the canonical inclusion $\mu \mapsto (\mu|$, together with a morphism of $\mathscr{P}$-algebras $A \to C^*A'$ induced by the structure map $A = \mathscr{P}^{(A)}(0) \to C^*A'$ of A'.

Conversely, every morphism of $\mathscr{P}$-algebras $f : A \to A'$ induces a $\mathscr{P}^{(A)}$-algebra structure over A' defined as follows:

$$(\mu; a_1, \dots, a_k|(b_1, \dots, b_m) := \mu_{A'}(f(a_1), \dots, f(a_k), b_1, \dots, b_m)$$

for $\mu \in \mathscr{P}(m+k)$, $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_m \in A'$. This proves that the category of $\mathscr{P}^{(A)}$-algebras is equivalent to the coslice category of $\mathscr{P}$-algebras over A (cf. [6, Lemma 1.7]).

4.2.1 The geometric tangent category of the enveloping operad

Theorem 4.12 establishes that Term and Slice form an adjunction and from our discussion on the enveloping operad we also know that Env and Init form an adjunction. We would like to compare Term with Init and Slice with Env. However, Term is a left adjoint, while Init is a right adjoint and similarly, Slice is a right adjoint and Env is a left adjoint. To solve this issue, we transpose the adjunction $\mathsf{Env} \dashv \mathsf{Init}$ to the opposite categories. To compare these functors, notice that Geom^* extends to operadic pairs as follows:

$$\begin{aligned}
&\mathsf{Geom}^*\colon \mathsf{OprPair}^{\mathsf{op}} \longrightarrow \mathsf{TngPair}\\
&\mathsf{Geom}^*(\mathscr{P};A) \colon= (\mathsf{Geom}(\mathscr{P});A)\\
&\mathsf{Geom}^*((f,\varphi)\colon (\mathscr{P};A) \longrightarrow (\mathscr{P}';A')) \colon=\\
&\qquad (\mathsf{Geom}^*(f) = (f^*,\alpha^*);\varphi^{\mathsf{op}}\colon A \leftarrow \varphi^*A')\colon (\mathsf{Geom}(\mathscr{P});A) \longrightarrow (\mathsf{Geom}(\mathscr{P}');A')
\end{aligned}$$

Note that, since $\mathsf{Alg}_{\mathscr{P}}$ is cocomplete and the tangent bundle functor is a left adjoint (and therefore it preserves all colimits), every morphism of $\mathsf{Geom}(\mathscr{P})$ is a $\mathrm{T}^{(\mathscr{P})}$-display map.

Lemma 4.14. *The following diagram:*

$$\begin{tikzcd}
\mathsf{Operad}^{\mathsf{op}} \arrow[r, "\mathsf{Init}"] \arrow[d, "\mathsf{Geom}^*"'] & \mathsf{OprPair}^{\mathsf{op}} \arrow[d, "\mathsf{Geom}^*"] \\
\mathsf{cTngCat} \arrow[r, "\mathsf{Term}"'] & \mathsf{TngPair}
\end{tikzcd}$$

commutes.

Proof. It is straightforward to see that, for an operad $\mathscr{P}$:

$$\mathsf{Geom}^*(\mathsf{Init}((\mathscr{P})) = (\mathsf{Geom}(\mathscr{P});\mathscr{P}(0)) = \mathsf{Term}(\mathsf{Geom}(\mathscr{P})) = \mathsf{Term}(\mathsf{Geom}^*(\mathscr{P}))$$

and for a morphism of operads $f : \mathscr{P} \to \mathscr{P}'$:

$$\begin{aligned}&\mathsf{Geom}^*(\mathsf{Init}(f)) = \mathsf{Geom}^*(f; !\colon f^*\mathscr{P}'(0) \leftarrow \mathscr{P}(0)) = \\ &= (f^*, \alpha^*; !\colon \mathscr{P}(0) \to f^*\mathscr{P}'(0)) = \mathsf{Term}(f^*, \alpha^*) = \mathsf{Term}(\mathsf{Geom}^*(f))\end{aligned}$$

So, the diagram commutes. □

Thanks to Lemma 4.14 we can now also compare the functors Env and Slice. Crucially, to do that we are going to use that Init ⊣ Env (on the opposite categories) and that Term ⊣ Slice form adjunctions. In general, given a square diagram as follows:

$$\begin{tikzcd} \bullet \arrow[r, shift left, "F'"] \arrow[d, "G"'] & \bullet \arrow[l, shift left, "U'"] \arrow[d, "H"] \\ \bullet \arrow[r, shift left, "F"] & \bullet \arrow[l, shift left, "U"] \end{tikzcd}$$

with $(\eta, \varepsilon)\colon F \dashv U$ and $(\eta', \varepsilon')\colon F' \dashv U'$ forming adjunctions, then if the diagram:

$$\begin{tikzcd} \bullet \arrow[r, "F'"] \arrow[d, "G"'] & \bullet \arrow[d, "H"] \\ \bullet \arrow[r, "F"] & \bullet \end{tikzcd}$$

commutes, then, by using mates (see [35]), we can define the following natural transformation:

$$G \circ U' \xrightarrow{\eta_{GU'}} U \circ F \circ G \circ U' = U \circ H \circ F' \circ U' \xrightarrow{UH\varepsilon'} U \circ H$$

A priori, there is no reason to conclude that such a natural transformation is a natural isomorphism. In order to prove that the natural transformation induced by the adjunctions Init ⊣ Env, Term ⊣ Slice, and by Lemma 4.14 is an isomorphism, we need to show that the counit of Init ⊣ Env induces a Cartesian morphism of tangent pairs over the geometric tangent pairs.

Lemma 4.15. *The counit, regarded as a morphism of* OprPair, $(C, \varepsilon)\colon (\mathscr{P}; A) \to \mathsf{Init}(\mathsf{Env}(\mathscr{P}; A)) = (\mathscr{P}^A; \mathscr{P}^{(A)}(0))$ *of the adjunction* Init ⊣ Env *induces a Cartesian morphism of tangent pairs:*

$$\mathsf{Geom}^*(C, \varepsilon)\colon \mathsf{Geom}^*(\mathscr{P}^{(A)}; \mathscr{P}^{(A)}(0)) \to \mathsf{Geom}^*(\mathscr{P}; A)$$

Proof. Let's start by recalling the definition of the counit. $C\colon \mathscr{P} \to \mathscr{P}^{(A)}$ is the morphism of operads which includes $\mathscr{P}$ into $\mathscr{P}^{(A)}$ by mapping $\mu \in \mathscr{P}(m)$ into $(\mu| \in \mathscr{P}^{(A)}(m)$. Moreover, $\varepsilon\colon A \to C^*\mathscr{P}^{(A)}(0)$ is the isomorphism $A \ni a \mapsto (1_{\mathscr{P}}; a| \in C^*\mathscr{P}^{(A)}(0)$, where $1_{\mathscr{P}} \in \mathscr{P}(1)$ is the unit of $\mathscr{P}$. To see that this is an isomorphism, notice that the generators of $\mathscr{P}^{(A)}(0)$ are all the symbols $(\mu; a_1, \dots, a_m|$ for every $\mu \in \mathscr{P}(m)$ and $a_1, \dots, a_m \in A$, but thanks to the relations (4.2.1) we also have:

$$(\mu; a_1, \dots, a_m| = (1_{\mathscr{P}}(\mu); a_1, \dots, a_m| = (1_{\mathscr{P}}; \mu(a_1, \dots, a_m)|$$

So, with the identification $a = (1_{\mathscr{P}}; a|$ we have that $\mathscr{P}^{(A)}(0)$ is equal to A. Notice also that, given a $\mathscr{P}^{(A)}$-algebra A', C^*A' is the $\mathscr{P}$-algebra over A' with structure map defined by:

$$\mu(b_1, \dots, b_m) := (\mu|_{A'}(b_1, \dots, b_m)$$

To distinguish between the different tangent structures, for this proof we adopt the following convention: we denote by $\mathbb{T}$ the geometric tangent structure of $\mathscr{P}$, by $\mathbb{T}^{(A')}$ the slice tangent structure on A', and by $\mathbb{T}_A$ the geometric tangent structure of $\mathscr{P}^{(A)}$.

The Cartesianity of $\mathsf{Geom}^*(C, \varepsilon)$ means that for a morphism $q\colon A' \to E$ of $\mathscr{P}^{(A)}$-algebras the diagrams in the category of $\mathscr{P}$-algebras:

$$\begin{tikzcd}
C^*T_AA' \arrow[r, "C^*\mathrm{T}_Aq"] \arrow[d, "C^*z_A"'] & C^*T_AE \arrow[d, "C^*\xi_q"] \\
C^*A' \arrow[r, "C^*q_*"'] & C^*(\mathrm{T}_A)^{(A')}E
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T}C^*A' \arrow[r, "\alpha^*"] \arrow[d, "\mathrm{T}C^*q"'] & C^*\mathrm{T}_AA' \arrow[d, "C^*\mathrm{T}_Aq"] \\
\mathrm{T}C^*E \arrow[r, "\alpha^*"'] & C^*\mathrm{T}_AE
\end{tikzcd}$$

are all pushout diagrams, where q_* is the morphism defined by the pushout diagram in $\mathsf{Alg}_{\mathscr{P}^{(A)}}$:

$$\begin{tikzcd}
T_AA' \arrow[r, "\mathrm{T}_Aq"] \arrow[d, "z_A"'] & T_AE \arrow[d, "\xi_q"] \\
A' \arrow[r, "q_*"'] \arrow[ur, phantom, "\ulcorner", very near end] & (\mathrm{T}_A)^{(A')}E
\end{tikzcd}$$

Let's then consider the first diagram. Under the identification $\mathsf{Alg}_{\mathscr{P}^{(A)}} \cong \mathsf{Alg}_{\mathscr{P}}/A$, the functor $C^*\colon \mathsf{Alg}_{\mathscr{P}^{(A)}} \cong \mathsf{Alg}_{\mathscr{P}}/A \to \mathsf{Alg}_{\mathscr{P}}$ coincides with the forgetful functor, which sends a morphism $f\colon A \to A'$ of $\mathscr{P}$-algebras to the $\mathscr{P}$-algebra A'. Notice that the forgetful functor $\mathbb{X}/M \to \mathbb{X}$ from the slice category of a category $\mathbb{X}$ over an object M of $\mathbb{X}$ preserves all connected limits, therefore, dually, the forgetful functor $M/\mathbb{X} \to \mathbb{X}$ preserves all connected colimits. In particular, pushouts are connected colimits and thus C^* preserves all pushouts. Thanks to this general fact, we conclude that the first diagram is a pushout.

Finally, let's prove that the diagram which expresses the naturality of α^* is also a pushout. The first step is to lift α^* to a morphism of $\mathscr{P}^{(A)}$-algebras $\overline{\alpha^*}$ so that $C^*(\overline{\alpha^*}) = \alpha^*$. Secondly, we are going to show that $\overline{\alpha^*}$ is a coequalizer morphism from direct inspection, and finally, we use that C^* preserves the universal property of $\overline{\alpha^*}$ to conclude our result.

Let's start by noticing that, since A' is a $\mathscr{P}^{(A)}$-algebra it corresponds to a morphism of $\mathscr{P}$-algebras $\beta\colon A \to C^*A'$. Moreover, using the projection we obtain a morphism $A \xrightarrow{\beta} C^*A' \xrightarrow{p} \mathrm{T}C^*A'$ of $\mathscr{P}$-algebras which defines a new $\mathscr{P}^{(A)}$-algebra $\overline{\mathrm{T}C^*A'}$. Concretely, this is the $\mathscr{P}^{(A)}$-algebra defined over $\mathrm{T}C^*A'$ whose structure map is defined by:

$$(\mu; a_1, \dots, a_k | (x_1, \dots, x_m) \colon= \mu_{\mathrm{T}C^*A'}(\beta(a_1), \dots, \beta(a_k), x_1, \dots, x_m)$$

Then, it is not hard to see that α^* can be lifted to a morphism of $\mathscr{P}^{(A)}$-algebras $\overline{\alpha^*}\colon \overline{\mathrm{T}C^*A'} \to \mathrm{T}_A A'$, which sends an element $y \in \mathrm{T}C^*A'$ to $\alpha^*(y) \in \mathrm{T}_A A'$. Recall also that, by construction, α^* sends the generators b and $\mathsf{d}b$ of $\mathrm{T}C^*A'$ to the corresponding generators b and $\mathsf{d}_A b$ of $C^*\mathrm{T}_A A'$.

By direct inspection we see that the $\mathscr{P}^{(A)}$-algebra $\mathrm{T}_A A'$ is generated by all $b \in A'$ and by symbols $\mathsf{d}_A b$ for $b \in A'$, satisfying the following properties:

$$\begin{aligned}
&(\mu; a_1, \dots, a_k |_{\mathrm{T}_A A'}(b_1, \dots, b_m) = (\mu; a_1, \dots, a_k |_{A'}(b_1, \dots, b_m) = \\
&\qquad = \mu_{C^*A'}(\beta(a_1), \dots, \beta(a_k), b_1, \dots, b_m) \\
&\mathsf{d}_A((\mu; a_1, \dots, a_k | (b_1, \dots, b_m)) = \sum_{j=1}^{m} (\mu; a_1, \dots, a_k | (b_1, \dots, \mathsf{d}_A b_j, \dots, b_m) \\
&= \sum_{j=1}^{m} \mu_{C^*\mathrm{T}_A A'}(\beta(a_1), \dots, \beta(a_k), b_1, \dots, \mathsf{d}_A b_j, \dots, b_m)
\end{aligned}$$

Similarly, it is not hard to see that $\overline{\mathrm{T}C^*A'}$ is also generated by $b \in A'$ and by symbols

$\mathsf{d}b$, for $b \in A'$, satisfying the following properties:

$$
\begin{aligned}
&(\mu; a_1, \dots, a_k|_{\overline{\mathrm{T}C^*A'}}(b_1, \dots, b_m) = \\
&\qquad = \mu_{\mathrm{T}C^*A'}(\beta(a_1), \dots, \beta(a_k), b_1, \dots, b_m) = \mu_{C^*A'}(\beta(a_1), \dots, \beta(a_k), b_1, \dots, b_m) \\
&\mathsf{d}(\mu(b_1, \dots, b_m)) = \sum_{j=1}^{m} \mu(b_1, \dots, \mathsf{d}b_j, \dots, b_m)
\end{aligned}
$$

It is clear from this that the relations of $\mathrm{T}_A A'$ imply the ones of $\overline{\mathrm{T}C^*A'}$. Since $\overline{\alpha^*}$ sends generators to corresponding generators, this implies that $\mathrm{T}_A A'$ can be represented as a quotient algebra of $\overline{\mathrm{T}C^*A'}$ over a specific ideal I, that is $\mathrm{T}_A A' \cong \overline{\mathrm{T}C^*A'}/I$, and that $\overline{\alpha^*}$ is the quotient map $\overline{\mathrm{T}C^*A'} \to \overline{\mathrm{T}C^*A'}/I$. Direct inspection shows that the ideal I is generated by all the $\mathsf{d}_A(\beta(a))$ for every $a \in A$, that is in $\mathrm{T}_A A'$, $\mathsf{d}_A(\beta(a)) = 0$.

Using a similar argument as the one we used to prove that the first diagram was a pushout, we conclude also that α^* is a quotient map $\mathrm{T}C^*A' \to C^*\mathrm{T}_A A'$, so that $C^*\mathrm{T}_A A'$ is a quotient algebra of $\mathrm{T}C^*A'$ over an ideal I generated by $\mathsf{d}_A(\beta(a)) = 0$.

Let's now come back to the naturality diagram and consider $g\colon \mathrm{T}C^*E \to K$ and $h\colon C^*\mathrm{T}_A A' \to K$ as follows:

$$
\begin{tikzcd}
\mathrm{T}C^*A' \arrow[r, "\alpha^*"] \arrow[d, "\mathrm{T}C^*q"'] & C^*\mathrm{T}_A A' \arrow[d, "C^*\mathrm{T}_A q"] \arrow[ddr, bend left, "h"] & \\
\mathrm{T}C^*E \arrow[r, "\alpha^*"'] \arrow[drr, bend right, "g"'] & C^*\mathrm{T}_A E & \\
& & K
\end{tikzcd}
$$

This implies that:

$$
\begin{aligned}
h(b) &= g(q(b)) \\
h(\mathsf{d}_A b) &= g(q(\mathsf{d}b)) = g(\mathsf{d}q(b))
\end{aligned}
$$

for every $b \in A'$. Notice that, since E is a $\mathscr{P}^{(A)}$-algebra, we can also define a morphism of $\mathscr{P}$-algebras $\gamma\colon A \to E$ and that since q is a morphism of $\mathscr{P}^{(A)}$-algebras we have that $q(\beta(a)) = \gamma(a)$. So, to lift g to $C^*\mathrm{T}_A E$ we need to show that $g(\mathsf{d}\gamma(a)) = 0$, however, we have the following:

$$
g(\mathsf{d}\gamma(a))
$$

$$\begin{aligned} &= g(\mathsf{d}f(\beta(a)) \\ &= g(\mathsf{d}h(\mathsf{d}_A\beta(a))) \\ &= 0 \end{aligned}$$

where we used that $\mathsf{d}_A\beta(a) = 0$. This proves that we can lift g to $\mathrm{T}C^*E/I = C^*\mathrm{T}_A E$, that is we find a morphism $\overline{[g,h]}\colon C^*\mathrm{T}_A A' \to K$. We leave it to the reader to prove that such a morphism is the unique morphism which makes commutative the following diagram:

$$\begin{tikzcd}
\mathrm{T}C^*A' \arrow[r, "\alpha^*"] \arrow[d, "\mathrm{T}C^*q"'] & C^*\mathrm{T}_A A' \arrow[d, "C^*\mathrm{T}_A q"] \arrow[ddr, bend left, "h"] & \\
\mathrm{T}C^*E \arrow[r, "\alpha^*"'] \arrow[drr, bend right, "g"'] & C^*\mathrm{T}_A E \arrow[dr, dashed, "{[g,h]}"] & \\
& & K
\end{tikzcd}$$

This concludes the proof. □

We can prove the main result of this chapter.

Proposition 4.16. *Consider the tangent morphism obtained as follows:*

$$\begin{aligned} &\mathsf{Geom}^* \circ \mathsf{Env} \xrightarrow{(U;\eta)_{\mathsf{Geom}^*\mathsf{Env}}} \mathsf{Slice} \circ \mathsf{Term} \circ \mathsf{Geom}^* \circ \mathsf{Env} \cong \\ &\cong \mathsf{Slice} \circ \mathsf{Geom}^* \circ \mathsf{Init} \circ \mathsf{Env} \xrightarrow{\mathsf{Slice}(\mathsf{Geom}^*(C,\varepsilon))} \mathsf{Slice} \circ \mathsf{Geom}^* \end{aligned}$$

This defines an equivalence of pseudofunctors which makes the following diagram commutative:

$$\begin{tikzcd}
\mathsf{Operad}^{\mathsf{op}} \arrow[d, "\mathsf{Geom}^*"'] & \mathsf{OprPair}^{\mathsf{op}} \arrow[l, "\mathsf{Env}"'] \arrow[d, "\mathsf{Geom}^*"] \\
\mathsf{cTngCat} & \mathsf{TngPair} \arrow[l, "\mathsf{Slice}"]
\end{tikzcd}$$

Proof. By Theorem 4.12, (U,η) is an equivalence of tangent categories. Moreover, thanks to Lemma 4.15, $\mathsf{Geom}^*(C,\varepsilon)$ is a Cartesian morphism of tangent pairs. By Lemma 4.11, Slice maps Cartesian morphisms into strong tangent morphisms.

Thus, $\mathsf{Slice}(\mathsf{Geom}^*(C,\varepsilon))$ is strong. Finally, thanks to [6, Lemma 1.7] the functorial component of $\mathsf{Slice}(\mathsf{Geom}^*(C,\varepsilon))$ is an isomorphism between the categories of $\mathscr{P}^{(A)}$-algebras and the coslice category of $\mathscr{P}$-algebras under A, i.e. the slice category $\mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}}/A$. Therefore, $\mathsf{Slice}(\mathsf{Geom}^*(C,\varepsilon))$ is an equivalence of tangent categories. □

Theorem 4.17. *Given an operad $\mathscr{P}$ and a $\mathscr{P}$-algebra A, the geometric tangent category of the enveloping operad $\mathscr{P}^{(A)}$ of $\mathscr{P}$ over A is equivalent, as a tangent category, to the slice tangent category over A of the geometric tangent category of $\mathscr{P}$. In formulas:*

$$\mathsf{Geom}(\mathscr{P}^{(A)}) = \mathsf{Geom}(\mathscr{P})/A$$

Thanks to this characterization, we can now understand the vector fields over a $\mathscr{P}^{(A)}$-algebra. For this purpose, recall that for a morphism of $\mathscr{P}$-algebras $\beta\colon A \to A'$ and a A'-module M (see Section 2.2.5 for details) a β-relative derivation is a derivation $\delta\colon A' \to M$, i.e. an R-linear morphism which satisfies the Leibniz rule:

$$\delta(\mu(b_1,\dots,b_m)) = \sum_{k=1}^{m} \mu(b_1,\dots,\delta(b_k),\dots,b_m)$$

and moreover $\delta \circ \beta = 0$.

Corollary 4.18. *For an operad $\mathscr{P}$, a $\mathscr{P}$-algebra A, and a $\mathscr{P}^{(A)}$-algebra A', the vector fields over A' in the geometric tangent category of $\mathscr{P}^{(A)}$ are in bijective correspondence with β-relative derivations, $\beta\colon A \to C^*A'$ being the morphism of $\mathscr{P}$-algebras corresponding to the $\mathscr{P}^{(A)}$-algebra A'.*

Proof. Recall that in [29, Corollary 4.5.3] it was proved that vector fields in a geometric tangent category of an operad correspond to derivations over the operadic algebras. Concretely, a vector field $v\colon \mathrm{T}A \to A$, regarded as a morphism of $\mathscr{P}$-algebras, corresponds to a derivation $\delta_v\colon A \to A$ defined by:

$$\delta_v(a) := v(\mathsf{d}a)$$

Viceversa, a derivation δ defines a vector field $v_\delta\colon \mathrm{T}A \to A$ by:

$$v(a) := a$$
$$v(\mathsf{d}a) := \delta(a)$$

Thanks to Theorem 4.17, we have that $\mathsf{Geom}(\mathscr{P}^{(A)}) \cong \mathsf{Geom}(\mathscr{P})/A$, thus, given a morphism $\beta\colon A \to C^*A'$, by definition of the slice tangent category, the tangent bundle functor $\mathrm{T}^{(A)}$ of $\mathsf{Geom}(\mathscr{P})/A$ is given by the coequalizer (in the category of $\mathscr{P}$-algebras):

$$\begin{tikzcd} \mathrm{T}C^*A \arrow[r, shift left, "\mathrm{T}\beta"] \arrow[r, shift right, "\mathrm{T}\beta z p"'] & \mathrm{T}C^*A' \arrow[r, dashed, "v_\beta"] & \mathrm{T}^{(A)}A' \end{tikzcd}$$

or equivalently, by the pushout diagram:

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}\beta"] \arrow[d, "z"'] & \mathrm{T}C^*A' \arrow[d, "v_\beta"] \\ A \arrow[r, "\beta_*"'] & \mathrm{T}^{(A)}A' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This implies that $\mathrm{T}^{(A)}A'$ is the quotient of $\mathrm{T}C^*A'$ by the ideal generated by $\mathsf{d}\beta(a)$, for every $a \in A$. Therefore, a vector field $v\colon \mathrm{T}^{(A)}A' \to A'$ corresponds to a derivation $\delta_v\colon A' \to A'$ defined by $\delta_v(b) := v(\mathsf{d}b)$, and satisfying the following:

$$\delta_v(\beta(a)) = v(\mathsf{d}\beta(a)) = 0$$

that is a β-relative derivation of A'. Conversely, a β-relative derivation $\delta\colon A' \to A'$ being a derivation over A', defines a vector field $v_\delta\colon \mathrm{T}C^*A' \to C^*A'$ over C^*A' by $v_\delta(b) := b$ and $v_\delta(\mathsf{d}b) = \delta(b)$, but since δ is β-relative, $v_\delta(\mathsf{d}\beta(a)) = \delta(\beta(a)) = 0$, thus v_δ lifts to $\mathrm{T}^{(A)}A' \to A'$. □

4.3 The classification of differential bundles

This section is dedicated to the classification of differential bundles in the geometric tangent category of a given algebraic operad. Our approach is in two main steps: first, we classify differential objects in the geometric tangent category of an arbitrary operad, and second, we employ that differential objects in the slice tangent category over a given object are precisely (display) differential bundles. Theorem 4.17 proves that the slice tangent category $\mathsf{Geom}(\mathscr{P})/A$ of the geometric tangent category of an operad $\mathscr{P}$, over an affine scheme A, is equivalent to the geometric tangent category of the enveloping operad $\mathscr{P}^{(A)}$ of $\mathscr{P}$ over A. Thus, differential bundles over A in $\mathsf{Geom}(\mathscr{P})$ are differential objects in $\mathsf{Geom}(\mathscr{P}^{(A)})$.

In Section 4.3.1 we prove that differential objects in $\mathsf{Geom}(\mathscr{P})$ are equivalent to left modules over the unital and associative ring $\mathscr{P}(1)$. This implies that differential bundles over an arbitrary operadic affine scheme A of an operad $\mathscr{P}$ are equivalent to modules over the $\mathscr{P}$-algebra A. We also show that linear morphisms (linear in the tangent category sense) between differential bundles correspond to linear morphisms (linear in the algebraic sense) between the corresponding modules in a contravariant fashion. In a nutshell, we prove the following equivalence of categories:

$$\mathsf{DBnd}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P}); A) \cong \mathsf{Mod}^{\mathsf{op}}(\mathscr{P}; A)$$

where on the left we denote the category of differential bundles of $\mathsf{Geom}(\mathscr{P})$ over A and linear morphisms and on the right the opposite of the category of modules over the operadic algebra A, in the operadic sense.

Finally, we show that this equivalence is also an equivalence between tangent categories, where the tangent structure of $\mathsf{Mod}^{\mathsf{op}}(\mathscr{P}; A)$ is the adjoint tangent structure of the one induced by biproducts, and the one of $\mathsf{DBnd}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P}); A)$ is the restriction to differential bundles and linear morphisms of the slice tangent structure of $\mathsf{Geom}(\mathscr{P})/A$.

4.3.1 The classification of differential objects

Intuitively speaking, differential objects, reviewed in Section 2.2.2, are objects in a tangent category which behave like Euclidean spaces in the category of smooth manifolds: they have a distinct point, the zero, they have a translation symmetry, axiomatized by the sum operation, and their tangent bundle is trivial.

In Section 3.4 we proved that the monad associated with an operad, assuming the base monoidal category to have biproducts as in Convention 3.20, is a coCartesian differential monad. We also discussed how a coCartesian differential monad is precisely a monad for which the associated coKleisli category, i.e. the opposite of the Kleisli category, is a Cartesian differential category. Since differential objects also form a Cartesian differential category, it is natural to wonder whether or not the differential objects of the geometric tangent category of an operad form precisely the coKleisli category of the associated coCartesian differential monad.

To investigate this question, first consider a tangent monad (S,α) over an adjunctable tangent category $(\mathbb{X},\mathbb{L})$ and whose category of algebras admits reflexive coequalizers, so that $\mathsf{Alg}_S^{\mathsf{op}}$ is a tangent category denoted by $\mathsf{Geom}(S)$. Therefore, the free functor $F\colon \mathbb{X} \to \mathsf{Alg}_S$ which sends objects A of the base category $\mathbb{X}$ to the corresponding free algebras $F(A) = (SA,\gamma)$, γ being the monad multiplication, extends to a strong tangent morphism $(F,\tau)\colon (\mathbb{X}^{\mathsf{op}},\mathbb{T}) \to \mathsf{Geom}(S)$ between the adjoint tangent categories.

To understand why this is the case, recall that the forgetful functor $U\colon \mathsf{Alg}(S) \to (\mathbb{X},\mathbb{L})$ is a strict tangent morphism. Employing Lemma 3.78 to the strict, so in particular strong, tangent morphism U we conclude that the free functor $F\colon \mathbb{X} \to \mathsf{Alg}_S^{\mathsf{op}}$, which is the left adjoint of U, extends to a strong tangent morphism over the adjoint tangent categories: $(F,\tau)\colon (\mathbb{X}^{\mathsf{op}},\mathbb{T}) \to \mathsf{Geom}(S)$, where we recall that $\mathsf{Geom}(S)$ denotes the adjoint tangent category of $\mathsf{Alg}(S)$. Strong tangent morphisms preserve differential objects so all free S-algebras generated by differential objects of the base tangent category $(\mathbb{X},\mathbb{L})$ are also differential objects of $\mathsf{Geom}(S)$.

Proposition 4.19. *If (S,α) is a tangent monad over an adjunctable tangent category $(\mathbb{X},\mathbb{L})$ and the category of algebras of S have reflexive coequalizers, the free functor $F\colon \mathbb{X} \to \mathsf{Alg}$ restricts to a functor $\mathsf{DObj}(\mathbb{X},\mathbb{T}) \to \mathsf{DObj}(\mathsf{Geom}(S))$.*

Theorem 4.20. *The free algebras of a coCartesian differential monad (S,∂) are differential objects in the geometric tangent category $\mathsf{Geom}(S)$ of (S,∂). In particular, free algebras of an operad $\mathscr{P}$ defined over a monoidal category which satisfies Convention 3.20, are differential objects in $\mathsf{Geom}(\mathscr{P})$.*

Proof. Thanks to Proposition 4.19 the free functor $F\colon \mathbb{X} \to \mathsf{Alg}_S$ extends to a strong tangent morphism $(\mathbb{X},\mathbb{L}) \to \mathsf{Geom}(S)$. Moreover, since $\mathbb{L}$ is the tangent structure induced by biproducts, every object of $\mathbb{X}$ is a differential object in $(\mathbb{X},\mathbb{L})$. In particular, $\mathsf{DObj}(\mathbb{X},\mathbb{T}) \cong \mathbb{X}$. Since strong tangent morphisms preserve differential objects, every free S-algebra is a differential object in $\mathsf{Geom}(S)$. $\square$

Unfortunately, this result does not guarantee the converse: there could be differential objects in $\mathsf{Geom}(\mathscr{P})$ which are not free algebras. Indeed, Example 3.14

constitutes a counterexample. We leave it to future work to classify for which tangent monads differential objects in their geometric tangent categories are precisely the free algebras.

To provide a complete classification of differential objects of the geometric tangent category of an operad we first recall that, in a Cartesian tangent category, differential objects can be regarded as differential bundles over the terminal object (cf. [11, Proposition 3.4]).

Proposition⋆ 4.21. *In a Cartesian tangent category, differential objects are precisely differential bundles over the terminal object.*

Interestingly, also (linear) morphisms of differential objects are carried over by the correspondence between differential objects and differential bundles over the terminal object and so it extends to an equivalence of categories. For this purpose, recall that a morphism of differential objects is linear if it preserves the differential projection and a morphism of differential bundles is linear if it preserves the vertical lift.

Proposition 4.22. *Given a Cartesian tangent category* $(\mathbb{X},\mathbb{T})$*, the category* $\mathsf{DObj}_{\mathsf{lnr}}(\mathbb{X},\mathbb{T})$ *of differential objects and linear morphisms is equivalent to the category* $\mathsf{DBnd}_{\mathsf{lnr}}((\mathbb{X},\mathbb{T});*)$ *of differential bundles over the terminal object* $*$ *and linear morphisms.*

The second step is to recall that, in the presence of negatives, differential bundles can be fully characterized as pre-differential bundles satisfying Rosický's universality condition, as shown by MacAdam in [47].

Definition⋆ 4.23. *A* ***pre-differential bundle*** *in a tangent category* $(\mathbb{X},\mathbb{T})$ *consists of a morphism* $q\colon E\to M$ *together with a section* $z_q\colon M\to E$*, called the* ***zero morphism*** *and morphism* $l_q\colon E\to\mathrm{T}E$*, called the* ***vertical lift****, for which the following axioms hold:*

$$
\begin{tikzcd}
M \arrow[r, "z_q"] \arrow[dr, equal] & E \arrow[d, "q"] \\
& M
\end{tikzcd}
\quad
\begin{tikzcd}
E \arrow[r, "l_q"] \arrow[d, "q"'] & \mathrm{T}E \arrow[d, "p"] \\
M \arrow[r, "z_q"'] & E
\end{tikzcd}
\quad
\begin{tikzcd}
E \arrow[r, "l_q"] & \mathrm{T}E \\
M \arrow[u, "z_q"] \arrow[r, "z_q"'] & E \arrow[u, "z"']
\end{tikzcd}
\quad
\begin{tikzcd}
\mathrm{T}E \arrow[r, "l"] & \mathrm{T}^2E \\
E \arrow[u, "l_q"] \arrow[r, "l_q"'] & \mathrm{T}E \arrow[u, "\mathrm{T}l_q"']
\end{tikzcd}
$$

*Moreover, in a Cartesian tangent category $(\mathbb{X}, \mathbb{T})$, a pre-differential bundle is **universal** if the n-fold tangent pullback of the projection q along itself exists and the following diagram:*

$$\begin{tikzcd} E \arrow[r, "l_q"] \arrow[d, "q"'] & \mathrm{T}E \arrow[d, "\langle \mathrm{T}q, p\rangle"] \\ M \arrow[r, "\langle z, z_q\rangle"'] & \mathrm{T}M \times E \end{tikzcd}$$

is a tangent pullback.

MacAdam proved (cf. [47, Corollary 2.2.5]) that, when a tangent category has negatives, it suffices to have a universal pre-differential bundle to define a differential bundle uniquely. In particular, a universal pre-differential bundle comes with a sum morphism $s_q \colon E_2 \to E$. Let's restate this important result.

Proposition* 4.24. *In a tangent category with negatives, differential bundles are equivalent to universal pre-differential bundles.*

Since the geometric tangent category of an algebraic operad has negatives, in our proof, we treat differential objects as universal pre-differential bundles over the terminal object. Recall also that the terminal object of $\mathsf{Geom}(\mathscr{P})$ is the initial $\mathscr{P}$-algebra $\mathscr{P}(0)$. With this in mind, let's start by showing that the functor $\mathsf{Free}_{\mathscr{P}(0)}$ sends modules over the initial $\mathscr{P}$-algebra $\mathscr{P}(0)$ to differential objects of $\mathsf{Geom}(\mathscr{P})$.

Proposition 4.25. *Let $M \in \mathsf{Mod}_{\mathscr{P}(0)}$ be a $\mathscr{P}(0)$-module (in the operadic sense, see Definition 3.57). Therefore, the $\mathscr{P}$-algebra $\mathsf{Free}_{\mathscr{P}(0)}M$ is a differential object in $\mathsf{Geom}(\mathscr{P})$.*

Proof. First note that the functor $\mathsf{Free}_{\mathscr{P}(0)} \colon \mathsf{Mod}_{\mathscr{P}(0)} \to \mathsf{Alg}_{\mathscr{P}}$ is well-defined since $\mathscr{P}(0)$ is a $\mathscr{P}$-algebra. Let's start by defining the zero-morphism of $\mathsf{Free}_{\mathscr{P}(0)}M$. Let's recall that $\mathsf{Free}_{\mathscr{P}(0)}M$ is the $\mathscr{P}$-algebra generated by pairs $(\alpha, x) \in \mathscr{P}(0) \times M$, satisfying the following relations:

$$(\mu; (\alpha_1, 0), \ldots, (\alpha_k, x), \ldots, (\alpha_n, 0)) = (\mu(\alpha_1, \ldots, \alpha_n), \mu(\alpha_1, \ldots, \alpha_{k-1}, x, \alpha_{k+1}, \ldots, \alpha_n)) \tag{4.3.1}$$

for every $\mu \in \mathscr{P}(n)$, $\alpha_1, \ldots, \alpha_n \in \mathscr{P}(0)$, $x \in M$, and any positive integer n. Let's consider the morphism:

$$\zeta \colon \mathsf{S}_{\mathscr{P}}(\mathscr{P}(0) \times M) \xrightarrow{\mathsf{S}_{\mathscr{P}}(\pi_1)} \mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)) \xrightarrow{\theta} \mathscr{P}(0)$$

where θ is the structure map of the $\mathscr{P}$-algebra $\mathscr{P}(0)$. This is a well-defined $\mathscr{P}$-algebra morphism since it is the composition of $\mathscr{P}$-algebra morphisms. Let's prove that this morphism lifts to $\mathsf{Free}_{\mathscr{P}(0)}M$ by showing that it is compatible with the relations (4.3.1):

$$\begin{aligned}
&\zeta(\mu;(\alpha_1,0),\dots,(\alpha_k,x),\dots,(\alpha_n,0))\\
=\;&\theta(S(\mathscr{P},\pi_1)(\mu;(\alpha_1,0),\dots,(\alpha_k,x),\dots,(\alpha_n,0)))\\
=\;&\theta(\mu;\alpha_1,\dots,\alpha_n)\\
=\;&\mu(\alpha_1,\dots,\alpha_n)\\
=\;&\zeta(\mu(\alpha_1,\dots,\alpha_1),\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))
\end{aligned}$$

This proves that ζ lifts to the quotient and therefore it provides a well-defined $\mathscr{P}$-algebra morphism $\mathsf{Free}_{\mathscr{P}(0)}M \to \mathscr{P}(0)$ that, abusing notation, will be also denoted by ζ. The second step is to provide a vertical lift for $\mathsf{Free}_{\mathscr{P}(0)}M$. To define this morphism, note that the tangent bundle functor T preserves colimits since it is a left-adjoint. This allows us to regard $\mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M$ as the $\mathscr{P}$-algebra generated by pairs $(\alpha,x)\in\mathscr{P}(0)\times M$ and symbols $\mathsf{d}(\alpha,x)$, for $(\alpha,x)\in\mathscr{P}(0)\times M$, satisfying the relations (4.3.1) and the following:

$$\begin{aligned}
&\mathsf{d}(\mu(\alpha_1,\dots,\alpha_n),\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))=\\
&=\sum_{i\neq k}(\mu;(\alpha_1,0),\dots,\mathsf{d}(\alpha_i,0),\dots,(\alpha_k,x),\dots,(\alpha_n,0))+\\
&\quad+(\mu;(\alpha_1,0),\dots,\mathsf{d}(\alpha_k,x),\dots,(\alpha_n,0))
\end{aligned}\tag{4.3.2}$$

Let's consider the morphism:

$$\begin{aligned}
&\lambda\colon \mathrm{T}\mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)\times M)\to \mathsf{Free}_{\mathscr{P}(0)}M\\
&\lambda(\alpha,x):=[(\alpha,0)]\\
&\lambda(\mathsf{d}(\alpha,x)):=[(0,x)]
\end{aligned}$$

We used the square brackets to indicate the equivalence classes in the quotient $\mathsf{Free}_{\mathscr{P}(0)}M$. For Theorem 4.20, the free algebra $\mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)\times M)$ is a differential object, so we conclude that $\mathrm{T}(\mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)\times M)\cong \mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)\times M)\times \mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)\times M)\cong \mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)\times M\times\mathscr{P}(0)\times M)$. Therefore, $\mathrm{T}\mathsf{S}_{\mathscr{P}}(\mathscr{P}(0)\times M)$ is a free $\mathscr{P}$-algebra and then λ is a well-defined $\mathscr{P}$-algebra morphism. The next step is to show that λ lifts to $\mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M$

by showing that it is compatible with the relations (4.3.1) and (4.3.2). Let's start by proving the compatibility with (4.3.1):

$$\begin{aligned}
& \lambda(\mu;(\alpha_1,0),\dots,(\alpha_k,x),\dots,(\alpha_n,0)) \\
= & [(\mu;(\alpha_1,0),\dots,(\alpha_n,0)] \\
= & [(\mu(\alpha_1,\dots,\alpha_n),0)] \\
= & \lambda(\mu(\alpha_1,\dots,\alpha_n),\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))
\end{aligned}$$

where in the second passage we used Equation (4.3.1). Let's prove the compatibility with the relations (4.3.2). First notice that:

$$\begin{aligned}
& \lambda(\mathsf{d}(\mu(\alpha_1,\dots,\alpha_n),\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))) = \\
& \quad = [(0,\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n)]
\end{aligned}$$

On the other hand:

$$\begin{aligned}
& \lambda\left(\sum_{i\neq k}(\mu;(\alpha_1,0),\dots,\mathsf{d}(\alpha_i,0),\dots,(\alpha_k,x),\dots,(\alpha_n,0))+\right. \\
& \qquad \left.+(\mu;(\alpha_1,0),\dots,\mathsf{d}(\alpha_k,x),\dots,(\alpha_n,0))\right) \\
= & \sum_{i\neq k}[(\mu;(\alpha_i,0),\dots,(0,0),\dots,(\alpha_k,0),\dots,(\alpha_n,0))]+ \\
& \qquad +[(\mu;(\alpha_1,0),\dots,(0,x),\dots,(\alpha_n,0))] \\
= & [(0,\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))]
\end{aligned}$$

This proves that λ is compatible with the relations of $\mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M$ and that it can be lifted to a morphism of $\mathscr{P}$-algebras $\mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M \to \mathsf{Free}_{\mathscr{P}(0)}M$, that, abusing notation, will be denoted by λ. The next step is to show that (A,ζ,λ) is a universal pre-differential bundle over $\mathscr{P}(0)$. Notice first that in $\mathsf{Geom}(\mathscr{P})$ every map is a $\mathrm{T}^{(\mathscr{P})}$-display map, so we do not need to prove the existence of the n-fold tangent pushout (regarding the maps as morphisms in $\mathsf{Alg}_{\mathscr{P}}$) of the projection along itself.

Let's start by showing the compatibility between the vertical lift l and λ, i.e. $\lambda\circ l=\lambda\circ\mathrm{T}\lambda$. Using the presentation of $\mathrm{T}^2\mathsf{Free}_{\mathscr{P}(0)}M$, we show the equivalence of the two morphisms on generators as follows:

$$\begin{aligned}
\lambda(l(\alpha,x)) &= \lambda(\alpha,x)=(\alpha,0) \\
\lambda(l(\mathsf{d}(\alpha,x))) &= \lambda(0,0)=(0,0)
\end{aligned}$$

$$\lambda(l(\mathsf{d}'(\alpha, x))) = \lambda(0, 0) = (0, 0)$$
$$\lambda(l(\mathsf{d}'\mathsf{d}(\alpha, x))) = \lambda(\mathsf{d}(\alpha, x)) = (0, x)$$

$$\lambda(\mathrm{T}\lambda(\alpha, x)) = \lambda(\alpha, x) = (\alpha, 0)$$
$$\lambda(\mathrm{T}\lambda(\mathsf{d}(\alpha, x))) = \lambda(0, x) = (0, 0)$$
$$\lambda(\mathrm{T}\lambda(\mathsf{d}'(\alpha, x))) = \lambda(\mathsf{d}(\alpha, 0)) = (0, 0)$$
$$\lambda(\mathrm{T}\lambda(\mathsf{d}'\mathsf{d}(\alpha, x))) = \lambda(\mathsf{d}(0, x)) = (0, x)$$

The next step is to prove the compatibility between the projection and λ, i.e. $\lambda \circ p = ! \circ \zeta$, where $! :: \mathscr{P}(0) \to \mathsf{Free}_{\mathscr{P}}(0)M$ is the unique $\mathscr{P}$-algebra morphism defined by $!\alpha := [(\alpha, 0)]$. Thus, on generators:

$$\lambda(p(\alpha, x)) = \lambda(\alpha, x) = (\alpha, 0) = !(\alpha) = !(\zeta(\alpha, x))$$

Let's show the compatibility between λ and the zero morphism, i.e. $\zeta \circ \lambda = \zeta \circ z$:

$$\zeta(\lambda(\alpha, x)) = \zeta(\alpha, 0) = \alpha$$
$$\zeta(\lambda(\mathsf{d}(\alpha, x))) = \zeta(0, x) = 0$$

$$\zeta(z(\alpha, x)) = \zeta(\alpha, x) = \alpha$$
$$\zeta(z(\mathsf{d}(\alpha, x))) = \zeta(0, 0) = 0$$

This proves that $(\mathsf{Free}_{\mathscr{P}(0)}M, \zeta, \lambda)$ is a pre-differential bundle over $\mathscr{P}(0)$. Let's prove the universality of the vertical lift, which corresponds to stating that the diagram:

$$\begin{tikzcd}
\mathscr{P}(0) \sqcup \mathsf{Free}_{\mathscr{P}(0)}M \arrow[r, "{[\mathrm{T}!, p]}"] \arrow[d, "{[z,\zeta]}"'] & \mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M \arrow[d, "\lambda"] \\
\mathscr{P}(0) \arrow[r, "!"'] & \mathsf{Free}_{\mathscr{P}(0)}M
\end{tikzcd}$$

is a pushout diagram. Since the tangent bundle functor is a left adjoint it follows directly that this is also a tangent pushout. For this purpose, consider two morphisms $f\colon \mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M \to X$ and $g\colon P(0) \to X$ of $\mathscr{P}$-algebras, making the following diagram commutative:

$$\begin{tikzcd}
\mathscr{P}(0) \sqcup \mathsf{Free}_{\mathscr{P}(0)}M \arrow[r, "{[\mathrm{T}!, p]}"] \arrow[d, "{[z,\zeta]}"'] & \mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M \arrow[d, "\lambda"] \arrow[ddr, bend left, "f"] & \\
\mathscr{P}(0) \arrow[r, "!"'] \arrow[drr, bend right, "g"'] & \mathsf{Free}_{\mathscr{P}(0)}M & \\
& & X
\end{tikzcd}$$

We want to provide a morphism $h\colon \mathsf{Free}_{\mathscr{P}(0)}M \dashrightarrow X$ of $\mathscr{P}$-algebras so that:

$$\begin{tikzcd}
\mathscr{P}(0) \sqcup \mathsf{Free}_{\mathscr{P}(0)}M \arrow[r, "{[\mathrm{T}!, p]}"] \arrow[d, "{[z,\zeta]}"'] & \mathrm{T}\mathsf{Free}_{\mathscr{P}(0)}M \arrow[d, "\lambda"] \arrow[ddr, bend left, "f"] & \\
\mathscr{P}(0) \arrow[r, "!"'] \arrow[drr, bend right, "g"'] & \mathsf{Free}_{\mathscr{P}(0)}M \arrow[dr, dashed, "h"] & \\
& & X
\end{tikzcd}$$

commutes. First, note that, since $f \circ [\mathrm{T}!, p] = g \circ [z, \zeta]$ we have that:

$$\begin{aligned} f(\alpha, x) &= g(\alpha) = f(\alpha, 0) \\ f(\mathsf{d}(\alpha, 0)) &= 0 \end{aligned}$$

Thus, let's define h on generators as follows:

$$h(\alpha, x) := g(\alpha) + f(\mathsf{d}(0, x))$$

Let's first prove that h is well-defined, i.e. that it is compatible with the relations (4.3.1):

$$\begin{aligned} & h(\mu; (\alpha_1, 0), \dots, (\alpha_k, x), \dots, (\alpha_n, 0)) \\ = \ & \mu(g(\alpha_1), \dots, g(\alpha_k) + f(\mathsf{d}(0, x)), \dots, g(\alpha_n)) \\ = \ & \mu(g(\alpha_1), \dots, g(\alpha_k), \dots, g(\alpha_n)) + \mu(g(\alpha_1), \dots, f(\mathsf{d}(0, x)), \dots, g(\alpha_n)) \end{aligned}$$

$$
\begin{aligned}
&= \mu(f(\alpha_1,0),\dots,f(\alpha_k,0),\dots,f(\alpha_n,0)) + \mu(f(\alpha_1,0),\dots,f(\mathsf{d}(0,x)),\dots,f(\alpha_n,0)) \\
&= f[(\mu;(\alpha_1,0),\dots,(\alpha_n,0)] + f[(\mu;(\alpha_1,0),\dots,\mathsf{d}(0,x),\dots,(\alpha_n,0))] \\
&= f[(\mu(\alpha_1,\dots,\alpha_n),0)] + f[\mathsf{d}(0,\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))] \\
&= g(\mu(\alpha_1,\dots,\alpha_n)) + f(\mathsf{d}(0,\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))) \\
&= h(\mu(\alpha_1,\dots,\alpha_n),\mu(\alpha_1,\dots,\alpha_{k-1},x,\alpha_{k+1},\dots,\alpha_n))
\end{aligned}
$$

This shows that h is a well-defined $\mathscr{P}$-algebra morphism. Let's now prove that $h \circ \lambda = f$:

$$
\begin{aligned}
&h(\lambda(\alpha,x)) = h(\alpha,0) = g(\alpha) = f(\alpha,x) \\
&h(\lambda(\mathsf{d}(\alpha,x))) = h(0,x) = f(\mathsf{d}(0,x)) = f(\mathsf{d}(\alpha,0)) + f(\mathsf{d}(0,x)) = f(\mathsf{d}(\alpha,x))
\end{aligned}
$$

Let's also prove the compatibility with !, i.e. that $h \circ ! = g$:

$$
h(!(\alpha)) = h(\alpha,0) = g(\alpha)
$$

The final step is to show that h is the unique morphism so that $h \circ \lambda = f$ and $h \circ ! = g$. Let's consider a second morphism h' satisfying these conditions, so that:

$$
h'(\alpha,x) = h'(\alpha,0) + h'(0,x) = h'(!(\alpha)) + h'(\lambda(\mathsf{d}(0,x))) = g(\alpha) + f(\mathsf{d}(0,x)) = h(\alpha,x)
$$

In conclusion, we constructed ζ and λ so that $(\mathsf{Free}_{\mathscr{P}(0)}M,\zeta,\lambda)$ is a universal pre-differential bundle over the terminal object of $\mathsf{Geom}(\mathscr{P})$. We conclude that $\mathsf{Free}_{\mathscr{P}(0)}M$ is a differential object. □

Proposition 4.26. *Let (A,σ,ζ,λ) be a differential object (regarded as a differential bundle over the terminal object) in the geometric tangent category $\mathsf{Geom}(\mathscr{P})$ of an algebraic operad $\mathscr{P}$ and let D_λ be the morphism so defined:*

$$
D_\lambda(a) := \lambda(\mathsf{d}a)
$$

Thus, $D_\lambda(A)$ is a $\mathscr{P}(0)$-module (in the operadic sense).

Proof. In order to prove that $D_\lambda(A)$ is a $\mathscr{P}(0)$-module we need to provide linear morphisms $\psi_{n+1} \colon \mathscr{P}(n+1) \otimes \mathscr{P}(0)^{\otimes^n} \otimes D_\lambda(A) \to D_\lambda(A)$:

$$
\psi_{n+1}(\mu;\alpha_1,\dots,\alpha_n,D_\lambda(a)) := D_\lambda(\mu(!\alpha_1,\dots,!\alpha_n,a))
$$

where $!\colon \mathscr{P}(0) \to A$ and show that ψ_n are compatible with the operadic composition. First, let's prove that ψ_n is well-defined. Note that from the Leibniz rule we have that:

$$D_\lambda(\mu(!\alpha_1,\dots,!\alpha_n,a)) = \sum_{k=1}^{n} \mu(!\alpha_1,\dots,D_\lambda(!\alpha_1),\dots,!\alpha_n,a) + \mu(!\alpha_1,\dots,!\alpha_n,D_\lambda(a))$$

However:

$$D_\lambda(!\alpha) = \lambda(\mathsf{d}(!\alpha)) = \lambda(\mathrm{T}!(\mathsf{d}\alpha)) =!(z(\mathsf{d}\alpha)) =!0 = 0$$

where we used that $\lambda \circ \mathrm{T}! =! \circ z$. Thus:

$$D_\lambda(\mu(!\alpha_1,\dots,!\alpha_n,a)) = \mu(!\alpha_1,\dots,!\alpha_n,D_\lambda(a))$$

This proves that ψ_n is well-defined. Let's show the compatibility between ψ_n and the operadic composition:

$$\begin{aligned}
& \psi_{n+1}(\mu;\alpha_1,\dots,\nu(\alpha_k,\dots,\alpha_{k+m}),\dots,\alpha_{n+m},D_\lambda(a)) \\
= \;& \mu(!\alpha_1,\dots,!\nu(\alpha_k,\dots,\alpha_{k+m}),\dots,!\alpha_{m+n},D_\lambda(a)) \\
= \;& \mu(!\alpha_1,\dots,\nu(!\alpha_k,\dots,!\alpha_{k+m}),\dots,!\alpha_{m+n},D_\lambda(a))(\mu \circ_k \nu)(!\alpha_1,\dots,!\alpha_{m+n},D_\lambda(a)) \\
= \;& \psi_{n+m+1}(\mu \circ_k \nu;\alpha_1,\dots,\alpha_{n+m},D_\lambda(a))
\end{aligned}$$

$$\begin{aligned}
& \psi_{n+1}(\mu;\alpha_1,\dots,\alpha_n,\psi_{m+1}(\nu;\alpha_{n+1},\dots,\alpha_{n+m+1},D_\lambda(a))) \\
= \;& \psi_{n+1}(\mu;\alpha_1,\dots,\alpha_n,\nu(!\alpha_{n+1},\dots,!\alpha_{n+m+1},D_\lambda(a))) \\
= \;& \mu(!\alpha_1,\dots,!\alpha_n,\nu(!\alpha_{n+1},\dots,!\alpha_{n+m+1},D_\lambda(a))) \\
= \;& (\mu \circ_{n+1} \nu)(!\alpha_1,\dots,!\alpha_{n+m+1},D_\lambda(a)) \\
= \;& \psi_{n+m+2}(\mu \circ_{n+1} \nu;\alpha_1,\dots,\alpha_{n+m+1},D_\lambda(a))
\end{aligned}$$

The compatibility with the symmetric action is left to the reader. This concludes the proof. □

Proposition 4.27. *Consider a $\mathscr{P}(0)$-module $M \in \mathsf{Mod}_{\mathscr{P}(0)}$. Then there exists an isomorphism of $\mathscr{P}(0)$-modules:*

$$M \to D_\lambda(\mathsf{Free}_{\mathscr{P}(0)}M)$$

Proof. Let's start by proving the existence of a morphism $M \to D_\lambda(\mathsf{Free}_{\mathscr{P}(0)}M)$. For this purpose, note first that, by definition of λ over $\mathsf{Free}_{\mathscr{P}(0)}M$ we have that:

$$D_\lambda(\alpha, x) = \lambda(\mathsf{d}(\alpha, x)) = (0, x)$$

Thus, on generators:

$$D_\lambda(\mu; (\alpha_1, x_1), \ldots, (\alpha_n, x_n)) = \sum_{k=1}^{n} (\mu; \lambda(\alpha_1, x_1), \ldots, \lambda(\mathsf{d}(\alpha_k, x_k)), \ldots, (\alpha_n, x_n)) =$$
$$= \sum_{k=1}^{n} (\mu; (\alpha_1, 0), \ldots, (0, x_k), \ldots, (\alpha_n, 0))$$

Let's define the following morphism:

$$\varphi \colon D_\lambda \mathsf{Free}_{\mathscr{P}(0)}M \to M$$

defined by:

$$\varphi(D_\lambda(\mu; (\alpha_1, x_1), \ldots, (\alpha_n, x_n))) := \sum_{k=1}^{n} \mu(\alpha_1, \ldots, \alpha_{k-1}, x_k, \alpha_{k+1}, \ldots, \alpha_n)$$

Note that:

$$\varphi(\alpha, x) = x$$

Let's first prove that φ is well-defined. First, notice that $\varphi(D_\lambda(\alpha, x)) = \varphi(0, x)$, thus φ does not depend on α. Moreover, it is compatible with the relations (4.3.1):

$$\varphi(D_\lambda(\mu; (\alpha_1, 0), \ldots, (\alpha_k, x), \ldots, (\alpha_n, 0))) = \mu(\alpha_1, \ldots, \alpha_{k-1}, x, \alpha_{k+1}, \ldots, \alpha_n)$$
$$\varphi(D_\lambda(\mu(\alpha_1, \ldots, \alpha_n), \mu(\alpha_1, \ldots, \alpha_{k-1}, x, \alpha_{k+1}, \ldots, \alpha_n))) =$$
$$= \mu(\alpha_1, \ldots, \alpha_{k-1}, x, \alpha_{k+1}, \ldots, \alpha_n)$$

Thus, φ is well-defined. Let's prove that φ is a $\mathscr{P}(0)$-module morphism:

$$\varphi([\psi_{n+1}(\mu; \alpha_1, \ldots, \alpha_n, D_\lambda(\alpha, x))]) = \varphi(D_\lambda([(\mu; (\alpha_1, 0), \ldots, (\alpha_n, 0), (\alpha, x))])) =$$
$$= \varphi(D_\lambda([(\mu(\alpha_1, \ldots, \alpha_n, \alpha), \mu(\alpha_1, \ldots, \alpha_n, x))])) = \mu(\alpha_1, \ldots, \alpha_n, x)$$

where we used the relations (4.3.1). The next step is to provide an inverse for φ. Consider the following morphism $\chi \colon M \to D_\lambda(\mathsf{Free}_{\mathscr{P}(0)}M)$ so defined:

$$\chi(x) := [(0, x)]$$

Thus:

$$\varphi(\chi(x)) = \varphi([(0,x)]) = x$$
$$\chi(\varphi(D_\lambda[(\alpha,x)])) = \chi(x) = [(0,x)] = D_\lambda[(\alpha,x)]$$

This proves that φ is a $\mathscr{P}(0)$-module isomorphism, as expected. □

Proposition 4.28. *Given a differential object (A,σ,ζ,λ) in the geometric tangent category $\mathsf{Geom}(\mathscr{P})$ of an algebraic operad $\mathscr{P}$, there is an isomorphism of $\mathscr{P}$-algebras:*

$$\mathsf{Free}_{\mathscr{P}(0)} D_\lambda A \to A$$

Proof. Consider the following morphism:

$$\psi : \mathsf{Free}_{\mathscr{P}(0)} D_\lambda(A) \to A$$

defined on generators by:

$$\psi(\alpha, D_\lambda(a)) := !\alpha + D_\lambda(a)$$

The first step is to prove that ψ is well-defined, i.e. it is compatible with the relations (4.3.1):

$$\begin{aligned}
&\psi(\mu; (\alpha_1,0),\dots,(\alpha_k,D_\lambda(a)),\dots,(\alpha_n,0)) = \\
&= \mu(!\alpha_1,\dots,!\alpha_k + D_\lambda(a),\dots,!\alpha_n) = \\
&= \mu(!\alpha_1,\dots,!\alpha_k,\dots,!\alpha_n) + \mu(!\alpha_1,\dots,D_\lambda(a),\dots,!\alpha_n) = \\
&= !\mu(\alpha_1,\dots,\alpha_n) + D_\lambda(\mu(\alpha_1,\dots,a,\dots,\alpha_n)) = \\
&= \psi(\mu(\alpha_1,\dots,\alpha_n), D_\lambda(\mu(\alpha_1,\dots,a,\dots,\alpha_n))) = \\
&= \psi(\mu(\alpha_1,\dots,\alpha_n), \mu(\alpha_1,\dots,D_\lambda(a),\dots,\alpha_n))
\end{aligned}$$

This proves that ψ is well-defined. The goal is to show that ψ is an isomorphism. In order to do that, consider a morphism $\delta : \mathrm{T}A \to \mathsf{Free}_{\mathscr{P}(0)} D_\lambda A$ so defined on generators by:

$$\delta(a) := [(\zeta(a), 0)]$$
$$\delta(da) := [(0, D_\lambda(a))]$$

Note that, since D_λ is a derivation of A, it follows that δ is a well-defined $\mathscr{P}$-algebra morphism. The next step is to consider the following diagram:

$$\begin{tikzcd}
\mathrm{T}\mathscr{P}(0) \sqcup A \arrow[r, "{[\mathrm{T}!,p]}"] \arrow[d, "{[z,\zeta]}"'] & \mathrm{T}A \arrow[d, "\lambda"] \arrow[ddr, bend left, "\delta"] & \\
\mathscr{P}(0) \arrow[r, "!"'] \arrow[drr, bend right, "!"'] & A \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & \mathsf{Free}_{\mathscr{P}(0)} D_\lambda A
\end{tikzcd}$$

Let's prove that the diagram commutes, i.e. that $\delta \circ [\mathrm{T}!, p] = ! \circ [z, \zeta]$. On generators:

$$\delta([T!, p](\beta; a)) = \delta(!\beta + a) = (\zeta(!\beta) + \zeta(a), 0) = (\beta + \zeta(a), 0)$$
$$\delta([T!, p](d\beta; a)) = \delta(d!\beta + a) = (0, D_\lambda(!\beta)) + (\zeta(a), 0) = (\zeta(a), 0)$$

$$!([z, \zeta](\beta; a)) = !(\beta + \zeta(a)) = (\beta + \zeta(a), 0)$$
$$!([z, \zeta](d\beta; a)) = !(\zeta(a)) = (\zeta(a), 0)$$

Employing the universality of the vertical lift, we obtain a $\mathscr{P}$-algebra morphism $\psi^{-1} \colon A \longrightarrow \mathsf{Free}_{\mathscr{P}(0)} D_\lambda A$. Let's prove that ψ^{-1} is an inverse for ψ. To do that, consider the diagram:

$$\begin{tikzcd}
\mathrm{T}\mathscr{P}(0) \sqcup A \arrow[r, "{[\mathrm{T}!,p]}"] \arrow[d, "{[z,\zeta]}"'] & \mathrm{T}A \arrow[d, "\lambda"] \arrow[dr, bend left, "\delta"] \arrow[ddrr, bend left, "\lambda"] & & \\
\mathscr{P}(0) \arrow[r, "!"'] \arrow[drr, bend right, "!"] \arrow[ddrrr, bend right, "!"'] & A \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\psi^{-1}"] & & \\
& & \mathsf{Free}_{\mathscr{P}(0)} D_\lambda A \arrow[dr, "\psi"] & \\
& & & A
\end{tikzcd}$$

We want to prove that $\psi \circ \delta = \lambda$. Notice that $\psi \circ ! = !$ comes for free from the universality of the initial algebra $\mathscr{P}(0)$. Thus, on generators:

$$\psi(\delta(a)) = \psi(\zeta(a), 0) = !\zeta(a) = \lambda(a)$$
$$\psi(\delta(da)) = \psi(0, D_\lambda(a)) = D_\lambda(a) = \lambda(da)$$

This proves that $\psi \circ \psi^{-1}$ is the unique morphism making commuting the following diagram:

$$\begin{tikzcd}
\mathrm{T}\mathscr{P}(0) \sqcup A \arrow[r, "{[\mathrm{T}!, p]}"] \arrow[d, "{[z,\zeta]}"'] & \mathrm{T}A \arrow[d, "\lambda"] \arrow[ddr, bend left, "\lambda"] & \\
\mathscr{P}(0) \arrow[r, "!"'] \arrow[drr, bend right, "!"'] & A \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\psi\circ\psi^{-1}"] & \\
& & A
\end{tikzcd}$$

However, so does the identity over A. Therefore, $\psi \circ \psi^{-1} = \mathsf{id}_A$. Let's finally show the converse:

$$\begin{aligned}
& \psi^{-1}(\psi(\alpha, D_\lambda(a))) \\
= \; & \psi^{-1}(!\alpha + D_\lambda(a)) \\
= \; & \psi^{-1}(!\alpha) + \psi^{-1}(\lambda(da)) \\
= \; & !\alpha + \delta(da) \\
= \; & (\alpha, 0) + (0, D_\lambda(a)) \\
= \; & (\alpha, D_\lambda(a))
\end{aligned}$$

This concludes the proof. □

Theorem 4.29. *There is an equivalence between the categories of differential objects and linear morphisms in the geometric tangent category* $\mathsf{Geom}(\mathscr{P})$ *of an algebraic operad* $\mathscr{P}$ *and the opposite of the category* $\mathscr{P}(0)$*-modules in the operadic sense and* $\mathscr{P}(0)$*-linear morphisms:*

$$\mathsf{DObj}_{\mathsf{lnr}}\mathsf{Geom}(\mathscr{P}) \cong \mathsf{Mod}^{\mathsf{op}}(\mathscr{P}; \mathscr{P}(0))$$

Proof. The existence of this correspondence between the objects of these two categories is a direct consequence of Propositions 4.27 and 4.28. Let's prove that this correspondence preserves linear morphisms. Consider a linear morphism

$f\colon A \to B$ of differential objects of $\mathsf{Geom}(\mathscr{P})$. The linearity of f expresses the commutativity of the diagram:

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "\lambda_A"'] & \mathrm{T}B \arrow[d, "\lambda_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

where λ_A and λ_B represent the lifts of A and B respectively. This implies that:

$$D_{\lambda_B}(f(a)) = \lambda_B(\mathsf{d}f(a)) = \lambda_B(\mathrm{T}f(\mathsf{d}a)) = f(\lambda_A(\mathsf{d}a)) = f(D_{\lambda_A}(a))$$

So, f restricts to the image of D_{λ_A}. Conversely, if $f\colon M \to N$ is a $\mathscr{P}(0)$-module morphism, then $\mathsf{Free}_{\mathscr{P}(0)}f$ is linear (in the sense of differential objects):

$$\begin{aligned}
&\mathsf{Free}_{\mathscr{P}(0)}f(\lambda[(\alpha,x)]) = \mathsf{Free}_{\mathscr{P}(0)}f[(\alpha,0)] = [(\alpha,0)]\\
&\mathsf{Free}_{\mathscr{P}(0)}f(\lambda(\mathsf{d}[(\alpha,x)])) = \mathsf{Free}_{\mathscr{P}(0)}f[(0,x)] = [(0,f(x))]
\end{aligned}$$

$$\begin{aligned}
&\lambda(\mathrm{T}f[(\alpha,x)]) = \lambda[(\alpha,f(x))] = [(\alpha,0)]\\
&\lambda(\mathrm{T}f(\mathsf{d}[(\alpha,x)])) = \lambda(\mathsf{d}[(\alpha,f(x))]) = [(0,f(x))]
\end{aligned}$$

This proves that $\mathsf{Free}_{\mathscr{P}(0)}f$ is linear. □

The enveloping algebra of a $\mathscr{P}$-algebra A is the associative and unital algebra $\mathscr{P}^{(A)}(1)$, denoted by $\mathscr{P}(A)$. It is not hard to see that $\mathscr{P}$ itself is the enveloping operad of the initial algebra $\mathscr{P}(0)$. Consequently, $\mathscr{P}(1)$ is the enveloping algebra of $\mathscr{P}(0)$. One of the main striking features of the enveloping algebra of a $\mathscr{P}$-algebra A is that modules over A in the operadic sense are equivalent to left modules over $\mathscr{P}(A)$. We advise the reader to consult [6, Theorem 1.10] for a proof of this result.

Lemma* 4.30. *The category $\mathsf{Mod}(\mathscr{P};A)$ of modules over a $\mathscr{P}$-algebra A is equivalent to the category $\mathsf{Mod}(\mathscr{P}(A))$ of left modules over the associative and unital algebra $\mathscr{P}(A) := \mathscr{P}^{(A)}(1)$, known as the* ***enveloping algebra*** *of A. Moreover, the enveloping algebra of the initial $\mathscr{P}$-algebra $\mathscr{P}(0)$ is $\mathscr{P}(1)$.*

Corollary 4.31. *The category* $\mathsf{DObj}_{\mathsf{lnr}}\mathsf{Geom}(\mathscr{P})$ *of differential objects and linear morphisms of the geometric tangent category* $\mathsf{Geom}(\mathscr{P})$ *of an algebraic operad* $\mathscr{P}$ *is equivalent to the opposite of the category of left* $\mathscr{P}(1)$*-modules:*

$$\mathsf{DObj}_{\mathsf{lnr}}\mathsf{Geom}(\mathscr{P}) \cong \mathsf{Mod}^{\mathsf{op}}(\mathscr{P}(1))$$

Example 4.32. For the operad $\mathscr{C}om$, $u\mathscr{C}om$, $\mathscr{A}ss$, $u\mathscr{A}ss$, and $\mathscr{L}ie$, differential objects in the corresponding geometric tangent categories are all equivalent to R-modules, since $\mathscr{C}om(1) = u\mathscr{C}om(1) = \mathscr{A}ss(1) = u\mathscr{A}ss(1) = \mathscr{L}ie(1) = R$. In particular, in these examples, differential objects coincide with the free algebras.

Example 4.33. Consider an associative and unital algebra A and the operad $A^\bullet$ whose entries are all trivial but $A^\bullet(1) = A$. Multiplication and unit of $A^\bullet$ are multiplication and unit of A. The algebras of $A^\bullet$ are precisely all left A-modules, since to give an $A^\bullet$-algebra is to give an R-module M with an action $A \otimes M \to M$. Free $A^\bullet$-algebras are all A-modules of the form $A \otimes M$ for a given R-module M. The geometric tangent category $\mathsf{Geom}(A^\bullet)$ is the adjoint tangent category $(\mathsf{Mod}_A^{\mathsf{op}}, \mathbb{T})$ of the tangent category $(\mathsf{Mod}_A, \mathbb{L})$ induced by biproducts. Consequently, $\mathsf{Geom}(A^\bullet)$ is a Cartesian differential category and all maps are linear, which means that the category of differential objects and linear morphisms coincide with the whole category. However, not every A-module is of the form $A \otimes M$ for a given R-module M. This provides an example of a coCartesian differential monad $\mathsf{S}_{A^\bullet}$ for which the differential objects in the corresponding geometric tangent category do not coincide with its free algebras.

Applying Example 4.33 to the associative and unital algebra $\mathscr{P}(1)$, we conclude that $\mathsf{Mod}^{\mathsf{op}}(\mathscr{P}(1))$ carries a tangent structure for which every morphism is linear. Since the correspondence between modules and differential objects preserves linear morphisms we conclude that this extends to an equivalence of tangent categories. In particular, this implies that $\mathsf{DObj}_{\mathsf{lnr}}\mathsf{Geom}(\mathscr{P})$ is the geometric tangent category of the operad $\mathscr{P}(1)^\bullet$.

Corollary 4.34. *The tangent category* $\mathsf{DObj}_{\mathsf{lnr}}\mathsf{Geom}(\mathscr{P})$ *of differential objects and linear morphisms of the geometric tangent category of an algebraic operad* $\mathscr{P}$*, whose tangent structure is the restriction of the tangent structure on differential objects, is equivalent to the geometric tangent category of the operad* $\mathscr{P}(1)^\bullet$*.*

4.3.2 Differential bundles are modules

We are finally in the position to prove the main result of this chapter: differential bundles over an affine scheme A in the geometric tangent category of an algebraic operad $\mathscr{P}$ are modules over A in the operadic sense. Given a $\mathscr{P}$-algebra A we denote by $\mathscr{P}(A)^\bullet$ the operad associated to the enveloping algebra $\mathscr{P}(A)$ of A.

Let's also denote by $\mathsf{DBnd}_{\mathsf{lnr}}(\mathscr{P};A)$ the tangent category of differential bundles and linear morphisms over a $\mathscr{P}$-affine scheme A in the geometric tangent category $\mathsf{Geom}(\mathscr{P})$ of an operad $\mathscr{P}$.

Theorem 4.35. *Let $\mathscr{P}$ be an operad and A a $\mathscr{P}$-affine scheme. Then the tangent category $\mathsf{DBnd}_{\mathsf{lnr}}(\mathscr{P};A)$ of differential bundles over A and linear morphisms in the geometric tangent category of $\mathscr{P}$ is equivalent to the geometric tangent category of the operad $\mathscr{P}(A)^\bullet$:*

$$\mathsf{DBnd}_{\mathsf{lnr}}(\mathscr{P};A) \cong \mathsf{Geom}(\mathscr{P}(A)^\bullet) \cong (\mathsf{Mod}_A^{\mathsf{op}}, \mathbb{T}_A)$$

In particular, differential bundles over A are equivalent to A-modules in the operadic sense and linear morphisms of differential bundles over A are equivalent to A-linear morphisms of A-modules (in the opposite of the category of A-modules).

Proof. Consider an operad $\mathscr{P}$ and a $\mathscr{P}$-algebra A. Then, the tangent category $\mathsf{DBnd}_{\mathsf{lnr}}(\mathscr{P};A)$ of differential bundles over A and linear morphisms in the geometric tangent category $\mathsf{Geom}(\mathscr{P})$ of $\mathscr{P}$ is equivalent to the tangent category $\mathsf{DObj}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P})/A)$ of differential objects and linear morphisms of the slice tangent category $\mathsf{Geom}(\mathscr{P})/A$. Thanks to Theorem 4.17, $\mathsf{Geom}(\mathscr{P})/A \cong \mathsf{Geom}(\mathscr{P}^{(A)})$, where $\mathscr{P}^{(A)}$ is the enveloping operad of $\mathscr{P}$ over A. By Corollary 4.34, differential objects over $\mathsf{Geom}(\mathscr{P}^{(A)})$ are $\mathscr{P}^{(A)}(1)$-left modules; in particular, $\mathsf{DObj}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P}^{(A)})) \cong \mathsf{Geom}(\mathscr{P}^{(A)}(1)^\bullet)$, but $\mathscr{P}^{(A)}(1)$ is the enveloping algebra of A (see Lemma 4.30), thus $\mathsf{Geom}(\mathscr{P}^{(A)}(1)^\bullet) \cong \mathsf{Geom}(\mathscr{P}(A)^\bullet)$:

$$\begin{aligned}
&\mathsf{DBnd}_{\mathsf{lnr}}(\mathscr{P};A) && \\
= \;&\mathsf{DBnd}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P});A) && \text{Diff. bundles are diff. objects in the slice tangent cat.}\\
\cong \;&\mathsf{DObj}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P})/A)) && \text{Theorem 4.17}\\
\cong \;&\mathsf{DObj}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P}^{(A)})) && \text{Corollary 4.34}\\
\cong \;&\mathsf{Geom}(\mathscr{P}^{(A)}(1)^\bullet) && \mathscr{P}^{(A)}(1) = \mathscr{P}(A)
\end{aligned}$$

$$\cong \quad \mathsf{Geom}(\mathscr{P}(A)^\bullet)$$

This concludes the proof. □

Example 4.36. Consider the geometric tangent category $\mathsf{Geom}(u\mathscr{C}om)$ of the operad $u\mathscr{C}om$. As pointed out in Example 3.69, this tangent category was originally introduced by Cockett and Cruttwell in [12]. Recently Cruttwell and Lemay have pointed out that this tangent category can be employed to study algebraic geometry of affine schemes (cf. [18]). In particular, they classified differential bundles in this tangent category over a given affine scheme and proved that $\mathsf{DBnd}_{\mathsf{lnr}}(\mathsf{Geom}(u\mathscr{C}om); A)$ is equivalent to the opposite of the category of modules over the commutative algebra A. Modules in the operadic sense over a $u\mathscr{C}om$-algebra A are precisely left modules over A, in the usual sense. Therefore, Cruttwell and Lemay's classification can be regarded as a special application of Theorem 4.35.

Example 4.37. Consider the operad $u\mathscr{A}ss$. Modules over a $u\mathscr{A}ss$-algebra A, in the operadic sense, correspond to bimodules over A (cf. [27, Examples 1.6.2] and [46, Section 12.3.1]). Therefore, differential bundles over a non-commutative affine scheme A, i.e. an associative and unital algebra, in the geometric tangent category $\mathsf{Geom}(u\mathscr{A}ss)$ are equivalent to bimodules over A.

Example 4.38. Consider the operad $\mathscr{L}ie$. Modules over a $\mathscr{L}ie$-algebra $\mathfrak{g}$, in the operadic sense, correspond to linear representations of $\mathfrak{g}$ (cf. [27, Examples 1.6.2] and [46, Section 12.3.1]). Therefore, differential bundles over a $\mathscr{L}ie$-affine scheme $\mathfrak{g}$ in the geometric tangent category $\mathsf{Geom}(\mathscr{L}ie)$ are equivalent to linear representations of $\mathfrak{g}$.

Chapter 5

A tangent category approach to deformation theory

Algebraic deformation theory, first introduced by Gerstenhaber in [23], aims to study extensions of algebraic structures of a certain type by "thickening" the original object. Intuitively speaking, a deformation of a mathematical object is a perturbation, a slight modification, of this object. Geometrically, the deformation of a space can be regarded as a new space obtained by deforming the initial space by adding some noise. Algebraically this can be done by perturbing the relations which present a given object. For instance, consider a polynomial $p(x,y)$ and its locus S. A deformation of S is then obtained by slightly changing the polynomial $p(x,y)$ by adding an extra polynomial term $q(x,y,t)$ which contains a new variable t. We then obtain the locus S_t of the polynomial $p(x,y)+q_t(x,y) = p(x,y)+q(x,y,t)$. By assuming that $q(x,y,t) = 0$ when $t = 0$, we can interpret S_t as a family of geometric spaces of the same type as S, e.g. polynomial loci, parametrized by t and for which S_0 is the original space S.

Gerstenhaber's work, developed in parallel with the work of Kodaira and Spencer [38] on the deformation of complex analytic structures, found interesting applications in mathematical physics with the theory of deformation quantization [5, 4]. The main idea of deformation quantization is to regard quantum mechanics as a non-commutative deformation of classical mechanics. In particular, in the quantization process via deformation, the commutative algebra of observables of a given classical system is deformed into a non-commutative associative algebra which represents the corresponding quantized system.

Newman in [50] and Nijenhuis together with Richardson in [51] extended Gerstenhaber's ideas to Lie algebras and subsequently, Kontsevich and Soibelman in [40] introduced a theory of deformation of operadic algebras for a given operad.

One can easily recognize geometric aspects underpinning deformation theory: the deformation of a given object S can be regarded as a family of similar objects S_t,

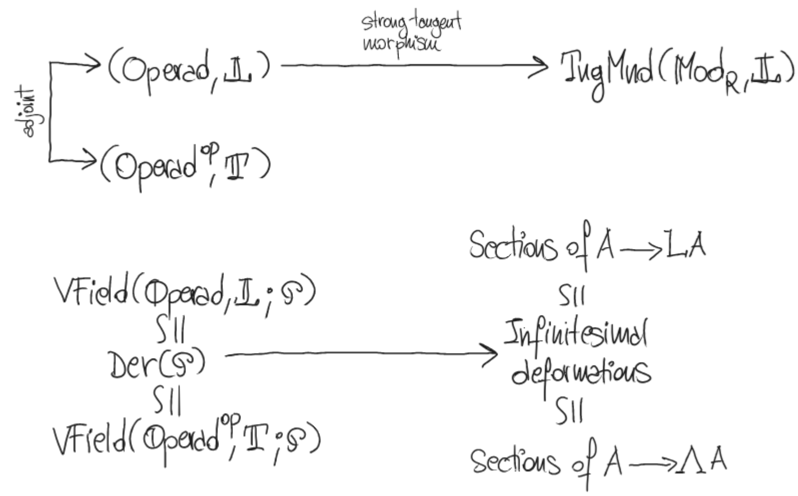
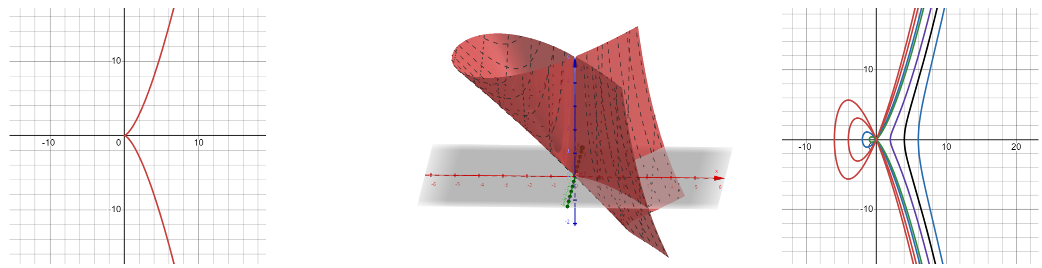


Figure 5.1: The concept map of the chapter

parametrized by a parameter t, in which the original space S corresponds to $S_{t=0}$. In this sense, a deformation can be regarded as a "smooth" path in some geometric space whose points are mathematical objects or mathematical structures. This is precisely the point of contact between deformation theory and moduli spaces, which are geometric spaces whose points are mathematical structures, like curves of a certain type, or associative multiplications over a given vector space. The intuition is to regard infinitesimal deformations of an object, which intuitively are deformations parametrized by small values of the parameter t, as tangent vectors of the moduli space. We advise the interested reader to consult [28].

From our perspective, it is natural to wonder if tangent category theory can be employed to give a geometric intuition of these spaces of mathematical objects, possibly interpreting infinitesimal deformations as legitimate vector fields over a given object. The main purpose of this chapter is to introduce some ideas to answer this question. The first main insight that drives this investigation is noticing that the category Operad of algebraic operads is itself a tangent category. Moreover, the vector fields of this tangent category are closely related to infinitesimal deformations. To put this in a slogan: *the geometry of operadic affine schemes is captured by the geometric tangent category of the given operad, while the deformation theory of operadic affine schemes is captured by the tangent structure on the category of operads*.



(a) The "bow-tie" affine scheme

(b) Deformation as an extension

(c) Deformation as a family of affine schemes

Figure 5.2: Deformation of an affine scheme: the bow-tie example

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We start in Section 5.1 with a brief introduction to algebraic deformation theory, recalling the ideas of Gerstenhaber and then extending them to operadic algebras, following [46, Section 12.2]. In Section 5.2 we prove that the category of operads is a tangent category whose vector fields are closely related to infinitesimal deformations. We also discuss the relationship between this tangent category and the one of tangent monads (Section 5.2.1). In Section 5.2.2 we show that the opposite of the category of operads is also a tangent category and discuss its relationship with deformation theory. In Section 5.3, we discuss how this approach does not capture relevant examples of deformations of algebras and we propose two solutions, one that involves the construction of a differential bundle whose sections classify all infinitesimal deformation of an algebra, and the other one which involves the construction of a tangent comonad for which the sections of the counit classify infinitesimal deformations.

Figure 5.1 displays the concept map of this chapter.

5.1 An introduction to algebraic deformation theory

Let us start with a basic example. Consider the polynomial:

$$p(x,y) = x^3 - x^2 - y^2$$

Figure 5.2a represents the locus of this polynomial. This polynomial is also associated with the affine scheme $R[x,y]/(p(x,y))$, which is the commutative and unital ring obtained by quotienting the free ring $R[x, y]$ generated by two generators x and y by the ideal generated by $p(x, y)$. From a geometric point of view, a deformation of the locus of $p(x, y)$ is the locus of another polynomial, obtained by twisting $p(x, y)$ with some extra terms, parametrized by an extra variable. One of these polynomials is the following polynomial in three variables:

$$\tilde{p}(x, y, t) = x^3 - x^2 - y^2 + tx^2$$

There are two geometric approaches to interpreting the new space as a deformation of the original one. The first approach considers the locus of $\tilde{p}(x, y, t)$, which is the surface represented in Figure 5.2b. In this sense, the deformation is conceptualized as an extension of the original object, e.g. the locus of $p(x, y)$, to a larger object of the same type, i.e. still an affine scheme but in an extra dimension. In the second approach, the new variable t is conceptualized as a parameter, whose values define different affine schemes associated with the polynomials $p_t(x, y) := \tilde{p}(x, y, t)$. Notice that we reobtain the original object when t is set to zero. This second approach regards deformations as paths in the space of mathematical objects of a certain type. Notice that this example captures the deformation of a commutative algebra into a new commutative algebra. However, this does not represent the general case. For example, in the context of deformation quantization, one deforms a commutative algebra into an associative noncommutative algebra (see for instance [39]).

Gerstenhaber's approach is closer to the first way of conceptualization; in the following, we mostly adopt this point of view. However, we suggest the reader keep the second approach in mind since it is closer to a geometric interpretation of deformations. The intuition of Gerstenhaber was to regard the deformation of an affine scheme from the point of view of the coordinate rings: perturbing the affine scheme of the polynomial $p(x, y)$ by adding the extra term tx^2 is equivalent to perturbing the multiplication of the associative algebra $A = R[x, y]/(p(x, y))$ and obtaining a new algebra $\tilde{A} = R[x, y, t]/(\tilde{p}(x, y, t))$.

Concretely, one starts with an associative R-algebra A and extends the associative algebra structure of A over the $R[t]$-module $A[t]$ of polynomials in the variable t with coefficients in A, obtaining an $R[t]$ algebra $\tilde{A}$. The multiplication map of $\tilde{A}$ is a map $\tilde{\mu} \colon \tilde{A} \otimes_{R[t]} \tilde{A} \to \tilde{A}$ fully determined by a sequence $\{\mu_n : A \otimes A \to A\}$ of binary operations:

$$\tilde{\mu}(a,b) = \sum_{n=0}^{\infty} \mu_n(a,b)t^n$$

for every $a, b \in A$. Notice that, since $\tilde{\mu}$ is $R[t]$-linear these maps are fully determined by their restriction to the elements of A. Then $A[t]$ equipped with $\tilde{\mu}$ is a deformation of A if (1) the term μ_0 is precisely the multiplication map μ of A and (2) $\tilde{\mu}$ is also associative. Condition (2) is equivalent to the following relations on the binary operations μ_n:

$$\sum_{j+k=n} \mu_j(\mu_k(a,b),c) - \mu_k(a,\mu_j(b,c)) = 0$$

which, in particular, for $n = 1$, implies:

$$a\nu(b,c) - \nu(ab,c) + \nu(a,bc) - \nu(a,b)c = 0 \tag{5.1.1}$$

for every $a, b, c \in A$, where we denoted the multiplication of A by juxtaposition and by ν the binary operation μ_1, known as the **infinitesimal deformation**.

Gerstenhaber's insight was to realize that the condition satisfied by ν is precisely the condition that a binary operation $\nu\colon A \otimes A \to A$ must satisfy in order to be a 2-cocycle in the Hochschild cohomology of the algebra A. In particular, such 2-cocycles form the space of infinitesimal deformations of A.

More generally, deformations of associative algebras over a given ring R are associative algebras over an augmented R-ring S which reduce to the original algebras. To clarify this definition, let's start by recalling the definition of an augmented R-ring. We refer the reader to [20] for this approach.

Definition⋆ 5.1. *Given a unital and commutative ring R, an **augmented R-ring** is a commutative and unital algebra S over the ring R equipped with a morphism of rings $e\colon S \to R$, called the **augmentation map** which preserves the unit, i.e. such that for every $r \in R$, $e(r\mathbb{1}_S) = r$. In the following, we denote an augmented R-ring S with augmentation map ε by (S, e).*

Example 5.2. The ring $R[\varepsilon]$ of dual numbers, obtained by quotienting $R[x]$ by the ideal x^2, is an augmented R-ring with augmentation map $e\colon R[\varepsilon] \to R$ which sends the variable ε to 0.

Example 5.3. The ring $R[t]$ of polynomials in the variable t is an augmented R-ring with augmentation map $e\colon R[t] \to R$ which sends the variable t to 0.

Example 5.4. The ring $R[\![t]\!]$ of formal power series in the variable t is an augmented R-ring with augmentation map $e\colon R[\![t]\!]$ which sends the variable t to 0.

The augmentation map $e\colon S \to R$ of an augmented ring S over R induces an adjunction:

$$\overline{(-)}\colon \mathsf{Alg}^{(S)} \leftrightarrows \mathsf{Alg}^{(R)}\colon \mathsf{Restr}$$

between the categories of associative algebras over S and over R in which the right adjoint Restr is the functor which restricts the scalars of an R-algebra along e, and the left adjoint $\overline{(-)}$ sends an S-algebra B to the R-algebra $B \otimes_S R$, where R is a left S-module via the augmentation map. Equivalently, $\overline{B}$ is the R-algebra:

$$\overline{B} = B/(\ker e \cdot B)$$

For an S-algebra B, the R-algebra $\overline{B}$ is sometimes called the **reduction** of B. To understand what the reduction does, let's apply the functor $\overline{(-)}$ to the $R[t]$-algebra $B := R[x,y,t]/(x^3 - x^2 - y^2 + tx^2)$. Let's start by noticing that the ideal $\ker \varepsilon \cdot B$ of B contains all polynomials of the form $\sum_{k=1}^{N} p_k(x,y)t^k$, which are those polynomials in the variable t with coefficients $p_k(x,y)$ in the ring B, in which each term is multiplied by a power t^k of t with $k > 0$. In particular, this ideal of B is generated by the polynomial t, so:

$$\overline{B} = B/(\ker e \cdot B) = \overline{B}/(t) = R[x,y,t]/(x^3 - x^2 - y^2 + tx^2, t) = R[x,y]/(x^3 - x^2 - y^2)$$

So the reduction kills the variable t.

Definition* 5.5. *Given an augmented R-ring (S,e), an S-**deformation** of an R-algebra A is an S-algebra $\tilde{A}$ whose reduction $\overline{B}$ is isomorphic to the original R-algebra A. In particular, when $S = R[\varepsilon]$ we call an $R[\varepsilon]$-deformation an **infinitesimal** deformation.*

In a nutshell, S-deformations of an R-algebra A are the objects of the fibre over A of the functor $\overline{(-)}$. From the previous discussion, we conclude that $B = R[x,y,t]/(x^3 - x^2 - y^2 + tx^2)$ is an $R[t]$-deformation of $A = R[x,y]/(x^3 - x^2 - y^2)$. The next step is to generalize this definition to operadic algebras.

The key to understanding how to extend the notion of deformation for operadic algebras consists of changing both the base ring from R to the augmented ring S and changing the operad itself. To justify the necessity of changing the operad, recall that for an associative algebra A, the multiplication map $\tilde{\mu}$ of an $R[t]$-deformation $\tilde{A}$ of A can be extended into a power series in the variable t of binary operations $\mu_n \colon A \otimes A \to A$. The intuition is to provide those binary operations μ_n as part of the n-ary operations of the operad.

As discussed in [46, Section 12.2], every algebraic operad $\mathscr{P}$ over the ring R can be extended to a new operad $\mathscr{P} \otimes S$ over the augmented R-ring S. Concretely, the n-th entry of $\mathscr{P} \otimes S$ is the S-module $\mathscr{P}(n) \otimes S$ (notice that $\otimes$ is the tensor product over the ring R). We denote the generators of $\mathscr{P}(n) \otimes S$ as $s\mu$ for $s \in S$ and $\mu \in \mathscr{P}(n)$. With this notation, the multiplication is induced by the one of $\mathscr{P}$ and the one of S as follows:

$$(s\mu)(s_1\mu_1, \ldots, s_n\mu_n) := (s \cdot s_1 \cdot \ldots \cdot s_n)(\mu(\mu_1, \ldots, \mu_n))$$

The unit of $\mathscr{P} \otimes S$ is $\mathbb{1}_S \mathbb{1}_{\mathscr{P}}$, i.e. $\mathbb{1}_{\mathscr{P}} \otimes \mathbb{1}_S$, where $\mathbb{1}_S$ denotes the unit of S. Finally, the symmetric group acts as follows:

$$(s\mu) \cdot \sigma = s(\mu \cdot \sigma)$$

for every $s \in S$, $\mu \in \mathscr{P}(n)$, and $\sigma \in \mathbb{S}_n$. Let's consider some relevant examples.

Example 5.6. Consider an algebraic operad $\mathscr{P}$ over a ring R and let S be the augmented ring $R[\varepsilon] = R[x]/(x^2)$ of dual numbers over R with the augmentation map which sends ε to 0. Let's denote by $\mathscr{P}[\varepsilon]$ the operad $\mathscr{P} \otimes R[\varepsilon]$. Concretely, the n-th entry of this operad consists of terms of the form $\mu + \varepsilon\nu$ for $\mu, \nu \in \mathscr{P}(n)$ and such that $\varepsilon^2 = 0$. This operad is equivalent to the operad $\mathscr{P} \ltimes \mathscr{P}$ defined as follows. The n-th entry of $\mathscr{P} \ltimes \mathscr{P}$ is given by the R-module $\mathscr{P}(n) \oplus \mathscr{P}(n)$. Let's denote by (μ, ν) the elements of this biproduct. Then, the multiplication map of $\mathscr{P} \ltimes \mathscr{P}$ is defined as follows:

$$\begin{aligned}&(\mu, \nu)((\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)) = \\ &\quad = \left(\mu(\mu_1, \ldots, \mu_n), \nu(\mu_1, \ldots, \mu_n) + \sum_{k=1}^{n} \mu(\mu_1, \ldots, \nu_k, \ldots, \mu_n)\right)\end{aligned}$$

Moreover, the unit of $\mathscr{P} \ltimes \mathscr{P}$ is given by $(\mathbb{1}_{\mathscr{P}}, 0)$ and the symmetric group acts on each term of the pair, i.e. $(\mu, \nu) \cdot \sigma = (\mu \cdot \sigma, \nu \cdot \sigma)$. It is not hard to see that the isomorphism between $\mathscr{P}[\varepsilon]$ and $\mathscr{P} \ltimes \mathscr{P}$ sends $\mu + \varepsilon\nu$ to (μ, ν).

Example 5.7. Consider an algebraic operad $\mathscr{P}$ over a ring R and let S be the augmented ring $R[t]$ of polynomials over R in the variable t, with the augmentation map which sends t to 0. Let's denote by $\mathscr{P}[t]$ the operad $\mathscr{P} \otimes R[t]$. Concretely, the elements of the n-th entry of this operad are polynomial terms of the form $\sum_{k=0}^{N} \mu_k t^k$ for some positive integer N and for which each μ_k belongs to $\mathscr{P}(n)$. The multiplication works as the multiplication of polynomials, i.e.:

$$\left(\sum_{k=0}^{N} \mu_k t^k\right)\left(\sum_{k_1=0}^{N_1} \mu_{k_1}^{(1)} t^{k_1}, \dots, \sum_{k_n=0}^{N_n} \mu_{k_n}^{(n)} t^{k_n}\right) = \sum_{k,k_1,\dots,k_n} \mu_k(\mu_{k_1}^{(1)}, \dots, \mu_{k_n}^{(n)}) t^{k+k_1+\dots+k_n}$$

where the sum on the right-hand side is understood over all the indices within the corresponding intervals, e.g. k_j runs from 0 to N_j. The unit is just the polynomial $\mathbb{1}_{\mathscr{P}}$ and finally, the symmetric group acts on each μ_k, i.e. $\left(\sum_{k=0}^{N} \mu_k t^k\right) \cdot \sigma = \sum_{k=0}^{N} (\mu_k \cdot \sigma) t^k$.

Example 5.8. Consider an algebraic operad $\mathscr{P}$ over a ring R and let S be the augmented ring $R[\![t]\!]$ of formal power series over R in the variable t, with the augmentation map which sends t to 0. Let's denote by $\mathscr{P}[\![t]\!]$ the operad $\mathscr{P} \otimes R[\![t]\!]$. Concretely, the elements of the n-th entry of this operad are formal power series terms of the form $\sum_{k=0}^{\infty} \mu_k t^k$, for which each μ_k belongs to $\mathscr{P}(n)$. The multiplication works as the multiplication of power series, i.e.:

$$\left(\sum_{k=0}^{\infty} \mu_k t^k\right)\left(\sum_{k_1=0}^{\infty} \mu_{k_1}^{(1)} t^{k_1}, \dots, \sum_{k_n=0}^{\infty} \mu_{k_n}^{(n)} t^{k_n}\right) = \sum_{k,k_1,\dots,k_n} \mu_k(\mu_{k_1}^{(1)}, \dots, \mu_{k_n}^{(n)}) t^{k+k_1+\dots+k_n}$$

where the sum on the right-hand side is understood over all the indices from 0 to ∞. The unit is just the polynomial $\mathbb{1}_{\mathscr{P}}$ and finally, the symmetric group acts on each μ_k, i.e. $\left(\sum_{k=0}^{\infty} \mu_k t^k\right) \cdot \sigma = \sum_{k=0}^{\infty} (\mu_k \cdot \sigma) t^k$.

Notice that the functor which sends an operad $\mathscr{P}$ over a ring R to the operad $\mathscr{P} \otimes S$ over S forms an adjunction with the functor Restr which restricts scalars of S to R via the inclusion map $\iota \colon R \to S$ which simply sends $\mathbb{1}_R$ to $\mathbb{1}_S$. In particular, this adjunction is the one induced by ι:

$$(-) \otimes S \colon \mathsf{Operad}^{(R)} \leftrightarrows \mathsf{Operad}^{(S)} \colon \mathsf{Restr}$$

Notice in particular that the composition $\mathsf{Restr} \circ ((-) \otimes S)$ induces a monad on the category of operads over the ring R. In the following, for the sake of simplicity, we abuse notation and denote by $\mathscr{P} \otimes S$ the R-operad image under this monad of the R-operad $\mathscr{P}$. In particular, we also adopt the notation $\mathscr{P}[\varepsilon]$ for the R-operad $\mathscr{P} \otimes R[\varepsilon]$ of Example 5.6. As mentioned before, the S-linearity implies that one can simply work with R-operads and then extend each construction by construction over the ring S. So, for example, the algebra structure of an algebra over the operad $\mathscr{P}[\varepsilon]$ over R can be extended to an algebra structure over $R[\varepsilon]$.

We can now introduce the main ingredient: the reduction functor. This is precisely the left adjoint of the adjunction induced by $\mathscr{P} \xrightarrow{\mathscr{P}\otimes e} \mathscr{P} \otimes S$, where e is the augmentation map:

$$\overline{(-)}\colon \mathsf{Alg}_{\mathscr{P}} \leftrightarrows \mathsf{Alg}_{\mathscr{P}\otimes S} : \mathsf{Restr}$$

We refer to the $\overline{(-)}$ functor as the **reduction** functor.

Remark 5.9. In the initial approach to the deformation of associative algebras of Definition 5.5 the reduction was the functor which kills the extra terms of the augmented ring in the associative algebra, so a deformation is seen as an extension of an associative algebra over the augmented ring S. Here, instead the reduction $\overline{(-)}$ kills the extra terms of the augmented ring in the *operad*. The operation of changing the operad instead of changing the ring is precisely aligned with Gerstenhaber's insight into conceptualizing a deformation as a perturbation of the original multiplication μ with extra terms $\mu_1 t, \mu_2 t^2, \ldots$. Changing the operad explicitly introduces these extra operations $\mu_1, \mu_2, \ldots$ directly into the operad. So, the reduction is the operation which removes these extra operations from the operad.

Definition* 5.10. *Given an operadic algebra A over a given operad $\mathscr{P}$ and an augmented ring (S, e) over R, an S-**deformation** of A is a $(\mathscr{P} \otimes S)$-algebra $\tilde{A}$ whose reduction $\overline{\tilde{A}}$ is isomorphic to A. We adopt the convention of calling **infinitesimal deformations** the S-deformations for which S is the augmented ring of dual numbers of Example 5.2.*

When we consider the operad $\mathscr{A}ss$, S-deformations of associative algebras according to Definition 5.10 coincide with S-deformations according to Definition 5.5. Moreover, when we consider the operad $\mathscr{L}ie$ we precisely obtain deformations as described in [23, Equation 2′].

5.2 The category of operads is itself a tangent category

According to Definition 5.10, an infinitesimal deformation of an operadic algebra A of an operad $\mathscr{P}$ consists of a $\mathscr{P}[\varepsilon]$-algebra $\tilde{A}$ whose reduction is isomorphic to A. The structure map of $\tilde{A}$ represents each n-ary operation $\mu + \varepsilon\nu$ of $\mathscr{P}[\varepsilon]$ as a concrete n-ary operation over $\tilde{A}$. Since the reduction of $\tilde{A}$ must be isomorphic to A, as a $\mathscr{P}$-algebra, the underlying R-module of $\tilde{A}$ must be precisely the same one as that of A. Furthermore, the n-ary operations of $\mathscr{P}[\varepsilon]$ of the form μ, i.e. with no ε part, must agree with the structure map of A, so, once A has been extended to $\tilde{A}$ by linearity over the augmented ring $R[\varepsilon]$, one can characterize the structure map of $\tilde{A}$ by:

$$(\mu + \nu\varepsilon)(a_1, \ldots, a_n) = \mu_A(a_1, \ldots, a_n) + \nu_{\tilde{A}}(a_1, \ldots, a_n)\varepsilon$$

for every $a_1, \ldots, a_n \in A$. Therefore, the infinitesimal deformations of A are fully captured by the terms $\nu_{\tilde{A}}$. One approach to classifying (some of the) infinitesimal deformations is to define a morphism of operads which sends each $\mu \in \mathscr{P}(n)$ to a new n-ary operation $\delta(\mu)$ which represents the infinitesimal deformation of the n-ary operation μ, so that the term $\nu_{\tilde{A}}(a_1, \ldots, a_n)$ becomes $\delta(\mu)_A(a_1, \ldots, a_n)$. Let's formalize this idea. Such a construction corresponds to choosing a morphism of operads of the form $v\colon \mathscr{P} \to \mathscr{P}[\varepsilon]$ which sends $\mu \in \mathscr{P}(n)$ to $\mu' + \delta(\mu)\varepsilon$ and for which $\mu' = \mu$. This last condition is equivalent to asking such a morphism v to be a section of the projection $p\colon \mathscr{P}[\varepsilon] \to \mathscr{P}$, which sends ε to 0, i.e. the operad morphism induced by the augmentation map.

Example 5.11. Let's consider the operad $\mathscr{A}ss$ and let's consider a section $v\colon \mathscr{A}ss \to \mathscr{A}ss[\varepsilon]$ of the projection $\mathscr{A}ss[\varepsilon] \to \mathscr{A}ss$. Let's employ the isomorphism $\mathscr{A}ss[\varepsilon] \cong \mathscr{A}ss \ltimes \mathscr{A}ss$ to characterize $\mathscr{A}ss[\varepsilon]$. v sends the generator $\mu \in \mathscr{A}ss(2)$, which corresponds to the associative multiplication map of an associative algebra, to a pair $(\mu', \delta_v(\mu)) \in \mathscr{A}ss \ltimes \mathscr{A}ss$ of binary operations. Since v is a section of p, $\mu' = \mu$, so the only new data is $\delta_v(\mu)$. Let's adopt the notation ν to denote $\delta_v(\mu)$. Recall that μ satisfies the relation $\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu)$. Then v must send $\mu(\mu, \mathbb{1})$ and $\mu(\mathbb{1}, \mu)$ to the same ternary operation $(\mu(\mathbb{1}, \mu), \delta_v(\mu(\mathbb{1}, \mu)))$. However, since v is an operad morphism, we also have:

$$(\mu(\mu, \mathbb{1}), \delta_v(\mu(\mu, \mathbb{1})))$$

$$
\begin{aligned}
&= v(\mu(\mu, \mathbb{1})) \\
&= (v(\mu))(v(\mu), v(\mathbb{1})) \\
&= (\mu, \nu)((\mu, \nu), (\mathbb{1}, \delta_v(\mathbb{1})))
\end{aligned}
$$

Notice that, since v must preserve the unit of the operad, $(\mathbb{1}, \delta_v(\mathbb{1})) = v(\mathbb{1}) = (\mathbb{1}, 0)$, so we have:

$$
\begin{aligned}
& (\mu(\mathbb{1}, \mu), \delta_v(\mu(\mathbb{1}, \mu)) \\
= \ & (\mu, \nu)((\mu, \nu), (\mathbb{1}, 0)) \\
= \ & (\mu(\mu, \mathbb{1}), \nu(\mu, \mathbb{1}) + \mu(\nu, \mathbb{1}) + \mu(\mu, 0)) \\
= \ & (\mu(\mu, \mathbb{1}), \nu(\mu, \mathbb{1}) + \mu(\nu, \mathbb{1}))
\end{aligned}
$$

On the other hand, v sends $\mu(\mathbb{1}, \mu)$ to:

$$
\begin{aligned}
& (\mu(\mathbb{1}, \mu), \delta_v(\mu(\mathbb{1}, \mu))) \\
= \ & v(\mu(\mathbb{1}, \mu)) \\
= \ & (v(\mu))(v(\mathbb{1}), v(\mu)) \\
= \ & (\mu, \nu)((\mathbb{1}, 0), (\mu, \nu)) \\
= \ & (\mu(\mathbb{1}, \mu), \nu(\mathbb{1}, \mu) + \mu(0, \mu) + \mu(\mathbb{1}, \nu)) \\
= \ & (\mu(\mathbb{1}, \mu), \nu(\mathbb{1}, \mu) + \mu(\mathbb{1}, \nu))
\end{aligned}
$$

Consequently, we obtain the equation:

$$
\nu(\mu, \mathbb{1}) + \mu(\nu, \mathbb{1}) = \nu(\mathbb{1}, \mu) + \mu(\mathbb{1}, \nu) \tag{5.2.1}
$$

Consider now an associative algebra A and let's denote by juxtaposition, i.e. ab the multiplication map $\mu(a, b)$ of A. The operad morphism $v\colon \mathscr{Ass} \to \mathscr{Ass}[\varepsilon]$ induces an adjunction:

$$
v_! \colon \mathsf{Alg} \leftrightarrows \mathsf{Alg}_{\mathscr{Ass}[\varepsilon]} \colon v^*
$$

The free functor $v_!$ sends an associative algebra A to the $\mathscr{Ass}[\varepsilon]$-algebra obtained by quotienting the free $\mathscr{Ass}[\varepsilon]$-algebra by the two different algebra structures, the one induced by the operad multiplication and the one induced by v. In particular, the $\mathscr{P}$-algebra structure of $v_!A$ is determined by:

$$
(\mu + \varepsilon\nu)(a, b) = ab + \nu_A(a, b)\varepsilon
$$

However, ν must satisfy Equation (5.2.1), therefore:

$$\nu(ab,c)+\nu(a,b)c=\nu(a,bc)+a\nu(b,c)$$

Rearranging we obtain:

$$a\nu(b,c)-\nu(ab,c)+\nu(a,bc)-\nu(a,b)c=0$$

which is precisely Equation 5.1.1! So, ν must be a 2-cocycle in the Hochschild cohomology of the algebra A.

Example 5.11 shows an important relationship between sections $v\colon \mathscr{P}\to\mathscr{P}[\varepsilon]$ of the projection map $p\colon\mathscr{P}[\varepsilon]\to\mathscr{P}$ and infinitesimal deformations. This phenomenon is similar to the equivalence proved in Proposition 3.36, between vector fields of the algebraic tangent category of an algebraic operad, i.e. sections of the projection $A[\varepsilon]=A\ltimes A\to A$, and derivations of the algebra A. This analogy is made precise by the next observation: sections v of $p\colon\mathscr{P}[\varepsilon]\to\mathscr{P}$ are equivalent to derivations of the operad $\mathscr{P}$. First, let's recall this definition (in [46, Section 6.3] one can find a similar notion in the context of graded operads).

Definition⋆ 5.12. *A* ***derivation*** *of an operad $\mathscr{P}$ consists of a morphism of symmetric sequences satisfying the Leibniz rule. Concretely, this is a family of R-linear morphisms $\delta_n\colon\mathscr{P}(n)\to\mathscr{P}(n)$ satisfying the following condition:*

$$\delta(\mu(\mu_1,\dots,\mu_n))=(\delta(\mu))(\mu_1,\dots,\mu_n)+\sum_{k=1}^n\mu(\mu_1,\dots,\delta(\mu_k),\dots,\mu_n)$$

and the equivariant condition, i.e.:

$$\delta(\mu\cdot\sigma)=\delta(\mu)\cdot\sigma$$

If $v\colon\mathscr{P}\to\mathscr{P}[\varepsilon]$ is a section of the projection $p\colon\mathscr{P}[\varepsilon]\to\mathscr{P}$, then v composed with the R-linear morphism $\mathscr{P}[\varepsilon]\to\mathscr{P}$ which sends ε to 1 and $\mathbb{1}$ to 0 defines a derivation $\delta_v\colon\mathscr{P}\to\mathscr{P}$. More concretely, v sends μ to $\mu+\varepsilon\delta_v(\mu)$, where δ_v is a derivation of $\mathscr{P}$. Conversely, a $\mathscr{P}$-derivation $\delta\colon\mathscr{P}\to\mathscr{P}$ defines a section $v_\delta\colon\mathscr{P}\to\mathscr{P}[\varepsilon]$ which sends μ to $\mu+\varepsilon\delta(\mu)$. Let's briefly recap this discussion in a lemma.

Lemma 5.13. *There is a bijection between derivations $\delta\colon \mathscr{P} \to \mathscr{P}$ over an operad $\mathscr{P}$ and sections $v\colon \mathscr{P} \to \mathscr{P}[\varepsilon]$ of the projection map $p\colon \mathscr{P}[\varepsilon] \to \mathscr{P}$.*

This relationship between sections of p and derivations together with the explicit use of the ring of dual numbers in the definition of $\mathscr{P}[\varepsilon]$ suggests the existence of a tangent structure on the category Operad of algebraic operads. Moreover, the similarity with the algebraic tangent structure of a given operad suggests that this tangent structure could also be the algebraic tangent structure of a suitable coCartesian differential monad.

To explore this idea, let's start by recalling that operads over a given symmetric monoidal category $\mathbb{E}$ are algebras of a particular monad on the category of symmetric sequences of $\mathbb{E}$. This construction can be found in [46, Section 5.6]. Recall first that a symmetric sequence over a symmetric monoidal category $\mathbb{E}$ is a sequence $\{E(n)\}$ of objects of $\mathbb{E}$ for which the symmetric groups $\mathbb{S}_n$ act on each entry $E(n)$ with right action. Morphisms of symmetric sequences are sequences $\{\varphi(n)\}\colon \{E(n)\} \to \{E'(n)\}$ of $\mathbb{E}$-morphisms $\varphi(n)\colon E(n) \to E'(n)$ satisfying an equivariant condition. Let's denote by $\mathsf{SymSeq}(\mathbb{E})$ the category of symmetric sequences and corresponding morphisms over the symmetric monoidal category $\mathbb{E}$. If $\mathbb{E}$ has biproducts and the biproducts are compatible with the monoidal structure as in Convention 3.20, then also $\mathsf{SymSeq}(\mathbb{E})$ has biproducts, which are defined pointwise, i.e. $\{E(n)\} \oplus \{E'(n)\} = \{E(n) \oplus E'(n)\}$.

To see operads as algebras of a monad, the idea is to consider all possible tree graphs whose vertices are represented by elements of $E(k)$. Let's recall this notion from graph theory.

Definition* 5.14. *An n-**rooted tree** is a tree graph, i.e. a connected graph with no cycles, with a distinct leaf, called the **root**, and with n other leaves labelled with n distinct labels. For an n-rooted tree τ, a **vertex** of τ is a vertex of the underlying graph which is not a leaf.*

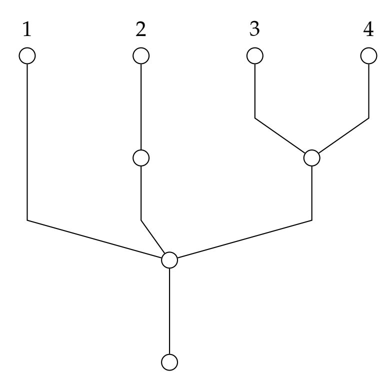
Every n-rooted tree can be made into an oriented tree as follows: an edge e between two distinct vertices u and v is oriented from u to v if v is connected with the root of τ via a path which does not include the edge e. We denote this as $e\colon u \to v$. This is a well-defined orientation. Indeed, suppose that $e\colon u \to v$ and also $e\colon v \to u$ for two distinct vertices u and v, so both u and v are connected to the root $*$ of τ via two paths which do not pass from the edge e. This implies the existence of a cycle in τ, which is impossible since τ is a tree. In particular, this orientation allows one to introduce the following notion: for a vertex v of a rooted tree τ, we denote by $\mathsf{in}(v)$ the number of inputs of v, i.e. edges with target v. Notice also that every vertex has a unique output, since if there were two distinct outputs $e\colon v \to w$ and $e'\colon v \to w'$ this would imply that both w and w', which must be distinct because e and e' are, are connected to the root $*$, implying the existence of a cycle.

Let's also denote by $\mathsf{Tree}(n)$ the set of all n-rooted trees. For a given symmetric sequence $E := \{E(n)\}$ and for an n-rooted tree $\tau \in \mathsf{Tree}(n)$, let's introduce:

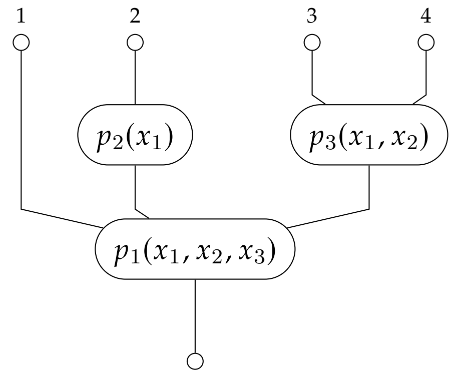
$$E(\tau) := \bigotimes_{v\in\tau} E(\mathsf{in}(v))$$

where we write $v \in \tau$ for a (internal) vertex v of τ and where the tensor product $\otimes_{v\in\tau}$ is made precise in [46, Section 5.1.20]. When $\mathbb{E}$ is the category of R-modules one can interpret the generators of $E(\tau)$ as n-rooted trees in which each vertex v with $k := \mathsf{in}(v)$ inputs is replaced by an element of $E(k)$. The interpretation of the elements of $E(k)$ is to think of them as k-rooted trees, so an element of $E(\tau)$ is a rooted tree obtained by composing rooted trees of the symmetric sequence.

Example 5.15. Consider the following n-rooted tree:



Consider now $E(n) := R\langle x_1, \ldots, x_n\rangle$ to be the ring of non-commutative polynomials in n variables, i.e. polynomials in which the variables do not commute with the other variables. Elements of $E(\tau)$ are finite sums of 4-rooted trees:



obtained by varying the polynomials $p_1(x_1, x_2, x_3)$, $p_2(x_1)$, and $p_3(x_1, x_2)$.

Consider now a morphism $\varphi\colon E \to E'$ of symmetric sequences and an n-rooted tree τ. Then, let:

$$\varphi(\tau)\colon E(\tau) = \bigotimes_{v\in\tau} E(\mathsf{in}(v)) \xrightarrow{\otimes_{v\in\tau}\varphi(\mathsf{in}(v))} \bigotimes_{v\in\tau} E'(\mathsf{in}(v)) = E'(\tau)$$

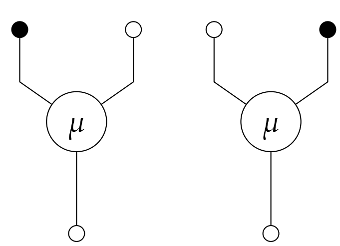
$\varphi(\tau)$ replaces the vertexes of each n-rooted tree decorated by elements of E with the corresponding one of E' via φ.

Let's define the following endofunctor:

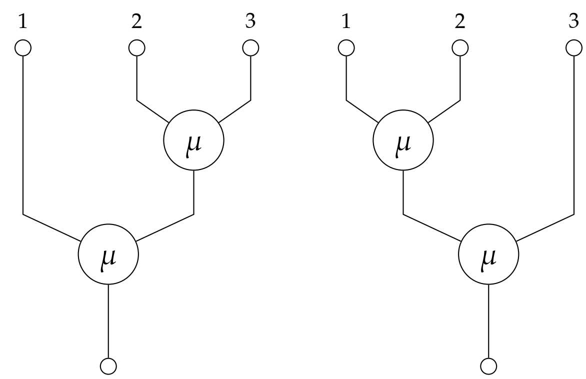
$$W\colon \mathsf{SymSeq} \longrightarrow \mathsf{SymSeq}$$
$$WE(n) := \bigoplus_{\tau\in\mathsf{Tree}(n)} E(\tau)$$
$$W(\varphi\colon E \longrightarrow E') := \bigoplus_{\tau\in\mathsf{Tree}(n)} \varphi(\tau)$$

For a symmetric sequence E over R-modules, $WE(n)$ is generated by all n-rooted trees decorated with the appropriate elements of E, in the right arity.

Example 5.16. A **binary n-rooted tree** is an n-rooted tree τ for which each vertex v has precisely 2 inputs, i.e. $\mathsf{in}(v) = 2$. Let B be the symmetric sequence whose n-entry is the free R-module generated by all binary n-rooted trees. In particular, $B(2)$ only contains two generators:



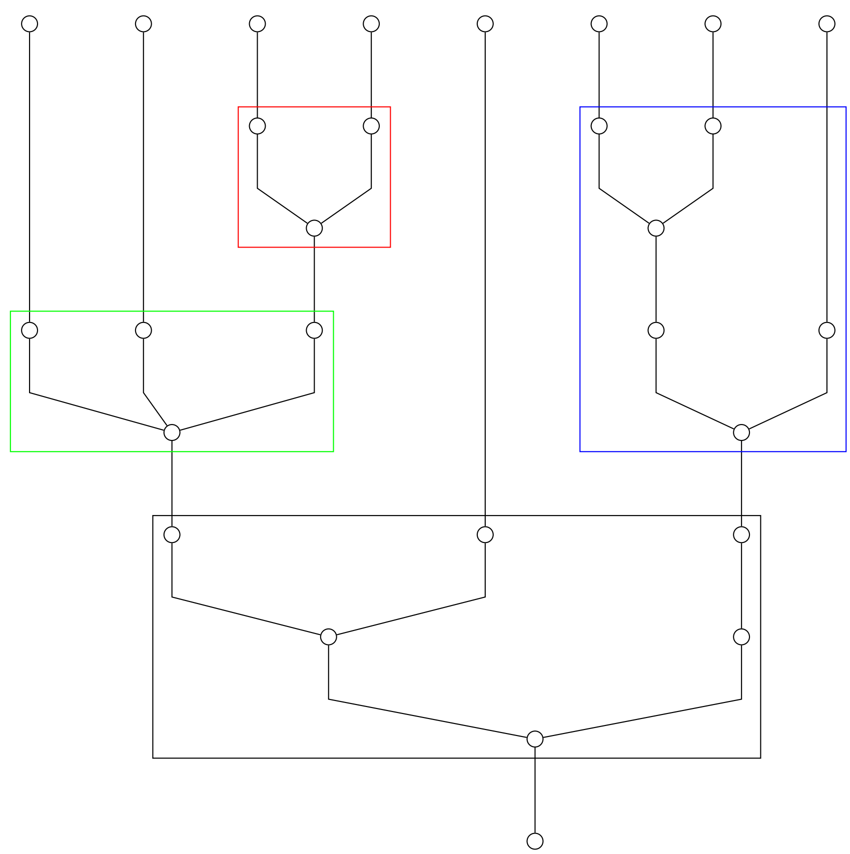
where the distinction is only on the labels, one white and the other one black. The generators of $B(3)$ are obtained by permuting the labels on the inputs of:



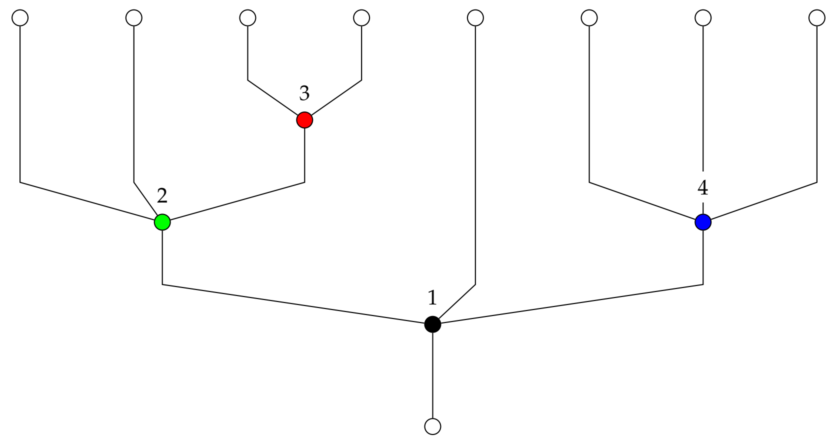
for a total of 12 generators. The symmetric group acts on the generators of $B(n)$ by shuffling the n inputs of each n-rooted tree. One can see that every binary n-rooted tree is simply generated by composing an elementary binary tree that we denote by μ. This simple fact is fully captured by the following statement. Consider the symmetric sequence E defined by $E(2) = R$ and all other entries are the trivial R-module 0. Intuitively, we are saying that E contains only a binary tree μ. Then, WE is isomorphic to B. To see this, notice that for every n-rooted tree τ $B(\tau)$ is zero if τ is not binary, and for each binary n-rooted tree τ $B(\tau)$ is generated by the tree τ decorated in each vertex by the binary operation μ of B. So, $WB(n)$ is precisely $E(n)$.

To see how to characterize operads as algebras of a monad, let's notice that we can characterize the operad $\mathscr{Ass}$ as the operad generated by a binary operad μ subject to the relation $\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu)$. This presentation can be made precise by introducing a morphism of operads from the binary tree operad $B = WE$ described in Example 5.16 to $\mathscr{Ass}$, which sends the generator $\mu \in E(2)$ to $\mu \in \mathscr{Ass}(2)$ and which sends both the generators $\mu(\mu, \mathbb{1})$ and $\mu(\mathbb{1}, \mu)$ of $B(3)$ to the same ternary operation $\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu) \in \mathscr{Ass}(3)$. This morphism is a quotient map which identifies the two ternary operations $\mu(\mu, \mathbb{1})$ and $\mu(\mathbb{1}, \mu)$. This suggests seeing $B = WE$ as a free operad and W as the free functor which sends a symmetric sequence E to the free operad WE.

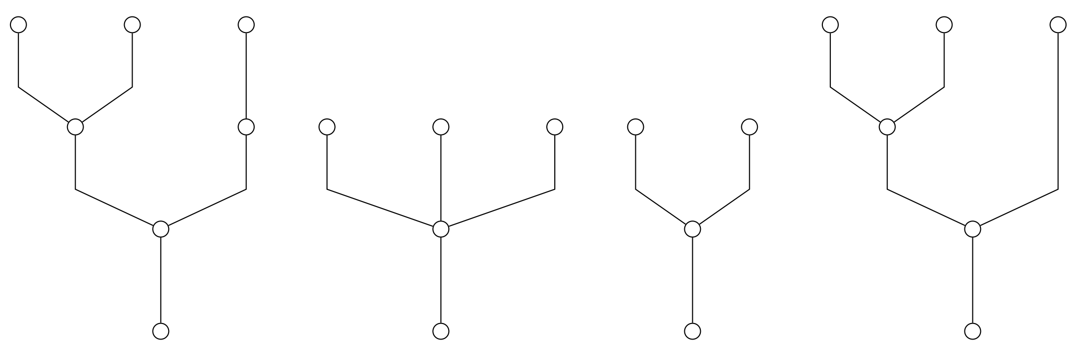
To see this, let's show that W comes equipped with a monad structure. Let's start by introducing the unit. Notice that, when τ is a corolla, i.e. an n-rooted tree with a unique vertex (remember that for vertices we always mean only the internal ones; see Definition 5.14), then $E(\tau) = E(n)$. Therefore, there is a morphism of symmetric sequences $E \longrightarrow WE$ which sends each $\mu \in E(n)$ to the corresponding n-corolla. The next step is to introduce the multiplication $W^2E \longrightarrow WE$. One can see the elements of $W^2E(n)$ as n-rooted trees whose vertices v with $\mathsf{in}(v)$-inputs are $\mathsf{in}(v)$-rooted trees. For example, consider the tree diagram:



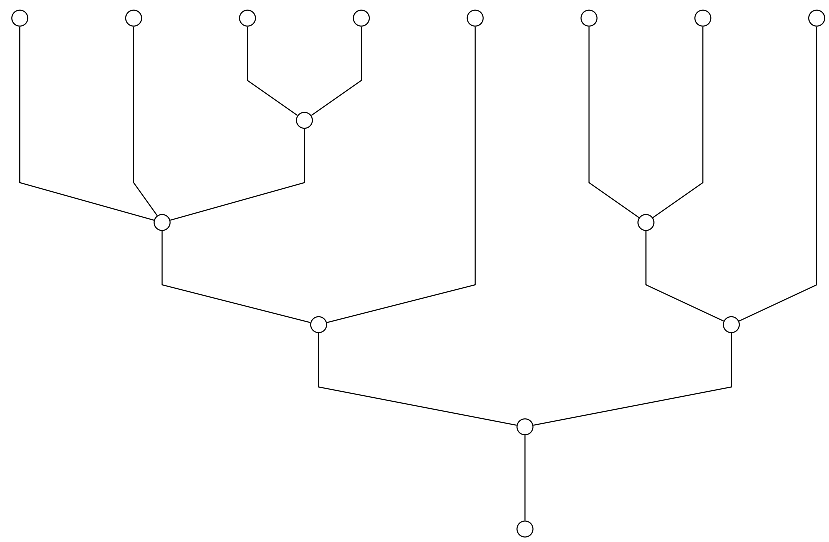
The rooted trees in the boxes represent the vertices of this rooted tree in W^2E. So, from the perspective of W^2E this tree looks like:



where the vertices labelled by $1, 2, 3$ and 4 (respectively, black, green, red, and blue) are the trees:



respectively. So, the multiplication of W consists of "unboxing" the vertices of W^2E and connecting the edges. This operation is known in the literature as *substitution of trees* (see for example [46, Section Section 5.6]). So, the graph of this example is sent to the graph:



Proposition⋆ 5.17. *The endofunctor* $W\colon \mathsf{SymSeq}(\mathbb{E}) \longrightarrow \mathsf{SymSeq}(\mathbb{E})$ *equipped with the morphism* $\eta\colon E \longrightarrow WE$ *which sends each* $\mu \in E(n)$ *to the* n*-corolla and with the morphism* $\gamma\colon W^2E \longrightarrow WE$ *which substitutes the rooted trees of* W^2E*, is a monad whose algebras are all the operads. So,* $\mathsf{Alg}_W = \mathsf{Operad}(\mathbb{E})$.

Remark 5.18. Kontsevich and Soibelman in [40, Proposition 1] showed that operads can be characterized as algebras of a coloured operad. For the sake of simplicity, we decided to ignore this characterization of the monad W in our discussion and adopt the approach of Loday and Vallette we explained. In future work, we are interested in reviewing this construction from the point of view of coloured operads. On this point, in an informal discussion, Sacha Ikonicoff suggested seeing whether or not the monad of a coloured operad is also a coCartesian differential monad. Provided Ikonicoff's suggestion is correct, this can be employed to characterize the category of operads as a tangent category. In future work, we are interested in comparing these two approaches.

We want to prove that the monad W comes equipped with a differential combinator so that W becomes a coCartesian differential monad on the category of symmetric sequences. First, we need a preliminary observation: if $\mathbb{E}$ has biproducts and satisfies Convention 3.20, so does $\mathsf{SymSeq}(\mathbb{E})$. Even though this fact is immediate to prove, it plays an important role in our story.

Lemma 5.19. *If $\mathbb{E}$ has biproducts and satisfies Convention 3.20, so does* $\mathsf{SymSeq}(\mathbb{E})$. *In particular, biproducts in* SymSeq *are defined pointwise, i.e.* $(E \oplus E')(n) := E(n) \oplus E'(n)$.

To understand how to define a suitable differential combinator $\partial\colon WE \longrightarrow W(E \oplus E)$ for the monad W, recall that the differential combinator of an algebraic operad $\mathscr{P}$ is the natural transformation $\partial_{\mathscr{P}}\colon \mathsf{S}_{\mathscr{P}}V \longrightarrow \mathsf{S}_{\mathscr{P}}(V \oplus V)$ defined as follows:

$$\partial_{\mathscr{P}}(\mu; x_1, \dots, x_n) = \sum_{k=1}^{n} (\mu; (x_1, 0), \dots, (0, x_k), \dots, (x_n, 0))$$

For simplicity, let's denote the elements of $\mathsf{S}_{\mathscr{P}}(V \oplus V)$ of the form $(x, 0)$ simply as x and the terms $(0, x)$ as $\mathsf{d}x$. Under this convention, $\partial_{\mathscr{P}}$ can be rewritten as follows:

$$\partial_{\mathscr{P}}(\mu; x_1, \dots, x_n) = \sum_{k=1}^{n} (\mu; x_1, \dots, \mathsf{d}x_k, \dots, x_n)$$

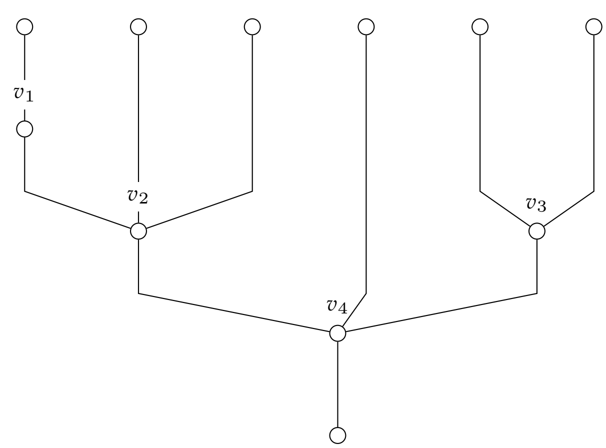
To understand how to define $\partial\colon WE \longrightarrow W(E \oplus E)$, let's start by considering a rooted tree of WE and suppose that $\mathbb{E}$ is the monoidal category of R-modules. Let's adopt the following notation: we denote the n-rooted tree τ of WE by $(\tau; \mu_1(v_1), \dots, \mu_N(v_N))$, where $\mu_k(v_k)$ represents the element μ_k of $E(\mathsf{in}(v_k))$ used to decorate the vertex v_k of τ. So, ∂ is defined as follows:

$$\partial(\tau; \mu_1(v_1), \dots, \mu_N(v_N)) := \sum_{k=1}^{N} (\tau; (\mu_1(v_1), 0), \dots, (0, \mu_k(v_k)), \dots, (\mu_N(v_N), 0))$$

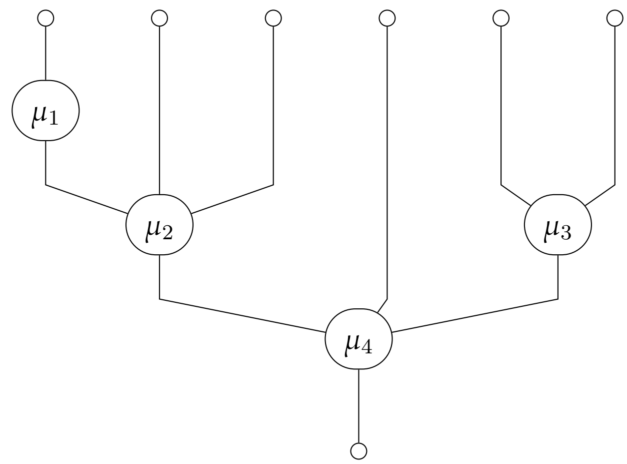
Employing the convention to denote by $\mu_i := (\mu_i, 0)$ and $\mathsf{d}\mu_i := (0, \mu_i)$ we can rewrite:

$$\partial(\tau; \mu_1(v_1), \dots, \mu_N(v_N)) := \sum_{k=1}^{N} (\tau; \mu_1(v_1), \dots, \mathsf{d}\mu_k(v_k), \dots, \mu_N(v_N))$$

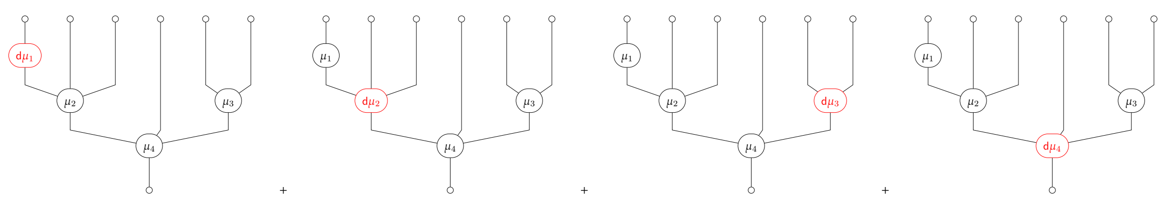
Example 5.20. Consider the rooted tree τ:



where we labelled the vertices from 1 to 4. Let's now consider the elements $\mu_1 \in E(1)$, $\mu_2 \in E(3)$, $\mu_3 \in E(2)$, and $\mu_4 \in E(3)$. Then, the rooted tree of WE denoted by $(\tau; \mu_1(v_1), \ldots, \mu_4(v_4))$ is the tree:



Then $\partial(\tau; \mu_1, \ldots, \mu_4) = \sum_{k=1}^{4} (\tau; \mu_1(v_1), \ldots, \mathsf{d}\mu_k(v_k), \ldots, \mu_4(v_4))$ corresponds to:



where on each term we coloured in red the vertex decorated with d for visual purposes only.

In the following, we restrict our attention to the algebraic case, i.e. when $\mathbb{E}$ is the monoidal category of R-modules. We leave the extension of this construction to the general case for future work.

Proposition 5.21. *The monad W equipped with the natural transformation $\partial \colon WE \to W(E \oplus E)$ is a coCartesian differential monad.*

Proof. In this proof, we simplify notation by removing the reference to the vertex of the tree, so for example, we denote by $(\tau; \mu_1, \ldots, \mu_N)$ the tree τ decorated with the terms $\mu_1, \ldots, \mu_N$ on the vertices. This notation can be employed in this context because all the morphisms involved in the proof do not change the vertices, but only their decoration.

Let's start by proving the zero rule, that is $\partial W(\pi_1) = 0$. This reduces to the following equations:

$$\begin{aligned}
& (\pi_1(\tau) \circ \partial)(\tau; \mu_1, \ldots, \mu_N) \\
= \;& \pi_1(\tau)\left(\sum_{k=1}^{N}(\tau; (\mu_1, 0), \ldots, (0, \mu_k), \ldots, (\mu_N, 0))\right) \\
= \;& \sum_{k=1}^{N}(\tau; \mu_1, \ldots, 0, \ldots, \mu_N) \\
= \;& 0
\end{aligned}$$

To prove the additive rule is to show that $\partial W(\mathsf{id} \oplus \Delta) = \partial\langle W(\mathsf{id} \oplus \iota_1), W(\mathsf{id} \oplus \iota_2)\rangle +$. The left-hand side reads as:

$$\begin{aligned}
& ((\mathsf{id} \oplus \Delta)(\tau) \circ \partial)(\tau; \mu_1, \ldots, \mu_N) \\
= \;& (\mathsf{id} \oplus \Delta)(\tau)\left(\sum_{k=1}^{N}(\tau; (\mu_1, 0), \ldots, (0, \mu_k), \ldots, (\mu_N, 0))\right) \\
= \;& \sum_{k=1}^{N}(\tau; (\mu_1, 0, 0), \ldots, (0, \mu_k, \mu_k), \ldots, (\mu_N, 0, 0))
\end{aligned}$$

Conversely, the right-hand side reads as:

$$\begin{aligned}
& (+ \circ \langle(\mathsf{id} \oplus \iota_1)(\tau), (\mathsf{id} \oplus \iota_2)(\tau)\rangle \circ \partial)(\tau; \mu_1, \ldots, \mu_N) \\
= \;& (+ \circ \langle(\mathsf{id} \oplus \iota_1)(\tau), (\mathsf{id} \oplus \iota_2)(\tau)\rangle)\left(\sum_{k=1}^{N}(\tau; (\mu_1, 0), \ldots, (0, \mu_k), \ldots, (\mu_N, 0))\right) \\
= \;& \qquad +\left(\sum_{k=1}^{N}(\tau; (\mu_1, 0, 0), \ldots, (0, \mu_k, 0), \ldots, (\mu_N, 0, 0)),\right. \\
= \;& \qquad \left.\sum_{k=1}^{N}(\tau; (\mu_1, 0, 0), \ldots, (0, 0, \mu_k), \ldots, (\mu_N, 0, 0))\right) \\
= \;& \sum_{k=1}^{N}(\tau; (\mu_1, 0, 0), \ldots, (0, \mu_k, \mu_k), \ldots, (\mu_N, 0, 0))
\end{aligned}$$

Let's now focus on the linear rule $\eta\partial = \iota_2\eta$. Recall that η sends $\mu \in E(n)$ to the n-corolla, denoted $(\kappa(n); \mu)$. However, by definition, a corolla has a unique (internal) vertex $*$, so the sum and the tensor product over the vertices reduces to a unique term:

$$\partial(\eta(\mu)) = \partial(\kappa(n); \mu) = (\kappa(n); (0, \mu)) = \eta((0, \mu)) = \eta(\iota_2(\mu))$$

The next step is to prove the chain rule $\partial_W W(\langle W(\iota_1), \partial\rangle)\gamma = \gamma\partial$. First, notice that the elements of W^2E are of the form:

$$\left(\tau; (\tau_1; \mu_1^{(1)}, \ldots, \mu_{k_1}^{(1)}), \ldots, (\tau_N; \mu_1^{(N)}, \ldots, \mu_{k_N}^{(N)})\right)$$

Let's simplify notation by denoting each $\left(\tau_i; \mu_1^{(i)}, \ldots, \mu_{k_i}^{(i)}\right)$ by $\vec{\tau}_i$ and by adopting the convention of writing $(\mu_i), 0)$ as μ_i and $(0, \mu_i)$ as $\mathsf{d}\mu_i$. We also introduce the following shorthand $\tau(\tau_1, \ldots, \tau_N)$ for $\gamma(\tau; \tau_1, \ldots, \tau_N)$, which is the rooted tree obtained by *sustituting* the subtrees $\tau_1, \ldots, \tau_N$. With this in mind, we can write:

$$\begin{aligned}
& (\gamma \circ W(\langle W(\iota_1), \partial\rangle) \circ \partial_W)(\tau; \vec{\tau}_1, \ldots, \vec{\tau}_N) \\
= \; & (\gamma \circ W(\langle W(\iota_1), \partial\rangle)) \left(\sum_{k=1}^{N} (\tau; \vec{\tau}_1, \ldots, \mathsf{d}\vec{\tau}_k, \ldots, \vec{\tau}_N)\right) \\
= \; & \gamma\left(\sum_{i=1}^{N} \left(\tau_1; \mu_1^{(1)}, \ldots, \mu_{k_1}^{(1)}\right), \ldots, \sum_{j_i=1}^{k_i} \left(\tau_i; \mu_1^{(i)}, \ldots, \mathsf{d}\mu_{j_i}^{(i)}, \ldots, \mu_{k_i}^{(i)}\right), \ldots \right. \\
& \quad \left. \ldots, \left(\tau_N; \mu_1^{(N)}, \ldots, \mu_{k_N}^{(N)}\right)\right) \\
= \; & \sum_{i=1}^{N} \sum_{j_i=1}^{k_i} \left(\tau(\tau_1, \ldots, \tau_N); \mu_1^{(1)}, \ldots, \mathsf{d}\mu_{j_i}^{(i)}, \ldots, \mu_{k_N}^{(N)}\right) \\
= \; & \sum_{l_1}^{k_1+\ldots+k_N} \left(\tau(\tau_1, \ldots, \tau_N); \mu_1, \ldots, \mathsf{d}\mu_l, \ldots, \mu_{k_1+\ldots+k_N}^{(N)}\right)
\end{aligned}$$

where in the last step we simply reindexed the terms. However:

$$\begin{aligned}
& \sum_{l_1}^{k_1+\ldots+k_N} \left(\tau(\tau_1, \ldots, \tau_N); \mu_1, \ldots, \mathsf{d}\mu_l, \ldots, \mu_{k_1+\ldots+k_N}^{(N)}\right) \\
= \; & \partial(\tau(\tau_1, \ldots, \tau_N); \mu_1, \ldots, \mu_{k_1+\ldots+k_N}^{(N)}) \\
= \; & (\partial \circ \gamma(\tau; \vec{\tau}_1, \ldots, \vec{\tau}_N)
\end{aligned}$$

The next step is to prove the lift rule $\partial\partial W(\pi_1 \oplus \pi_4) = \partial$:

$$\begin{aligned}
& (W(\pi_1 \oplus \pi_4) \circ \partial \circ \partial)(\tau; \mu_1, \ldots, \mu_N) \\
= \; & (W(\pi_1 \oplus \pi_4) \circ \partial)\left(\sum_{k=1}^{N} (\tau; (\mu_1, 0), \ldots, (0, \mu_k), \ldots, (\mu_N), 0)\right) \\
= \; & W(\pi_1 \oplus \pi_4)\left(\sum_{k=1}^{N} \sum_{j \neq k} (\tau; (\mu_1, 0, 0, 0), \ldots, (0, 0, \mu_j, 0), \ldots, (0, \mu_k, 0, 0), \ldots\right.
\end{aligned}$$

$$\ldots,(\mu_N,0,0,0)) + \sum_{k=1}^{N}(\tau;(\mu_1,0,0,0),\ldots,(0,0,0,\mu_k),\ldots,(\mu_N,0,0,0)\Bigg)$$

$$\begin{aligned}
&= \sum_{k=1}^{N}(\tau;(\mu_1,0),\ldots,(0,\mu_k),\ldots,(\mu_N),0) \\
&= \partial(\tau;\mu_1,\ldots,\mu_N)
\end{aligned}$$

Finally, the symmetry rule $\partial\partial W(\mathsf{id}\oplus\tau\oplus\mathsf{id}) = \partial\partial$ reads as follows:

$$\begin{aligned}
& (W(\mathsf{id}\oplus\tau\oplus\mathsf{id})\circ\partial\circ\partial)(\tau;\mu_1,\ldots,\mu_N) \\
={}& (W(\mathsf{id}\oplus\tau\oplus\mathsf{id})\circ\partial)\left(\sum_{k=1}^{N}(\tau;(\mu_1,0),\ldots,(0,\mu_k),\ldots,(\mu_N),0)\right) \\
={}& W(\mathsf{id}\oplus\tau\oplus\mathsf{id})\Bigg(\sum_{k=1}^{N}\sum_{j\neq k}(\tau;(\mu_1,0,0,0),\ldots,(0,0,\mu_j,0),\ldots \\
& \quad \ldots,(0,\mu_k,0,0),\ldots,(\mu_N,0,0,0))+ \\
& \quad \sum_{k=1}^{N}(\tau;(\mu_1,0,0,0),\ldots,(0,0,0\mu_k),\ldots,(\mu_N,0,0,0)\Bigg) \\
={}& \sum_{k=1}^{N}\sum_{j\neq k}(\tau;(\mu_1,0,0,0),\ldots,(0,\mu_j,0,0),\ldots,(0,0,\mu_k,0),\ldots,(\mu_N,0,0,0))+ \\
& \quad \sum_{k=1}^{N}(\tau;(\mu_1,0,0,0),\ldots,(0,0,0,\mu_k),\ldots,(\mu_N,0,0,0) \\
={}& (\partial\circ\partial)(\tau;\mu_1,\ldots,\mu_N)
\end{aligned}$$

This concludes the proof. □

Thanks to Proposition 5.21, the monad W comes equipped with the structure of a coCartesian differential monad. Thanks to Proposition 3.24, this implies that W is a tangent monad over the tangent category of symmetric sequences equipped with the tangent structure induced by biproducts. Consequently, the category of algebras of W comes with a tangent structure.

Theorem 5.22. *The category* Operad *of algebraic operads comes equipped with a tangent structure denoted by* $\mathbb{L}$ *so defined:*

tangent bundle functor *The tangent bundle functor* $\mathrm{L}\colon \mathsf{Operad} \to \mathsf{Operad}$ *sends an operad* $\mathscr{P}$ *to the operad* $\mathscr{P}[\varepsilon] \cong \mathscr{P} \ltimes \mathscr{P}$ *and a morphism* $\varphi\colon \mathscr{P} \to \mathscr{P}'$ *of operads to* $\varphi[\varepsilon]$*, which sends* $(\mu+\varepsilon\nu) \in \mathscr{P}[\varepsilon]$ *to* $(\varphi(\mu)+\varepsilon\varphi(\nu)) \in \mathscr{P}'[\varepsilon]$*;*

projection *The projection* $p^{(\mathbb{L})}\colon \mathbb{L} \Rightarrow \mathsf{id}_{\mathsf{Operad}}$ *is the natural transformation so defined:*

$$p^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon] \to \mathscr{P}$$
$$p^{(\mathbb{L})}(\mu + \varepsilon\nu) = \mu$$

zero morphism *The zero morphism* $z^{(\mathbb{L})}\colon \mathsf{id}_{\mathsf{Operad}} \Rightarrow \mathbb{L}$ *is the natural transformation so defined:*

$$z^{(\mathbb{L})}\colon \mathscr{P} \to \mathscr{P}[\varepsilon]$$
$$z^{(\mathbb{L})}(\mu) = \mu$$

n**-fold pullback** *The* n*-fold pullback of the projection along itself is the functor* $\mathbb{L}_n\colon \mathsf{Operad} \to \mathsf{Operad}$ *which sends an operad* $\mathscr{P}$ *to* $\mathscr{P}[\varepsilon_1, \ldots, \varepsilon_n]$*, which is the operad* $\mathscr{P} \otimes R[\varepsilon_1, \ldots, \varepsilon_n]$*, where* $R[\varepsilon_1, \ldots, \varepsilon_n] := R[x_1, \ldots, x_n]/(x_i x_j, i, j = 1, \ldots, n)$*. Moreover,* $\mathbb{L}_n$ *sends a morphism* $\varphi\colon \mathscr{P} \to \mathscr{P}'$ *to* $\varphi[\varepsilon_1, \ldots, \varepsilon_n] := \varphi \otimes R[\varepsilon_1, \ldots, \varepsilon_n]$*. The* k*-th projection* $\pi_k\colon \mathbb{L}_n \Rightarrow \mathbb{L}$ *is the natural transformation so defined:*

$$\pi_k\colon \mathscr{P}[\varepsilon_1, \ldots, \varepsilon_n] \Rightarrow \mathscr{P}[\varepsilon]$$
$$\pi_k(\mu + \varepsilon_1\nu_1 + \ldots + \varepsilon_n\nu_n) = \mu + \varepsilon_k\nu_k$$

sum morphism *The sum morphism* $s^{(\mathbb{L})}\colon \mathbb{L}_2 \Rightarrow \mathbb{L}$ *is the natural transformation so defined:*

$$s^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon_1, \varepsilon_2] \to \mathscr{P}[\varepsilon]$$
$$s^{(\mathbb{L})}(\mu + \varepsilon_1\nu_1 + \varepsilon_2\nu_2) = \mu + \varepsilon(\nu_1 + \nu_2)$$

vertical lift *The vertical lift* $l^{(\mathbb{L})}\colon \mathbb{L} \Rightarrow \mathbb{L}^2 2$ *is the natural transformation so defined:*

$$l^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon] \to \mathscr{P}[\varepsilon][\varepsilon']$$
$$l^{(\mathbb{L})}(\mu + \varepsilon\nu) = \mu + \varepsilon'\varepsilon\nu$$

canonical flip *The canonical flip* $c^{(\mathbb{L})}\colon \mathbb{L}^2 \Rightarrow \mathbb{L}^2 2$ *is the natural transformation so defined:*

$$c^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon][\varepsilon'] \to \mathscr{P}[\varepsilon][\varepsilon']$$
$$c^{(\mathbb{L})}(\mu + \varepsilon\nu + \varepsilon'\mu' + \varepsilon'\varepsilon\nu') = \mu + \varepsilon\mu' + \varepsilon'\nu + \varepsilon'\varepsilon\nu'$$

Moreover, (Operad, $\mathbb{L}$) *is Cartesian and it has negatives with negation so defined:*

negation *The negation* $n^{(\mathbb{L})}\colon \mathbb{L} \Rightarrow \mathbb{L}$ *is the natural transformation so defined:*

$$n^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon] \to \mathscr{P}[\varepsilon]$$
$$n^{(\mathbb{L})}(\mu + \varepsilon\nu) = \mu - \varepsilon\nu$$

Proof. Thanks to Proposition 5.21, the monad W is a coCartesian differential monad, so, by Proposition 3.24, W is also a tangent monad on the tangent category SymSeq induced by biproducts. Therefore, the category of algebras $\mathsf{Alg}_W \cong$ Operad of W is a tangent category with negatives (where negatives come from the additivity of Mod_R). Finally, Operad has Cartesian products and it is not hard to see that the tangent bundle functor preserves them, so the tangent category is Cartesian. □

From the description of $\mathbb{L}$ over the category of algebraic operads provided by Theorem 5.22, one can notice that the action of the ring of dual numbers over the category of operads determines the entire tangent structure. More precisely:

- the projection $p^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon] \to \mathscr{P}$ is determined by the augmentation map $p := e\colon R[\varepsilon] \to R$ so, $p^{(\mathbb{L})} = \mathsf{id}_{\mathscr{P}} \otimes p$;

- the zero morphism $z^{(\mathbb{L})}$ is determined by the inclusion $z\colon R \to R[\varepsilon]$ so, $z^{(\mathbb{L})} = \mathsf{id}_{\mathscr{P}} \otimes z$;

- the n-fold pullbacks are induced by the n-fold pullbacks $R[\varepsilon_1, \dots, \varepsilon_n]$ of $p\colon R[\varepsilon] \to R$ along itself;

- the sum morphism $s^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon_1, \varepsilon_2] \to \mathscr{P}[\varepsilon]$ is induced by the sum morphism $s\colon R[\varepsilon_1, \varepsilon_2] \to R[\varepsilon]$ which sends ε_1 and ε_2 both to ε, so $s^{(\mathbb{L})} = \mathsf{id}_{\mathscr{P}} \otimes s$;

- the vertical lift $l^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon] \to \mathscr{P}[\varepsilon][\varepsilon']$ is induced by the morphism $l\colon R[\varepsilon] \to R[\varepsilon][\varepsilon']$ which sends ε to $\varepsilon'\varepsilon$,so $l^{(\mathbb{L})} = \mathsf{id}_{\mathscr{P}} \otimes l$;

- the canonical flip $c^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon][\varepsilon'] \to \mathscr{P}[\varepsilon][\varepsilon']$ is induced by $c\colon R[\varepsilon][\varepsilon'] \to R[\varepsilon][\varepsilon']$ which sends ε to ε' and viceversa, so $c^{(\mathbb{L})} = \mathsf{id}_{\mathscr{P}} \otimes c$;

- finally, the negation $n^{(\mathbb{L})}\colon \mathscr{P}[\varepsilon] \to \mathscr{P}[\varepsilon]$ is induced by $n\colon R[\varepsilon] \to R[\varepsilon]$ which sends ε to $-\varepsilon$, so $n^{(\mathbb{L})} = \mathsf{id}_{\mathscr{P}} \otimes n$.

This is justified by the following observation. First, recall that an actegory, a.k.a. a module category, over a monoidal category $\mathbb{E}$, consists of a category $\mathbb{X}$ together with a functor $\cdot\colon \mathbb{E}\times\mathbb{X}\to\mathbb{E}$ and two natural isomorphisms $\alpha\colon (X\cdot A)\cdot B\to X\cdot(A\otimes B)$ and $\eta\colon X\cdot\mathbb{1}\to X$ satisfying some conditions (see [32]).

Lemma 5.23. *The action that sends a Weil algebra $W\in\mathsf{Weil}_1$ (see Section 2.3) over the base ring R and an operad $\mathscr{P}$ to the operad $\mathscr{P}\cdot W := \mathscr{P}\otimes W$ makes the category* Operad *of algebraic operads into a* Weil_1*-actegory. Moreover, the action preserves all pullbacks of* Weil_1.

Proof. We leave it to the reader to check the details of this proof. □

In particular, if $\mathbb{X}$ is an $\mathbb{E}$-actegory, one can define a strong monoidal functor $\mathbb{E}\to\mathsf{End}(\mathbb{X})$ where $\mathsf{End}(\mathbb{X})$ denotes the monoidal category of endofunctors over $\mathbb{X}$. When the base monoidal category $\mathbb{E}$ is the category Weil_1 and the strong monoidal functor $\mathsf{Weil}_1\to\mathsf{End}(\mathbb{X})$ preserves certain pullback diagrams, as discussed in 2.3, this defines a tangent structure on the category $\mathbb{X}$. This is precisely an equivalent characterization of the tangent category $(\mathsf{Operad},\mathbb{L})$ of Theorem 5.22.

Proposition 5.24. *The tangent structure $\mathbb{L}$ over the category of algebraic operads described in Theorem 5.22 is precisely the tangent structure corresponding to the strong monoidal functor* $\mathsf{Weil}_1\to\mathsf{End}(\mathsf{Operad})$ *which sends a Weil algebra W over the ring R to the endofunctor* $(-)\otimes W\colon \mathsf{Operad}\to\mathsf{Operad}$.

5.2.1 The tangent category of tangent monads

Each algebraic operad $\mathscr{P}$ is associated with a tangent monad $(\mathsf{S}_{\mathscr{P}},\alpha_{\mathscr{P}})$ and this operation extends also to morphisms, functorially. In the previous section, we showed that the category of operads comes equipped with a tangent structure. In this section, we show that so does the category of tangent monads over a fixed tangent category. We then show that the functor $\mathsf{S}\colon(\mathsf{Operad},\mathbb{L})\to\mathsf{TngMnd}(\mathsf{Mod}_R,\mathbb{L})$ extends to a strong tangent morphism.

In this section, we fix a base tangent category $(\mathbb{X},\mathbb{T})$ and we denote by $\mathsf{TngMnd}(\mathbb{X},\mathbb{T})$ the category of tangent monads over this base tangent category. The main tool we need to explore this construction is the concept of distributive law. We invite the reader to consult [54] (notice that *triples* is the dated name for monads). Let's recall this definition for formal monads.

Definition* **5.25.** *A* ***distributive law*** *between two monads S and S' over an object $\mathbb{X}$ in a strict 2-category $\mathbf{C}$ consists of a 2-morphism $\kappa\colon S'\circ S\Rightarrow S\circ S'$ of $\mathbf{C}$ compatible with the multiplication morphisms γ and γ' of S and S', respectively, as follows:*

$$\begin{tikzcd}
S'^2\circ S \arrow[r, "S'\kappa"] \arrow[d, "\gamma' S"'] & S'\circ S\circ S' \arrow[r, "\kappa S'"] & S\circ S'^2 \arrow[d, "S\gamma'"] \\
S'\circ S \arrow[rr, "\kappa"'] & & S\circ S'
\end{tikzcd}$$

$$\begin{tikzcd}
S'\circ S^2 \arrow[r, "\kappa S"] \arrow[d, "S'\gamma"'] & S\circ S'\circ S \arrow[r, "S\kappa"] & S^2\circ S' \arrow[d, "\gamma S'"] \\
S'\circ S \arrow[rr, "\kappa"'] & & S\circ S'
\end{tikzcd}$$

and compatible with the unit η of S and the unit η' of S', as follows:

$$\begin{tikzcd}
S'\circ S \arrow[r, "\kappa"] & S\circ S' \\
S \arrow[u, "\eta' S"] \arrow[r, equal] & S \arrow[u, "S\eta'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S'\circ S \arrow[r, "\kappa"] & S\circ S' \\
S' \arrow[u, "S'\eta"] \arrow[r, equal] & S' \arrow[u, "\eta S'"']
\end{tikzcd}$$

Distributive laws between two monads allow one to compose the underlying 1-morphisms of the monads to obtain a new monad. Concretely, given a distributive law $\kappa\colon S'\circ S\Rightarrow S\circ S'$, we can define the monad $S\circ_\kappa S'$ over $\mathbb{X}$ whose underlying 1-morphism is given by the composition $S\circ S'\colon\mathbb{X}\to\mathbb{X}$ and whose multiplication and unit are given by:

$$\gamma_\kappa\colon S\circ S'\circ S\circ S'\xrightarrow{S\kappa S'}S^2\circ S'^2\xrightarrow{\gamma S'^2}S\circ S'^2\xrightarrow{S\gamma}S\circ S'$$
$$\eta_\kappa\colon \mathsf{id}_{\mathbb{X}}\xrightarrow{\eta}S\xrightarrow{S\eta'}S\circ S'$$

respectively.

Recall also that tangent monads are formal monads in the 2-category TngCat of tangent categories. Therefore, we can define a distributive law between two tangent monads over the base tangent category $(\mathbb{X}, \mathbb{T})$ as a distributive law between the two formal monads in TngCat.

Street in [56] showed that giving a distributive law $\kappa\colon S' \circ S \Rightarrow S' \circ S$ between two formal monads S and S' over the same object $\mathbb{X}$ of a strict 2-category $\mathbf{C}$ is precisely the same as giving a monad $(S, \kappa)\colon S \to S$ in the 2-category $\mathsf{Mnd}(\mathbf{C})$ of monads of $\mathbf{C}$ over S.

In Section 2.4 we proved that $\mathsf{Tng}(\mathsf{Mnd}(\mathbf{C})) \cong \mathsf{Mnd}(\mathsf{Tng}(\mathbf{C}))$ which implies:

$$\mathsf{Mnd}(\mathsf{Mnd}(\mathsf{Tng}(\mathbf{C}))) \cong \mathsf{Tng}(\mathsf{Mnd}(\mathsf{Mnd}(\mathbf{C})))$$

So, distributive laws of tangent monads are distributive laws of the underlying monads which are also 2-morphisms between the underlying lax tangent morphisms. Let's make this definition precise.

Definition 5.26. *A **distributive law of tangent monads** κ between two tangent monads (S', α') and (S, α) over $(\mathbb{X}, \mathbb{T})$ consists of a distributive law $\kappa\colon S' \circ S \Rightarrow S \circ S'$ between the underlying monads which make the following diagram commutative:*

$$\begin{tikzcd}
S' \circ S \circ \mathrm{T} \arrow[r, "S'\alpha"] \arrow[d, "\kappa\mathrm{T}"'] & S' \circ \mathrm{T} \circ S \arrow[r, "\alpha' S"] & \mathrm{T} \circ S' \circ S \arrow[d, "\mathrm{T}\kappa"] \\
S \circ S' \circ \mathrm{T} \arrow[r, "S\alpha'"'] & S \circ \mathrm{T} \circ S' \arrow[r, "\alpha S'"'] & \mathrm{T} \circ S \circ S'
\end{tikzcd}$$

In the following, to denote κ we adopt the notation : $(S', \alpha') \circ (S, \alpha) \Rightarrow (S, \alpha) \circ (S', \alpha')$.

A consequence of the composability of formal monads in the presence of a distributive law and of the fact that tangent monads are formal monads in the 2-category TngCat is that distributive laws of tangent monads $\kappa\colon (S', \alpha') \circ (S, \alpha) \Rightarrow (S, \alpha) \circ (S', \alpha')$ allow one to compose the tangent monads to obtain a new tangent monad $(S, \alpha) \circ_\kappa (S', \alpha')$. Concretely, the underlying monad is precisely $S \circ_\kappa S'$ and the distributive law between $S \circ_\kappa S'$ and the tangent bundle functor T is the natural transformation:

$$\alpha \circ_\kappa \alpha'\colon S \circ S' \circ \mathrm{T} \xrightarrow{S\alpha'} S \circ \mathrm{T} \circ S' \xrightarrow{\alpha S'} \mathrm{T} \circ S \circ S'$$

Lemma 5.27. *Given a distributive law $\kappa\colon (S', \alpha') \circ (S, \alpha) \Rightarrow (S, \alpha) \circ (S', \alpha')$ of tangent monads over $(\mathbb{X}, \mathbb{T})$, the monad $S \circ_\kappa S'$ equipped with the natural transformation $\alpha \circ_\kappa \alpha'$ is a tangent monad on $(\mathbb{X}, \mathbb{T})$, denoted by $(S, \alpha) \circ_\kappa (S', \alpha')$.*

Cockett and Cruttwell in [12, Section 3.2] showed that the tangent bundle functor $\mathrm{T}\colon \mathbb{X} \to \mathbb{X}$ comes equipped with the structure of a monad. Moreover, they also showed that the canonical flip $c\colon \mathrm{T} \circ \mathrm{T} \Rightarrow \mathrm{T} \circ \mathrm{T}$ is a distributive law of the monad T with itself, making $\mathrm{T} \circ_c \mathrm{T}$ into a new monad. However, since c is a distributive law between the monad T and the tangent bundle functor T which is compatible with the tangent structure, (T, c) becomes a tangent monad on $(\mathbb{X}, \mathbb{T})$. We record this observation in the following lemma.

Lemma 5.28. *The tangent bundle functor* $\mathrm{T}\colon \mathbb{X} \to \mathbb{X}$ *of a tangent category* $(\mathbb{X}, \mathbb{T})$ *equipped with the multiplication map defined by:*

$$\gamma^{(\mathrm{T})}\colon \mathrm{T}^2 \xrightarrow{\langle \mathrm{T}p, p\mathrm{T}\rangle} \mathrm{T}_2 \xrightarrow{s} \mathrm{T}$$

and with the unit map $\eta^{(\mathrm{T})} := z\mathsf{id}_{\mathbb{X}} \to \mathrm{T}$ *becomes a monad on* $\mathbb{X}$*. Moreover,* T *equipped with* $c\colon \mathrm{T} \circ \mathrm{T} \to \mathrm{T} \circ \mathrm{T}$ *becomes a tangent monad* $(\mathrm{T}, c)\colon (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$*. Furthermore, for every positive integer* n*, also* T_n *is a tangent monad, whose multiplication and unit maps are respectively defined by:*

$$\gamma_n^{(\mathrm{T})} := \langle \mathrm{T}_n\pi_1\pi_{1\mathrm{T}}\gamma^{(\mathrm{T})}, \dots, \mathrm{T}_n\pi_n\pi_{n\mathrm{T}}\gamma^{(\mathrm{T})}\rangle\colon \mathrm{T}_n \circ \mathrm{T}_n \Rightarrow \mathrm{T}_n$$
$$\eta_n^{(\mathrm{T})} := \langle z, \dots, z\rangle$$

and whose distributive law with T *is defined by:*

$$c_n := \langle \pi_{1\mathrm{T}}c, \dots, \pi_{n\mathrm{T}}c\rangle\colon \mathrm{T}_n \circ \mathrm{T} \Rightarrow \mathrm{T} \circ \mathrm{T}_n$$

For a given tangent monad (S, α) over $(\mathbb{X}, \mathbb{T})$, there always exists a distributive law between (S, α) and (T, c). To see this, notice that $\kappa := \alpha\colon S \circ \mathrm{T} \Rightarrow \mathrm{T} \circ S$ satisfies the axioms of a distributive law between a monad and an endofunctor. To conclude that $\kappa := \alpha$ is a distributive law between tangent monads one needs to show that the following diagrams commute:

$$\begin{tikzcd}
S \circ \mathrm{T}^2 \arrow[r, "\kappa\mathrm{T}"] \arrow[d, "S\gamma^{(\mathrm{T})}"'] & \mathrm{T} \circ S \circ \mathrm{T} \arrow[r, "\mathrm{T}\kappa"] & \mathrm{T}^2 \circ S \arrow[d, "\gamma^{(\mathrm{T})}S"] \\
S \circ \mathrm{T} \arrow[rr, "\kappa"'] & & \mathrm{T} \circ S
\end{tikzcd}
\qquad
\begin{tikzcd}
S \circ \mathrm{T} \arrow[r, "\kappa"] & \mathrm{T} \circ S \\
S \arrow[u, "S\eta^{(\mathrm{T})}"] \arrow[r, equal] & S \arrow[u, "\eta^{(\mathrm{T})}S"']
\end{tikzcd}$$

$$\begin{tikzcd}
S\circ \mathrm{T}\circ \mathrm{T} \arrow[r, "Sc"] \arrow[d, "\kappa\mathrm{T}"'] & S\circ \mathrm{T}\circ \mathrm{T} \arrow[r, "\alpha\mathrm{T}"] & \mathrm{T}\circ S\circ \mathrm{T} \arrow[d, "\mathrm{T}\kappa"] \\
\mathrm{T}\circ S\circ \mathrm{T} \arrow[r, "\mathrm{T}\alpha"'] & \mathrm{T}\circ \mathrm{T}\circ S \arrow[r, "cS"'] & \mathrm{T}\circ \mathrm{T}\circ S
\end{tikzcd}$$

However, the commutativity of the first diagram is a consequence of the compatibility between α and the projection and the sum morphism; the second diagram establishes precisely the compatibility between α and the zero morphism; finally, the last diagram is the compatibility between α and the canonical flip.

Lemma 5.29. *Given a tangent monad (S,α) over $(\mathbb{X},\mathbb{T})$, the lax distributive law α defines a distributive law $\alpha\colon (S,\alpha)\circ(\mathrm{T},c)\Rightarrow(\mathrm{T},c)\circ(S,\alpha)$ of tangent monads. In particular, this defines a new tangent monad $\overline{\mathrm{T}}(S,\alpha) := (\mathrm{T},c)\circ_\alpha(S,\alpha) = (\mathrm{T}\circ_\alpha S, \mathrm{T}\circ S\circ\mathrm{T}\xrightarrow{\mathrm{T}\alpha}\mathrm{T}\circ\mathrm{T}\circ S\xrightarrow{cS}\mathrm{T}\circ\mathrm{T}\circ S)$.*

Similarly, for each positive integer n, $\alpha_n := \langle S\pi_1\alpha,\dots,S\pi_n\alpha_n\rangle\colon S\circ\mathrm{T}_n\Rightarrow\mathrm{T}_n\circ S$ constitutes a distributive law $(\mathrm{T}_n,c_n)\circ(S,\alpha)\Rightarrow(S,\alpha)\circ(\mathrm{T}_n,c_n)$ of tangent monads. In particular, this defines a new tangent monad $\overline{\mathrm{T}}_n(S,\alpha) := (\mathrm{T}_n,c_n)\circ_{\alpha_n}(S,\alpha) = (\mathrm{T}_n\circ_{\alpha_n} S, \mathrm{T}_n\alpha c_{nS})$.

Theorem 5.30. *The category $\mathsf{TngMnd}(\mathbb{X},\mathbb{T})$ of tangent monads over $(\mathbb{X},\mathbb{T})$ is a tangent category with the following tangent structure:*

tangent bundle functor *The tangent bundle functor $\overline{\mathrm{T}}\colon \mathsf{TngMnd}(\mathbb{X},\mathbb{T})\to\mathsf{TngMnd}(\mathbb{X},\mathbb{T})$ sends a tangent monad (S,α) to $\overline{\mathrm{T}}(S,\alpha) := (\mathrm{T},c)\circ_\alpha(S,\alpha)$ and it sends a morphism $\varphi\colon(S,\alpha)\to(S',\alpha')$ of tangent monads to $\overline{\mathrm{T}}\varphi := \mathrm{T}\varphi$;*

projection *The projection $\overline{p}\colon\overline{\mathrm{T}}\Rightarrow\mathsf{id}_{\mathsf{TngMnd}(\mathbb{X},\mathbb{T})}$ is the natural transformation so defined:*

$$\overline{p}\colon(\mathrm{T}\circ_\alpha S,\mathrm{T}\alpha c_S)\xrightarrow{pS}(S,\alpha)$$

zero morphism *The zero morphism $z\colon\mathsf{id}_{\mathsf{TngMnd}(\mathbb{X},\mathbb{T})}\Rightarrow\overline{\mathrm{T}}$ is so defined:*

$$\overline{z}\colon(S,\alpha)\xrightarrow{zS}(\mathrm{T}\circ_\alpha S,\mathrm{T}\alpha c_S)$$

n-fold pullback *the n-fold pullback of the projection along itself is the functor $\overline{\mathrm{T}}_n$ which sends a tangent monad (S,α) to $\overline{\mathrm{T}}_n(S,\alpha) := (\mathrm{T}_n,c_n)\circ_{\alpha_n}(S,\alpha)$ and a morphism*

$\varphi\colon (S,\alpha) \to (S',\alpha')$ *of tangent monads to* $\overline{\mathrm{T}}_n\varphi \colon= \mathrm{T}_n\varphi$. *Moreover, the k-th projection* $\pi_k\colon \overline{\mathrm{T}}_n \Rightarrow \overline{\mathrm{T}}$ *is the natural transformation so defined:*

$$\pi_k\colon (\mathrm{T}_n \circ_{\alpha_n} S, \mathrm{T}_n\alpha c_{nS}) \xrightarrow{\pi_k S} (\mathrm{T} \circ_\alpha S, \mathrm{T}\alpha c_S)$$

sum morphism *The sum morphism* $\overline{s}\colon \overline{\mathrm{T}}_2 \Rightarrow \overline{\mathrm{T}}$ *is the natural transformation so defined:*

$$\overline{s}\colon (\mathrm{T}_2 \circ_{\alpha_2} S, \mathrm{T}_2\alpha c_{2S}) \xrightarrow{sS} (\mathrm{T} \circ_\alpha S, \mathrm{T}\alpha c_S)$$

The lift $\overline{l}\colon \overline{\mathrm{T}} \Rightarrow \overline{\mathrm{T}}^2$ *is the natural transformation so defined:*

$$\overline{l}\colon (\mathrm{T} \circ_\alpha S, \mathrm{T}\alpha c_S) \xrightarrow{lS} (\mathrm{T}^2 \circ_\alpha S, \mathrm{T}^2\alpha \mathrm{T} c_S c_{\mathrm{T}S})$$

The canonical flip $\overline{c}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$ *is the natural transformation so defined:*

$$\overline{c}\colon (\mathrm{T}^2 \circ_\alpha S, \mathrm{T}^2\alpha \mathrm{T} c_S c_{\mathrm{T}S}) \xrightarrow{cS} (\mathrm{T}^2 \circ_\alpha S, \mathrm{T}^2\alpha \mathrm{T} c_S c_{\mathrm{T}S})$$

Finally, if $(\mathbb{X},\mathbb{T})$ *has negatives with negation* $n\colon \mathrm{T} \Rightarrow \mathrm{T}$ *so does* $\mathsf{TngMnd}(\mathbb{X},\mathbb{T})$ *with negation defined as follows:*

negation *The negation* $\overline{n}\colon \overline{\mathrm{T}} \Rightarrow \overline{\mathrm{T}}$ *is the natural transformation so defined:*

$$\overline{n}\colon (\mathrm{T} \circ_\alpha S, \mathrm{T}\alpha c_S) \xrightarrow{nS} (\mathrm{T} \circ_\alpha S, \mathrm{T}\alpha c_S)$$

Proof. Let's give a sketch of the proof and leave it to the reader to complete all the tedious details. First, as noticed in [11, Example 2.2(vii)], given any tangent category $(\mathbb{X},\mathbb{T})$ and any other category $\mathbb{Y}$, the category of functors $\mathsf{Cat}(\mathbb{Y},\mathbb{X})$ comes equipped with a tangent structure with tangent bundle functor:

$$\overline{\mathrm{T}}(F\colon \mathbb{Y} \to \mathbb{X}) \colon= \mathrm{T} \circ F\colon \mathbb{Y} \to \mathbb{X}$$

and natural transformations are defined as in $\mathbb{T}$. In particular, this implies that the category $\mathsf{End}(\mathbb{X},\mathbb{T})$ of endofunctors of a tangent category $(\mathbb{X},\mathbb{T})$ is also a tangent category. Notice that $\mathsf{TngMnd}(\mathbb{X},\mathbb{T})$ is a subcategory of this tangent category and that the tangent bundle functor $\overline{\mathrm{T}}$ is a restriction of the tangent bundle functor of $\mathsf{Cat}(\mathbb{X},\mathbb{X})$. Therefore, to prove that $\mathsf{TngMnd}(\mathbb{X},\mathbb{T})$ is a tangent category, it suffices to show that (1) the natural transformations of the tangent structure on $\mathsf{Cat}(\mathbb{X},\mathbb{X})$

restrict to $\mathsf{TngMnd}(\mathbb{X},\mathbb{T})$ and (2) that the universality of the lift also holds in the category of tangent monads. We leave it to the reader to check the numerous diagrams required to prove (1) and instead, we focus on showing the universality of the vertical lift.

First, notice that the morphism $\overline{v}\colon = (\bar{l} \times \overline{z}_{\overline{\mathrm{T}}})\overline{\mathrm{T}}\overline{s}\colon \overline{\mathrm{T}}_2 \Rightarrow \overline{\mathrm{T}}^2$ is the natural transformation so defined:

$$\overline{v}\colon (\mathrm{T}_2 \circ_{\alpha_2} S, \mathrm{T}_2\alpha c_S) \xrightarrow{(lS\times z\mathrm{T}S)\mathrm{T}sS} (\mathrm{T}^2 \circ_\alpha S, \mathrm{T}^2\alpha\mathrm{T}c_S c_{\mathrm{T}S})$$

With this in mind, let's consider a morphism $\varphi\colon (S',\alpha') \to \overline{\mathrm{T}}^2(S,\alpha)$ for which $\varphi\overline{\mathrm{T}}\overline{p} = \varphi\overline{\mathrm{T}}p_{\overline{\mathrm{T}}}\overline{pz}$. Concretely, φ is a natural transformation $\varphi\colon S' \to \mathrm{T}^2S$, compatible with the monad structures and with the tangent structures, and for which:

$$\begin{tikzcd}
S' \arrow[rrr, "\varphi"] \arrow[d, "\varphi"'] & & & \mathrm{T}^2 \circ S \arrow[d, "\mathrm{T}pS"] \\
\mathrm{T}^2 \circ S \arrow[r, "p\mathrm{T}S"'] & \mathrm{T} \circ S \arrow[r, "pS"'] & S \arrow[r, "zS"'] & \mathrm{T} \circ S
\end{tikzcd}$$

By the universality of the lift in $\mathsf{End}(\mathbb{X},\mathbb{T})$ we obtain a unique natural transformation $\psi\colon (S',\alpha') \to \overline{\mathrm{T}}_2(S,\alpha)$ such that $\psi\overline{v} = \varphi$. To prove that ψ is indeed a morphism of tangent monads one employs the universality of the equalizer morphism v. Here, we only show how to prove the compatibility with the multiplication maps and we leave it to the reader to check unitality and the compatibility with the distributive laws of the tangent monads. We want to show the commutativity of the following diagram:

$$\begin{tikzcd}
S'^2 \arrow[r, "S'\psi"] \arrow[dddd, "\gamma'"'] & S'\mathrm{T}_2S \arrow[r, "\psi\mathrm{T}_2S"] & \mathrm{T}_2S\mathrm{T}_2S \arrow[d, "\mathrm{T}_2\alpha_2S"] \\
& & \mathrm{T}_2^2S^2 \arrow[d, "\gamma_2^{(\mathrm{T})}S^2"] \\
& & \mathrm{T}_2S^2 \arrow[d, "\mathrm{T}_2\gamma"] \\
& & \\
S' \arrow[rr, "\psi"'] & & \mathrm{T}_2S
\end{tikzcd}$$

In order to do so, we first post compose by ν_S as follows:

$$\begin{tikzcd}
S'^2 \arrow[r, "S'\varphi"] \arrow[dd, equal] & S'\circ\mathrm{T}^2\circ S \arrow[rrr, "\varphi\mathrm{T}^2S"] & & & \mathrm{T}^2\circ S\circ\mathrm{T}^2\circ S \arrow[dd, "\mathrm{T}^2\alpha\mathrm{T}^2"] \\
& & & \mathrm{T}_2\circ S\circ\mathrm{T}^2\circ S \arrow[ur, "\nu S\mathrm{T}^2S"] & \\
S'^2 \arrow[r, "S'\psi"] \arrow[dddd, "\gamma'"'] & S'\circ\mathrm{T}_2\circ S \arrow[uu, "S'\nu S"'] \arrow[r, "\psi\mathrm{T}_2S"] & \mathrm{T}_2\circ S\circ\mathrm{T}_2\circ S \arrow[ur, "\mathrm{T}_2S\nu S"] \arrow[d, "\mathrm{T}_2\alpha_2S"] & & \mathrm{T}^3\circ S\circ\mathrm{T}\circ S \arrow[d, "\mathrm{T}^4\alpha S"] \\
& & \mathrm{T}_2^2\circ S^2 \arrow[r, "\mathrm{T}_2\nu S^2"] \arrow[dd, "\gamma_2^{(\mathrm{T})}S^2"] & \mathrm{T}_2\circ\mathrm{T}^2\circ S^2 \arrow[r, "\nu\mathrm{T}^2S^2"] & \mathrm{T}^4\circ S^2 \arrow[d, "\gamma^{(\mathrm{T})}\mathrm{T}^2S^2"] \\
& & & & \mathrm{T}^3S\circ S^2 \arrow[d, "\mathrm{T}\gamma^{(\mathrm{T})}S^2"] \\
& & \mathrm{T}_2\circ S^2 \arrow[rr, "\nu S^2"] \arrow[d, "\mathrm{T}_2\gamma"] & & \mathrm{T}^2\circ S^2 \arrow[dd, "\mathrm{T}^2\gamma"] \\
S' \arrow[rr, "\psi"'] \arrow[d, equal] & & \mathrm{T}_2\circ S \arrow[drr, "\nu S"'] & & \\
S' \arrow[rrrr, "\varphi"'] & & & & \mathrm{T}^2\circ S
\end{tikzcd}$$

Thanks to this and the universality of ν we conclude that the previous diagram commutes as well. □

An interesting corollary of Theorem 5.30 is that for every integer n, the functor T^n defines a tangent monad.

Corollary 5.31. *For a given tangent category $(\mathbb{X},\mathbb{T})$ and an integer n, the endofunctor T^n defines a tangent monad $(\mathrm{T}^n, c^{(n)})$, where the tangent distributive law $c^{(n)}$ is so defined:*

$$c^{(n)} := (\mathrm{T}^n c)(\mathrm{T}^{n-1}c_{\mathrm{T}})\dots(\mathrm{T}^{n-k}c_{\mathrm{T}^k})\dots(\mathrm{T}c_{\mathrm{T}^{n-1}})(c_{\mathrm{T}^n})$$

Proof. Lemma 5.28 establishes that (T,c) is a tangent monad. By induction, let $(\mathrm{T}^{n-1}, c^{(n-1)})$ be a tangent monad and let's prove that so does $(\mathrm{T}^n, c^{(n)})$. However, this is immediate from $(\mathrm{T}^n, c^{(n)}) = \overline{\mathrm{T}}(\mathrm{T}^{n-1}, c^{(n-1)})$. □

The goal of this section is to show (1) that the category of tangent monads over a fixed tangent category is itself a tangent category and (2) that the functor $\mathsf{S}\colon \mathsf{Operad} \to \mathsf{TngMnd}(\mathsf{Mod}_R, \mathbb{L})$ which sends an operad to its corresponding tangent monad on the tangent category of R-modules induced by biproducts preserves the tangent structure described in 5.22.

Theorem 5.30 accomplishes the first goal. Let's now focus on (2). First of all, let's unwrap the monad structure of the tangent monad $\mathbb{L}$ whose underlying functor is the tangent bundle functor over Mod_R.

Lemma 5.32. *In the tangent category* $(\mathsf{Mod}_R, \mathbb{L})$*, the monad structure associated with the tangent bundle functor given by Lemma 5.28 is defined as follows:*

$$\gamma^{(\mathbb{L})}\colon \mathbb{L}^2 M \to \mathbb{L}M$$
$$\gamma^{(\mathbb{L})}(x, y, z, t) := (x, z + y)$$

$$\eta^{(\mathbb{L})}\colon M \to \mathbb{L}M$$
$$\eta^{(\mathbb{L})}(x) := (x, 0)$$

Proof. By definition, the unit of $\mathbb{L}$ is the zero morphism so, let's focus on the multiplication morphism. $\gamma^{(\mathbb{L})}$ is the morphism:

$$\gamma^{(\mathbb{L})}\colon \mathbb{L}^2 M \xrightarrow{\langle \mathbb{L}p, p\mathbb{L}\rangle} \mathbb{L}_2 \xrightarrow{s} \mathbb{L}$$

So:

$$\begin{aligned} & \gamma^{(\mathbb{L})}(x, y, z, t) \\ = \;& (s \circ \langle \mathbb{L}p, p\mathbb{L}\rangle)(x, y, z, t) \\ = \;& s(x, z; x, y) \\ = \;& (x, z + y) \end{aligned}$$

This concludes the proof. □

For the next result, we employ the convention of regarding $\mathbb{L}\mathscr{P}$ as $\mathscr{P} \ltimes \mathscr{P}$, so $\mathbb{L}\mathscr{P}(n)$ is the R-module $\mathscr{P}(n) \oplus \mathscr{P}(n)$ and the multiplication and the unit of $\mathscr{P} \ltimes \mathscr{P}$ are defined as in Example 5.6.

Theorem 5.33. *The functor* $\mathsf{S}\colon \mathsf{Operad} \to \mathsf{TngMnd}(\mathsf{Mod}_R, \mathbb{L})$ *equipped with the natural transformation* $\chi\colon \mathsf{S}_{\mathbb{L}\mathscr{P}} \to \overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}}$ *defined as follows:*

$$\chi\colon \mathsf{S}_{\mathbb{L}\mathscr{P}}A = \bigoplus_n \mathbb{L}\mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \to \mathbb{L}\left(\bigoplus_n \mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}\right) = \overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}}A$$
$$\chi((\mu, \nu); a_1, \dots, a_n) := ((\mu; a_1, \dots, a_n), (\nu; a_1, \dots, a_n))$$

becomes a strong tangent morphism $(\mathsf{S}, \chi)\colon (\mathsf{Operad}, \mathbb{L}) \to \mathsf{TngMnd}(\mathsf{Mod}_R, \mathbb{L})$.

Proof. In order to prove that χ makes S into a strong tangent morphism, we need to show that (1)χ is compatible with the units of the two tangent monads $\mathsf{S}_{\mathbb{L}\mathscr{P}}$ and $\overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}}$; (2) that χ is compatible with the multiplication morphisms of the tangent monads; (3) that χ is compatible with the distributive laws between the two tangent monads and the tangent bundle functor $\overline{\mathbb{L}}$; (4) that χ is invertible; finally, (5) we leave to the reader to prove the compatibility between χ and the tangent structures.

Let's start with (1). This amounts to prove the commutativity of the following diagram:

$$\begin{tikzcd}
\mathsf{S}_{\mathbb{L}\mathscr{P}} \arrow[r, "\chi"] & \overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}} \\
\mathsf{id} \arrow[u, "\eta\mathbb{L}"] \arrow[r, equal] & \mathsf{id} \arrow[u, "\overline{\mathbb{L}}\eta^{(\mathbb{L})}"']
\end{tikzcd}$$

Let's start by recalling that $\eta_{\mathscr{P}}\colon A \to \mathsf{S}_{\mathscr{P}}A$ sends $a \in A$ to $(\mathbb{1}_{\mathscr{P}}; a)$ where $\mathbb{1}_{\mathscr{P}} \in \mathscr{P}(1)$ is the unit of the operad $\mathscr{P}$. so, we have:

$$\begin{aligned}
& (\chi \circ \eta_{\mathbb{L}})(a) \\
= \;& \chi(\mathbb{1}_{\mathbb{L}\mathscr{P}}; a) \\
= \;& \chi((\mathbb{1}_{\mathscr{P}}, 0); a) \\
= \;& ((\mathbb{1}_{\mathscr{P}}; a), (0; a)) \\
= \;& ((\mathbb{1}_{\mathscr{P}}; a), 0) \\
= \;& (\eta_{\mathscr{P}}(a), 0) \\
= \;& z(\eta_{\mathscr{P}}(a)) \\
= \;& \eta^{(\mathbb{L})}(a)
\end{aligned}$$

Let's now show (2) the compatibility with the multiplication maps, expressed by the commutativity of the diagram:

$$
\begin{tikzcd}
\mathsf{S}^2_{\mathbb{L}\mathscr{P}} \arrow[r, "\mathsf{S}_{\mathbb{L}\mathscr{P}}\chi"] \arrow[dddd, "\gamma_{\mathbb{L}\mathscr{P}}"'] & \mathsf{S}_{\mathbb{L}\mathscr{P}} \circ \overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}} \arrow[r, "\chi\overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}}"] & \overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}} \circ \overline{\mathbb{L}}\mathsf{S}_{\mathscr{P}} \arrow[d, "\mathbb{L}\alpha\mathsf{S}_{\mathscr{P}}"] \\
 & & \mathbb{L}^2\mathsf{S}^2_{\mathscr{P}} \arrow[d, "\gamma^{(\mathbb{L})}\mathsf{S}^2_{\mathscr{P}}"] \\
 & & \mathbb{L}\mathsf{S}^2_{\mathscr{P}} \arrow[d, "\mathbb{L}\gamma_{\mathscr{P}}"] \\
 & & \\
\mathsf{S}_{\mathbb{L}\mathscr{P}} \arrow[rr, "\chi"'] & & \mathbb{L}\mathsf{S}_{\mathscr{P}}
\end{tikzcd}
$$

First, notice that the generic element of $\mathsf{S}^2_{\mathbb{L}\mathscr{P}}A$ is of the form:

$$\left((\mu,\nu); \left((\mu_1,\nu_1); a^{(1)}_1, \dots, a^{(1)}_{k_1}\right), \dots, \left((\mu_n,\nu_n); a^{(n)}_1, \dots, a^{(n)}_{k_n}\right)\right)$$

For this proof, the elements of A do not play any role, therefore, we simplify the notation by denoting each tuple $a^{(i)}_1, \dots, a^{(i)}_{k_i}$ by $\vec{a}_i$. Adopting this notation, the generic element of $\mathsf{S}^2_{\mathbb{L}\mathscr{P}}$ is now denoted by:

$$\left((\mu,\nu); \left((\mu_1,\nu_1); \vec{a}_1\right), \dots, \left((\mu_n,\nu_n); \vec{a}_n\right)\right)$$

We also write $\vec{a}$ for $\vec{a}_1, \dots, \vec{a}_n$. With this in mind, we can write:

$$
\begin{aligned}
& \left((\mathbb{L}\gamma_{\mathscr{P}}) \circ (\gamma^{(\mathbb{L})}_{\mathsf{S}^2_{\mathscr{P}}}) \circ (\mathbb{L}\alpha_{\mathsf{S}_{\mathscr{P}}}) \circ (\chi_{\mathbb{L}\mathsf{S}_{\mathscr{P}}})(\mathsf{S}_{\mathbb{L}\mathscr{P}}\chi)\right) \\
& \qquad \left((\mu,\nu); \left((\mu_1,\nu_1); \vec{a}_1\right), \dots, \left((\mu_n,\nu_n); \vec{a}_n\right)\right) \\
= \; & \left((\mathbb{L}\gamma_{\mathscr{P}}) \circ (\gamma^{(\mathbb{L})}_{\mathsf{S}^2_{\mathscr{P}}}) \circ (\mathbb{L}\alpha_{\mathsf{S}_{\mathscr{P}}}) \circ (\chi_{\mathbb{L}\mathsf{S}_{\mathscr{P}}})\right) \\
& \qquad \left((\mu,\nu); \left((\mu_1;\vec{a}_1),(\nu_1;\vec{a}_1)\right), \dots, \left((\mu_n;\vec{a}_n),(\nu_n;\vec{a}_n)\right)\right) \\
= \; & \left((\mathbb{L}\gamma_{\mathscr{P}}) \circ (\gamma^{(\mathbb{L})}_{\mathsf{S}^2_{\mathscr{P}}}) \circ (\mathbb{L}\alpha_{\mathsf{S}_{\mathscr{P}}})\right) \left(\left(\mu; \left((\mu_1;\vec{a}_1),(\nu_1;\vec{a}_1)\right), \dots, \left((\mu_n;\vec{a}_n),(\nu_n;\vec{a}_n)\right)\right)\right., \\
& \qquad \left.\left(\nu; \left((\mu_1;\vec{a}_1),(\nu_1;\vec{a}_1)\right), \dots, \left((\mu_n;\vec{a}_n),(\nu_n;\vec{a}_n)\right)\right)\right) \\
= \; & \left((\mathbb{L}\gamma_{\mathscr{P}}) \circ (\gamma^{(\mathbb{L})}_{\mathsf{S}^2_{\mathscr{P}}})\right) \left(\left(\mu; (\mu_1;\vec{a}_1), \dots, (\mu_n;\vec{a}_n)\right)\right., \\
& \qquad \sum_{k_1=1}^{n} \left(\mu; (\mu_1;\vec{a}_1), \dots, (\nu_k,\vec{a}_k), \dots, (\mu_n;\vec{a}_n)\right),
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \left(\nu;(\mu_1;\vec{a}_1),\ldots,(\mu_n;\vec{a}_n)\right), \sum_{k_1=1}^{n}\left(\nu;(\mu_1;\vec{a}_1),\ldots,(\nu_k,\vec{a}_k),\ldots,(\mu_n;\vec{a}_n)\right)\Bigg) \\
= \;& (\perp\gamma_{\mathscr{P}})\Bigg(\left(\mu;(\mu_1;\vec{a}_1),\ldots,(\mu_n;\vec{a}_n)\right), \\
&\qquad \left(\nu;(\mu_1;\vec{a}_1),\ldots,(\mu_n;\vec{a}_n)\right) + \sum_{k_1=1}^{n}\left(\mu;(\mu_1;\vec{a}_1),\ldots,(\nu_k,\vec{a}_k),\ldots,(\mu_n;\vec{a}_n)\right)\Bigg) \\
= \;& \Bigg(\left(\mu(\mu_1,\ldots,\mu_n);\vec{a}\right), \left(\nu(\mu_1,\ldots,\mu_n);\vec{a}\right) + \sum_{k_1=1}^{n}\left(\mu(\mu_1,\ldots,\nu_k,\ldots,\mu_n);\vec{a}\right)\Bigg) \\
= \;& \chi\left(\left(\mu(\mu_1,\ldots,\mu_n),\nu(\mu_1,\ldots,\mu_n) + \sum_{k=1}^{n}\mu(\mu_1,\ldots,\nu_k,\ldots,\mu_n)\right);\vec{a}\right) \\
= \;& (\chi\circ\gamma_{\perp\mathscr{P}})\left((\mu,\nu);\left((\mu_1,\nu_1);\vec{a}_1\right),\ldots,\left((\mu_n,\nu_n);\vec{a}_n\right)\right)
\end{aligned}
$$

Let's now prove (3). This amounts to show the following:

$$
\begin{tikzcd}
\mathsf{S}_{\perp\mathscr{P}}\circ\perp \arrow[r, "\chi\perp"] \arrow[d, "\alpha_{\perp\mathscr{P}}"'] & \overline{\perp}\mathsf{S}_{\mathscr{P}}\circ\perp \arrow[r, "\overline{\perp}\alpha_{\mathscr{P}}"] & \overline{\perp}^2\mathsf{S}_{\mathscr{P}} \arrow[d, "c\mathsf{S}_{\mathscr{P}}"] \\
\overline{\perp}\mathsf{S}_{\perp\mathscr{P}} \arrow[rr, "\overline{\perp}\chi"'] & & \overline{\perp}^2\mathsf{S}_{\mathscr{P}}
\end{tikzcd}
$$

$$
\begin{aligned}
&((c\mathsf{S}_{\mathscr{P}})\circ(\perp\alpha)\circ(\chi\perp))\left((\mu,\nu);(a_1,b_1),\ldots,(a_n,b_n)\right) \\
= \;& ((c\mathsf{S}_{\mathscr{P}})(\perp\alpha))\left((\mu;(a_1,b_1),\ldots,(a_n,b_n)),(\nu;(a_1,b_1),\ldots,(a_n,b_n))\right) \\
= \;& c\mathsf{S}_{\mathscr{P}}\Bigg((\mu;a_1,\ldots,a_n),\sum_{k=1}^{n}(\mu;a_1,\ldots,b_k,\ldots,a_n), \\
&\qquad (\nu;a_1,\ldots,a_n),\sum_{k=1}^{n}(\nu;a_1,\ldots,b_k,\ldots,a_n)\Bigg) \\
= \;& \Bigg((\mu;a_1,\ldots,a_n),(\nu;a_1,\ldots,a_n), \\
&\qquad \sum_{k=1}^{n}(\mu;a_1,\ldots,b_k,\ldots,a_n),\sum_{k=1}^{n}(\nu;a_1,\ldots,b_k,\ldots,a_n)\Bigg) \\
= \;& \perp\chi\left(((\mu,\nu);a_1,\ldots,a_n),\sum_{k=1}^{n}((\mu,\nu);a_1,\ldots,b_k,\ldots,a_n)\right) \\
= \;& (\perp\chi\circ\alpha_{\perp\mathscr{P}})\left((\mu,\nu);(a_1,b_1),\ldots,(a_n,b_n)\right)
\end{aligned}
$$

Finally, (4) notice that the natural transformation:

$$\overline{\perp}\mathsf{S}_{\mathscr{P}}\to\mathsf{S}_{\perp\mathscr{P}}$$

$$((\mu; a_1, \ldots, a_n), (\nu; b_1, \ldots, b_n)) \mapsto ((\mu, 0); a_1, \ldots, a_n) + ((0, \nu); b_1, \ldots, b_n)$$

inverts χ. □

5.2.2 The adjoint tangent category of operads

Theorem 5.22 proves that the category Operad of algebraic operads comes equipped with a tangent structure, generated by a coCartesian differential monad (W, ∂). Furthermore, Theorem 5.33 shows that the functor $\mathsf{S}\colon \mathsf{Operad} \to \mathsf{TngMnd}(\mathsf{Mod}_R, \mathbb{L})$ strongly preserves this tangent structure so, $\mathbb{L}$ on Operad is compatible with the tangent structure $\overline{\mathbb{L}}$ of tangent monads described by Theorem 5.30.

Before going back to the relationship between $(\mathsf{Operad}, \mathbb{L})$ and deformation theory introduced in Example 5.11, we dedicate this section to investigate the following question: is this tangent category adjunctable (Definition 2.46)? In particular, is $\mathsf{Operad}^{\mathsf{op}}$ also a tangent category? To answer this question, notice that the category Operad is cocomplete (cf. [24, Theorem 1.13]), thus, by Lemma 2.49, $(\mathsf{Operad}, \mathbb{L})$ is adjunctable if the tangent bundle functor admits a left adjoint. In this section, we prove a stronger result: we show that L is corepresentable, which implies the existence of a left adjoint $\mathrm{T} \dashv \mathrm{L}$. To prove this, we show that the operad $R^\bullet[\varepsilon] := R[\varepsilon]^\bullet$ (see Example 3.6) is an *infinitesimal object* of $\mathsf{Operad}^{\mathsf{op}}$.

Infinitesimal objects are discussed in [12, Section 5.2]. In particular, Cockett and Cruttwell showed that a Cartesian category has a representable tangent structure $\mathbb{T}$, i.e. a tangent structure whose tangent bundle functor is representable, if and only if it has an infinitesimal object D; in that case, $\mathrm{T} = (-)^D$. Showing that $\mathsf{Operad}^{\mathsf{op}}$ has a representable tangent structure is the same as establishing that Operad has a corepresentable tangent structure. Let's unwrap this definition.

Recall that a **coexponential object** in a coCartesian category $\mathbb{X}$, i.e. an exponential object in the opposite category, is an object D for which the functor $(-) + D$ admits a left adjoint $(-)^D$.

Definition 5.34. *A tangent structure $\mathbb{L}$ over a coCartesian category (i.e. a category which admits all finite coproducts, denoted by $+$) $\mathbb{X}$ is* ***corepresentable*** *whenever there is an object D of $\mathbb{X}$, which is* ***coinfinitesimal****, i.e. the following conditions hold:*

1. *The tangent bundle functor L is isomorphic to the functor $(-) + D$;*

2. *For every positive integer n, $D^n := D + \ldots + D$ is a coexponential object;*

3. *For every positive integer n, D_n which is the object which corepresents $\mathbb{L}_n \cong (-)+D_n$ is a coexponential object.*

When the category $\mathbb{X}$ has enough colimits, one can apply Lemma 2.49 and prove that a tangent structure $\mathbb{L}$ is corepresentable if $\mathbb{L} \cong (-) + D$ for some object D and that D is a coexponential object. Indeed, if this is the case, then $\mathbb{L}$ admits a left adjoint $\mathbb{T} = (-)^D$, which in the opposite category is, by construction representable.

Lemma 5.35. *If the category $\mathbb{X}$ is cocomplete then, a tangent structure $\mathbb{L}$ over $\mathbb{X}$ is corepresentable if and only if the tangent bundle functor L is isomorphic to $(-) + D$ for a coexponential object D.*

The goal is to show that the tangent structure $\mathbb{L}$ of Theorem 5.22 is corepresentable. We divide this problem into two steps. First, we show that the tangent bundle functor L is precisely the functor $(-) + D$, where $D := R^\bullet[\varepsilon] := R[\varepsilon]^\bullet$ (see Example 3.6). Second, we show that the functor L admits a left adjoint, which implies that D is a coexponential object and therefore a coinfinitesimal object.

Lemma 5.36. *For an operad $\mathscr{P}$, the operad $\mathscr{P}[\varepsilon] = \mathscr{P} \otimes R[\varepsilon]$ is isomorphic to the coproduct between $\mathscr{P}$ and the operad $R^\bullet[\varepsilon]$. In particular, the tangent bundle functor L is isomorphic to $(-) + D$ with $D := R^\bullet[\varepsilon]$.*

Proof. In order to show that $\mathscr{P}[\varepsilon]$ is the coproduct between $\mathscr{P}$ and $R^\bullet[\varepsilon]$, we need to show that, for any pair of morphisms $\varphi\colon \mathscr{P} \to \mathscr{P}'$ and $\psi\colon R^\bullet[\varepsilon] \to \mathscr{P}'$ of operads, there is a unique morphism $\xi\colon \mathscr{P}[\varepsilon] \to \mathscr{P}'$ of operads for which the diagram:

$$
\begin{tikzcd}
\mathscr{P} \arrow[r, "\iota_1"] \arrow[dr, "\varphi"'] & P[\varepsilon] \arrow[d, dashed, "\xi"] & R^\bullet[\varepsilon] \arrow[l, "\iota_2"'] \arrow[dl, "\psi"] \\
& \mathscr{P}' &
\end{tikzcd}
$$

commutes; where ι_1 and ι_2 are defined as follows:

$$\iota_1\colon \mathscr{P} \to \mathscr{P}[\varepsilon]$$
$$\iota_1(\mu) := \mu$$

$$\begin{aligned}
\iota_2 &\colon R^\bullet[\varepsilon] \to \mathscr{P}[\varepsilon] \\
\iota_2(a + b\varepsilon) &:= a\mathbb{1}_{\mathscr{P}} + b\varepsilon\mathbb{1}_{\mathscr{P}}
\end{aligned}$$

Let's define ξ as follows:

$$\begin{aligned}
\xi &\colon \mathscr{P}[\varepsilon] \to \mathscr{P}' \\
\xi(\mu + \varepsilon\nu) &:= \varphi(\mu) + \psi(\varepsilon)\varphi(\nu)
\end{aligned}$$

Let's show that ξ is a well-defined morphism of operads:

$$\begin{aligned}
& \xi(\mu + \varepsilon\nu)\,(\xi(\mu_1 + \varepsilon\nu_1), \dots, \xi(\mu_n + \varepsilon\nu_n)) \\
= \;& (\varphi(\mu) + \psi(\varepsilon)\varphi(\nu))(\varphi(\mu_1) + \psi(\varepsilon)\varphi(\nu_1), \dots, \varphi(\mu_n) + \psi(\varepsilon)\varphi(\nu_n)) \\
= \;& \varphi(\mu)(\varphi(\mu_1), \dots, \varphi(\mu_n)) + \psi(\varepsilon)\,(\varphi(\nu)(\varphi(\mu_1), \dots, \varphi(\mu_n)) + \\
& \qquad + \sum_{k=1}^{n} \varphi(\mu)(\varphi(\mu_1), \dots, \varphi(\nu_k), \dots, \varphi(\mu_n))\Bigg) \\
= \;& \varphi(\mu(\mu_1, \dots, \mu_n) + \psi(\varepsilon)\left(\varphi(\nu(\mu_1, \dots, \mu_n) + \sum_{k=1}^{n} \mu(\mu_1, \dots, \nu_k, \dots, \mu_n)\right) \\
= \;& \xi((\mu + \varepsilon\nu)\,(\mu_1 + \varepsilon\nu_1, \dots, \mu_n + \varepsilon\nu_n)
\end{aligned}$$

where we used that $\psi(\varepsilon)\psi(\varepsilon) = \psi(\varepsilon^2) = 0$. Moreover:

$$\begin{aligned}
& \xi(\mathbb{1} + \mathscr{P}[\varepsilon]) \\
= \;& \xi(\mathbb{1}_{\mathscr{P}}) \\
= \;& \varphi(\mathbb{1}_{\mathscr{P}}) \\
= \;& \mathbb{1}_{\mathscr{P}'}
\end{aligned}$$

So, ξ is an operad morphism. Finally, suppose that $\xi' \colon \mathscr{P}[\varepsilon] \to \mathscr{P}'$ is another morphism of operads for which $\iota_1\xi' = \varphi$ and $\iota_2\xi' = \psi$. Then:

$$\begin{aligned}
& \xi'(\mu + \varepsilon\nu) \\
= \;& \xi'(\mu) + \xi'(\varepsilon\mathbb{1}_{\mathscr{P}})\xi'(\nu) \\
= \;& \xi'(\iota_1(\mu)) + \xi'(\iota_2(\varepsilon))\xi'(\iota_1(\nu)) \\
= \;& \varphi(\mu) + \psi(\varepsilon)\varphi(\nu) \\
= \;& \xi(\mu + \varepsilon\nu)
\end{aligned}$$

This proves that ξ is the unique morphism which makes the above diagram commuting. Thus, $\mathscr{P}[\varepsilon] \cong \mathscr{P} + R^\bullet[\varepsilon]$. Extending the same argument to morphisms, one can easily see that $\mathbb{L} \cong (-) + R^\bullet[\varepsilon]$. $\square$

The next step is to show that the functor $\mathbb{L}$ admits a left adjoint $\mathbb{T}$.

Lemma 5.37. *The functor* $\mathbb{L}\colon \mathsf{Operad} \to \mathsf{Operad}$ *admits a left adjoint* $\mathbb{T}\colon \mathsf{Operad} \to \mathsf{Operad}$ *which sends an operad* $\mathscr{P}$ *to the operad* $\mathbb{T}\mathscr{P}$ *generated by all* $\mu \in \mathscr{P}(n)$*, for every positive integer* n *and by symbols* $\mathsf{d}\mu$*, for each* $\mu \in \mathscr{P}(n)$*, such that the following relations hold:*

$$\begin{aligned}
&\mu_{\mathbb{T}\mathscr{P}}(\mu_1, \dots, \mu_n) = \mu_{\mathscr{P}}(\mu_1, \dots, \mu_n)\\
&\mathsf{d}(r\mu + s\nu) = r\mathsf{d}\mu + s\mathsf{d}\nu\\
&\mathsf{d}(\mu(\mu_1, \dots, \mu_n)) = (\mathsf{d}\mu)(\mu_1, \dots, \mu_n) + \sum_{k=1}^{n} \mu(\mu_1, \dots, \mathsf{d}\mu_k, \dots, \mu_n)
\end{aligned}$$

Moreover, $\mathbb{T}$ *sends a morphism* $\varphi\colon \mathscr{P} \to \mathscr{P}'$ *of operads to the morphism which sends each* $\mu \in \mathbb{T}\mathscr{P}$ *to* $\varphi(\mu) \in \mathbb{T}\mathscr{P}'$ *and each* $\mathsf{d}\mu \in \mathbb{T}\mathscr{P}$ *to* $\mathsf{d}\varphi(\mu) \in \mathbb{T}\mathscr{P}'$*. In particular, the unit* η *and the counit* ε *of the adjunction* $\mathbb{T} \dashv \mathbb{L}$ *are defined as follows:*

$$\begin{aligned}
&\eta\colon \mathscr{P} \to \mathbb{L}\mathbb{T}\mathscr{P}\\
&\eta(\mu) := (\mu, \mathsf{d}\mu)
\end{aligned}$$

$$\begin{aligned}
&\varepsilon\mathbb{T}\mathbb{L}\mathscr{P} \to \mathscr{P}\\
&\varepsilon(\mu, \nu) := \mu\\
&\varepsilon(\mathsf{d}(\mu, \nu)) := \nu
\end{aligned}$$

Proof. Let's start by showing that η is a morphism of operads:

$$\begin{aligned}
&\eta(\mu(\mu_1, \dots, \mu_n))\\
= \ &(\mu(\mu_1, \dots, \mu_n), \mathsf{d}(\mu(\mu_1, \dots, \mu_n)))\\
= \ &\left(\mu(\mu_1, \dots, \mu_n), (\mathsf{d}\mu)(\mu_1, \dots, \mu_n) + \sum_{k=1}^{n} \mu(\mu_1, \dots, \mathsf{d}\mu_k, \dots, \mu_n)\right)\\
= \ &(\mu, \mathsf{d}\mu)((\mu_1, \mathsf{d}\mu_1), \dots, (\mu_n, \mathsf{d}\mu_n))\\
= \ &(\eta(\mu))(\eta(\mu_1), \dots, \eta(\mu_n))
\end{aligned}$$

Moreover, $\eta(\mathbb{1}_{\mathscr{P}}) = (\mathbb{1}_{\mathscr{P}}, \mathsf{d}\mathbb{1}_{\mathscr{P}})$. However, since:

$$\mathsf{d}\mathbb{1}_{\mathscr{P}} = \mathsf{d}(\mathbb{1}_{\mathscr{P}}(\mathbb{1}_{\mathscr{P}})) = (\mathsf{d}(\mathbb{1}_{\mathscr{P}})(\mathbb{1}_{\mathscr{P}}) + \mathbb{1}_{\mathscr{P}}(\mathsf{d}\mathbb{1}_{\mathscr{P}}) = \mathsf{d}\mathbb{1}_{\mathscr{P}} + \mathsf{d}\mathbb{1}_{\mathscr{P}}$$

we obtain $\mathsf{d}\mathbb{1}_{\mathscr{P}} = 0$, thus $\eta(\mathbb{1}_{\mathscr{P}}) = (\mathbb{1}_{\mathscr{P}}, 0) = \mathbb{1}_{\mathbb{L}\mathscr{P}}$. Let's now show that also ε is well-defined. To do so, we need to show that ε is compatible with the relations which define $\mathrm{T}\mathbb{L}\mathscr{P}$:

$$\begin{aligned}
& \varepsilon((\mu,\nu)((\mu_1,\nu_1),\ldots,(\mu_n,\nu_n)) \\
= \ & \varepsilon\left(\mu(\mu_1,\ldots,\mu_n), \nu(\mu_1,\ldots,\mu_n) + \sum_{k=1}^{n} \mu(\mu_1,\ldots,\nu_k,\ldots,\mu_n)\right) \\
= \ & \mu(\mu_1,\ldots,\mu_n) \\
= \ & (\varepsilon(\mu,\nu))(\varepsilon(\mu_1,\nu_1),\ldots,\varepsilon(\mu_n,\nu_n))
\end{aligned}$$

$$\begin{aligned}
& \varepsilon(\mathsf{d}(r(\mu_1,\nu_1) + s(\mu_2,\nu_2))) \\
= \ & r\nu_1 + s\nu_2 \\
= \ & \varepsilon(r\mathsf{d}(\mu_1,\nu_1) + s\mathsf{d}(\mu_2,\nu_2))
\end{aligned}$$

$$\begin{aligned}
& \varepsilon(\mathsf{d}((\mu,\nu)((\mu_1,\nu_1),\ldots,(\mu_n,\nu_n)))) \\
= \ & \varepsilon\left(\mathsf{d}\left(\mu(\mu_1,\ldots,\mu_n), \nu(\mu_1,\ldots,\mu_n) + \sum_{k=1}^{n} \mu(\mu_1,\ldots,\nu_k,\ldots,\mu_n)\right)\right) \\
= \ & \nu(\mu_1,\ldots,\mu_n) + \sum_{k=1}^{n} \mu(\mu_1,\ldots,\nu_k,\ldots,\mu_n) \\
= \ & (\varepsilon(\mathsf{d}(\mu,\nu)))(\varepsilon(\mu_1,\nu_1),\ldots,\varepsilon(\mu_n,\nu_n)) + \\
& \quad + \sum_{k=1}^{n} (\varepsilon(\mu,\nu))(\varepsilon(\mu_1,\nu_1),\ldots,\varepsilon(\mathsf{d}(\mu_k,\nu_k)),\ldots,\varepsilon(\mu_n,\nu_n)) \\
= \ & \varepsilon\left((\mathsf{d}(\mu,\nu))((\mu_1,\nu_1),\ldots,(\mu_n,\nu_n)) + \sum_{k=1}^{n} (\mu,\nu)((\mu_1,\nu_1),\ldots,\mathsf{d}(\mu_k,\nu_k),\ldots,(\mu_n,\nu_n))\right)
\end{aligned}$$

Let's now prove the triangle identities. Let's start by showing that $\mathrm{T}\eta\varepsilon\mathrm{T} = \mathsf{id}_{\mathrm{T}}$. To do so, let's evaluate $\mathrm{T}\eta\varepsilon\mathrm{T}$ on the generators:

$$\begin{aligned}
& \varepsilon_{\mathrm{T}}(\mathrm{T}\eta(\mu)) \\
= \ & \varepsilon_{\mathrm{T}}(\eta(\mu))
\end{aligned}$$

$$\begin{aligned} &= \varepsilon_{\mathrm{T}}(\mu, \mathsf{d}\mu) \\ &= \mu \end{aligned}$$

$$\begin{aligned} &\varepsilon_{\mathrm{T}}(\mathrm{T}\eta(\mathsf{d}\mu)) \\ &= \varepsilon_{\mathrm{T}}(\mathsf{d}\eta(\mu)) \\ &= \varepsilon_{\mathrm{T}}(\mathsf{d}(\mu, \mathsf{d}\mu)) \\ &= \mathsf{d}\mu \end{aligned}$$

Let's now show the second triangle identity $\eta\mathrm{L}\mathrm{L}\varepsilon = \mathsf{id}_{\mathrm{L}}$:

$$\begin{aligned} &\mathrm{L}\varepsilon(\eta_{\mathrm{L}}(\mu, \nu)) \\ &= \mathrm{L}\varepsilon((\mu, \nu), \mathsf{d}(\mu, \nu)) \\ &= (\varepsilon(\mu, \nu), \varepsilon(\mathsf{d}(\mu, \nu))) \\ &= (\mu, \nu) \end{aligned}$$

This proves that $(\eta, \varepsilon) \colon \mathrm{T} \dashv \mathrm{L}$ forms an adjunction. □

Theorem 5.38. *The tangent category* $(\mathsf{Operad}, \mathbb{L})$ *is corepresentable and* $R^\bullet[\varepsilon]$ *is a coinfinitesimal object of* Operad.

Corollary 5.39. *The tangent category* $(\mathsf{Operad}, \mathbb{L})$ *is adjunctable; in particular, the adjoint tangent structure* $\mathbb{T}$ *over the opposite category is defined as follows. For the sake of simplicity, all morphisms are regarded as morphisms of operads:*

tangent bundle functor *The tangent bundle functor* $\mathrm{T} \colon \mathsf{Operad} \to \mathsf{Operad}$, *regarded as an endofunctor of* Operad, *is the left adjoint of* L *described by Lemma 5.37. In particular,* $\mathrm{T}\mathscr{P}$ *is the operad generated by all* $\mu \in \mathscr{P}(n)$*, for every positive integer* n *and by symbols* $\mathsf{d}\mu$*, for each* $\mu \in \mathscr{P}(n)$*, such that the following relations hold:*

$$\mu_{\mathrm{T}\mathscr{P}}(\mu_1, \dots, \mu_n) = \mu_{\mathscr{P}}(\mu_1, \dots, \mu_n)$$

$$\mathsf{d}(r\mu + s\nu) = r\mathsf{d}\mu + s\mathsf{d}\nu$$

$$\mathsf{d}(\mu(\mu_1, \dots, \mu_n)) = (\mathsf{d}\mu)(\mu_1, \dots, \mu_n) + \sum_{k=1}^{n} \mu(\mu_1, \dots, \mathsf{d}\mu_k, \dots, \mu_n)$$

Moreover, T *sends a morphism* $\varphi \colon \mathscr{P} \to \mathscr{P}'$ *of operads to the morphism which sends each* $\mu \in \mathrm{T}\mathscr{P}$ *to* $\varphi(\mu) \in \mathrm{T}\mathscr{P}'$ *and each* $\mathsf{d}\mu \in \mathrm{T}\mathscr{P}$ *to* $\mathsf{d}\varphi(\mu) \in \mathrm{T}\mathscr{P}'$;

projection *The projection* $p^{(\mathrm{T})}\colon \mathsf{id}_{\mathsf{Operad}} \Rightarrow \mathrm{T}$ *is the natural transformation defined as follows:*

$$
\begin{aligned}
&p^{(\mathrm{T})}\colon \mathscr{P} \longrightarrow \mathrm{T}\mathscr{P}\\
&p^{(\mathrm{T})}(\mu) := \mu
\end{aligned}
$$

zero morphism *The zero morphism* $z^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathsf{id}_{\mathsf{Operad}}$ *is the natural transformation defined on generators as follows:*

$$
\begin{aligned}
&z^{(\mathrm{T})}\colon \mathrm{T}\mathscr{P} \longrightarrow \mathscr{P}\\
&z^{(\mathrm{T})}(\mu) := \mu\\
&z^{(\mathrm{T})}(\mathsf{d}\mu) := 0
\end{aligned}
$$

n**-fold pullback** *The* n*-fold pushot (in the category* Operad*) of the projection along itself is the functor* $\mathrm{T}_n\colon \mathsf{Operad} \longrightarrow \mathsf{Operad}$ *which sends an operad* $\mathscr{P}$ *to the operad* $\mathrm{T}_n\mathscr{P}$ *generated by all* $\mu \in \mathscr{P}(n)$ *and by symbols* $\mathsf{d}_1\mu, \ldots, \mathsf{d}_n\mu$*, for each* $\mu \in \mathscr{P}(n)$*, for every positive integer* n*, satisfying the follwing relations:*

$$
\begin{aligned}
&\mu_{\mathrm{T}_n\mathscr{P}}(\mu_1, \ldots, \mu_n) = \mu_{\mathscr{P}}(\mu_1, \ldots, \mu_n)\\
&\mathsf{d}_i(r\mu + s\nu) = r\mathsf{d}_i\mu + s\mathsf{d}_i\nu\\
&\mathsf{d}_i(\mu(\mu_1, \ldots, \mu_n)) = (\mathsf{d}_i\mu)(\mu_1, \ldots, \mu_n) + \sum_{k=1}^{n} \mu(\mu_1, \ldots, \mathsf{d}_i\mu_k, \ldots, \mu_n)
\end{aligned}
$$

for every $i = 1, \ldots, n$*. Moreover,* T_n *sends a morphism* $\varphi\colon \mathscr{P} \longrightarrow \mathscr{P}'$ *of operads to the morphism* $\mathrm{T}_n\varphi$ *which sends the generator* $\mu \in \mathrm{T}_n\mathscr{P}$ *to* $\varphi(\mu) \in \mathrm{T}_n\mathscr{P}'$ *and* $\mathsf{d}_k\mu \in \mathrm{T}_n\mathscr{P}$ *to* $\mathsf{d}_k\varphi(\mu) \in \mathrm{T}\mathscr{P}'$*. Finally, the k-th injection* $\iota_k\colon \mathrm{T}\mathscr{P} \longrightarrow \mathrm{T}_n\mathscr{P}$ *is the natural transformation which sends each* $\mu \in \mathrm{T}\mathscr{P}$ *to itself and each* $\mathsf{d}\mu \in \mathrm{T}\mathscr{P}$ *to* $\mathsf{d}_k\mu \in \mathrm{T}_n\mathscr{P}$*;*

sum morphism *The sum morphism* $s^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}_2$ *is the natural transformation defined on generators as follows:*

$$
\begin{aligned}
&s^{(\mathrm{T})}\colon \mathrm{T}\mathscr{P} \longrightarrow \mathrm{T}_2\mathscr{P}\\
&s^{(\mathrm{T})}(\mu) := \mu\\
&s^{(\mathrm{T})}(\mathsf{d}\mu) := \mathsf{d}_1\mu + \mathsf{d}_2\mu
\end{aligned}
$$

vertical lift *The vertical lift* $l^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}$ *is the natural transformation defined on generators as follows:*

$$\begin{aligned}
&l^{(\mathrm{T})}\colon \mathrm{T}^2\mathscr{P} \to \mathrm{T}\mathscr{P}\\
&l^{(\mathrm{T})}(\mu) := \mu\\
&l^{(\mathrm{T})}(\mathsf{d}\mu) := 0\\
&l^{(\mathrm{T})}(\mathsf{d}'\mu) := 0\\
&l^{(\mathrm{T})}(\mathsf{d}'\mathsf{d}\mu) := \mathsf{d}\mu
\end{aligned}$$

canonical flip *The canonical flip* $c^{(\mathrm{T})}\colon \mathrm{T}^2 \Rightarrow \mathrm{T}^2$ *is the natural transformation defined on generators as follows:*

$$\begin{aligned}
&c^{(\mathrm{T})}\colon \mathrm{T}^2\mathscr{P} \to \mathrm{T}^2\mathscr{P}\\
&c^{(\mathrm{T})}(\mu) := \mu\\
&c^{(\mathrm{T})}(\mathsf{d}\mu) := \mathsf{d}'\mu\\
&c^{(\mathrm{T})}(\mathsf{d}'\mu) := \mathsf{d}\mu\\
&c^{(\mathrm{T})}(\mathsf{d}'\mathsf{d}\mu) := \mathsf{d}\mu
\end{aligned}$$

Moreover, $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$ *has negatives with negation:*

negation *The negation* $n^{(\mathrm{T})}\colon \mathrm{T} \Rightarrow \mathrm{T}$ *is the natural transformation defined on generators as follows:*

$$\begin{aligned}
&n^{(\mathrm{T})}\colon \mathrm{T}\mathscr{P} \to \mathrm{T}\mathscr{P}\\
&n^{(\mathrm{T})}(\mu) := \mu\\
&n^{(\mathrm{T})}(\mathsf{d}\mu) := -\mathsf{d}\mu
\end{aligned}$$

Finally, $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$ *is representable and Cartesian.*

Remark 5.40. In analogy with the description of the geometric tangent functor $\mathrm{T}^{(\mathscr{P})}$ of Theorem 3.68, one would expect the operad $\mathrm{T}\mathscr{P}$ to be described as the free $\mathscr{P}$-operad of the module of Kähler differentials over $\mathscr{P}$. In our knowledge, a definition for the module of Kähler differentials of an operad is missing from the literature. We expect this module to represent the functor of derivations of a given operad. We believe such a description is possible. However, we leave this as a future work.

Example 5.41. Let's consider the operad $\mathscr{A}ss$ described in Example 3.7. Recall that $\mathscr{A}ss$ is generated by a binary tree μ which satisfies $\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu)$. Then, the operad $\mathrm{T}\mathscr{A}ss$ is generated by two binary trees μ and $\nu := \mathsf{d}\mu$. μ satisfies the usual relation $\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu)$. Thanks to the relations which define d we obtain the following:

$$\begin{aligned}
& \mathsf{d}(\mu(\mu, \mathbb{1})) \\
= \ & (\mathsf{d}\mu)(\mu, \mathbb{1}) + \mu(\mathsf{d}\mu, \mathbb{1}) + \mu(\mu, \mathsf{d}\mathbb{1}) \\
= \ & (\mathsf{d}\mu)(\mu, \mathbb{1}) + \mu(\mathsf{d}\mu, \mathbb{1}) \\
= \ & \nu(\mu, \mathbb{1}) + \mu(\nu, \mathbb{1})
\end{aligned}$$

where we used that $\mathsf{d}\mathbb{1} = 0$. Similarly:

$$\begin{aligned}
& \mathsf{d}(\mu(\mathbb{1}, \mu)) \\
= \ & (\mathsf{d}\mu)(\mathbb{1}, \mu) + \mu(\mathsf{d}\mathbb{1}, \mu) + \mu(\mathbb{1}, \mathsf{d}\mu) \\
= \ & (\mathsf{d}\mu)(\mathbb{1}, \mu) + \mu(\mathbb{1}, \mathsf{d}\mu) \\
= \ & \nu(\mathbb{1}, \mu) + \mu(\mathbb{1}, \nu)
\end{aligned}$$

So, we obtain:

$$\nu(\mathbb{1}, \mu) + \mu(\mathbb{1}, \nu) = \mathsf{d}(\mu(\mathbb{1}, \mu)) = \mathsf{d}(\mu(\mu, \mathbb{1})) = \nu(\mu, \mathbb{1}) + \mu(\nu, \mathbb{1})$$

Rearranging:

$$\mu(\mathbb{1}, \nu) - \nu(\mu, \mathbb{1}) + \nu(\mathbb{1}, \mu) - \mu(\nu, \mathbb{1}) = 0$$

which is precisely the 2-cocycle condition of Equation (5.1.1). So, an algebra of $\mathrm{T}\mathscr{A}ss$ is an associative algebra A with associative multiplication $\mu_A \colon A \otimes A \longrightarrow A$ together with a binary operation $\nu_A \colon A \otimes A \longrightarrow A$. In particular, extending ν_A by $R[\varepsilon]$-linearity over the $R[\varepsilon]$-module $A[\varepsilon] = A \otimes R[\varepsilon]$, we obtain a new associative algebra $A[\varepsilon]$ with associative multiplication defined by $\mu_A + \varepsilon\nu_A$. In particular, $A[\varepsilon]$ is an infinitesimal deformation of A.

Conversely, if B is an infinitesimal deformation of an associative algebra A, then the associative multiplication of B is of the form $\mu_A + \varepsilon\nu_A$ for a binary operation $\nu_A \colon A \otimes A \longrightarrow A$ which satisfies precisely the relation of the generator ν of $\mathrm{T}\mathscr{A}ss$. So, B, regarded as an R-module, is a $\mathrm{T}\mathscr{A}ss$-algebra.

Let's now consider the unital associative operad $u\mathcal{A}ss$, also described in Example 3.7. $u\mathcal{A}ss$ is generated by a binary operation μ satisfying the same associative relation of $\mu \in \mathcal{A}ss$, and by a 0-ary operation $\eta \in u\mathcal{A}ss(0)$ satisfying $\mu(\eta, \mathbb{1}) = \mathbb{1} = \mu(\mathbb{1}, \eta)$. Applying d on both sides of this equation we get:

$$\nu(\eta, \mathbb{1}) + \mu(\mathsf{d}\eta, \mathbb{1}) = \mathsf{d}\mathbb{1} = 0 = \nu(\mathbb{1}, \eta) + \mu(\mathbb{1}, \mathsf{d}\eta)$$

where we used $\nu := \mathsf{d}\mu$. So,$\mathrm{T}u\mathcal{A}ss$ is generated by two binary operations μ and ν satisfying:

$$\begin{aligned}
&\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu) \\
&\mu(\mathbb{1}, \nu) - \nu(\mu, \mathbb{1}) + \nu(\mathbb{1}, \mu) - \mu(\nu, \mathbb{1}) = 0
\end{aligned}$$

and by two 0-ary operations η and $\theta := \mathsf{d}\eta$ satisfying:

$$\begin{aligned}
\mu(\eta, \mathbb{1}) &= \mathbb{1} \\
\mu(\mathbb{1}, \eta) &= \mathbb{1} \\
\nu(\eta, \mathbb{1}) &= -\mu(\theta, \mathbb{1}) \\
\nu(\mathbb{1}, \eta) &= -\mu(\mathbb{1}, \theta)
\end{aligned}$$

However, we also have:

$$\begin{aligned}
& \theta \\
= \;& (\mathbb{1})\theta \\
= \;& \mu(\eta, \mathbb{1})(\theta) \\
= \;& \mu(\eta, \theta) \\
= \;& \mu(\mathbb{1}, \theta)(\eta) \\
= \;& -\nu(\mathbb{1}, \eta)(\eta) \\
= \;& -\nu(\eta, \eta)
\end{aligned}$$

Moreover, $\theta = -\nu(\eta, \eta)$ implies $\nu(\eta, \mathbb{1}) = -\mu(\theta, \mathbb{1})$. To see this, consider the following:

$$\begin{aligned}
& -\mu(\theta, \mathbb{1}) \\
= \;& -\mu(-\nu(\eta, \eta), \mathbb{1})
\end{aligned}$$

$$
\begin{aligned}
&= \mu(\nu(\eta,\eta),\mathbb{1}) \\
&= \mu(\nu,\mathbb{1})(\eta,\eta,\mathbb{1}) \\
&= (\mu(\mathbb{1},\nu) - \nu(\mu,\mathbb{1}) + \nu(\mathbb{1},\mu))(\eta,\eta,\mathbb{1}) \\
&= \mu(\eta,\nu(\eta,\mathbb{1})) - \nu(\mu(\eta,\eta),\mathbb{1}) + \nu(\eta,\mu(\eta,\mathbb{1})) \\
&= \mu(\eta,\mathbb{1})\nu(\eta,\mathbb{1}) - \nu(\mu(\eta,\mathbb{1})(\eta),\mathbb{1}) + \nu(\eta,\mu(\eta,\mathbb{1})) \\
&= \nu(\eta,\mathbb{1}) - \nu(\eta,\mathbb{1}) + \nu(\eta,\mathbb{1}) \\
&= \nu(\eta,\mathbb{1})
\end{aligned}
$$

Similarly, one can also show that $\theta = -\nu(\eta,\eta)$ implies $\nu(\mathbb{1},\eta) = -\mu(\mathbb{1},\theta)$. Therefore, the presentation of $\mathrm{T}\mathscr{uAss}$ can be simplified as follows: $\mathrm{T}\mathscr{uAss}$ is generated by 0-ary operation η and by two binary operations μ and ν, satisfying the following relations:

$$
\begin{aligned}
&\mu(\mu,\mathbb{1}) = \mu(\mathbb{1},\mu) \\
&\mu(\mathbb{1},\nu) - \nu(\mu,\mathbb{1}) + \nu(\mathbb{1},\mu) - \mu(\nu,\mathbb{1}) = 0 \\
&\mu(\eta,\mathbb{1}) = \mathbb{1} \\
&\mu(\mathbb{1},\eta) = \mathbb{1}
\end{aligned}
$$

Example 5.42. Consider the operad $\mathscr{Com}$ described in Example 3.8. Recall that $\mathscr{Com}$ is generated by a binary operation $\mu \in \mathscr{Com}(2)$ which satisfies:

$$
\begin{aligned}
&\mu(\mu,\mathbb{1}) = \mu(\mathbb{1},\mu) \\
&\mu \cdot \tau = \mu
\end{aligned}
$$

τ being the permutation $(1 \quad 2)$. Then, $\mathrm{T}\mathscr{Com}$ is generated by two binary operations μ and $\nu := \mathrm{d}\mu$ satisfying:

$$
\begin{aligned}
&\mu(\mu,\mathbb{1}) = \mu(\mathbb{1},\mu) \\
&\mu \cdot \tau = \mu \\
&\mu(\mathbb{1},\nu) - \nu(\mu,\mathbb{1}) + \nu(\mathbb{1},\mu) - \mu(\nu,\mathbb{1}) = 0 \\
&\nu \cdot \tau = \nu
\end{aligned}
$$

Similarly, $\mathscr{uCom}$, also described in Example 3.8, is generated by μ and by a 0-ary operation η for which $\mu(\eta,\mathbb{1}) = \mathbb{1}$. Thus, $\mathrm{T}\mathscr{uCom}$ is generated by two binary operations μ and ν and by a 0-ary operation η for which:

$$\begin{aligned}
&\mu(\mu, \mathbb{1}) = \mu(\mathbb{1}, \mu)\\
&\mu \cdot \tau = \mu\\
&\mu(\mathbb{1}, \nu) - \nu(\mu, \mathbb{1}) + \nu(\mathbb{1}, \mu) - \mu(\nu, \mathbb{1}) = 0\\
&\nu \cdot \tau = \nu\\
&\mu(\eta, \mathbb{1}) = \mathbb{1}
\end{aligned}$$

Example 5.43. Consider the operad $\mathscr{Lie}$ described in Example 3.9. Recall that $\mathscr{Lie}$ is generated by a binary operation μ satisfying the following relations:

$$\begin{aligned}
&\mu + \mu \cdot \tau = 0\\
&\mu(\mu, \mathbb{1}) + \mu(\mu, \mathbb{1}) \cdot \sigma + \mu(\mu, \mathbb{1}) \cdot \sigma^2 = 0
\end{aligned}$$

where σ denotes the permutation $(1 \quad 2 \quad 3)$ and $\tau = (1 \quad 2)$. Then, $\mathrm{T}\mathscr{Lie}$ is generated by two binary operations μ and $\nu := \mathsf{d}\mu$ satisfying the following conditions:

$$\begin{aligned}
&\nu + \nu \cdot \tau = 0\\
&\nu(\mu, \mathbb{1}) + \mu(\nu, \mathbb{1}) + \nu(\mu, \mathbb{1}) \cdot \sigma + \mu(\nu, \mathbb{1}) \cdot \sigma + \nu(\mu, \mathbb{1}) \cdot \sigma^2 + \mu(\nu, \mathbb{1}) \cdot \sigma^2 = 0
\end{aligned}$$

or equivalently:

$$\begin{aligned}
&\nu + \nu \cdot \tau = 0\\
&\nu(\mu, \mathbb{1}) + \nu(\mu, \mathbb{1}) \cdot \sigma + \nu(\mu, \mathbb{1}) \cdot \sigma^2 = \mu(\mathbb{1}, \nu) + \mu(\mathbb{1}, \nu) \cdot \sigma + \mu(\mathbb{1}, \nu) \cdot \sigma^2
\end{aligned}$$

Example 5.44. Consider a unital associative algebra A over R and the corresponding operad $A^\bullet$, described in Example 3.6. Then the operad $\mathrm{T}A^\bullet$ is generated by all $a \in A^\bullet(1) = A$ and by symbols $\mathsf{d}a$, for each $a \in A$. Moreover:

$$\mathsf{d}(ab) = \mathsf{d}(a(b)) = (\mathsf{d}a)b + a(\mathsf{d}b)$$

for every $a, b \in A$; where we denoted the associative multiplication of A by juxtaposition. So, $\mathrm{T}A^\bullet$ is precisely $(\mathrm{T}A)^\bullet$, where $\mathrm{T}A$ denotes the geometric tangent bundle over the associative affine scheme A.

Example 5.44 suggests an interesting relationship between operads and algebras.

Proposition 5.45. *The functor* $(-)^\bullet \colon \mathsf{Alg} \to \mathsf{Operad}$ *which sends an associative and unital algebra A to the operad $A^\bullet$ described in Example 3.6 and which sends a morphism $f \colon A \to B$ of algebras to the corresponding morphism $f^\bullet$ such that $f^\bullet(1) = f$, extends to a strict tangent morphism* $(-)^\bullet \colon \mathsf{Alg}(u\mathscr{A}ss) \to (\mathsf{Operad}, \mathbb{L})$*. Moreover, (the opposite of) the same functor extends also to a strong tangent morphism* $(-)^\bullet \colon \mathsf{Geom}(u\mathscr{A}ss) \to (\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$*.*

Proof. Since the proof is straightforward, we leave it to the reader to complete the details of this proof. □

It is also possible to send an operad $\mathscr{P}$ to the associative and unital algebra $\mathscr{P}(1)$. It is straightforward to see that this extends to a functor $(-)(1) \colon \mathsf{Operad} \to \mathsf{Alg}$. The next proposition shows that $(-)(1)$ is compatible with the tangent structures $\mathbb{L}$ and $\mathbb{T}$.

Proposition 5.46. *The functor* $(-)(1) \colon \mathsf{Operad} \to \mathsf{Alg}$ *extends to a strict tangent morphism* $(-)(1) \colon (\mathsf{Operad}, \mathbb{L}) \to \mathsf{Alg}(u\mathscr{A}ss)$*. Moreover, (the opposite of) the same functor extends also to a strong tangent morphism* $(-)(1) \colon (\mathsf{Operad}^{\mathsf{op}}, \mathbb{T}) \to \mathsf{Geom}(u\mathscr{A}ss)$*.*

Proof. Since the proof is straightforward, we leave it to the reader to complete the details of this proof. □

In the exploration of the possible relationships between the tangent categories of operads and the ones of operadic algebras, the functor which sends a $\mathscr{P}$-algebra A to the corresponding enveloping operad $\mathscr{P}^{(A)}$ is one of the most interesting. So, it is natural to wonder if this functor is compatible with the tangent structures.

Recall (see Section 4.2.1), that the enveloping operad $\mathscr{P}^{(A)}$ is the operad generated by tuples $(\mu; a_1, \dots, a_k|$ formed by $a_1, \dots, a_k \in A$ and by $\mu \in \mathscr{P}(n+k)$, for every positive integers n and k, which are R-linear in each entry, and that satisfy the following relations:

$$(\mu; a_1, \dots, \nu(a_k, \dots, a_{i+n}), \dots, a_k| = (\mu \circ_i \nu; a_1, \dots, a_k|$$

where $\mu \circ_k \nu$ denotes $\mu(\mathbb{1}, \dots, \nu, \dots, \mathbb{1})$.

Proposition 5.47. *The functor* $\mathscr{P}^{(-)} \colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Operad}$ *which sends a $\mathscr{P}$-algebra A to the corresponding enveloping operad $\mathscr{P}^{(A)}$ and a morphism $f \colon A \to B$ to the morphism of* *operads* $f^{(A)}\colon \mathscr{P}^{(A)} \to \mathscr{P}^{(B)}$ *defined on generators as follows:*

$$f^{(A)}(\mu; a_1, \dots, a_k| := (\mu; f(a_1), \dots, f(a_k)|$$

equipped with the distributive law $\xi_{\mathscr{P}}\colon \mathscr{P}^{(\mathbb{L}A)} \to \mathbb{L}\mathscr{P}^{(A)}$ *defined on generators as follows:*

$$\xi_{\mathscr{P}}(\mu; (a_1, b_1), \dots, (a_k, b_k)| := \left((\mu; a_1, \dots, a_k|, \sum_{i=1}^{k}(\mu; a_1, \dots, b_i, \dots, a_k|\right)$$

becomes a lax tangent morphism $(\mathscr{P}^{(-)}, \xi_{\mathscr{P}})\colon \mathsf{Alg}(\mathscr{P}) \to (\mathsf{Operad}, \mathbb{L})$. *Moreover, by mating, (the opposite of) the functor* $\mathscr{P}^{(-)}$ *also extends to a lax tangent morphism:*

$$(\mathscr{P}^{(-)}, \omega_{\mathscr{P}})\colon \mathsf{Geom}(\mathscr{P}) \to (\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$$

Proof. The first step is to show that $\xi_{\mathscr{P}}$ is well-defined by proving that is compatible with the relations of the enveloping operad:

$$\begin{aligned}
&\xi_{\mathscr{P}}(\mu; (a_1, b_1), \dots, \nu((a_i, b_i), \dots, (ai+n, b_{i+n})), \dots, (a_k, b_k)| \\
= \;& \xi_{\mathscr{P}}\left(\mu; (a_1, b_1), \dots, \left(\nu(a_i, \dots, ai+n), \sum_{j=i}^{i+n}\nu(a_i, \dots, b_j, \dots, a_{i+n}\right), \dots, (a_k, b_k)\right| \\
= \;& ((\mu; a_1, \dots, \nu(a_i, \dots, a_{i+n}), \dots, a_k|, \\
&\quad \sum_{l\neq i,\dots,i+n}(\mu; a_1, \dots, b_l, \dots, \nu(a_i, \dots, a_{i+n}), \dots, a_k| + \\
&\quad \left. + \sum_{j=i}^{i+n}(\mu; a_1, \dots, \nu(a_i, \dots, b_j, \dots, a_{i+n}), \dots, a_k|\right) \\
= \;& ((\mu \circ_i \nu; a_1, \dots, a_k|, \\
&\quad \sum_{l\neq i,\dots,i+n}(\mu \circ_i \nu; a_1, \dots, b_l, \dots, a_k| + \\
&\quad \left. + \sum_{j=i}^{i+n}(\mu \circ_i \nu; a_1, \dots, b_j, \dots, a_k|\right) \\
= \;& \left((\mu \circ_i \nu; a_1, \dots, a_k|, \sum_{j-1}^{k}(\mu \circ_i \nu; a_1, \dots, b_j, \dots, a_k|\right) \\
= \;& \xi_{\mathscr{P}}(\mu \circ_i \nu; (a_1, b_1), \dots, (a_k, b_k)|
\end{aligned}$$

This shows that $\xi_{\mathscr{P}}$ is well-defined. To check the compatibility with the tangent structures, recall first that the distributive law $\alpha_{\mathscr{P}}\colon \mathsf{S}_{\mathscr{P}} \circ \mathbb{L} \Rightarrow \mathbb{L} \circ \mathsf{S}_{\mathscr{P}}$ between the monad associated with the operad $\mathscr{P}$ and the tangent bundle functor $\mathbb{L}\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$ induced by biproducts is defined as follows:

$$\alpha_{\mathscr{P}}(\mu;(a_1,b_1),\ldots,(a_n,b_n)) = \left((\mu;a_1,\ldots,a_n),\sum_{k=1}^{n}(\mu;a_1,\ldots,b_k,\ldots,a_n)\right)$$

$\xi_{\mathscr{P}}$ is defined on generators in a similar fashion. From this observation, one immediately concludes that $(\mathscr{P}^{(-)},\xi_{\mathscr{P}})$ is a lax tangent morphism. Finally, by employing Proposition 3.75, we conclude that $(\mathscr{P}^{(-)},\xi_{\mathscr{P}})^{\mathsf{op}}\colon \mathsf{Geom}(\mathscr{P}) \to (\mathsf{Operad}^{\mathsf{op}},\mathbb{T})$ is also a lax tangent moprhism. □

By definition, the enveloping algebra $\mathsf{Env}_{\mathscr{P}}(A)$ of a $\mathscr{P}$-algebra A is the unital and associative algebra $\mathscr{P}^{(A)}(1)$ (see for example [6, Definition 1.11]). Thus, from Propositions 5.46 and 5.47 we conclude that the functor $\mathsf{Env}_{\mathscr{P}}\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}$ which sends a $\mathscr{P}$-algebra A to its enveloping algebra $\mathsf{Env}_{\mathscr{P}}(A)$ is compatible with the algebraic and geometric tangent structures. Moreover, so does the functor $\mathsf{Env}^\bullet_{\mathscr{P}}\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Operad}$ which sends a $\mathscr{P}$-algebra to the operad $\mathsf{Env}^\bullet_{\mathscr{P}}(A) := (\mathsf{Env}_{\mathscr{P}}(A))^\bullet$.

Corollary 5.48. *The functors* $\mathsf{Env}_{\mathscr{P}}\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}$ *and* $\mathsf{Env}^\bullet_{\mathscr{P}}\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Operad}$ *extend to lax tangent morphisms* $\mathsf{Env}_{\mathscr{P}}\colon \mathsf{Alg}(\mathscr{P}) \to \mathsf{Alg}(u\mathscr{A}ss)$ *and* $\mathsf{Env}^\bullet_{\mathscr{P}}\colon \mathsf{Alg}(\mathscr{P}) \to (\mathsf{Operad},\mathbb{L})$. *Moreover, (the opposite of) the same functors extend to the lax tangent morphisms* $\mathsf{Env}_{\mathscr{P}}\colon \mathsf{Geom}(\mathscr{P}) \to \mathsf{Geom}(u\mathscr{A}ss)$ *and* $\mathsf{Env}^\bullet_{\mathscr{P}}\colon \mathsf{Geom}(\mathscr{P}) \to (\mathsf{Operad}^{\mathsf{op}},\mathbb{T})$.

Remark 5.49. In Examples 3.51 and 3.82 we claimed that the functor which sends a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $\mathsf{Env}_{\mathscr{L}ie}(\mathfrak{g})$ extends to two tangent morphisms $\mathsf{Env}_{\mathscr{L}ie}\colon \mathsf{Alg}(\mathscr{L}ie) \to \mathsf{Alg}(u\mathscr{A}ss)$ and $\mathsf{Env}_{\mathscr{L}ie}\colon \mathsf{Geom}(\mathscr{L}ie) \to \mathsf{Geom}(u\mathscr{A}ss)$. This was shown as a consequence of $\mathsf{Env}_{\mathscr{L}ie}$ being induced by a morphism of operads $u\mathscr{A}ss \to \mathscr{L}ie$. This result can also be seen as a special case of Corollary 5.48.

We conclude this section with two important consequences of $(\mathsf{Operad}^{\mathsf{op}},\mathbb{T})$ being the geometric tangent category of the coCartesian differential monad (W,∂). Applying Proposition 3.73 we classify vector fields and applying Theorem 4.20, we prove that every free operad is a differential object in $(\mathsf{Operad}^{\mathsf{op}},\mathbb{T})$.

Theorem 5.50. *Vector fields* $v\colon \mathrm{T}\mathscr{P} \to \mathscr{P}$ *over an operad* $\mathscr{P}$ *(regarded as operad morphisms) in the tangent category* $(\mathsf{Operad}^{\mathsf{op}},\mathbb{T})$ *are in bijective correspondence with vector* *fields $\mathscr{P} \to \mathbb{L}\mathscr{P}$ over $\mathscr{P}$ in the tangent category* $(\mathsf{Operad}, \mathbb{L})$*, which are in bijective correspondence with derivations over $\mathscr{P}$. Moreover, these bijections preserve the Lie brackets, defined by the two tangent structures (see Section 2.2.1) and by the commutator between derivations, respectively.*

Concretely, the equivalence between vector fields $v\colon \mathscr{P} \to \mathbb{L}\mathscr{P}$ and $v^\sharp\colon \mathrm{T}\mathscr{P} \to \mathscr{P}$ is defined as follows. Every $v\colon \mathscr{P}\mathbb{L}\mathscr{P}$ sends $\mu\mathscr{P}(n)$ to $(\mu, \delta_v(\mu)) \in \mathbb{L}\mathscr{P}$. Then $v^\sharp\colon \mathrm{T}\mathscr{P} \to \mathscr{P}$ sends μ to itself and $\mathsf{d}\mu$ to $\delta_v\mu$. Conversely, a vector field $u\colon \mathrm{T}\mathscr{P} \to \mathscr{P}$ sends each μ to μ and $\mathsf{d}\mu$ to $\delta_u(\mu) \in \mathscr{P}$, so $u^\flat\colon \mathscr{P} \to \mathbb{L}\mathscr{P}$ sends μ to $(\mu, \delta_u(\mu))$.

Theorem 5.51. *Free operads are differential objects in* $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$*.*

We leave it to future work to classify all differential objects and differential bundles of $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$. However, we conjecture that differential bundles in this tangent category could be operad bimodules in which the left action is linearized (see [46, Section 5.2.2]).

5.3 Towards a local approach to deformations

In Example 5.11 we showed that vector fields $v\colon \mathscr{A}\!s\!s \to \mathbb{L}\mathscr{A}\!s\!s$ over $\mathscr{A}\!s\!s$ of $(\mathsf{Operad}, \mathbb{L})$, or equivalently, vector fields $v^\sharp\colon \mathrm{T}\mathscr{A}\!s\!s \to \mathscr{A}\!s\!s$ of the adjoint tangent category $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$, define for every associative algebra A an infinitesimal deformation of A. More generally, given a derivation $\delta\colon \mathscr{P} \to \mathscr{P}$ of an operad $\mathscr{P}$, we can define a functor $\delta_!\colon \mathsf{Alg}_{\mathscr{P}} \to \mathsf{Alg}_{\mathbb{L}\mathscr{P}}$ which sends every $\mathscr{P}$-algebra A to an infinitesimal deformation $\delta_! A$ of A.

However, this does not guarantee that every infinitesimal deformation of a $\mathscr{P}$-algebra A can be defined via this construction. Let's consider again the case with $\mathscr{P} = \mathscr{A}\!s\!s$. Every derivation δ of $\mathscr{A}\!s\!s$ is fully determined by $\delta(\mu) \in \mathscr{A}\!s\!s(2)$, where μ denotes the usual generator of $\mathscr{A}\!s\!s$ (see Example 3.7). However, $\mathscr{A}\!s\!s(2) = R^2$, thus $\delta(\mu) = r\mu + s\mu^{\mathsf{op}}$ for some $r, s \in R$. However, from the 2-cocycle condition and the associativity of μ we conclude that s must be zero. Therefore, all derivations δ of $\mathscr{A}\!s\!s$ are specified by $\delta(\mu) = r\mu$. Unfortunately, this only accounts for the trivial infinitesimal deformations, since, for an associative algebra A, $r\mu_A$ is precisely the 2-coboundary in the Hochschild cohomology of A associated to the 1-cocycle $r\mathsf{id}_A\colon A \to A$.

In this section, we would like to investigate some ideas to employ the tangent categories $(\mathsf{Operad}, \mathbb{L})$ and $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$ to capture *all* infinitesimal deformations of $\mathscr{P}$-algebras. We develop two main approaches: the first focuses on defining a module which classifies all infinitesimal deformations of a given algebra via a universal property, in a similar fashion as the module of Kähler differentials classifies all derivations of an algebra. With the second approach, we try to take advantage of the tangent category $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$ to construct a functor which associates each $\mathscr{P}$-algebra A with a bundle $A \to \Lambda A$ whose sections in the category of affine schemes are in bijection with infinitesimal deformations. Finally, we compare these two approaches.

The main intuition underpinning our efforts in this subject is the idea of introducing on the category of $\mathscr{P}$-affine schemes a "quasi-tangent structure" which captures all infinitesimal deformations of a $\mathscr{P}$-algebra A as sections of the "quasi-tangent bundle" of A in a similar way as derivations of a $\mathscr{P}$-algebra A are sections of the geometric tangent bundle over A. To make sense of this idea, first, recall why the tangent bundle $p\colon A \to \mathrm{T}A$ classifies derivations of A. Recall that $\mathrm{T}A$ is the free A-algebra of the A-bimodule of Kähler differentials of A. The important ingredient is the module of Kähler differentials ΩA of A. In particular, ΩA is precisely the A-bimodule representing the functor $\mathsf{Der}_A\colon \mathsf{Mod}_A \to \mathsf{Mod}_R$ of derivations over A. So, for any A-bimodule M a derivation $\delta\colon A \to M$ splits along the universal derivation $\mathsf{d}\colon A \to \Omega A$, i.e. there is a unique morphism of A-bimodules $\overline{\delta}\colon \Omega A \to M$ such that:

$$
\begin{tikzcd}
A \arrow[r, "\mathsf{d}"] \arrow[dr, "\delta"'] & \Omega A \arrow[d, dashed, "\overline{\delta}"] \\
& M
\end{tikzcd}
$$

Can we classify infinitesimal deformations of a $\mathscr{P}$-algebra similarly? First, let's focus on the associative case, i.e. with $\mathscr{P} = \mathscr{A}\!ss$ and let's recall that, given an associative algebra A, and an A-bimodule M, a 2-cocycle of the Hochschild cohomology of A with coefficients in M is an R-linear morphism:

$$\xi\colon A \otimes A \to M$$

which satisfies the following condition:

$$a\xi(b,c) - \xi(ab,c) + \xi(a,bc) - \xi(a,b)c = 0$$

for every $a, b, c \in A$, where the juxtaposition indicates the left and the right action of A on M. In the following, we simply refer to such a ξ as a 2-cocycle. Notice also that, given a morphism $f\colon M \longrightarrow N$ of A-bimodules and a 2-cocycle $\xi\colon A \otimes A \longrightarrow M$, the morphism $f_*\xi\colon A \otimes A \xrightarrow{\xi} M \xrightarrow{f} N$ is also a 2-cocycle. So, the operation which sends an A-bimodule to the R-module $\mathsf{Def}_A(M)$ which contains all 2-cocycles $A \otimes A \longrightarrow M$ extends to a functor:

$$\mathsf{Def}_A\colon \mathsf{BiMod}_A \longrightarrow \mathsf{Mod}_R$$

We want to prove that Def_A is a representable functor. Let's define the following A-bimodules.

Definition 5.52. *For an associative algebra A, the **bimodule of infinitesimal deformators** of A is the free A-bimodule ΞA generated by all symbols $\nu(a,b)$, called infinitesimal **deformators**, for each $a, b \in A$, subject to the relations:*

$$\nu(ra + sb, c) = r\nu(a,c) + s\nu(b,c)$$
$$\nu(c, ra + sb) = r\nu(c,a) + s\nu(c,b)$$
$$a\nu(b,c) - \nu(ab,c) + \nu(a,bc) + \nu(a,b)c = 0$$

Proposition 5.53. *The bimodule of infinitesimal deformators ΞA of an associative algebra A represents the functor $\mathsf{Def}_A\colon \mathsf{BiMod}_A \longrightarrow \mathsf{Mod}_R$. Concretely, this means that for any A-bimodule M a 2-cocycle $\xi\colon A \otimes A \longrightarrow M$ splits in a unique way along the **universal 2-cocycle** $\nu\colon A \otimes A \longrightarrow \Xi A$, which sends $a \otimes b$ to $\nu(a,b) \in \Xi A$. In particular, there is a unique morphism of A-bimodules $\overline{\xi}\colon \Xi A \longrightarrow M$ such that:*

$$\begin{tikzcd} A \otimes A \arrow[r, "\nu"] \arrow[dr, "\xi"'] & \Xi A \arrow[d, dashed, "\overline{\xi}"] \\ & M \end{tikzcd}$$

Proof. First, thanks to the relations which define ΞA, ν is indeed a 2-cocycle. The second step is to show the universal property. Consider a 2-cocycle $\xi\colon A \otimes A \longrightarrow M$

and let's define $\overline{\xi}$ as the morphism which sends $\nu(a,b)$ in $\xi(a,b)$. Since ξ is a 2-cocycle, $\overline{\xi}$ is compatible with the relations of ΞA, so $\overline{\xi}$ is a well-defined morphism of A-bimodules for which, by construction, $\xi = \nu\overline{\xi}$. Let's now take a second morphism $\overline{\xi}'\colon \Xi A \to M$ of A-bimodules for which $\xi = \nu\overline{\xi}'$. Then, $\overline{\xi}'\nu(a,b) = \overline{\xi}$. However, since $\overline{\xi}'$ and $\overline{\xi}$ agree on the generators of ΞA, we conclude that they must be the same morphism. □

Let $\mathrm{L}\colon \mathsf{Alg} \to \mathsf{Alg}$ be the functor which sends an associative algebra A to the A-tensor algebra of the bimodule of infinitesimal deformators of A, i.e.:

$$\mathrm{L}A := \mathsf{Tens}_A \Xi A$$

Moreover, L sends a morphism $f\colon A \to B$ of algebras to the morphism $\mathrm{L}f\colon \mathrm{L}A \to \mathrm{L}B$ which sends each generator $\nu(a,b) \in \mathrm{L}A$ to $\nu(f(a), f(b)) \in \mathrm{L}B$. Concretely, $\mathrm{L}A$ is the associative algebra generated by all $a \in A$ and by symbols $\nu(a,b)$ for each $a,b \in A$ such that the multiplication between the elements of A behaves like the multiplication in A and $\nu(a,b)$ satisfy the same relations which define ΞA.

Proposition 5.54. *Consider the following morphisms:*

projection $q\colon A \to \mathrm{L}A$ *which sends a to itself;*

zero morphism $z_q\colon \mathrm{L}A \to A$ *which sends a to itself and $\nu(a,b)$ to 0;*

n-fold pullbacks *the n-pushout of q along itself are the associative algebras $\mathrm{L}_n A$ generated by all elements a of A for which the multiplication is defined as in A, and symbols $\nu_k(a,b)$ each bilinear and satisfying the 2-cocycle relation, for each $k = 1, \dots, n$;*

sum morphism $s_q\colon \mathrm{L}A \to \mathrm{L}_2 A$ *which sends a to itself and $\nu(a,b)$ to $\nu_1(a,b) + \nu_2(a,b)$;*

vertical lift $l_q\colon \mathrm{T}\mathrm{L}A \to \mathrm{L}A$*, where T denotes the geometric tangent bundle functor in the tangent category $\mathsf{Geom}(\mathscr{Ass})$, which sends a to itself, $\mathsf{d}a$ and $\nu(a,b)$ to 0, and $\mathsf{d}\nu(a,b)$ to $\nu(a,b)$.*

For each associative algebra A, $\mathrm{L}A := (\mathrm{L}A, q, z_q, s_q, l_q)$ is a differential bundle over A in the geometric tangent category $\mathsf{Geom}(\mathscr{Ass})$. Moreover, L extends to a functor:

$$\mathrm{L}\colon \mathsf{Geom}(\mathscr{Ass}) \to \mathsf{DBnd}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{Ass}))$$

Proof. This proof is a direct consequence of the classification of differential bundles in the geometric tangent category of an operad expressed by Theorem 4.35, which establishes that modules in the operadic sense over an operadic algebra are equivalent to differential bundles over the corresponding operadic affine scheme in the geometric tangent category of the operad. Moreover, linear morphisms correspond to linear morphisms between the corresponding differential bundles, in a contravariant way.

In particular, for $\mathscr{P} = \mathscr{A}ss$, $\mathrm{L}A$ equals $\mathsf{Free}_A \Xi A$ where ΞA is an A-bimodule in the operadic sense, so for the equivalence between modules and differential objects, $\mathrm{L}A$ becomes a differential bundle. $\square$

The next proposition proves that the differential bundle $A \to \mathrm{L}A$ classifies infinitesimal deformations of A.

Proposition 5.55. *2-cocycles $\xi\colon A \otimes A \to A$ of A are in bijective correspondence with sections of the differential bundle $q\colon A \to \mathrm{L}A$. In particular, each $\xi\colon A \otimes A \to A$ defines a morphism of $\mathscr{P}$-algebras $u_\xi\colon \mathrm{L}A \to A$ which sends each generator $a \in A$ to itself and each infnitesimal deformator $\nu(a,b)$ to $\xi(a,b)$. Conversely, each section $u\colon \mathrm{L}A \to A$ defines a 2-cocycle $\xi_u\colon A \otimes A \to A$ defined by $\xi_u(a,b) := u(\nu(a,b))$.*

Proof. The proof follows directly from the adjunction $\mathsf{Free}_A \dashv \mathsf{Restr}_A$ of Lemma 3.61 and by the universality of the bimodule of infinitesimal deformators established by Proposition 5.53. $\square$

The next step is to extend this construction for any operadic algebra for a given algebraic operad. Let $\mathscr{P}$ be an algebraic operad. In Examples 5.41, 5.42, 5.43, and 5.44 we showed the relationship between the algebras of $\mathrm{T}\mathscr{P}$ and corresponding infinitesimal deformations. In particular, these examples suggest that given an n-ary operation μ of an operad $\mathscr{P}$, the corresponding n-ary operation $\mathsf{d}\mu \in \mathrm{T}\mathscr{P}(n)$ represents an infinitesimal deformation of μ.

This insight suggests defining the module ΞA of infinitesimal deformators for a $\mathscr{P}$-algebra A as the A-module (in the operadic sense) generated by symbols $\mathsf{d}\mu(a_1,\dots,a_n)$ for every $\mu \in \mathscr{P}(n)$ and every $a_1,\dots,a_n \in A$. Indeed, when $\mathscr{P} = \mathscr{A}ss$ and $\mu \in \mathscr{A}ss(2)$ is the usual generator of $\mathscr{A}ss$, $\mathsf{d}\mu = \nu$ is precisely the infinitesimal deformation of μ. In the following, we adopt the following notation: we denote by

$\vec{a}$ a tuple $a_1, \dots, a_n$ of elements of a $\mathscr{P}$-algebra A. So, for instance, we write $\mathsf{d}\mu(\vec{a})$ for $\mathsf{d}\mu(a_1, \dots, a_n)$. We also write $\vec{a}_1, \dots, \vec{a}_n$ for $a_1^{(1)}, \dots, a_{k_1}^{(1)}, \dots, a_1^{(n)}, \dots, a_{k_n}^{(n)}$.

Definition 5.56. *The* ***module of infinitesimal deformators*** *of a $\mathscr{P}$-algebra A is the A-module ΞA freely generated by symbols $\mathsf{d}\mu(a_1, \dots, a_n) = \mathsf{d}\mu(\vec{a})$, called* ***infinitesimal deformators****, for each $\mu \in \mathscr{P}(n)$ and $\vec{a} := a_1, \dots, a_n \in A$ subject to the following relations:*

$$\begin{aligned}
&\mathsf{d}\mu(a_1, \dots, ra_k + sb_k, \dots, a_n) = r\mathsf{d}\mu(a_1, \dots, a_n) + s\mathsf{d}\mu(a_1, \dots, b_k, \dots, a_n)\\
&\mathsf{d}(r\mu + s\nu)(\vec{a}) = r\mathsf{d}\mu(\vec{a}) + s\mathsf{d}\nu(\vec{a})\\
&\mathsf{d}(\mu(\mu_1, \dots, \mu_n))(\vec{a}_1, \dots, \vec{a}_n) = \mathsf{d}\mu(\mu_1(\vec{a}_1), \dots, \mu_n(\vec{a}_n)) +\\
&\qquad + \sum_{k=1}^{n} \mu(\mu_1(\vec{a}_1), \dots, \mathsf{d}\mu_k(\vec{a}_k), \dots, \mu_n(\vec{a}_n))
\end{aligned}$$

The first equation of Definition 5.56 establishes that the infinitesimal deformators are R-linear in each A-entry; the second equation establishes that deformators are also R-linear in the μ-entry; finally, the third equation establishes that $\mathsf{d}\mu$ is an infinitesimal deformation of the n-ary operation μ. In particular, when $\mathscr{P} = \mathscr{A}ss$, the third equation, following a similar argument as in Example 5.41, implies:

$$\begin{aligned}
&\mathsf{d}\mu(a, \mu(b, c)) + \mu(a, \mathsf{d}\mu(b, c)) = \mathsf{d}(\mu(\mathbb{1}, \mu))(a, b, c) =\\
&= \mathsf{d}(\mu(\mu, \mathbb{1}))(a, b, c) = \mathsf{d}\mu(\mu(a, b), c) + \mu(a, \mathsf{d}\mu(b, c))
\end{aligned}$$

Denoting the multiplication of A and the left and the right actions of A on ΞA by juxtaposition and $\mathsf{d}\mu(a, b)$ by $\nu(a, b)$ we rewrite:

$$a\nu(b, c) - \nu(ab, c) + \nu(a, bc) - \nu(a, b)c = 0$$

In particular, for $\mathscr{P} = \mathscr{A}ss$, Definition 5.56 agrees with Definition 5.52.

Proposition 5.57. *When $\mathscr{P} = \mathscr{A}ss$, the module ΞA of Definition 5.56 is precisely the bimodule of Definition 5.52.*

Since, by construction, ΞA is an A-module, thanks to Theorem 4.35, $\mathrm{L}A :=$ $\mathsf{Free}_A \Xi A$ becomes a differential bundle in the geometric tangent category of $\mathscr{P}$. Moreover, $q\colon \mathrm{L}A \to A$ classifies infinitesimal deformations of A. To see this, recall that the $\mathscr{P}$-algebra structure of an infinitesimal deformation $\tilde{A}$ of a $\mathscr{P}$-algebra A is fully specified by:

$$\mu_{\tilde{A}}(a_1, \dots, a_n) = \mu_A(a_1, \dots, a_n) + \varepsilon\mu'_A(a_1, \dots, a_n)$$

for each $\mu \in \mathscr{P}(n)$ and $a_1, \ldots, a_n \in A$. So, the $\mathscr{P}$-algebra structure of $\tilde{A}$ is specified by the $\mathscr{P}$-algebra structure of A together with a morphism $\Xi A \to A$ which sends each infinitesimal deformator $\mathsf{d}\mu(a_1, \ldots, a_n)$ to $\mu'_A(a_1, \ldots, a_n)$.

Theorem 5.58. *Let* L *be the functor which sends each* $\mathscr{P}$*-algebra* A *to the free* A*-algebra over the module of infinitesimal deformators of* A*, i.e.* $\mathrm{L}A \colon= \mathsf{Free}_A \Xi A$*. Consider the following morphisms:*

projection $q \colon A \to \mathrm{L}A$ *which sends* a *to itself;*

zero morphism $z_q \colon \mathrm{L}A \to A$ *which sends* a *to itself and each deformator* $\mathsf{d}\mu(\vec{a})$ *to* 0*;*

n**-fold pullbacks** *the* n*-pushouts of* q *along itself are the associative algebras* $\mathrm{L}_n A$ *generated by all elements* a *of* A *for which the algebra structure is defined as in* A*, and infinitesimal deformators* $\mathsf{d}_k\mu(\vec{a})$*, for each* $k = 1, \ldots, n$*;*

sum morphism $s_q \colon \mathrm{L}A \to \mathrm{L}_2 A$ *which sends* a *to itself and* $\mathsf{d}\mu(\vec{a})$ *to* $\mathsf{d}_1\mu(\vec{a}) + \mathsf{d}_2\mu(\vec{a})$*;*

vertical lift $l_q \colon \mathrm{TL}A \to \mathrm{L}A$*, where* T *denotes the geometric tangent bundle functor in the tangent category* $\mathsf{Geom}(\mathscr{P})$*, which sends* a *to itself,* $\mathsf{d}a$ *and* $\mathsf{d}\mu(\vec{a})$ *to* 0*, and* $\mathsf{d}\mu(\vec{a})$ *to* $\mathsf{d}\mu(\vec{a})$*.*

For each associative algebra A*,* $\mathrm{L}A \colon= (\mathrm{L}A, q, z_q, s_q, l_q)$ *is a differential bundle over* A *in the geometric tangent category* $\mathsf{Geom}(\mathscr{P})$*. Moreover,* L *extends to a functor:*

$$\mathrm{L} \colon \mathsf{Geom}(\mathscr{P}) \to \mathsf{DBnd}_{\mathsf{lnr}}(\mathsf{Geom}(\mathscr{P}))$$

Finally, $q \colon A \to \mathrm{L}A$ *classifies infinitesimal deformations of* A*, i.e. there is a bijective correspondence between sections* $u \colon \mathrm{L}A \to A$ *of* q *(in* $\mathsf{Geom}(\mathscr{P})$*) and infinitesimal deformations of* A*. In particular, for each section* $u \colon \mathrm{L}A \to A$ *of* q*, the* n*-ary operation* $\mathsf{d}\mu_A \colon A^{\otimes n} \to A$ *defined by:*

$$\mathsf{d}\mu_A(a_1, \ldots, a_n) \colon= u(\mathsf{d}\mu(a_1, \ldots, a_n))$$

defines an infinitesimal deformation of the n*-ary operation* μ_A*.*

The symbol d in the definition of the differential bundle $q \colon A \to \mathrm{L}A$ of Theorem 5.58 is reminiscent of the analogous symbol d employed in the definition of the tangent bundle functor T on $\mathsf{Operad}^{\mathsf{op}}$. The rest of this section is dedicated to exploring this relationship. The main intuition consists of noticing that the free $\mathrm{T}\mathscr{P}$-algebra over a $\mathscr{P}$-algebra contains elements of the form $(\mathsf{d}\mu; a_1, \ldots, a_n)$, however, it also contains terms of the form $(\mu; a_1, \ldots, a_n)$, for $\mu \in \mathscr{P}(n)$. Given a $\mathscr{P}$-algebra A, to remove these terms, we can make the algebra structure of the free $\mathrm{T}\mathscr{P}$-algebra $\mathsf{S}_{\mathrm{T}\mathscr{P}}A$ over A to agree with the $\mathscr{P}$-algebra structure of A. Now, recall that a morphism of operads $\varphi\colon \mathscr{P} \to \mathscr{P}'$ induces an adjunction:

$$\varphi_!\colon \mathsf{Alg}_{\mathscr{P}} \leftrightarrows \mathsf{Alg}_{\mathscr{P}'}\colon \varphi^*$$

In particular, the projection $p\colon \mathscr{P} \to \mathrm{T}\mathscr{P}$ of $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$ induces an adjunction:

$$p_!\colon \mathsf{Alg}_{\mathscr{P}} \leftrightarrows \mathsf{Alg}_{\mathrm{T}\mathscr{P}}\colon p^*$$

By direct inspection and recalling the definition of the left adjoint $\varphi_!$ (see Section 3.5.2), the $\mathrm{T}\mathscr{P}$-algebra $p_!A$, for a $\mathscr{P}$-algebra A, is the free $\mathrm{T}\mathscr{P}$-algebra $\mathsf{S}_{\mathrm{T}\mathscr{P}}A$ quotiented by the ideal generated by terms $(\mu; a_1, \ldots, a_n) - \mu_A(a_1, \ldots, a_n)$.

Lemma 5.59. *The $\mathrm{T}\mathscr{P}$-algebra $p_!A$ is the free $\mathrm{T}\mathscr{P}$-algebra generated by all $a \in A$ and by symbols $\mathsf{d}\mu(\vec{a}) = \mathsf{d}\mu(a_1, \ldots, a_n)$ subject to the following relations:*

$$\begin{aligned}
&\mu_{p_!A}(a_1, \ldots, a_n) = \mu_A(a_1, \ldots, a_n)\\
&\mathsf{d}\mu(a_1, \ldots, ra_k + sb_k, \ldots, a_n) = r\mathsf{d}\mu(a_1, \ldots, a_n) + s\mathsf{d}\mu(a_1, \ldots, b_k, \ldots, a_n)\\
&\mathsf{d}(r\mu + s\nu)(\vec{a}) = r\mathsf{d}\mu(\vec{a}) + s\mathsf{d}\nu(\vec{a})\\
&\mathsf{d}(\mu(\mu_1, \ldots, \mu_n))(\vec{a}_1, \ldots, \vec{a}_n) = \mathsf{d}\mu(\mu_1(\vec{a}_1), \ldots, \mu_n(\vec{a}_n)) +\\
&\qquad + \sum_{k=1}^{n} \mu(\mu_1(\vec{a}_1), \ldots, \mathsf{d}\mu_k(\vec{a}_k), \ldots, \mu_n(\vec{a}_n))
\end{aligned}$$

Notice that Lemma 5.59 establishes that $p_!A$ is generated by terms $\mathsf{d}\mu(\vec{a})$ *as a $\mathrm{T}\mathscr{P}$-algebra* not as an A-module, like in Definition 5.56. This implies that $p_!A$ also contains terms of the form $\mathsf{d}\mu(\mathsf{d}\mu_1, \ldots, \mathsf{d}\mu_n)(\vec{a}_1, \ldots, \vec{a}_n)$. These terms are not present in $\mathrm{L}A$, since $\mathrm{L}A$ does not contain terms in which d appears twice. Notice also, that in $p_!A$ terms of the form $\mathsf{d}\mu(\mu_1, \ldots, \mathsf{d}\mu_k, \ldots, \mu_n)(\vec{a}_1, \ldots, \vec{a}_n)$ are identified with terms of the form $\mathsf{d}\mu((\mu_1)_A(\vec{a}_1), \ldots, \mathsf{d}\mu_k(\vec{a}_k), \ldots, (\mu_n)_A(\vec{a}_n))$. Consequently, the $\mathscr{P}$-algebra $\Lambda A := p^*p_!A$ is the $\mathscr{P}$-algebra generated by all $a \in A$ and by terms

$(\tau; \mathsf{d}\mu_1, \ldots, \mathsf{d}\mu_n)(a_1, \ldots, a_n)$ where $(\tau; \mathsf{d}\mu_1, \ldots, \mathsf{d}\mu_n)$ is an n-rooted tree whose vertices are decorated by terms $\mathsf{d}\mu_k$, each satisfying the relations of an infinitesimal deformator.

Since $p_! \dashv p^*$ form an adjunction, $\Lambda \colon= p^* p_!$ is a monad on $\mathsf{Alg}_{\mathscr{P}}$, thus a comonad on $\mathsf{Alg}_{\mathscr{P}}^{\mathsf{op}}$. Moreover, both $p_!$ and p^* are lax tangent morphisms so $p_! \colon \mathsf{Geom}(\mathscr{P}) \leftrightarrows \mathsf{Geom}(\mathrm{T}\mathscr{P}) \colon p^*$ form an adjunction in the 2-category of tangent categories. In particular, Λ becomes a tangent comonad, i.e. a comonad which is also a lax tangent morphism, over $\mathsf{Geom}(\mathscr{P})$. The sections of the counit $A \to \Lambda A$ of Λ are morphisms $u \colon \Lambda A \to A$ of $\mathscr{P}$-algebras which send each a to itself and each $\mathsf{d}\mu(\vec{a})$ to $\mu'_A(a_1, \ldots, a_n) \colon= u(\mathsf{d}\mu(\vec{a}))$. So, it is not hard to see that the sections of the counit of Λ classify infinitesimal deformations.

Theorem 5.60. *The tangent comonad $\Lambda \colon \mathsf{Geom}(\mathscr{P}) \to \mathsf{Geom}(\mathscr{P})$ classifies infinitesimal deformations of $\mathscr{P}$-algebras. In particular, sections of the counit $A \to \Lambda A$ are in bijective correspondence with the infinitesimal deformations of A.*

It is important to realize that ΛA is not a differential bundle. To understand why this is not the case, suppose, by contradiction that ΛA is indeed a differential bundle. According to Theorem 4.35, ΛA is then isomorphic to the free algebra under A of an A-module M, i.e. $\Lambda A \cong \mathsf{Free}_A M$. By looking at ΛA as an R-module, one concludes that M should be the A-module generated by all trees whose internal vertices are operations $\mathsf{d}\mu$, for some μ of $\mathscr{P}$ and whose leaves are elements of A. However, since M is an A-module and not a $\mathscr{P}$-algebra, any two of such trees will be *independent* generators. Take, for example, the two elements of M:

$$\mathsf{d}\mu(b, c)$$
$$\mathsf{d}\mu(a, \mathsf{d}\mu(b, c))$$

and consider a section $v \colon \mathsf{Free}_A M \to A$ of the inclusion morphism $A \to \mathsf{Free}_A M$ (regarded as morphisms in the opposite category). v sends the first tree, $\mathsf{d}\mu(b, c)$ to an element x of A and the second tree $\mathsf{d}\mu(a, \mathsf{d}\mu(b, c))$ to another element y of A. However, $v(\mathsf{d}\mu(a, \mathsf{d}\mu(b, c))$ does not have to agree with $v(\mathsf{d}\mu(a, x))$, since these two trees are independent in M. Conversely, in ΛA, these two trees are related, since the former is used in the definition of the latter. This implies that, given a section $u\colon \Lambda A \to A$ of the counit $A \to \Lambda A$ (regarded as morphisms in the opposite category):

$$u(\mathsf{d}\mu(a, \mathsf{d}\mu(b, c))) = u(\mathsf{d}\mu(a, u(\mathsf{d}\mu(b, c))))$$

In particular, this implies that ΛA and $\mathrm{L}A$ are not isomorphic algebras. However, there is a close relationship between these two bundles, since their sections are in bijection. In future work, we aim to further investigate their connection.

Chapter 6

Conclusions

This final chapter is dedicated to recalling this thesis's story, highlighting the main results, and exploring some ideas for future work.

6.1 What this thesis is about

In this thesis, we explored the interaction between the theory of operads and the one of tangent categories. The initial motivation for this work was to test whether or not tangent category theory was capable of capturing some important geometric aspects of noncommutative geometry. We showed that the opposite of the category of algebras of an operad carries a tangent structure which captures some key geometric features of operadic affine schemes. In particular, this applies to associative algebras, providing the first model of a tangent category for noncommutative geometry.

Ginzburg's work was a crucial inspiration for this work. In particular, Ginzburg's idea of a theory of operadic geometry inspired the research that led to this thesis. In some sense, our work formalizes Ginzburg's intuition for a common language of operadic geometry. On the other hand, this thesis was also fundamentally inspired by Cruttwell and Lemay's idea of employing tangent category theory to study algebraic geometry. In some sense, our work is a generalization of Cruttwell and Lemay's paper [18]. In particular, in Chapter 4 we extended their classification of differential bundles for affine schemes, to the operadic setting.

In the last chapter, we also explored some ideas to study deformation theory by employing tangent category theory. In particular, we showed how the category of operads and its opposite comes equipped with a tangent structure closely related to the infinitesimal deformations of operadic algebras. This connection between operad theory, tangent category, and deformation theory shows a deep relationship between these three distinct worlds and suggests a geometric interpretation for deformations.

The main results of this thesis can be listed as follows:

Definition 2.24 We introduced tangent display maps to avoid the use of display systems;

Theorem 2.73 We proved that the tangent category of algebras of a tangent monad represents indeed the Eilenberg-Moore object of the given tangent monad, in the sense of Street [56];

Theorem 3.26 We proved that the monad associated with an algebraic operad is a coCartesian differential monad, and therefore a tangent monad. Consequently, the category of algebras of a given operad is a tangent category, that we call, the algebraic tangent category of the operad;

Theorem 3.36 We classified vector fields of the algebraic tangent category of an operad as derivations;

Propositions 3.42 and 3.48 We showed that the operation which associates an operad to its algebraic tangent category extends to a contravariant and a covariant functor;

Theorem 3.68 We showed that the algebraic tangent category of an operad is adjunctable and consequently, the opposite of the category of algebras of an operad comes with a tangent structure to form the geometric tangent category of the given operad;

Theorem 3.74 We classified vector fields of the geometric tangent category of an operad as derivations;

Proposition 3.80 We showed that the operation which associates an operad to its geometric tangent category extends to a contravariant and a covariant functor;

Theorem 4.12 We gave a new characterization of the functor Slice which sends a pair formed by a tangent category $(\mathbb{X},\mathbb{T})$ and one of its objects A to the slice tangent category $(\mathbb{X},\mathbb{T})/A$. In particular, we showed that Slice is a right adjoint of Term;

Theorem 4.17 We showed that the geometric tangent category of the enveloping operad of a $\mathscr{P}$-algebra A is equivalent to the slice tangent category of the geometric tangent category of $\mathscr{P}$ over A;

Theorem 4.35 We classified differential bundles in the geometric tangent category of an operad as modules over the operadic algebras;

Theorem 5.22 We showed that the category of algebraic operads itself is a tangent category;

Theorem 5.38 We showed that the tangent category of operads is corepresentable and that consequently, the opposite of the category of operads also comes with a tangent structure;

Theorems 5.58 and 5.60 We proposed two alternative approaches to classifying all infinitesimal deformations of a given operadic algebra, one that involves a differential bundle $\mathrm{L}A$ and the second one which involves a tangent comonad Λ.

6.2 Future work

In this section, we explore some of the directions that this work could lead to. These are organized into four categories.

6.2.1 Operadic constructions for tangent categories

We believe we have just started to explore the relationship between tangent category theory and operad theory. In particular, we are interested in employing operad theory to explore new constructions of tangent categories. Many questions should be addressed, among which we list the following ones:

1. Koszul duality is an important operation in operad theory [46, Chapter 7]. What does Koszul duality represent for tangent category theory? Is there a similar notion of duality for tangent categories? What are the algebraic and geometric tangent categories of the Koszul dual of an operad? Can we

classify the algebraic and geometric tangent categories of Koszul operads, i.e. operads whose Koszul complex is acyclic?

2. The Hadamard product between two operads is a well-defined operation of operads [46, Section 5.3.2]. Can we define a Hadamard product for tangent categories? What are the algebraic and the geometric tangent categories of the Hadamard product of two operads?

3. A Hopf operad is a comonoid in the category of operads with respect to the Hadamard product [46, Section 5.3.3]. In particular, the tensor product of two algebras of a Hopf operad is again an algebra of the same operad. Can we classify the tangent categories associated with Hopf operads? What kind of structure does the comonoid structure add to the corresponding tangent categories?

4. A symmetric operad is associated with a Lie algebra [46, Section 5.4.3]. Could the Lie algebra of an operad $\mathscr{P}$ be related to the Lie algebra of vector fields of $\mathscr{P}$ in the tangent category $(\mathsf{Operad}, \mathbb{L})$ or equivalently, $(\mathsf{Operad}^{\mathsf{op}}, \mathbb{T})$?

5. Are cooperads also associated with tangent categories?

6. Given an operad and a cooperad one can define the convolution operad of this pair [46, Section 6.4.1]. What are the corresponding tangent categories?

7. Given two operads, one can define a new operad provided there is a distributive law between the former two operads [46, Section 8.6]. What are the corresponding tangent categories? In particular, the operad $\mathscr{Pois}$ which generates Poisson algebras is obtained from a distributive law between the operads $\mathscr{Lie}$ of Lie algebras and $\mathscr{Ass}$ of associative algebras. We are interested to see what relationship exists between the tangent categories of these three operads.

8. Ikonicoff in [31] showed that differential algebras, i.e. algebras equipped with a derivation, of a given operad $\mathscr{P}$ can be seen as algebras of another operad $\mathscr{P}'$ obtained from a distributive law between $\mathscr{P}$ and an operad D. Our classification of vector fields in the algebraic and in the geometric tangent

categories of an operad implies that the category of differential algebras of $\mathscr{P}$ is also equivalent to the category of vector fields of these two tangent categories. So, it is natural to wonder if the algebraic and the geometric tangent categories of $\mathscr{P}'$ are equivalent to the tangent categories of vector fields of the algebraic and geometric tangent categories of $\mathscr{P}$, respectively.

6.2.2 Tangent constructions for operads

We are not just interested in applications of operad theory in the context of tangent categories. In future work, we are also interested in exploring applications of tangent category theory for operads. Here, we list some of these ideas:

1. In [10], Cockett and Cruttwell introduced connections over differential bundles in the context of tangent categories. Given an operadic affine scheme, can we classify its affine connections in the geometric tangent category of the operad? What do connections tell us about a module over an operadic algebra?

2. In this thesis we classified differential bundles and vector fields in the geometric tangent category of an operad. One can also define two cohomology theories, the cohomology of differential forms and the cohomology of sector forms, for objects in a tangent category (see [17]). On the other hand, operadic algebras also admit a cohomology theory (see [46, Section 12.3.11]). Does the latter correspond to either of the former two?

3. In [14], Cockett and Cruttwell introduced a notion of ordinary differential equations in the context of tangent categories (cf. [14]). An important ingredient required to define the solutions of a differential equation is represented by a curve object. It is not clear whether or not the geometric tangent category, or more realistically a suitable tangent subcategory, admits a curve object. In an informal discussion, Cruttwell pointed out that there might be a curve object which is "infinitesimal", in the sense that it captures infinitesimal paths. Can we distinguish between "infinitesimal curve objects" and "real curve objects"?

4. In this thesis, we also show that the category of operads itself and its opposite come equipped with tangent structures. We already noticed that vector fields

in these two tangent categories are equivalent to derivations of operads. In future work, we are interested in classifying differential bundles of these two tangent categories. We conjecture they could be related to a suitable notion of bimodules over operads. A similar question regards the classification of connections, the cohomology, and the differential equations.

6.2.3 Towards a formal theory of tangent objects

In Section 2.3 we introduced the notion of tangent objects in a strict 2-category. This approach was partially inspired by the formal approach to monad theory presented by Street in [56]. In future work, we plan to investigate some ideas that could lead to a formal approach to tangent category theory. Here is a list of some of the directions of research we would like to pursue:

1. The arrow category of a tangent category comes equipped with a tangent structure. In the 2-category of categories, the arrow category is a comma object (see [55]). Is the arrow tangent category a comma object in the 2-category of tangent categories? When does the 2-category of tangent objects admit comma objects?

2. We introduced tangent objects in the context of a strict 2-category. Can we extend this notion to a bicategory or a double category?

3. The category of differential bundles of a tangent category form another tangent category (see [11]). Can we define the tangent object of differential bundles of a given tangent object? What are differential bundles of tangent monads, regarded as tangent objects? What about this notion applied to other classes of tangent objects, like tangent fibrations?

4. The category of vector fields and the one of affine connections also form tangent categories (see [14] and [8]). Can we extend these constructions for tangent objects?

5. There is a canonical inclusion 2-functor which sends an object $\mathbb{X}$ of a 2-category $\mathbf{C}$ to the trivial tangent object $(\mathbb{X}, \mathbb{1})$, i.e. the tangent object with the

trivial tangent structure. Does this 2-functor admit a right adjoint? What is the meaning of such a right adjoint?

6. In [22], tangent categories are interpreted as suitable enriched categories. What is the relationship between the enrichment point of view and the tangent object approach?

6.2.4 Deformation theory

In Chapter 5 we explored some ideas to employ tangent category theory to study deformation theory. In future work, we would like to explore this intuition further. Formal deformations are deformations of algebras over the R-augmented ring $S = R[\![x]\!]$ of formal power series. In particular, for associative algebras, the deformed associative multiplication can be expanded into a power series like:

$$a \star b = \sum_{n=0}^{\infty} \mu_n(a, b)t^n$$

Obstruction theory explored by Gerstenhaber in [23] establishes conditions for whether or not an infinitesimal deformation can be expanded into a formal deformation. Our intuition suggests an analogy with geodesic completeness in differential geometry. Our approach to deformation theory would like to interpret infinitesimal deformations of an algebra as vector field-like objects. So, the process of finding a formal deformation could be regarded as solving a differential equation whose initial values are established by the given infinitesimal deformation. This could be related to the problem of finding global solutions for geodesics.

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