EXISTENTIALLY CLOSED PROPERTY IN DIRECTED INFINITE GRAPHS

by

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Abstract

Graph theory abounds with applications inside mathematics itself, and in computer science, and engineering. One direction of research within graph theory is the topic of infinite graphs, which is the focus of this thesis.

We review results on existentially closed (or e.c.) graphs and directed graphs. Properties of e.c. graphs, including isomorphism results, universality, and connections with probability are discussed. We develop new results in infinite random directed graphs. We first give the definition of two types of directed graphs used in our thesis. We define directed e.c of two types I and II. We explore the properties of such graphs.

We review the LARG model for random geometric graphs, and the definition of δ-g.e.c. We recall the infinite random one-dimensional random geometric graph and summarize its properties. We then extend our study to directed g.e.c. graphs. We first define δ-d.g.e.c. graphs of types I and II. We define directed random graphs with probabilities for an edge of either directions between two vertices. We find that with probability 1, directed random graphs are of directed e.c. of type I or type II. Then we define the directed LARG graphs, the DLARG model. We show that DLARG graphs are linked to directed random graphs of the same threshold. The final topic focuses on directed geometric e.c. graphs with asymmetric thresholds of influence. We characterize when such graphs are isomorphic, and study the ratio of the radii of intervals and connect this with limits of the graph distance.

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Chapter 1

Introduction

1.1 Background

The study of infinite graphs is an interesting, but often untouched, part of graph theory. There are typical kinds of phenomena that will always appear when graphs are infinite. These can lead to deep and fascinating problems.

Perhaps the most typical such phenomena occur when they have only countably many vertices. This is not surprising: after all, some of the most basic structural features of graphs, such as paths, are intrinsically countable.

Problems that become interesting only for uncountable graphs tend to be interesting for reasons that have more to do with sets than with graphs, and are studied in combinatorial set theory. This, too, is a charming field, but not the topic of this thesis.

The problems we shall consider will be of interest for countable graphs, and set-theoretic problems will not arise. The terminology we need is exactly the same as for finite graphs, except when we wish to describe an aspect of infinite graphs that has no finite counterpart.

Since the rise of computer networks in the last decade, the study of large-scale and complex networks has emerged. Examples include, the study of the web graphs [2], which consist of web pages and the hyperlinks and the friendship network on Facebook. We are interested in the area of using random graph models to represent such computer networks. In this chapter, we introduce the basic concepts of graph theory, probability theory and mathematical analysis.

In the real world, these networks often have underlying metric spaces which influence their structure. For an example, for the World Wide Web, the web sites and pages are organized in a high dimensional topic space where similar contents are positioned close together in the underlying metric space. In the area of stochastic graph modeling, geometric graph models are also applied. In this line of modeling, we use points from metric spaces as vertices. The edges between these vertices are determined by the distance between vertices in the metric space. Geometric graph models are useful in many real networks [4], including wireless networks [10, 16, 22, 23].

In the following chapters, we start with the introductory concepts in the areas of mathematical analysis, probability and graph theory. In the chapter of introductory concepts in graph theory, we include a number of relevant results cited explicitly or implicitly by our study.

1.2 Introductory concepts in graph theory

We first define graphs.

Definition 1. *A* graph *is a collection of points and lines connecting a pair of them. The points of a graph are most commonly known as* vertices*, but may also be called vertices or simply points. Similarly, the lines connecting the vertices of a graph are most commonly known as* edges*, but may also be called arcs or lines.*

A *simple* graph has no loops or multipled edges. If multiple edges are allowed between vertices, then the graph is known as a *multigraph*. If we allow loops and multipled edges, then we obtain *pseudographs*.

Recall that a *countable set* is a set with the same cardinality as some subset of the set of natural numbers. A countable set is either a finite set or a countably infinite set.

Definition 2. *A graph $G(V, E)$ is called a countable graph if its vertex set V is countable.*

We give the following definitions. A directed graph differs from an ordinary or undirected graph, in that the latter is defined in terms of unordered pairs.

Definition 3. *A* directed graph *is graph where all the edges are directed from one vertex to another. A directed graph is sometimes called a* digraph*. A directed graph having no multiple edges or loops is called a simple directed graph. A directed graph*

Figure 1.1: Examples of simple graph (left), multigraph (center) and pseudograph (right).

having no symmetric pair of directed edges (that is, no bidirected edges) is called an oriented graph.

We give a few examples in Figures 1.2 and 1.3.

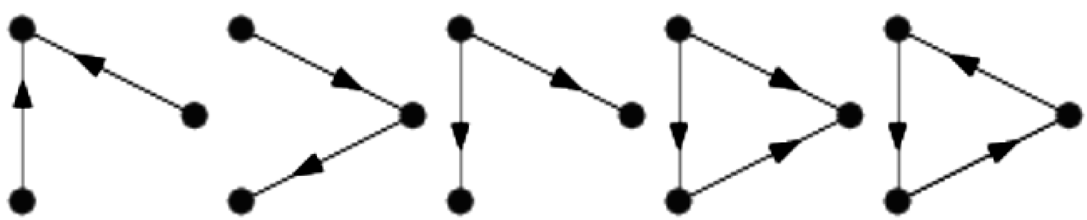

Figure 1.2: Examples of oriented graphs.

Definition 4. *A graph H is a* subgraph *of graph G if its graph vertices and graph edges form subsets of the graph vertices and graph edges of G. If H is a subgraph of G, then G is said to be a* supergraph *of H.*

Definition 5. *A* vertex-induced *subgraph (sometimes simply called an* induced subgraph*) is a subset of the vertices of a graph G together with any edges whose endpoints are both in this subset.*

Figure 1.4 illustrates the subgraph induced on the complete graph K_{10} by the vertex subset $\{1, 2, 3, 5, 7, 10\}$.

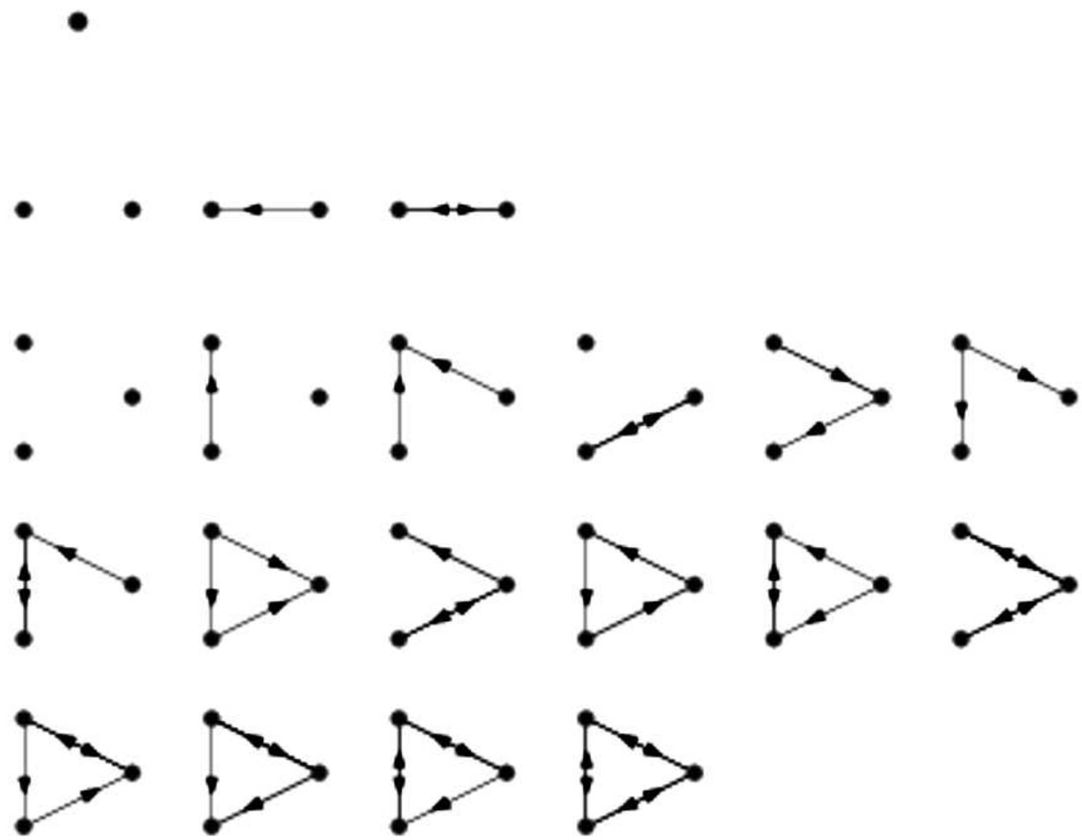

Figure 1.3: Examples of simple directed graphs.

A random graph is a graph that is obtained by randomly sampling from a collection of graphs. This collection may be characterized by certain graph parameters having fixed values.

Definition 6. *There are two closely related variants of the Erdős–Rényi (ER) random graph model.*

1. *In the $G(n, M)$ model, a graph is chosen uniformly at random from the collection of all graphs which have n vertices and M edges.*

2. *In the $G(n, p)$ model, a graph is constructed by connecting vertices randomly. Each edge is included in the graph with probability p independent from every other edge.*

In this thesis, we only use the $G(n, p)$ model. The $G(n, p)$ model was first introduced by Gilbert in a 1959 paper studying the connectivity threshold [15]. The

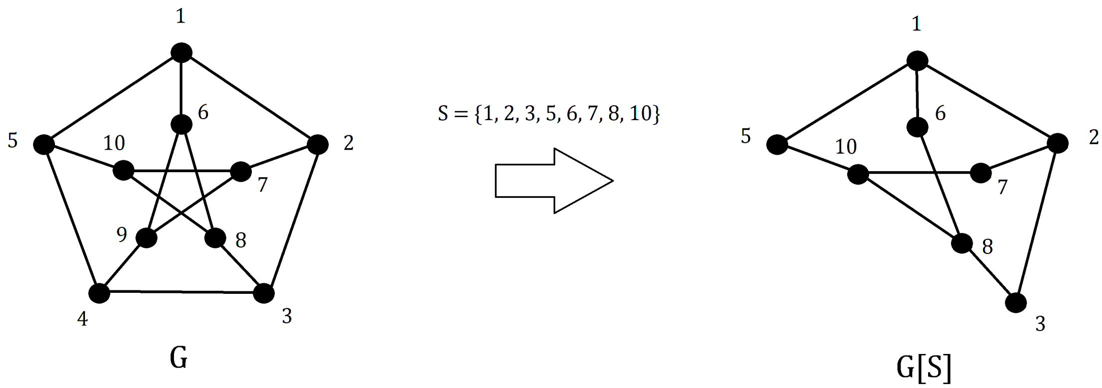

Figure 1.4: Example of induced subgraph.

$G(n, M)$ model was introduced by Erdős and Rényi in 1959. As with Gilbert, their first investigations were as to the connectivity of $G(n, M)$, with the more detailed analysis following in 1960.

In an area closely related to the study by stochastic graph models, we may consider infinite limit graphs. It is a well practiced scientific model to use the infinite limit. This is especially helpful applied to large-scale networks and long term behaviors [25].

Next, we recall the concepts of graph isomorphism and isotype. Two graphs which contain the same number of graph vertices connected in the same way are said to be *isomorphic.*

Definition 7. *Two graphs G and H with graph vertices $V_n = \{1, 2, \ldots, n\}$ are said to be* isomorphic *if there is a permutation p of V_n such that edge $\{u, v\}$ is in the set of graph edges $E(G)$ if and only if edge $\{p(u), p(v)\}$ is in the set of graph edges $E(H)$.*

Definition 8. *An isotype is an isomorphsm type of graphs. We denote isomorphic graphs G and H by $G \cong H$.*

Unlike finite graphs, infinite graphs offer the possibility to represent an entire graph property P by just one specimen.

Definition 9. *A* universal *graph is an infinite graph that contains every countable graph as an induced subgraph.*

More precisely, if $\leq$ is a graph relation (such as subgraph or induced subgraph), we call a countable graph G^* *universal* for $\mathbb{P}$ (for $\leq$) if $G^* \in \mathbb{P}$ and $G \leq G^*$ for every

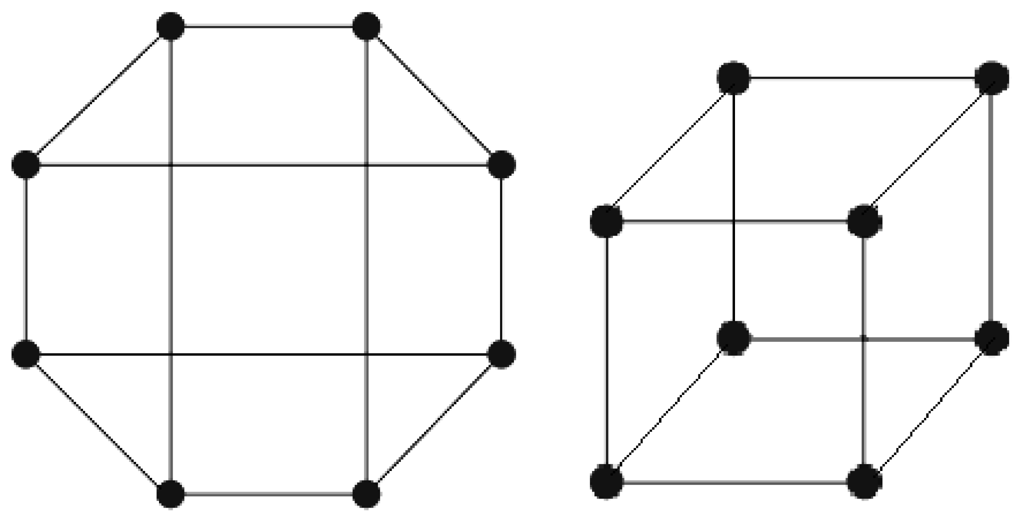

Figure 1.5: Examples of isomorphic graphs.

countable graph $G \in \mathbb{P}$. In this thesis, all graphs considered are simple, undirected and countable unless otherwise stated.

1. If S is a set of vertices in G, notation $G[S]$ represents the subgraph of G induced by S.

2. We write $H \leq G$ to represent the case when H is an induced subgraph of G.

3. $\mathbb{N}$ denotes the non-negative natural numbers.

4. $\mathbb{N}^+$ denotes the positive natural numbers.

Importantly, we have the definition for idf, δdf and (δ_1, δ_2)df properties used in Chapters 4, 5 and 6. The concept of idf is first introduced by Bonato and Janssen [4]. This thesis extends the definition to δdf and (δ_1, δ_2)df, where δ, δ_1 and δ_2 are real numbers. These are used to prevent boundary condition from invalidating our results. For an example of idf, Definition 28's condition number two requires the distance between two vertices to be less than the threshold δ. If there is a vertex at exact distance δ, the case becomes intractable for a few theorems in [4]. Hence, idf property is introduced and mentioned when needed.

Definition 10. *Fix a set S of real numbers and a metric of S. Let δ, δ_1 and δ_2 be real numbers. If the set S has* idf *property, it means the metric distance between any two* *distinct elements in S is not integer. The set S with* δdf *property means the metric distance between any two distinct elements is not a multiple of δ. The set S with* (δ_1, δ_2)df *property means for all pairs of vertices of the graph, denoted u, v, we have that*

$$\mathrm{d}(u, v) \notin \{s\delta_1 + t\delta_2, s, t \in \mathbb{Z}\}.$$

In this thesis, we have three different types of distances. There is the distance in the underlying metric space of real numbers. There is the graph distance of the undirected graphs, and there is the graph distance in the directed graphs. To differentiate the various types of distances, we make the following clarification by giving each a unique notation which we use in our writing.

Recall that a *path* is a *trail* in which all vertices (except possibly the first and last) are distinct. A *trail* is a *walk* in which all edges are distinct. A *walk* of length k in a graph is an alternating sequence of vertices and edges, $v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$, which begins and ends with vertices. If the graph is undirected, then the endpoints of e_i are v_i and v_{i+1}. If the graph is directed, then e_i is an arc from v_i to v_{i+1}. In a directed graph, a *directed path* (sometimes called *dipath*) is sequence of edges which connect a sequence of vertices, but with the added restriction that the edges all be directed in the same direction.

Definition 11. *Let $G(V, E)$ be a directed graph and let u, v be vertices of G.*

1. *The undirected distance between two vertices u and v is the minimum length of the paths connecting them. If no such path exists, then the distance is set equal to ∞. We let $dist(u, v)$ denote the distance.*

2. *The directed distance between two vertices u and v is the minimum length of the dipaths from u to v. If no such path exists, then the distance is set equal to ∞. We let* $\mathrm{dgd}(u, v)$ *denote the distance.*

1.3 Introductory concepts on metric spaces

In mathematics, specifically the area of analysis, a metric space is a set for which distances between all members of the set are defined. Those distances, taken together, are called a metric on the set.

Definition 12. *A metric space is a set X together with a function d (called a metric or* distance function*) which assigns a real number* $\mathrm{d}(x, y)$ *for every pair $x, y \in X$, satisfying the following properties for all $x, y, z \in X$:*

1. $\mathrm{d}(x, y) \geq 0$ *and* $\mathrm{d}(x, y) = 0$ *if and only if* $x = y$,
2. $\mathrm{d}(x, y) = \mathrm{d}(y, x)$,
3. $\mathrm{d}(x, y) + \mathrm{d}(y, z) \geq \mathrm{d}(x, z)$.

The concept of a ball is an important one in a metric space.

Definition 13. *Let S be a metric space and d its distance function, define the* (open) ball *of radius δ around x by $B_\delta(x) = \{u \in S : \mathrm{d}(u, x) < \delta\}$*

Next, we recall the definition of a density and closure. We use these concepts for the vertex sets of our graphs.

Definition 14. *A subset V is dense in S if for every point $x \in S$, every ball around x contains at least another point from V.*

Definition 15. *The closure of a set V is the set V together with all of its limit points. The closure of V is denoted by $\overline{V}$.*

We lastly recall the definition of a Euclidean metric. This is one of the metric spaces that we consider in our proofs.

Definition 16. *The Euclidean metric is the function $d : \mathbb{R}^n \times \mathbb{R}^n$ that assigns to any two vectors in Euclidean n-space, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ the number*

$$\mathrm{d}(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

As defined above, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. The points $\mathbf{x}$ and $\mathbf{y}$ lie in n-dimensional space. For a vector $\mathbf{z}$, let $\mathbf{z}[i]$ denote its i^{th} components. For a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

Definition 17. *Let p be a positive integer. The ℓ_p metric is the function* $\mathrm{d}_p : \mathbb{R}^n \times \mathbb{R}^n$ *for the distance between* $\mathbf{x}$ *and* $\mathbf{y}$*:*

$$\mathrm{d}_p(\mathbf{x}, \mathbf{y}) = \Big(\sum_{j=1}^{n} |\mathbf{x}[j] - \mathbf{y}[j]|^p \Big)^{\frac{1}{p}}.$$

Note that d_2 is the usual notation for the Euclidean distance between the two points $\mathbf{x}$ and $\mathbf{y}$.

Definition 18. *The ℓ_∞ metric is the function* $\mathrm{d}_\infty : \mathbb{R}^n \times \mathbb{R}^n$ *for the distance between* $\mathbf{x}$ *and* $\mathbf{y}$*:*

$$\mathrm{d}_\infty(\mathbf{x}, \mathbf{y}) = \max\{|\mathbf{x}[j] - \mathbf{y}[j]|, j = 1, 2, \ldots, n\}.$$

For an example of ℓ_∞ metric, let $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (3, 12)$ in the plane of real numbers. Then, we have

$$\begin{aligned} \mathrm{d}_\infty(\mathbf{a}, \mathbf{b}) &= \max\{|\mathbf{a}[1] - \mathbf{b}[1]|, |\mathbf{a}[2] - \mathbf{b}[2]|\} \\ &= \max\{|1 - 3|, |2 - 12|\} \\ &= \max\{2, 10\} \\ &= 10. \end{aligned}$$

1.4 Introductory concepts in probability

Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner. They have applications in many disciplines including sciences such as biology, chemistry, ecology, neuroscience and physics as well as technology and engineering fields such as image processing, signal processing, information theory, computer science, cryptography and telecommunications. Furthermore, seemingly random changes in financial markets have motivated the extensive use of stochastic processes in finance.

The study of the network models through stochastic models are mostly using asymptotic results. This is the case for situations when the number of participating vertices or vertices in the graph is very large. Eventually, we extend the idea of a very large set of vertices to an infinite set.

In probability theory, a probability space or a probability triple of $(\Omega, \mathcal{F}, p)$ is a mathematical construct that models a real-world process (or experiment) consisting of states that occur randomly. A probability space is constructed with a specific kind of situation or experiment in mind. One proposes that each time a situation of that kind arises, the set of possible outcomes is the same and the probabilities are also the same.

Definition 19. *A probability space consists of three parts:*

1. *A sample space, Ω, which is the set of all possible outcomes.*
2. *A set of events $\mathcal{F}$, where each event is a set containing zero or more outcomes.*
3. *The assignment of probabilities to the events; that is, a function p from events to probabilities.*

We use the following property called the *union bound*.

Theorem 1. [26] *For a countable set of events $A_1, A_2, A_3, \ldots,$ in a probability space with probability measure P, we have that*

$$P\Big(\bigcup_i A_i\Big) \leq \sum_i P(A_i).$$

Once the probability space is established, it is assumed that nature makes its move and selects a single outcome, ω, from the sample space Ω. All the events in $\mathcal{F}$ that contain the selected outcome ω (recall that each event is a subset of Ω) are said to have occurred. The selection performed by nature is done in such a way that if the experiment were to be repeated an infinite number of times, the relative frequencies of occurrence of each of the events would coincide with the probabilities prescribed by the function p.

Definition 20. *A* random variable*, usually written X, is a variable whose possible values are numerical outcomes of a random phenomenon.*

There are two types of *random variables*, discrete and continuous. A *discrete random variable* is one which may take on only a countable number of distinct values such as $0, 1, 2, 3, 4, \ldots$. Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete.

A *continuous random variable* takes all values in a given interval of numbers. Continuous random variables are usually measurements. Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.

Definition 21. *A* discrete probability function*, $p(x)$, is a function that satisfies the following properties.*

1. *The probability that x can take a specific value is $p(x)$. That is*

$$P[X = x] = p(x) = p_x$$

2. *$p(x)$ is non-negative for all real x.*

3. *The sum of $p(x)$ over all possible values of x is 1, that is*

$$\sum_j p_j = 1$$

   where j represents all possible values that x can have and p_j is the probability at x_j.

Discrete probability functions *are referred to as* probability mass functions*.*

The relevant topics in this section are summarized by several publications, such as [9, 17, 24, 26].

1.5 Outline of the thesis

In the introductory chapter of this thesis, we reviewed concepts from graph theory and probability theory. In Chapter 2 and 3, we review results on the Rado graph.

We review the concepts of e.c. property. We show that a graph with e.c. property is infinite, and has diameter 2. We show that any e.c. graphs are isomorphic to each other. The method used is the well known "back-and-forth". Then, we review the definition of a Rado graph R. We review the definition of $G(\mathbb{N}, p)$ and the theorem that all such graphs are isomorphic to R. We also review the properties of R. We review the definition of the $LARG$ model, and the definition of δ-g.e.c. We define GR and show that it is δ-g.e.c. We review the findings on the exact forms of distances between any two vertices in a geometric δ-graph.

In Chapter 4, we develop new results in directed random graph. We first give the definition of two types of directed graphs used in our thesis. We define directed e.c of two types I and II. We explore the properties of such graphs. We find these graphs are infinite of infinite incoming or outgoing degrees. We further find results on isomorphisms. If two oriented graphs are directed existentially closed, then they are isomorphic. Similar result is found for the other type of directed graphs. We define the oriented random graph and the type II random graph. We define the isotypes for them. We show both types of directed random graphs contain every finite oriented graph as their subgraph.

In Chapter 5, we extend our study to directed g.e.c. We first define δ-d.g.e.c. of types I and II. We define $G(\mathbb{N}, p, \alpha)$ and $G(\mathbb{N}, p, \beta, \alpha)$. We find that with probability 1, $G(\mathbb{N}, p, \alpha)$ is d.e.c. I, and $G(\mathbb{N}, p, \beta, \alpha)$ is d.e.c. II. Then we define $DLARG$ model. We show that $DLARG(V, \delta_1, \delta_2, p)$ is linked to δ_2-d.g.e.c of type I and II.

In Chapter 6, we study d.g.e.c with different thresholds on real numbers. We define (δ_1, δ_2)-d.g.e.c. We show that $DLARG(V, \delta_1, \delta_2, p)$ is a (δ_1, δ_2) linear graph. We give the exact formula for distances of vertices in a (δ_1, δ_2) linear graph. We find the equivalent condition for two such graphs to be isomorphic. We determine isomorphism of two graphs from the ratio of their thresholds δ_1, δ_2 in each graph. In Chapter 7, we conclude this work and post a few open problems.

Chapter 2

General results of Rado graphs

The concept of a finite random graph is readily extended to an infinite random graph. We introduce the definition of the Rado graph using randomness. We note that the Rado graph can also be constructed non-randomly, by symmetrizing the membership relation of the hereditarily finite sets, by applying the bit predicate to the binary representations of the natural numbers, or as an infinite Paley graph that has edges connecting pairs of prime numbers congruent to $1 \mod 4$ when one is a quadratic residue modulo the other.

The Rado graph is named after Richard Rado who gave one of its earliest explicit definitions. We give the original construction by Richard Rado in 1964.

Definition 22. *The Rado graph (also called the Erdős–Rényi graph or infinite random graph) is a countably infinite graph that can be constructed as follows.*

1. *The vertex set of the graph is the set* $\mathbb{N}$ *of natural numbers including* 0.

2. *Given two vertices* x *and* y*, with* $x < y$*, we join* x *to* y *if, when* y *is written as binary number, its* x*-th digit is* 1.

3. *No other edge is added to the graph.*

The same Rado graph can be generated by choosing independently at random for each pair of natural numbers (vertices) whether to connect the vertices by an edge with any probability $p \in (0, 1)$. The Rado graph is one of the most important objects of study in this thesis.

2.1 The e.c. property

We define the existentially closed property as follows.

Definition 23. *A graph G is existentially closed (or e.c.) if for all finite disjoint sets of vertices A and B (one or both of which may be empty), there is a vertex $z \notin A \cup B$ adjacent to all vertices of A and to no vertex of B. We say that z is correctly joined to A and B.*

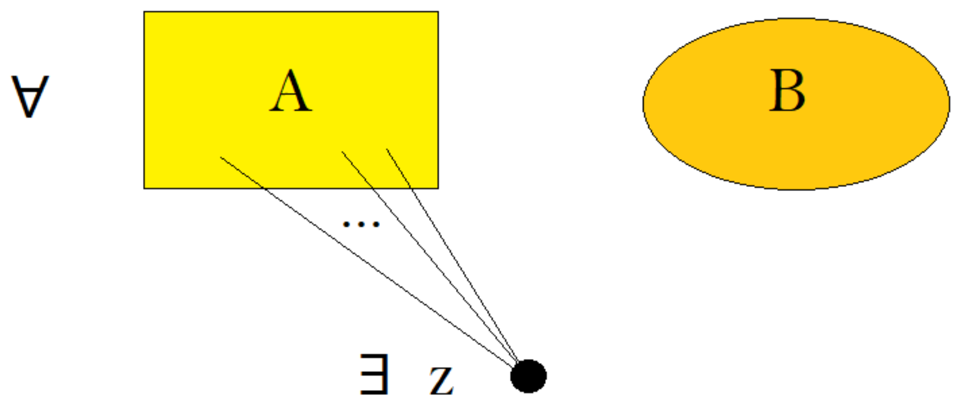

Figure 2.1: The e.c. property.

We can show that a graph with e.c. property is infinite.

Lemma 1. [14] *A graph G with e.c. property is infinite.*

Proof. We use proof by contradiction. Suppose there is a graph G which is a finite graph whose vertex set V is finite. Using Definition 23: we let $A = V$ and $B = \emptyset$. Then there is no vertex of the graph G correctly joined to A and B. Therefore, G does not have e.c. property, a contradiction. Thus, a graph G with e.c. property is an infinite graph. $\square$

The next result discusses the diameter of R. We show that the diameter is at most 2 and greater than 1.

Lemma 2. [14] *A graph R with e.c. property has diameter* 2*.*

Proof. We show that R cannot be a clique of diameter 1. If R is a clique, then for any nonempty finite set B from Definition 23, we cannot find any vertex not joined to B. Therefore, R does not have e.c. property, a contradiction.

Lastly, we show that R cannot have diameter larger than 2. Suppose there are two vertices x and y of R of distance at least 3. Then let set A from Definition 23 contains both vertices x and y. Then we cannot find any vertex joined to A, because the distance between x and y is at least 3.Therefore, R does not have e.c. property, a contradiction. Then, we can conclude that a graph R with e.c. property has diameter exactly 2. □

It turns out that the following theorem was first proven by Roland Fraïssé [13]. The method of this proof is called the "back and forth" method.

Theorem 2. *If graphs G and H are e.c., then G and H are isomorphic.*

Proof. We have G and H be two countable graphs satisfying e.c. property. Suppose that f is a map from a finite set $\{x_1, \ldots, x_n\}$ of vertices of G to H, which is an isomorphism of induced subgraphs, and x_{n+1} is another vertex of G. We show that f can be extended to x_{n+1}. Let U be the set of neighbors of x_{n+1} within $\{x_1, \ldots, x_n\}$, and $V = \{x_1, \ldots, x_n\} \setminus U$. A potential image of x_{n+1} must be a vertex of H adjacent to every vertex in $f(U)$ and nonadjacent to every vertex in $f(V)$. Now property e.c. (for the graph H) guarantees that such a vertex exists.

Now we use a proof technique called "back-and-forth". Enumerate the vertices of G and H as $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ respectively. We build finite isomorphisms f_n as follows. Start with $f_0 = \emptyset$. Suppose that f_n has been constructed. If n is even, let m be the smallest index of a vertex of G not in the domain of f_n; then extend f_n (as above) to a map f_{n+1} with x_m in its domain. (To avoid the use of the Axiom of Choice, select the correctly-joined vertex of H with smallest index to be the image of x_m.) If n is odd, then we work backwards. Let m be the smallest index of a vertex of H which is not in the range of f_n; extend f_n to a map f_{n+1} with y_m in its range (using the e.c. property for G).

Take f to be the union of all these partial maps. By going alternately back and forth, we guaranteed that every vertex of G is in the domain, and every vertex of H

is in the range of f. Hence, f is the required isomorphism. $\square$

The proof technique "back-and-forth" is often attributed to Cantor [8], in his characterization of the rationals as countable dense ordered set without endpoints. However, as Plotkin [54] has shown, it was not used by Cantor; it was discovered by Huntington [20] and popularized by Hausdorff [19].

2.2 The random graph

Definition 23 yields a unique isomorphism type which we call the Rado graph. The isotype is of all countably infinite e.c. graph. It is the same as infinite random graph. We denote this isomorphism type of the Rado graph R. Research on R has been conducted for many years [5, 6].

In graph theory, the Rado graph, Erdős–Rényi graph, or infinite random graph is a countably infinite graph that can be constructed (with probability one) by choosing independently at random for each pair of its vertices whether to connect the vertices by an edge. The method of constructing a bijection in alternating steps, as in the uniqueness part of the proof, is known as the "back-and-forth" technique.

We define a probability space $G(\mathbb{N}, p)$.

Definition 24. *$G(\mathbb{N}, p)$ consists of graphs with vertices of the set of natural numbers, and each distinct pair of natural numbers is adjacent independently with a fixed probability p, where $p \in (0, 1)$.*

The Rado graph R is unique in another interesting respect. If we generate a countably infinite random graph by admitting its pairs of vertices as edges independently with some fixed positive probability $p \in (0, 1)$, then with probability 1, the resulting graph has the e.c. property, and is hence, isomorphic to R. Erdős and Rényi [12] have proved that, with probability 1, all $G \in G(\mathbb{N}, p)$ are isomorphic; and thus, they are the Rado graph.

Theorem 3. [3] *With probability 1, $G(\mathbb{N}, p)$ is e.c. and so isomorphic to R.*

Proof. We have to show that the event that e.c. property fails has probability 0; that is, the set of graphs not satisfying e.c. property is a null set. By Definition 23, we let $A = \{u_1, u_2, \ldots u_m\}$ and $B = \{v_1, u_2, \ldots v_m\}$ be distinct sets of vertices. It is enough to show that the set of graphs for which e.c. property fails for some given vertices $u_1, \ldots, u_m, v_1, \ldots, v_n$ is null.

For this deduction, we use an elementary lemma from measure theory: the union of countably many null sets is null. There are only countably many values of m and n, and for each pair of values, only countably many choices of the vertices $u_1, \ldots, u_m, v_1, \ldots, v_n$.

Now we can calculate the probability of this set. Let $z_1, \ldots, z_N$ be vertices distinct from $u_1, \ldots, u_m, v_1, \ldots, v_n$. Let p_{z_i} denote the probability that any z_i is not correctly joined.

$$p_{z_i} = 1 - \frac{1}{2^{m+n}}.$$

Since these events are independent (for different z_i), the probability that none of $z_1, \ldots, z_N$ is correctly joined is

$$\bigcap_{i=1}^{N} p_{z_i} = \left(1 - \frac{1}{2^{m+n}}\right)^N. \tag{2.1}$$

Equation 2.1 tends to 0 as $N \to \infty$; so the event that no vertex is correctly joined does have probability 0. By Theorem 2, $G(\mathbb{N}, p)$ is e.c. and so is isomorphic to R. $\square$

Next, we introduce a construction of Rado graph using number theory. An integer q is called a *quadratic residue* modulo n if it is congruent to a perfect square modulo n: that is, if there exists an integer x such that:

$$x^2 \equiv q \pmod{n}.$$

Recall that given an odd prime p and an integer q, then the *Legendre symbol* is given by:

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } q \text{ is a quadratic residue mod } p \\ -1 & \text{otherwise.} \end{cases}$$

Using these, we have the following lemma.

Lemma 3. [7] *Take as vertices the set* $\mathbb{P}$ *of primes congruent to* $1 \mod 4$. *By quadratic reciprocity, if* $p, q \in \mathbb{P}$, *then* $\left(\frac{p}{q}\right) = 1$ *if and only if* $\left(\frac{q}{p}\right) = 1$. *Here* $\left(\frac{p}{q}\right) = 1$ *means that* p *is a quadratic residue modulo* q. *We declare* p *and* q *adjacent if* $\left(\frac{p}{q}\right) = 1$. *The constructed graph is e.c.*

Proof. Let $u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbb{P}$. Choose a fixed quadratic residue $a_i \mod u_i$, for an example $a_i = 1$, and a fixed non-residue $b_j \pmod{v_j}$. Since we choose vertices the set $\mathbb{P}$ of primes congruent to $1 \mod 4$, by the Chinese Remainder Theorem [21], the congruences

$$x \equiv 1 (\text{mod } 4),$$
$$x \equiv a_i (\text{mod } u_i),$$
$$x \equiv b_j (\text{mod } v_j).$$

have a unique solution

$$x \equiv x_0 (\text{mod } 4u_1 \ldots u_m v_1 \ldots v_n). \tag{2.2}$$

Recall Dirichlet's Theorem [18]: given an arithmetic progression of terms $an+b$, for $n = 1, 2, \ldots$, the series contains an infinite number of primes if a and b are relatively prime. In (2.2), each u_i or v_j is a prime congruent to $1(\text{mod } 4)$. Since $x \equiv 1(\text{mod } 4)$, then $x_0 \equiv 1(\text{mod } 4)$ and $x_0 \equiv 1(\text{mod } 4u_1 \ldots u_m v_1 \ldots v_n)$. Apply Dirichlet's Theorem to (2.2), let b be x_0, and let a be $4u_1 \ldots u_m v_1 \ldots v_n$. Hence, there is a prime x call it z satisfying (2.2). The u_i's form set A, and the v_i's form set B, as in Definition 23. We can find z correctly connected to any sets A and B. So the e.c. property holds. $\square$

2.3 Properties of R

In 1964, Richard Rado published a construction of a countable graph which was universal. This means that every finite or countable graph occurs as an induced subgraph of Rado's graph. This is the next theorem.

Theorem 4. *R is universal.*

Proof. Indeed, in order to embed a given countable graph G in R we just map its vertices $v_1, v_2, \ldots$ to R inductively, making sure that v_n gets mapped to a vertex $v \in R$ adjacent to the images of all the neighbors of v_n in $G[v_1, \ldots, v_n]$ but not adjacent to the image of any non-neighbor of v_n in $G[v_1, \ldots, v_n]$. Hence, this map is an isomorphism between G and the subgraph of R induced by its image. □

As one would expect of a random graph, the Rado graph shows a high degree of uniformity. One aspect of this is its resilience against small changes: the deletion of finitely many vertices or edges, and similar local changes, leave it unchanged and result in just another copy of R.

The following robust aspect of uniformity, however, is still valid: no matter how we partition the vertex set of R into two parts, at least one of the parts will induce another isomorphic copy of R. Trivial examples aside, the Rado graph is the only countable graph with this property, and hence, unique in yet another respect [5].

Definition 25. *The graph $K^{\aleph_0}$ is the* countably infinite clique.

Theorem 5. [5] *The Rado graph is the only countable graph G other than $K^{\aleph_0}$ and $\overline{K^{\aleph_0}}$ such that, no matter how $V(G)$ is partitioned into two parts, one of the parts induces an isomorphic copy of G.*

Proof. We first show that the Rado graph R has the partition property:

Let $\{V_1, V_2\}$ be a partition of the vertex set of the graph $V(R)$. If the property fails in both $R[V_1]$ and $R[V_2]$, say for sets of vertices U_1, W_1 and U_2, W_2, respectively. Then the property fails for the sets $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$ in R. This is a contradiction.

Next, we show that the Rado graph R has uniqueness. To show the uniqueness of R, let $G = (V, E)$ be a countable graph with the partition property. Let V_1 be the set of isolated vertices of G and V_2 be the rest of the vertices of G. If $V_1 \neq \emptyset$, then $G \ncong G[V_2]$. This is because that G has isolated vertices but $G[V_2]$ does not. Hence, $G = G[V_1] \cong \overline{K^{\aleph_0}}$.

When G has a vertex adjacent to all other vertices, let V_2 be the set of vertices forming a clique in G and V_1 be the rest of the vertices of G. We know $V_2 \neq \emptyset$, then $G \ncong G[V_1]$. This is because that G has no isolated vertices but $G[V_1]$ does. Hence, $G = G[V_2] \cong K^{\aleph_0}$.

Assume now that G has no isolated vertex and no vertex joined to all other vertices. If G is not the Rado graph then there are sets U, W for which the property fails in G. Choose these with $|U \cup W|$ minimum. Assume first that $U \neq \emptyset$, and pick $u \in U$. Let V_1 consist of u and all vertices outside $U \cup W$ that are not adjacent to u, and let V_2 contain the remaining vertices.

As u is isolated in $G[V_1]$, we have $G \ncong G[V_1]$, and hence, $G \cong G[V_2]$ as well. By the minimality of $|U \cup W|$, there is a vertex $v \in G[V_2] - U - W$ such that v is adjacent to every vertex in $U \setminus \{u\}$ and v is adjacent to no vertex in W. But v is also adjacent to u, because it lies in V_2. So U, W and v satisfy the property for G. This is contrary to the assumption.

In the last step, we assume that $U = \emptyset$. Then $W \neq \emptyset$. We choose $w \in W$, and consider the partition $\{V_1, V_2\}$ of V where V_1 consists of w and all its neighbors outside W.

As before, $G \ncong G[V_1]$ and hence, $G \cong G[V_2]$. Therefore, U and $W \setminus \{w\}$ satisfy the property in $G[V_2]$, with $v \in V_2 \setminus W$ say, and then U, W, v satisfy the property in G. $\square$

A second piece of evidence of the high degree of uniformity in the structure of the Rado graph is its large automorphism group. We first review the relevant definitions and then look at the previous results.

The graph R is vertex-transitive: given any two vertices x and y, there is an automorphism of R mapping x to y. In fact, much more is true: using the back-and-forth technique, one can show that the Rado graph is homogeneous (any isomorphism between finite induced subgraphs extends to an automorphism of the graph) [7]: every isomorphism homogeneous between two finite induced subgraphs can be extended to an automorphism of the entire graph. Next, we look at the question: which other countable graphs are homogeneous?

The complete graph $K^{\aleph_0}$ and its complement are again obvious examples. Moreover, for every integer $r > 3$ there is a homogeneous K^r-free graph R^r, constructed as follows.

Let R^r_0 be K^1, and let R^r_{n+1} be obtained from R^r_n by joining, for every subgraph $H \ncong K^{r-1}$ of R^r_n, a new vertex v_H to every vertex in H. Then let R^r be $\bigcup_{n\in\mathbb{N}} R^r_n$. Clearly, as the new vertices v_H of R^r_{n+1} are independent, there is no K^r in R^r_{n+1} if there was none in R^r_n, so $R^r \not\supseteq K^r$ by induction on n. Just like the Rado graph, R^r is universal among the K^r-free countable graphs, and it is clearly homogeneous.

By the following deep theorem of Lachlan and Woodrow, the countable homogeneous graphs we have seen so far are essentially all:

Theorem 6. *[27] Every countably infinite homogeneous graph is one of the following:*

1. *a disjoint union of complete graphs of the same order, or the complement of such a graph;*

2. *the graph R^r or its complement, for some $r> 3$;*

3. *the Rado graph R.*

Chapter 3

Random geometric graphs

The term *geometric graph theory* is often used to refer to a body of research related to graphs defined by geometric means. We focus on infinite random geometric graphs. We define a random geometric model that plays an important role in the thesis.

Definition 26. *A random geometric graph is constructed by randomly placing N vertices in some metric space (according to a specified probability distribution) and having two vertices adjacent if and only if their distance is in a given range; for example, smaller than a certain neighborhood radius, $r > 0$.*

Random geometric graphs have been well studied [1, 11, 29]. One area of study is through the stochastic process.

Random geometric graphs resemble real human social networks in a number of ways. For instance, they spontaneously demonstrate community structure; that is, clusters of vertices with high modularity. Modularity was designed to measure the strength of division of a network into modules (also called clusters). Networks with high modularity have dense connections between the nodes within modules but sparse connections between nodes in different modules. Other random graph generation algorithms, such as those generated using the Erdős–Rényi model or Barabási–Albert (BA) model do not create this type of structure. Additionally, random geometric graphs display degree associativity: popular vertices (those with high degree) are particularly likely to be linked to other popular vertices. Random geometric graphs has also been used in the modeling of ad hoc networks [28]. Next, we define the concept of a Local Area Random Graph LARG(V, δ, p).

Definition 27. *Let S be a metric space, and d be the distance function: $d : S \times S \to \mathbb{R}$. The Local Area Random Graph LARG(V, δ, p) has vertices V, where $V \subseteq S$. For each pair of vertices u and v with $\mathrm{d}(u, v) < \delta$, an edge is added independently with probability p.*

We note that the LARG model can be used to generalizes many classes of random graphs: the random geometric graph arises from LARG with $p = 1$, and the binomial random graph $G(n, p)$ arises from LARG when S has finite diameter d_s and $\delta \geq d_s$. Here we identify $G(n, p)$ by its full name the binomial random graph. In this thesis, we do not use the other type of random graph, the uniform random graph.

When the set V of LARG is infinite, [3] shows that with probability 1, graphs in LARG(V, δ, p) satisfy a certain adjacency property [2]. In Theorem 7, the concept δ-g.e.c. is given immediately next in Definition 28.

Theorem 7. [2] *Let (S, d) be a metric space and V a countable subset of S which is dense in itself. If $\delta > 0$ and $p \in (0, 1)$, then with probability 1, LARG(V, δ, p) is δ-g.e.c.*

The unique isotype of the infinite random graph R is characterized by the e.c. property. Here, we introduce the concept of geometrically e.c.

Definition 28. *Let $G = (V, E)$ be a graph whose vertices are points in the metric space S with metric d. The graph G is geometrically e.c. at level δ (or δ-g.e.c.) if for all δ' so that $0 < \delta' < \delta$, for all $x \in V$, and for all disjoint finite sets A and B so that $A \cup B \in B_\delta(x)$, there exists a vertex $z \notin A \cup B \cup \{x\}$ so that*

1. *z is correctly joined to A and B,*

2. *for all $u \in A \cup B, \mathrm{d}(u, z) < \delta$, and*

3. *$\mathrm{d}(x, z) < \delta'$.*

The definition of geometric e.c property closely resembles the e.c. property. The definition of geometrically e.c. at level δ (or δ-g.e.c.) implies that V is dense in itself. If G is δ-e.c, then G is δ'-e.c for all $0 < \delta' < \delta$.

The difference of the definitions of the geometric e.c property and the e.c. property is that the correctly joined vertex must exists only for sets A and B which are contained in an open ball with radius δ and center x. The definition of geometric e.c requires that it must be possible to choose the vertex z correctly joined to A and B arbitrarily close to x.

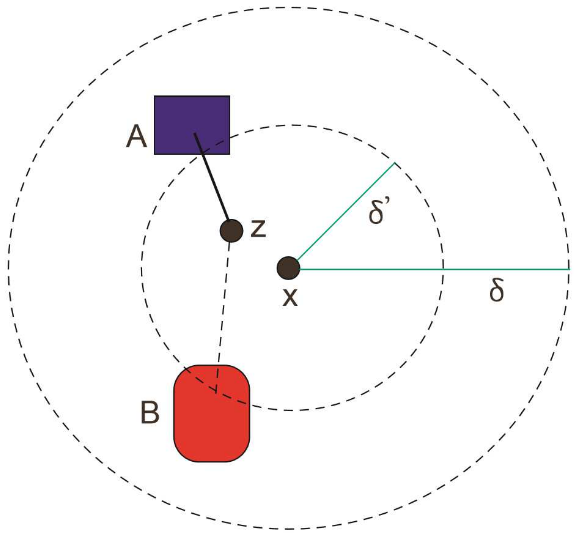

Figure 3.1: [2] The δ-g.e.c. property.

Let a graph $G = (V, E)$ have vertices from the metric space (S, d). Let δ be a *threshold* for the edges of G. In other words, for all edges $uv \in E$, $\mathrm{d}(u, v) < \delta$. We have the following definition of a geometrically g.e.c. graph.

Definition 29. *A geometric δ-graph is a geometrically g.e.c. graph at level δ and has threshold δ.*

By definition, a graph G generated by $LARG(V, \delta, p)$ has threshold δ.

If V is countable and dense in itself, then a graph G generated by $LARG(V, \delta, p)$ is a geometric δ-graph. Therefore, we claim that the random graph model generates geometric δ-graphs.

Next, we recapture the graph GR(V, δ, σ) [2]. This definition gives a inductive construction of the δ-g.e.c. graph. The limit of the process, when $t \to \infty$, is a δ-g.e.c. graph.

Definition 30. *Let $\delta > 0$. Let V be a countable set of vertices which is dense in itself. Let $\sigma : \mathbb{N} \to V$ be a linear ordering. Define GR(V, δ, σ) as the limit of a chain of finite graphs R_t, where $R_t \le R_{t+1}$ for any $t > 1$, and $\{\sigma(i) : 1 \le i \le t\} \subseteq V(R_t)$.*

1. *Let R_1 be the trivial graph with vertex set $\sigma(1)$.*

2. *Let R_t be defined and $\{\sigma(i) : 1 \le i \le t\} \subseteq V(R_t)$.*

3. *To construct R_{t+1}, we first enumerate all pairs (A, x), so that the following conditions are satisfied: $A \subseteq V(R_t)$ and $x \in V(R_t) \setminus A$ and, so that $A \subseteq B_\delta(x)$, via a lexicographic ordering based on σ. The enumeration is possible because the set of vertices is countable, and dense in itself.*

4. *For each pair (A, x), in order, choose $z = z_{A,x}$ to be the least index point in V (according to σ) such that z has not been chosen for any previous pairs (A, x), $B_\delta(z) \cap V(R_t) = B_\delta(x) \cap V(R_t)$ and $\mathrm{d}(z, x) < \min\{\frac{1}{t}, \delta\}$.*

5. *Join z to all vertices in A and to no other vertices of R_t. Allow $\sigma(t+1)$ to be an isolated vertex to form the graph R_{t+1}.*

In Definition 30, the vertex z is guaranteed to always exist. This is due to the requirement that the set of vertice V of the graph is a dense set, and R_t is a finite set. By this Definition 30, GR(V, δ, σ) is a δ-threshold graph.

Theorem 8. [2] *The graph GR(V, δ, σ) is δ-g.e.c.*

The work in this paper [2] supplies the following result.

Theorem 9. [2] *Let S be a metric space. Let $\delta \in \mathbb{R}$. Let $U \subseteq S$ be so that $U \subseteq B_\delta(x)$ for some $x \in U$. Then a δ-g.e.c. graph with vertex set U is e.c., and so is isomorphic to R.*

Theorem 9 is not an if and only if statement. Its converse is: let S be a metric space. Let $\delta \in \mathbb{R}$. Let $U \subseteq S$ be so that $U \subseteq B_\delta(x)$ for some $x \in U$. If a graph with vertex set U is e.c. and isomorphic to R, then it is a δ-g.e.c. graph.

For a counterexample to the converse of Theorem 9, consider the metric space $(\mathbb{R}, d)$, where d is the Euclidean metric, and let $\delta = 1$. Fix U an infinite clique in R, and let $U' = V(R) \setminus U$. Embed the vertices of U in $\mathbb{R}$ so that they form a set that is dense in $B_{\frac{1}{2}}(0)$. Embed the vertices of U' so that they form a set that is dense in $B_1(0) \setminus B_{\frac{1}{2}}(0)$.

Now choose $y \in U$ so that $\mathrm{d}(0, y) < \frac{1}{4}$, and let $A = \emptyset$, and $B = \{b\}$, where $b \in U \setminus \{y\}$. Let $\delta' = \frac{1}{4}$. Note that $A \cap B \subseteq B_1(y)$. The embedding of the vertices of R is such that all vertices in $B_{\delta'}(y)$ are in U, so they are all adjacent to b. Thus, $B_{\delta'}(y)$ does not contain any vertex correctly joined to A and B. Hence, this embedding of R is not δ-g.e.c.

The work in [2] shows that there exists a close relationship between the distance in graph and the metric distance in any graph that is δ-g.e.c. Subsequently, we define a step isometry and provide a theorem based on it.

Theorem 10. [2] *Let $G = (V, E)$ be a geometric δ-graph, and let $\overline{V}$ (closure of V) be a convex set. If $u, v \in V$ so that $\mathrm{d}(u, v) > \delta$ then the graph distance between u and v in G is given by*

$$\left\lfloor \frac{\mathrm{d}(u, v)}{\delta} \right\rfloor + 1.$$

This theorem yields the following corollary.

Corollary 1. *If $\overline{V}$ and $\overline{W}$ are convex, and there is a δ-g.e.c. graph with vertices V and a γ-g.e.c. graph with vertex set W which are isomorphic via f, then for every pair of vertices $u, v \in V$,*

$$\left\lfloor \frac{\mathrm{d}(u, v)}{\delta} \right\rfloor = \left\lfloor \frac{\mathrm{d}(f(u), f(v))}{\gamma} \right\rfloor.$$

The definition of a step-isometry is given in [2].

Definition 31. *Given metric spaces (S, d_S) and (T, d_T), sets $V \subseteq S$ and $W \subseteq T$, and positive real numbers δ and γ, a step-isometry at level (δ, γ) from V to W is a surjective map $f : V \to W$ with the property that for every pair of vertices $u, v \in V$,*

$$\left\lfloor \frac{d_S(u, v)}{\delta} \right\rfloor = \left\lfloor \frac{d_W(f(u), f(v))}{\gamma} \right\rfloor.$$

We also have these terms associated with the definitions.

Definition 32. *Fix $\delta > 0$ and $v_0 \in \mathbb{R}$. Each $v \in \mathbb{R}$ can be uniquely represented as*

$$v = v_0 + q(v)\delta + r(v)$$

where $q(v) = \lfloor \frac{v - v_0}{\delta} \rfloor$ and $0 \leq r(v) \leq \delta$.

1. *We call δ the* offset.

2. *We call v_0 as the* anchor.

3. $r(v)$ *denotes the* representative *of* v*.*

4. $q(v)$ *denotes the* quotient.

We have the following theorem.

Theorem 11. [2] *Let V and W be two countable dense subsets of $\mathbb{R}$. Let δ and γ be two non-negative real numbers. If G is geometric δ-graph with vertex set V and H is geometric γ-graph with vertex set W, then $G \simeq H$.*

In $\mathbb{R}$, we have the following lemma.

Lemma 4. [2] *Let V and W be subsets of $\mathbb{R}$. Let δ and γ be two non-negative real numbers. A surjective function $f : V \to W$ is a step-isometry at level (δ, γ) if and only if the following two conditions hold.*

1. *For every $u, v \in V$, $r(u) \leq r(v)$ if and only if $r(f(u)) \leq r(f(v))$.*

2. *For every $u \in V$, $q(u) = q(f(u))$.*

Chapter 4

Directed infinite random graphs

We continue our investigation of infinite graph theory from the previous chapters. Our study extends the study of the Rado graph into directed graphs. A directed graph differs from an undirected graph, in that the latter is defined in terms of unordered pairs of vertices. This work follows the work of [13, 30]. We introduce the two types of directed graphs that we study in this thesis. For d.e.c. type II graphs, we study their isomorphism properties.

The definition of a directed graph is given in Definition 3. We further clarify the subjects of our study next.

Definition 33. *An* oriented graph (type I directed graph) $G(V, E)$ *is a directed graph having no symmetric pair of directed edges. For all* $u, v \in V, u \neq v$*, if* $(u, v) \in E$ *then* $(v, u) \notin E$*.*

Definition 34. *A* type II directed graph *is a directed graph allowing symmetric pairs of directed edges.*

Based on this definition, the set of type II directed graphs contains oriented graphs. We have the following lemma.

Lemma 5. *Any oriented graph (type I directed graph) is a type II directed graph.*

Our thesis mainly concentrates on type II directed graph and explores relevant results on this type of directed graphs. Note that we omit the study of directed graphs with loops.

4.1 Basic definitions

We extend the concepts of e.c. as stated in Definition 23. We extend it from simple graphs to oriented graphs and type II directed graphs. In the concept of e.c. in the previous chapter there are only two cases: two vertices are either adjacent or not adjacent. In directed graphs, we have the additional complexity: the directions of the arcs between the vertices in the graph.

Next we define the concepts of *o.e.c.* and *i.e.c.* These are new definitions we developed for this thesis.

Definition 35. *Let $G(V, E)$ be a directed graph. We say G is out-existentially closed (o.e.c.), if for all finite disjoint sets of vertices A and B, $A, B \in V$, there exists a vertex $z \notin A \cup B$, such that z is correctly joined to A and B: there is an outgoing arc from z to every vertex in A, but there is no outgoing arc from z to any vertex in B.*

Definition 36. *Let $G(V, E)$ be a directed graph. We say G is in-existentially closed (i.e.c.), if for all finite disjoint sets of vertices A and B, $A, B \in V$, there exists a vertex $z \notin A \cup B$, such that z is correctly joined to A and B: there is an incoming arc to z from any vertex in A, but there is no incoming arc to z from any vertex in B.*

We introduce the important concept of directed-existentially closed of type I (d.e.c. I). This type of directed-existentially closed applies mainly to oriented graphs but it also applies to type II digraphs. This is a key subject of our study. Definitions 37 and 38 are first constructed in this thesis.

Definition 37. *G is directed-existentially closed of type I (d.e.c. I) if, for all finite disjoint sets of vertices A, B and C, there exists a vertex $z \notin A \cup B \cup C$, such that, there is an outgoing arc from z to any vertex in A, there is an incoming arc to z from any vertex in B, but there is no arc between z and any vertex in C, we say z is directed-correctly joined of type I to A, B and C.*

For completeness of the cases, we introduce the concept of directed-existentially closed of type II (d.e.c. II). This type of directed-existentially closed applies only to a type II directed graph.

Definition 38. *G is directed-existentially closed of type II (d.e.c. II) if, for all finite disjoint sets of vertices A, B, C and D, there exists a vertex $z \notin A \cup B \cup C \cup D$, such that, there is only an outgoing arc from z to any vertex in A, there is only an incoming arc to z from any vertex in B, there are both incoming and outgoing arcs from z to any vertex in C, but there is no arc between z and any vertex in D, we say z is directed-correctly joined of type II to A, B, C and D.*

4.2 Relevant results

The results in this section are all new although they are not extremely difficult to derive from [13, 30]. We have the following result immediately with regard to the cardinality of the vertex set.

Lemma 6. *If a directed graph $G = (V, E)$ (oriented or of type II) is either o.e.c. i.e.c. d.e.c. I, or d.e.c II, then G must be infinite.*

The proof of Lemma 6 is trivial. We omit it from here. We next have the following lemma.

Lemma 7. *Let a directed graph $G(V, E)$ be oriented or a type II directed graph. The following statements are true.*

1. *If G is o.e.c., then each vertex has infinite incoming degree.*

2. *If G is i.e.c., then each vertex has infinite outgoing degree.*

Proof. Since G is oriented or a type II directed graph, we have G is infinite, by Lemma 6.

We use proof by contradiction. For part (1) let G be o.e.c., suppose part (1) of the lemma is false, and let $a \in V$ be a vertex of finite incoming degree for a graph G being o.e.c. Let a be the only element in set A in Definition 35. Let the closed neighborhood (finite, denoted at $N^{-}(a)$) of incoming arcs of vertex a be set B. Then

G cannot be o.e.c. since there is no more vertex available outside B that can be correctly joined to A. This gives a contradiction.

For the case when G is i.e.c., suppose part (2) of the lemma is false. Let b be a vertex of finite outgoing degree for a graph G being i.e.c. By the definition of i.e.c. Let the closed neighborhood (finite, denoted at $N^+(b)$) of outgoing arcs of vertex b be set A. Let b be the only element in set B. then G cannot be i.e.c. since there is no more vertex available outside A that can be correctly joined to B. This gives a contradiction. □

Next, we look at the properties of degrees of the vertices of a directed graph that is d.e.c.

Lemma 8. *Let a directed graph $G(V, E)$ be d.e.c. Then each vertex has infinitely many one direction arcs, either outgoing or incoming; and the number of neighbors sharing two arcs of both directions is also infinite.*

Proof. When G is a directed graph of type II, the definition of d.e.c we use for this case is Definition 38. Since G is oriented or a type II directed graph, we have G is infinite, by Lemma 6.

Since G is d.e.c., then G is i.e.c and o.e.c by definition. By Lemma 7, each vertex has infinitely many one direction arcs, either outgoing or incoming.

Let c be a vertex with finite neighbors with arcs of both directions. That is $N^{\pm}(c) = N^+(c) \cup N^-(c)$ is finite. By the definition of d.e.c. of type II, let $N^{\pm}(c)$ be set A of Definition 38. Let c be the only element in set C. Then G cannot be d.e.c. since there is no more vertex available outside A that can be correctly joined to C. This gives a contradiction. □

Note that by Lemma 5, an oriented graph is a special case of the type II directed graph. Hence, we have the following corollary.

Corollary 2. *Let a directed graph $G(V, E)$ be d.e.c. If G is a oriented graph, then for each vertex, any of its one direction arcs, outgoing or incoming, are infinite.*

4.3 Isomorphism results

We have the following isomorphism result. For d.e.c. type II, we have the following isomorphism result.

Theorem 12. *Let two directed graphs G and H be countable and directed graphs. If G and H are directed type II graphs and they are both d.e.c II, then $G \simeq H$.*

Proof. Let G and H be both directed type II graphs. Let V denote the vertices of G. Let W denote the vertices of H. Let G and H both be of d.e.c. II. We want to show that they are isomorphic $G \simeq H$. We build a partial isomorphism by induction.

We use the "back-and-forth" method. Since we have that both G and H are countable, we can enumerate the vertices of G and H, as $x_1, x_2, \ldots$, and $y_1, y_2, \ldots$, respectively. We build infinite isomorphisms f_n, from $G[X_n]$ to $H[Y_n]$, as follows.

The induction hypothesis is that f_i is an isomorphism, so that for all $i \geq 1, v_i \in X_i, w_i \in Y_i, X_i \subseteq X_{i+1}$ and $Y_i \subseteq Y_{i+1}$, and f_{i+1} extends f_i. Here, v_i is a vertex in the graph $G[X_i]$. w_i is a vertex in the graph $G[Y_i]$

Start with $f_0 = \emptyset$. The induction hypothesis is true for the base case. Suppose that f_n has been constructed satisfying our induction hypothesis. Let x_{n+1} be another vertex of G. Next, we show that f can be extended to x_{n+1}.

We first identify all the neighbors of x_{n+1} in the finite set X_n of vertices of G, and properly group them. Let A be the set of outgoing neighbors of x_{n+1} within X_n. Here X_n contains $\{x_1, \ldots, x_n\}$ but possibly more vertices. Let B be the set of incoming neighbors of of x_{n+1} within X_n. Let C be the set of neighbors have arcs of both incoming and outgoing arc of x_{n+1} within X_n. Let $D = X_n \setminus (A \cup B \cup C)$.

A potential image of x_{n+1} of G must be a vertex of H adjacent by outgoing arc to every vertex in $f_n(A)$, and adjacent by incoming arc to every vertex in $f_n(B)$, and adjacent by both incoming and outgoing arc to every vertex in $f_n(C)$, and nonadjacent to every vertex in $f_n(D)$. Since G and H are both d.e.c. type II, we guarantee that such a vertex exists.

Next, then we work backwards. In similar way, we extends f_n to a map f_{n+1} with y_{n+1} in its range.

Take f to be the union of all these partial maps. By going alternately back and forth, we guaranteed that every vertex of G is in the domain and every vertex of H

is in the range of f. Hence, G and H are isomorphic via f. □

By Lemma 5, an oriented graph is a special case of the type II directed graph. Hence, we have the following corollary.

Corollary 3. *Let two directed graphs G and H be countable and directed graphs. If G and H are both oriented graphs and they are both d.e.c. I, then they are isomorphic.*

Corollary 3 says that all oriented graphs of d.e.c type I are isomorphic. Theorem 12 says that all directed type II graphs of d.e.c. type II are isomorphic. We give the definition of directed R based on it which is the corresponding concept of a Rado graph, or R on simple graphs.

Definition 39. *We call the isotype of all oriented graphs of d.e.c. type I, the* oriented random graph. *We call the isotype of all directed type II graphs of d.e.c. type II, the* directed type II random graph. *We call the oriented random graph and the directed type II random graph, the* directed random graph, *the* directed R, *the* directed Rado graph.

We explore the properties of the type II directed graph of d.e.c. type II. We have the following results.

Theorem 13. *If G is a type II directed graph of d.e.c. type II, then it contains every countable directed graph of type II as its induced subgraph.*

Proof. Let G be a directed type II graph that is d.e.c type II. We use the going forward technique.

For each countable directed type II graph H, we can specify an order of its vertices $\{x_1, x_2, \ldots\}$, and incrementally create a mapping (subgraph) in G that is isomorphic to H. Let V denote the vertices of G. Let W denote the vertices of H.

Suppose that f_n is a map from a finite set W_n of vertices of H to the vertices of G. We build finite isomorphisms f_n as follows.

1. We set the induction hypothesis that, f_n, a map from $H[W_n]$ to $G[V_n]$, is an isomorphism.

2. Start with $f_0 = \emptyset$. Suppose that f_n has been constructed.

3. Let m be the smallest index of a vertex of G not in the domain of f_n; then extend f_n to a map f_{n+1} with x_m in its domain. Select the directly-correctly joined vertex of G with smallest index, not in V_n, to be the image of x_m.

4. Since H_n is finite, we guaranteed that every vertex of H_n is in the domain and corresponding vertex of G is in the range, of f_n, and f_n is the required isomorphism.

Let x_{n+1} is another vertex of H not in the set $\{x_1, \ldots, x_n\}$. We show that f can be extended to x_{n+1}.

We first identify all the neighbors of x_{n+1} in the finite set $\{x_1, \ldots, x_n\}$ of vertices of H. Let A be the set of outgoing neighbors of of x_{n+1} within $\{x_1, \ldots, x_n\}$. Let B be the set of incoming neighbors of of x_{n+1} within $\{x_1, \ldots, x_n\}$. Let C be the set of neighbors have arcs of both incoming and outgoing arc of x_{n+1} within $\{x_1, \ldots, x_n\}$. Let $D = \{x_1, \ldots, x_n\} \setminus (A \cup B \cup C)$. We can see that vertices in set D are not adjacent to any vertices in sets A, B or C.

A potential image of x_{n+1} must be a vertex of G adjacent by outgoing arc to every vertex in $f(A)$, adjacent by incoming arc to every vertex in $f(B)$, adjacent by both incoming and outgoing arcs to every vertex in $f(C)$, and nonadjacent to every vertex in $f(D)$. Since G is d.e.c. II, such a vertex exists. $\square$

By Lemma 5, we have the following Corollary.

Corollary 4. *If G is an oriented graph of d.e.c type I, then it contains every finite oriented graph as an induced subgraph.*

4.4 Random directed graph models

We vary the classical random graph model $G(\mathbb{N}, p)$ to give random directed graphs (Definition 40). We enumerate the vertices of G and give each a natural number matching its order in the enumeration. This natural number associated with a vertex is used for comparison. Suppose there is an arc between two vertices, we assume the probability of an arc coming from a smaller vertex to a larger vertex is $\alpha \in (0,1)$. Then the previous $G(\mathbb{N}, p)$ can be extended to $G(\mathbb{N}, p, \alpha)$. Here $p \in (0,1)$ still refers to the probability that an edge is included in the graph independently from every other edge. Based on the definition, suppose there is an arc between two vertices, we note that the probability of an arc coming from a larger vertex to a smaller vertex is $1-\alpha$. The probability that there is an arc coming from a smaller vertex to a larger vertex is $p\alpha$. The probability that there is an arc coming from a larger vertex to a smaller vertex is $p(1-\alpha)$.

Definition 40. *The* Random Oriented Graph $G(\mathbb{N}, p, \alpha)$*,* $0 \leq p, \alpha \leq 1$*, is defined by the following random process. Given a set of vertices* $\mathbb{N}$*, an oriented graph is constructed by connecting vertices randomly and independently. Each arc is added with probability* p *independent from every other edge. Further, if an arc is added between two vertices, the probability that it comes from a smaller vertex to a larger vertex is* α *independent from the insertion of every other edge and its direction.*

We extend the definition to generate directed type II random graphs. We start from the classic random graph model $G(\mathbb{N}, p)$. For simplicity, we consider the two arcs of opposite directions between a pair of vertices one bidirectional arc. That is, there are three types of arcs between two vertices.

1. One arc from a smaller vertex to a larger vertex.

2. One arc from a larger vertex to a smaller vertex.

3. One bidirectional arc.

If an arc is added between two vertices, we assume the probability that it is bidirectional is $\beta \in (0,1)$. We also assume the probability of a sole arc coming from a smaller vertex to a larger vertex is α. Hence, the previous $G(\mathbb{N}, p)$ is extended to $G(\mathbb{N}, p, \beta, \alpha)$, $0 \leq p, \alpha, \beta \leq 1$. Specifically, p still refers to the probability that an edge is included in the graph independently from every other edge. From this definition, we note that the probability of a sole arc coming from a larger vertex to a smaller vertex is $p(1-\beta-\alpha)$.

Definition 41. *The* Random directed type II graph $G(\mathbb{N}, p, \beta, \alpha)$, $0 \leq p, \alpha, \beta \leq 1$, *is defined by a random process. Given a set of vertices* $\mathbb{N}$*, a directed type II graph is constructed by connecting vertices randomly as follows.*

1. *Each arc is included in the graph with probability* p *independent from every other edge.*

2. *If an arc is added between two vertices, the probability that it comes from a smaller vertex to a larger vertex is* α *independent from the insertion of every other edge and its direction.*

3. *If an arc is added between two vertices, the probability that it is bidirectional is* β *independent from the insertion of every other edge and its direction.*

We have the following result linking directed type II random graphs to the properties of d.e.c II.

Theorem 14. *With probability* 1*, a directed type II random graph* $G(\mathbb{N}, p, \beta, \alpha)$ *is d.e.c II, when* $\beta \in (0, 1)$*,* $\alpha \in (0, 1)$ *and* $p \in (0, 1)$*.*

Proof. For all finite disjoint sets of vertices A, B, C and D of a directed graph of d.e.c. II, G, there must exist a vertex $z \notin A \cup B \cup C \cup D$, that is correctly joined to A, B, C, and D.

Based on the definition of $G(\mathbb{N}, p, \beta, \alpha)$, we derive the probability of a vertex not correctly joined to sets A, B, C and D. For ease of proof, we only consider those z that have numbers larger than the number of any vertex in the finite disjoint sets A, B, C and D. Since G has vertex set $\mathbb{N}$, the options for z are countably infinite. Also note that z must be directed-correctly joined of type II to A, B, C and D. Since z has a larger number, each arc between any vertex in A and z is an arc directed from vertices of larger to smaller numbers. Each arc between any vertex in B and z is an arc directed from vertices of smaller to larger numbers.

The probability of a vertex not correctly joined to sets A,B and C is as follows. Here the probability of a vertex correctly joined to sets A,B and C is $((1 - \beta - \alpha)p)^{|A|}(\alpha p)^{|B|}(\beta p)^{|C|}(1 - p)^{|D|}$. Then, a vertex fail to correctly join to sets A,B and C is 1 minus the probability that it correctly joins to the sets. That is:

$$1-((1-\beta-\alpha)p)^{|A|}(\alpha p)^{|B|}(\beta p)^{|C|}(1-p)^{|D|}.$$

Since $\beta, \alpha \in (0,1)$ and $p \in (0,1)$, we have $0 < ((1-\beta-\alpha)p)^{|A|} < 1$ when $|A| > 0$; $0 < (\alpha p)^{|B|} < 1$ when $|B| \neq 0$; $(\beta p)^{|C|} < 1$ when $|C| \neq 0$ and $0 < (1-p)^{|D|} < 1$ when $|D| \neq 0$. Since $|A| + |B| + |C| + |D| > 0$, we have that

$$0 < ((1-\beta-\alpha)p)^{|A|}(\alpha p)^{|B|}(\beta p)^{|C|}(1-p)^{|D|} < 1, \text{ and}$$

$$0 < 1-((1-\beta-\alpha)p)^{|A|}(\alpha p)^{|B|}(\beta p)^{|C|}(1-p)^{|D|} < 1.$$

For convenience of notation, we identify the set of all candidates from z, and we name this set Z. Let P_Z denote the probability that no vertex z exists such that z is correctly to A, B, C and D. We have that

$$P_Z = \left(1-((1-\beta-\alpha)p)^{|A|}(\alpha p)^{|B|}(\beta p)^{|C|}(1-p)^{|D|}\right)^{|Z|}.$$

Our graph is infinite. The carnality of Z is the same as all natural numbers. Hence, we have that,

$$\begin{aligned}\lim_{|Z|\to\infty} P_Z &= \lim_{|Z|\to\infty} \left(1-((1-\beta-\alpha)p)^{|A|}(\alpha p)^{|B|}(\beta p)^{|C|}(1-p)^{|D|}\right)^{|Z|} \\ &= 0.\end{aligned}$$

Hence, it is guaranteed that we can find a vertex z that is correctly joined to A, B, C and D. Hence, with probability 1, an oriented random graph $G(\mathbb{N}, p, \beta, \alpha)$ is of d.e.c II, when $\beta \in (0,1)$, $\alpha \in (0,1)$ and $p \in (0,1)$. □

Notice Lemma 5, an oriented graph is a special case of the type II directed graph. Then d.e.c. I is a special case of d.e.c. II. Hence, we have the following corollary.

Corollary 5. *With probability* 1*, an oriented random graph* $G(\mathbb{N}, p, \alpha)$ *is of d.e.c I, when* $\alpha \in (0,1)$ *and* $p \in (0,1)$.

By Theorem 14 and Corollary 5, we know that the directed graphs generated by our random models match Definition 39.

Chapter 5

Directed geometric random graph of type two

In this chapter, we extend the important concept of g.e.c. property and explore results in directed type II graphs. We allow the edges of geometric random graphs to have directions.

5.1 Basic definitions

We first introduce the g.e.c. property extended to oriented graphs. Recall that the concept of directed-correctly joined for oriented graphs and directed type II graphs is introduced in Definitions 37 and 38.

Definition 42. *Let $G = (V, E)$ be an oriented graph whose vertices are points in the metric space S with metric d. The graph G is directed geometrically e.c. of type I at level δ (δ-d.g.e.c. I) if for all δ' so that $0 < \delta' < \delta$, for all points $x \in V$, and for all disjoint finite sets of vertices A, B and C, so that $A \cup B \cup C \subset B_\delta(x)$, there exists a vertex $z \notin A \cup B \cup C \cup \{x\}$ so that*

1. *the vertex z is directed-correctly joined of type I to A, B and C.*
2. *for all $u \in A \cup B \cup C, \mathrm{d}(u, z) < \delta$, and*
3. $\mathrm{d}(x, z) < \delta'$.
4. *if the metric distance between any two vertices of G is no smaller than δ, then there is no edge connecting the two vertices.*

Next we introduce the version of g.e.c. property defined on directed type II graphs, which allow two arcs of opposite directions between any two vertices. Note that any directed type I graph is also a directed type II graph. Hence, we say a graph is type I only when it is not type II.

Definition 43. *Let $G = (V, E)$ be a directed graph of type II whose vertices are points in the metric space S with metric d. The graph G is directed geometrically e.c. of type II at level δ (δ-d.g.e.c. II) if for all δ' so that $0 < \delta' < \delta$, for all points $x \in V$, and for all disjoint finite sets of vertices A, B, C and D, so that $A \cup B \cup C \cup D \subset B_\delta(x)$, there exists a vertex $z \notin A \cup B \cup C \cup D \cup \{x\}$ so that*

1. *the vertex z is directed-correctly joined of type II to A, B, C and D.*
2. *for all $u \in A \cup B \cup C \cup D, \mathrm{d}(u, z) < \delta$, and*
3. $\mathrm{d}(x, z) < \delta'$.
4. *if the metric distance between any two vertices of G is no smaller than δ, then there is no edge connecting the two vertices.*

If a directed graph G is d.g.e.c of type I or II, then we refer to it as being d.g.e.c. If G is δ-d.g.e.c. I or II, then we refer to it as being δ-d.g.e.c.

5.2 Relevant results

We set the space S be $\mathbb{R}$. Parallel to the LARG random graph model for the simple infinite graphs, we introduce the new concept of a DLARG graph. The next definition is for a general metric space (S, d).

Definition 44. *Consider a metric space S with distance function $d : S \times S \to \mathbb{R}^+$, and $\delta_1, \delta_2 \in \mathbb{R}^+$. We define a Directed Local Area Random Graph DLARG$(V, \delta_1, \delta_2, p_1, p_2)$.*

1. *$V \subseteq S$ can be finite or infinite;*
2. *V is a totally ordered set;*
3. *$\forall u, v \in V, u < v$, if $d(u, v) < \delta_1$, an arc from u to v is added independently with probability p_1;*

4. $\forall u, v \in V, u < v$, *if* $d(u,v) < \delta_2$, *an arc from* v *to* u *is added independently with probability* p_2*.*

For the general case $\delta_1 = \delta_2 = \delta$ *and* $p_1 = p_2$*, we denote it* $DLARG(V, \delta, p)$*.*

Note that, Definition 44 treats each bidirectional edge as two arcs of opposite directions, such that each of the two arcs can be added independently. Hence Definition 44 cannot generate oriented graphs. We require that the event of adding an edge must be independent from adding any other edge. For an oriented graph, if an edge of one direction exists between two vertices, then we cannot add another edge of opposite direction between the two vertices. This imposes a special case making the process of adding an edge not always an independent event from adding another edge.

Lemma 9. *Let* (S, d) *be a metric space and* V *a countable subset of* S *which is dense in itself. If* $\delta > 0$ *and* $p \in (0, 1)$*, then with probability* 1*, a graph generated by* $DLARG(V, \delta, p_1, p_2)$ *is* δ*-d.g.e.c of type II.*

Proof. Arbitrarily choose but fix the following: $x \in V$ and disjoint finite subsets A, B, C and D in $B_\delta(x) \cap (V \setminus \{x\})$ Let

$$\beta = \max\{\mathrm{d}(x, v) : v \in A \cup B \cup C \cup D\}.$$

We have $\beta < \delta$ since disjoint finite subsets A, B, C and D are in $B_\delta(x)$ and they are finite. Let

$$\epsilon = \min\{\delta - \beta, \delta'\}.$$

Consider the set $Z = B_\epsilon(x) \cap V$. Note that ϵ is chosen so that for any $z \in Z$, such that $\mathrm{d}(z, x) < \delta'$, and for all $u \in A \cup B \cup C$,

$$\mathrm{d}(u, z) < \mathrm{d}(u, x) + \mathrm{d}(x, z) < \beta + \epsilon \le \delta.$$

For any directed type II graph G in $DLARG(V, \delta, p)$, the probability that any vertex $z \in Z$ is correctly joined of type II (Definition 38) to sets A, B, C and D equals:

$$p^{|A|+|B|+2|C|}(1-p)^{|A|+|B|+2|D|}$$

.

The probability that no vertex in Z is direct-correctly joined of type II to sets A, B, C and D equals

$$P = \prod_{z \in Z} \left[1 - \left(p^{|A|+|B|+2|C|}(1-p)^{|A|+|B|+2|D|}\right)\right].$$

Note that V is a countable subset of S, of the metric space (S, d), and V is dense in itself. Therefore Z contains infinitely many points; hence, $P = 0$. As there are only countably many choices for x, A, B, C and D, and a countable union of measure 0 sets is measure 0, the proof follows. □

Next, we will show the relationship between the graph distance and metric distance of two vertices in graph G of δ-d.g.e.c property.

Theorem 15. *Let G has δ-d.g.e.c property in a metric space (S, d). We require that S is convex. Let u and v be vertices of the graph, $u, v \in V$ and $\mathrm{d}(u, v) > \delta$. We then have that* $\mathrm{dgd}(u, v) = \lfloor \frac{\mathrm{d}(u,v)}{\delta} \rfloor + 1$.

Proof. Let u and v be real numbers, u,$v \in V(G)$. We show that $\mathrm{dgd}(u, v) = \lfloor \frac{\mathrm{d}(u,v)}{\delta} \rfloor + 1$.

Let $k = \lfloor \frac{\mathrm{d}(u,v)}{\delta} \rfloor + 1$. For $\mathrm{d}(u, v) > \delta$, we always have $k \geq 2$. Note that the choice of k supplies that

$$(k - 1)\delta \leq \mathrm{d}(u, v) < k\delta.$$

Let $\ell = \mathrm{dgd}(u, v)$, the length of the shortest directed path from vertex u to vertex v. By Definitions 42 and 43, every edge is shorter than the threshold. Since $\mathrm{d}(u, v) > \delta$, we have that ℓ is at least 2, or $\ell > 1$. This is because that graph G has threshold δ for any arc directed consistently between two vertices. Here, this threshold is specified by the definition of d.g.e.c. property.

In the next steps, we first show that $\ell \geq k$. Then we show that $\ell \leq k$. In the end we conclude that $\ell = k$.

Let $v_0 v_1 \ldots v_\ell$ where $v_0 = u, v_\ell = v$ be a shortest directed path in G from u to v. Since G has d.g.e.c. property, there is threshold δ for arcs from u to v. We have

$d(v_{i-1}, v_i) < \delta$ for $i = 1, \ldots, \ell$. Therefore,

$$\begin{aligned}(k-1)\delta &\leq d(u, v) \\ &\leq \sum_{i=1}^{\ell} d(v_{i-1}, v_i) \\ &< \ell\delta.\end{aligned}$$

Hence, we conclude that $\ell \geq k$. In the next part of the proof, we want to show that $\ell \leq k$. We prove this by constructing a directed path of length k from u to v in G. Let $\epsilon = \frac{k\delta - d(u,v)}{k}$. That is, $d(u, v) = k(\delta - \epsilon)$.

Since S is convex, we have the property that, for any x and y, there exists a $z \in S$, such that $d(x, z) + d(z, y) = d(x, y)$. Using this property, we can obtain a sequence of vertices between u and v whose successive distances add up to $d(u, v)$, and which are at most $\frac{\epsilon}{4}$ apart. Then we choose numbers $x_1, \ldots, x_{k-1}$ from the sequence so that $d(x_i, x_{i+1}) < \delta - \frac{3\epsilon}{4}$ for $i = 0, \ldots, k-1$, and where $x_0 = u, x_k = v$. Note that each x_i is in S. For $1 \leq i < k$, we can find $w_i \in V$ so that $d(w_i, x_i) < \frac{\epsilon}{8}$. Let $w_0 = u, w_k = v$, then we have that, for $i = 0, \ldots, k-1$,

$$\begin{aligned}d(w_i, w_{i+1}) &\leq d(w_i, x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, w_{i+1}) \\ &< \delta - \frac{3\epsilon}{4} + \frac{2\epsilon}{8} \\ &< \delta - \frac{\epsilon}{2}.\end{aligned}$$

Let $v_0 = w_0 = u$. By successively applying the property of δ-d.g.e.c., we can choose $v_i \in V$ such that

1. $d(v_i, w_i) < \frac{\epsilon}{2}$, and

2. there is an arc from v_{i-1} to v_i in G.

Here we explain the process. Suppose vertices $v_0, \ldots, v_i$ are already chosen so that $v_0, \ldots, v_i$ is a directed path and $d(v_i, w_i) < \frac{\epsilon}{2}$. To match the names used in Definitions 42 and 43, let x be w_{i+1} and choose $\delta' = \frac{\epsilon}{2}$. We will chose a correctly joined point z and set it to be v_{i+1}. Let $B = \{v_i\}$. Because we are looking for z correctly connected to v_i, this means the set B satisfies Definitions 42 and 43. We can let sets A, C and D be $\emptyset$. By the δ-d.g.e.c. property, we are guaranteed to have such a vertex

z or v_{i+1} that is correctly connected to the sets A, B, C and D. Specifically, there is an arc from v_i to $v_i + 1$. Since $\mathrm{d}(v_i, v_{i+1}) < \delta$, $\mathrm{d}(v_i, w_i) < \frac{\epsilon}{2}$ and we have shown that $\mathrm{d}(w_i, w_{i+1}) < \delta - \frac{\epsilon}{2}$. We have $\mathrm{d}(v_{i+1}, w_{i+1}) < \frac{\epsilon}{2}$. We demonstrate this process in Figure 5.1

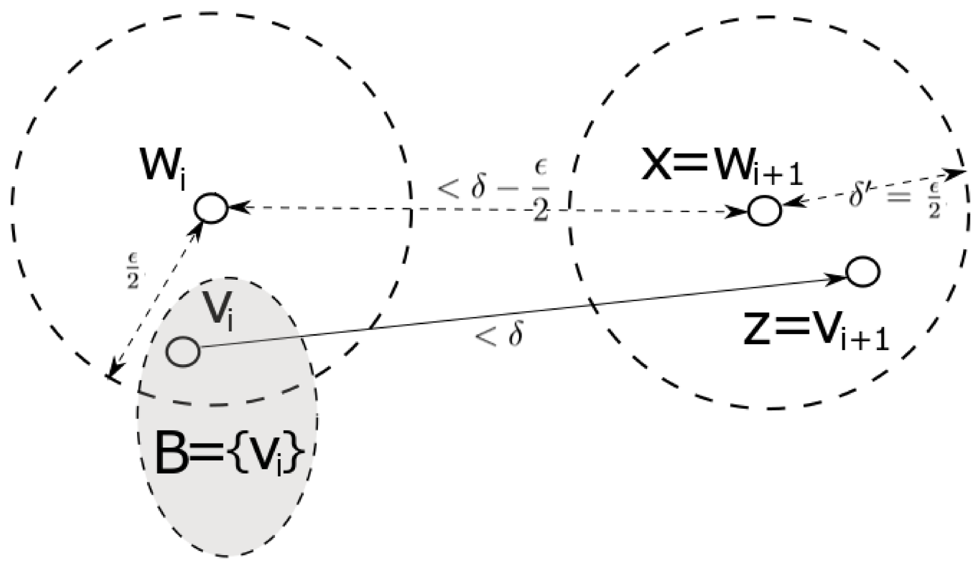

Figure 5.1: Using δ-g.e.c. property finding v_{i+1}

To choose the second to last vertex v_{k-1} in the path, let $v_k = w_k = v$. Without loss of generality, let $v_{k-2} < w_{k-1} < v_k$. Then $\mathrm{d}(v_{k-2}, w_{k-1}) < \delta$ and $\mathrm{d}(w_{k-1}, v_k) < \delta$. That is $\{v_{k-2}, v_k\} \subseteq B_\delta(w_{k-1})$. Let $A = \{v_k\}$ and $B = \{v_{k-2}\}$. We can let sets C and D be $\emptyset$. Let $\delta' = \frac{\epsilon}{2}$. By the δ-d.g.e.c. property, we are guaranteed to have such a vertex z or v_{k-1} that is correctly connected to the sets A, B, C and D. We can find vertex v_{k-1}, with an arc from v_{k-2} to it, and from it another arc to $v_k = v$. We demonstrate this process in Figure 5.2.

Therefore $\ell \leq k$ and we conclude that $\ell = k$. $\square$

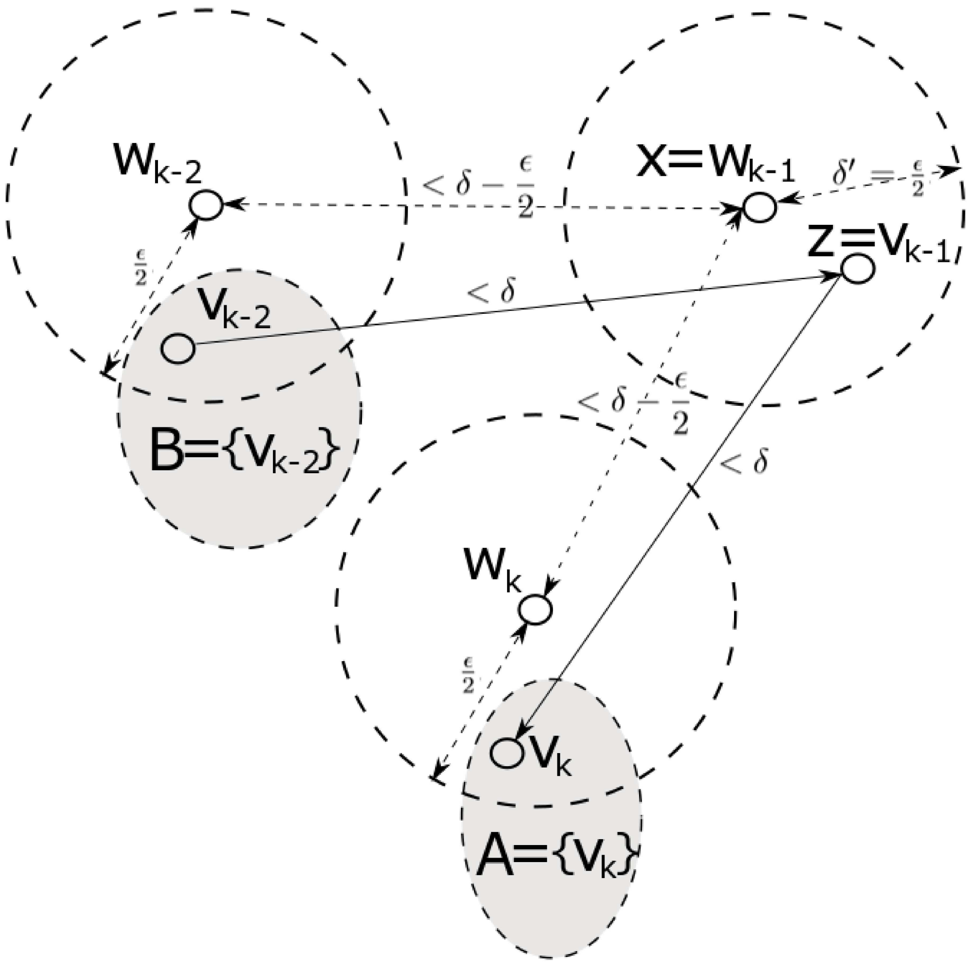

Figure 5.2: Using δ-g.e.c. property finding v_{k-1}

5.3 Relevant isomorphism results in ℓ_∞^1

We move from the general metric (S, d) to $\mathbb{R}$. In our set up, the metric space of real numbers $\mathbb{R}$ is ℓ_∞^1. Specifically for $\mathbb{R}$ of DLARG, we have the following definition.

Definition 45. *Consider the metric space as the real numbers, $\mathbb{R}$ with distance function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, and $\delta \in \mathbb{R}^+$. We define Directed Local Area Random Graph $DLARG(V, \delta, p_1, p_2)$, $0 \leq p_1, p_2 \leq 1$.*

1. *$V \subseteq \mathbb{R}$ is countable;*

2. *for all $u, v \in V, u < v$, if $\mathrm{d}(u, v) < \delta$, an arc from u to v is added independently with probability p_1; an arc from v to u is added independently with probability p_2.*

For the general case $p_1 = p_2$, we denote this random process by $DLARG(V, \delta, p)$.

We first show that for certain dense sets of vertices, with any threshold δ or link probability p, the DLARG model generates graphs that are isomorphic with probability 1. For ℓ_∞^1, we require the vertex set V be dense in $\mathbb{R}$ and it has the δ-df property (that is, the idf property when $\delta = 1$).

Recall that given metric spaces (S, d_S) and (T, d_T), sets $V \subseteq S$ and $W \subseteq T$, and positive real numbers δ and γ, a *step-isometry* at level (δ, γ) from V to W is a surjective map $f : V \to W$ with the property that for every pair of vertices $u, v \in V$,

$$\left\lfloor \frac{\mathrm{d}_S(u, v)}{\delta} \right\rfloor = \left\lfloor \frac{\mathrm{d}_W(f(u), f(v))}{\gamma} \right\rfloor .$$

Note every isometry is a step-isometry, but the converse is not true.

For each $x \in \mathbb{R}$ and let $\delta \in \mathbb{Q}$, let the *quotient* be $q(x) = \left\lfloor \frac{x}{\delta} \right\rfloor$. Let the *remainder* be $r(x) = x\delta - q(x)$.

Without loss of generality, we assume the thresholds δ and γ are 1. We can do this because the graph can be scaled. Suppose a graph G of threshold δ is defined on vertices dense in $\mathbb{R}$. The graph is defined in metric ℓ_∞^1. Picture G is drawn on the line $\mathbb{R}$. If we scale the line of the graph by $\frac{1}{\delta}$ (multiply each number by $\frac{1}{\delta}$), then the resulting graph H has no change from G. That is, $G \simeq H$. But the threshold of H

becomes 1. Previous δ-df property becomes idf property. Previous δ-d.e.c becomes 1-d.e.c. After this assumption, we have for each $x \in \mathbb{R}$, the *quotient* $q(x) = \left\lfloor \frac{x}{1} \right\rfloor = \lfloor x \rfloor$. The *remainder* $r(x) = x - q(x)$.

We have the lemma that will be useful later.

Lemma 10. [3] *Let V and W be substes of $\mathbb{R}$ with idf property. Then a bijection map $f : V \to W$ is a step-isometry if the next two statements are true.*

1. *for all $u, v \in V$, if $r(u) < r(v)$, then $r(f(u)) < r(f(v))$.*

2. *for all $u \in V$, $q(u) = q(f(u))$*

Using Lemma 10, we are ready to prove the theorem next. This is an important isomorphism result.

Theorem 16. *Let V and W be two countable dense subsets of $\mathbb{R}$ with idf property and containing zero. If G has vertex set V and satisfies the 1-d.g.e.c. property, H has vertex set W and satisfies the 1-d.g.e.c. property, then $G \simeq H$.*

Proof. We use a variation of the back-and-forth method. Let $V = \{v_i : i \geq 0\}$ and $W = \{w_i : i \geq 0\}$. For $i \geq 0$, we inductively construct a sequence of pairs of sets (V_i, W_i) and isomorphisms $f_i : G[V_i] \to H[W_i]$.

The induction hypothesis is that f_i is an isomorphism, so that for all $i \geq 1, v_i \in V_i, w_i \in W_i, V_i \subseteq V_{i+1}$ and $W_i \subseteq W_{i+1}$, and f_{i+1} extends f_i. Plus we require that f_i is a step-isometry from V_i to W_i at level $(1, 1)$.

It follows that

$$\bigcup_{i \in \mathbb{N}} f_i : G \to H$$

is an isomorphism.

Also as part of the induction hypothesis, we maintain the two conditions of items (1) and (2) from Lemma 10 in the underlying un-directed graph of G and H. Thus, f_i is a step isomorphism.

Let $V_0 = \{v_0\}$, $W_0 = \{w_0\}$, and define f_0 by $f_0(v_0) = w_0$. Then $q(v_0) = q(w_0) = 0$ and $r(v_0) = r(w_0) = 0$, so the base case of the induction follows. For the induction step, fix $i \geq 0$. To construct f_{i+1} from f_i, we first go forth by finding an image of v_{i+1}.

In the following, for simplification of notation, let f be f_i, and let v be v_{i+1}. Define

$$\begin{aligned} a &= \max\{r(f(u)) : u \in V_i \text{ and } r(u) \leq r(v)\}, \\ b &= \min\{r(f(u)) : u \in V_i \text{ and } r(u) > r(v)\}. \end{aligned}$$

We claim that $a < b$. Namely, let u_a and u_b be the elements in V_i for which the maximum and minimum that define a and b are attained, respectively. Thus $r(f(u_a)) = a$ and $r(f(u_b)) = b$. By definition, $r(u_a) \leq r(v) \leq r(u_b)$. By the induction hypothesis, this implies that $a = r(f(u_a)) < r(f(u_b)) = b$.

In order to maintain the induction hypothesis, $r(f(v))$ must lie in $[a, b)$, and $q(f(v))$ must equal to $q(v)$. Let $k = q(v)$, and consider the interval

$$I = (k + a, k + b).$$

Any vertex in I will qualify as a candidate for $f(v)$, so that f_{i+1} satisfies the part of the induction hypothesis that f_{i+1} satisfies (1) and (2) of Lemma 10. We must then find a vertex in I that will also guarantee that f is an isomorphism, by making sure it has the correct neighbors. For this, we apply the 1-d.g.e.c. property of H.

To apply the 1-d.g.e.c. property of H, we need to ensure that the images of all neighbors of v in V_i lie in a 1-ball. Since G has threshold 1, we consider all vertices of V_i that lie in a 1-ball around v. Let $Y = B_1(v) \cap V_i$, and fix $x \in I \cap W$. Such a vertex x exists since W is dense in $\mathbb{R}$. By definition of $I, q(x) = k$. We claim that

$$f(Y) \subseteq B_1(x). \tag{5.1}$$

To prove this, let $u \in Y$. Since $q(v) = k$ and $\mathrm{d}(u, v) < 1$, it follows that $|q(u) - k| \leq 1$. Hence $q(u)$ is one of $k, k-1$ or $k+1$.

If $q(u) = k$, then $q(f(u)) = k$ by induction hypothesis, so $\mathrm{d}(f(u), x) < 1$. If $q(u) = k - 1$, then $r(u) > r(v)$, so $r(f(u)) > b$ by definition of b. Hence

$$\begin{aligned} \mathrm{d}(f(u), x) &= x - f(u) \\ &< k + b - (k - 1) - r(f(u)) \\ &< 1. \end{aligned}$$

If $q(u) = k+1$, then $r(f(u)) \leq a$ so we have that

$$\begin{aligned} \mathrm{d}(f(u), x) &= f(u) - x \\ &< (k+1) + r(f(u)) - k - a \\ &\leq 1. \end{aligned}$$

In all three cases, $f(u) \subseteq B_1(x)$ and (5.1) follows.

Since G has threshold 1, $N(v) \cap V_i \subseteq Y$. Now let $A \cup B \cup C = f(N(v) \cap V_i)$ and $D = (W_i \cap B_1(x) \setminus A)$. Then $A \cap B \subseteq B_1(x) \cap W_i$. Let $\epsilon > 0$ be chosen such that $B_\epsilon(x) \subseteq I$. We now use the 1-d.g.e.c. property of H to find a point $z \in B_\epsilon(x)$ which is adjacent to all vertices in A and no other vertices of the finite set W_i. Thus we can add z to W_i to form W_{i+1} and add v to V_i to form V_{i+1} and set $f_{i+1}(v) = z$. Observe that f_{i+1} is an isomorphism.

To finish the induction step, if $w_{i+1} \notin W_{i+1}$ then we may go back, by finding an image $z = f_{i+1}^{-1}(w_{i+1})$ in an analogous fashion. We then add z to V_{i+1} and maintain that f_{i+1} is an isomorphism. □

We have the following corollaries.

Corollary 6. *Let V and W be two countable dense subsets of $\mathbb{R}$ with δdf and γdf property respectively. If G has vertex set V and satisfies the δ-d.g.e.c. property, H has vertex set W and satisfies the γ-d.g.e.c. property, then $G \simeq H$.*

Corollary 7. *For all countable dense subsets V of $\mathbb{R}$, $\delta > 0$, and $p \in (0,1)$, with probability 1, there is a unique isotype of graph, written DGR_1, in $DLARG(V, \delta, p)$.*

We obtain similar unique isomorphism types of graphs in all dimensions, as proven in the following result. For higher dimensions, we need to extend the definition of idf. Given a set $V \subseteq \mathbb{R}^n$, denote the i-th component set of V as:

$$V_i = \{x_i : x \in V\},$$

where $x = (x_1, x_2, \ldots, x_d)$. A set $V \in \mathbb{R}^d$ is *idf* if the coordinate sets $V_1, \ldots, V_d$ are all idf.

For clarity we let δ and γ be 1 in the following theorem.

Theorem 17. *Consider the metric space ℓ_∞^d. Let V and W be two countable dense idf sets in $\mathbb{R}^d$. If graph G with vertex set V has 1-d.g.e.c property; and graph H with vertex set W has 1-d.g.e.c property, and in G and H there is no edge between any two vertices of distance larger than 1, then $G \simeq H$. In particular, for all choices of dense idf vertex set V, there is a unique isomorphism type of d.g.e.c. graphs in ℓ_∞^d written DGR_d.*

The proof of Theorem 17 is largely identical to the proof of Theorem 16. To avoid repetition, we only sketch5 the proof here.

For higher dimensions, we have no complete characterization of step isometries. We use Lemma 10 to obtain sufficient conditions. Precisely, a bijection function $f : V \to W$ is a step-isometry if the following two conditions hold for all $u, v \in V$ and for all $i, 1 \leq i \leq d$:

$$r(u_i) < r(v_i) \text{ if and only if } r(f(u)_i) < r(f(v)_i),$$
$$q(u_i) = q(f(u)_i).$$

To prove the theorem, we construct an isomorphism between sets V_i and W_i much as in the one-dimensional case. We now explain how to extend an isomorphism $f : V_i \to W_i$ to a new vertex $v = v_{i+1}$.

For all $j, 1 \leq j \leq d$, define

$$a_j = \max\{r(f(u)_j) : u \in V_i \text{ and } r(u_j) \leq r(v_j)\},$$
$$b_j = \min\{r(f(u)_j) : u \in V_i \text{ and } r(u_j) > r(v_j)\}.$$

Note that $a_j < b_j$ for all j. Namely, if not there must exist $1 \leq j \leq d$ and two points $u, v \in W_i$ so that $r(u_j) = r(w_j)$. This contradicts that fact that W is idf.

In order to maintain the induction hypothesis, for all $j, r(f(v))_j$ should lie in interval $[a_j, b_j)$, and $q(f(v)_j)$ should be equal to $q(v_j)$. Let $k_j = q(v_j)$, and consider the product set

$$I = \prod_{1 \leq j \leq d} (q(v_j) + a_j, q(v_j) + b_j).$$

Any vertex in I qualifies as a candidate for $f(v)$ so that f satisfies conditions. To complete the proof, we use the fact that W is 1-d.g.e.c. to show that I contains a vertex that is correctly joined to the vertices in W_i so that f remains an isomorphism.

We have the following corollary.

Corollary 8. *For each dimension d, there exists a unique isotype of graph, written DGR_d such that for all countable dense subsets V of $\mathbb{R}$, so that V is idf, for all $p \in (0,1)$, and for all $\delta > 0$, the graph $DLARG(V, \delta, p)$ is isomorphic to DGR_d.*

We name DGR_d the infinite direct random geometric graph of dimension d. Note that GDR_d has infinite diameter for all $d \geq 1$ unlike R which has diameter 2.

Theorem 18. *Let V and W be two countable subets of $\mathbb{R}$, and let $\delta = \gamma = 1$. Lef F be a bijective step-isometry from V to W at level $(1,1)$. If graph G with vertex set V has 1-d.g.e.c property, and graph H with vertex set W has 1-d.g.e.c property, then $G \simeq H$.*

We only sketch the proof here. Let $V = \{v_i : i \geq 0\}$ and $W = \{w_i : i \geq 0\}$ where $w_i = F(v_i)$. We inductively construct a sequence of pairs of sets $(V_i, W_i)(i \geq 0)$ and isomorphisms $f_i : G[V_i] \to H[W_i]$, so that for all $i \geq 1, v_i \in V_i, w_i \in W_i, V_i \subseteq V_{i+1}$ and $W_i \subseteq W_{i+1}$ and f_{i+1} extends f_i. As an additional part of the induction hypothesis, we require that f_i satisfies the following three conditions.

1. For every $u, v \in V, r(u) \leq r(v)$ if and only if $r(f(u)) \leq r(f(v))$.
2. For every $u, v \in V, r(u) \leq r(v)$ if and only if $r(f(u)) \leq r(F(v))$.
3. For every $u \in V, q(u) = q(f(u))$.

The first two conditions imply that f_i is a step-isometry by Lemma 10. We can also conclude from Lemma 10 that for all $u, v \in V, r(u) \leq r(v)$ if and only if $r(F(u)) \leq r(F(v))$.

Let $V_0 = \{v_0\}$ and $W_0 = \{w_0\}$, and set $f_0(v_0) = w_0$. Conditions (1) and (3) follow as in the proof of Theorem 16. Condition (2) follows from the fact that $w_0 = F(v_0) = f(v_0)$. For the induction step, fix $i \geq 0$. We construct f_{i+1} from f_i by first finding an image of v_{i+1}.

In the following, f refers to f_i and v refers to v_{i+1}. Let

$$M_a = \{u : u \in V_i \text{ and } r(F(u)) \leq r(F(v))\},$$
$$M_b = \{u : u \in V_i \text{ and } r(F(u)) > r(F(v))\},$$

and

$$a = \max\{x : x = r(f(u)) \text{ or } x = r(F(u)) \text{ where } u \in M_a\},$$
$$b = \max\{x : x = r(f(u)) \text{ or } x = r(F(u)) \text{ where } u \in M_b\}.$$

We have that $a < b$, since the order of the representatives of vertices in V_i is preserved under f and under F. This is the same argument as in Theorem 16.

In order to maintain conditions (1) and (2) of the induction hypothesis, $r(f(v))$ should lie in $[a, b)$ and because of condition (3), $q(f(v))$ must equal $q(v)$. Let $k = q(v)$, and consider the interval $I = (k\gamma + a, k\gamma + b)$. From the definition of a and b it follows $F(v) \in I$.

The remainder of the proof is now analogous to the proof of Theorem 16. Hence we describe it next. Let $x = F(v)$. We can show that $f(B_\delta(v) \cap V_i) \subseteq B_\gamma(x)$. We can then invoke the γ-d.g.e.c. condition of H and find a vertex z in I which is correctly jointed to the vertices in W_i so that an isomorphism is maintained if we set $f(v) = w$. Finally, we finish the induction step by going back and finding a suitable image $f^{-1}(w_{i+1})$.

We apply Theorem 17 to yield a result about isomophisms between graphs with vertex sets in $\mathbb{R}^n$ if there exists a special type of map between the sets.

Theorem 19. *Consider the metric space ℓ_∞^d. Let V and W be two countable idf sets in $\mathbb{R}^d$. Assume that for all $1 \leq i \leq d$, there exists a step-isometry from V_i to W_i. If graph G with vertex set V has 1-d.g.e.c property; and graph H with vertex set W has 1-d.g.e.c property; then $G \simeq H$.*

The proof of the theorem is an extension of the proof of Theorem 18. To avoid repetition, we omit it here. In Theorem 19, V and W are not required to be dense in $\mathbb{R}^n$, but only in a compact subset of $\mathbb{R}^n$.

Chapter 6

Directed geometric random graphs with different thresholds

In this chapter, we study a special case of the type II directed graphs. Specifically, we concentrate on the directed geometric random graphs defined on vertices on the real line. For different theorems, we require V to have a number of additional properties including: dense, countable and the (δ_1, δ_2)df property in Definition 10. If applicable, then we include the relevant definitions and results for oriented graphs, for completeness.

In such specific graphs, we expand the threshold parameter in the δ-d.g.e.c. property to δ_1 and δ_2. The basic idea is that we differentiate the threshold for arcs: those from a larger vertex to a smaller vertex versus those from a smaller vertex to a larger vertex. This creates a stage that gives our results which we explore in this chapter.

6.1 Linear geometric graphs with two thresholds

Let u and v be two vertices of our graph. We allow different thresholds for the cases when $u > v$ and when $u < v$. We set the space S to be $\mathbb{R}$.

Parallel to the LARG random graph model for the simple infinite graphs, we introduce the new concept of a DLARG graph.

Definition 46. *Consider ℓ_∞^1 and $\delta_1, \delta_2 \in \mathbb{R}^+$. We define Directed Local Area Random Graph DLARG$(V, \delta_1, \delta_2, p_1, p_2)$.*

1. *The set $V \subseteq \mathbb{R}$ is countable;*

2. *For all $u, v \in V, u < v$, if $\mathrm{d}(u, v) < \delta_1$, an arc from u to v is added independently with probability p_1;*

3. *for all* $u, v \in V, u < v$, *if* $\mathrm{d}(v, u) < \delta_2$, *an arc from* v *to* u *is added independently with probability* p_2.

For the general case $\delta_1 = \delta_2 = \delta$ *and* $p_1 = p_2$, *we denote this random process by* $DLARG(V, \delta, p)$.

Note that the model in Definition 46 cannot generated oriented graphs. We have the following result.

Lemma 11. *Let* (S, d) *be a metric space and* V *a countable subset of* S *which is dense in itself. If* $\delta_1 \geq \delta_2 > 0$ *and* $p \in (0, 1)$, *then with probability 1, a graph generated by* $DLARG(V, \delta_1, \delta_2, p)$ *is* δ_2*-d.g.e.c of type II.*

Proof. Arbitrarily choose but fix the following: $x \in V$ and disjoint finite subsets A, B, C and D in $B_{\delta_2}(x) \cap (V \setminus \{x\})$, and $0 < \delta' < \delta_2 \leq \delta_1$. Let

$$\beta = \max\{\mathrm{d}(x, v) : v \in A \cup B \cup C \cup D\}.$$

We have $\beta < \delta_2$ since disjoint finite subsets A, B, C and D are in $B_{\delta_2}(x) \cap (V \setminus \{x\})$. Let

$$\epsilon = \min\{\delta_2 - \beta, \delta'\}.$$

Consider the set $Z = B_\epsilon(x) \cap V$. Note that ϵ is chosen so that for any $z \in Z$, such that $\mathrm{d}(z, x) < \delta'$, and for all $u \in A \cup B \cup C$,

$$\mathrm{d}(u, z) < \mathrm{d}(u, x) + \mathrm{d}(x, z) < \beta + \epsilon \leq \delta_2.$$

For any directed type II graph G in $DLARG(V, \delta_1, \delta_2, p)$, the probability that any vertex $z \in Z$ is direct-correctly joined of type II to sets A, B, C and D equals $p^{|A|+|B|+2|C|}(1-p)^{|D|}$.

The probability that no vertex in Z is direct-correctly joined of type II to sets A, B, C and D equals

$$P = \prod_{z \in Z} 1 - \left(p^{|A|+|B|+2|C|}(1-p)^{|D|}\right).$$

Note that V is a countable subset of S, of the metric space (S, d), and V is dense in itself. Therefore Z contains infinitely many points; hence, $P = 0$. As there are only countably many choices for x, A, B, C and D, and a countable union of measure 0 sets is measure 0, the proof follows. $\square$

As in the previous chapter, we consider graphs where the vertex set consists of real numbers. The metric d on $\mathbb{R}$ is the usual metric defined by: $\mathrm{d}(x, y) = |x - y|$, for all $x, y \in V$ where $V \subseteq \mathbb{R}$. We have the following definitions. For convenience of proofs, without loss of generality, we let $\delta_1 \geq \delta_2$ in this chapter.

Definition 47. *Let $G = (V, E)$ be a directed type II graph whose vertices are points in the metric space the real numbers $\mathbb{R}$ with metric d. The graph G is directed geometrically e.c. at level δ_1, δ_2 ((δ_1, δ_2)-d.g.e.c. II) if for all δ' so that $0 < \delta' < \delta_2$, for all points $x \in V$, and for all disjoint finite sets of vertices $A \subseteq (x - \delta_2, x + \delta_1)$, $B \subseteq (x - \delta_1, x + \delta_2)$, $C \subseteq (x - \delta_2, x + \delta_2)$, and $D \subseteq (x - \delta_1, x + \delta_1)$ there exists a vertex $z \notin A \cup B \cup C \cup D \cup \{x\}$ so that*

1. *the vertex z is directed-correctly joined of type II to A, B, C and D.*

2. *for all $u \in A$, if $u < z$, then $\mathrm{d}(u, z) < \delta_2$, and if $u > z$, then $\mathrm{d}(u, z) < \delta_1$.*

3. *for all $u \in B$, if $u < z$, then $\mathrm{d}(u, z) < \delta_1$, and if $u > z$, then $\mathrm{d}(u, z) < \delta_2$.*

4. *for all $u \in C$, $\mathrm{d}(u, z) < \delta_2$.*

5. *for all $u \in D$, $\mathrm{d}(u, z) < \delta_1$.*

6. *$\mathrm{d}(x, z) < \delta'$*

Note that in Definition 47, if δ_1 and δ_2 are equal, then this property is the same as the property defined in Definition 43. Hence, the (δ_1, δ_2)-d.g.e.c. property can be seen as an extension of the δ-d.g.e.c. property of type II.

In this chapter, we often assume the vertex set V to be a countable dense set in $\mathbb{R}$. We always require that there is no edge from a smaller vertex to a larger vertex longer than δ_1, and there is no edge from a larger vertex to a smaller vertex longer than δ_2.

Definition 48. *A directed type II graph G is called a (δ_1, δ_2)-linear graph, if the following three conditions are satisfied. Let $u, v \in G$ and $u < v$.*

1. *G is (δ_1, δ_2)-d.g.e.c.;*

2. *if there is an arc from u to v in G, then $|u - v| < \delta_1$.*

3. if there is an arc from v to u in G, then $|u - v| < \delta_2$;

In a (δ_1, δ_2)-linear graph, let u, v be arbitrary vertices in V and $u < v$. There is no edge from u to v if $\mathrm{d}(u, v) > \delta_1$. This is because by condition (3) of Definition 48, if there is an arc from u to v in G, then $|u - v| < \delta_1$. Hence, there is no edge from u to v if $\mathrm{d}(u, v) > \delta_1$. By condition (2), it holds for threshold δ_2 in the other direction as well. There is no edge from v to u if $\mathrm{d}(u, v) > \delta_2$.

6.2 The *DLARG* model with two thresholds

We first link the graphs generated by the $DLARG(V, \delta_1, \delta_2, p)$ model to the (δ_1, δ_2)-linear graphs. As noted in Definition 45, $DLARG(V, \delta_1, \delta_2, p)$ can only generate directed type II graphs. The reason $DLARG$ is limited to directed type II graphs is because the event of adding each edge is required to be independent. For oriented graphs, if there is already an arc between two vertices, then we are forbidden to add the edge of opposite direction. In this case, adding a new edge is clearly not an independent event. Hence, the focus of our study is the directed type II graphs in this chapter.

Theorem 20. *Let vertex set V be a countable dense subset of $\mathbb{R}$, and suppose V to have the (δ_1, δ_2)df property. With probability 1, a directed type II graph generated via $DLARG(V, \delta_1, \delta_2, p)$ is a (δ_1, δ_2)-linear graph.*

Proof. We want to show that a directed type II graph generated via $DLARG(V, \delta_1, \delta_2, p)$ is a (δ_1, δ_2)-linear graph. Recall our Definition 48 for (δ_1, δ_2)-linear graph. Our proof follows the three conditions in the definition: (1) G is (δ_1, δ_2)-d.g.e.c.; and (2) if there is an arc from v to u in G, then $|u - v| < \delta_2$; and (3) if there is an arc from u to v in G, then $|u - v| < \delta_1$.

Note that items (2) and (3) in the definition are clearly satisfied since these match the defined process of DLARG. They match the thresholds as in Definition 45 for $DLARG(V, \delta_1, \delta_2, p)$. Threshold δ_1 is for arcs connecting smaller to larger vertices; and threshold δ_2 is for arcs connecting larger to smaller vertices.

To show that a directed type II graph generated via $DLARG(V, \delta_1, \delta_2, p)$ is a (δ_1, δ_2)-linear graph, we are yet to show item (1) that G is (δ_1, δ_2)-d.g.e.c. Based on the definition of $DLARG(V, \delta_1, \delta_2, p)$, we can derive the probability of a vertex not being correctly joined to sets A, B, C and D in this case. We proceed as follows.

Arbitrarily choose but fix the following: $x \in V$; disjoint finite subsets A, B, C and D such that $A \subseteq (x - \delta_2, x + \delta_1)$, $B \subseteq (x - \delta_1, x + \delta_2)$, $C \subseteq (x - \delta_2, x + \delta_2)$, and $D \subseteq (x - \delta_1, x + \delta_1)$. Let δ' be such that $0 < \delta' < \delta_2 \leq \delta_1$.

We first look at set A. Let

$$\beta_1 = \max\{\mathrm{d}(x, v) : v \in A \text{ and } v > x)\},$$
$$\beta_2 = \max\{\mathrm{d}(x, v) : v \in A \text{ and } v < x)\}.$$

We have $0 < \beta_1 < \delta_1$ and $0 < \beta_2 < \delta_2$. Let

$$\epsilon_1 = \min\{\delta_1 - \beta_1, \delta'\},$$
$$\epsilon_2 = \min\{\delta_2 - \beta_2, \delta'\}.$$

Consider the set $Z_A = (x - \epsilon_2, x + \epsilon_1)$. Note that ϵ_1 is chosen so that for any $z \in Z_A$, $\mathrm{d}(z, x) < \delta'$ and for all $u \in A, u > x$,

$$\mathrm{d}(u, z) < \mathrm{d}(u, x) + \mathrm{d}(x, z) < \beta_1 + \epsilon_1 \leq \delta_1.$$

Also note that ϵ_2 is chosen so that for any $z \in Z_A$, $\mathrm{d}(z, x) < \delta'$ and for all $u \in A, u < x$,

$$\mathrm{d}(u, z) < \mathrm{d}(u, x) + \mathrm{d}(x, z) < \beta_2 + \epsilon_2 \leq \delta_2.$$

Notice $Z_A \neq \emptyset$. Similar to Z_A, for set B, let

$$\beta_3 = \max\{\mathrm{d}(x, v) : v \in B \text{ and } v > x)\},$$
$$\beta_4 = \max\{\mathrm{d}(x, v) : v \in B \text{ and } v < x)\},$$
$$\epsilon_3 = \min\{\delta_2 - \beta_3, \delta'\}, \text{ and}$$
$$\epsilon_4 = \min\{\delta_1 - \beta_4, \delta'\}.$$

We have that $Z_B = (x - \epsilon_4, x + \epsilon_3)$. $Z_B \neq \emptyset$. For set C, let

$$\beta_5 = \max\{\mathrm{d}(x, v) : v \in C \text{ and } v > x)\},$$
$$\beta_6 = \max\{\mathrm{d}(x, v) : v \in C \text{ and } v < x)\},$$
$$\epsilon_5 = \min\{\delta_2 - \beta_5, \delta'\}, \text{ and}$$
$$\epsilon_6 = \min\{\delta_2 - \beta_6, \delta'\}.$$

We have $Z_C = (x - \epsilon_6, x + \epsilon_5)$. For set D, let

$$\beta_7 = \max\{\mathrm{d}(x, v) : v \in D \text{ and } v > x)\},$$
$$\beta_8 = \max\{\mathrm{d}(x, v) : v \in D \text{ and } v < x)\},$$
$$\epsilon_7 = \min\{\delta_1 - \beta_7, \delta'\}, \text{ and}$$
$$\epsilon_8 = \min\{\delta_1 - \beta_8, \delta'\}.$$

We have that $Z_D = (x - \epsilon_8, x + \epsilon_7)$. The sets Z_A, Z_B, Z_C and Z_D are open intervals centered at x. Hence, their intersection is a nonempty interval. Let $Z = Z_A \cap Z_B \cap Z_C \cap Z_D$. For any directed type II graph G generated via the process of $DLARG(V, \delta_1, \delta_2, p)$, the probability that any vertex $z \in Z$ is direct-correctly joined of type II to sets A, B, C and D equals $p^{|A|+|B|+2|C|}(1-p)^{|D|}$.

The probability that no vertex in Z is direct-correctly joined of type I to sets A, B, C and D equals

$$P = \prod_{z \in Z \cap V} \left(1 - \left(p^{|A|+|B|+2|C|}(1-p)^{|D|}\right)\right).$$

Note that V is a dense subset of the real numbers $\mathbb{R}$ and Z is an open interval. Therefore $Z \cap V$ contains infinitely many points; hence, $P = 0$. As there are only countably many choices for x, A, B, C and D, and a countable union of measure 0 sets is measure 0. With probability 1, there exists z that is correctly joined. Moreover, by construction z satisfies property (2) through (6) of Definition 47. □

6.3 Isomorphism of (δ_1, δ_2)-linear graphs

We can bound the directed graph distance (dgd) based on the metric distance (d) between any two vertices of our graph. The exact bound for graph distance is an important theorem. It sets a foundation for proofs of subsequent theorems.

Theorem 21. *Let $G = (V, E)$ be a (δ_1, δ_2)-linear graph. Let V be a dense subset in $\mathbb{R}$. Let u and v be vertices of the graph, $u, v \in V$ and $u < v$. We have that*

1. *If* $\mathrm{d}(u, v) > \delta_1$*, then* $\mathrm{dgd}(u, v) = \lfloor \frac{\mathrm{d}(u,v)}{\delta_1} \rfloor + 1$*, and*
2. *if* $\mathrm{d}(v, u) > \delta_2$*, then* $\mathrm{dgd}(v, u) = \lfloor \frac{\mathrm{d}(u,v)}{\delta_2} \rfloor + 1$.

Proof. Let u and v be real numbers, u,$v \in V(G)$ and $u < v$. It suffices to show that $\mathrm{dgd}(u, v) = \lfloor \frac{\mathrm{d}(u,v)}{\delta_1} \rfloor + 1$. By symmetry, the proof is similar for the case $\mathrm{dgd}(v, u) = \lfloor \frac{\mathrm{d}(u,v)}{\delta_2} \rfloor + 1$.

Thus we show that $\mathrm{dgd}(u, v) = \lfloor \frac{\mathrm{d}(u,v)}{\delta_1} \rfloor + 1$. Let $k = \lfloor \frac{\mathrm{d}(u,v)}{\delta_1} \rfloor + 1$. By assumption, $\mathrm{d}(u, v) > \delta_1$, so we always have $k \geq 2$. Note that the choice of k implies that

$$(k - 1)\delta_1 \leq \mathrm{d}(u, v) < k\delta_1.$$

Let $\ell = \mathrm{dgd}(u, v)$, the length of the shortest path of arcs following the direction from vertex u to vertex v. In the next steps, we first show that $\ell \geq k$. Then we show that $\ell \leq k$. In the end we conclude that $\ell = k$.

Let $v_0 v_1 \ldots v_\ell$ where $v_0 = u, v_\ell = v$ be a shortest directed path in G from u to v. Since G is a (δ_1, δ_2)-linear graph, there is threshold δ_1 for arcs. From smaller to larger points, by the definition of (δ_1, δ_2)-linear graph ,we have $\mathrm{d}(v_{i-1}, v_i) < \delta_1$ for $i = 1, \ldots, \ell$. We cannot guarantee that the points are sequential from smaller to larger ones. There may be arcs go from larger to smaller vertices. This is addressed in the second line below. We have that

$$\begin{aligned}(k - 1)\delta_1 &\leq \mathrm{d}(u, v) \\ &\leq \sum_{i=1}^{\ell} [v_i - v_{i-1}]_+ \\ &< \ell\delta_1.\end{aligned}$$

Here we utilize the following, for $x \in \mathbb{R}$:

$$[x]_+ = \begin{cases} x, & \text{if } x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we conclude that $\ell > k - 1$, and thus, $\ell \geq k$. We next show that $\ell \leq k$. We prove this by constructing a directed path of length k from u to v in G. Let $\epsilon = \frac{k\delta_1 - \mathrm{d}(u,v)}{k}$. That is, $\mathrm{d}(u, v) = k(\delta_1 - \epsilon)$.

Note that in our domain of vertices, we have the property that there exists a vertex $z \in \mathbb{R}$, such that $\mathrm{d}(x,z)+\mathrm{d}(z,y) = \mathrm{d}(x,y)$. Using this property, we can obtain a sequence of real numbers between u and v whose successive distances add up to $\mathrm{d}(u,v)$, and which are at most $\frac{\epsilon}{4}$ apart. There are many vertices in this sequence. Then we choose $x_1, \ldots, x_{k-1}$ from the sequence so that $\mathrm{d}(x_i, x_{i+1}) < \delta_1 - \frac{3\epsilon}{4}$ for $i = 0, \ldots, k-1$, and where $x_0 = u, x_k = v$. Note that $x_i, 1 < i < k$ are in $\mathbb{R}$ but not guaranteed in V. This path of length k is achievable because we have shown that $\mathrm{d}(u,v) = k(\delta_1 - \epsilon)$. For $1 \le i < k$, since V is dense in $\mathbb{R}$, we can find $w_i \in V$ so that $\mathrm{d}(w_i, x_i) < \frac{\epsilon}{8}$. Let $w_0 = u, w_k = v$, then we have, for $i = 0, \ldots, k-1$,

$$\begin{aligned}\mathrm{d}(w_i, w_{i+1}) &= \mathrm{d}(w_i, x_i) + \mathrm{d}(x_i, x_{i+1}) + \mathrm{d}(x_{i+1}, w_{i+1}) \\ &< \delta_1 - \frac{3\epsilon}{4} + \frac{2\epsilon}{8} \\ &< \delta_1 - \frac{\epsilon}{2}.\end{aligned}$$

Let $v_0 = w_0 = u$. By successively applying the property of (δ_1, δ_2)-linear graph, we can choose $v_i \in V$ such that

1. $\mathrm{d}(v_i, w_i) < \frac{\epsilon}{2}$, and

2. there is an arc from v_{i-1} to v_i in G.

Here, we explain the process. Suppose vertices $v_0, \ldots, v_i$ are already chosen so that $v_0, \ldots, v_i$ is a directed path and $\mathrm{d}(v_i, w_i) < \frac{\epsilon}{2}$. To match the names used in Definition 47, let x be w_{i+1} and choose $\delta' = \frac{\epsilon}{2}$. We will chose a correctly joined point z and set it to be v_{i+1}. Let $B = \{v_i\}$. Because we are looking for z correctly connected to v_i, this means the set B satisfies Definition 47. We confirm this by noticing that $\mathrm{d}(v_i, w_i) < \frac{\epsilon}{2} < \delta_1, \delta_2$. In Definition 47, this means $B = \{v_i\} \subseteq (x - \delta_1, x + \delta_2)$ where x is w_i in this case. We can let sets A, C and D be $\emptyset$. By the (δ_1, δ_2)-d.g.e.c. property, we are guaranteed to have such a vertex z or v_{i+1} that is correctly connected to the sets A, B, C and D. Specifically, there is an arc from v_i to $v_i + 1$. We have $\mathrm{d}(v_{i+1}, w_{i+1}) = d(x,z) < \delta' = \frac{\epsilon}{2}$, where x, z are defined in Definition 47.

To choose the second to last vertex v_{k-1} in the path, let $v_k = w_k = v$. Without loss of generality, let $v_{k-2} < w_{k-1} < v_k$. Then $\mathrm{d}(v_{k-2}, w_{k-1}) \le \mathrm{d}(v_{k-2}, w_{k-2}) + \mathrm{d}(w_{k-2}, w_{k-1}) < \delta_1 - \frac{\epsilon}{2} + \frac{\epsilon}{2} = \delta_1$. By the same argument, we have $\mathrm{d}(w_{k-1}, v_k) < \delta_1$.

Let $A = \{v_k\}$ and $B = \{v_{k-2}\}$. Hence $A \subseteq (w_{k-1}, w_{k-1} + \delta_1) \subset (x - \delta_2, x + \delta_1)$ where x of Definition 47 is w_{k-1}. We also have $B \subseteq (w_{k-1} - \delta_1, w_{k-1}) \subset (x - \delta_1, x + \delta_2)$. We can let sets C and D be $\emptyset$. Let $\delta' = \frac{\epsilon}{2}$. By the (δ_1, δ_2)-d.g.e.c. property, we are guaranteed to have such a vertex z or v_{k-1} that is correctly connected to the sets A, B, C and D. We can find vertex v_{k-1}, with an arc from v_{k-2} to it, and from it another arc to $v_k = v$. Therefore, $\ell \leq k$ and we conclude that $\ell = k$. □

We first give a theorem on a useful property of a (δ_1, δ_2)-linear graph. We can retrieve the ratio of the thresholds δ_1 and δ_2 by taking a limit of dgd between vertices in the graph.

Theorem 22. *Let $G = (V, E)$ be a (δ_1, δ_2)-linear graph. Let $u, v \in V$ and $u < v$ be arbitrarily selected. Let $n \in \mathbb{N}$. We have that*

$$\frac{\delta_2}{\delta_1} = \lim_{n\to\infty} \inf \left\{ \frac{\operatorname{dgd}(u,v)}{\operatorname{dgd}(v,u)} \middle| u, v \in V, \operatorname{dgd}(u,v) \geq n \right\}.$$

Proof. Let u and v, $u < v$, be vertices of G. Let $\mathrm{d}(u, v) = t$. Let G be a (δ_1, δ_2)-linear graph. Without loss of generality, let $\mathrm{d}(u, v) > \delta_1$, $n > 1$ and recall we assume $\delta_1 \geq \delta_2$. By Lemma 21, we have that $\operatorname{dgd}(u,v) = \left\lfloor \frac{\mathrm{d}(u,v)}{\delta_1} \right\rfloor + 1$. $\operatorname{dgd}(v,u) = \left\lfloor \frac{\mathrm{d}(u,v)}{\delta_2} \right\rfloor + 1$.

$$\operatorname{dgd}(u,v) - 1 = \left\lfloor \frac{t}{\delta_1} \right\rfloor,$$
$$\operatorname{dgd}(u,v) - 1 \leq \frac{t}{\delta_1} < \operatorname{dgd}(u,v).$$

Hence

$$\frac{t}{\delta_1} < \operatorname{dgd}(u,v) \leq \frac{t}{\delta_1} + 1, \tag{6.1}$$
$$\frac{t}{\delta_2} < \operatorname{dgd}(v,u) \leq \frac{t}{\delta_2} + 1. \tag{6.2}$$

Now, use both equations (6.1) and (6.2). Note t, $\operatorname{dgd}(u,v)$ and $\operatorname{dgd}(v,u)$ are not zero. We have that

$$\frac{\frac{t}{\delta_1}}{\frac{t}{\delta_2} + 1} < \frac{\operatorname{dgd}(u,v)}{\operatorname{dgd}(v,u)} < \frac{\frac{t}{\delta_1} + 1}{\frac{t}{\delta_2}},$$

$$\frac{\delta_2}{\delta_1} \left(\frac{1}{1 + \frac{\delta_2}{t}} \right) < \frac{\operatorname{dgd}(u,v)}{\operatorname{dgd}(v,u)} < \frac{\delta_2}{\delta_1} \left(1 + \frac{\delta_1}{t} \right). \tag{6.3}$$

Suppose that $\mathrm{dgd}(u,v) \geq n$, we have that

$$n - 1 \leq \mathrm{dgd}(u,v) - 1 \leq \frac{t}{\delta_1},$$

$$\frac{\delta_1}{t} \leq \frac{1}{n-1}. \tag{6.4}$$

Similarly, we have that

$$\mathrm{dgd}(v,u) - 1 \leq \frac{t}{\delta_2} < \mathrm{dgd}(v,u),$$

$$\frac{\delta_2}{t} \leq \frac{1}{n-1}. \tag{6.5}$$

Apply (6.4) in the right side inequality in (6.3), we have that

$$\frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)} < \frac{\delta_2}{\delta_1}\left(1 + \frac{1}{n-1}\right). \tag{6.6}$$

Apply (6.5) in the left side inequality in (6.3), we have that

$$\frac{\delta_2}{\delta_1}\left(\frac{1}{1+\frac{1}{n-1}}\right) < \frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)},$$
$$\frac{\delta_2}{\delta_1}\left(\frac{n-1}{n}\right) < \frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)},$$

That is,

$$\frac{\delta_2}{\delta_1}\left(1 - \frac{1}{n}\right) < \frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)}. \tag{6.7}$$

Combine (6.6) and (6.7), we have that

$$\frac{\delta_2}{\delta_1}\left(1 - \frac{1}{n}\right) < \frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)} < \frac{\delta_2}{\delta_1}\left(1 + \frac{1}{n-1}\right). \tag{6.8}$$

Note that both the left and the right sides of equation 6.8 approaches $\frac{\delta_2}{\delta_1}$ as $n \to \infty$. Also, since V is dense in $\mathbb{R}$, we can always find vertices u and v whose graph distance $\mathrm{dgd}(u,v)$ is large enough. The limits of both the left and the right sides of equation 6.8 are well defined, and so we derive that

$$\lim_{n\to\infty} \frac{\delta_2}{\delta_1}\left(\frac{1}{1+\frac{1}{n-1}}\right) = \frac{\delta_2}{\delta_1}, \tag{6.9}$$

$$\lim_{n\to\infty} \frac{\delta_2}{\delta_1}\left(1+\frac{1}{n-1}\right) = \frac{\delta_2}{\delta_1}. \tag{6.10}$$

Equations (6.9) and (6.10) hold for all pairs of vertices hence also for infimum of any pair of vertices. For $\delta_1 > \delta_2$, a sequence with increasing index n can be formed by this generic element s_n,

$$s_n = \inf\left\{\frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)}\middle| u,v\in V, \mathrm{dgd}(u,v)\geq n\right\},$$

Sequence s_n is non-decreasing and bounded from below by (6.9) and bounded from above by (6.10). Hence, we derive from equations (6.9) and (6.10) and have

$$\lim_{n\to\infty} \inf\left\{\frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)}\middle| u,v\in V, \mathrm{dgd}(u,v)\geq n\right\} = \frac{\delta_2}{\delta_1}.$$

This result is true if $\delta_1 = \delta_2$. □

Next, we introduce an isomorphism result of the (δ_1, δ_2)-linear graphs. The proof of this theorem is a variation of the back-and-forth method. We require that δ_1 and δ_2 be integers for this theorem.

Theorem 23. *If $G = (V, E_G)$ and $H = (W, E_H)$ are both (δ_1, δ_2)-linear graphs, where $\delta_1, \delta_2 \in \mathbb{Z}$ and $\gcd(\delta_1, \delta_2) = 1$, V and W are countable dense subsets of $\mathbb{R}$ with idf property, then $G \simeq H$.*

Proof. Note that $\gcd(\delta_1, \delta_2) = 1$. Hence, there exists $s, t \in \mathbb{Z}$ such that $1 = s\delta_1 + t\delta_2$. That is, any integer can be written as a linear integer combination of δ_1 and δ_2.

We use a variation of the back-and-forth method. Let $V = \{v_i : i \geq 0\}$ and $W = \{w_i : i \geq 0\}$. For $i \geq 0$, we inductively construct a sequence of pairs of sets (V_i, W_i) and isomorphisms $f_i : G[V_i] \to H[W_i]$, so that for all $i \geq 1, v_i \in V_i, w_i \in W_i, V_i \subseteq V_{i+1}$ and $W_i \subseteq W_{i+1}$, and f_{i+1} extends f_i. It follows that

$$\bigcup_{i\in\mathbb{N}} f_i : G \to H$$

is an isomorphism.

Base case: let $V_0 = \{v_0\}$, $W_0 = \{w_0\}$, and define f_0 by $f_0(v_0) = w_0$. This is the base case.

Induction hypothesis: for integers up to i, we have that

1. $v_i \in V_i, w_i \in W_i$.

2. f_i is an isomorphism from $G[V_i]$ to $H[W_i]$.

3. For all $u, v \in V_i, u < v$, $\lfloor \mathrm{d}(u,v) \rfloor = \lfloor \mathrm{d}(f_i(u), f_i(v)) \rfloor$ and $f_i(u) < f_i(v)$.

Note that (3) implies that for all $u, v \in V_i$ and for all $s, t \in \mathbb{Z}$, if $\lfloor \mathrm{d}(u,v) \rfloor = s\delta_1 + t\delta_2$ then $\lfloor \mathrm{d}(f(u), f(v)) \rfloor = s\delta_1 + t\delta_2$. The required properties are true for the base case.

For the induction step, first let $i > 0$ and assume V_i, W_i and f_i satisfy the inductive hypothesis. Set $f = f_i$.

We define an interval $Z = (z^-, z^+)$, where

$$\begin{aligned} z^- &= \max\{f(u) + \lfloor \mathrm{d}(u, v_{i+1}) \rfloor : u < v_{i+1}, u \in V_i\} \\ &\qquad \cup \{f(u) - \lfloor \mathrm{d}(u, v_{i+1}) \rfloor - 1 : u > v_{i+1}, u \in V_i\}, \\ z^+ &= \min\{f(u) + \lfloor \mathrm{d}(u, v_{i+1}) \rfloor + 1 : u < v_{i+1}, u \in V_i\} \\ &\qquad \cup \{f(u) - \lfloor \mathrm{d}(u, v_{i+1}) \rfloor : u > v_{i+1}, u \in V_i\}. \end{aligned}$$

To show that Z is not empty, we show that $z^- < z^+$. Suppose $z^- \geq z^+$, then there exists two vertices $s, t \in W_i$ such that at least one of the following four equations is true.

$$f(s) + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor \geq f(t) - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor, t > v_{i+1} > s, \tag{6.11}$$

$$f(t) - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor - 1, t > v_{i+1} > s \geq f(s) + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1, \tag{6.12}$$

$$f(s) + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor \geq f(t) + \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1, s, t < v_{i+1}, \tag{6.13}$$

$$f(s) - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor - 1 \geq f(t) - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor, s, t > v_{i+1}. \tag{6.14}$$

For the case of (6.11), we have that $t > v_{i+1} > s$,

$$\begin{aligned} f(s) + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor &\geq f(t) - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor, \\ f(t) - f(s) &\leq \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + \lfloor \mathrm{d}(t, v_{i+1}) \rfloor. \end{aligned}$$

Because of the idf property, we have that

$$f(t) - f(s) < \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + \lfloor \mathrm{d}(t, v_{i+1}) \rfloor. \tag{6.15}$$

By definition of (δ_1, δ_2)-linear graph, we have that

$$\begin{aligned} t - s &= (t - v_{i+1}) + (v_{i+1} - s) \\ &\geq \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor. \end{aligned}$$

Because of the idf property, we have that

$$t - s > \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor. \tag{6.16}$$

Note that $\lfloor \mathrm{d}(t, v_{i+1}) \rfloor + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor$ is an integer. By using equations (6.15) and (6.16), we have a contradiction to the induction hypothesis (3). Hence, the case of (6.11) cannot occur.

For the case of (6.12), we have that $t > v_{i+1} > s$,

$$\begin{aligned} f(s) + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1 &\leq f(t) - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor - 1, \\ f(t) - f(s) &\geq \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 2. \end{aligned}$$

Because of the idf property, we have that

$$f(t) - f(s) > \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 2. \tag{6.17}$$

By definition of (δ_1, δ_2)-linear graph, we have that

$$\begin{aligned} t - s &= (t - v_{i+1}) + (v_{i+1} - s) \\ &\leq (\lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1) + (\lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1) \\ &= \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 2. \end{aligned}$$

Because of the idf property, we have that

$$t - s < \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 2. \tag{6.18}$$

Note that $\lfloor \mathrm{d}(t, v_{i+1}) \rfloor + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 2$ is an integer. By using equations (6.17) and (6.18), we have a contradiction to the induction hypothesis (3). Hence, the case of (6.12) cannot occur.

For the case of (6.13), we have that $s, t < v_{i+1}$.

$$f(s) + \lfloor \mathrm{d}(s, v_{i+1}) \rfloor \geq f(t) + \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1.$$

If $v_{i+1} > s > t$, then by (3) of induction hypothesis, $f(s) > f(t)$. We have that

$$f(s) - f(t) \geq \lfloor \mathrm{d}(t, v_{i+1}) \rfloor - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1.$$

Because of the idf property, we have that

$$f(s) - f(t) > \lfloor \mathrm{d}(t, v_{i+1}) \rfloor - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1. \tag{6.19}$$

By definition of (δ_1, δ_2)-linear graph, we have that

$$\begin{aligned} s - t &= (v_{i+1} - t) - (v_{i+1} - s) \\ &\leq (\lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1) - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor \\ &= \lfloor \mathrm{d}(t, v_{i+1}) \rfloor - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1. \end{aligned}$$

Because of the idf property, we have that

$$t - s < \lfloor \mathrm{d}(t, v_{i+1}) \rfloor - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1. \tag{6.20}$$

Note that $\lfloor \mathrm{d}(t, v_{i+1}) \rfloor - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1$ is an integer. By using equations (6.19) and (6.20), we have a contradiction to the induction hypothesis (3).

If $v_{i+1} > t > s$, then an analogous argument holds.

For the case of (6.14), we have that $s, t > v_{i+1}$.

$$f(s) - \lfloor \mathrm{d}(s, v_{i+1}) \rfloor - 1 \geq f(t) - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor.$$

If $v_{i+1} < t < s$, then by (3) of induction hypothesis, $f(t) < f(s)$. We have that

$$f(s) - f(t) \geq \lfloor \mathrm{d}(s, v_{i+1}) \rfloor - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1.$$

Because of the idf property, we have that

$$f(s) - f(t) > \lfloor \mathrm{d}(s, v_{i+1}) \rfloor - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1. \tag{6.21}$$

By definition of (δ_1, δ_2)-linear graph, we have that

$$\begin{aligned} s - t &= (s - v_{i+1}) - (t - v_{i+1}) \\ &\leq (\lfloor \mathrm{d}(s, v_{i+1}) \rfloor + 1) - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor \\ &= \lfloor \mathrm{d}(s, v_{i+1}) \rfloor - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1. \end{aligned}$$

Because of the idf property, we have that

$$s - t < \lfloor \mathrm{d}(s, v_{i+1}) \rfloor - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1. \tag{6.22}$$

Note that $\lfloor \mathrm{d}(s, v_{i+1}) \rfloor - \lfloor \mathrm{d}(t, v_{i+1}) \rfloor + 1$ is an integer. By using equations (6.21) and (6.22), we have a contradiction to the induction hypothesis (3).

If $v_{i+1} < s < t$, then an analogous argument holds. Next, to extend f_i to f_{i+1} we define the following sets in V_i:

$$\begin{aligned}
N^+(v_{i+1}) &= \{u \in V_i : \text{ there is an arc from } v_{i+1} \text{ to } u\}, \\
N^-(v_{i+1}) &= \{u \in V_i : \text{ there is an arc from } u \text{ to } v_{i+1}\}, \\
N^\pm(v_{i+1}) &= \{u \in V_i : \text{ there are two arcs of both directions between } v_{i+1} \text{ and } u\}, \\
N^0(v_{i+1}) &= \{u \in V_i : \text{ there is no arc between } v_{i+1} \text{ and } u\}.
\end{aligned}$$

We define the following sets in W_i:

$$\begin{aligned}
A &= f(N^+(v_{i+1})), \\
B &= f(N^-(v_{i+1})), \\
C &= f(N^\pm(v_{i+1})), \\
D &= f(N^0(v_{i+1})).
\end{aligned}$$

We select a vertex x from interval Z. The vertex x maintains induction hypothesis (3): that for all $x \in (z^-, z^+)$ and any $u \in V_i$, we have that $\lfloor \mathrm{d}(u, v_{i+1}) \rfloor = \lfloor \mathrm{d}(f(u), x)) \rfloor$ and $f_i(u) < x$ if and only if $u < v_{i+1}$ (and $f_i(u) > x$ if and only if $u > v_{i+1}$.)

We claim that $A \subseteq (x - \delta_2, x + \delta_1)$. For any $a \in A$, we have $a = f(u), u \in N^+(v_{i+1})$. If $u < v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_2$. By construction that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_2$, and $f(u) < x$. There is no equality due to the idf property. If $u > v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_1$. by construction that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_1$ and $f(u) < x$. There is no equality due to idf property.

We claim that $B \subseteq (x - \delta_1, x + \delta_2)$. For any $b \in B$, we have $b = f(u), u \in N^-(v_{i+1})$. If $u < v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_1$. So that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_1$. There is no equality due to idf property. If $u > v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_2$. Hence, we have that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_2$. There is no equality due to idf property.

We claim that $C \subseteq (x - \delta_2, x + \delta_2)$. For any $c \in C$, we have $c = f(u), u \in N^\pm(v_{i+1})$. If $u < v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_2$. So that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_2$. There is no equality due to idf property. If $u > v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_2$. So that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_2$. There is no equality due to the idf property.

We claim that $D \subseteq (x - \delta_1, x + \delta_1)$. For any $d \in D$, we have $d = f(u), u \in N^+(v_{i+1})$. If $u < v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_1$. So that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_1$.

There is no equality due to idf property. If $u > v_{i+1}$, then $\mathrm{d}(u, v_{i+1}) < \delta_1$. So that $\lfloor \mathrm{d}(f(u), x) \rfloor = \lfloor \mathrm{d}(u, v_{i+1}) \rfloor < \delta_1$. There is no equality due to idf property.

Let $\delta' = \min\{\frac{x-z^-}{2}, \frac{z^+-x}{2}\}$. By the definition of (δ_1, δ_2)-linear graph, we can find a vertex $z \in Z$, such that z is correctly connected to A, B, C and D. Set $f_{i+1}(v_{i+1})$ to be z, then z successfully extends the mapping f_i to f_{i+1} and maintains the inductive hypothesis. Observe that f_{i+1} is an isomorphism.

To finish the induction step, if $w_{i+1} \notin W_{i+1}$ then we may go back, by finding an image $z = f_{i+1}^{-1}(w_{i+1})$ in an analogous fashion. We then add z to V_{i+1} and maintain that f_{i+1} is an isomorphism. □

Corollary 9. *Two DLARG$(V, \delta_1, \delta_2, p)$ graphs, with $\delta_1, \delta_2 \in \mathbb{Z}$ and $\gcd(\delta_1, \delta_2) = 1$, are isomorphic with probability 1.*

We understand that the thresholds of (δ_1, δ_2)-linear graphs are very important. They are the defining properties of these graphs. Next, we introduce our last result in this section. When the vertex sets are dense subsets of real numbers, we show that we can determine the isomorphism of (δ_1, δ_2)-linear graphs by the ratio of their thresholds. The equality of the ratio of the thresholds of two (δ_1, δ_2)-linear graphs is equivalent to the isomorphism of the two graphs.

Theorem 24. *Let $G = (V, E)$ be a (δ_1, δ_2)-linear graph, so that $\frac{\delta_1}{\delta_2}$ is rational. Let $H = (W, F)$ be a (γ_1, γ_2)-linear graph, so that $\frac{\gamma_1}{\gamma_2}$ is rational. Suppose that V has (δ_1, δ_2)df property, and W has (γ_1, γ_2)df property.*

1. *If $\frac{\delta_1}{\delta_2} = \frac{\gamma_1}{\gamma_2}$, then $G \simeq H$.*

2. *Otherwise, the graphs G and H are not isomorphic.*

Proof. We first prove that if $\frac{\delta_1}{\delta_2} = \frac{\gamma_1}{\gamma_2}$, then $G \simeq H$. We scale the graph $H(W, F)$ to $H'(W', F')$ as follows.

1. $W' = \{\frac{\delta_1}{\gamma_1} w, w \in W\}$.

2. For F', if there is an arc from u to v in H then there is an arc from $\frac{\delta_1}{\gamma_1} u$ to $\frac{\delta_1}{\gamma_1} v$ in H'.

It follows that H and H' are isomorphic. Since $\frac{\delta_1}{\delta_2} = \frac{\gamma_1}{\gamma_2}$ and W has (γ_1, γ_2)df property, W' is guaranteed to have (δ_1, δ_2)df property. Next we show that H' is a (δ_1, δ_2)-linear graph.

We first show that H' is (δ_1, δ_2)-d.g.e.c. Since H is a (γ_1, γ_2)-linear graph, it is (γ_1, γ_2)-d.g.e.c. From H to H', γ_1 is scaled to $\gamma_1 \frac{\delta_1}{\gamma_1} = \delta_1$. γ_2 is scaled to $\gamma_2 \frac{\delta_1}{\gamma_1} = \delta_2$. We see that, for H, the sets A, B, C and D, using thresholds γ_1 and γ_2 in Definition 47 properly transform to sets A', B', C' and D' using thresholds δ_1 and δ_2 for H'. Hence, H' inherits (δ_1, δ_2)-d.g.e.c. property from H.

Let u, v in W, $u < v$. Let $u' = \frac{\delta_1}{\gamma_1} u$ and $v' = \frac{\delta_1}{\gamma_1} v$. If there is an arc from u to v in H, then $|u - v| < \gamma_1$. After scaling in H', we have that

$$\begin{aligned} |u - v| &< \gamma_1, \\ \frac{\delta_1}{\gamma_1}|u - v| &< \frac{\delta_1}{\gamma_1}\gamma_1, \\ |u' - v'| &< \delta_1. \end{aligned}$$

The result holds for the other direction as well. We conclude that H' is a (δ_1, δ_2)-linear graph.

Then, when $\delta_1, \delta_2 \in \mathbb{Z}$ and $\gcd(\delta_1, \delta_2) = 1$, we conclude that the graphs G and H' are isomorphism to each other by Theorem 23. Since each isomorphism is a bijiection, we have $G \simeq H$.

Then, we conclude that this result holds for graphs G and H as long as $\frac{\delta_1}{\delta_2}$ is rational. We can do this because the graph can be scaled. Suppose a graph I of threshold δ is defined on vertices dense in $\mathbb{R}$. If we multiply the real line of the graph by $\frac{1}{\delta}$, then the resulting graph J has no change from I. That is $I \simeq J$. But the threshold of J becomes 1. As long as $\frac{\delta_1}{\delta_2}$ is rational, we can scale the graph to meet the condition for Theorem 23.

It remains to show that if $\frac{\delta_1}{\delta_2} \neq \frac{\gamma_1}{\gamma_2}$, then the directed type II graphs G and H are not isomorphic. We prove the contrapositive of the statement: if the directed type II graphs G and H are isomorphic, then $\frac{\delta_1}{\delta_2} = \frac{\gamma_1}{\gamma_2}$. Let mapping $f : V \to W$ be the isomorphism between G and H. We have that

$$\mathrm{dgd}_G(u, v) = \mathrm{dgd}_H(f(u), f(v)), \text{ for all vertices } u, v \in V.$$

We omit the subscript of dgd when there is no confusion. By Theorem 22, we have that

$$\frac{\delta_2}{\delta_1} = \lim_{n\to\infty} \inf \left\{ \frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)} \middle| u, v \in V, \mathrm{dgd}(u,v) \geq n \right\}. \tag{6.23}$$

Since G and H are isomorphic by an isomorphism bijection mapping f, we can transform $\mathrm{dgd}(u,v)$ to $\mathrm{dgd}(f(u), f(v))$ in (6.23). Let $n \in \mathbb{N}$. We have that

$$\begin{aligned}
\frac{\delta_2}{\delta_1} &= \lim_{n\to\infty} \inf \left\{ \frac{\mathrm{dgd}(u,v)}{\mathrm{dgd}(v,u)} \middle| u, v \in V, \mathrm{dgd}(u,v) \geq n \right\}, \\
&= \lim_{n\to\infty} \inf \left\{ \frac{\mathrm{dgd}(f(u),f(v))}{\mathrm{dgd}(f(v),f(u))} \middle| u, v \in V, \mathrm{dgd}(u,v) \geq n \right\}, \\
&= \lim_{n\to\infty} \inf \left\{ \frac{\mathrm{dgd}(f(u),f(v))}{\mathrm{dgd}(f(v),f(u))} \middle| f(u), f(v) \in V, \mathrm{dgd}(f(u),f(v)) \geq n \right\}, \\
&= \frac{\gamma_2}{\gamma_1}.
\end{aligned}$$

The last step in the previous equation is guaranteed by Theorem 22. Hence, we have that if the directed type II graphs G and H are isomorphic, then $\frac{\delta_2}{\delta_1} = \frac{\gamma_2}{\gamma_1}$.

We have proven the contrapositive of the statement: (2) if $\frac{\delta_1}{\delta_2} \neq \frac{\gamma_1}{\gamma_2}$, then the directed type II graphs G and H are not isomorphic. □

Note that when $\frac{\delta_1}{\delta_2}$ is irrational, the set $\{s\delta_1 + t\delta_2 : s, t \in \mathbb{Z}\}$ is the entire real line $\mathbb{R}$. Then we cannot have a (δ_1, δ_2)-df set for vertices of G. Therefore, this requirement is necessary and minimum in our set up.

Chapter 7

Conclusion

In this thesis, we only explored a small fraction of the problems in the domain. The majority of problems are still to be looked at in the areas of directed infinite random graphs and directed geometric random graphs. We considered, in Chapter 6, a one dimensional situation. Even in this case, many other problems are open.

7.1 Conclusion

Our study extended the study of the Rado graph and geometric graphs to directed graphs. In Chapter 4, we defined type I and type II directed graphs. Type II directed graph is a superset of the oriented graph (type I). We defined the concepts of d.e.c. of type I and II. In Chapter 4, we found that: if a directed graph is either o.e.c. i.e.c. d.e.c. I, or d.e.c II, then it must be an infinite graph. We then presented our isomorphism results. Two countable directed type I or II graphs are isomorphic. We showed that a type I or II directed graph contains every finite directed graph of type I or II as its subgraph. We linked directed type I or II random graphs to d.e.c. I or II properties.

In Chapter 5, we presented our results in the directed geometric random graphs. We first introduced the g.e.c. property extended to oriented and type II directed graphs. Parallel to the $LARG$ random graph model for the simple infinite graphs, we introduced the new concept of a $DLARG$ graph. We found the correlation between graphs generated by $DLARG$ and the property of d.g.e.c. of type II. We found the graph distance of any two vertices in a graph of d.g.e.c. property. Then we presented relevant isomorphism results. We defined the unique isotype DGR_1 coming from the DLARG model. Finally, we extended our isomorphism results to higher dimensions.

We explored the directed g.e.c. with different thresholds in Chapter 6. We expanded the threshold parameter of δ-d.g.e.c. to two: δ_1 and δ_2. We defined what is a (δ_1, δ_2)-linear graph. We showed the link between $DLARG$ generated graphs and the (δ_1, δ_2)-linear graphs. We can bound the exact dgd distance between any two vertices of (δ_1, δ_2)-linear graphs. We then presented our isomorphism results. We showed that the proportion of the thresholds of two such graphs is determinant to their isomorphism.

7.2 Open problems

We finish by stating several open problems. The first open problem is to look at expanding the d.g.e.c. graphs of different thresholds studied in Chapter 6 into higher dimensions, and find isomorphism results there.

The second open problem is to extend the study into different types of metric spaces. For example, in $\mathbb{R}^2$ with the Euclidean metric, are directed graphs generated by the DLARG model almost surely isomorphic?

The third open problem is to look beyond the problem of isomorphism. For infinite graphs, many of the graph parameters are infinite. For the infinite digraphs we discovered in this thesis, it would be interesting, for example, to determine their chromatic number, domination number, and cop number.

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