SPLIT RELIABILITY

by

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Abstract

The split reliability of a graph G is the probability that if every edge is independently operational with probability p, every vertex is always operational, and we specify two vertices s and t, that every vertex can communicate with exactly one of s or t. The split reliability is a polynomial. First, we find explicit formulas for the split reliability for various families of graphs. We show that finding split reliability polynomials is intractable. We prove that split reliability polynomials are always alternating in sign. We also find some lower and upper bounds for the split reliability polynomial. Finally, we prove that the value of p that maximizes the probability of split reliability and the maximum probability of split reliability are both dense in the interval $[0, 1]$, though this is not the case for all families of graphs.

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Chapter 1

Introduction

1.1 Graph Theory and Reliability Polynomials Background

A graph $G = (V, E)$ consists of a finite vertex set V and a finite multiset E consisting of unordered pairs and singletons of V (the singletons in E are called *loops*). The set of edges of G that have the same endpoints as edge e of G is denoted by $[e]$ and $|[e]|$ is called the *multiplicity* of $[e]$ (we often refer to $[e]$ as a *bundle*). A graph without loops and where every edge has multiplicity one is called *simple.* If $H = (V, E')$ is a subgraph of G where E' consists of one edge from each bundle and no loops, then we call H the *underlying simple graph* of G.

The *order* and *size* of a graph G are $|V|$ and $|E|$ respectively. If $G = (V, E)$ is a graph, and e is an element of E, then $G - e = (V, E - \{e\})$, and $G * e$ is the graph formed from $G - e$ by identifying the endpoints of e into a new vertex v_e (these two operations are called the *deletion of* e and the *contraction of* e, respectively). We extend this definition to $G - [e]$ and $G * [e]$ in the natural way, and observe that if G is loopless then $G * [e]$ (and $G - [e]$) is loopless as well (though $G * e$ may have loops even if G does not).

A *walk* is a finite alternating sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_k, v_k$ where the ends of e_i are v_{i-1} and v_i for $i = 1, \ldots, k$; the *length* of the walk is k, the number of edges in the walk. A *trail* is a walk where every edge is distinct, and a *path* is a walk where every vertex is distinct. We say that two vertices are *connected* if there exists a path between them. A subgraph $G' = (V', E')$ of G, where E' consists of all edges in E between the vertices in V', is called a *component* if every pair of vertices in V' is connected (in G'), and no vertex in V' is adjacent in G to any vertex in $V - V'$. If G only has one component, we say that G is *connected.*

A cycle is a non-empty trail in which the only repeated vertices are the first and last vertices. A tree is a connected graph that contains no cycles, and a tree of order n always has size $n - 1$. Moreover, removing any $k \leq |E|$ edges from a tree will result

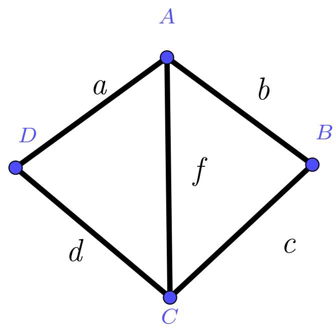

Figure 1.1: Graph G_1

in the tree being split into $k+1$ components [7, pg. 69].

In this thesis, I will be exploring a modified version of a reliability problem on a network. In the original problem, you would have a graph G (for our purposes, always finite and undirected, possibly with loops and multiple edges), and we would analyze the probability of the graph being in a specific "working" (or "operational") state, with each edge independently having a probability p of being "up", while all vertices are always up. For example, the *all-terminal reliability* problem is the probability that all vertices in G can communicate, and in the *two-terminal reliability* problem we would choose two vertices s and t and find the probability that s and t can communicate. In a more general setting, let G have vertex set V and edge (multi)set E, and let K be a non-empty subset of vertices. Then the *K-terminal reliability* of G is given by

$$\mathrm{Rel}_K(G;p) \;=\; \sum_{E'} p^{|E'|}(1-p)^{|E-E'|}, \tag{1.1}$$

where the sum is over all subsets E' of edges of G that contain paths between all pairs of vertices in K. Clearly the all-terminal reliability and two terminal reliability correspond to when $K=V$ and $|K|=2$, respectively (we always use $\mathrm{Rel}_V(G;p)$ for the former, and for the latter, if $K=\{s,t\}$, we simply write $\mathrm{Rel}_{s,t}(G;p)$ for $\mathrm{Rel}_{\{s,t\}}(G;p)$). Note that from (1.1), K-terminal reliability is identically 1 if $|K|=1$, and more general is always a polynomial in p of degree at most $m=|E(G)|$.

Consider the graph G_1 in Figure 1.1. To calculate the all-terminal reliability, we can count all of the possible operational states with i up edges, and multiply by the probability of each state occurring (as the probability of a particular state occurring only depends on its number of edges). There is one operational state with 5 edges, five with 4 edges, eight with 3 edges, and none with fewer than 3 edges (as you need at least a spanning tree up). Thus we have

$$\begin{aligned}\mathrm{Rel}_V(G_1;p) &= p^5+5p^4(1-p)+8p^3(1-p)^2\\ &= 4p^5-11p^4+8p^3\end{aligned}$$

as the all-terminal reliability of this graph.

To calculate the two-terminal reliability, we can use the same process of counting operational states, except that now an operational state is when our choices of s and t can communicate. For G_1, consider D as s and B as t. There are two operational states with 2 edges, eight with 3 edges, five with 4 edges (if we only remove one edge, no matter what edge we choose, s and t can still communicate), and one with all edges active. Any fewer edges active and s and t cannot communicate since the distance between s and t is 2. Thus we have

$$\begin{aligned}\mathrm{Rel}_{D,B}(G_1;p) &= p^5+5p^4(1-p)+8p^3(1-p)^2+2p^2(1-p)^3\\ &= 2p^5-5p^4+2p^3+2p^2\end{aligned}$$

as the two-terminal reliability of this graph in terms of B and D.

Another method we will be using to calculate all forms of reliability is explained in the following theorem, called the **Factor Theorem for Reliability** [6]. Note that the theorem is stated in terms of choosing a non-loop edge, as loops can be freely deleted without changing the reliability.

Theorem 1.1. *Consider an undirected graph G and a subset of vertices K. Let $e=\{x,y\}$ be any non-loop edge of G. Then*

$$Rel_K(G;p) = p\cdot Rel_{K\bullet e}(G*e;p)+(1-p)\cdot Rel_K(G-e;p), \tag{1.2}$$

where $K\bullet e$ is K unless both endpoints of e are in K, in which case we remove both of the endpoints of e from K and add in v_e (so $K\bullet e=(K-\{x,y\})\cup v_e$). □

The theorem follows naturally since this is considering the probability of reliability holding in two mutually exclusive situations, one where e is up and one where e is not.

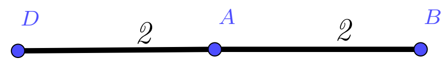

Figure 1.2: $G_1 * f$, the numbers on each edge are the number of edges in the bundle.

The resulting graphs $G * e$ and $G - e$ depend on our specific choice of e, as deleting or contracting different edges in G will in general result in different graphs (as well, even if G is a simple graph, the contraction of e may yield multiple edges). It may be possible in certain situations to strategically choose a specific edge ordering when using Theorem 1.1 repeatedly in order to make reliability easier to calculate.

For example, consider again all-terminal reliability on G_1, focusing on edge f. The graphs $G_1 * f$ and $G_1 - f$ are shown in Figures 1.2 and 1.3, respectively. Note that $\mathrm{Rel}_{V*f}(G_1 * f; p) = (1-(1-p)^2)^2$ as we need at least one edge active between D and A and at least one edge active between A and B in order for all of the vertices to communicate. Since both (A, B) and (D, A) have two edges between them, the probability of none of the edges being active in both of these cases is $(1-p)^2$, so the probability of at least one edge being active between (D, A) or (A, B) is $1-(1-p)^2$. Thus the probability of all of the vertices communicating in $G_1 * f$ is $(1-(1-p)^2)^2$. We also note that $\mathrm{Rel}_V(G_1 - f; p) = p^4 + 4p^3(1-p)$ as we can either have all edges active or all but one edge active (in the latter case, the resulting subgraph is a path of length 3 and there are 4 choices of an edge to remove). If we remove any more edges from $G_1 - f$ it will result in a subgraph with 4 vertices and 2 edges, thus being disconnected. From Theorem 1.1 we derive that

$$\begin{aligned}\mathrm{Rel}_V(G_1; p) &= p(1-(1-p)^2)^2 + (1-p)(p^4 + 4p^3(1-p)) \\ &= 4p^5 - 11p^4 + 8p^3,\end{aligned}$$

which concurs with our previous calculation.

We should note that in the calculation above we introduced multiple edges into our graphs where none existed to begin with. Also, any loops that form when we contract e can always be removed from the graph without affecting the split reliability, as we always assume that vertices can communicate with themselves.

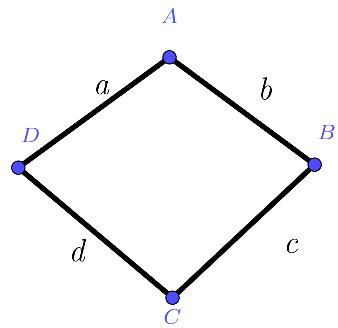

Figure 1.3: $G_1 - f$

Now let's consider a two-terminal reliability problem on our graph with A as s and C as t (i.e. $K = \{s, t\}$). Using Theorem 1.1 with f as e, we first note that

$$K \bullet f = (\{A, C\} - \{A, C\}) \cup \{v_e\} = \{v_e\},$$

so $K \bullet f$ is a singleton, and $\mathrm{Rel}_{v_1}(G_1 * f; p) = 1$. We also see that $\mathrm{Rel}_{\{A,C\}}(G_1 - f; p) = p^4 + 4p^3(1-p) + 2p^2(1-p)^2$ since in order for A and C to communicate in $G_1 - f$, we can either have all edges active, 3 edges active with any 1 edge down, or only 2 edges up where the edges active form one of the two paths between A and C. From Theorem 1.1 we derive that

$$\begin{aligned}\mathrm{Rel}_{A,C}(G_1; p) &= p(1) + (1-p)(p^4 + 4p^3(1-p) + 2p^2(1-p^2)) \\ &= p^5 - p^4 - 2p^3 + 2p^2 + p,\end{aligned}$$

Now consider a different choice of s and t. Let D be s and let B be t, using the same method as before with edge f once again acting as our e, we first note that in order for B and D to communicate in $G_1 * f$, at least one edge in each bundle needs to be active. We found the likelihood of this happening when we found the all-terminal reliability of G_1. So $\mathrm{Rel}_{\{B,D\}}(G_1 * f; p) = (1 - (1-p)^2)^2$. Next, we see that to calculate $\mathrm{Rel}_{B,D}(G_1 - f; p)$, we need the probability that two vertices on opposite sides of C_4 can communicate. However, this is the same as $\mathrm{Rel}_{A,C}(G_1 - f; p)$ which was calculated in the last example as $p^4 + 4p^3(1-p) + 2p^2(1-p)^2$. From Theorem 1.1, we derive that the two terminal reliability is

$$\begin{aligned}\mathrm{Rel}_{B,D}(G_1; p) &= p(1-(1-p)^2)^2 + (1-p)(p^4 + 4p^3(1-p) + 2p^2(1-p)^2) \\ &= 2p^5 - 5p^4 + 2p^3 + 2p^2,\end{aligned}$$

Figure 1.4: Graph G_2

so we see that the value of the two-terminal reliability is dependent on our choices of s and t.

It will also be useful to consider two families of graphs, trees and cycles. As an example of a of tree, let's consider the path G_2 in Figure 1.4. We can see that for the graph G_2, all vertices can communicate if and only if all edges are up. It can easily be seen that this extends to all trees, since in order for two vertices in a tree to be able to communicate, the unique path between them in the tree must have all edges up. So if a tree has n vertices and $n-1$ edges, the all-terminal reliability of the tree must be p^{n-1}. This logic also extends to two-terminal reliability of trees. Since for the two terminals to communicate, the unique path between them must be up, so if the path between s and t has length k, the two-terminal reliability in this case is p^k (the status of the other edges is inconsequential).

The graph G_3 in Figure 1.5 is an example of a cycle. We can see that in order for all vertices in a cycle to communicate, we can have all edges up or any one edge down (in this case we end up with a path like the last example). Any more edges being down would lead to the graph being separated into at least two different components. So if a cycle has n edges (and vertices), the all-terminal reliability of that cycle is

$$p^n + np^{n-1}(1-p).$$

For the two-terminal reliability of a cycle with vertices s and t, let a shortest path P between s and t have length r, so the longer path is of length $m-r$. For two-terminal reliability to hold here we need either the shorter or longer path to be active. So the two-terminal reliability is

$$p^r + (1-p^r)p^{n-r} = p^r + p^{n-r} - p^n.$$

as p^r is the probability that P is up and $(1-p^r)p^{n-r}$ is the probability that P is not up but the other path is.

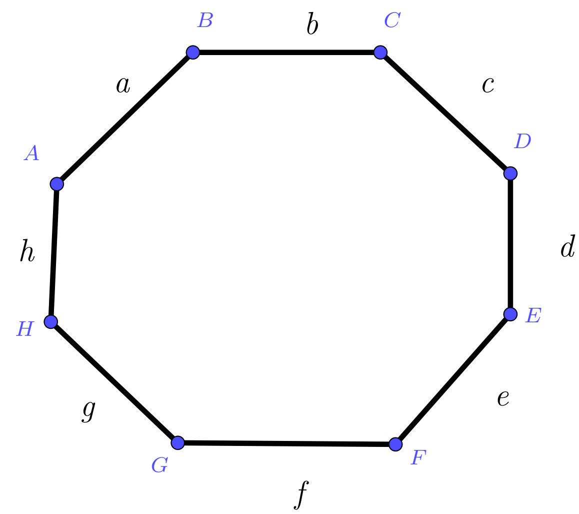

Figure 1.5: Graph G_3

Reliability polynomials have applications in measuring the robustness of the structure of networks. For example, if you consider the vertices to be computers and the edges to be the connections between these computers, the all-terminal reliability will yield the probability that all of these computers can communicate, given that every computer always works and every connection has an equal but independent probability of failing; the two-terminal reliability will be the probability that two specific computers can communicate. You can also change certain aspects of the problem in order to make it more applicable to real life; for example, you can make the probabilities of edges being up different or dependent on the status of one another. However, such choices would require alterations to our models of reliability.

It is important to note that both all-terminal and two-terminal reliability functions have been shown to have specific analytic properties. For both reliability problems, it is known that they are increasing on the interval $(0, 1)$ (as we increase the likelihood of edges being up, it can only increase the probability that vertices can communicate). Assuming that the reliability polynomial is not identically 0 or 1, we have that at $p = 0$ the polynomial is 0 and at $p = 1$ the polynomial is 1 (if no edges are active it is impossible for the required vertices to communicate and if the polynomial is not identically 0, there must be a subgraph that connects all of the vertices, so if all of the edges are active, the specified vertices must be able to communicate). In most instances, all-terminal reliability is concave up near 0 and concave down near 1, and has a unique fixed point in $(0,1)$ (see [1, 3]). Moreover, the coefficients of an all-terminal reliability polynomial are always (strictly) alternating in sign[3].

1.2 Introducing Split Reliability

Now that we have covered some basics of reliability problems, we can talk about the new form of reliability we are considering. Specify two vertices s and t in our graph G. Given that edges are independently up with probability p (but the vertices are always up), the *split reliability* of G with terminals s and t is the likelihood that every vertex in the graph can communicate with either s or t but not both (that is, the likelihood that the graph will be split into exactly two components with one containing s and one containing t). We will use $\mathrm{splitRel}_{s,t}(G;p)$ to represent the split reliability polynomial of G with a specific choice of s and t. Again, the resulting function is indeed a polynomial in p, as one can enumerate and add up the probabilities of each of the working states with respect to this new measure of operability.

For an example of counting the operational states, we can calculate the split reliability of the graph C_4 as seen in Figure 1.3 with A as s and D as t. We can see that we have three operational states, all with only two edges active. We either have edges b and c active, edges c and d active, or edges b and d active. Any other combination of two edges will lead to A and D being able to communicate, any fewer edges will leave either B or C unable to communicate with A and D, and any more edges will lead to A and D communicating. So we have three operational states, all of which have probability $p^2(1-p)^2$ of being active so

$$\begin{aligned} \mathrm{splitRel}_{A,D}(C_4;p) &= 3p^2(1-p)^2 \\ &= 3p^4 - 6p^3 + 3p^2 \end{aligned}$$

is the split reliability of this graph in terms of A and D. A list of all the different split reliability polynomials for all small simple graphs is given in Table 1.1.

Split reliability opens up some interesting new applications. If we consider using graphs to model computer networks, the split reliability of the graph would be a way of giving the probability of two different computers sending information to all other

order n	split reliability polynomials
2	$-p+1$
3	$-2p^2+2p, -p^2+p, 2p^3-4p^2+2p$
4	$-3p^3+3p^2, -2p^3+2p^2, -p^3+p^2, 2p^4-5p^3+3p^2, 2p^4-4p^3+2p^2,$ $3p^4-6p^3+3p^2, 4p^4-9p^3+5p^2, 4p^4-8p^3+4p^2, -6p^5+20p^4-22p^3+8p^2,$ $-4p^5+12p^4-12p^3+4p^2, -4p^5+13p^4-14p^3+5p^2,$ $6p^6-26p^5+42p^4-30p^3+8p^2$
5	$-4p^4+4p^3, -3p^4+3p^3, -2p^4+2p^3, -p^4+p^3, 2p^5-5p^4+3p^3,$ $2p^5-4p^4+2p^3, 3p^5-7p^4+4p^3, 3p^5-6p^4+3p^3, 4p^5-10p^4+6p^3,$ $4p^5-9p^4+5p^3, 4p^5-8p^4+4p^3, 6p^5-14p^4+8p^3, 6p^5-13p^4+7p^3,$ $6p^5-12p^4+6p^3, 7p^5-15p^4+8p^3, -10p^6+32p^5-34p^4+12p^3,$ $-10p^6+33p^5-36p^4+13p^3, -10p^6+35p^5-41p^4+16p^3,$ $-8p^6+24p^5-24p^4+8p^3, -8p^6+26p^5-28p^4+10p^3,$ $-8p^6+27p^5-31p^4+12p^3, -8p^6+28p^5-33p^4+13p^3,$ $-8p^6+28p^5-32p^4+12p^3, -7p^6+22p^5-23p^4+8p^3,$ $-6p^6+18p^5-18p^4+6p^3, -6p^6+19p^5-20p^4+7p^3,$ $-6p^6+20p^5-22p^4+8p^3, -4p^6+12p^5-12p^4+4p^3,$ $-4p^6+13p^5-14p^4+5p^3, -4p^6+14p^5-16p^4+6p^3,$ $-4p^6+15p^5-19p^4+8p^3, 6p^7-30p^6+57p^5-49p^4+16p^3,$ $6p^7-26p^6+42p^5-30p^4+8p^3, 8p^7-36p^6+60p^5-44p^4+12p^3,$ $8p^7-36p^6+61p^5-46p^4+13p^3, 8p^7-35p^6+58p^5-43p^4+12p^3,$ $8p^7-34p^6+54p^5-38p^4+10p^3, 8p^7-32p^6+48p^5-32p^4+8p^3,$ $10p^7-45p^6+76p^5-57p^4+16p^3, 10p^7-43p^6+69p^5-49p^4+13p^3,$ $10p^7-42p^6+66p^5-46p^4+12p^3, 12p^7-56p^6+99p^5-79p^4+24p^3,$ $12p^7-54p^6+91p^5-68p^4+19p^3, 12p^7-54p^6+92p^5-70p^4+20p^3,$ $12p^7-52p^6+84p^5-60p^4+16p^3, 14p^7-64p^6+110p^5-84p^4+24p^3,$ $14p^7-62p^6+103p^5-76p^4+21p^3,$ $-18p^8+100p^7-222p^6+246p^5-136p^4+30p^3,$ $-18p^8+102p^7-232p^6+265p^5-152p^4+35p^3,$ $-14p^8+76p^7-165p^6+179p^5-97p^4+21p^3,$ $-14p^8+78p^7-174p^6+194p^5-108p^4+24p^3,$ $-12p^8+64p^7-136p^6+144p^5-76p^4+16p^3,$ $-12p^8+66p^7-146p^6+162p^5-90p^4+20p^3,$ $-12p^8+66p^7-145p^6+159p^5-87p^4+19p^3,$ $-12p^8+68p^7-155p^6+178p^5-103p^4+24p^3,$ $18p^9-120p^8+334p^7-497p^6+417p^5-187p^4+35p^3,$ $18p^9-118p^8+322p^7-468p^6+382p^5-166p^4+30p^3,$ $24p^9-162p^8+456p^7-686p^6+582p^5-264p^4+50p^3,$ $-24p^{10}+186p^9-618p^8+1142p^7-1268p^6+846p^5-314p^4+50p^3$

Table 1.1: The split reliability polynomials of all connected simple graphs, with any choice of s and t, of order n (that is, with n vertices).

computers in the network such that all computers get one of the messages, but no two computers get both sets of information (this might be imposed as the two messages might be somewhat conflicting). If we take all-terminal reliability into account, we can also model the probability that at least one of the computers in the network will get neither of the messages ($1 - \mathrm{Rel}_V(G;p) - \mathrm{splitRel}_{s,t}(G;p)$). Split reliability has been previously introduced to study roots of all-terminal reliability polynomials. To be more specific, if we have the all-terminal reliability of a loopless graph G of size m, and we want the all-terminal reliability of the graph formed from G by replacing each edge e by a fixed graph H with a fixed pair of terminals s and t (by identifying s and t in a 1-1 way with the endpoints of e), finding the split reliability of H can aid in that calculation. In particular [2, pg. 1293], if $G[H(s,t)]$ is the resulting graph, then

$$\begin{aligned}\mathrm{Rel}_V(G[H(s,t)];p) &= \left(\mathrm{Rel}_V(H;p) + \mathrm{splitRel}_{s,t}(H;p)\right)^m \cdot \\ &\quad \mathrm{Rel}_V\left(G; \frac{\mathrm{Rel}_V(H;p)}{\mathrm{Rel}_V(H;p) + \mathrm{splitRel}_{s,t}(H;p)}\right).\end{aligned}$$

In terms of calculating split reliability, it is not hard to see that we get a theorem analogous to that of Theorem 1.1, which we state in the following more useful, general form.

Theorem 1.2 (The Factor Theorem for Split Reliability). *Consider a loopless undirected graph G and distinct vertices s and t of G. Let e be an edge of G with multiplicity $\nu \geq 1$. If $e = \{s,t\}$, then*

$$\mathit{splitRel}_{s,t}(G;p) = (1-p)^{\nu}\,\mathit{splitRel}_{s,t}(G-[e];p),$$

and otherwise

$$\begin{aligned}\mathit{splitRel}_{s,t}(G;p) &= (1-(1-p)^{\nu}) \cdot \mathit{splitRel}_{s,t}(G*[e];p) + \\ &\quad (1-p)^{\nu} \cdot \mathit{splitRel}_{s,t}(G-[e];p).\end{aligned}$$

Proof. The same argument that holds for Theorem 1.1 holds here as well. The one difference, is if we are considering contracting an edge that is connecting s and t –

if this edge is up, s and t can communicate, so $\text{splitRel}_{s,t}(G * [e]; p) = 0$. Thus if $[e]$ connects s and t, then all edges in $[e]$ must be down, so

$$\text{splitRel}_{s,t}(G; p) \quad = \quad (1-p)^{\nu} \cdot \text{splitRel}_{s,t}(G - [e]; p).$$

Now assume that not both ends of e are in $\{s, t\}$. Let $[e]$ contain exactly $\nu \geq 1$ edges. For this proof, we will need the following result: Let A and B be events. Then

$$Prob(A) \quad = \quad Prob(B)Prob(A|B) + Prob(\bar{B})Prob(A|\bar{B}).$$

(For a proof see [4, pg. 80].) If we let A be the event that every vertex in G communicates with exactly one of s and t, and let B be the event that at least one edge from $[e]$ is up, we can see that

$$\begin{aligned} Prob(A) &= \text{splitRel}_{s,t}(G; p) \\ Prob(B) &= 1 - (1-p)^{\nu} \\ Prob(\bar{B}) &= (1-p)^{\nu} \\ Prob(A|B) &= \text{splitRel}_{s,t}(G * [e]; p) \\ Prob(A|\bar{B}) &= \text{splitRel}_{s,t}(G - [e]; p) \end{aligned}$$

So, if $[e]$ is not between s and t, we have that

$$\begin{aligned} \text{splitRel}_{s,t}(G; p) \quad = \quad & (1 - (1-p)^{\nu}) \cdot \text{splitRel}_{s,t}(G * [e]; p) + \\ & (1-p)^{\nu} \cdot \text{splitRel}_{s,t}(G - [e]; p). \end{aligned}$$

$\square$

Having this theorem will be a very useful tool for calculating split reliability as we can take complicated graphs and simplify by working instead with two simpler (and smaller) graphs (the base cases for the algorithm are for edgeless graphs, which have split reliability of 1 if G has order 2, and 0 otherwise, although in practice we can often stop earlier than that).

For example, let's look back at our graph G_1 in Figure 1.1, with A as s and C as t. We notice that edge f must be down since we do not want s and t to communicate. So the split reliability of G_1 is $(1-p)$ times the split reliability of a cycle of size 4 with s and t on opposite sides. Like before, we can list the possible operational states for split reliability. That is, both of the other vertices communicate with s only, both communicate with t only, or one communicates with s only and the other communicates with t only. Listing the operational states, we get

$$p^2(1-p)^2 + p^2(1-p)^2 + 2p^2(1-p)^2 \quad = \quad 4p^2(1-p)^2$$

So we see that the split reliability is

$$\begin{aligned} \mathrm{splitRel}_{s,t}(G_1; p) &= (1-p)(4p^2(1-p)^2) \\ &= -4p^5 + 12p^4 - 12p^3 + 4p^2. \end{aligned}$$

Alternatively, let's choose B as s and D as t. We can again use Theorem 1.2 on edge f and then list the operational states for both of the new graphs we create. If we contract f, we have a path of length 2 with a bundle of 2 edges between each adjacent vertex and s and t at the endpoints of the path. So in order to split the graph into two components with one containing s and one containing t, we must have one of the bundles down and one of the bundles with at least one edge up. The probability of all edges being down in a bundle of 2 edges is $(1-p)^2$ so the probability of at least one edge being up in the bundle is $1-(1-p)^2$. Thus the probability of split reliability holding with f active is $2(1-(1-p)^2)(1-p)^2$. The graph with f deleted is a cycle with four edges with nonadjacent terminals, its split reliability, $4p^2(1-p)^2$, was calculated earlier. Therefore the split reliability is

$$\begin{aligned} \mathrm{splitRel}_{s,t}(G_1; p) &= p(2(1-(1-p)^2)(1-p)^2) + (1-p)(4p^2(1-p)^2) \\ &= -6p^5 + 20p^4 - 22p^3 + 8p^2. \end{aligned}$$

Thus the split reliability of a graph, like two-terminal reliability, is dependent of our choices of s and t.

Looking back at the other families of graphs we discussed before, let's consider the split reliability of trees. We need that every vertex must communicate with s or t. If no edges are down, s and t can communicate; if two or more are down the tree will be split into more than two components and we must have a vertex that cannot communicate with s or t. So we must have only one edge down, and in order for s and t to be in two different components, the edge down must be on the unique path between s and t. Thus if a tree has n vertices (and $n-1$ edges) and the path between s and t has length k, the split reliability of a tree is $kp^{n-2}(1-p)$.

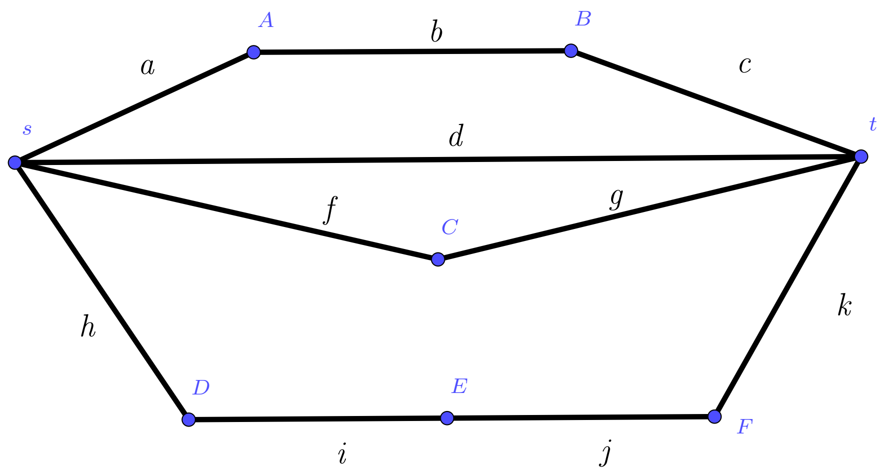

Figure 1.6: Graph G_4

As for cycles, for any choice of s or t we have two paths between s and t. So we must have one edge down on each path in order for s and t to be in separate components. Thus, for split reliability to hold, we need to have exactly two edges down, one on each path between s and t. Suppose our cycle has n edges (and thus n vertices), with k as the length of a shorter path between s and t. We have $k(n-k)$ choices for the two edges we take down (one on each path). So the split reliability of a cycle is $k(n-k)p^{n-2}(1-p)^2$.

Consider our previous calculations for the split reliability of a cycle of size 4 with s and t on opposite sides of the cycle. Our preceding remarks show that the split reliability polynomial is $4p^2(1-p)^2$ which is equal to $(2)(4-2)p^{4-2}(1-p)^2$. We extend our argument to a more general family of graphs. Consider two vertices s and t; join these by l internally disjoint paths. For an example of this kind of graph, see G_4 in Figure 1.6. Such a graph is called a generalized θ-graph, $\theta(m_1, \ldots, m_l)$, where the paths lengths are $m_1, \ldots, m_l$ (so $G_4 = \theta(3, 1, 2, 4)$).

To calculate the split reliability of such a graph with terminals s and t, we see that every vertex on each path must communicate with either s or t but not both. So, we must have exactly one edge down on each path. Any less and s and t can communicate, any more and we must have that a vertex cannot communicate with s or t. Let the graph have n vertices and m edges, with L as the number of paths between s and t. We will number each of these paths from 1 to l and let m_i be the length of path i. We see that the split reliability of this graph is

$$\left(\prod_{i=1}^{l} m_i\right) p^{m-l}(1-p)^l.$$

This formula extends that for cycles since a cycle is this kind of graph with exactly two paths between s and t.

Finally, we should note that all of these examples above are for connected graphs, but what if we are dealing with a disconnected graph? If a graph has three or more components, no matter what our choice of s or t is, $\text{splitRel}_{s,t}(G;p) = 0$ since we must have one vertex in a component that can never connect to s or t. This is the same if our graph has two components with s and t in the same component. If we have two components with s and t in different components, the split reliability becomes the probability that all vertices in each component can communicate, so the split reliability becomes the product of the all-terminal reliability of both components. The most interesting case is therefore the one left, namely when the graph is connected.

Now that we have a good understanding of how split reliability works, we can begin discussing more specific properties of split reliability polynomials.

Chapter 2

The Split Reliability Polynomial

We begin by showing that calculating split reliability is intractable. We then consider the coefficients of split reliability polynomials, and prove that they are alternating in sign. Finally, we introduce a new formula for split reliability polynomials for connected graphs (as well as using this formula to find formulas for specific families of graphs).

2.1 The Intractability of Calculating Split Reliability

Clearly determining the split reliability is easy for $p = 0$ – it is 0 unless the only vertices of G are s and t (in which case the split reliability is 1). Likewise, the value of the split reliability at $p = 1$ is 0 unless G is disconnected, with exactly two components, one containing s the other containing t (and in which case the value is 1). What about other values? What about the value of split reliability in general? We have seen how to use the Factor Theorem for split reliability (Theorem 1.2) but it generates two graphs for every one, and thus yields an exponential algorithm for split reliability.

It is known that the following problem:

ALL-TERMINAL RELIABILITY
Input: A graph G
Output: $\mathrm{Rel}_V(G; p)$

is $\#P$-complete (for details about the class $\#P$ and the argument, see [5]). We can use this to show that

SPLIT RELIABILITY
Input: A graph G and distinct vertices s and t of G

Output: $\mathrm{splitRel}_{s,t}(G; p)$

is also $\#P$-complete.

Theorem 2.1. *Calculating split reliability is $\#P$-complete.*

Proof. The problem is in $\#P$ as, like the argument for ALL-TERMINAL RELIABILITY, any state of the graph (that is, any spanning subgraph) can be verified as "operational" (i.e. one in which every vertex can communicate with exactly one of s and t) or not in polynomial time for the following reason. We start with the vertex s then use breadth-first search to find the component containing s. If t belongs to this component, then the state is not operational. Otherwise, use the same process to find the component containing t. If there exists a vertex v that is not connected to s or t then split reliability fails, otherwise the state is operational.

Now suppose we take an instance G for ALL-TERMINAL RELIABILITY; without loss, G is connected (as otherwise the all-terminal reliability of G is identically 0). Using G we can make a new graph, $\hat{G}$, that is composed of two copies of graph G with s in one copy of G and t in the other. The construction of $\hat{G}$ can clearly be done in polynomial time. We now have $\hat{G}$ as a graph with two components, one containing s and the other containing t. From chapter 1, we know that the split reliability of $\hat{G}$ is the product of the all-terminal reliabilities of each component. So, as

$$\mathrm{splitRel}_{s,t}(\hat{G}; p) = (\mathrm{Rel}_V(G; p))^2,$$

if we had a polynomial time algorithm for split reliability, we would be able to find $\mathrm{Rel}_V(G; p)$ in polynomial time. Thus SPLIT RELIABILITY must be $\#P$-hard.

We can also show that calculating split reliability is $\#P$-hard even when we restrict ourselves to connected graphs, by again reducing to ALL-TERMINAL RELIABILITY. Again, we will take any connected graph G. In this graph we will label any vertex s, and attach a new vertex t to s via an edge. Call this new graph G^* (clearly G^* can be constructed from G in polynomial time). Looking at this new graph, we can see that

$$\mathrm{splitRel}_{s,t}(G^*; p) = (1 - p) \cdot \mathrm{Rel}_V(G; p),$$

since we see that in G^*, every vertex communicates with exactly one of s and t if and only if every vertex of G communicates with s and the edge between s and t fails.

Thus

$$\text{splitRel}_{s,t}(G^*;p) = (1-p)\cdot \text{Rel}_V(G;p),$$

that is,

$$\text{Rel}_V(G;p) = \frac{\text{splitRel}_{s,t}(G^*;p)}{1-p}.$$

So if we can find split reliability in polynomial time, we can find the all-terminal reliability of G in polynomial time. Thus SPLIT RELIABILITY must still be $\#P$-complete even when we restrict ourselves to connected graphs. □

2.2 The Alternating Sign Theorem

Even though computing split reliability is difficult, there still can be much we can say about the function. We start with a fascinating property that holds for all split reliability polynomials. We begin with an elementary result about *alternating (in sign) polynomials* (that is, those where between the highest and lowest degree of the polynomial there are no zero coefficients, and the nonzero coefficients alternate in sign).

Lemma 2.2. *The product of two alternating polynomials is an alternating polynomial.*

□

Proof. Suppose

$$f(x) = \sum_{i=k}^{c} a_i x^i$$

and

$$g(x) = \sum_{j=l}^{d} b_j x^j$$

are two alternating polynomials (so all of a_is and b_js are nonzero). Then as

$$f(x)g(x) = a_k b_l x^{k+l}\left((1+\sum_{i=1}^{c-k} a_i x^i)(1+\sum_{j=1}^{d-l} b_j x^j)\right),$$

we can assume that

$$f(x) = \sum_{i=0}^{c} a_i x^i$$

and

$$g(x) = \sum_{j=0}^{d} b_j x^j,$$

where $a_0 = b_0 = 1$, so that for all i and j, the sign of a_i and b_j are $(-1)^i$ and $(-1)^j$, respectively. However, then the coefficient of x^k in $f(x)g(x)$ is

$$\sum_{i+j=k} a_i b_j,$$

each summand of which has sign $(-1)^k$. It follows that $f(x)g(x)$ is also alternating in sign. $\square$

We are now ready to prove our main result (as loops do not affect split reliability, we can remove them first and assume all graphs under question are loopless).

Theorem 2.3. *If G is a loopless graph of order $n \geq 2$ and size m, then $splitRel_{s,t}(G;p)$ is an alternating polynomial, and if G is connected*

$$splitRel_{s,t}(G;p) \quad = \quad \sum_{i=n-2}^{m} (-1)^{n+i} a_i p^i, \tag{2.1}$$

where each a_i is a positive integer.

Proof. We will consider two different cases, one where G is connected and one where it is not.

If G is not connected, we already know from the first chapter that the split reliability is either 0 (and trivially alternating) or the product of two all-terminal reliability polynomials. It is well known that all-terminal reliability polynomials are strictly alternating and have integer coefficients (see, for example, [3]), so by Lemma 2.2, the product is alternating as well. Thus if G is not connected the split reliability polynomial is alternating with integer coefficients.

We can therefore assume that G is connected. We will use double induction on $n \geq 2$ and $\hat{m} - n + 1$, where $\hat{m}$ is the number of edges in the underlying simple graph $\hat{G}$ of G, to prove that the polynomial is alternating. Now as $\hat{G}$ is connected and on n vertices, we know that $\hat{m} \geq n - 1$, with equality if and only if $\hat{G}$ is a tree. So the smallest value for $\hat{m}$ is $\hat{m} = n - 1$, i.e. $\hat{m} - n + 1 = 0$, and in this case G is a *bundled tree*, that is, a tree with each edge e replaced by a bundle of say $l_e \geq 1$ edges. The base cases are $n = 2$ and $\hat{m} - n + 1 = 0$, and observing that the base case $n = 2$ implies that G is also a bundled tree, it suffices, for our base case, to prove the result when G is a bundled tree.

Base Case: For $n = 2$, $\text{splitRel}_{s,t}(G;p) = (1-p)^m$, which is clearly alternating in sign with integer coefficients, starting with a positive constant and of degree m. So we assume $n \geq 3$. To find the split reliability of a bundle tree of order n, let the distance between vertex s and vertex t on our graph be k, with bundles of sizes $b_1, \ldots, b_k$ on the path joining s and t, and let $b_{k+1}, \ldots, b_{n-1}$ be the sizes of the other bundles. In order for every vertex of G to communicate with exactly one of s and t, every bundle that is not on our path needs to have at least one edge up and the path between s and t needs to have exactly one bundle down . So our function looks like this:

$$\text{splitRel}_{s,t}(G;p) \;=\; \prod_{l=k+1}^{n-1}\left(1-(1-p)^{b_l}\right)\left(\sum_{i=1}^{k}\left((1-p)^{b_i}\prod_{j\leq k,\ j\neq i}\left(1-(1-p)^{b_j}\right)\right)\right).$$

We need to show that this function alternates in sign, starting at the lower end with a positive coefficient. To begin, note that the expansion of $1-(1-p)^b$ for $b \geq 1$ is an alternating polynomial by the binomial theorem:

$$(1-(1-p)^b) \;=\; (-1)^{b+1}p^b + (-1)^b\binom{b}{1}p^{b-1} + \ldots + (-1)^2\binom{b}{b-1}p.$$

It follows by Lemma 2.2,

$$\prod_{l=k+1}^{n-1}\left(1-(1-p)^{b_l}\right)$$

alternates in sign, starting with a positive coefficient with power p^{n-k-1}.

For fixed $i \in \{1, \ldots, k\}$, consider the term

$$(1-p)^{b_i}\prod_{j\leq k,\ j\neq i}\left(1-(1-p)^{b_j}\right).$$

Again, by Lemma 2.2, this polynomial alternates in sign, with nonconstant term, and the coefficient of p^{k-1} being positive. It follows that

$$\sum_{i=1}^{k}\left((1-p)^{b_i}\prod_{j\leq k,\ j\neq i}\left(1-(1-p)^{b_j}\right)\right)$$

has the same properties, and so

$$\prod_{l=k+1}^{n-1}\left(1-(1-p)^{b_l}\right)\left(\sum_{i=1}^{k}\left((1-p)^{b_i}\prod_{j\leq k,\ j\neq i}\left(1-(1-p)^{b_j}\right)\right)\right)$$

alternates in sign, starting with a positive coefficient for p^{n-2}, and having degree $\sum_{i=1}^{n-1} b_i = m$, and we have proved the base case.

Induction: Now that we have our two base cases, assume a graph G has parameters $n \geq 3$ and $\hat{m}-n+1 \geq 1$. Since $\hat{m}-n+1 \geq 1$, we know that $\hat{G}$, the underlying simple graph of G, is not a tree, and since we know that G (and hence $\hat{G}$) is connected, $\hat{G}$ must have a cycle C. Let e be any edge of C that is not adjacent to both s and t, and let e have multiplicity $b \geq 1$ in G. Call the corresponding bundle of edges in G $[e]$. We already know from Theorem 1.2 that

$$\begin{aligned}\operatorname{splitRel}_{s,t}(G;p) &= (1-(1-p)^b)\cdot\operatorname{splitRel}_{s,t}(G*[e];p)\\ &+ (1-p)^b\cdot\operatorname{splitRel}_{s,t}(G-[e];p).\end{aligned}\tag{2.2}$$

Note that since e belongs to cycle C and e is not adjacent to both s and t, we know that both $G*[e]$ and $G-[e]$ are connected with no loops.

To start, we will look at the first term in our sum on the right-hand side of (2.2):

$$(1-(1-p)^b)\cdot\operatorname{splitRel}_{s,t}(G*[e];p)$$

We can see that the connected graph $G*[e]$ has one less vertex and b less edges than G. By induction, $\operatorname{splitRel}_{s,t}(G*[e];p)$ alternates in sign, starting with a positive coefficient for the first nonzero coefficient, that of $p^{(n-1)-2} = p^{n-3}$, and has degree $m-b$. It follows that $(1-(1-p)^b)\cdot\operatorname{splitRel}_{s,t}(G*[e];p)$ alternates in sign, starting with a positive coefficient for the first nonzero coefficient, that of p^{n-2}, and has degree m. On the other hand, consider the second term in our sum on the right-hand side of (2.2),

$$(1-p)^b\cdot\operatorname{splitRel}_{s,t}(G-[e];p).$$

We can see that the connected graph $G-[e]$ also has b less edges than G (but the same number of vertices). By induction, $\operatorname{splitRel}_{s,t}(G-[e];p)$ alternates in sign, starting with a positive coefficient for the first nonzero coefficient, that of p^{n-2}, and has degree

$m-b$. It follows that $(1-p)^b \cdot \text{splitRel}_{s,t}(G-[e];p)$ alternates in sign, starting with a positive coefficient for the first nonzero coefficient, that of p^{n-2}, and has degree m. Putting these together, we see that

$$\begin{aligned} \text{splitRel}_{s,t}(G;p) &= (1-(1-p)^b) \cdot \text{splitRel}_{s,t}(G*[e];p) \\ &\quad +(1-p)^b \cdot \text{splitRel}_{s,t}(G-[e];p) \end{aligned}$$

alternates in sign, starting with a positive coefficient for the first nonzero coefficient, that of p^{n-2}, and has degree m.

On top of that, notice that all of these polynomials are found by adding or multiplying polynomials that must have integer coefficients and start with the same sign. Since the positive integers are closed under addition, we must have that all of the coefficients in the split reliability polynomial are nonzero integers, and thus the proof is complete. □

Theorem 2.4. *For any connected graph G of order at least 2, m non-loop edges, and any distinct vertices s and t of G,*

$$\mathit{splitRel}_{s,t}(G;p) = \sum_{i=n-2}^{m} (-1)^{n+i} a_i p^i, \tag{2.3}$$

where each a_i is a positive integer.

We will consider (2.3) the *S-form* of the split reliability polynomial. Table 1.1 displays the S-forms of split reliability polynomials for graphs of small order.

Alternating in sign is a very interesting property since it holds for all-terminal reliability but not two-terminal reliability (for the latter, see the example of the two-terminal reliability of graph G_1 in Section 1.1)). Due to split reliability's similarities with two-terminal reliability (choosing two specific vertices to look at specifically) it seems odd that this property would not be shared with two-terminal reliability, but this makes the study of split reliability all the more interesting.

2.3 Forms of the Split Reliability Polynomials

It is worthwhile to look at different ways to write out the split reliability function; polynomials, after all, form a vector space, and the choice of basis is arbitrary – we need not choose the standard basis! We have already seen that via Theorem 1.2 we can calculate the split reliability polynomial, and by expanding out fully into a polynomial in p, we get what we call the *S-form* of split reliability (see Theorem 2.3),

$$\text{splitRel}_{s,t}(G;p) \;=\; \sum_{i=n-2}^{m} (-1)^{n+i} a_i p^i,$$

where each a_i is a positive integer.

Another way we can write a formula for the split reliability of a graph G with n vertices and m edges is to consider the number of subgraphs of G that contain all of the vertices in G which have two components with s and t in different components. This form of split reliability looks like this,

$$\text{splitRel}_{s,t}(G;p) \;=\; \sum_{i=0}^{m} N_i p^i (1-p)^{m-i},$$

where N_i is the number of operational subgraphs with i edges up. We will call this form of the split reliability polynomial the *N-form*. In fact, we can start the sum off at $i = n-2$.

Proposition 2.5. *For any graph G of order at least 2 and any distinct vertices s and t of G,*

$$\textit{splitRel}_{s,t}(G;p) \;=\; \sum_{i=n-2}^{m} N_i p^i (1-p)^{m-i},$$

with $N_{n-2} \neq 0$ if and only if either G consists of two components with s and t in different components, or G is connected.

Proof. In any operational state for split reliability, we need there to be exactly two components with s and t in different components. Let the first component have n_1 vertices and the second component have n_2 vertices $(n_1 + n_2 = n)$. Each component contains a spanning tree, so such an operational state has at least $n_1 - 1 + n_2 - 1 = n-2$ edges. So, if i is less than $n_1 - 1 + n_2 - 1 = n-2$, then $N_i = 0$.

Moreover, N_{n-2} must count the number of spanning subgraphs that consists of two trees, one containing s, the other containing t. If G has at least three components or two components with s and t in the *same* component, then this is impossible (and $N_{n-2} = 0$). If G consists of two components with s and t in *different components*, we can take the union of spanning trees in each to find an operational state with $n-2$ edges. Finally, if G is connected, there is at least one such operational subgraph with $n-2$ edges. To prove this, consider a spanning tree of G. Let the length of the path between s and t in this spanning tree be k. We see that removing any one of the edges on this path will result in the graph being split into two components with one containing s and the other containing t which has a total of $n-2$ edges active. Thus $N_{n-2} \geq k$. So, in these last two cases, $N_{n-2} > 0$. □

From here on in this section, we will be assuming G is a connected graph, as that is the more interesting scenario since it is not directly related to all-terminal reliability. We can see that if G is connected and $n \geq 2$, then $N_m = 0$ since if all edges are active, s and t will be able to communicate in a connected graph. This leads us to consider how we can find a value (or the largest value, if we can!) d such that $N_{m-j} = 0$ for all $0 \leq j < d$. This index d, unlike the lower index of $n-2$ we found, changes depending on the structure of G. However, surprisingly, we can determine the value of d (even in polynomial time). To show this, we will relate this value d of a specific graph G to the minimum cardinality of certain cutsets.

In reliability problems, a *cutset* is a collection of edges whose removal causes the reliability we are considering to fail (this generalizes the usual definition of a cutset in graph theory, which refers to a set of edges whose removal disconnects a connected graph). For example, a cutset for all-terminal reliability is a collection of edges whose removal disconnects the graph. *Mincuts* are cutsets of the smallest possible size. With this in mind, let c be the minimum cardinality of an s,t-cutset, that is, the cardinality of a mincut for the two-terminal reliability of G with s and t as our specified vertices. So c is the minimum number of edges whose removal disconnects s and t. Thus if fewer edges are down, s and t can always communicate, so $N_{m-i} = 0$ for all $0 \leq i < c$. This gives us that $d \geq c$. We can prove that, in fact, $d = c$.

Proposition 2.6. *For any graph G of order at least 2 and any distinct vertices s and t of G,*

$$\mathit{splitRel}_{s,t}(G;p) \;=\; \sum_{i=n-2}^{m-c} N_i p^i (1-p)^{m-i},$$

where c is the minimum cardinality of an s,t-cutset.

Proof. We have already argued that $N_i = 0$ for $i > m - c$, so all that needs to be show is that $N_{m-c} > 0$. Let E' be a set of c edges in G such that $G - E'$ has s and t in different components (i.e. E' is an s,t-mincut). If $G - E'$ has two components, then $G - E'$ is an operational state for split reliability. Otherwise, there are more than two components, so we can specify a component C in $G - E'$ that does not contain s or t. Since we are assuming that G is a connected graph, there must exist an edge $e \in E'$ that connects C to another component. However, then $E' - e$ is an s,t-cutset, since placing e back into the graph will at most allow C to connect with s or t but not both. This contradicts the fact that E' is an s,t-mincut. Thus $G - E'$ must have the property that every vertex can communicate with exactly one of s and t, that is, $G - E'$ is an operational state for split reliability. On the other hand, if E' is a set of c edges that is not an s,t-cutset, then $G - E'$ contains a path between s and t, and hence $G - E'$ is not operational for split reliability. We conclude that N_{m-c} is precisely the number of s,t-mincuts (and hence is nonzero). □

We point out that we can find c for any graph G and any two distinct vertices s and t in polynomial time using network flows (see, for example, [3, pg. 51]).

Thus, if C_i is the number of s,t-cutsets of size i, the proof of Proposition 2.6 shows that

$$C_c \quad = \quad N_{m-c}, \tag{2.4}$$

where $c = c(G, s, t)$ is the minimum cardinality of an s,t-cutset in G. Finding C_c is $\#P$-complete [5]. So this tells us that despite the fact that we can find the largest i such that $N_i > 0$ in polynomial time, solving for the corresponding N_i is likely very difficult.

We prove one more thing about the N_is — they have no *internal zeros.*

Theorem 2.7. *Let c be the minimum cardinality of an (s,t)-cutset in G. If*

$$splitRel_{s,t}(G;p) \quad = \quad \sum_{i=n-2}^{m-c} N_i p^i (1-p)^{m-i},$$

then

$$N_{m-i} \geq \binom{m-n-c+2}{i-c} \geq 1$$

for $c \leq i \leq m-n+2$.

Proof. To begin, consider a specific mincut of our graph G; we know that this mincut is of size c and we will call it E'. $G - E'$ will then have exactly two components with the first one containing s and the other containing t. Let the first component have n_1 vertices and m_1 edges while the second component has n_2 vertices and m_2 edges. So the following hold:

$$\begin{aligned} m_1 &\geq n_1 - 1 \\ n_2 &= n - n_1 \\ m_2 &= m - c - m_1 \geq n_2 - 1 \\ n_1 - 1 + n_2 - 1 &= n - 2 \end{aligned}$$

Now, we know that each of these components must contain at least one spanning tree. We will choose one specific spanning tree for each component, T_1 being the edges of the tree in component 1 and T_2 being the edges of the tree in component 2. Notice that $|T_1| = n_1 - 1$ and $|T_2| = n_2 - 1$. We can see that the union of these sets of edges is an operational state for split reliability. So, we can add any of the edges not in E', T_1, or T_2 back into the graph and also get an operational state for split reliability.

Suppose we specify that all edges in T_1 and T_2 are active and all edges in E' are not. If we want to know how many operational states there are in this case with i edges inactive ($c \leq i \leq m - n + 2$), all we have to do is find how many ways we can choose $i - c$ edges from the remaining edges. Since no matter which edges we remove, T_1 and T_2 are still active so every vertex can communicate with s or t and since E' is down, s and t cannot communicate. So it turns out the number of operational states with T_1 and T_2 active, E' inactive and $i - c$ other edges inactive is precisely

$$\begin{aligned} & \binom{(m_1 - n_1 + 1) + (m_2 - n_2 + 1)}{i - c} \\ = & \binom{(m_1 + m_2) - (n_1 + n_2) + 2}{i - c} \\ = & \binom{(m - c) - n + 2}{i - c} \end{aligned}$$

We see that

$$N_{m-i} \;\; \geq \;\; \binom{m-n-c+2}{i-c}.$$

Finally, note that since $c \leq i \leq m-n+2$, $0 \leq i-c \leq m-n-c+2$. So the bottom of the binomial coefficient is always less than or equal to the top. Thus

$$N_{m-i} \;\; \geq \;\; \binom{m-n-c+2}{i-c} \geq 1.$$

and we have that the sequence of N_is have no *internal zeros* (all $N_{m-i} > 0$ when $c \leq i \leq m-n+2$ so all i that cause $N_{m-i} = 0$ must be outside of this interval). $\square$

We will also define two other lower bounds for the split reliability polynomials of connected graphs.

Proposition 2.8. *Suppose G is a connected graph of order n and size m with terminals s and t, and let c be the minimum cardinality of an (s,t)-cutset. Then for all $p \in [0,1]$,*

$$\mathit{splitRel}_{s,t}(G;p) \;\; \geq \;\; p^{n-2}(1-p)^c.$$

Proof. Since we know that c is the minimum cardinality of an (s,t)-cutset, we can select as set C of c specific edges to disconnect the graph into two components with s and t in separate components. Choosing a spanning tree in each component, we have $n-2$ specific edges not in C that, when they are the only edges active, result in a subgraph that consists of two components that are each trees, one containing s and the other containing t. If we have those c edges down and those $n-2$ edges up, split reliability must hold since s and t cannot communicate but every other vertex can communicate with s or t. The probability of those c edges being down and those $n-2$ edges being up is

$$p^{n-2}(1-p)^c.$$

Since this is only one situation where split reliability holds, the probability of this case occurring is less than or equal to the probability that split reliability holds. So we see that

$$\mathrm{splitRel}_{s,t}(G;p) \;\; \geq \;\; p^{n-2}(1-p)^c.$$

when $p \in [0,1]$. $\square$

In fact, we can improve this lower bound by using the distance between s and t.

Proposition 2.9. *Suppose G is a connected graph of order n and size m with terminals s and t, let c be the minimum cardinality of an (s,t)-cutset and let the distance between s and t be d. Then*

$$splitRel_{s,t}(G;p) \quad \geq \quad (d-1)p^{n-2}(1-p)^{m-n+2} + p^{n-2}(1-p)^c$$

when $p \in [0,1]$.

Proof. To begin, we find a spanning tree T that contains the path between s and t of length d. (To do this, begin with the spanning subgraph that only contains the edges in the path. From there we add in edges that are adjacent to this path as long as they do not form a cycle with other edges that are already in the subgraph; continue until every vertex is connected and we have a spanning tree containing the path.) This leaves us with $n-1$ edges in T and d choices of edges $e_1, ..., e_d$ to remove to disconnect the graph into two components, one containing s and the other containing t. Label $T - e_i$ as F_i, for $i = 1, \ldots, d$; the F_is are distinct operational states for the split reliability of G.

Now, let C be an (s,t)-mincut (with c edges). As shown in the proof of Proposition 2.6, there is a forest F in $G - C$ with two components, one containing s, the other t. Let $\mathscr{F} = \{S \subseteq E(G) - C : S \supseteq F\}$. Note that all elements of $\mathscr{F}$ are operational, since all of them contain the forest F; and the sum of the probabilities of all states in $\mathscr{F}$ is precisely $p^{n-2}(1-p)^c$. Note as well that C must contain some edge e_k from the path of length d between s and t, for otherwise C would not be an (s,t)-cutset. Now, consider $\{F_j : j \neq k\} \cup \mathscr{F}$. All of these are operational states and no two coincide, as e_k belongs to every F_j $(j \neq k)$ but no $S \in \mathscr{F}$ (as $e_k \in C$). Each F_j has probability $p^{n-2}(1-p)^{m-n+2}$ of occurring, and the probability of some $F \in \mathscr{F}$ occurring is $\sum_{S \in \mathscr{F}} Prob(S) = p^{n-2}(1-p)^c$, as such a state occurs if and only if every edge in C (which has size c) is down and every edge in F (which has size $n-2$) is up. Thus for all $p \in [0,1]$,

$$\begin{aligned} \text{splitRel}_{s,t}(G;p) &\geq \sum_{j \neq k} Prob(F_j) + \sum_{S \in \mathscr{F}} Prob(S) \\ &= (d-1)p^{n-2}(1-p)^{m-n+2} + p^{n-2}(1-p)^c \end{aligned}$$

$\square$

Now we have two different forms of the split reliability polynomial. The S-form from Theorem 2.3, and the N-form from Theorem 2.7. Here, we will be considering how we can convert one form into the other. We can see this by expanding the N-form:

$$\begin{aligned} & \sum_{i=n-2}^{m} N_i p^i (1-p)^{m-i} \\ = & \sum_{i=n-2}^{m} N_i p^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} p^k \right) \\ = & \sum_{i=n-2}^{m} \left(\sum_{k=0}^{m-i} (-1)^k N_i \binom{m-i}{k} p^{k+i} \right) \end{aligned}$$

If we rearrange the terms to group together specific powers of p we get

$$\sum_{i=n-2}^{m} N_i p^i (1-p)^{m-i} = \sum_{i=n-2}^{m} \sum_{k=n-2}^{i} \left((-1)^{i-k} \binom{m-k}{i-k} N_k \right) p^i,$$

giving us a formula for each coefficient of the probability polynomial in terms of the N_is, (note that we have to include N_is when $i > m-c$ here in order to find all of the a_is since the a_is are nonzero up to m).

We will also consider how to calculate the N-form of split reliability from the S-form. If the N-form of $\text{splitRel}_{s,t}(G;p)$ is given by

$$\text{splitRel}_{s,t}(G;p) = \sum_{i=n-2}^{m-c} N_i p^i (1-p)^{m-i},$$

then we define the generating function

$$N(z) = \sum_{i=n-2}^{m-c} N_i z^i.$$

First, to show the connection between $N(z)$ and $\text{splitRel}_{s,t}(G;p)$, consider

$$\begin{aligned} N\left(\frac{p}{1-p}\right) &= \sum_{i=n-2}^{m-c} N_i p^i (1-p)^{-i} \\ \Rightarrow (1-p)^m N\left(\frac{p}{1-p}\right) &= \sum_{i=n-2}^{m-c} N_i p^i (1-p)^{m-i} \\ \Rightarrow \text{splitRel}_{s,t}(G;p) &= (1-p)^m N\left(\frac{p}{1-p}\right). \end{aligned}$$

If we set

$$z = \frac{p}{1-p},$$

then

$$p = \frac{z}{1+z}.$$

it follows that

$$\begin{aligned} \mathrm{splitRel}_{s,t}\left(G; \frac{z}{1+z}\right) &= \frac{1}{(1+z)^m} N(z) \\ \Rightarrow N(z) &= (1+z)^m \mathrm{splitRel}_{s,t}\left(G; \frac{z}{1+z}\right). \end{aligned}$$

Now, using our S-form of the split reliability polynomial,

$$\begin{aligned} N(z) &= (1+z)^m \sum_{i=n-2}^{m} (-1)^{n+i} a_i \left(\frac{z}{1+z}\right)^i \\ \Rightarrow \sum_{i=n-2}^{m-c} N_i z^i &= \sum_{i=n-2}^{m} (-1)^{n+i} a_i z^i (1+z)^{m-i} \\ \Rightarrow \sum_{i=n-2}^{m-c} N_i z^i &= \sum_{i=n-2}^{m} (-1)^{n+i} a_i \sum_{k=0}^{m-i} \binom{m-i}{k} z^{i+k}. \end{aligned}$$

So in order to find the value of N_i for each possible value i, we need to know the coefficient of z^i. But this value is the sum of all the possible ways that $i + k = j$. Rearranging, we see that

$$N_i = \sum_{j=0}^{i} (-1)^{n+j} a_j \binom{m-j}{i-j}.$$

This gives us a formula for each N_i. Keeping in mind that $a_i = 0$ when $i < n-2$, we have that

$$\mathrm{splitRel}_{s,t}(G; p) = \sum_{i=n-2}^{m-c} \left(\sum_{j=n-2}^{i} (-1)^{n+j} \binom{m-j}{i-j} a_j \right) p^i (1-p)^{m-i}$$

For an example of changing S-form and N-form and back again, as well as using our formulas for lower bounds, consider the graph G_5 in Figure 2.1. If we let n be the order and m be the size of G_5, we have that $n = 6$ and $m = 9$. We will be using s and

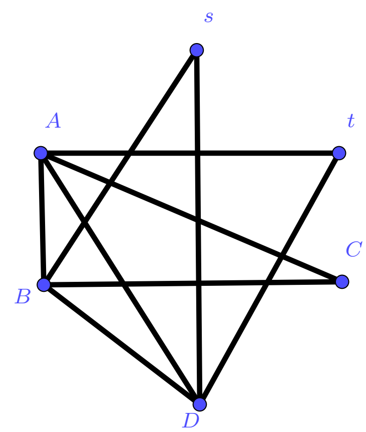

Figure 2.1: Graph G_5

t as they are labelled in Figure 2.1. We see that $c \leq 2$ since we can remove the two edges connected to s and get an operational state for split reliability (so $N_{9-2} > 0$). Moreover, if we only remove one edge there is no way we can disconnect s and t since there are two edge disjoint paths between s and t, namely (s, D, t) and (s, B, A, t), so $c \geq 2$ (so $N_{9-1} = 0$). Thus $c = 2$.

Using Maple to calculate the split reliability, we find that the S- and N-form of the split reliability of G_5 are, respectively,

$$-28p^9 + 162p^8 - 376p^7 + 438p^6 - 256p^5 + 60p^4$$

and

$$2p^7(1-p)^2 + 14p^6(1-p)^3 + 44p^5(1-p)^4 + 60p^4(1-p)^5$$

so $N_1 = N_2 = N_3 = 0$, $N_4 = 60$, $N_5 = 44$, $N_6 = 14$, $N_7 = 2$ and $a_1 = a_2 = a_3 = 0$, $a_4 = 60$, $a_5 = 256$, $a_6 = 438$, $a_7 = 376$, $a_8 = 162$, $a_9 = 28$. If we use our formula to change N-form to S-form, we get the following calculation

$$\begin{aligned}
\mathrm{splitRel}_{s,t}(G_5;p) &= \sum_{i=4}^{9}\left(\sum_{k=4}^{i}(-1)^{i-k}\binom{9-k}{i-k}N_k\right)p^i \\
&= (-1)^0N_4p^4+\Big[(-1)^1 5N_4+(-1)^0N_5\Big]p^5 \\
&\quad +\Big[(-1)^2 10N_4+(-1)^1 4N_5+(-1)^0N_6\Big]p^6 \\
&\quad +\Big[(-1)^3 10N_4+(-1)^2 6N_5+(-1)^1 3N_6+(-1)^0N_7\Big]p^7 \\
&\quad +\Big[(-1)^4 5N_4+(-1)^3 4N_5+(-1)^2 3N_6+(-1)^1 2N_7\Big]p^8 \\
&\quad +\Big[(-1)^5 N_4+(-1)^4 N_5+(-1)^3 N_6+(-1)^2 N_7\Big]p^9 \\
&= 60p^4+(-300+44)p^5+(600-176+14)p^6 \\
&\quad +(-600+264-42+2)p^7+(300-176+42-4)p^8 \\
&\quad +(-60+44-14+2)p^9 \\
&= -28p^9+162p^8-376p^7+438p^6-256p^5+60p^4.
\end{aligned}$$

This is the S-form of the split reliability polynomial. Now if we use our formula to change S-form to N-form, we get this calculation,

$$\begin{aligned}
\mathrm{splitRel}_{s,t}(G_5;p) &= \sum_{i=4}^{7}\left(\sum_{j=0}^{i}(-1)^{6+j}\binom{9-j}{i-j}a_j\right)p^j(1-p)^{9-j} \\
&= (-1)^{10}a_4p^4(1-p)^5+\Big[(-1)^{10}5a_4+(-1)^{11}a_5\Big]p^5(1-p)^4 \\
&\quad +\Big[(-1)^{10}10a_4+(-1)^{11}4a_5+(-1)^{12}a_6\Big]p^6(1-p)^3 \\
&\quad +\Big[(-1)^{10}10a_4+(-1)^{11}6a_5+(-1)^{12}3a_6+(-1)^{13}a_7\Big]p^7(1-p)^2 \\
&= 60p^4(1-p)^5+[300-256]p^5(1-p)^4 \\
&\quad +[600-1024+438]p^6(1-p)^3 \\
&\quad +[600-1536+1314-376]p^7(1-p)^2 \\
&= 2p^7(1-p)^2+14p^6(1-p)^3+44p^5(1-p)^4+60p^4(1-p)^5.
\end{aligned}$$

This is the N-form of the split reliability polynomial.

Now we explore our methods of finding lower bounds of the split reliability polynomial for G_5, (even though we can calculate it directly). We begin by observing that since we know the value c for G_5, we know which N_is are greater than 0. So we can find a lower bound by assuming all of the non-zero N_is are equal to 1. This is a rather poor lower bound since it will likely be quite distant from the actual value of the split reliability, but in this case it looks like this

$$\begin{aligned} & p^7(1-p)^2 + p^6(1-p)^3 + p^5(1-p)^4 + p^4(1-p)^5 \\ = \;& 2p^8 - 6p^7 + 7p^6 - 4p^5 + p^4. \end{aligned}$$

A better method for lower bounding the split reliability is the method discussed in Theorem 2.7. This lower bound is

$$\begin{aligned} & \sum_{i=4}^{7} \binom{3}{7-i} p^i(1-p)^{9-i} \\ = \;& \binom{3}{3} p^4(1-p)^5 + \binom{3}{2} p^5(1-p)^4 + \binom{3}{1} p^6(1-p)^3 + \binom{3}{0} p^7(1-p)^2 \\ = \;& p^6 - 2p^5 + p^4. \end{aligned}$$

Finally, we will use the method we discussed in Proposition 2.9. Since we can tell from the graph that the distance between s and t is 2, we see that this lower bound is

$$\begin{aligned} & p^{6-2}(1-p)^5 + p^{6-2}(1-p)^2 \\ = \;& -p^9 + 5p^8 - 10p^7 + 11p^6 - 7p^5 + 2p^4. \end{aligned}$$

We can see all of these graphs if Figure 2.2. We see that in this case our lower bounds are not that useful in estimating the true value of the polynomial due to the distance between the estimations and the true plot, but we do know at least that the true plot is greater than all of these estimations.

These lower bound formulas are very useful for estimating the split reliability of a graph that is too complicated to calculate the split reliability normally. For an example, consider G_6 in Figure 2.3. We see that this graph has 12 vertices and 33 edges. With so many vertices and edges, it will take a computer a very long time to calculate the split reliability with s and t as labelled in the image (let alone calculating it by hand). We can use our methods of bounding the split reliability to estimate it.

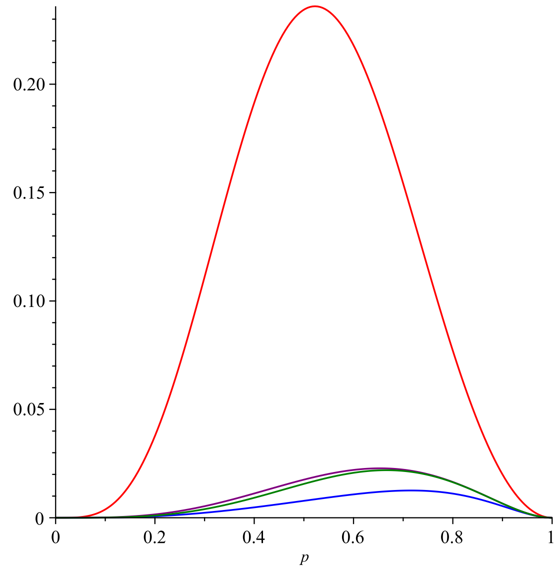

Figure 2.2: The plot of our lower bounds of the split reliability of G_5. The blue plot are when all N_is are equal to 1, the green line is the method from Theorem 2.7, the purple line is from Proposition 2.9, and the red line is the true value for the split reliability.

Using Maple, we can find that the value of c for this question is 4 (remember that calculating c can be done in polynomial time [3, pg. 51]). So our estimations for $\mathrm{splitRel}_{s,t}(G_6;p)$ are as follows.

We start by considering the lower bound where all non-zero N_is are equal to 1. This looks like

$$\begin{aligned}
& \sum_{i=10}^{29} p^i(1-p)^{33-i} \\
= \quad & 10p^{32} - 130p^{31} + 945p^{30} - 4840p^{29} + 18832p^{28} - 57882p^{27} + 144111p^{26} \\
- \quad & 295680p^{25} + 505912p^{24} - 727672p^{23} + 884236p^{22} - 909960p^{21} + 793092p^{20} \\
- \quad & 584082p^{19} + 361711p^{18} - 186880p^{17} + 79612p^{16} - 27492p^{15} + 7506p^{14} \\
- \quad & 1560p^{13} + 232p^{12} - 22p^{11} + p^{10}
\end{aligned}$$

Now, using the method discussed in Theorem 2.7, we have a better lower bound of

$$\begin{aligned}
& \sum_{i=10}^{29} \binom{19}{29-i} p^i(1-p)^{33-i} \\
= \quad & p^{14} - 4p^{13} + 6p^{12} - 4p^{11} + p^{10}
\end{aligned}$$

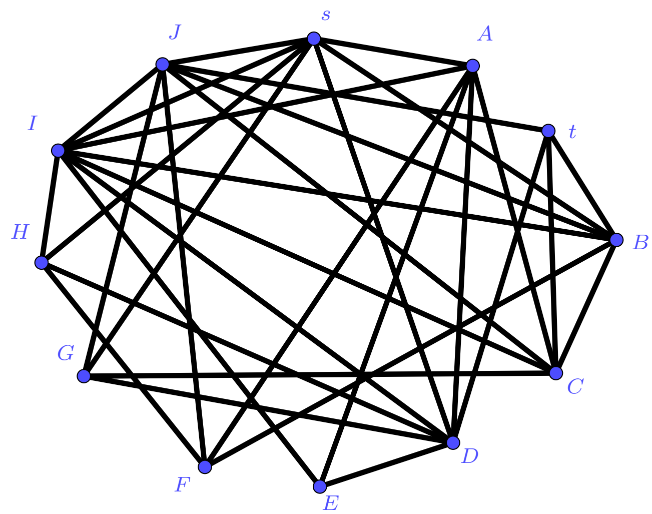

Figure 2.3: Graph G_6

Finally, using the method from Proposition 2.9 with $d = 2$ as the shortest path between s and t (the path is (s, B, t)) we have yet another lower bound on the split reliability of G_6:

$$\begin{aligned}
& p^{12-2}(1-p)^{33-12+2} + p^{12-2}(1-p)^4 \\
= \;& p^{10}(1-p)^{23} + p^{10}(1-p)^4 \\
= \;& -p^{33} + 23p^{32} - 253p^{31} + 1771p^{30} - 8855p^{29} + 33649p^{28} - 100947p^{27} \\
+ \;& 245157p^{26} - 490314p^{25} + 817190p^{24} - 1144066p^{23} + 1352078p^{22} - 1352078p^{21} \\
+ \;& 1144066p^{20} - 817190p^{19} + 490314p^{18} - 245157p^{17} + 100947p^{16} - 33649p^{15} \\
+ \;& 8856p^{14} - 1775p^{13} + 259p^{12} - 27p^{11} + 2p^{10}
\end{aligned}$$

We can see the graphs of these lower bounds in Figure 2.4. The plot of the bound from the method in Theorem 2.7 is not visible on the plot as it is too close to the plot of the bound from Proposition 2.9. You can see how close they are by examining Figure 2.5 where we see both plots on a smaller scale.

When we begin to consider how difficult if is to calculate the split reliability of certain graphs, we can consider whether certain values of N_i are easier to calculate than others. This leads us to the following theorem.

Theorem 2.10. *In the N-form of the split reliability polynomial for a graph G with specific vertices s and t, N_{n-2} can be calculated in polynomial time.*

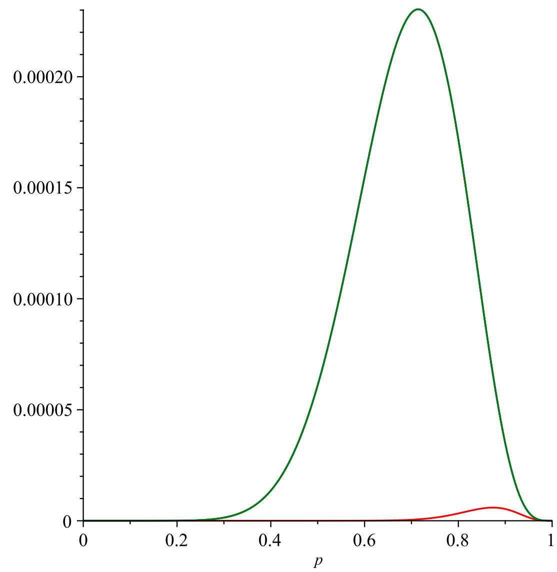

Figure 2.4: The plot of our lower bounds of the split reliability of G_6. The red plot are when all N_is are equal to 1, and the green line is the method from Proposition 2.9

Proof. To prove this, we will link the problem of finding N_{n-2} to a problem that we already know can be done in polynomial time, namely finding the number of spanning trees of a graph.

To calculate N_{n-2}, we approach it like so (thanks to W. Myrvold for providing this insight). We know that every operational state for split reliability with $n-2$ edges active takes the form of two disconnected trees one containing s and the other containing t. If we took this subgraph and condensed s and t into the same vertex, we claim that the resulting graph will be a spanning tree for the graph G with s and t condensed on one new vertex st (we'll call this new graph $G * st$). Let $\phi : E(G) \to E(G * st)$ be the natural edge mapping, where any vertex s or t in an edge e of G is replaced by the vertex st (edges between s and t become loops at st). Clearly, ϕ is a bijection. We will show that ϕ induces a bijection between the operational states of G with $n-2$ edges active and the spanning trees of $G * st$.

First, we prove that the map takes the operational states for split reliability with $n-2$ edges active to spanning trees of $G * st$. Without loss of generality, suppose $e_{j_1}, ..., e_{j_{n-2}}$ are $n-2$ edges active in an operational state for split reliability, so that they connect every vertex of G to exactly one of s and t. If we condense s and t into one new vertex st, we have that $G * st$ has order $n-1$. All we have to show now

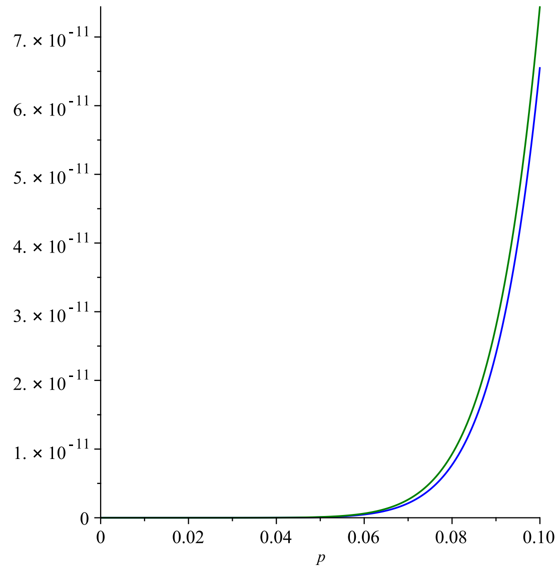

Figure 2.5: These are the plots of the lower bounds of the split reliability of G_6 where the blue plot is from Theorem 2.7 and the green plot is from Proposition 2.9.

is that the subgraph with edges $e_{j_1}, ..., e_{j_{n-2}}$ active form a spanning tree of $G * st$. However, this is clear as the $n-2$ edges $e_{j_1}, ..., e_{j_{n-2}}$ connect all vertices in $G * st$, and $G * st$ has order $n - 1 = (n-2) + 1$.

It is clear that our mapping is injective, so we have to prove that the mapping is surjective. Say we have a spanning tree of $G * st$ (with $n-2$ edges). We know that with these edges, every unlabelled vertex can communicate with either s or t since in $G * st$ they can communicate with st. So the only way this is not an operational state for split reliability in G is if s and t can communicate. Say we have a path in our subgraph of G of length l made of edges $e_1, ..., e_l$. If $l > 1$, we know that these edges being active in $G * st$ will result in a path in our tree beginning and ending at st, which is clearly impossible for trees. If $l = 1$ the tree will have a loop on st, which also cannot happen in trees. So we must have that s and t cannot communicate in our subgraph of G. Thus our subgraph must be an operational state for split reliability with $n-2$ edges, and so our operation is surjective. It follows that ϕ induces a bijection from the operational states of G with $n-2$ edges and the spanning trees of $G * st$.

Since we have a bijection between the operational states with $n-2$ edges active and the number of trees in $G * st$, we must have that these two groups have the same

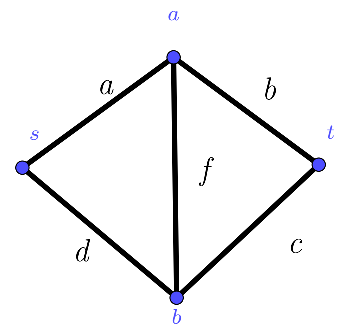

Figure 2.6: Graph G_1

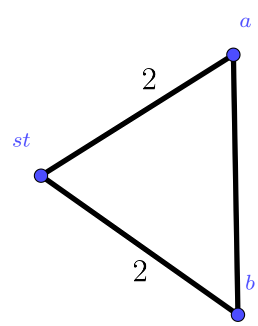

Figure 2.7: Graph $G_1 * st$

size. So N_{n-2} equals the number of spanning trees in $G * st$, which we can find in polynomial time by Kirchhoff's well known Matrix Tree Theorem via the *Laplacian matrix* of $G * st$ (see for example, [7, pg. 86]). (The Laplacian matrix $L = L(G)$ of a graph G with vertices $v_1, \ldots, v_n$, where $L_{i,j}$ is the degree of vertex v_i if $i = j$, and is the negative of the number of edges between v_i and v_j otherwise.

Theorem 2.11 (Kirchhoff's Matrix Tree Theorem)**.** *Let $L(H)$ be the Laplacian matrix of graph H. Then the number of spanning trees of H is the value of any cofactor of $L(H)$.*

□

For an example of this at work, let's consider our graph G_1 from earlier. We can see G_1 in Figure 2.6, and we can see $G_1 * st$ in Figure 2.7.

We have that the Laplacian of G_1 is

$$L_1 = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

and the Laplacian of $G_1 * st$ is

$$\hat{L}_1 = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

So, if we take the cofactor $C_{1,1}$ of $\hat{L}_1$ (by removing the first row and the first column), we get

$$\begin{aligned} & (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} \\ = \; & (3)(3) - (-1)(-1) \\ = \; & 8 \end{aligned}$$

By the Matrix Tree Theorem, $G_1 * st$ has 8 spanning trees and for G_1, $N_{4-2} = 8$. This matches our calculations from earlier where we found the split reliability in this case to be $-6p^5 + 20p^4 - 22p^3 + 8p^2$. Using our formula to convert S-form to N-form, we find that N_{4-2} is

$$\begin{aligned} N_2 &= \sum_{j=0}^{2} (-1)^{4+j} a_j \binom{5-j}{2-j} \\ &= 8. \end{aligned}$$

Finally, notice that when we took the (1,1) cofactor of the Laplacian of $G_1 * st$, we removed all of the entries corresponding to edges coming out of st. This only leaves entries corresponding to edges between vertices that are not st and the order of vertices that are not st. We can get the same result by looking at the Laplacian of G_1 and removing the rows and columns corresponding to s and t.

$$\begin{aligned}
& \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \\
\Rightarrow & \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} \\
= & \ (3)(3) - (-1)(-1) \\
= & \ 8
\end{aligned}$$

By the same argument, we can see that we can do a "double" deletion of the s and t rows and columns in the Laplacian of G rather than deleting the st row and column of the Laplacian of $G*st$. So to calculate N_{n-2}, all we need to do is find the Laplacian of our graph G, remove the rows and columns corresponding to s and t, and then take the determinant.

With this idea of $G * st$ as defined above, we can also find some upper bounds for the split reliability polynomial. First, note that for any graph G, a good state for split reliability is a good state for the all-terminal reliability of $G * st$ since all vertices would be able to communicate with s or t in G and thus would be able to communicate with st in $G * st$. Thus

$$\mathrm{splitRel}_{s,t}(G;p) \quad \leq \quad \mathrm{Rel}_V(G * st; p).$$

Now, we note that every good state for split reliability is a bad state of two-terminal reliability. So

$$\mathrm{splitRel}_{s,t}(G;p) \quad \leq \quad 1 - \mathrm{Rel}_{s,t}(G;p).$$

The best way to find an upper bound for the split reliability polynomial in this case is to consider the lower of these two polynomials on $[0, 1]$. For example, consider again our graph G_1 as it is in Figure 2.6. We have from Chapter 1 that

$$\begin{aligned}
\mathrm{Rel}_{s,t}(G_1;p) &= 2p^5 - 5p^4 + 2p^3 + 2p^2 \\
\Rightarrow 1 - \mathrm{Rel}_{s,t}(G_1;p) &= -2p^5 + 5p^4 - 2p^3 - 2p^2 + 1,
\end{aligned}$$

and by examining Figure 2.7 and using Theorem 1.1 with $e = \{a, b\}$, we find that

$$\begin{aligned} \mathrm{Rel}_V(G_1 * st; p) &= p(1-(1-p)^4) + (1-p)(1-(1-p)^2)^2 \\ \Rightarrow \mathrm{Rel}_V(G_1 * st; p) &= -2p^5 + 9p^4 - 14p^3 + 8p^2, \end{aligned}$$

giving us two upper bounds for the split reliability polynomial (for the plot see Figure 2.8).

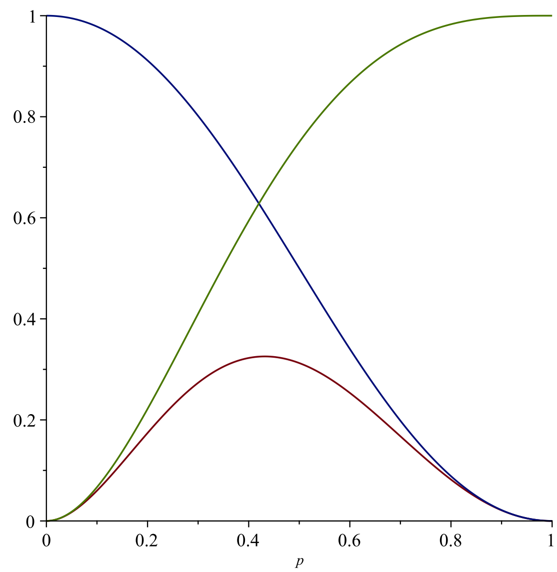

Figure 2.8: These are the plots of the upper bounds of the split reliability of G_1 where the green plot is $\mathrm{Rel}_V(G * st; p)$, the blue plot is $1 - \mathrm{Rel}_{\{s,t\}}(G; p)$, and the red plot is the actual value of the split reliability.

Now, if we have that the graph is disconnected, $G * st$ becomes even more useful.

Theorem 2.12. *If G is disconnected, $splitRel_{s,t}(G; p) = Rel_V(G * st; p)$*

Proof. If G has more than 2 components or has 2 components with s and t in the same component, $\mathrm{splitRel}_{s,t}(G; p) = \mathrm{Rel}_V(G_1 * st; p) = 0$ since $G * st$ has more than one component. If G has 2 components with s and t in different components, we already know that $\mathrm{splitRel}_{s,t}(G; p) \leq \mathrm{Rel}_V(G * st; p)$, but the only case where an operational state for all-terminal reliability of $G * st$ is not an operational state for split reliability in G is if s and t can communicate, which is impossible if they are in different components. $\square$

These theorems show some very interesting properties of and bounds for split reliability polynomials that in many cases are not shared by all-terminal and two-terminal reliability polynomials.

Chapter 3

Functional Properties of Split Reliability Polynomials

With all-terminal or two-terminal reliability of connected graphs of order at least 2, at $p = 0$ the polynomial is 0 since it is impossible for either form of reliability to hold when all edges are down, and at $p = 1$ the value of the polynomial is 1 since here all vertices can always communicate. However, with split reliability the function is 0 at both $p = 0$ (since none of the vertices can communicate) and at $p = 1$ (since s and t can communicate). For example, consider Figures 3.1 and 3.2. Both are reliability polynomials for C_4, but Figure 3.1 is all-terminal reliability and Figure 3.2 is split reliability with s and t nonadjacent.

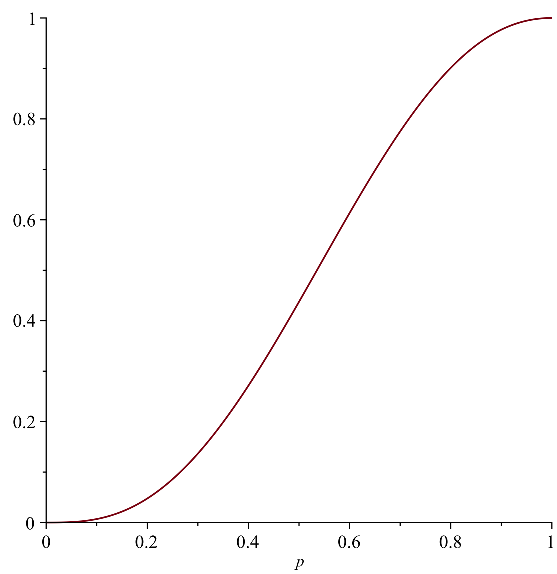

Figure 3.1: Plot of the all-terminal reliability of C_4

With this in mind, we can discuss some interesting functional properties of split reliability polynomials. (In this chapter we assume all graphs have order at least two and are connected.)

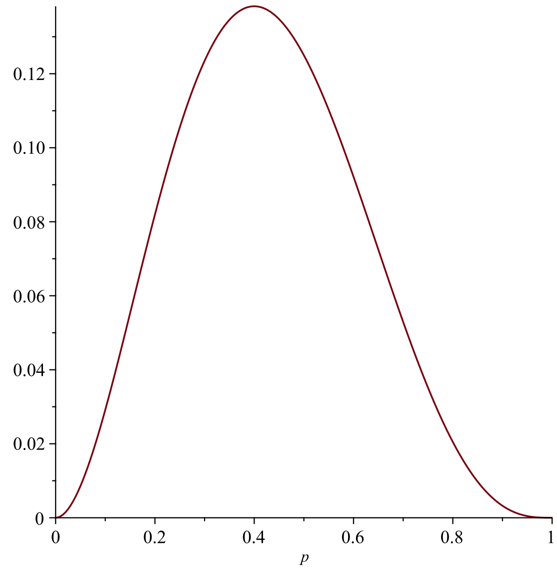

Figure 3.2: Plot of the split reliability of C_4 with antipodal terminals.

3.1 The Set of the Locations of the Maxima of Split Reliability Polynomials is Dense in $[0, 1]$.

We first begin with where the maxima of these functions occur in the interval $[0, 1]$. Of course, such a maximum occurs in the interior of the interval, as the function is 0 at the ends and positive in between. In Figure 3.3, Figure 3.4, and Figure 3.5, we have plotted the p values where the maxima of the split reliability occur for all simple graphs of order 5, 6, and 7 respectively (with all possible choices of terminals). We see that none of these p values go past 0.85, and it is unclear if we can "fill up" the whole interval $[0, 1]$. However, we can show that for a family of graphs, the set of locations where the maxima of the split reliability occur in $[0, 1]$ indeed "fills" the interval.

Theorem 3.1. *The values of p where the split reliability has a maximum in $[0, 1]$ is dense in $[0, 1]$.*

Proof. Let $P_{n,b}$ be formed from the path of order $n \geq 3$ by replacing each edge by a bundle of b edges. Let s and t be the end points of the path.

We'll calculate the split reliability $\text{splitRel}_{s,t}(P_{n,b}; p)$ and find the only critical point as a function of n and b. Then, we'll prove that if you take any $r \in (0, 1)$ and any $\epsilon > 0$ you can choose n and b such that the p value where the maximum of

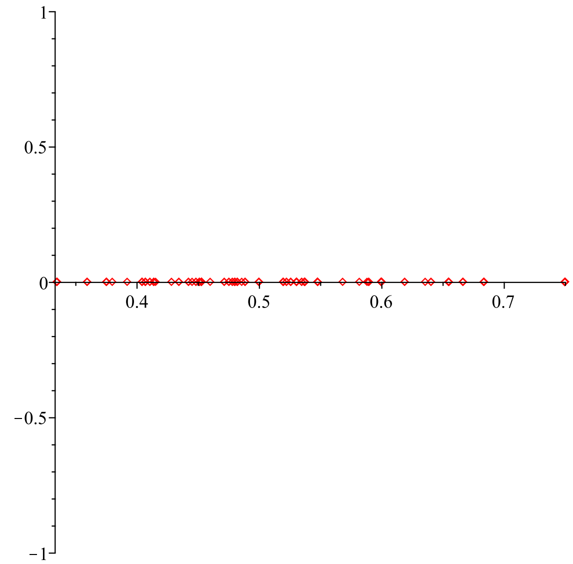

Figure 3.3: The p values that maximize split reliability polynomials for all simple connected graphs of order 5.

$\mathrm{splitRel}_{s,t}(P_{n,b}; p)$ on $[0, 1]$ occurs in the interval $(r - \epsilon, r + \epsilon)$. This will be sufficient to prove our theorem.

From our formula for trees in Chapter 1, we can write the function for the split reliability of $P_{n,b}$ as

$$f = \mathrm{splitRel}_{s,t}(P_{n,b}; p) = (n-1)(1-p)^b(1-(1-p)^b)^{n-2}.$$

We know that the maximum of this function on $[0, 1]$ occurs in the interior of the interval. If we take the derivative of this function we get

$$\begin{aligned} f' &= (n-1)\Big[(-1)b(1-p)^{b-1}(1-(1-p)^b)^{n-2} + \\ &\quad (1-p)^b(n-2)(1-(1-p)^b)^{n-3}b(1-p)^{b-1}\Big] \\ &= (n-1)b(1-p)^{b-1}(1-(1-p)^b)^{n-3}\Big[(-1)(1-(1-p)^b) + (1-p)^b(n-2)\Big]. \end{aligned}$$

So $f' = 0$ in $[0, 1]$ when $p = 0$, 1 or when $(-1)(1-(1-p)^b) + (1-p)^b(n-2) = 0$, that is, when

$$p = 1 - \frac{1}{(n-1)^{\frac{1}{b}}}$$

Since $n - 1$ and b are positive integers, we know that $0 < 1 - \frac{1}{(n-1)^{\frac{1}{b}}} < 1$ so this must be where the polynomial is largest on $[0, 1]$.

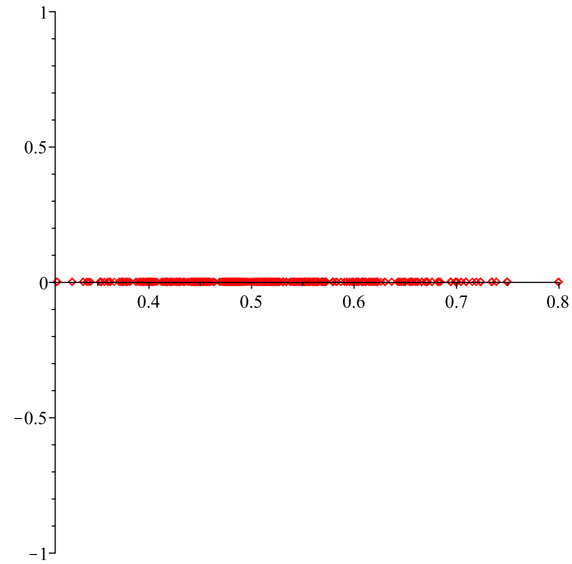

Figure 3.4: The p values that maximize split reliability polynomials for all simple connected graphs of order 6.

Now we will show that for any $r \in (0,1)$ and any $\epsilon > 0$ you can choose n and b such that $1 - \frac{1}{(n-1)^{\frac{1}{b}}}$ is in the interval $(r-\epsilon, r+\epsilon)$. Without loss of generality, assume that $0 < r-\epsilon < r+\epsilon < 1$. Consider the two values $S_1 = \frac{1}{1-r+\epsilon}$ and $S_2 = \frac{1}{1-r-\epsilon}$. Now with our conditions on r and ϵ, we know that $1 < S_1 < S_2$. We claim that by picking b to be a large enough positive integer, we can make it so that the difference between S_1^b and S_2^b is greater than 1. To prove this, take the difference between these two values:

$$\begin{aligned} S_2^b - S_1^b &= (S_2 - S_1)(S_2^{b-1} + S_2^{b-2}S_1 + ... + S_1^{b-1}) \\ &\geq b(S_2 - S_1) \end{aligned}$$

So, by choosing $b \geq \frac{1}{S_2 - S_1}$, we can make the difference between these two values greater than 1 (in fact, we can choose b large enough so that interval (S_1^b, S_2^b) is as wide as we like). For such a value of b there must be a positive integer between S_1^b and S_2^b; we will denote such an integer value as $n-1$ (which will be the length of our

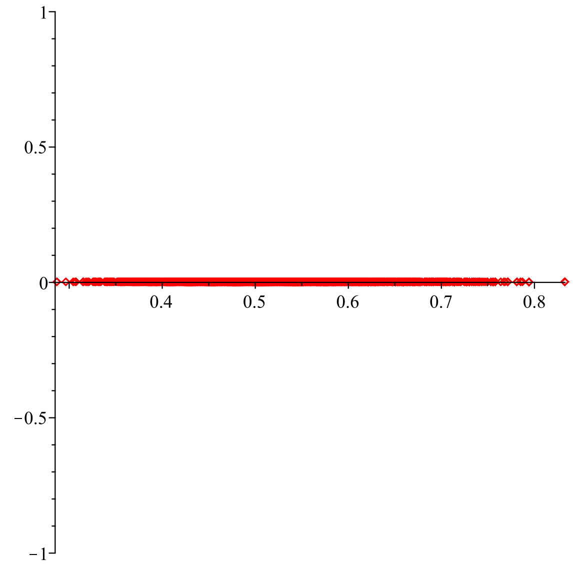

Figure 3.5: The p values that maximize split reliability polynomials for all simple connected graphs of order 7.

path). Now, we can see that the following inequalities hold:

$$\begin{array}{rlcll}
 & S_1^b & < & n-1 & < \quad S_2^b \\
\Rightarrow & \left(\dfrac{1}{1-r+\epsilon}\right)^b & < & n-1 & < \quad \left(\dfrac{1}{1-r-\epsilon}\right)^b \\
\Rightarrow & \dfrac{1}{1-r+\epsilon} & < & (n-1)^{1/b} & < \quad \dfrac{1}{1-r-\epsilon} \\
\Rightarrow & 1-r-\epsilon & < & \dfrac{1}{(n-1)^{1/b}} & < \quad 1-r+\epsilon \\
\Rightarrow & r-\epsilon & < & 1-\dfrac{1}{(n-1)^{1/b}} & < \quad r+\epsilon
\end{array}$$

Thus the location in $[0,1]$ where such a $\mathrm{splitRel}_{s,t}(P_{n,b};p)$ has a maximum will be in the interval $(r-\epsilon, r+\epsilon)$. This proves that the set points where the split reliabilities of the graphs $P_{n,b}$ (with terminals at the end of the paths) have their maxima is dense in $[0,1]$. $\square$

To illustrate the previous proof, suppose we want to find a graph $P_{n,b}$ (with terminals s and t at the ends of the path) such that the probability p where the maximum of the split reliability occurs is within a distance of $\epsilon = 0.01$ of $r = \frac{1}{3}$. We set

$$\begin{aligned} S_1 &= \frac{1}{1-r+\epsilon} = \frac{300}{203}, \text{ and} \\ S_2 &= \frac{1}{1-r-\epsilon} = \frac{300}{197}. \end{aligned}$$

If we choose $b = 23$, then $(S_1^b, S_2^b) \approx (7968.6, 15887.9)$, so we can choose $n = 10000$. Using the proof above, we have that the maximum of $\mathrm{splitRel}_{s,t}(P_{10000,23}; p)$ occurs at

$$p = 1 - \frac{1}{(10000-1)^{\frac{1}{23}}} \approx 0.32998,$$

which is within our specified interval, $(1/3 - 0.01, 1/3 + 0.01)$.

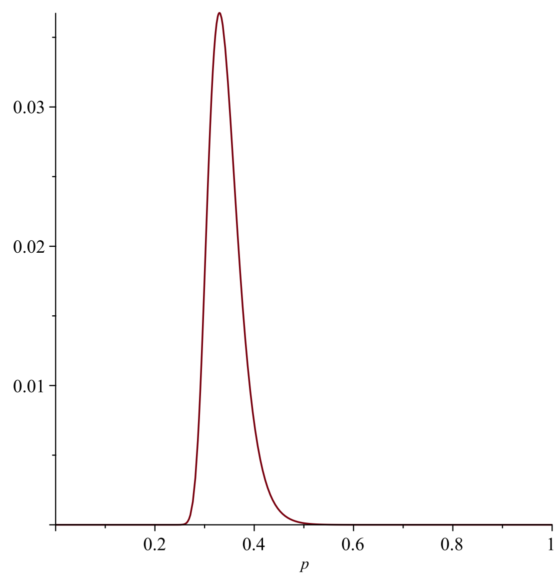

Figure 3.6: Plot of Example 3.1

The use of bundles paths is not required; we can use cycles too, via a similar argument.

Theorem 3.2. *The values of* p *where the split reliability of the class of bundles cycles* $C_{n,b}$ *of order* n *and each edge with multiplicity* b *has a maximum in* $[0, 1]$ *is dense in* $[0, 1]$.

Proof. Let $C_{n,b}$ be formed from the cycle of order $n \geq 3$ by replacing each edge by a bundle of b edges. Let s and t be two distinct points in the cycle.

Choose any two terminals s and t in the bundled cycle $C_{n,b}$, and let k denote the (shortest) distance between s and t. So, for split reliability, one bundle of edges on each path between s and t must fail and for all other bundles of edges at least one edge needs to stay up. so so the split reliability is

$$\begin{aligned} f &= k(1-p)^b(1-(1-p)^b)^{k-1}(n-k)(1-p)^b(1-(1-p)^b)^{n-k-1} \\ &= k(n-k)(1-p)^{2b}(1-(1-p)^b)^{n-2} \end{aligned}$$

Taking the derivative of f, we get

$$f' = k(n-k)b(1-p)^{2b-1}(1-(1-p)^b)^{n-3}\Big[-2(1-(1-p)^b)+(1-p)^b(n-2)\Big].$$

We can see that f' is 0 at $p = 0$, 1 and when $-2(1-(1-p)^b)+(1-p)^b(n-2) = 0$, that is, when

$$p = 1 - \left(\frac{2}{n}\right)^{\frac{1}{b}}.$$

Since $n \geq 3$ and b are positive integers, this value must belong to $(0,1)$. Now, to show that for any $r \in (0,1)$ and any $\epsilon > 0$ you can choose n and b such that $1-(\frac{2}{n})^{1/b}$ occurs on the interval $(r-\epsilon, r+\epsilon)$. Without loss of generality, assume that $0 < r-\epsilon < r+\epsilon < 1$. We will be using the same values S_1 and S_2 from the proof of the previous theorem and again choose b such that the difference between S_1^b and S_2^b is at least 1. Let the positive integer between these two values be $\frac{n}{2}$ (by choosing b large enough, we can ensure that $n/2 > 2$, i.e. $n \geq 4$). Now we see that

$$\begin{aligned} & S_1^b && < \frac{n}{2} && < S_2^b \\ \Rightarrow\ & \left(\frac{1}{1-r+\epsilon}\right)^b && < \frac{n}{2} && < \left(\frac{1}{1-r-\epsilon}\right)^b \\ \Rightarrow\ & (1-r-\epsilon)^b && < \frac{2}{n} && < (1-r+\epsilon)^b \\ \Rightarrow\ & r-\epsilon-1 && < -\left(\frac{2}{n}\right)^{1/b} && < r+\epsilon-1 \\ \Rightarrow\ & r-\epsilon && < 1-\left(\frac{2}{n}\right)^{1/b} && < r+\epsilon \end{aligned}$$

Thus, we can see that the locations where the split reliability polynomials for bundles cycles has a maximum in $[0,1]$ is dense in $[0,1]$. □

We remark that not every family of graphs has the location of their maxima dense in $[0,1]$. For example, consider graph G_n with s and t adjacent, with $n-2$ other

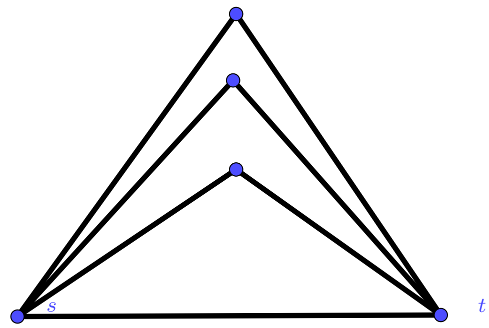

Figure 3.7: An example of G_n where $n = 5$.

vertices that are neighbours of both s and t (see Figure 3.7). G_n is a generalized $\theta-$graph.

We can see that

$$\begin{aligned}\text{splitRel}_{s,t}(G_n; p) &= (1-p)(2(1-p)p)^{n-2} \\ &= 2^{n-2}(1-p)^{n-1}p^{n-2}.\end{aligned}$$

Taking the derivative of this function we find that

$$f' = 2^{n-2}(1-p)^{n-2}p^{n-3}[-(n-1)p + (1-p)(n-2)].$$

This is 0 at $p = 0$, 1 and when $-(n-1)p + (1-p)(n-2) = 0$, that is, when

$$p = \frac{n-2}{2n-3} < \frac{1}{2}.$$

So no matter how much we increase n, we will not find any peaks where p is greater than $\frac{1}{2}$.

What about replacing each edge by a bundle of b edges? We observe the following:

Proposition 3.3. *For any graph G, if G' is the graph formed from G by replacing each edge by a bundle of edges of size b, then the maximum value of splitRel$_{s,t}(G; p)$ on $[0, 1]$ is equal to the maximum value of splitRel$_{s,t}(G'; p)$ on $[0, 1]$.*

Proof. Let

$$f(p) = \text{splitRel}_{s,t}(G; p).$$

From above, we have that

$$\begin{aligned}\text{splitRel}_{s,t}(G'; p) &= \text{splitRel}_{s,t}(G; 1-(1-p)^b) \\ &= f(1-(1-p)^b)\end{aligned}$$

The function $g(p) = 1-(1-p)^b$ is a bijection on $[0,1]$, so if the maximum of $\text{splitRel}_{s,t}(G;p)$ on $[0,1]$ is at $p = p'$, then the maximum of $\text{splitRel}_{s,t}(G';p)$ on $[0,1]$ is where $1-(1-p)^b = p'$, that is, at $p = 1-(1-p')^{1/b}$. Both maximums are $f(p')$ and are thus equal. □

This proposition implies that all $\text{splitRel}_{s,t}(G_n;p)$ will have the location of their maxima to the left of $1/2$ as well.

3.2 The Set of Maximum Values of Split Reliability Polynomials is Dense in $[0,1]$.

We have shown that the location of the maximum values of split reliabilities in $[0,1]$ is dense in the interval. However, what about the actual maximum values themselves? In Figure 3.8, Figure 3.9, and Figure 3.10 we plotted all of the maximum values of the split reliability of simple graphs of order 5, 6, and 7 (and all choices of terminals). From these diagrams, it seems to be the case that the maximum values may fill up the lower half of the interval $[0,1]$ but no point goes past the value 0.5. However, the next result shows that indeed they fill up the *entire* interval $[0,1]$.

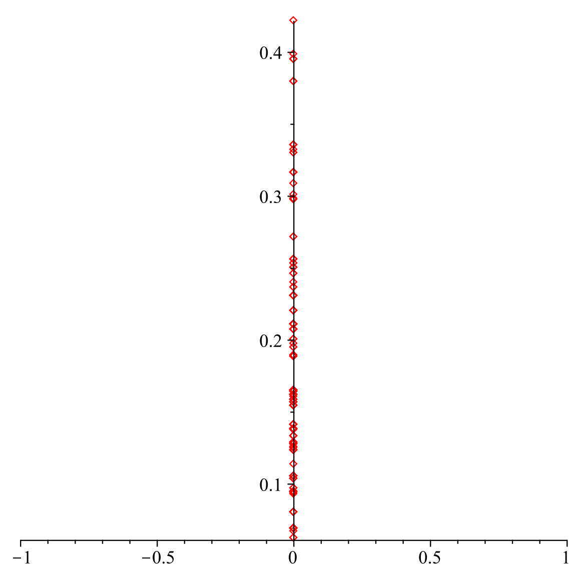

Figure 3.8: The maximum values of split reliability polynomials for all simple connected graphs of order 5.

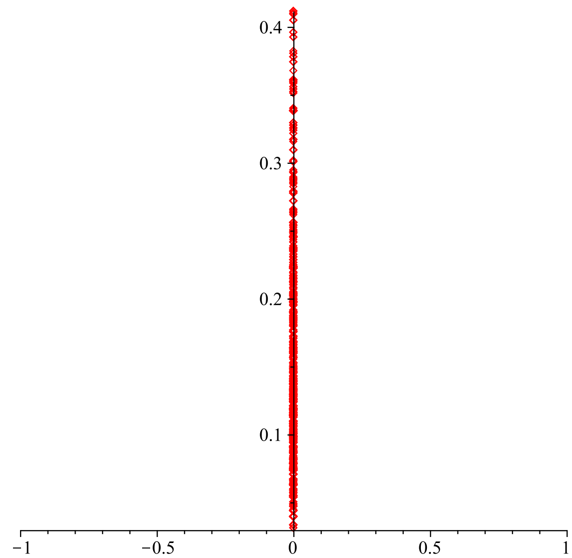

Figure 3.9: The maximum values of split reliability polynomials for all simple connected graphs of order 6.

Theorem 3.4. *The maximum values of split reliability over the interval* $[0, 1]$ *is dense in* $[0, 1]$.

Proof. Consider a graph $H_{k,b}$ such that s and t are connected by a bundle of k edges and t is connected to another vertex u by a bundle of $b \geq 1$ edges (See Figure 3.11). The split reliability of this graph is

$$\mathrm{splitRel}_{s,t}(H_{k,b}; p) = f(p) = (1-p)^k(1-(1-p)^b) = (1-p)^k - (1-p)^{b+k}.$$

To start, we will take the derivative of our split reliability polynomial to find our maximum.

$$\begin{aligned} f' &= -k(1-p)^{k-1} + (b+k)(1-p)^{b+k-1} \\ &= (1-p)^{k-1}((b+k)(1-p)^b - k) \end{aligned}$$

We have a zero at $p = 1$ and when $(b+k)(1-p)^b - k = 0$, that is, when

$$p = 1 - \left(\frac{k}{b+k}\right)^{1/b}.$$

As there is only one critical value in $(0, 1)$, and the function is 0 at the ends but positive in between, f must have its maximum at this critical value. If we plug this

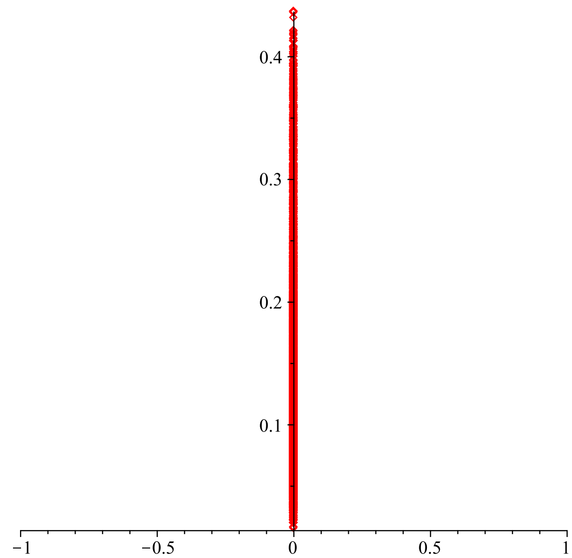

Figure 3.10: The maximum values of split reliability polynomials for all simple connected graphs of order 7.

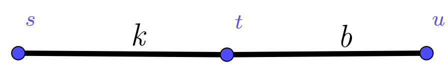

Figure 3.11: Graph $H_{k,b}$

value into the split reliability to find what the maximum value for the polynomial is, we get:

$$\begin{aligned} f\left(1-\left(\frac{k}{b+k}\right)^{1/b}\right) &= \left(1-\left(1-\left(\frac{k}{b+k}\right)^{1/b}\right)\right)^{k} - \\ &\quad \left(1-\left(1-\left(\frac{k}{b+k}\right)^{1/b}\right)\right)^{b+k} \\ &= \left(\frac{k}{b+k}\right)^{k/b} - \left(\frac{k}{b+k}\right)^{(b+k)/b} \end{aligned}$$

Now, to show that the maximum of the split reliability polynomials for this collection of graphs is dense in $[0, 1]$, set $s = \frac{k}{b}$; the set S of all such s (over all choices of $k, b \geq 1$) is clearly the positive rationals, which is dense in $[0, \infty)$.

Rewriting the maximum value above in terms of s, we get

$$\begin{aligned}
\left(\frac{k}{b+k}\right)^{k/b} - \left(\frac{k}{b+k}\right)^{(b+k)/b} &= \left(\frac{sb}{sb+b}\right)^{s} - \left(\frac{sb}{sb+b}\right)^{s+1} \\
&= \left(\frac{s}{s+1}\right)^{s} - \left(\frac{s}{s+1}\right)^{s+1} \\
&= \left(\frac{s}{s+1}\right)^{s}\left(1 - \frac{s}{s+1}\right) \\
&= \frac{s^s}{(s+1)^{s+1}}
\end{aligned}$$

We will prove that if we take the limits as s goes to 0 and as s goes to ∞, we get that the maximum value goes to 1 and 0, respectively. To prove this, we will use the variable x instead of s since we will be considering the limit with the real numbers, but it will be the same limit as if we were only considering the rationals. Clearly the function $g(x) = x^x/(x+1)^{x+1}$ is continuous on $(0, \infty)$, as $f(x) = x^x$ and $h(x) = (x+1)^{x+1}$ are.

We will begin by calculating the limit as x goes to ∞.

$$\begin{aligned}
&\lim_{x\to\infty} g(x) \\
= &\lim_{x\to\infty} \frac{x^x}{(x+1)^{x+1}} \\
= &\lim_{x\to\infty} \frac{x^x}{(x+1)^x(x+1)} \\
= &\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{x}\left(\frac{1}{x+1}\right)
\end{aligned}$$

Since we are considering when x is very large (and positive), we can see that both $(\frac{x}{x+1})^x$ and $\frac{1}{x+1}$ are in the interval $(0, 1)$ (both are positive fractions with larger denominator than numerator and one is raised to a large positive value). So the following inequality holds

$$0 \leq \left(\frac{x}{x+1}\right)^{x}\left(\frac{1}{x+1}\right) \leq \left(\frac{1}{x+1}\right),$$

and we can see that as x goes to infinity, both the left and right sides go to 0. So by the Squeeze Theorem, we must have that

$$\lim_{x\to\infty} \frac{x^x}{(x+1)^{x+1}} = 0.$$

Now, we will calculate the limit as x goes to 0^+.

$$\begin{aligned} & \lim_{x\to 0^+} g(x) \\ = & \lim_{x\to 0^+} \frac{x^x}{(x+1)^{x+1}} \\ = & \frac{\lim_{x\to 0^+} x^x}{\lim_{x\to 0^+}(x+1)^{x+1}} \end{aligned}$$

Just by plugging $x = 0$ into the denominator, we find that the limit of the denominator is 1 (this and the ensuing argument that the limit of the numerator exists, justifies the last equality above). The limit of the numerator is more complicated to find, so we will use logarithms. Let $y = x^x$.

$$\begin{aligned} & \ln y = x \ln x \\ \Rightarrow & \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln x \\ = & \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \end{aligned}$$

We see that both the numerator and the denominator of this limit go to plus or minus infinity as x approaches 0^+. So we can use L'Hôpital's Rule:

$$\begin{aligned} \lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0^+} (-x) \\ &= 0 \end{aligned}$$

So we have $\lim_{x\to 0^+} \ln y = 0$, and we can find the limit of y.

$$\begin{aligned} \lim_{x\to 0^+} y &= \lim_{x\to 0^+} e^{\ln y} \\ &= e^{\lim_{x\to 0^+} \ln y} \\ &= e^0 = 1 \end{aligned}$$

Thus $\lim_{x\to 0^+} x^x = 1$ and finally

$$\begin{aligned} \lim_{x\to 0^+} \frac{x^x}{(x+1)^{x+1}} &= \frac{\lim_{x\to 0^+} x^x}{\lim_{x\to 0^+}(x+1)^{x+1}} \\ &= \frac{1}{1} \\ &= 1. \end{aligned}$$

$\square$

We also have that $g(x)$ is decreasing on the interval $(0, \infty)$. We can see this by taking the derivative:

$$\begin{aligned} & \frac{d}{dx}\left(\frac{x^x}{(x+1)^{x+1}}\right) \\ = & \frac{x^x(\ln(x)+1)}{(x+1)^{x+1}} - \frac{x^x(\ln(x+1)+1)}{(x+1)^{x+1}} \\ = & \frac{x^x\ln(x)+x^x-x^x\ln(x+1)-x^x}{(x+1)^{x+1}} \\ = & \frac{x^x(\ln(x)-\ln(x+1))}{(x+1)^{x+1}} \\ = & \frac{x^x\ln(\frac{x}{x+1})}{(x+1)^{x+1}} \end{aligned}$$

We see that if x is positive, $0 < \frac{x}{x+1} < 1$ so $\ln(\frac{x}{x+1})$ is negative, x^x must be positive since it is a positive value raised to a positive value, and the same goes for $(x+1)^{x+1}$. This gives us that the derivative is negative on the interval $(0, \infty)$ and so $g(x)$ is decreasing on the interval $(0, \infty)$.

As $g(x)$ is decreasing on $(0, \infty)$, $\lim_{x \to 0^+} g(x) = 1$ and $\lim_{x \to \infty} g(x) = 0$, we see that $g(x)$ is a continuous function on $(0, \infty)$ whose image is $(0, 1)$. As S (the set of positive rationals) is dense in $(0, \infty)$ and the image of a dense set under a continuous function is dense in the image, we see that $g(S)$ is dense in $(0, 1)$. It follows that the closure of the set of maximum values of the split reliability polynomials of $H_{k,b}$ is $[0, 1]$, and we are done. $\square$

Also, something else interesting to note. Since we can define the maximum value of the split reliability polynomials for this family of graphs $\{H_{k,b} : k, b \geq 1\}$ as a function of the value $s = \frac{k}{b}$, we can see that the maximum probability of the split reliability holding depends on the fraction $\frac{b}{k}$ and not the individual values of b and k (i.e. if we have two graphs in this collection such that k_1 and b_1 are the values for one and k_2 and b_2 are the values for the other, if $\frac{k_1}{b_1} = \frac{k_2}{b_2}$, then the maximum values for both graphs on the interval $[0, 1]$ are equal). As an example, consider $H_{1,2}$ and $H_{2,4}$ as described in the proof of the above theorem (with the terminals s and t mentioned there). We have that

$$\text{splitRel}_{s,t}(H_{1,2}; p) = (1-p) - (1-p)^3$$

and

$$\text{splitRel}_{s,t}(H_{2,4}; p) = (1-p)^2 - (1-p)^6.$$

The maximum value of the split reliability for $H_{1,2}$ on $[0, 1]$ is

$$\left(\frac{1}{3}\right)^{1/2} - \left(\frac{1}{3}\right)^{3/2}$$

and the maximum value of the split reliability for $H_{2,4}$ on $[0, 1]$ is

$$\left(\frac{2}{6}\right)^{2/4} - \left(\frac{2}{6}\right)^{6/4} = \left(\frac{1}{3}\right)^{1/2} - \left(\frac{1}{3}\right)^{3/2}.$$

So, the maximum value for both split reliability polynomials on $[0, 1]$ is the same, as was expected from the proof above.

Chapter 4

Future Work

When we talked about split reliability in this thesis, we only ever chose two vertices s and t and examined the probability that every other vertex can communicate with s or t but not both. In the future, we could see what happens if we specify more than 2 vertices to examine in this way. If we only choose one vertex, the polynomial is the all-terminal reliability polynomial, if we specify 2, it is the split reliability polynomial, but suppose we select 3. If we choose 3 vertices, call them r, s, t, we can consider the probability that every vertex can communicate with exactly one of r, s, or t. This will give us a new polynomial. We can also note that if the number of vertices we select is equal to the order of the graph, then every edge needs to be down in order for no vertices to be able to communicate. It could be interesting to consider what new theorems we could find for split reliability polynomials if we consider choosing more than 2 vertices.

We could also consider the split reliability for different families of graphs. We did not consider the split reliability of bipartite graphs in this thesis, and we may be able to develop more theorems about split reliability of bipartite graphs with s and t in different parts or in the same part. We could also consider the split reliability of digraphs, where we specify that we want there to be a path from every vertex to precisely one of s or t or vice versa.

We could also look for better lower and upper bounds for the split reliability polynomial. Specifically, our upper bounds are based on finding the two-terminal or all-terminal reliability of graphs. We also have lower bounds for both all-terminal and two-terminal reliability polynomials. With this in mind, it may be possible to find better upper bounds of the split reliability polynomial by examining the lower bounds of two-terminal and all-terminal reliability.

Overall, there are still many properties of split reliability that have yet to be explored and I intend to look further into this field in future work.

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