ALGEBRAIC PROPERTIES OF MONOMIAL IDEALS

by

Ali Alilooee

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

at

Dalhousie University
Halifax, Nova Scotia
August 2015

© Copyright by Ali Alilooee, 2015

Table of Contents

List of Figures

Abstract

In this thesis we first study a special class of squarefree monomial ideals, namely, path ideals. We give a formula to compute all graded Betti numbers of the path ideal of a cycle and a path. As a consequence we can give new and short proofs for the known formulas of regularity and projective dimensions of path ideals of path graphs and cycles. We also study the Rees algebra of squarefree monomial ideals. In 1995 Villarreal gave a combinatorial description of the equations of Rees algebras of quadratic squarefree monomial ideals. His description was based on the concept of closed even walks in a graph. In this thesis we will generalize his results to all squarefree monomial ideals by using a definition of even walks in a simplicial complex. We show that simplicial complexes with no even walks have facet ideals that are of linear type, generalizing Villarreal's work.

List of Abbreviations and Symbols Used

$\binom{[n]}{i}$	set of all subsets of size i of $[n] = \{1, 2, \ldots, n\}$, 55
$\mathrm{V}(\mathit{\Delta})$	set of vertices of a simplicial complex Δ, 4
$\mathit{\Delta}_{\mathcal{X}}{}^{c}$	complement of a simplicial complex Δ, 5
$I_t(G)$	path ideal of length t of graph G, 23
$I(G)$	edge ideal of graph G, 14
$\mathit{\Delta}_t(G)$	path complex of length t of graph G, 23
$\mathcal{N}(I)$	Stanly-Reisner complex of an ideal I, 6
$\mathcal{E}(s_1, s_2, \ldots, s_r)$	complement of run sequence $s_1, s_2, \ldots, s_r$, 27
$\mathcal{N}(\Delta)$	Stanly-Reisner ideal of a simplicial complex Δ, 6
$\mathbb{K}[\mathit{\Delta}](\Delta)$	Stanly-Reisner ring of a simplicial complex Δ, 7
$\mathcal{F}(I)$	facet complex of an ideal I, 6
$\mathcal{F}(\Delta)$	facet ideal of a simplicial complex Δ, 7
$\mathit{\Delta}^*$	Alexander Dual of a simplicial complex Δ, 7
$\mathcal{C}_i(\Delta)$	set of all i-dimensional faces of a simplicial complex Δ, 5
$\mathrm{Facets}(\Delta)$	set of facets of a simplicial complex Δ, 5
$\widetilde{H}_i(\mathit{\Delta}, \mathbb{K})$	i-th reduced homology module of a simplicial complex Δ, 9
$\widetilde{H}^i(\mathit{\Delta}, \mathbb{K})$	i-th reduced cohomology module of a simplicial complex Δ, 10
$\mathit{\Delta}_1 * \mathit{\Delta}_2$	join of Δ_1 and Δ_2, 11
$Cn(\Delta)$	cone of Δ, 11
$\beta_{i,j}(M)$	i- Betti number of degree j of module M, 12
$Syz(f_1, \ldots, f_g)$	Syzygy module of $f_1, f_2, \ldots, f_q$, 13
$pd(R/I)$	projective dimension of R/I, 13
$reg(R/I)$	regularity of R/I, 13
$R[It]$	Rees algebra of I, 15
$\mathrm{Matrix}_{p\times q}(R)$	The set of $p \times q$ matrices over a ring R, 16
$deg_{\alpha}(x)$	α-degree for a vertex x, 66

Acknowledgements

First I would like to express my special gratitude to my advisor Prof. Sara Faridi for her excellent guidance, caring, patience, and providing me with an excellent atmosphere for doing research.

I would like to thank my readers for their patience to read my thesis and for their helpful comments and questions. I would also like to thank my family and my friends for all their help and support.

Finally, and most importantly, I would like to thank my wife Maryam for her support, encouragement, quiet patience and unwavering love were undeniably the bedrock upon which the past thirteen years of my life have been built.

Chapter 1

Introduction

Let $\mathbb{K}$ be a field and consider a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over $\mathbb{K}$. A monomial in R is a polynomial f of the form

$$f = {x_1}^{\alpha_1} \cdots {x_n}^{\alpha_n} \quad \text{for} \quad (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

If all $a_i \in \{0, 1\}$, the polynomial f is called a **squarefree monomial**. An ideal I in R is called a **(squarefree) monomial ideal** if I is generated by (squarefree) monomials. A monomial ideal is called **pure** if the degrees of its generators are the same.

Monomial ideals have been investigated by many authors from several points of view. One of the important aspects of these ideals is their application in exchanging information between commutative algebra and combinatorics. In fact we can assign a squarefree monomial ideal to a graph or a simplicial complex to make a dictionary between their algebraic and combinatorial properties.

This thesis is divided into three parts. In Chapter 2 we introduce some basic theorems and well-known results of finite free resolutions of finitely generated modules over polynomial rings and Rees algebras of squarefree monomials. We also introduce some combinatorial notions like simplicial complexes and their homology modules and graphs.

In Chapter 3 and Chapter 4 we study the minimal free resolution of a special class of squarefree monomial ideals (path ideals). After that, in Chapter 5 we investigate the Rees algebras of squarefree monomial ideals.

1.1 Path Ideals

Path ideals of a graph were first introduced by Conca and De Negri [14] in the context of monomial ideals of linear type. Simply, path ideals are ideals whose monomial generators correspond to vertices in paths of a given length in a graph. More precisely the path ideal of a graph G, denoted by $I_t(G)$, is defined as an ideal of R generated by the monomials of the form $x_{i_1}x_{i_2}\cdots x_{i_t}$ where $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$ is a path in G. The case $t = 2$ is called the **edge ideal** of G, first introduced by Villarreal in [45].

In this thesis we are interested in the free resolutions of path ideals. In 2010 Bouchat, Hà and O'Keefe [11] and He and Van Tuyl [21] studied invariants related to resolutions of path ideals of certain graphs. Since edge ideals of graphs are a special class of path ideals one approach may be to first consider edge ideals. In his thesis, Jacques [28] used beautiful techniques to compute Betti numbers of edge ideals of several classes of graphs. In this thesis we extend Jacques's techniques to higher dimensions to compute Betti numbers of path ideals of cycles and paths.

In Chapters 3 and 4 we consider the path ideal of a graph as a disjoint union of connected components. We then use homological methods to glue these components back together, and use Hochster's formula (Corollary 3.1.2) to compute all top-degree graded Betti numbers (Theorem 4.1.1).

We then use purely combinatorial arguments to give an explicit formula for all the graded Betti numbers of path ideals of path graphs and cycles. As a consequence we can give new and short proofs for the known formulas of regularity and projective dimensions of path ideals of path graphs. Also, we can find the projective dimension and regularity of path ideals of cycles.

1.2 Rees Algebras of Squarefree Monomial Ideals

Rees algebras are of special interest in algebraic geometry and commutative algebra because they describe the blowing up of the spectrum of a ring along the subscheme defined by an ideal. The Rees algebra of an ideal can also be viewed as a quotient of a polynomial ring. If I is an ideal of a ring R, we denote the Rees algebra of I by $R[It]$, and we can represent $R[It]$ as S/J where S is a polynomial ring over R. The ideal J is called the **defining ideal** of $R[It]$.

Finding generators of J is difficult and crucial for a better understanding of $R[It]$. Many authors have worked to get a better insight into these generators in special classes of ideals, such as those with special height, special embedding dimension and so on (cf. Fouli and Lin [18], Morey [34], Muiños and Planas-Vilanova [35], Ulrich and Vasconcelos [41], Villarreal [47], Vasconcelos [44]).

One of these classes of ideals is squarefree monomial ideals. An interesting approach for these ideals is to give a combinatorial criterion for minimal generators of J for a squarefree monomial ideal I. The simplest case of a squarefree monomial ideal is an edge ideal, which is equivalent to a quadratic squarefree monomial ideal (squarefree monomial generated by monomials of degree 2). In 1995 Villarreal gave a combinatorial characterization of irredundant generators of J for edge ideals of graphs by attributing irredundant generators of J_s to closed even walks in the graph G (cf. Villarreal [47]).

In Chapter 5, motivated by this work, we define **simplicial closed even walks**. We prove that if $T_{\alpha,\beta}(I)$ is an irredundant generator of J_s, then the generators of I involved in $T_{\alpha,\beta}(I)$ form a simplicial even walk. We show that the class of simplicial even walks includes even special cycles (cf. Berge [9], Herzog, Hibi, Trung and Zheng [23]), as they are known in hypergraph theory.

By using the concept of the simplicial closed even walks we can give a necessary condition for a squarefree monomial ideal to be of linear type (I is called of linear type if J can be generated by its degree one elements). We also show that every simplicial closed even walk contains a simplicial cycle. By using this new result we can conclude that every simplicial tree is of linear type. This fact can also be deduced by the concept of M-sequences from the work of Conca, De Negri [14] and Soleyman Jahan, Zheng [29].

The results of this thesis appear in [1], [2] and [3].

Chapter 2

Background

Throughout, we assume that $\mathbb{K}$ is a field and $R = \mathbb{K}[x_1, \ldots, x_n]$ is a polynomial ring in n variables.

2.1 Simplicial Complexes and Monomial Ideals

Let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ be an integer vector. We set $x^a = x_1^{a_1} \cdots x_n^{a_n}$. A polynomial f in R of the form $f = x^a$ where $a \in \mathbb{N}^n$ is called a **monomial** in R. The monomial f is called **squarefree** if $a \in \{0, 1\}^n$. An ideal I in R is called a **squarefree monomial ideal** if I is generated by squarefree monomials.

One of the most useful techniques applied to connect commutative algebra to combinatorics is assigning a squarefree monomial ideal to a graph or a simplicial complex to make a dictionary between their algebraic and combinatorial properties. Here we need some definitions of simplicial complex terminology.

Definition 2.1.1 (simplicial complex)**.** An abstract **simplicial complex** on the vertex set $\mathcal{X} = \{x_1, \ldots, x_n\}$ is a collection Δ of subsets of $\mathcal{X}$ satisfying

- $F \in \Delta, G \subset F \Longrightarrow G \in \Delta$.

We set $\mathrm{V}(\Delta) = \{x \in \mathcal{X} : \{x\} \in \Delta\}$ and the elements of $\mathrm{V}(\Delta)$ are called the **vertices** of Δ. The elements of Δ are called **faces** of Δ and the maximal faces under inclusion are called **facets** of Δ. An element $F \in \Delta$ of cardinality $i + 1$ is called an i**-dimensional face** of Δ. Note that $\emptyset$ is the only -1-dimensional face of Δ. We set

$$\dim(\Delta) = \max\{\dim F : F \text{ is a face of } \Delta\}.$$

A simplicial complex is called **pure** if all dimensions of its facets are the same.

We denote the simplicial complex Δ with facets $F_1, \ldots, F_s$ by $\langle F_1, \ldots, F_s \rangle$. We call $\{F_1, \ldots, F_s\}$ the facet set of Δ, and it is denoted by Facets(Δ). For $i \in \mathbb{Z}$, let $\mathcal{C}_i(\Delta)$

denotes the set of all i-dimensional faces of Δ.

Example 2.1.2. The simplicial complex Δ with $\mathrm{V}(\Delta) = \{x_1, x_2, x_3, x_4\}$ and $\mathrm{Facets}(\Delta) = \{F_1, F_2, F_3\}$ where $F_1 = \{x_1, x_2, x_3\}$, $F_2 = \{x_2, x_4\}$, $F_3 = \{x_3, x_4\}$ is pictured below.

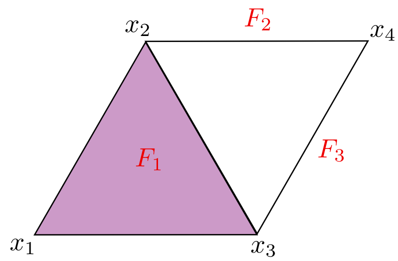


Figure 2.1: A simplicial complex with three facets

Here we also need the notation of subcollections of a simplicial complex and complement of a simplicial complex.

Definition 2.1.3 (subcollection of a simplicial complex)**.** A **subcollection** of a simplicial complex Δ with vertex set $\mathcal{X}$ is a simplicial complex whose facet set is a subset of the facet set of Δ. For $\mathcal{Y} \subseteq \mathcal{X}$, the **induced subcollection** of Δ on $\mathcal{Y}$, denoted by $\Delta_{\mathcal{Y}}$, is the simplicial complex whose vertex set is a subset of $\mathcal{Y}$ and facet set is

$$\{F \in \mathrm{Facets}(\Delta) \ : \ F \subseteq \mathcal{Y}\}.$$

Definition 2.1.4. Let $\Delta = \langle F_1, \ldots, F_s \rangle$ be a simplicial complex over the vertex set $\mathcal{X}$. If F is a face of Δ and $W \subset \mathcal{X}$, we define the **complement** of F in Δ with respect to W to be $F^c_W = W \setminus F$ and also the complement of Δ with respect to W is defined as

$$\Delta^c_W = \langle (F_1)^c_W, \ldots, (F_s)^c_W \rangle.$$

Also if $W \subsetneq \mathcal{X}$, then $\Delta^c_W = (\Delta_W)^c_W$ where Δ_W is the induced subcollection of Δ on W.

Remark 2.1.5. When $W = \mathcal{X}$ we will use Δ^c to denote Δ^c_W.

Example 2.1.6. In Example 2.1.2 simplicial complex $\Gamma = \langle F_1, F_3 \rangle$ is a subcollection of Δ, such that $\mathrm{V}(\Gamma) = \mathrm{V}(\Delta)$. However Γ is not an induced subcollection on its vertex set. Also the following graph is Δ^c.

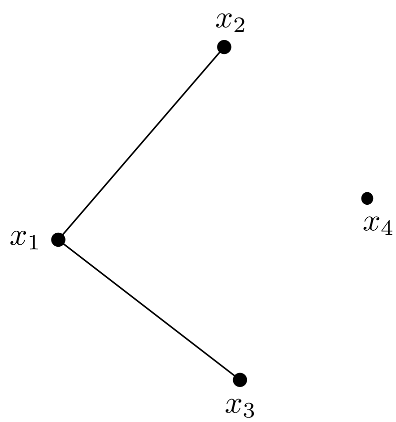


Figure 2.2: Complement

For every squarefree monomial ideal we can assign two simplicial complexes.

Definition 2.1.7. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, and I an ideal in R minimally generated by squarefree monomials $m_1, \ldots, m_s$. One can associate two simplicial complexes to I.

i. The **Stanley-Reisner complex** $\mathcal{N}(I)$ associated to I has vertex set $V = \{x_i \,:\, x_i \notin I\}$ and is defined as

$$\mathcal{N}(I) = \{\{x_{i_1}, \ldots, x_{i_k}\} \,:\, i_1 < i_2 < \cdots < i_k, \text{where } x_{i_1} \cdots x_{i_k} \notin I\}.$$

ii. The **facet complex** $\mathcal{F}(I)$ associated to I has vertex set $\{x_1, \ldots, x_n\}$ and is defined as

$$\mathcal{F}(I) = \langle F_1, \ldots, F_s \rangle \text{ where } F_i = \{x_j \,:\, x_j | m_i,\ 1 \leq j \leq n\} \text{ for } 1 \leq i \leq s.$$

Conversely to a simplicial complex Δ one can associate two monomial ideals.

Definition 2.1.8. Let Δ be a simplicial complex with vertex set $x_1, \ldots, x_n$ and $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$.

i. The **Stanley-Reisner ideal** of Δ is defined as

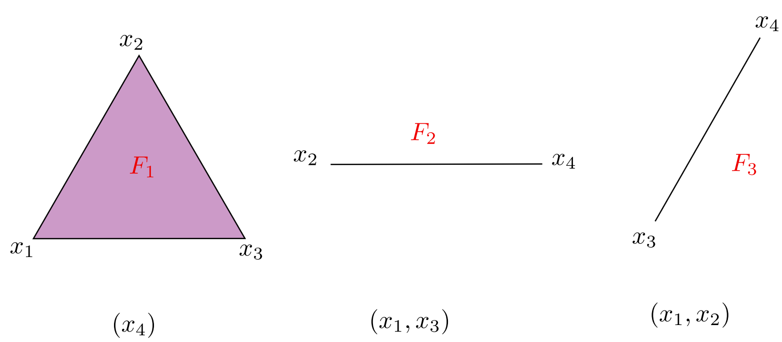
$$\mathcal{N}(\Delta) = \left(\prod_{x \in F} x \,:\, \text{F is not a face of } \Delta \right).$$

The quotient ring $R/\mathcal{N}(\Delta)$ is called the **Stanley-Reisner ring** of Δ and is denoted by $\mathbb{K}[\Delta]$.

ii. The **facet ideal** of Δ is defined as

$$\mathcal{F}(\Delta) = \left(\prod_{x \in F} x \ : \quad F \text{ is a facet of} \Delta \right).$$

Example 2.1.9. Consider the simplicial complex Δ in Example 2.1.2. $\mathcal{N}(\Delta)$ and $\mathcal{F}(\Delta)$ have been computed below.



$$\mathcal{N}(\Delta) = (x_4) \cap (x_2, x_3) \cap (x_1, x_2) = (x_1x_4, x_2x_3x_4)$$

$$\mathcal{F}(\Delta) = (x_1x_2x_3, x_2x_4, x_3x_4)$$

Figure 2.3: Facet ideal and Stanley-Reisner ideal

Since we have

- $\mathcal{F}(\mathcal{F}(I)) = I$ and $\mathcal{N}(\mathcal{N}(I)) = I$, for each squarefree monomial ideal I;
- $\mathcal{F}(\mathcal{F}(\Delta)) = \Delta$ and $\mathcal{N}(\mathcal{N}(\Delta)) = \Delta$, for each simplicial complex Δ

there is a one-to-one correspondence between monomial ideals and simplicial complexes via each of these methods.

We can assign a dual to each simplicial complex. To prove some of our results we recall the definition of this.

Definition 2.1.10 (Alexander dual)**.** Let Δ be a simplicial complex with vertex set $\mathcal{X}$. The

Alexander dual Δ^* is defined to be the simplicial complex with faces

$$\Delta^* = \{F^c_{\mathcal{X}} \ : \ F \text{ is not a face of } \Delta\}.$$

For the simplicial complex Δ in Example 2.1.2, we have

$$\Delta^* = \langle \{x_1\}, \{x_2, x_3\} \rangle.$$

2.2 Simplicial Homology, Cohomology and Mayer-Vietoris Sequence

Simplicial homology modules of a simplicial complex are $\mathbb{K}$-vector spaces that provide information about the number of holes (cycles) contained in a complex.

Definition 2.2.1 (simplicial homology module)**.** Let Δ be a d-dimensional simplicial complex on $\mathcal{X}$. For each $\Gamma \in \mathcal{C}_i(\Delta)$, let e_Γ denote the corresponding basis vector in the $\mathbb{K}$-vector space, $\mathbb{K}^{\mathcal{C}_i(\Delta)}$ (it is a $\mathbb{K}$-vector space generated by $\mathcal{C}_i(\Delta)$). Consider the following sequence

$$0 \to \mathbb{K}^{\mathcal{C}_d(\Delta)} \xrightarrow{\delta_d} \cdots \mathbb{K}^{\mathcal{C}_i(\Delta)} \xrightarrow{\delta_i} \mathbb{K}^{\mathcal{C}_{i-1}(\Delta)} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_0} \mathbb{K}^{\mathcal{C}_{-1}(\Delta)} \to 0 \tag{2.2.1}$$

where, for all $i = 0, 1, \ldots, d$, and $\Gamma \in \mathcal{C}_i(\Delta)$,

$$\delta_i(e_\Gamma) = \sum_{j \in \Gamma} sign(j, \Gamma) e_{\Gamma \setminus j}.$$

If $i > d$ or $i < -1$, then $\mathbb{K}^{\mathcal{C}_i(\Delta)} = 0$, and we define $\delta_i = 0$. Take $sign(j, \Gamma) = (-1)^{i-1}$ if j is the i-th element of Γ when the elements of Γ are listed in increasing order.

Since $\delta_i \delta_{i+1} = 0$, the sequence (2.2.1) is a chain complex, which is called the **simplicial chain complex** of Δ over $\mathbb{K}$, and is denoted by $\mathcal{SC}(\Delta)$.

Example 2.2.2. For Δ in Example 2.1.2 since we have $\dim(\Delta) = 2$ and

$$\begin{aligned}
\mathcal{C}_2(\Delta) &= \{\{x_1, x_2, x_3\}\} \\
\mathcal{C}_1(\Delta) &= \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\} \\
\mathcal{C}_0(\Delta) &= \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\} \\
\mathcal{C}_{-1}(\Delta) &= \emptyset
\end{aligned}$$

we have the following simplicial chain complex.

$$0 \longrightarrow \mathbb{K} \xrightarrow{\begin{pmatrix} 1\\ -1\\ 1\\ 0\\ 0\end{pmatrix}} \mathbb{K}^5 \xrightarrow{\begin{pmatrix} -1 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & -1 & 0\\ 0 & 1 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 1\end{pmatrix}} \mathbb{K}^4 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 1\end{pmatrix}} \mathbb{K} \longrightarrow 0$$

For $i \in \mathbb{Z}$, the i-**th reduced homology module** of Δ over $\mathbb{K}$ is the $\mathbb{K}$-vector space

$$\widetilde{H}_i(\Delta; \mathbb{K}) = \operatorname{kernel}(\delta_i)/\operatorname{image}(\delta_{i+1}).$$

Elements of $\operatorname{kernel}(\delta_i)$ are called i-cycles and elements of $\operatorname{image}(\delta_{i+1})$ are called i-boundaries. When $\mathbb{K}$ is clear from the context we use $\widetilde{H}_i(\Delta)$ to denote $\widetilde{H}_i(\Delta; K)$.

Theorem 2.2.3 (Proposition 5.2.3, [46]). *The dimension of* $\widetilde{H}_0(\Delta; \mathbb{K})$ *as a* $\mathbb{K}$*-vector space is one less than the number of connected components of* Δ .

Note that i-th reduced homology module of a simplicial complex Δ when $i > 1$ provides information about the number of i-cycles contained in Δ.

Example 2.2.4. The simplicial complex in Example 2.1.2 is connected. Therefore from Theorem 2.2.3 $\widetilde{H}_0(\Delta; \mathbb{K}) = 0$. Also since the only non-zero cycle is a 1-cycle there is only one non-zero homology module which is $\widetilde{H}_1(\Delta; \mathbb{K}) = \mathbb{K}$.

Now the question is how we can compute the homology modules of simplicial complexes. One method that we use frequently is the **Mayer-Vietoris sequence**. This method consists of splitting a simplicial complex into two subcollections, for which the homology module might be more easily computed. The Mayer-Vietoris sequence is a long exact sequence that relates the homology module of the simplicial complex to the homology module of its subcollections. For more details see Hatcher [20] Chapter 2.

Theorem 2.2.5 (Mayer-Vietoris sequence). *Suppose* Δ *is a simplicial complex and* Δ_1 *and* Δ_2 *are two subcollections such that* $\Delta = \Delta_1 \cup \Delta_2$. *We have the exact sequence of the chain complexes*

$$0 \to \mathcal{SC}(\Delta_1 \cap \Delta_2) \to \mathcal{SC}(\Delta_1) \oplus \mathcal{SC}(\Delta_2) \to \mathcal{SC}(\Delta) \to 0$$

which produces the following long exact sequence, called the **Mayer-Vietoris sequence**

$$\cdots \to \widetilde{H}_i(\Delta_1 \cap \Delta_2) \to \widetilde{H}_i(\Delta_1) \oplus \widetilde{H}_i(\Delta_2) \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\Delta_1 \cap \Delta_2) \to \dots .$$

Definition 2.2.6 (Simplicial cohomology module)**.** Let Δ be a d-dimensional simplicial complex on $\mathcal{X}$ with the simplicial chain complex $\mathcal{SC}(\Delta)$ as follows

$$0 \to \mathbb{K}^{\mathcal{C}_d(\Delta)} \xrightarrow{\delta_d} \cdots \mathbb{K}^{\mathcal{C}_i(\Delta)} \xrightarrow{\delta_i} \mathbb{K}^{\mathcal{C}_{i-1}(\Delta)} \xrightarrow{\delta_{i-1}} \cdots \xrightarrow{\delta_0} \mathbb{K}^{\mathcal{C}_{-1}(\Delta)} \to 0 \tag{2.2.2}$$

We define the dual of $\mathcal{SC}(\Delta)$ as follows and called it **simplicial cochain complex** of Δ,

$$0 \leftarrow \left(\mathbb{K}^{\mathcal{C}_d(\Delta)}\right)^* \xleftarrow{\delta_d{}^*} \cdots \left(\mathbb{K}^{\mathcal{C}_i(\Delta)}\right)^* \xleftarrow{\delta_i^*} \left(\mathbb{K}^{\mathcal{C}_{i-1}(\Delta)}\right)^* \xleftarrow{\delta_{i-1}{}^*} \cdots \xleftarrow{\delta_0{}^*} \left(\mathbb{K}^{\mathcal{C}_{-1}(\Delta)}\right)^* \leftarrow 0$$

where $\left(\mathbb{K}^{\mathcal{C}_i(\Delta)}\right)^* = \mathrm{Hom}_{\mathbb{K}}\left(\mathbb{K}^{\mathcal{C}_i(\Delta)}, \mathbb{K}\right)$ is the set of linear transformations from $\mathbb{K}^{\mathcal{C}_i(\Delta)}$ to $\mathbb{K}$. For each $\phi \in \left(\mathbb{K}^{\mathcal{C}_{i-1}(\Delta)}\right)^*$ the map $\delta_i^*(\phi)$ is defined as the composition of $\mathbb{K}^{\mathcal{C}_i(\Delta)} \xrightarrow{\delta_i} \mathbb{K}^{\mathcal{C}_{i-1}(\Delta)} \xrightarrow{\phi} \mathbb{K}$.

It is straightforward to show that $\delta_{i+1}{}^*\delta_i{}^* = 0$, so we can define $\widetilde{H}^i(\Delta;\mathbb{K})$, the i-th **reduced cohomology module** of Δ, as the quotient

$$\widetilde{H}^i(\Delta;\mathbb{K}) = \mathrm{kernel}(\delta_{i+1}{}^*)/\,\mathrm{image}(\delta_i{}^*).$$

Since we work with homology module and cohomology module with coefficients in a field, by using the **universal coefficient theorem for cohomology module** (see [20] Theorem 3.2) we have the following theorem.

Theorem 2.2.7 ([20], Page 198)**.** *Let Δ be a simplicial complex and $\mathbb{K}$ be a field. Then we have*

$$\widetilde{H}^i(\Delta;\mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(\widetilde{H}_i(\Delta;\mathbb{K}), \mathbb{K}) \cong \widetilde{H}_i(\Delta;\mathbb{K}).$$

Lemma 2.2.8 (Lemma 5.5.3, [12])**.** *Let $\mathbb{K}$ be a field and $\Delta \subset \Sigma$ be simplicial complexes where Σ is a simplex on the vertex set $\mathcal{X}$, $|\mathcal{X}| = d$. Then*

$$\widetilde{H}_i(\Delta;\mathbb{K}) \cong \widetilde{H}^{d-3-i}(\Delta^*;\mathbb{K}). \tag{2.2.3}$$

To prove some of our results we need the definition of a cone and its properties.

Definition 2.2.9. Let Δ_1 and Δ_2 be two simplicial complexes on disjoint vertex sets V and W. The **join** $\Delta_1 * \Delta_2$ is the simplicial complex on $V \bigsqcup W$ with faces $F \cup G$ where $F \in \Delta_1$ and $G \in \Delta_2$.

The **cone** $Cn(\Delta)$ of Δ is the simplicial complex $\omega * \Delta$, where ω is a new vertex.

Proposition 2.2.10 (Proposition 5.2.5, [46]). *If Δ is a simplicial complex we have*

$$\widetilde{H}_i(Cn(\Delta)) = 0 \quad \textit{for all } i.$$

2.3 Minimal Free Resolutions and Betti Numbers

Definition 2.3.1 (graded rings and modules). A ring R with a decomposition $R = \bigoplus_{i \in \mathbb{Z}} R_i$ of $\mathbb{Z}$-submodules of R is called a **graded ring** if

$$R_i R_j \subset R_{i+j} \quad \text{for all } i, j \in \mathbb{Z}.$$

For the graded ring R, the R-module M with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ of $\mathbb{Z}$-submodules of M is called a **graded module** if

$$R_i M_j \subset M_{i+j} \quad \text{for all } i, j \in \mathbb{Z}.$$

Every element $x \in M_i$ is called **homogeneous of degree** i. A submodule N in M is called **homogeneous** if N is generated by homogeneous elements. The graded module M is called **positive** if $M_i = 0$ for every $i < 0$. Every R_i and M_i is an R_0-module.

Example 2.3.2. Let $\mathbb{K}$ be a field. The polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ in which we have

$$R = \bigoplus_{i=0}^{\infty} R_i$$

where R_i is the $\mathbb{K}$-vector space generated by monomials of degree i is a positive graded ring .

Definition 2.3.3 (graded maps). Let R be a graded ring and M, N be graded R-modules. The R-homomorphism $\psi : M \longrightarrow N$ is called **graded of degree** j if for each $i \in \mathbb{Z}$, the map ψ sends every homogeneous element of degree i in M to a homogeneous element in N of degree $i + j$.

If ψ is graded of degree 0 then $\operatorname{kernel}\psi$ and $\operatorname{image}\psi$ are graded R-modules.

Definition 2.3.4. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a positive graded ring and $a \in \mathbb{N}$. The graded R-module obtained by a shift in the graduation of R is given by

$$R(-a) = \bigoplus_{i=a}^{\infty} R(-a)_i.$$

where i-th graded component of $R(-a)$ is $R(-a)_i = R_{-a+i}$.

Proposition 2.3.5 (Proposition 2.5.5 and Theorem 2.5.6, [46])**.** *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a positive graded polynomial ring over a field $\mathbb{K}$ with maximal ideal $m = R_+ = \bigoplus_{i=1}^{\infty} R_i$ and M be a finitely generated positive graded R-module. Then there is an exact sequence, of finite length, of graded free R-modules*

$$0 \to \bigoplus_{d_p} R(-d_p)^{\beta_{p,d_p}} \xrightarrow{\delta_p} \cdots \xrightarrow{\delta_2} \bigoplus_{d_1} R(-d_1)^{\beta_{1,d_1}} \xrightarrow{\delta_1} \bigoplus_{d_0} R(-d_0)^{\beta_{0,d_0}} \xrightarrow{\delta_0} M \quad (2.3.1)$$

where δ_i is graded of degree 0 and for each $i = 1, 2, \ldots, p$ we have

$$\operatorname{image}(\delta_i) \subset mR^{b_{i-1}} \quad \textit{for } b_{i-1} = \sum_{d_{i-1}} \beta_{i-1,d_{i-1}}.$$

Definition 2.3.6 (graded minimal free resolution)**.** Let R be a positive graded polynomial ring over a field $\mathbb{K}$ and M be a finitely generated positive graded R-module. Then, the graded resolution of M by free modules described in Proposition 2.3.5 is called an $\mathbb{N}$-**graded minimal free resolution** of M.

We can see that a minimal graded free resolution is unique up to chain complexes isomorphism (cf. Villarreal [46], Corollary 2.5.7), so we have the following corollary.

Corollary 2.3.7 (Betti numbers)**.** *The numbers β_{i,d_i} in the minimal free resolution (2.3.1), which we shall refer to as the i-th $\mathbb{N}$-**graded Betti numbers** of degree d_i of M, are independent of the choice of graded minimal finite free resolutions.*

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a positive graded polynomial ring over a field $\mathbb{K}$ and $I \subset R$ be a homogeneous ideal in R. There are two important invariants attached to I, which are defined in terms of the minimal graded free resolution of R/I.

- **projective dimension** of R/I is defined as

$$pd(R/I) = \max\{i : \beta_{i,j}(R/I) \neq 0 \text{ for some } j\}.$$

- **regularity** of I is defined as

$$reg(I) = \max\{j - i : \beta_{i,j}(R/I) \neq 0 \text{ for some } j\}.$$

Definition 2.3.8 (syzygies)**.** Let $f_1, \ldots, f_q \in R$ and F be a free R-module and $\{e_1, \ldots, e_q\}$ be a basis of F. Consider the following homomorphism

$$F \longrightarrow (f_1, \ldots, f_q), \ e_i \mapsto f_i.$$

The kernel of this homomorphism is called the **syzygy** module of $f_1, \ldots, f_q$ and we denote it by $Syz(f_1, \ldots, f_q)$.

Example 2.3.9. Let $I = (x^2, y)$ be an ideal in the polynomial ring $R = \mathbb{K}[x, y]$. The following is the minimal free resolution of R/I.

$$0 \longrightarrow R(-3) \xrightarrow{\begin{pmatrix} -y \\ x^2 \end{pmatrix}} R(-2) \oplus R(-1) \xrightarrow{\begin{pmatrix} x^2 & y \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

where $\beta_{23} = \beta_{11} = \beta_{12} = 1$. From this resolution we can see that $pd(R/I) = 2$ and $reg(R/I) = 1$.

2.4 Graphs

A simple graph is a graph without multiple edges and loops. We give some definitions and theorems from graph theory. For the most part we follow West [48].

Definition 2.4.1. Let $G = (\mathrm{V}, E)$ be a graph where V is a nonempty set of vertices and E is a set of edges. A **walk** in G is a list $e_1, e_2, \ldots, e_n$ of edges such that

$$e_i = \{x_i, x_{i+1}\} \in E \qquad \text{for each } i \in \{1, \ldots, n-1\}.$$

A walk is called **closed** if its endpoints are the same i.e. $x_1 = x_n$. The number of edges of a walk $\mathcal{W}$ is called **length of** $\mathcal{W}$ and is denoted by $\ell(\mathcal{W})$. A **trial** is a walk with no repeated edges. A **path** in G is a walk with no repeated vertices or edges allowed. A closed path is called a **cycle**.

Lemma 2.4.2 (Lemma 1.2.15 and Remark 1.2.16, [48])**.** *Let G be a simple graph. Then we have*

- *Every closed odd walk contains a cycle.*

- *Every closed even walk that has at least one non-repeated edge contains a cycle.*

Definition 2.4.3 (unicyclic graphs and trees)**.** A connected graph without a cycle is called a **tree**. A graph which contains only one cycle is called a **unicyclic graph**.

We will make use of the following theorem. (cf. West [48]).

Theorem 2.4.4 (Theorem 2.1.4 and Corollary 2.1.5, [48])**.** *Let G be a connected graph with $|V(G)| = n$ and $|E(G)| = q$. Then we have*

- *G is a tree if and only if $q = n - 1$.*

- *G is a unicyclic graph if and only if $q = n$.*

Theorem 2.4.5 (Euler's theorem)**.** *[Theorem 1.2.26, [48]] If G is a connected graph, then the edges of G form a closed walk with no repeated edges if and only if the degree of every vertex of G is even.*

We can associate a quadratic squarefree monomial ideal to each simple connected graph.

Definition 2.4.6 (edge ideal)**.** Let G be a graph on the vertex set $\mathrm{V} = \{x_1, \ldots, x_n\}$ with edge set $E = \{e_1, \ldots, e_q\}$. The **edge ideal** of G is defined as follows over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$:

$$I(G) = (x_i x_j : \{i, j\} \in E).$$

Note that if we consider the graph G as a 1-dimensional simplicial complex, then we can easily imply that the edge ideal of G is the facet ideal of G as a simplicial complex.

Edge ideals were first defined by Villarreal in [45], and can be used to make a link between combinatorial properties of a graph and algebraic properties of its edge ideal. For instance the edge ideal of the following graph is

$$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5, x_1x_4, x_3x_5).$$

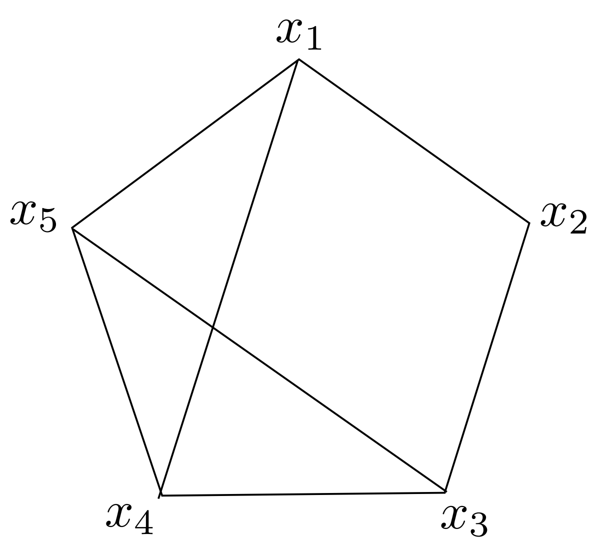


Figure 2.4: Edge ideal

2.5 Rees Algebras and Symmetric Algebras

First we need to recall the definition of Rees algebras. For more details see Huneke and Swanson [27].

Definition 2.5.1. Let R be a Noetherian ring, $I \subset R$ be an ideal and t be an indeterminate over R. The following subring of $R[t]$ is called **Rees algebra** of I.

$$R[It] = \left\{ \sum_{i=0}^{n} a_i t^i \ : \ n \in \mathbb{N}, a_i \in I^i \right\} = \bigoplus_{i=0}^{\infty} I^i t^i.$$

Definition 2.5.2 (defining ideal of the Rees algebras)**.** Let R be a Noetherian ring, $I = (f_1, \ldots, f_q)$ be an ideal in R and t be an indeterminate. Consider the polynomial ring

$S = R[T_1, \ldots, T_q]$ where $T_1, \ldots, T_q$ are indeterminates. For the Rees algebra $R[It] = R[f_1t, \ldots, f_qt]$ we can defined the following homomorphism of algebras

$$\psi : R[T_1, \ldots, T_q] \longrightarrow R[It],\ T_i \mapsto f_it.$$

Let J be the kernel of ψ and then $R[It] = S/J$. The map ψ is graded of degree 0, so J is graded and we have

$$J = \bigoplus J_i$$

The ideal J is called the **defining ideal** of $R[It]$ and its minimal generators are called the **Rees equations** of I.

These equations carry a lot of information about $R[It]$; see for example Vasconcelos [42] for more details.

We consider the linear homogeneous polynomials in J (i.e., J_1). These polynomials can be obtained from a presentation of I. To see this we need to state the definition of symmetric algebras.

Definition 2.5.3. Let R be a Noetherian ring and I be an ideal in R which is given by the following presentation.

$$R^p \xrightarrow{\phi} R^q \to I \to 0, \quad \phi = (a_{ij}) \in \mathrm{Matrix}_{q\times p}(R).$$

The **symmetric algebra** of I, denoted by $S(I)$, is the quotient ring of the polynomial ring $R[T_1, \ldots, T_q]$ by the ideal A generated by the following linear polynomials

$$F_i = a_{1i}T_1 + \cdots + a_{qi}T_q \quad \text{for } 1 \leq i \leq p$$

It can be seen that $A = (J_1) \subset J$ (cf. Vasconcelos [43]). Therefore when $J = (J_1)$, the Rees algebra and the symmetric algebra coincide.

Definition 2.5.4 (ideals of linear type)**.** The ideal I is called to be **of linear type** if $J = (J_1)$; in other words, the defining ideal of $R[It]$ is generated by linear forms in the variables $T_1, \ldots, T_q$.

Ideals of linear type have been investigated by many authors (cf. Conca and De Negri [14], Costa [15], Fouli and Lin [18], Huneke [25] and [26]). Because of the following proposition complete intersection ideals are perhaps the most obvious class of ideals of linear type.

Proposition 2.5.5 (Example 1.2, [43])**.** *Let R be a Noetherian ring and I be an ideal in R. If the ideal I is generated by a regular sequence $f_1, \ldots, f_q$ the Rees algebra of I is*

$$R[It] \cong R[T_1, \ldots, T_q]/I_2 \begin{pmatrix} T_1 & \ldots & T_q \\ f_1 & \ldots & f_q \end{pmatrix}$$

where $I_2(\Delta)$ is an ideal which is generated by all 2×2 minors of Δ.

Another famous class of ideals of linear type is the class of ideals generated by d-sequences which was introduced by Fiorentini [17] and Huneke [26].

We have the following useful result from Herzog, Simis and Vasconcelos [22].

Theorem 2.5.6 (Proposition 2.4, [22])**.** *Let I be an ideal in a Noetherian ring R and let I be of linear type. Then for every prime ideal $\mathfrak{p}$ containing I, the ideal $I_\mathfrak{p}$ can be generated by $ht(p)$ elements (where $ht(p)$ is height of $\mathfrak{p}$ which is the maximum of the lengths of the chains of prime ideals contained in $\mathfrak{p}$).*

Corollary 2.5.7. *Let $I = (f_1, \ldots, f_q)$ be a squarefree monomial ideal in $R = \mathbb{K}[x_1, \ldots, x_n]$. If I is of linear type, we have $q \leq n$.*

From Corollary 2.5.7 and Theorem 2.4.4 we can deduce the following corollary.

Corollary 2.5.8. *Let G be a connected graph and $I(G)$ be its edge ideal. If $I(G)$ is of linear type, then G is either a tree or a unicyclic graph.*

Proof. Let $V(G) = \{x_1, \ldots, x_n\}$ be the vertex set of G and $I(G) = (f_1, \ldots, f_q)$ where $|E(G)| = q$. Since $I(G)$ is of linear type from Corollary 2.5.7 we have $q \leq n$. On the other hand since G is connected we can conclude $q \geq n-1$ so $q \in \{n-1, n\}$. The claim follows from Theorem 2.4.4. □

Villareal showed that the converse of this corollary is also correct [47]. Here we prove the converse in Corollary 5.3.10 with a different method.

2.6 Simplicial Trees and Good Leaves

Good leaves were first introduced by X. Zheng in her PhD thesis [50]. To state the definition of a good leaf we first need to recall the definition of a leaf. The following definition was given by Faridi [16].

Definition 2.6.1 (leaves and simplicial forests)**.** Let Δ be a simplicial complex and F be a facet of Δ. The facet F is called a **leaf** of Δ if either F is the only facet of Δ or else there exists a facet G with $G \neq F$ such that for all facets $H \neq F$ we have $H \cap F \subset G$. The facet G is called a **joint** of F.

A simplicial complex Δ is called a **simplicial forest** if each of its subcollections has a leaf. A connected simplicial forest is called a **simplicial tree**.

Example 2.6.2. In Figure 2.5 F_1, F_3 are leaves of Δ and their joint is F_2.

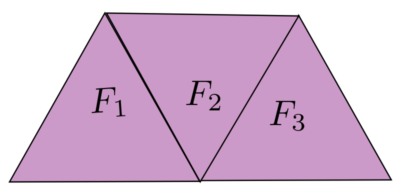


Figure 2.5: A simplicial tree

We now state the definition of a good leaf.

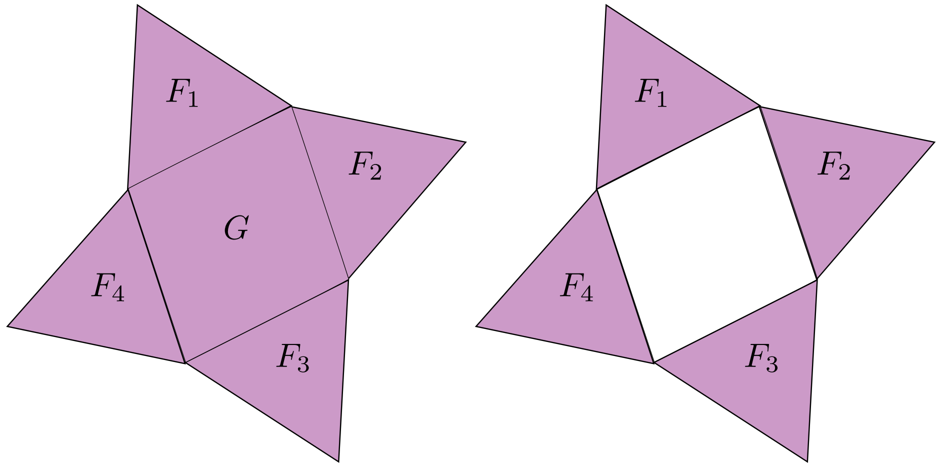
Definition 2.6.3 (Good Leaf)**.** Let Δ be a simplicial complex. A facet F of Δ is a **good leaf** of Δ if F is a leaf of all subcollections of Δ which contain F.

We also need the following useful property of a good leaf.

Lemma 2.6.4 (Lemma 3.10, [50])**.** *Let Δ be a simplicial complex and F be a facet of Δ. The following conditions are equivalent*

- *F is a good leaf of Δ;*
- *The set $\{H \cap F; H \in \text{Facets}(\Delta)\}$ is totally ordered by inclusion.*

Example 2.6.5. In Figure 5.1a, $F_1, \ldots, F_4$ are leaves of Δ and their joint G is a solid tetrahedron. However, since the subcollection Γ has no leaves, then $F_1, \ldots, F_4$ are not good leaves.



(a) $\Delta = \langle F_1, F_2, F_3, F_4, G\rangle$ (b) $\Gamma = \langle F_1, F_2, F_3, F_4\rangle \subset \Delta$

Figure 2.6: Good leaf

The existence of a good leaf in every simplicial tree was proved by Herzog, Hibi, Trung and Zheng in [23].

Theorem 2.6.6 (Corollary 3.4, [23])**.** *Every simplicial tree contains a good leaf.*

2.7 Higher Dimensional Cycles

We end this chapter by introducing three higher dimension cycles.

Definition 2.7.1. Let Δ be a simplicial complex with at least three facets, ordered as $F_1, \ldots, F_q$. Suppose $\bigcap F_i = \emptyset$. With respect to this order Δ can be one of the following:

(i) **extended trail** if we have

$$F_i \cap F_{i+1} \neq \emptyset \qquad i = 1, \ldots, q \quad \text{mod } q.$$

Extended trails come from the definition of a higher dimension cycle which is defined by Berge [10].

(ii) **special cycle** [23] if Δ is an extended trail in which we have

$$F_i \cap F_{i+1} \not\subset \bigcup_{j \notin \{i,i+1\}} F_j \quad i = 1, \ldots, q \quad \text{mod } q.$$

Special cycles come from hypergraph theory and were first introduced by Lovász in 1979 (cf. [30]).

(iii) **simplicial cycle** [13] if Δ is an extended trail in which we have

$$F_i \cap F_j \neq \emptyset \Leftrightarrow j \in \{i+1, i-1\} \quad i = 1, \ldots, q \quad \text{mod } q.$$

Simplicial cycles are related to simplicial trees and were first introduced by Caboara, Faridi and Selinger [13].

We say that Δ is an extended trail (or special or simplicial cycle) if there is an order on the facets of Δ such that the specified conditions hold on that order. Note that

$$\{\text{Simplicial Cycles}\} \subseteq \{\text{Special Cycles}\} \subseteq \{\text{Extended Trails}\}.$$

In the case of graphs the special and simplicial cycles are the ordinary cycles, but an extended trail in our definition is neither a cycle nor a trail (a walk without repeated edges) in the case of graphs (but they are walks in the graph terminology). For instance, the graph in Figure 2.7 is an extended trail, which is neither a cycle nor a trail, but contains one cycle.

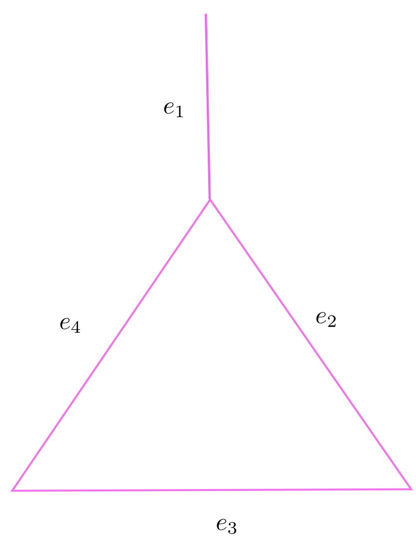


Figure 2.7: Extended trail in a graph

Chapter 3

Resolutions of Path Ideals of Cycles and Paths

In this chapter we consider the path ideal of a graph as a disjoint union of connected components. We then use homological methods to glue these components back together, and use Hochster's formula. In the next chapter we compute all graded Betti numbers, projective dimension and regularity of path ideals of paths and cycles.

3.1 Betti Numbers of Squarefree Monomial Ideals

Every squarefree ideal can be viewed as a Stanley-Reisner ideal of a simplicial complex. Then for computing the $\mathbb{N}$-graded Betti numbers of a squarefree monomial ideal we only need to use Hochster's formula for Betti numbers of simplicial complexes (Betti numbers of the Stanley-Reisner ring). We now state Hochster's theorem.

Theorem 3.1.1 (Theorem 5.1, [24])**.** *Let Δ be a simplicial complex. For $i > 0$ the Betti numbers $\beta_{i,d}$ of Δ are given by*

$$\beta_{i,d}(\mathbb{K}[\Delta]) = \sum_{\substack{W \subset \mathrm{V}(\Delta) \\ |W|=d}} \dim_{\mathbb{K}} \widetilde{H}_{d-i-1}(\Delta_{[W]}; \mathbb{K})$$

where $\Delta_{[W]} = \{F \in \Delta : F \subset W\}$.

Here we use an equivalent form of Hochster's formula (cf. Corollary 5.12 of Miller and Sturmfels [33]).

Theorem 3.1.2. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, and I be a pure square-free monomial ideal in R. Then the $\mathbb{N}$-graded Betti numbers of R/I are given by*

$$\beta_{i,d}(R/I) = \sum_{\substack{\Gamma \subset \mathcal{F}(I) \\ |\mathrm{V}(\Gamma)|=d}} \dim_{\mathbb{K}} \widetilde{H}_{i-2}(\Gamma^c_{\mathrm{V}(\Gamma)})$$

where the sum is taken over the induced subcollections Γ of $\mathcal{F}(I)$ which have d vertices.

Proof. Hochster's formula says

$$\beta_{i,d}(R/I) = \sum_{\substack{W \subset \mathrm{V}(\mathcal{N}(I)) \\ |W|=d}} \dim_{\mathbb{K}} \widetilde{H}_{d-i-1}(\mathcal{N}(I)_{[W]})$$

where $\mathcal{N}(I)_{[W]} = \{F \in \mathcal{N}(I) \ : \ F \subset W\}$. On the other hand from Theorem 2.2.7 and Lemma 2.2.8 we have

$$\widetilde{H}_{d-i-1}(\mathcal{N}(I)_{[W]}) \cong \widetilde{H}^{i-2}({\mathcal{N}(I)_{[W]}}^*) \cong \widetilde{H}_{i-2}({\mathcal{N}(I)_{[W]}}^*). \tag{3.1.1}$$

Suppose $m_1, m_2, \ldots, m_r$ is a minimal monomial generating set for I and correspondingly, $\mathcal{F}(I) = \langle F_1, \ldots, F_r \rangle$. We now claim ${\mathcal{N}(I)_{[W]}}^* = (\mathcal{F}(I)_W)^c_W$ for $W \subset \mathrm{V}(\mathcal{N}(I))$.

$$\begin{aligned} F \in {\mathcal{N}(I)_{[W]}}^* &\iff W \setminus F \notin \mathcal{N}(I)_{[W]} \\ &\iff W \backslash F \notin \mathcal{N}(I) \\ &\iff \prod_{x \in W \backslash F} x \in I = (m_1, m_2, \ldots, m_r) \\ &\iff m_s | \prod_{x \in W \backslash F} x, \text{ for some } s \in \{1, \ldots, r\} \\ &\iff F_s \subset W \backslash F \subset W, \text{ for some } s \in \{1, \ldots, r\} \\ &\iff F \subset W \backslash F_s \in (\mathcal{F}(I)_W)^c_W, \text{ for some } s \in \{1, \ldots, r\}. \end{aligned}$$

Now Hochster's formula and (3.1.1) imply that

$$\beta_{i,d}(R/I) = \sum_{\substack{W \subset \mathrm{V}(\mathcal{F}(I)) \\ |W|=d}} dim_{\mathbb{K}} \widetilde{H}_{i-2}((\mathcal{F}(I)_W)^c_W).$$

If we assume $\mathrm{V}(\mathcal{F}(I)_W) \neq W$ then it is straightforward to show that $(\mathcal{F}(I)_W)^c_W$ is a cone and by Proposition 2.2.10 it contributes 0 to the sum. So we have

$$\beta_{i,d}(R/I) = \sum_{\substack{\Gamma \subset \mathcal{F}(I) \\ |\mathrm{V}(\Gamma)|=d}} dim_{\mathbb{K}} \widetilde{H}_{i-2}(\Gamma^c_{\mathrm{V}(\Gamma)}).$$

where the sum is taken over the induced subcollections Γ of $\mathcal{F}(I)$ which have d vertices.

□

Based on Theorem 3.1.2, from here on all induced subcollections $\Gamma = \Delta_{\mathcal{Y}}$ of a simplicial complex Δ that we consider will have the property that $\mathcal{Y} = \mathrm{V}(\Gamma)$.

3.2 Path Ideals and Runs

Definition 3.2.1 (path ideal and path complex)**.** Let $G = (\mathcal{X}, E)$ be a finite simple graph and t be an integer such that $t \geq 2$. We define the **path ideal** of G, denoted by $I_t(G)$ to be the ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials of the form $x_{i_1}x_{i_2}\cdots x_{i_t}$ where $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$ is a path in G. The facet complex of $I_t(G)$, denoted by $\Delta_t(G)$, is called the **path complex** of the graph G.

We will be considering in this thesis a **cycle** C_n, or a **path graph** L_n on vertices $\{x_1, \ldots, x_n\}$.

$$C_n = \langle x_1x_2, \ldots, x_{n-1}x_n, x_nx_1\rangle \text{ and } L_n = \langle x_1x_2, \ldots, x_{n-1}x_n\rangle.$$

Example 3.2.2. Consider the cycle C_7 with vertex set $\mathcal{X} = \{x_1, \ldots, x_7\}$

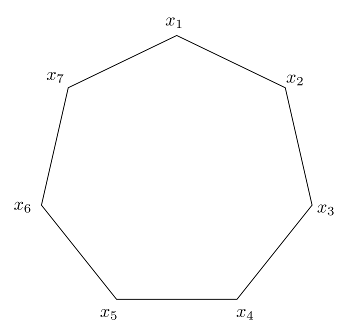


Figure 3.1: Cycle with 7 vertices

Then we have

$$I_4(C_7) = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_6, x_4x_5x_6x_7, x_1x_5x_6x_7, x_1x_2x_6x_7, x_1x_2x_3x_7)$$

$$\Delta_4(C_7) = \langle\{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\}, \{x_3, x_4, x_5, x_6\}, \{x_4, x_5, x_6, x_7\}, \{x_1, x_5, x_6, x_7\}, \{x_1, x_2, x_6, x_7\}, \{x_1, x_2, x_3, x_7\}\rangle.$$

We now focus on path ideals, path complexes, and their structures.

Notation 3.2.3. Let i and k be two positive integers. For (a set of) labeled objects we use the notation $\mod k$ to denote

$$x_i \mod k = \{x_j : 1 \leq j \leq k, i \equiv j \mod k\}$$

and

$$\{x_{u_1}, x_{u_2}, \dots, x_{u_t}\} \mod k = \{x_{u_j} \mod k : j = 1, 2, \dots, k\}.$$

Note 3.2.4. Let C_n be a cycle on vertex set $\mathcal{X} = \{x_1, \dots, x_n\}$ and $t < n$. The facets of the path complex $\Delta_t(C_n) = \langle F_1, \dots, F_n \rangle$ can be labeled as

$$F_1 = \{x_1, \dots, x_t\}, \dots, F_{n-(t-1)} = \{x_{n-(t-1)}, \dots, x_n\}, \dots, F_n = \{x_1, \dots, x_{t-1}, x_n\}$$

such that all indices are considered to be mod n and $F_i = \{x_i, x_{i+1}, \dots, x_{i+t-1}\}$ for all $1 \leq i \leq n$. This labeling is called the **standard labeling** of $\Delta_t(C_n)$.

Since for each $1 \leq i \leq n$ we have

$$F_{i+1} \setminus F_i = \{x_{t+i}\} \quad \text{and} \quad F_i \setminus F_{i+1} = \{x_i\},$$

it follows that $|F_i \setminus F_{i+1}| = 1 \quad \text{and} \quad |F_{i+1} \setminus F_i| = 1 \qquad \text{for all } 1 \leq i \leq n-1.$

Note 3.2.5. In this chapter and the next chapter when we work with the cycle C_n all indices are considered to be mod n.

It is straightforward to show that each induced subgraph of a graph cycle is a disjoint union of paths. Borrowing the terminology from Jacques [28], we call the path complex of a path a "run", and show that every induced subcollection of the path complex of a cycle is a disjoint union of runs.

Definition 3.2.6. We define a **run** to be the path complex of a path graph. A run which has p facets is called a **run of length** p and corresponds to $\Delta_t(L_{p+t-1})$ for some t. Therefore a run of length p has $p + t - 1$ vertices, for some t.

Example 3.2.7. Consider the cycle C_7 on vertex set $\mathcal{X} = \{x_1, \dots x_7\}$ and the simplicial complex $\Delta_4(C_7)$. The following induced subcollections are two runs in $\Delta_4(C_7)$

$$\begin{aligned} \Delta_1 &= \langle \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\} \rangle \\ \Delta_2 &= \langle \{x_1, x_2, x_6, x_7\}, \{x_1, x_2, x_3, x_7\}, \{x_1, x_2, x_3, x_4\} \rangle. \end{aligned}$$

Proposition 3.2.8 below shows that every proper induced subcollection of a path complex of a cycle is a disjoint union of runs.

Proposition 3.2.8. *Let C_n be a cycle with vertex set $\mathcal{X} = \{x_1, \ldots . x_n\}$ and $2 \leq t < n$. Let Γ be a non-empty proper induced connected subcollection of $\Delta_t(C_n)$ on $U \subsetneqq \mathcal{X}$. Then Γ is of the form $\Delta_t(L_{|\Gamma|})$, where $L_{|\Gamma|}$ is the path graph on $|\Gamma|$ vertices.*

Proof. Suppose $\Delta_t(C_n) = \langle F_1, \ldots, F_n \rangle$ has the standard labeling and $\Gamma = \langle F_{i_1}, \ldots, F_{i_r} \rangle$. Note that there exists a facet $F_a \in \Gamma$ for $1 \leq a \leq n$ such that $F_{a+1} \notin \Gamma$, because otherwise $\Gamma = \Delta_t(C_n)$. Therefore from Note 3.2.4 we have

$$\{x_a, x_{a+1}, \ldots, x_{a+t-1}\} \subset U \quad \text{and} \quad x_{a+t} \notin U. \tag{3.2.1}$$

Let r be the largest non-negative integer such that $x_{a-i} \in U$ for $0 \leq i \leq r$ so that

$$\underbrace{x_{a-r-1},}_{\notin U} \underbrace{x_{a-r}, \ldots, x_{a-1}, x_a, x_{a+1}, \ldots, x_{a+t-1},}_{\in U} \underbrace{x_{a+t}}_{\notin U}. \tag{3.2.2}$$

It follows that since Γ is an induced subcollection of $\Delta_t(C_n)$ on U, $F_{a-r}, F_{a-r+1}, \ldots, F_a \in \Gamma$. We now show that

$$F_i \notin \Gamma \text{ for all } i \notin \{a-r, a-r+1, \ldots, a\}.$$

This follows from the fact that Γ is connected: if any F_i (except for $a-r \leq i \leq a$) intersects some of the facets $F_{a-r}, \ldots, F_a$, then it must contain x_{a-r-1} or x_{a+t} (as otherwise it would be one of $F_{a-r}, \ldots, F_a$), and hence $F_i \notin \Gamma$.

We have therefore shown that

$$\Gamma = \langle F_{a-r}, F_{a-r+1}, \ldots, F_a \rangle.$$

Next we prove $\Gamma = \Delta_t(L_{|U|})$. Without loss of generality we can assume that $a - r = 1$, so that $\Gamma = \langle F_1, \ldots, F_{r+1} \rangle$. Since $\Gamma \neq \Delta_t(C_n)$ we can say that $r + 1 < n$ and therefore we have

$$\mathrm{V}(\Gamma) = \{x_1, x_2, \ldots, x_{r+t}\}.$$

Since Γ is induced and proper we have $r + t < n$, and therefore we can conclude that

$\Gamma = \Delta_t(L_{\{x_1,x_2,\ldots,x_{t+r}\}})$. $\square$

Lemma 3.2.9. *Let Γ and Λ be two induced subcollections of $\Delta_t(C_n) = \langle F_1, F_2, \ldots, F_n\rangle$ each of which is a disjoint union of runs of lengths $s_1, \ldots, s_r$. Then Γ and Λ are isomorphic as simplicial complexes. In particular the two simplicial complexes Γ^c and Λ^c are isomorphic and have the same reduced homology modules.*

Proof. First we suppose $\Delta_t(C_n) = \langle F_1, F_2, \ldots, F_n\rangle$ has the standard labeling. If we denote each run of length s_i in Γ and Λ by R_i and R_i', respectively, we have

$$\Gamma = \langle R_1, R_2, \ldots, R_r\rangle \text{ and } \Lambda = \langle {R_1}', {R_2}', \ldots, {R_r}'\rangle$$

where, using the standard labeling, for $1 \leq j_i, h_i \leq n$ we have

$$R_i = \langle F_{j_i}, F_{j_i+1}, \ldots, F_{j_i+s_i-1}\rangle \text{ and } {R_i}' = \langle F_{h_i}, F_{h_i+1}, \ldots, F_{h_i+s_i-1}\rangle.$$

Then we have

$$\mathrm{V}(\Gamma) = \bigcup_{i=1}^{r} \mathrm{V}(R_i) = \bigcup_{i=1}^{r} \{x_{j_i}, x_{j_i+1}, \ldots, x_{j_i+s_i+t-2}\}$$

and

$$\mathrm{V}(\Lambda) = \bigcup_{i=1}^{r} \mathrm{V}({R_i}') = \bigcup_{i=1}^{r} \{x_{h_i}, x_{h_i+1}, \ldots, x_{h_i+s_i+t-2}\}.$$

Now we define the function $\varphi : \mathrm{V}(\Gamma) \longrightarrow \mathrm{V}(\Lambda)$ where

$$\varphi(x_{j_i+u}) = x_{h_i+u} \text{ for } 0 \leq u \leq s_i + t - 2.$$

Since φ is a bijective map between vertex set Γ and Λ which preserves faces, we can conclude Γ and Λ are isomorphic. Therefore, the two simplicial complexes Γ^c and Λ^c are isomorphic as well and have the same reduced homology module. $\square$

Therefore, in light of Proposition 3.2.8, Theorem 3.1.2 and Lemma 3.2.9 all the information we need to compute the Betti numbers of $\Delta_t(C_n)$, or equivalently the homology modules of induced subcollections of $\Delta_t(C_n)$, depend on the number and the lengths of the runs.

Definition 3.2.10. For a fixed integer $t \geq 2$, let the pure $(t-1)$-dimensional simplicial complex $\Gamma = \langle F_1, \ldots, F_s \rangle$ be a disjoint union of runs of length $s_1, \ldots, s_r$. Then the sequence of positive integers $s_1, \ldots, s_r$ is called a **run sequence** on $\mathcal{Y} = \mathrm{V}(\Gamma)$, and we use the notation

$$E(s_1, \ldots, s_r) = \Gamma^c_{\mathcal{Y}} = \langle (F_1)^c_{\mathcal{Y}}, \ldots, (F_s)^c_{\mathcal{Y}} \rangle.$$

3.3 Reduced Homologies for Betti Numbers

Let $I = I_t(C_n)$ be the path ideal of the cycle C_n for some $t \geq 2$. By applying Hochster's formula (Theorem 3.1.2), we see that to compute the Betti numbers of R/I, we need to compute the reduced homology modules of complements of induced subcollections of Δ which by Proposition 3.2.8 are disjoint unions of runs. This section is devoted to complex homological calculations. The results here will allow us to compute all Betti numbers of R/I (and more) in the sections that follow.

We make a basic observation.

Lemma 3.3.1. *Let $E_1, \ldots, E_m$ be subsets of the finite set V where $m \geq 2$ and suppose that $\mathcal{E} = \langle (E_1)^c_V, (E_2)^c_V, ..., (E_m)^c_V \rangle$.*

i. Suppose $V \setminus \bigcup_{j=2}^{m} E_j \neq \emptyset$. If $\mathcal{E}_1 = \langle (E_1)^c_V \rangle$ and $\mathcal{E}_2 = \langle (E_2)^c_V, ..., (E_m)^c_V \rangle$ then for each i

$$\begin{aligned}\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_i(\mathcal{E}_1 \cup \mathcal{E}_2) &\cong \widetilde{H}_{i-1}(\mathcal{E}_1 \cap \mathcal{E}_2)\\ &= \widetilde{H}_{i-1}(\langle (E_1 \cup E_2)^c_V, ..., (E_1 \cup E_m)^c_V \rangle)\\ &= \widetilde{H}_{i-1}(\langle (E_2)^c_{(V \setminus E_1)}, ..., (E_m)^c_{(V \setminus E_1)} \rangle).\end{aligned}$$

ii. If $E_a \subset E_b$ for some $a \neq b$, then $\mathcal{E} = \langle (E_1)^c_V, \ldots, \widehat{(E_b)}^c_V, \ldots, (E_m)^c_V \rangle$.

The decomposition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ described in part (i) is called a **standard decomposition** of $\mathcal{E}$.

Proof. The proof of (ii) is trivial so we shall only prove (i). Since $\mathcal{E}_1$ is a simplex, we have $\widetilde{H}_i(\mathcal{E}_1) = 0$. Also since $V \setminus \bigcup_{i=2}^{s} E_i \neq \emptyset$ we have $\mathcal{E}_2$ is a cone, so from Proposition 2.2.10 we conclude

$$\widetilde{H}_i(\mathcal{E}_2) = 0 \text{ for all } i.$$

By applying the Mayer-Vietoris sequence we reach the following exact sequence

$$\cdots \longrightarrow \widetilde{H}_i(\mathcal{E}_1)\oplus\widetilde{H}_i(\mathcal{E}_2) \longrightarrow \widetilde{H}_i(\mathcal{E}) \longrightarrow \widetilde{H}_{i-1}(\mathcal{E}_1\cap\mathcal{E}_2) \longrightarrow \widetilde{H}_{i-1}(\mathcal{E}_1)\oplus\widetilde{H}_{i-1}(\mathcal{E}_2) \longrightarrow \cdots$$

which from $\mathcal{E}_1 \cap \mathcal{E}_2 = \langle (E_1 \cup E_2)_V^c, ..., (E_1 \cup E_m)_V^c \rangle$, implies that

$$\begin{aligned}\widetilde{H}_i(\mathcal{E}_1 \cup \mathcal{E}_2) \cong &\widetilde{H}_{i-1}(\mathcal{E}_1 \cap \mathcal{E}_2)\\ =&\widetilde{H}_{i-1}(\langle (E_1 \cup E_2)_V^c, ..., (E_1 \cup E_m)_V^c \rangle)\end{aligned}$$

and this settles our claim. □

Proposition 3.3.2. *Let $\Gamma = \langle E_1, \ldots, E_m \rangle$ be a pure simplicial complex of dimension $t-1$ on the vertex set $V = \{x_1, \ldots, x_n\}$ where $2 \le t \le n$. Suppose the connected components of Γ are runs of lengths $s_1, \ldots, s_r$, and $\mathcal{E} = E(s_1, \ldots, s_r)$. Let $s_j = (t+1)p_j + d_j$ where $p_j \ge 0$ and $0 \le d_j \le t$ and $1 \le j \le r$. Then for all i, we have*

$i.$ If $s_j \ge t+2$ then $\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-2}(E(s_1, \ldots, s_j - (t+1), \ldots, s_r))$;

$ii.$ If $d_j \ne 1, 2$ then $\widetilde{H}_i(\mathcal{E}) = 0$;

$iii.$ If $s_j = 2$ and $r \ge 2$ then $\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-2}(E(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_r))$;

$iv.$ If $s_j = 1$ and $r \ge 2$ then $\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-1}(E(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_r))$.

Proof. We assume without loss of generality that $E_1, \ldots, E_m$ are ordered such that $E_1, \ldots, E_{s_j}$ are the facets of the run of length s_j, and they have the standard labeling

$$E_1 = \{x_1, x_2, \ldots, x_t\}, E_2 = \{x_2, x_3, \ldots, x_{t+1}\}, \ldots, E_{s_j} = \{x_{s_j}, x_{s_j+1}, \ldots, x_{s_j+t-1}\}.$$

We have $\mathcal{E} = \langle (E_1)_V^c, (E_2)_V^c, ..., (E_m)_V^c \rangle$. Since $x_1 \in V \setminus \bigcup_{i=2}^m E_i$ there is a standard decomposition

$$\mathcal{E} = \langle (E_1)_V^c \rangle \cup \langle (E_2)_V^c, \ldots, (E_m)_V^c \rangle.$$

From Lemma 3.3.1 (i), setting $V' = V \setminus \{x_1, x_2, \ldots, x_t\}$, we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-1}(\langle (E_2)_{V'}^c, \ldots, (E_m)_{V'}^c \rangle). \tag{3.3.1}$$

If $s_j \geq t+2$ from (3.3.1) we have

$$\widetilde{H}_i(\mathcal{E})) = \widetilde{H}_{i-1}(\langle \{x_{t+1}\}^c_{V'}, (E_{t+2})^c_{V'}, \ldots, (E_m)^c_{V'}\rangle) \tag{3.3.2}$$

and since the following is a standard decomposition

$$\langle \{x_{t+1}\}^c_{V'}\rangle \cup \langle (E_{t+2})^c_{V'}, \ldots, (E_{s_j})^c_{V'}, \ldots, (E_m)^c_{V'}\rangle$$

from (3.3.2), Lemma 3.3.1 (i) and by setting $V'' = V \setminus \{x_1, x_2, \ldots, x_{t+1}\}$, we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-2}(\langle (E_{t+2})^c_{V''}, \ldots, (E_{s_j})^c_{V''}, \ldots, (E_m)^c_{V''}\rangle).$$

Now note that the connected components of $\langle E_{t+2}, \ldots, E_m\rangle$ are runs of the following lengths $s_1, \ldots, s_j - (t+1), \ldots, s_r$, and therefore we can conclude that for all i

$$\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-2}(E(s_1, \ldots, s_j - (t+1), \ldots, s_r)).$$

This settles Part (i) of the proposition. Now suppose $1 \leq s_j < t+2$. In this case by (3.3.1) and Lemma 3.3.1 (i) and (ii) we see that

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-1}(\langle \{x_{t+1}\}^c_{V'}, (E_{s_j+1})^c_{V'}, \ldots, (E_m)^c_{V'}\rangle) \quad \text{for all } i. \tag{3.3.3}$$

1. If $s_j \geq 3$ since $x_{s_j+t-1} \in V' \setminus (\bigcup_{i=s_j+1}^m E_i \cup \{x_{t+1}\})$ the simplicial complex

   $$\langle \{x_{t+1}\}^c_{V'}, (E_{s_j+1})^c_{V'}, \ldots, (E_m)^c_{V'}\rangle$$

   is a cone and by Proposition 2.2.10 and (3.3.3) we have $\widetilde{H}_i(\mathcal{E}) = 0$ for all i.

2. If $s_j = 2$ and $r \geq 2$, since $x_{t+1} \in V' \setminus (\bigcup_{i=s_j+1}^m E_i)$ we have

   $$\langle \{x_{t+1}\}^c_{V'}\rangle \cup \langle (E_{s_j+1})^c_{V'}, \ldots, (E_m)^c_{V'}\rangle$$

is a standard decomposition and then by Lemma 3.3.1 and (3.3.3) we have

$$\begin{aligned}\widetilde{H}_i(\mathcal{E}) &\cong \widetilde{H}_{i-2}(\langle (E_{s_j+1})^c_{V''}, \dots, (E_m)^c_{V''} \rangle) \\ &= \widetilde{H}_{i-2}(E(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_r)) \quad \text{for all } i.\end{aligned}$$

This settles Part (iii).

3. If $s_j = 1$ and $r \geq 2$ since $E_1 \cap E_h = \emptyset$ for $1 < h \leq m$, and from (3.3.1) we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-1}(E(s_1, \dots s_{j-1}, s_{j+1}, \dots, s_r)) \text{ for all } i.$$

This settles Part (iv).

To prove (ii), we use induction on p_j. If $p_j = 0$, then $d_j = s_j \geq 1$. From above we know that $\widetilde{H}_i(\mathcal{E}) = 0$ if $3 \leq s_j \leq t$, and we are done. Now suppose $p_j \geq 1$ and the statement holds for all values less than p_j. We have two cases:

1. If $s_j < t+2$, then since $p_j \geq 1$, we must have $p_j = 1$, $d_j = 0$, and $s_j = t+1$. It was proved above (under the case $3 \leq s_j < t+2$) that $\widetilde{H}_i(\mathcal{E}) = 0$.
2. If $s_j \geq t+2$, by (i) we have

$$\widetilde{H}_i(\mathcal{E}) \cong \widetilde{H}_{i-2}(E(s_1, \dots, (t+1)(p_j-1)+d_j, \dots, s_r)) = 0 \text{ when } d_j \neq 1, 2.$$

This proves (ii) and we are done. □

We conclude that for computing the homology module of the induced subcollections of path complexes of cycles or paths the only cases which have to be considered are those of runs of length 1 or 2. We now set about computing these.

Proposition 3.3.3. *Let $t \geq 2$ be an integer, $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$ and consider*

$$\mathcal{E} = E((t+1)p_1+1, \dots, (t+1)p_\alpha+1, (t+1)q_1+2, \dots, (t+1)q_\beta+2)$$

for nonnegative integers $p_1, \dots, p_\alpha$ and $q_1, \dots, q_\beta$. Then

$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} \mathbb{K} & i = 2(P+Q) + 2\beta + \alpha - 2 \\ 0 & \textit{otherwise} \end{cases}$$

where $P = \sum_{i=1}^{\alpha} p_i$ *and* $Q = \sum_{i=1}^{\beta} q_i$.

From here on, we use the notation $E(1^\alpha, 2^\beta)$ to denote the complex $\mathcal{E}$ described in the statement of Proposition 3.3.3 in the case where all the p's and q's are zero; i.e. the case of α runs of length one and β runs of length two.

Proof. First we prove the two cases $\alpha = 0, \beta = 1$ and $\alpha = 1, \beta = 0$.

1. If $\alpha = 1$, $\beta = 0$, then $\mathcal{E} \cong \langle V \setminus \{x_1, x_2, \ldots, x_t\}\rangle = \{\emptyset\}$ where $V = \{x_1, \ldots, x_t\}$, and therefore
$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} \mathbb{K} & i = -1 \\ 0 & \text{otherwise.} \end{cases}$$

2. If $\alpha = 0, \beta = 1$, then $\mathcal{E} \cong \langle (\{x_1, x_2, \ldots, x_t\})_V^c, (\{x_2, \ldots, x_{t+1}\})_V^c\rangle = \langle \{x_{t+1}\}, \{x_1\}\rangle$ where $V = \{x_1, \ldots, x_{t+1}\}$. Since $\mathcal{E}$ is disconnected and the number of connected components is 2, we have
$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} \mathbb{K} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To prove the statement of the proposition, we use repeated applications of Proposition 3.3.2 (i), p_1 times to the first run, p_2 times to the second run, and so on till q_β times to the last run as follows.

$$\begin{aligned}
\widetilde{H}_i(\mathcal{E}) &= \widetilde{H}_i(E((t+1)p_1+1, \ldots, (t+1)p_\alpha+1, (t+1)q_1+2, \ldots, (t+1)q_\beta+2)) \\
&\cong \widetilde{H}_{i-2}(E((t+1)(p_1-1)+1, \ldots, (t+1)p_\alpha+1, (t+1)q_1+2, \ldots, (t+1)q_\beta+2)) \\
&\;\vdots \\
&\cong \widetilde{H}_{i-2(P+Q)}(E(1^\alpha, 2^\beta)) && \text{apply Proposition 3.3.2 (iv)} \\
&\cong \widetilde{H}_{i-2(P+Q)-\alpha}(E(2^\beta)) && \text{apply Proposition 3.3.2 (iii)} \\
&\cong \widetilde{H}_{i-2(P+Q)-\alpha-2\beta+2}(E(2)) && \text{apply Case 2 above} \\
&= \begin{cases} \mathbb{K} & i = 2(P+Q)+\alpha+2\beta-2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

so we are done. $\square$

An immediate consequence of the above calculations is the homology module of the complement of a run, or equivalently, a path complex of any path graph.

Corollary 3.3.4. *Let t, p and d be integers such that $t \geq 2$, $p \geq 0$, and $0 \leq d \leq t$. Then*

$$\widetilde{H}_i(E((t+1)p+d)) = \begin{cases} \mathbb{K} & d=1,\ i=2p-1 \\ \mathbb{K} & d=2,\ i=2p \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Proposition 3.3.2 (ii), if $d \neq 1,2$ the homology module is zero. In the cases where $d = 1,2$, the result follows directly from Proposition 3.3.3. $\square$

We end this section with the calculation of the homology module of the complement of the whole path complex of a cycle; this will give us the top degree Betti numbers of the path ideal of a cycle. We will first need a technical lemma.

Lemma 3.3.5. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, and suppose $\Delta_t(C_n) = \langle F_1, F_2, \ldots, F_n \rangle$ is the path complex of a cycle C_n with standard labeling. Let $a, k, s, t \in \{1, \ldots, n\}$ be such that $k \leq t$, and $a+s+t-1 < n$. Suppose $s = (t+1)p+d$ where $p \geq 0$ and $0 \leq d < t+1$. Set $V = \{x_a, x_{a+1}, \ldots, x_{a+s+t-1}\}$ and*

$$\mathcal{E} = \langle (F_a)^c_V, \ldots, (F_{a+s-1})^c_V, \{x_{a+s+t-k}, x_{a+s+t-k+1}, \ldots, x_{a+s+t-1}\}^c_V \rangle.$$

Then for all i we have

$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} \mathbb{K} & d=1, i=2p \\ \mathbb{K} & k<t, d=k+1, i=2p+1 \\ \mathbb{K} & k=t, d=0, i=2p-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Suppose $a, k, s, t \in \{1, \ldots, n\}$. Without loss of generality we can assume $a = 1$ so that $V = \{x_1, \ldots, x_{s+t}\}$ and

$$\mathcal{E} = \langle (F_1)^c_V, \ldots, (F_s)^c_V, \{x_{s+t-k+1}, \ldots, x_{s+t}\}^c_V \rangle.$$

Since $x_{s+t} \notin F_h$ for $1 \le h \le s$, $\mathcal{E}$ has standard decomposition

$$\mathcal{E} = \langle (F_1)^c_V, (F_2)^c_V, \ldots, (F_s)^c_V \rangle \cup \langle \{x_{s+t-k+1}, x_{s+t-k+2}, \ldots, x_{s+t}\}^c_V \rangle$$

and then from Lemma 3.3.1 (i) and (ii), setting $V_1 = V \setminus \{x_{s+t-k+1}, x_{s+t-k+2}, \ldots, x_{s+t}\}$, we have

$$\begin{aligned}\widetilde{H}_i(\mathcal{E}) \cong &\widetilde{H}_{i-1}(\langle (F_1)^c_{V_1}, (F_2)^c_{V_1}, \ldots, (F_{s-k})^c_{V_1}, \{x_{s-k+1}, \ldots, x_{s+t-k}\}^c_{V_1}, \{x_{s-k+2}, \ldots, x_{s+t-k}\}^c_{V_1}, \\ &, \ldots, \{x_{s-1}, \ldots, x_{s+t-k}\}^c_{V_1}, \{x_s, \ldots, x_{s+t-k}\}^c_{V_1} \rangle) \\ = &\widetilde{H}_{i-1}(\langle (F_1)^c_{V_1}, (F_2)^c_{V_1}, \ldots, (F_{s-k})^c_{V_1}, \{x_s, \ldots, x_{s+t-k}\}^c_{V_1} \rangle) \qquad (3.3.4)\end{aligned}$$

We prove our statement by induction on $|V| = s + t = (t+1)p + d + t$. The base case is $|V| = d + t$, in which case $p = 0$ and $d = s \ge 1$. There are two cases to consider.

1. If $1 \le d \le k$, then $s \le k$, and so by (3.3.4)

$$\widetilde{H}_i(\mathcal{E}) = \widetilde{H}_{i-1}(\langle \{x_s, \ldots, x_{s+t-k}\}^c_{V_1} \rangle).$$

   The simplex $\{x_s, \ldots, x_{s+t-k}\}^c_{V_1}$ is not empty unless $s = d = 1$, and hence we have

$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} \mathbb{K} & d = 1, i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. If $d > k$, we use (3.3.4) to note that since $x_{s+t-k} \notin F_1 \cup \ldots \cup F_{s-k}$, the following is a standard decomposition

$$\langle (F_1)^c_{V_1}, (F_2)^c_{V_1}, \ldots, (F_{s-k})^c_{V_1}, \{x_s, \ldots, x_{s+t-k}\}^c_{V_1} \rangle.$$

   Using Lemma 3.3.1 and (3.3.4) along with the fact that $s = d \le t$, we find that if $V_2 = V \setminus \{x_s, \ldots, x_{s+t}\}$, then

$$\begin{aligned}\widetilde{H}_i(\mathcal{E}) &\cong \widetilde{H}_{i-2}(\langle \{x_1, \ldots, x_{s-1}\}^c_{V_2}, \{x_2, \ldots, x_{s-1}\}^c_{V_2}, \ldots, \{x_{s-k}, \ldots, x_{s-1}\}^c_{V_2} \rangle) \\ &\cong \widetilde{H}_{i-2}(\langle \{x_{s-k}, \ldots, x_{s-1}\}^c_{V_2} \rangle).\end{aligned}$$

   Now the simplex $\{x_{s-k}, \ldots, x_{s-1}\}^c_{V_2}$ is nonempty unless $s-k = 1$, or in other words,

$d = s = k + 1$. Therefore

$$\widetilde{H}_i(\mathcal{E}) = \begin{cases} \mathbb{K} & d = k+1, i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This settles the base case of the induction. Now suppose $|V| = s + t > d + t$ and the theorem holds for all the cases where $|V| < s + t$. Since $|V_1| = (s - k) + t < |V|$ we shall apply (3.3.4) and use the induction hypothesis on V_1, now with the following parameters: $k_1 = t - k + 1$, $s_1 = s - k = (t+1)p + d - k$ and

$$d_1 = \begin{cases} d - k & d \geq k \\ d - k + t + 1 & d < k \end{cases} \quad \text{and} \quad p_1 = \begin{cases} p & d \geq k \\ p - 1 & d < k. \end{cases}$$

Applying the induction hypothesis on V_1 we see that $\widetilde{H}_i(\mathcal{E}) = 0$ unless one of the following scenarios happens, in which case $\widetilde{H}_i(\mathcal{E}) = \mathbb{K}$.

1. $d_1 = 1$ and $i - 1 = 2p_1$.
   (a) When $d \geq k$ and $k \neq t$, this means that $d = k + 1$ and $i = 2p + 1$.
   (b) When $d < k$, this means $d - k + t + 1 = 1$ which implies that $0 \leq d = k - t \leq 0$, and hence $d = 0$, $t = k$ and $i = 2p_1 + 1 = 2p - 1$.
2. $d_1 = k_1 + 1$ and $i - 1 = 2p_1 + 1$.
   (a) When $d \geq k$, this means that $d - k = t - k + 1 + 1$ and so $d = t + 2$ which is not possible, as we have assumed $d < t + 1$.
   (b) When $d < k$, this means that $d - k + t + 1 = t - k + 1 + 1$ and so $d = 1$ and $i = 2p_1 + 2 = 2p$.

We conclude that $\widetilde{H}_i(\mathcal{E}) = \mathbb{K}$ only when $d = 1$ and $i = 2p$, or $d = k + 1$ and $i = 2p + 1$ and $k < t$, or $d = 0$ and $i = 2p - 1$ and $k = t$ and $\widetilde{H}_i(\mathcal{E}) = 0$ otherwise. $\square$

Theorem 3.3.6. *Let $2 \leq t \leq n$ and $\Delta = \Delta_t(C_n)$ be the path complex of a cycle C_n with vertex set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. Suppose $n = (t+1)p + d$ where $p \geq 0$, $0 \leq d \leq t$. Then* *for all* i

$$\widetilde{H}_i(\Delta^c_{\mathcal{X}}) = \begin{cases} \mathbb{K}^t & d = 0,\ i = 2p-2,\ p > 0 \\ \mathbb{K} & d \neq 0,\ i = 2p-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $p = 0$, then $n = t$ and our claim is true, so we assume that $p \geq 1$ and therefore $n \geq t+1$. We define the following simplicial complexes

$$\begin{aligned} E_0 &= \langle (F_1)^c_{\mathcal{X}}, (F_2)^c_{\mathcal{X}}, \ldots, (F_{n-t+1})^c_{\mathcal{X}} \rangle = E(n-t+1) \\ E_k &= E_{k-1} \cup \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle \text{ for } k = 1, 2, \ldots, t-1. \end{aligned} \tag{3.3.5}$$

Note that $\Delta^c_{\mathcal{X}} = E_{t-1}$. We start with E_0 and apply the Mayer-Vietoris sequence repeatedly to calculate the homology modules of the E_k. Since $E_0 = E(n-t+1)$, we find

$$n - t + 1 = (t+1)p + d - t + 1 = \begin{cases} (t+1)p + 1 & d = t \\ (t+1)p & d = t-1 \\ (t+1)(p-1) + d + 2 & d < t-1 \end{cases}$$

which by Corollary 3.3.4 implies that

$$\widetilde{H}_i(E_0) = \begin{cases} \mathbb{K} & d = 0,\ i = 2p-2 \\ \mathbb{K} & d = t,\ i = 2p-1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.6}$$

In order to to find the homology modules of E_{t-1} we shall recursively apply the Mayer-Vietoris sequence as follows. For a fixed $1 \leq k \leq t-1$ we have the following exact sequence,

$$\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle) \to \widetilde{H}_i(E_{k-1}) \to \widetilde{H}_i(E_k) \to \widetilde{H}_{i-1}(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle) \tag{3.3.7}$$

We claim that for $0 \leq k \leq t-2$,

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle) = \begin{cases} \mathbb{K} & d = 0,\ i = 2p-3 \\ \mathbb{K} & d = t-k-1,\ i = 2p-2 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.8}$$

Setting $\mathcal{X}' = \mathcal{X} \setminus F_{n-k} = \{x_{t-k}, \ldots, x_{n-k-1}\}$ we can write

$$E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle (F_1)^c_{\mathcal{X}'}, \ldots, (F_{n-t+1})^c_{\mathcal{X}'}, (F_{n-k+1})^c_{\mathcal{X}'}, \ldots, (F_n)^c_{\mathcal{X}'} \rangle. \tag{3.3.9}$$

We now compute the $(F_h)^c_{\mathcal{X}'} = \{x_h, \ldots, x_{h+(t-1)}\}^c_{\mathcal{X}'}$ appearing in (3.3.9).

- When $1 \leq h \leq t-k$, it is straightforward to show that

$$(F_h)^c_{\mathcal{X}'} = \{x_{t-k}, x_{t-k+1}, \ldots, x_{t+h-1}\}^c_{\mathcal{X}'}.$$

- When $t-k+1 \leq h \leq n-t-k-1$, then $2t-k \leq h+t-1 \leq n-k-2$, and so

$$(F_h)^c_{\mathcal{X}'} = \{x_{t-k}, \ldots, x_{h-1}, x_{h+t}, \ldots, x_{n-k-1}\}.$$

- When $n-k-t \leq h \leq n-t+1$, then $n-k-1 \leq h+t-1 \leq n$, and therefore

$$(F_h)^c_{\mathcal{X}'} = \{x_h, \ldots, x_{n-k-1}\}^c_{\mathcal{X}'}.$$

- When $h = n-j$ for $0 \leq j \leq k-1$, then $t-k \leq -j+(t-1) \leq t-1$ and so we have

$$F_{n-j} = \{x_{n-j}, \ldots, x_{n-j+(t-1)}\} = \{x_{n-j}, \ldots, x_n, x_1, \ldots, x_{t-j-1}\}$$

  which implies that

$$(F_{n-j})^c_{\mathcal{X}'} = \{x_{t-k}, x_{t-k+1}, \ldots, x_{t-j-1}\}^c_{\mathcal{X}'}.$$

From the observations above, (3.3.9) and Lemma 3.3.1 (ii) we see that

$$E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle \{x_{t-k}\}, F_{t-k+1}, \ldots, F_{n-t-k-1}, \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}'}. \tag{3.3.10}$$

We now consider the following scenarios.

1. Suppose $p = 1$. In this situation, $n = t+d+1 \leq 2t+1$ which implies that

$n-t-k-1 \leq t-k$. Therefore, (3.3.10) becomes

$$E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle \{x_{t-k}\}, \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}'}. \tag{3.3.11}$$

(a) If $d \leq t-k-2$, then $n-t+1 = t+d+1-t+1 = d+2 \leq t-k$. As well, since $n \geq t+1$, we have $n-k-1 \geq t-k$.

It follows that in this situation, $x_{t-k} \in \{x_{n-t+1}, \ldots, x_{n-k-1}\}$ which means that (3.3.11) becomes $E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle = \langle \{x_{t-k}\}^c_{\mathcal{X}'} \rangle$. Also note that $\mathcal{X}' = \{x_{t-k}\}$ only when $d = 0$. It follows that

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c \rangle) \cong \begin{cases} \mathbb{K} & d = 0,\ i = -1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $d > t-k-2$. In this situation, $x_{t-k} \notin \{x_{n-t+1}, \ldots, x_{n-k-1}\}$ which means that we can apply Lemma 3.3.1 (i), with $\mathcal{X}'' = \mathcal{X}' \setminus \{x_{t-k}\}$ to find that for all i

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c \rangle) = \widetilde{H}_{i-1}(\langle \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}''}).$$

Moreover, $\mathcal{X}'' = \{x_{n-t+1}, \ldots, x_{n-k-1}\}$ only when $d = t-k-1$, and so we have

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c \rangle) \cong \begin{cases} \mathbb{K} & d = t-k-1,\ i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Suppose $p \geq 2$. In this case it is straightforward to see that $n-t-k-1 > t-k$ and $n-t+1 > t-k$.

Therefore, we can apply Lemma 3.3.1 (i) with $\mathcal{X}'' = \mathcal{X}' \setminus \{x_{t-k}\}$ to (3.3.10) to conclude that for all i

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle) = \widetilde{H}_{i-1}(\langle F_{t-k+1}, \ldots, F_{n-t-k-1}, \{x_{n-t+1}, \ldots, x_{n-k-1}\} \rangle^c_{\mathcal{X}''}).$$

Now we use Lemma 3.3.5 with values $a = t-k+1$ and $s = n-2t-1 =$

$(p-2)(t+1)+d+1$ to conclude that

$$\widetilde{H}_i(E_k \cap \langle (F_{n-k})^c_{\mathcal{X}} \rangle) = \begin{cases} \mathbb{K} & d=0,\ i=2p-3 \\ \mathbb{K} & d=t-k-1,\ i=2p-2 \\ 0 & \text{otherwise} \end{cases}$$

This settles (3.3.8). We now return to finding $\widetilde{H}_i(E_{t-1})$ by recursively using the Mayer-Vietoris sequence to find $\widetilde{H}_i(E_k)$.

1. If $0<d<t$, then by (3.3.8) we know that $\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle)$ is nonzero only when $i=2p-2$ and $k=t-d$. We apply this observation and (3.3.6) to the exact sequence (3.3.7) to see that

   $$\widetilde{H}_i(E_k) = \widetilde{H}_i(E_{k-1}) = \widetilde{H}_i(E_0) = 0 \text{ for } 1 \le k \le t-d-1.$$

   Once again we use (3.3.7) to observe that

   $$\begin{aligned} \widetilde{H}_i(E_k) &= \begin{cases} 0 & 1 \le k \le t-d-1 \\ \widetilde{H}_{i-1}(E_{k-1} \cap \langle (F_{n-k+1})^c \rangle) & k=t-d \\ \widetilde{H}_i(E_{t-d}) & t-d<k \le t-1. \end{cases} \\ &= \begin{cases} \mathbb{K} & k \ge t-d,\ i=2p-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

   We can conclude that in this case

   $$\widetilde{H}_i(E_{t-1}) = \begin{cases} \mathbb{K} & i=2p-1 \\ 0 & \text{otherwise.} \end{cases}$$

2. If $d=t$, then by (3.3.8) we know that $\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle)$ is always zero. We apply this fact along with (3.3.6) to the sequence in (3.3.7) to observe that

   $$\widetilde{H}_i(E_k) \cong \widetilde{H}_i(E_0) = \begin{cases} \mathbb{K} & i=2p-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k \in \{1,2,\ldots,t-1\}.$$

3. If $d=0$, then by (3.3.8) we know that $\widetilde{H}_i(E_{k-1} \cap \langle (F_{n-k+1})^c_{\mathcal{X}} \rangle)$ is zero unless

$i = 2p-3$, and from (3.3.6) we know $\widetilde{H}_i(E_0)$ is zero unless $i = 2p-2$. Applying these facts to (3.3.7) we see that

$$\widetilde{H}_i(E_k) = \widetilde{H}_i(E_0) = 0 \quad \text{for } i \neq 2p-2.$$

When $i = 2p-2$, the sequence (3.3.7) produces an exact sequence

$$0 \longrightarrow \overbrace{\widetilde{H}_{2p-2}(E_0)}^{\mathbb{K}} \longrightarrow \widetilde{H}_{2p-2}(E_1) \longrightarrow \overbrace{\widetilde{H}_{2p-3}(E_0 \cap \langle (F_n)^c \rangle)}^{\mathbb{K}} \longrightarrow 0.$$

Therefore

$$\widetilde{H}_i(E_1) = \begin{cases} \mathbb{K}^2 & i = 2p-2 \\ 0 & \text{otherwise.} \end{cases}$$

We repeat the above method, recursively , for values $k = 2, 3, \ldots, t-1$

$$0 \longrightarrow \overbrace{\widetilde{H}_{2p-2}(E_{k-1})}^{\mathbb{K}^k} \longrightarrow \widetilde{H}_{2p-2}(E_k) \longrightarrow \overbrace{\widetilde{H}_{2p-3}(E_{k-1} \cap \langle (F_{n-k+1})^c \rangle)}^{\mathbb{K}} \longrightarrow 0$$

and conclude that for $1 \leq k \leq t-1$

$$\widetilde{H}_i(E_k) = \begin{cases} \mathbb{K}^{k+1} & i = 2p-2 \\ 0 & \text{otherwise.} \end{cases}$$

We put this all together

$$\widetilde{H}_i(E_{t-1}) = \begin{cases} \mathbb{K}^t & d = 0,\ i = 2p-2,\ p > 0 \\ \mathbb{K} & d \neq 0,\ i = 2p-1 \\ 0 & \text{otherwise} \end{cases}$$

and this proves the statement of the theorem. $\square$

Chapter 4

Graded Betti Numbers of Path Ideals of Cycles and Paths

In this chapter we use combinatorial techniques to calculate Betti numbers of path ideals of paths and cycles. We also compute projective dimension and regularity of path ideals of cycles and paths. By Theorem 3.1.2 we only need to count induced subcollections.

4.1 The Top Degree Betti Numbers for Cycles

We are now ready to apply the homological calculations from the previous section to compute the top degree Betti numbers of path ideals. If I is the degree t path ideal of a cycle, then

$$\beta_{i,j}(R/I) = 0 \text{ for all } i \geq 1 \text{ and } j > ti; \tag{4.1.1}$$

see Jacques [28], Theorem 3.3.4, for a proof.

By Theorem 3.1.2, to compute the Betti numbers of I of degree less than n, we should consider the complements of proper induced subcollections of $\Delta = \Delta_t(C_n)$. For degree n we should consider Δ^c.

Theorem 4.1.1 (top degree Betti numbers for cycles)**.** *Let p, t, n, d be integers such that $n = (t+1)p + d$, where $p \geq 0$, $0 \leq d \leq t$, and $2 \leq t \leq n$. If C_n is a cycle over n vertices, then*

$$\beta_{i,n}(R/I_t(C_n)) = \begin{cases} t & d = 0,\ i = 2\left(\dfrac{n}{t+1}\right) \\[2ex] 1 & d \neq 0,\ i = 2\left(\dfrac{n-d}{t+1}\right) + 1 \\[2ex] 0 & \textit{otherwise}. \end{cases}$$

Proof. Suppose $\Delta = \Delta_t(C_n)$. By Theorem 3.1.2 $\beta_{i,n}(R/I_t(C_n)) = \dim_{\mathbb{K}} \widetilde{H}_{i-2}(\Delta^c_{\mathcal{X}})$ and the result now follows directly from Theorem 3.3.6. $\square$

4.2 Eligible Subcollections

From Hochster's formula we see that computing Betti numbers of degree less than n comes down to counting induced subcollections of certain kinds.

Definition 4.2.1. Let i and j be positive integers. We call an induced subcollection Γ of $\Delta_t(C_n)$ an (i,j)**-eligible subcollection** of $\Delta_t(C_n)$ if Γ is composed of disjoint runs of lengths

$$(t+1)p_1+1,\ldots,(t+1)p_\alpha+1,(t+1)q_1+2,\ldots,(t+1)q_\beta+2 \tag{4.2.1}$$

for nonnegative integers $\alpha,\beta,p_1,p_2,\ldots,p_\alpha,q_1,q_2,\ldots,q_\beta$, which satisfy the following conditions

$$\begin{aligned} j &= (t+1)(P+Q)+t(\alpha+\beta)+\beta \\ i &= 2(P+Q)+2\beta+\alpha, \end{aligned}$$

where $P=\sum_{i=1}^{\alpha} p_i$ and $Q=\sum_{i=1}^{\beta} q_i$.

The next theorem is similar to a statement proved for the edge ideal of a cycle by Jacques [28].

Theorem 4.2.2. *Let $I=\mathcal{F}(\Lambda)$ be the facet ideal of an induced subcollection Λ of $\Delta_t(C_n)$. Suppose i and j are integers with $i\leq j<n$. Then the $\mathbb{N}$-graded Betti number $\beta_{i,j}(R/I)$ is the number of (i,j)-eligible subcollections of Λ.*

Proof. Since $\mathcal{F}(I)=\Lambda$ from Theorem 3.1.2 we have

$$\beta_{i,j}(R/I)=\sum_{\Gamma\subset\Lambda,|\mathrm{V}(\Gamma)|=j}\dim_{\mathbb{K}}\widetilde{H}_{i-2}(\Gamma^c_{\mathrm{V}(\Gamma)})$$

where $\mathrm{V}(\Gamma)$ is the vertex set of Γ and the sum is taken over induced subcollections Γ of Λ.

It is straightforward to show that each induced subcollection of Λ is an induced subcollection of $\Delta_t(C_n)$, and can therefore be written as a disjoint union of runs. So from Proposition 3.3.2 we can conclude the only Γ whose complements have nonzero homology module are those corresponding to run sequences of the form (4.2.1). Such subcollections have j vertices where by Definition 3.2.6

$$\begin{aligned} j &= ((t+1)p_1 + t) + \cdots + ((t+1)p_\alpha + t) + ((t+1)q_1 + t + 1) + \cdots + ((t+1)q_\beta + t + 1) \\ &= (t+1)(P+Q) + t(\alpha+\beta) + \beta. \end{aligned} \tag{4.2.2}$$

So

$$\Gamma^c_{\mathbf{V}(\Gamma)} = E((t+1)p_1 + 1, \ldots, (t+1)p_\alpha + 1, (t+1)q_1 + 2, \ldots, (t+1)q_\beta + 2)$$

and by Proposition 3.3.3 we have

$$\dim_{\mathbb{K}}(\widetilde{H}_{i-2}(\Gamma^c_{\mathbf{V}(\Gamma)})) = \begin{cases} 1 & i = 2(P+Q) + 2\beta + \alpha \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.3}$$

From (4.2.2) and (4.2.3) we see that each induced subcollection Γ corresponding to a run sequence as in (4.2.1) contributes 1 unit to $\beta_{i,j}$ if and only if

$$\begin{aligned} j &= (t+1)(P+Q) + t(\alpha+\beta) + \beta \\ i &= 2(P+Q) + 2\beta + \alpha \end{aligned}$$

it settles the claim. $\square$

Theorem 4.2.2 holds in particular for $\Lambda = \Delta_t(L_m)$ and $\Lambda = \Delta_t(C_n)$ for any integers m, n. The following corollary is a special case of Theorem 4.2.2.

Corollary 4.2.3. *Let $I = \mathcal{F}(\Lambda)$ be the facet ideal of an induced subcollection Λ of $\Delta_t(C_n)$. Then for every i, $\beta_{i,ti}(R/I)$, is the number of induced subcollections of Λ which are composed of i runs of length one.*

Proof. From Theorem 4.2.2 we have $\beta_{i,ti}(R/I)$ is the number of (i, ti)-eligible subcollections of Λ. With notation as in Definition 4.2.1 we have

$$\begin{cases} ti = (t+1)(P+Q) + t(\alpha+\beta) + \beta \\ i = 2(P+Q) + (\alpha+\beta) + \beta \end{cases} \quad \Rightarrow ti = 2t(P+Q) + t(\alpha+\beta) + t\beta$$

Putting the two equations for ti together, we conclude that $(t-1)(P+Q+\beta) = 0$. But β,

$P, Q \geq 0$ and $t \geq 2$, so we must have

$$\beta = P = Q = 0 \Rightarrow p_1 = p_2 = \cdots = p_\alpha = 0.$$

So $\alpha = i$ and Γ is composed of i runs of length one. $\square$

Theorem 4.2.2 holds in particular for $\Lambda = \Delta_t(L_m)$ and $\Lambda = \Delta_t(C_n)$ for any integers m, n. Our next statement is in a sense a converse to Theorem 4.2.2.

Proposition 4.2.4. *Let t and n be integers such that $2 \leq t \leq n$ and $I = \mathcal{F}(\Lambda)$ be the facet ideal of Λ where Λ is an induced subcollection of $\Delta_t(C_n)$. Then for each $i, j \in \mathbb{N}$ with $i \leq d < n$, if $\beta_{i,j}(R/I) \neq 0$, there exist nonnegative integers ℓ, d such that*

$$\begin{cases} i &= \ell + d \\ j &= t\ell + d \end{cases}$$

Proof. From Theorem 4.2.2 we know $\beta_{i,j}$ is equal to the number of (i, j)-eligible subcollections of Λ, where with notation as in Definition 4.2.1 we have

$$\begin{cases} j = (t+1)(P+Q) + t(\alpha+\beta) + \beta \\ i = 2(P+Q) + (\alpha+\beta) + \beta. \end{cases}$$

It follows that

$$j - i = (t-1)(P+Q+\alpha+\beta) \quad \text{and} \quad ti - j = (t-1)(P+Q+\beta). \tag{4.2.4}$$

We now show that there exist positive integers ℓ, d such that $i = \ell + d$ and $j = t\ell + d$.

$$\begin{cases} i = \ell + d \\ j = t\ell + d \end{cases} \Rightarrow \ \ell = \frac{j-i}{t-1} \quad \text{and} \quad d = \frac{ti-j}{t-1} \ .$$

From (4.2.4) we can see that i and j as described above are nonnegative integers. $\square$

Theorem 4.2.5. *Let i, j be integers and $i \leq j < n$ and $j \leq it$. Also suppose $n = (t+1)p+d$ and $d < t+1$. If $\beta_{i,j}(R/I_t(C_n)) \neq 0$ we have $j - i \leq (t-1)p$, and $i < 2p$ for $d = 0$ and $i \leq 2p+1$ for $d \neq 0$.*

Proof. By using Theorem 4.2.2 we know $\beta_{i,j}(R/I_t(C_n)$ is equal to the number of (i,j)-eligible subcollections of $\Delta_t(C_n)$. So if we assume $\beta_{i,j}(R/I_t(C_n)) \neq 0$ we can conclude there exists a (i,j)-eligible subcollection C of $\Delta_t(C_n)$ which is composed of runs of lengths as described in (4.2.1). Therefore

$$j - i = (t-1)(P + Q + \alpha + \beta) \quad \text{and} \quad ti - j = (t-1)(P + Q + \beta). \tag{4.2.5}$$

It follows that $j - i \geq ti - j$ so

$$i(t+1) \leq 2j \Rightarrow i \leq 2\left(\frac{j}{t+1}\right) < 2\left(\frac{p(t+1)+d}{t+1}\right)$$

so if $d = 0$ it follows that $i < 2p$ and if $d \neq 0$ it follows that $i \leq 2p + 1$.

On the other hand since $\Delta_t(C_n)$ has n facets and since there must be at least t facets between every two runs in C, we have

$$n \geq (t+1)P + (t+1)Q + \alpha + 2\beta + t\alpha + t\beta \geq (t+1)(P + Q + \alpha + \beta) = \left(\frac{t+1}{t-1}\right)(j - i)$$

which implies that

$$\frac{j-i}{t-1} \leq p + \frac{d}{t+1}$$

and since from (4.2.5) we have $(j-i)/(t-1)$ is an integer the formula follows. $\square$

We end this section with the computation of the projective dimension and regularity of path ideals of cycles. The case $t = 2$ is the case of graphs which appears in Jacques [28].

Corollary 4.2.6 (projective dimension and regularity of path ideals of cycles)**.** *Let n, t, p and d be integers such that $n \geq 2$, $2 \leq t \leq n$ and $n = (t+1)p + d$, where $p \geq 0$ and $0 \leq d \leq t$. Then*

i. The projective dimension of the path ideal of a graph cycle C_n is given by

$$pd(R/I_t(C_n)) = \begin{cases} 2p+1 & d \neq 0 \\ 2p & d = 0 \end{cases}$$

ii. The regularity of the path ideal of the graph cycle C_n is given by

$$reg(R/I_t(C_n)) = \begin{cases} (t-1)p+d-1 & \textit{for} \quad d \neq 0 \\ (t-1)p & \textit{for} \quad d = 0 \end{cases}.$$

Proof. i. This follows from Theorem 4.1.1 and Theorem 4.2.5.

ii. By definition, the regularity of a module M is $\max\{j - i \ : \ \beta_{i,j}(M) \neq 0\}$. By Theorem 4.2.5, and the observation above, if $d = 0$ then $reg(R/I_t(C_n))$ is

$$\max\{n-2p, (t-1)p\} = \max\{(t+1)p-2p, (t-1)p\} = (t-1)p$$

and if $d \neq 0$ then $reg(R/I_t(C_n))$ is

$$\max\{n-2p-1, (t-1)p\} = \max\{(t+1)p+d-2p-1, (t-1)p\} = (t-1)p+d-1.$$

The formula now follows. $\square$

4.3 Some Combinatorics

Theorem 4.2.2 tells us that to compute Betti numbers of induced subcollections of $\Delta_t(C_n)$ we need to count the number of its induced subcollections which consist of disjoint runs of lengths one and two. The next few pages are dedicated to counting such subcollections. We use some combinatorial methods to generalize a helpful formula which can be found in Stanley's book [39] on page 73.

Lemma 4.3.1. *Consider a collection of k points arranged on a line. The number of ways of coloring all points red or green so that for all m red points there are at least t green points on the line between each two consecutive red points is*

$$\binom{k-(m-1)t}{m}.$$

Proof. First label the points from $1, 2, \ldots, k$ from left to right, and let $a_1 < a_2 < \cdots < a_m$ be the red points. For $1 \leq i \leq m-1$, we define x_i to be the number of points, including

a_i, which are between a_i and a_{i+1}, and x_0 to be the number of points which exist before a_1, and x_m the number of points, including a_m, which are after a_m.

$$\begin{array}{cccccc} \overbrace{\cdots}^{x_0} & \overbrace{\bullet\cdots}^{x_1} & \overbrace{\bullet\cdots}^{x_2} & \bullet\cdots & \overbrace{\bullet\cdots}^{x_{m-1}} & \overbrace{\bullet\cdots}^{x_m} \\ 1 & a_1 & a_2 & a_3 & a_{m-1} & a_m \;\; n \end{array}$$

If we consider the sequence $x_0, x_1, \ldots, x_m$ it is not difficult to see that there is a one to one correspondence between the positive integer solutions of the following equation and the ways of coloring red m points of k points on the line with at least t green points between each two consecutive red points.

$$x_0 + x_1 + \cdots + x_m = k \quad x_0 \geq 0, x_i > t, \text{ for } 1 \leq i \leq m-1, \text{ and } x_m \geq 1.$$

So we only need to find the number of positive integer solutions of this equation. Consider the following equation

$$(x_0 + 1) + (x_1 - t) + \cdots + (x_{m-1} - t) + x_m = k - (m-1)t + 1$$

where $x_0 + 1 \geq 1$, $x_i - t \geq 1$, for $i = 0, \ldots, m-1$ and $x_m \geq 1$. The number of positive integer solution of this equation is (cf. Grimaldi [19] page 29) $\binom{k-(m-1)t}{m}$. □

Corollary 4.3.2. *Let C_n be a graph cycle and let R_k be a run of length k of $\Delta_t(C_n)$. The number of induced subcollections of R_k which are composed of m runs of length one is*

$$\binom{k-(m-1)t}{m}.$$

Proof. The run R_k has k facets, which following the standard labeling on the facets of $\Delta_t(C_n)$ we can arrange in a line from left to right. To compute the number of induced subcollections of R_k which are composed of m runs of length one, it is enough to compute the number of ways which we can color m points of these k arranged points red with at least t green points between each two consecutive red points. Therefore, by Lemma 4.3.1 we have the number of induced subcollections of R_k which are composed of m runs of length one is $\binom{k-(m-1)t}{m}$. □

Corollary 4.3.3. *Let C_n be a graph cycle and with the standard labeling let Γ be a proper* *subcollection of* $\Delta_t(C_n)$ *with* k *facets* $F_a, \ldots, F_{a+k-1}$. *The number of induced subcollections of* Γ *which are composed of* m *runs of length one is*

$$\binom{k-(m-1)t}{m}.$$

Proof. To compute the number of induced subcollections of Γ which are composed of m runs of length one, it is enough to consider the facets $F_a, \ldots, F_{a+k-1}$ as points arranged on a path and compute the number of ways which we can color m points of these k arranged points with at least t uncolored points between each two consecutive colored points. Therefore, by Lemma 4.3.1 we have the number of induced subcollections of Γ which are composed of m runs of length one is $\binom{k-(m-1)t}{m}$. $\square$

Proposition 4.3.4. *Let* C_n *be a graph cycle with vertex set* $\mathcal{X} = \{x_1, \ldots, x_n\}$. *The number of induced subcollections of* $\Delta_t(C_n)$ *which are composed of* m *runs of length one is*

$$\frac{n}{n-mt}\binom{n-mt}{m}.$$

Proof. Recall that $\Delta_t(C_n) = \langle F_1, \ldots, F_n \rangle$ with standard labeling. First we compute the number of induced subcollections of $\Delta_t(C_n)$ which consist of m runs of length one and do not contain the vertex x_i for $i \in \{1, 2, \ldots, n\}$.

There are t facets of $\Delta_t(C_n)$ which contain x_i, the remaining facets are $F_{i+1}, \ldots, F_{i+n-t}$, and so by Corollary 4.3.3 the number we are looking for is

$$\binom{n-t-(m-1)t}{m} = \binom{n-mt}{m}. \tag{4.3.1}$$

For each $i \in \{1, 2, \ldots, n\}$, there is an induced subcollection which is composed of m runs of length one not containing x_i. Since from (4.3.1) for each i the number of these induced subcollections is $\binom{n-mt}{m}$ so there are $n\binom{n-mt}{m}$ subcollections in all.

On the other hand for every subcollection there are $n-mt$ vertices which do not belong to it, so each subcollection is counted $n-mt$ times. Hence the number of subcollections which are composed of m runs of length one is $\frac{n}{n-mt}\binom{n-mt}{m}$. $\square$

We apply these counting facts to find Betti numbers in specific degrees; the formula in (iii) below (that of a path graph) was also computed by Bouchat, Hà and O'Keefe [11] using Eliahou-Kervaire techniques.

Corollary 4.3.5. *Let $n \geq 2$ and t be an integer such that $2 \leq t \leq n$. Then we have*

i. For the cycle C_n we have

$$\beta_{i,it}(R/I_t(C_n)) = \frac{n}{n-it}\binom{n-it}{i}.$$

ii. For any proper induced subcollection Λ of $\Delta_t(C_n)$ with k facets we have

$$\beta_{i,it}(R/\mathcal{F}(\Lambda)) = \binom{k-(i-1)t}{i}.$$

iii. For the path graph L_n, we have

$$\beta_{i,it}(R/I_t(L_n)) = \binom{n-it+1}{i}.$$

Proof. From Corollary 4.2.3 we have $\beta_{i,it}(R/I)$ in each of the three cases (i), (ii) and (iii) is the number of induced subcollections of $\Delta_t(C_n)$, Λ and $\Delta_t(L_n)$, respectively, which are composed of i runs of length 1. Case (i) now follows from Proposition 4.3.4, while (ii) and (iii) follow directly from Corollary 4.3.3. $\square$

4.4 Induced Subcollections Which Are Eligible

The following Lemma is the core of our counting later on in this section.

Lemma 4.4.1. *Let $\Delta_t(C_n) = \langle F_1, F_2, \ldots, F_n\rangle$, $2 \leq t \leq n$, be the standard labeling of the path complex of a cycle C_n on vertex set $\mathcal{X} = \{x_1, \ldots, x_n\}$. Let i be a positive integer and $\Gamma = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i}\rangle$ be an induced subcollection of $\Delta_t(C_n)$ consisting of i runs of length 1, with $1 \leq c_1 < c_2 < \cdots < c_i \leq n$. Suppose Σ is the induced subcollection on $V(\Gamma) \cup \{x_{c_u+t}\}$ for some $1 \leq u \leq i$. Then*

$$|\Sigma| = \begin{cases} |\Gamma| + t & u < i \ \textit{and} \quad c_{u+1} = c_u + t + 1 \\ |\Gamma| + 1 & u = i \ \textit{or} \quad\ \ c_{u+1} > c_u + t + 1 \end{cases}$$

Proof. Since Γ consists of runs of length one and each $F_{c_u} = \{x_{c_u}, x_{c_u+1}, \dots, x_{c_u+t-1}\}$ we must have $c_{u+1} > c_u + t$ for $u \in \{1, 2, \dots, i-1\}$. There are two ways that x_{c_u+t} could add facets to Γ to obtain Σ.

1. If $c_{u+1} = c_u + t + 1$ then $F_{c_u}, F_{c_u+1}, \dots, F_{c_u+t+1} = F_{c_{u+1}} \in \Sigma$ or in other words, we have added t new facets to Γ.

2. If $c_{u+1} > c_u + t + 1$ or $u = i$ then $F_{c_u+1} \in \Sigma$, and therefore one new facet is added to Γ.

So we are done. $\square$

The following propositions, which generalize Lemma 7.4.22 in Jacques [28], will help us compute the remaining Betti numbers.

Proposition 4.4.2. *Let $\Delta_t(C_n) = \langle F_1, F_2, \dots, F_n \rangle$, $2 \leq t \leq n$, be the standard labeling of the path complex of a cycle C_n on vertex set $\mathcal{X} = \{x_1, \dots, x_n\}$. Also let i, j be positive integers such that $j \leq i$ and $\Gamma = \langle F_{c_1}, F_{c_2}, \dots, F_{c_i} \rangle$ be an induced subcollection of $\Delta_t(C_n)$ consisting of i runs of length 1, with $1 \leq c_1 < c_2 < \cdots < c_i \leq n$. Suppose $W = V(\Gamma) \cup A \subsetneq \mathcal{X}$ for some subset A of $\{x_{c_1+t}, \dots, x_{c_i+t}\}$ with $|A| = j$. Then the induced subcollection Σ of $\Delta_t(C_n)$ on W is an $(i+j, ti+j)$-eligible subcollection.*

Proof. Since Γ consists of runs of length one and each $F_{c_u} = \{x_{c_u}, x_{c_u+1}, \dots, x_{c_u+t-1}\}$ we must have $c_{u+1} > c_u + t$ for $u \in \{1, 2, \dots, i-1\}$. The runs (or connected components) of Σ are of the form $\Sigma' = \Sigma_U$ where $U \subseteq W$, and can have one of the following possible forms.

a. For some $a \leq i$:

$$U = F_{c_a},$$

and therefore $\Sigma' = \langle F_{c_a} \rangle$ is a run of length 1.

b. For some $a \leq i$:

$$U = F_{c_a} \cup \{x_{c_a+t}\},$$

and therefore $c_{a+1} > c_a + t + 1$, so from Lemma 4.4.1 we have $\Sigma' = \langle F_{c_a}, F_{c_a+1} \rangle$ is a run of length 2.

c. For some $a \leq i$:

$$U = F_{c_a} \cup F_{c_{a+1}} \cup \cdots \cup F_{c_{a+r}} \cup \{x_{c_a+t}, x_{c_{a+1}+t}, \ldots, x_{c_{a+r-1}+t}\}$$

and $F_{c_{a+j}} = F_{c_a+j(t+1)}$ for $j = 0, 1, \ldots, r$ and $r \geq 1$. Then from Lemma 4.4.1 above we know Σ' is a run of length $r + 1 + tr = (t+1)r + 1$.

d. For some $a \leq i$:

$$U = F_{c_a} \cup F_{c_{a+1}} \cup \cdots \cup F_{c_{a+r}} \cup \{x_{c_a+t}, x_{c_{a+1}+t}, \ldots, x_{c_{a+r}+t}\}$$

and $F_{c_{a+j}} = F_{c_a+j(t+1)}$ for $j = 0, 1, \ldots, r$ and $r \geq 1$, and $c_{a+r+1} > c_{a+r} + t + 1$ or $a + r = i$. Then from Lemma 4.4.1 we have Σ' is a run of length $r + 1 + tr + 1 = (t+1)r + 2$.

So we have shown that Σ consists of runs of length 1 and $2 \mod t+1$.

Suppose the runs in Σ are of the form described in (4.2.1). By Definition 3.2.6 we have

$$\begin{aligned}
|\mathrm{V}(\Sigma)| &= (t+1)p_1 + t + \cdots + (t+1)p_\alpha + t + (t+1)q_1 + t + 1 + \cdots + (t+1)q_\beta + t + 1 \\
&= (t+1)P + t\alpha + (t+1)Q + t\beta + \beta \\
&= (t+1)(P+Q) + t(\alpha+\beta) + \beta.
\end{aligned}$$

On the other hand by the definition of Σ we know that, Σ has $ti + j$ vertices and therefore

$$ti + j = (t+1)(P+Q) + t(\alpha+\beta) + \beta.$$

It remains to show that $i + j = 2(P+Q) + (\alpha+\beta) + \beta$. Note that if $j = 0$ then $\beta = P = Q = 0$ and hence

$$j = 0 \implies P + Q + \beta = 0. \tag{4.4.1}$$

Moreover each vertex $x_{c_v+t} \in A$ either increases the length of a run in Γ by one and hence increases β (the number of runs of length 2 in Γ) by one, or increases the length of a run by $t+1$, in which case $P+Q$ increases by 1. We can conclude that if we add j vertices to Γ, $P+Q+\beta$ increases by j. From this and (4.4.1) we have $j = P + Q + \beta$. Now we solve the following system

$$\begin{cases} ti + j &= (t+1)(P+Q) + t(\alpha+\beta) + \beta & \implies & ti = t(P+Q) + t(\alpha+\beta) \\ j &= P+Q+\beta & \implies & i = P+Q+\alpha+\beta \end{cases}$$

$$\implies \begin{cases} i &= P+Q+\alpha+\beta \\ j &= P+Q+\beta \end{cases} \implies i+j = 2(P+Q) + (\alpha+\beta) + \beta.$$

It follows the claim. □

Proposition 4.4.3. *Let C_n be a cycle, $2 \leq t \leq n$, and i and j be positive integers. Suppose Σ is an $(i+j, ti+j)$-eligible subcollection of $\Delta_t(C_n)$, $2 \leq t \leq n$. Then with notation as in Definition 4.2.1, there exists a unique induced subcollection Γ of $\Delta_t(C_n)$ of the form $\langle F_{c_1}, F_{c_2}, \ldots, F_{c_i}\rangle$ with $1 \leq c_1 < c_2 < \cdots < c_i \leq n$ consisting of i runs of length 1, and a subset A of $\{x_{c_1+t}, \ldots, x_{c_i+t}\}$, with $|A| = j$ such that $\Sigma = \Delta_t(C_n)_W$ where $W = V(\Gamma) \cup A$. Moreover if $\mathcal{R} = \langle F_h, F_{h+1}, \ldots, F_{h+m}\rangle$ is a run in Σ with $|\mathcal{R}| = 2 \mod (t+1)$, then $F_{h+m} \notin \Gamma$.*

Proof. Suppose Σ consists of runs $R'_1, R'_2, \ldots, R'_{\alpha+\beta}$ where for $k = 1, 2, \ldots, \alpha+\beta$

$$R'_k = \langle F_{h_k}, F_{h_k+1}, \ldots, F_{h_k+m_k-1}\rangle$$

$$\mathrm{V}(R'_k) = \{x_{h_k}, x_{h_k+1}, \ldots, x_{h_k+m_k+t-2}\}$$

$$h_{k+1} \geq t + h_k + m_k$$

and

$$m_k = \begin{cases} (t+1)p_k + 1 & \text{for} \quad k = 1, 2, \ldots, \alpha \\ (t+1)q_{k-\alpha} + 2 & \text{for} \quad k = \alpha+1, \alpha+2, \ldots, \alpha+\beta. \end{cases} \tag{4.4.2}$$

For each k, we remove the following vertices from $\mathrm{V}(R'_k)$

$$\begin{array}{ll} x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+p_kt+(p_k-1)} & \text{if } 1 \leq k \leq \alpha \text{ and } p_k \neq 0 \\ x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+(q_{k-\alpha}+1)t+q_{k-\alpha}} & \text{if } \alpha+1 \leq k \leq \alpha+\beta \end{array} \tag{4.4.3}$$

Let $\Gamma = \langle R_1, R_2, \ldots R_{\alpha+\beta}\rangle$ be the induced subcollection on the remaining vertices of Σ, where

$$R_k = \begin{cases} \langle F_{h_k}, F_{h_k+t+1}, \ldots, F_{h_k+(t+1)p_k} \rangle & \text{for } 1 \leq k \leq \alpha \\ \langle F_{h_k}, F_{h_k+t+1}, \ldots, F_{h_k+(t+1)q_{k-\alpha}} \rangle & \text{for } \alpha+1 \leq k \leq \alpha+\beta. \end{cases} \tag{4.4.4}$$

In other words Γ has facets

$$F_{h_1}, F_{h_1+t+1}, \ldots, F_{h_1+(t+1)p_1}, F_{h_2}, F_{h_2+t+1}, \ldots, F_{h_2+(t+1)p_2}, \ldots, F_{h_{\alpha+\beta}}, \ldots, F_{h_{\alpha+\beta}+(t+1)q_\beta}.$$

It is straightforward to show that each R_k consists of runs of length one. Since Γ is a subcollection of Σ, no runs of R_k and $R_{k'}$ are connected to one another if $k \neq k'$, and hence we can conclude Γ is an induced subcollection of $\Delta_t(C_n)$ which is composed of runs of length one. From (4.4.4) we have the number of runs of length 1 in Γ (or the number of facets of Γ) is equal to

$$(p_1+1)+(p_2+1)+\cdots+(p_\alpha+1)+(q_1+1)+\cdots+(q_\beta+1) = P+Q+\alpha+\beta = i.$$

Therefore, Γ is an induced subcollection of $\Delta_t(C_n)$ which is composed of i runs of length 1. We relabel the facets of Γ as $\Gamma = \langle F_{c_1}, \ldots, F_{c_i} \rangle$. Now consider the following subset of $\{x_{c_1+t}, \ldots, x_{c_i+t}\}$ as A

$$\bigcup_{k=1, p_k \neq 0}^{\alpha} \{x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+p_k t+(p_k-1)}\} \cup \bigcup_{k=\alpha+1}^{\alpha+\beta} \{x_{h_k+t}, x_{h_k+2t+1}, \ldots, x_{h_k+(q_{k-\alpha}+1)t+q_{k-\alpha}}\}$$

by (4.4.3) we have:

$$|A| = (p_1+p_2+\cdots+p_\alpha)+(q_1+1\cdots+q_\beta+1) = P+Q+\beta = j.$$

Then if we set

$$W = (\bigcup_{h=1}^{i} F_{c_h}) \cup A$$

we have $\Sigma = (\Delta_t(C_n))_W$. This proves the existence of Γ, we now prove its uniqueness. Let $\Lambda = \langle F_{s_1}, F_{s_2}, \ldots, F_{s_i} \rangle$ be an induced subcollection of $\Delta_t(C_n)$ which is composed of i runs of length 1 such that $1 \leq s_1 < s_2 < \cdots < s_i \leq n$. Also let B be a j- subset of the set

$\{x_{s_1+t}, x_{s_2+t}, \ldots, x_{s_i+t}\}$ such that

$$\Sigma = (\Delta_t(C_n))_{\mathrm{V}(\Lambda)\cup B}. \tag{4.4.5}$$

Suppose $\Lambda = \langle S_1, S_2, \ldots, S_{\alpha+\beta}\rangle$, such that for $k = 1, 2, \ldots, \alpha+\beta$, S_k is an induced subcollection of R'_k which consists of y_k runs of length one. By (4.4.5) we have $y_k \neq 0$ for all k. Now we prove the following claims for each $k \in \{1, 2, \ldots, \alpha+\beta\}$.

a. $F_{h_k} \in \Lambda$. Suppose $1 \leq k \leq \alpha+\beta$. If $p_k = 0$ we are done, so consider the case $p_k \neq 0$.

   Assume $F_{h_k} \notin \Lambda$. Since F_{h_k} is the only facet of Σ which contains x_{h_k} we can conclude $x_{h_k} \notin \mathrm{V}(\Lambda)$. From (4.4.5), it follows that $x_{h_k} \in \{x_{s_1+t}, x_{s_2+t}, \ldots, x_{s_i+t}\}$, so

$$x_{h_k} = x_{s_a+t} \text{ for some } a. \tag{4.4.6}$$

   On the other hand we know

$$\begin{aligned} F_{s_a} &= \{x_{s_a}, x_{s_a+1}, \ldots, x_{s_a+t-1}\} \\ F_{s_a+1} &= \{x_{s_a+1}, x_{s_a+2}, \ldots, x_{s_a+t}\}. \end{aligned}$$

   Since R'_k is an induced connected component of Σ, by (4.4.6) we can conclude $x_{h_k} \in F_{s_a+1}$ and $F_{s_a}, F_{s_a+1} \in R'_k$. However, we know F_{h_k} is the only facet of R'_k which contains x_{h_k} and so $F_{s_a+1} = F_{h_k}$ and then $s_a + 1 = h_k$. This and (4.4.6) imply that $t = 1 \mod n$, which contradicts our assumption $2 \leq t \leq n$.

$$\begin{cases} y_k = p_k + 1 & \text{for} \quad k = 1, 2, \ldots, \alpha \\ y_k = q_{k-\alpha} + 1 & \text{for} \quad k = \alpha+1, \alpha+2, \ldots, \alpha+\beta. \end{cases}$$

   We know that

$$\begin{aligned} y_1 + y_2 + \cdots + y_{\alpha+\beta} &= i \\ &= P + Q + \alpha + \beta \\ &= (p_1+1) + \cdots + (p_\alpha+1) + (q_1+1) + \cdots + (q_\beta+1). \end{aligned} \tag{4.4.7}$$

Since we know for all $k = 1, 2, \ldots, \alpha + \beta$ there exist m_k facets in each R'_k, by Corollary 4.3.3 we can conclude the number of ways of selecting y_k runs of length one in R'_k is equal to $\binom{m_k-(y_k-1)t}{y_k}$. Then for all such k we have

$$\binom{m_k - (y_k - 1)t}{y_k} \neq 0 \Leftrightarrow m_k - (y_k - 1)t \geq y_k \Leftrightarrow m_k + t \geq y_k(t + 1) \quad (4.4.8)$$

and hence by (4.4.2) and (4.4.8) we have

$$\text{There exists } y_k \text{ runs of length one in } R'_k \iff \begin{cases} y_k \leq p_k + 1 & \text{for} \quad k = 1, 2, \ldots, \alpha \\ y_k \leq q_{k-\alpha} + 1 & \text{for} \quad k = \alpha + 1, \alpha + 2, \ldots, \alpha + \beta. \end{cases}$$

Putting this together with (4.4.7), we can conclude that

$$\begin{cases} y_k = p_k + 1 & \text{for} \quad k = 1, 2, \ldots, \alpha \\ y_k = q_{k-\alpha} + 1 & \text{for} \quad k = \alpha + 1, \alpha + 2, \ldots, \alpha + \beta. \end{cases} \quad (4.4.9)$$

b. *If $F_u \in S_k$ for some u and $F_{u+t+1} \in R'_k$, then $F_{u+t+1} \in S_k$.* Assume $F_{u+t+1} \notin S_k$ and $F_{u+t+1} \in R'_k$. Let

$$r_0 = \min\{r : r > u, F_r \in S_k \mod n\}.$$

Since S_k consists of runs of length one we can conclude $r_0 \geq u + t + 1$. Since $r_0 \neq u + t + 1$ we have $r_0 \geq u + t + 2$. But then

$$x_{u+t+1} \notin \mathrm{V}(\Lambda) \cup \{x_{s_1+t}, x_{s_2+t}, \ldots, x_{s_i+t}\}$$

and therefore $x_{u+t+1} \notin \mathrm{V}(\Sigma)$ which is a contradiction.

Now for each k, by (a) we have $F_{h_k} \in \Lambda$ and from repeated applications of (b) we find that

$$F_{h_k+f(t+1)} \in S_k \quad \text{for } f = \begin{cases} 1, 2, \ldots, p_k & 1 \leq k \leq \alpha \\ 1, 2, \ldots, q_{k-\alpha} & \alpha + 1 \leq k \leq \alpha + \beta. \end{cases}$$

So $R_k \subseteq S_k$. On the other hand S_k consists of runs of length one, so no other facet of R'_k can be added to it, and therefore $S_k = R_k$ for all k. We conclude that $\Lambda = \Gamma$ and we are therefore done. The last claim of the proposition is also apparent from this proof. □

4.5 Betti Numbers of Degree Less Than n

We are now ready to compute the remaining Betti numbers.

Theorem 4.5.1. *Let n, i, j and t be integers such that $n \geq 2$, $2 \leq t \leq n$, and $ti + j < n$. Then*

i. For the cycle C_n

$$\beta_{i+j,ti+j}(R/I_t(C_n)) = \frac{n}{n-it}\binom{i}{j}\binom{n-it}{i}$$

ii. For the path graph L_n

$$\beta_{i+j,ti+j}(R/I_t(L_n)) = \binom{i}{j}\binom{n-it}{i} + \binom{i-1}{j}\binom{n-it}{i-1}$$

Proof. If $I = I_t(C_n)$ (or $I = I_t(L_n)$), from Theorem 4.2.2, $\beta_{i+j,ti+j}(R/I)$ is the number of $(i+j, ti+j)$-eligible subcollections of $\Delta_t(C_n)$ (or $\Delta_t(L_n)$). We consider two separate cases for C_n and for L_n.

i. For the cycle C_n, suppose $\mathcal{R}_{(i)}$ denotes the set of all induced subcollections of $\Delta_t(C_n)$ which are composed of i runs of length one. By propositions 4.4.2 and 4.4.3 there exists a one to one correspondence between the set of all $(i+j, ti+j)$-eligible subcollections of $\Delta_t(C_n)$ and the set

$$\mathcal{R}_{(i)} \times \binom{[i]}{j}$$

where $\binom{[i]}{j}$ is the set of all j-subsets of a set with i elements. By Corollary 4.2.3 we have $|\mathcal{R}_{(i)}| = \beta_{i,ti}$ and since $|\binom{[i]}{j}| = \binom{i}{j}$ and so we apply Corollary 4.3.5 to observe that

$$\beta_{i+j,ti+j}(R/I_t(C_n)) = \binom{i}{j}\beta_{i,ti}(R/I_t(C_n)) = \frac{n}{n-it}\binom{i}{j}\binom{n-it}{i}.$$

ii. For the path graph L_n, recall that

$$\Delta_t(L_n) = \langle F_1, \ldots, F_{n-t+1} \rangle.$$

Let $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i} \rangle$ be the induced subcollection of $\Delta_t(L_n)$ which is composed of i runs of length 1 and $\mathrm{V}(\Lambda) \subset \mathcal{X} \setminus \{x_n\}$, so that it is also an induced subcollection of $\Delta_t(L_{n-1})$. Also let A be a j - subset of $\{x_{c_1+t}, x_{c_2+t}, \ldots, x_{c_i+t}\}$. So by Propositions 4.4.2 and Proposition 4.4.3 the induced subcollections on $\mathrm{V}(\Lambda) \cup A$ are $(i+j, ti+j)$-eligible and if one denotes these induced subcollections by $\mathcal{B}$ we have the following bijection

$$\mathcal{B} \rightleftharpoons \binom{[i]}{j} \times \{\Gamma \subset \Delta_t(L_{n-1}) : \Gamma \quad \text{is composed of } i \text{ runs of length 1}\}. \quad (4.5.1)$$

We make the following claim:

Claim: *Let Γ be an $(i+j, ti+j)$-eligible subcollection of $\Delta_t(L_n)$ which contains a run $\mathcal{R}$ with $F_{n-t+1} \in \mathcal{R}$. Then $\Gamma \in \mathcal{B}$ if and only if $|\mathcal{R}| = 2 \mod t+1$.*

Proof of Claim. Let $\Gamma \in \mathcal{B}$ and assume that $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i} \rangle$ is the subcollection of $\Delta_t(L_{n-1})$ used to build Γ as described above. Then we must have $c_i = n - t$. Now, the run $\mathcal{R}$ contains F_{n-t+1} and F_{n-t}.

If $|\mathcal{R}| > 2$, then $c_{i-1} = n - 2t - 1$ and $x_{c_{i-1}+t} = x_{n-t-1} \in A$ and from Lemma 4.4.1 we can see that another $t+1$ facets $F_{n-2t-1}, \ldots, F_{n-t-1}$ are in $\mathcal{R}$. If we have all elements of $\mathcal{R}$, we stop, and otherwise, we continue the same way. At each stage $t+1$ new facets are added to $\mathcal{R}$ and therefore in the end $|\mathcal{R}| = 2 \mod t+1$.

Conversely, if $|\mathcal{R}| = (t+1)q + 2$ then let $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_i} \rangle$ be the unique subcollection of $\Delta_t(L_n)$ consisting of i runs of length one from which we can build Γ. Since $F_{n-t-1} \in \mathcal{R}$, we must have $c_i = n - t$ or $c_i = n - t + 1$.

If $c_i = n - t$, then we are done, since Λ will be subcollection of $\Delta_t(L_{n-1})$ and so $\Gamma \in \mathcal{B}$. If $c_i = n - t + 1$, then R has one facet F_{n-t+1} and if $x_{c_{i-1}+t} \in A$, then by Lemma 4.4.1 $\mathcal{R}$ gets an additional $t+1$ facets. And so on: for each c_u either 0 or $t+1$ facets are contributed to $\mathcal{R}$. Therefore, for some p, $|\mathcal{R}| = (t+1)p + 1$ which is a contradiction. This settles our claim. $\square$

We now denote the set of remaining $(i+j, ti+j)$-eligible induced subcollections of $\Delta_t(L_n)$ by $\mathcal{C}$. First we note that $\mathcal{C}$ consists of those induced subcollections which contain F_{n-t+1} and are not in $\mathcal{B}$. Also, if $j=i$, then a $(2i,(t+1)j)$-eligible subcollection Γ of $\Delta_t(L_n)$ would have no runs of length 1, as the equations in Definition 4.2.1 would give $\alpha=0$. So $\Gamma \in \mathcal{C}$ and we can assume from now on that $j<i$.

We consider $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_{i-1}} \rangle \subset \Delta_t(L_n)$ which is composed of $i-1$ runs of length 1 with $\mathrm{V}(\Lambda) \subset \mathcal{X} \setminus F_{n-t+1} \cup \{x_{n-t}\}$. If A is a j-subset of the set $\{x_{c_1+t}, x_{c_2+t}, \ldots, x_{c_{i-1}+t}\}$, we claim that the induced subcollection Γ on $\mathrm{V}(\Lambda) \cup A \cup F_{n-t+1}$ belongs to $\mathcal{C}$.

Suppose $\mathcal{R}$ is the run in Γ which includes F_{n-t+1}. If $|\mathcal{R}| \neq 1$ then $c_{i-1}+t = n-t$ which implies that $c_{i-1} = n-2t$. By Lemma 4.4.1 we see that $t+1$ facets $F_{n-2t}, F_{n-2t+1}, \ldots, F_{n-t}$ are added to $\mathcal{R}$. If these facets are not all the facets of $\mathcal{R}$ then with the same method we can see that in each step $t+1$ new facets will be added to $\mathcal{R}$ and since $F_{n-t+1} \in \mathcal{R}$ we can conclude $|\mathcal{R}| = 1 \mod t+1$. Therefore $\Gamma \notin \mathcal{B}$.

Now we only need to show that Γ is an $(i+j, ti+j)$-eligible induced subcollection. By Proposition 4.4.2 the induced subcollection Γ' on $\mathrm{V}(\Lambda) \cup A$ is an $(i-1+j, t(i-1)+j)$-eligible induced subcollection. Suppose Γ' is composed of runs $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{\alpha'+\beta'}$ and then we have

$$\begin{cases} t(i-1)+j & = (t+1)(P'+Q') + t(\alpha'+\beta') + \beta' \\ i-1+j & = 2(P'+Q') + 2\beta' + \alpha' \end{cases} \Longrightarrow \begin{cases} i-1 & = P'+Q'+\alpha'+\beta' \\ j & = P'+Q'+\beta' \end{cases} \tag{4.5.2}$$

So Γ consists of all or all but one of the runs $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{\alpha'+\beta'}$ as well as $\mathcal{R}$ where $\mathcal{R}$ is the run which includes F_{n-t+1}.

As we have seen $|\mathcal{R}| = 1 \mod t+1$. If we suppose $|\mathcal{R}| = 1$ then we can claim that Γ is composed of $\alpha = \alpha'+1$ runs of length 1 and $\beta = \beta'$ runs of length $2 \mod t+1$, and with $P = P'$ and $Q = Q'$, by (4.5.2) we have Γ is an $(i+j, ti+j)$-eligible induced subcollection. Now assume $|\mathcal{R}| = (t+1)p+1$, so we have $F_{n-2t} \in \Lambda$ and $x_{n-t} \in A$. Let $\mathcal{R}'$ be the induced subcollection on $\mathrm{V}(\mathcal{R}) \setminus F_{n-t+1}$. Then we have $\mathcal{R}'$ is a run in Γ' and since the only facets which belong to $\mathcal{R}$ but not to $\mathcal{R}'$ are the t

facets $F_{n-2t+2}, \ldots, F_{n-t+1}$ we have

$$|\mathcal{R}'| = (t+1)p + 1 - t = (t+1)(p-1) + 2 \tag{4.5.3}$$

Therefore we have shown the run in Γ which includes F_{n-t+1} has been generated by a run of length $2 \mod (t+1)$ in Γ'. Using (4.5.2) we can conclude Γ consists of $\alpha = \alpha' + 1$ runs of length 1 and $\beta = \beta' - 1$ runs of length $2 \mod t+1$. We set $P = P' + p$ and $Q = Q' - (p-1)$, and use (4.5.3) to conclude that

$$\begin{cases} P + Q + \alpha + \beta = (P' + p) + (Q' - p + 1) + (\alpha' + 1) + (\beta' - 1) = i \\ P + Q + \beta = (P' + p) + (Q' - p + 1) + (\beta' - 1) = j. \end{cases}$$

Therefore $\Gamma \in \mathcal{C}$ as we had claimed.

Conversely, let $\Gamma \in \mathcal{C}$ then one can consider the induced subcollection Γ' on

$\mathrm{V}(\Gamma) \backslash F_{n-t+1}$. Assume Γ is composed of runs $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{\alpha+\beta}$, so that $\mod t+1$, $\mathcal{R}_h$ is a run of length 1 if $h \leq \alpha$ and length 2 otherwise.

Suppose $\mathcal{R}_h$ is the run which includes F_{n-t+1}. By our assumption we have $|\mathcal{R}_h| = 1 \mod t+1$, so $h \leq \alpha$. If $|\mathcal{R}_h| = 1$ then $\mathcal{R}_h \notin \Gamma'$ and therefore we delete one run of length one from Γ to obtain Γ', in which case Γ' is $(i-1+j, t(i-1)+j)$-eligible.

If $|\mathcal{R}_h| = (t+1)p_h + 1 > 1$ then the t facets $F_{n-2t+2}, \ldots, F_{n-t+1} \in \mathcal{R}_h$ do not belong to Γ'. So Γ' consists of $\alpha + \beta$ runs $\mathcal{R}_1, \ldots, \widehat{\mathcal{R}_h}, \ldots, \mathcal{R}_{\alpha+\beta}, \mathcal{R}'_h$ where

$$|\mathcal{R}'_h| = (t+1)p_h + 1 - t = (t+1)(p_h - 1) + 2.$$

Setting $\alpha' = \alpha - 1$, $\beta' = \beta + 1$, $P' = P - p_h$ and $Q' = Q + p_h - 1$ it follows that Γ' is $(i-1+j, t(i-1)+j)$-eligible. By Proposition 4.4.3 there exists a unique induced subcollection $\Lambda = \langle F_{c_1}, F_{c_2}, \ldots, F_{c_{i-1}} \rangle$ of $\Delta_t(L_{n-t-1})$ which is composed of $i-1$ runs of length one and a j subset A of $\{x_{c_1+t}, \ldots, x_{c_{i-1}+t}\}$ such that Γ' equals to induced subcollection on $\mathrm{V}(\Lambda) \cup A$. So Γ is the induced subcollection on $\mathrm{V}(\Lambda) \cup A \cup F_{n-t+1}$. Therefore there is a one to one correspondence between elements of $\mathcal{C}$ and

$$\binom{[i-1]}{j} \times \{\Gamma \subset \Delta_t(L_{n-t-1}) : \Gamma \quad \text{is composed of } i-1 \text{ runs of length } 1\} \tag{4.5.4}$$

By (4.5.1), (4.5.4) and Corollary 4.3.5 (iii) we have

$$\begin{aligned}
\beta_{i+j,ti+j}(R/I) &= |\mathcal{B}| + |\mathcal{C}| \\
&= \binom{i}{j}\beta_{i,ti}(R/I_t(L_{n-1})) + \binom{i-1}{j}\beta_{i-1,t(i-1)}(R/I_t(L_{n-t-1}) \\
&= \binom{i}{j}\binom{n-it}{i} + \binom{i-1}{j}\binom{n-it}{i-1}.
\end{aligned}$$

So we are done. □

Finally, we put together Theorem 4.2.2, Proposition 4.2.4, Theorem 5.1 of [1] and Theorem 3.1.2. Note that the case $t = 2$ is the case of graphs which appears in Jacques [28]. Also note that $\beta_{i,j}(R/I_t(C_n)) = 0$ for all $i \geq 1$ and $j > ti$ (cf. Jacques [28] 3.3.4).

Theorem 4.5.2 (Betti numbers of path ideals of paths and cycles)**.** *Let n, t, p and d be integers such that $n \geq 2$, $2 \leq t \leq n$, $n = (t+1)p + d$, where $p \geq 0$, $0 \leq d \leq t$. Then*

i. The $\mathbb{N}$-graded Betti numbers of the path ideal of the graph cycle C_n are given by

$$\beta_{i,j}(R/I_t(C_n)) = \begin{cases} t & j = n,\ d = 0,\ i = 2\left(\dfrac{n}{t+1}\right) \\[2ex] 1 & j = n,\ d \neq 0,\ i = 2\left(\dfrac{n-d}{t+1}\right) + 1 \\[2ex] \dfrac{n}{n - t\left(\frac{j-i}{t-1}\right)}\dbinom{\frac{j-i}{t-1}}{\frac{ti-j}{t-1}}\dbinom{n - t\left(\frac{j-i}{t-1}\right)}{\frac{j-i}{t-1}} & \begin{cases} j < n,\ i \leq j \leq ti, \text{ and} \\[2ex] 2p \geq \dfrac{2(j-i)}{t-1} \geq i \end{cases} \\[2ex] 0 & \text{otherwise.} \end{cases}$$

ii. The $\mathbb{N}$-graded Betti numbers of the path ideal of the path graph L_n are nonzero and equal to

$$\beta_{i,j}(R/I_t(L_n)) = \binom{\frac{j-i}{t-1}}{\frac{ti-j}{t-1}}\binom{n-t\left(\frac{j-i}{t-1}\right)}{\frac{j-i}{t-1}} + \binom{\frac{j-i}{t-1}-1}{\frac{ti-j}{t-1}}\binom{n-t\left(\frac{j-i}{t-1}\right)}{\frac{j-i}{t-1}-1}$$

if and only if

(a) $j \le n$ and $i \le j \le ti$;

(b) If $d < t$ then $p \ge \dfrac{j-i}{t-1} \ge i/2$;

(c) If $d = t$ then $(p+1) \ge \dfrac{j-i}{t-1} \ge (i+1)/2$.

Proof. We only need to make sure we have the correct conditions for the Betti numbers to be nonzero.

i. When $j < n$, $\beta_{i,j}(R/I_t(C_n)) \neq 0 \iff$

$$\iff \begin{cases} \dfrac{j-i}{t-1} \ge \dfrac{ti-j}{t-1} \\ n - \dfrac{t(j-i)}{t-1} \ge \dfrac{j-i}{t-1} \end{cases}$$

$$\iff \begin{cases} 2j \ge (t+1)i \\ n \ge \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases}$$

$$\iff \begin{cases} 2j \ge (t+1)i \\ (t+1)p + d \ge \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} \tag{4.5.5}$$

$$\iff \begin{cases} 2j \ge (t+1)i \\ p + \dfrac{d}{t+1} \ge \dfrac{j-i}{t-1} \end{cases}$$

$$\iff 2p \ge \frac{2(j-i)}{t-1} \ge i \qquad \text{as } d < t+1$$

ii. $\beta_{i,j}(R/I_t(L_n)) \neq 0 \iff$

$$\iff \begin{cases} \dfrac{j-i}{t-1} \geq \dfrac{ti-j}{t-1} \\ n - \dfrac{t(j-i)}{t-1} \geq \dfrac{j-i}{t-1} \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{j-i}{t-1} \geq \dfrac{ti-j}{t-1} + 1 \\ n - \dfrac{t(j-i)}{t-1} \geq \dfrac{j-i}{t-1} - 1 \end{cases}$$

$$\iff \begin{cases} 2j \geq (t+1)i \\ n \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} \quad \text{or} \quad \begin{cases} 2j \geq (t+1)(i+1) \\ n+1 \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases}$$

$$\iff \begin{cases} 2j \geq (t+1)i \\ (t+1)p + d \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} \quad \text{or} \quad \begin{cases} 2j \geq (t+1)(i+1) \\ (t+1)p + d + 1 \geq \left(\dfrac{t+1}{t-1}\right)(j-i) \end{cases} \tag{4.5.6}$$

$$\iff \begin{cases} 2j \geq (t+1)i \\ p + \dfrac{d}{t+1} \geq \dfrac{j-i}{t-1} \end{cases} \quad \text{or} \quad \begin{cases} 2j \geq (t+1)(i+1) \\ p + \dfrac{d+1}{t+1} \geq \dfrac{j-i}{t-1} \end{cases}$$

$$\iff p + \frac{d}{t+1} \geq \frac{(j-i)}{t-1} \geq i/2 \quad \text{or} \quad p + \frac{d+1}{t+1} \geq \frac{(j-i)}{t-1} \geq \frac{i+1}{2}$$

Then since $d < t+1$ and $(j-i)/(t-1)$ is an integer we can conclude that $i \leq 2p$ when $d \neq t$ and $i \leq 2p+1$ for $d = t$. Also we have $j - i \leq (t-1)p$ for $d \neq t$ and $j - i \leq (t-1)(p+1)$ for $d = t$. □

We can now easily derive the projective dimension and regularity of path ideals of paths, which were known before. The projective dimension of paths (Part i below) was computed by He and Van Tuyl in [21] using different methods. The case $t = 2$ is the case of graphs which appears in Jacques [28]. Part ii of the following Corollary reproves Theorem 5.3 in Bouchat, Hà and O'Keefe [11] which computes the Castelnuovo-Mumford regularity of path ideal of a path. The case of cycles was done in section 4.1.

Corollary 4.5.3 (projective dimension and regularity of path ideals of paths)**.** *Let n, t, p and d be integers such that $n \geq 2$, $2 \leq t \leq n$, $n = (t+1)p + d$, where $p \geq 0$, $0 \leq d \leq t$. Then* *i. The projective dimension of the path ideal of a path L_n is given by*

$$pd(R/I_t(L_n)) = \begin{cases} 2p & d \neq t \\ \\ 2p+1 & d = t \end{cases}$$

ii. The regularity of the path ideal of a path L_n is given by

$$reg(R/I_t(L_n)) = \begin{cases} p(t-1) & d < t \\ \\ (p+1)(t-1) & d = t \end{cases}$$

Proof. By using Theorem 4.5.2 we know that if $\beta_{i,j}(R/I_t(L_n) \neq 0$ then $i \leq 2p+1$ when $d = t$ and therefore $pd(R/I_t(L_n)) \leq 2p+1$. On the other hand by applying Theorem 4.5.2 we have

$$\beta_{2p+1,n}(R/I_t(L_n)) = \binom{p+1}{p}\binom{p}{p+1} + \binom{p}{p}\binom{p}{p} = 1 \neq 0.$$

Then we can conclude that $pd(R/I_t(L_n)) = 2p+1$.

Now we suppose $d \neq t$. From (4.5.6) we can see that if $\beta_{i,j}(R/I_t(L_n)) \neq 0$ then $2p \geq i$ and therefore $pd(R/I_t(L_n)) \leq 2p$. On the other hand, by applying Theorem 4.5.2 again, we can see that

$$\beta_{2p,p(t+1)}(R/I_t(L_n)) = \binom{p}{p}\binom{p+d}{p} + \binom{p-1}{p}\binom{p}{p} = \binom{p+d}{p} \neq 0.$$

Therefore $pd(R/I_t(L_n)) \geq 2p$ and we have $pd(R/I_t(L_n)) = 2p$.

By definition, the regularity of a module M is $\max\{j - i \ : \ \beta_{i,j}(M) \neq 0\}$. By Theorem 4.5.2, we know exactly when the graded Betti numbers of $R/I_t(L_n)$ are nonzero, and the formula follows directly from (4.5.6). $\square$

Chapter 5

Rees Algebras of Squarefree Monomial Ideals

In this chapter we investigate Rees equations of squarefree monomial ideals. We give a criterion for examining ideals of linear type. Throughout this chapter we are working with the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$.

5.1 Rees Algebras of Squarefree Monomial Ideals and Their Equations

Definition 5.1.1. For integers $s, q \geq 1$ we define

$$\mathcal{I}_s = \{(i_1, \ldots, i_s) : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq q\} \subset \mathbb{N}^s.$$

Let $\alpha = (i_1, \ldots, i_s) \in \mathcal{I}_s$ and $f_1, \ldots, f_q$ be monomials in R and $T_1, \ldots, T_q$ be variables. We use the following notation for the rest of this paper. If $t \in \{1, \ldots, s\}$

- $Supp\ (\alpha) = \{i_1, \ldots, i_s\}$;
- $\widehat{\alpha}_{i_t} = (i_1, \ldots, \widehat{i}_t, \ldots, i_s) = (i_1, \ldots, i_{t-1}, i_{t+1}, \ldots, i_s)$;
- $T_\alpha = T_{i_1} \cdots T_{i_s}$ and $Supp\ (T_\alpha) = \{T_{i_1}, \ldots, T_{i_s}\}$;
- $f_\alpha = f_{i_1} \cdots f_{i_s}$;
- $\widehat{f}_{\alpha_t} = f_{i_1} \cdots \widehat{f}_{i_t} \cdots f_{i_s} = \dfrac{f_\alpha}{f_{i_t}}$;
- $\widehat{T}_{\alpha_t} = T_{i_1} \cdots \widehat{T}_{i_t} \cdots T_{i_s} = \dfrac{T_\alpha}{T_{i_t}}$;
- $\alpha_t(j) = (i_1, \ldots, i_{t-1}, j, i_{t+1}, \ldots, i_s)$, for $j \in \{1, 2, \ldots, q\}$ and $s \geq 2$.

Theorem 5.1.2. *(D. Taylor [40]) Let $f_1, \ldots, f_q$ be the monomials in R. Then we have*

$$Syz(f_1, \ldots, f_q) = \left\langle \left(\frac{\operatorname{lcm}(f_i, f_j)}{f_i}\right) e_i - \left(\frac{\operatorname{lcm}(f_i, f_j)}{f_j}\right) e_j : 1 \leq i < j \leq q \right\rangle.$$

Definition 5.1.3. Let $I = (f_1, \ldots, f_q)$ be a monomial ideal, $s \geq 2$ and $\alpha, \beta \in \mathcal{I}_s$. We define

$$T_{\alpha,\beta}(I) = \left(\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha}\right) T_\alpha - \left(\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\beta}\right) T_\beta. \tag{5.1.1}$$

When I is clear from the context we use $T_{\alpha,\beta}$ to denote $T_{\alpha,\beta}(I)$.

Since products of monomials are again monomials the following theorem follows from Taylor's theorem.

Theorem 5.1.4. *Let $I = (f_1, \ldots, f_q)$ be a monomial ideal in R and J be the defining ideal of $R[It]$. Then for $s \geq 2$ we have*

$$J_s = \langle T_{\alpha,\beta}(I) : \alpha, \beta \in \mathcal{I}_s \rangle . \tag{5.1.2}$$

Moreover, if $m = \gcd(f_1, \ldots, f_q)$ and $I' = (f_1/m, \ldots, f_q/m)$, then for every $\alpha, \beta \in \mathcal{I}_s$ we have

$$T_{\alpha,\beta}(I) = T_{\alpha,\beta}(I')$$

and hence $R[It] = R[I't]$.

Proof. Let $s \geq 1$ we set $|\mathcal{I}_s| = N$. If $\eta \in J_s$ from the definition of defining ideals we have

$$\eta = \sum_{i=1}^{N} a_i T_{\alpha_i} \in J_s \tag{5.1.3}$$

where $(a_1, \ldots, a_N) \in Syz(f_\alpha : \alpha \in \mathcal{I}_s)$. On the other hand by using Theorem 5.1.2 we have

$$(a_1, \ldots, a_N) = \sum_{i=1}^{N} a_i e_i = \sum_{1 \leq i < j \leq N} \lambda_{ij} \left(\frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_i}} e_i - \frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_j}} e_j \right)$$

for $\lambda_{ij} \in R$. Then we have

$$a_i = \sum_{j=i+1}^{N} \lambda_{ij} \left(\frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_i}} \right) - \sum_{j=1}^{i-1} \lambda_{ji} \left(\frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_i}} \right) \qquad \text{for } i = 1, 2, \ldots, N$$

(empty sums are zero). Thus we have

$$\begin{aligned}\sum_{i=1}^{N} a_i T_{\alpha_i} &= \sum_{i=1}^{N}\sum_{j=i+1}^{N} \lambda_{ij}\left(\frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_i}}\right) T_{\alpha_i} - \sum_{i=1}^{N}\sum_{j=1}^{i-1} \lambda_{ji}\left(\frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_i}}\right) T_{\alpha_i} \\ &= \sum_{1\le i<j\le N} \lambda_{ij}\left(\frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_i}} T_{\alpha_i} - \frac{\operatorname{lcm}(f_{\alpha_i}, f_{\alpha_j})}{f_{\alpha_j}} T_{\alpha_j}\right) \\ &= \sum_{1\le i<j\le N} \lambda_{ij} T_{\alpha_i,\alpha_j}.\end{aligned}$$

Then by (5.1.3) we can conclude

$$\eta \in \langle T_{\alpha,\beta} : \alpha, \beta \in \mathcal{I}_s \rangle .$$

If $I' = (f_1/m, \dots, f_q/m)$ then it follows that $T_{\alpha,\beta}(I) = T_{\alpha,\beta}(I')$ for every $\alpha, \beta \in \mathcal{I}_s$ since

$$\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha} = \frac{\operatorname{lcm}(f_\alpha/m, f_\beta/m)}{f_\alpha/m}.$$

It settles the claim. $\square$

In light of Theorem 5.1.4, we will always assume that if $I = (f_1, \dots, f_q)$ then

$$\gcd(f_1, \dots, f_q) = 1.$$

We will also assume $\mathit{Supp}\,(\alpha) \cap \mathit{Supp}\,(\beta) = \emptyset$, since otherwise $T_{\alpha,\beta}$ reduces to those with this property. This is because if $t \in \mathit{Supp}\,(\alpha) \cap \mathit{Supp}\,(\beta)$ then we have $T_{\alpha,\beta} = T_t T_{\widehat{\alpha}_t,\widehat{\beta}_t}$.

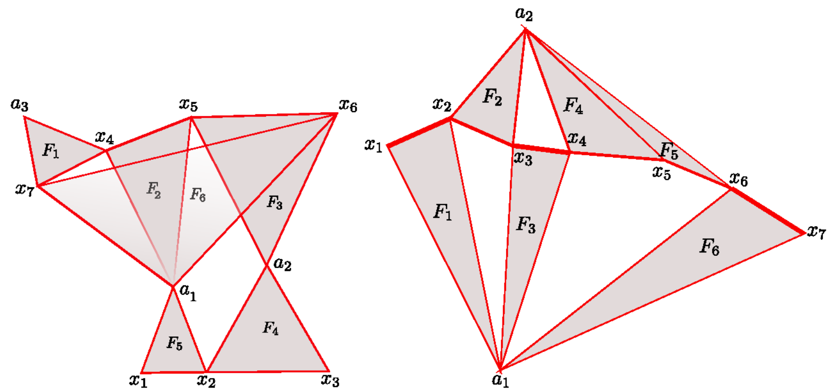
For this reason we define

$$J_s = \langle T_{\alpha,\beta}(I) : \alpha, \beta \in \mathcal{I}_s,\ \mathit{Supp}\,(\alpha) \cap \mathit{Supp}\,(\beta) = \emptyset \rangle \tag{5.1.4}$$

as an R-module. We have $J = J_1 S + J_2 S + \cdots$.

Definition 5.1.5. Let $I = (f_1, \dots, f_q)$ be a squarefree monomial ideal in R and J be the defining ideal of $R[It]$, $s \geq 2$, and $\alpha = (i_1, \dots, i_s), \beta = (j_1, \dots, j_s) \in \mathcal{I}_s$. We call $T_{\alpha,\beta}$ **redundant** if it is a redundant generator of J, i.e.

$$T_{\alpha,\beta} \in J_1 S + \cdots + J_{s-1} S.$$



(a) Even walk (b) Not an even walk

Figure 5.1: Examples of simplicial even walk

5.2 Simplicial Even Walks

By using the concept of closed even walks in graph theory Villarreal [47] classified all Rees equations of edge ideals of graphs in terms of closed even walks. In this section our goal is to define an even walk in a simplicial complex in order to classify all irredundant Rees equations of squarefree monomial ideals.

Definition 5.2.1 (degree). Let $\Delta = \langle F_1, \ldots, F_q \rangle$ be a simplicial complex, $\mathcal{F}(\Delta) = (f_1, \ldots, f_q)$ be its facet ideal and $\alpha = (i_1, \ldots, i_s) \in \mathcal{I}_s$, $s \geq 1$. We define the α-**degree for a vertex** x of Δ to be

$$deg_\alpha(x) \;\; = \max\{m : x^m | f_\alpha\} = |\{j \in \ Supp\ (\alpha) : x \in F_j\}|.$$

Example 5.2.2. Consider Figure [5.1a] where

$$F_1 = \{x_4, x_7, a_3\}, F_2 = \{x_4, x_5, a_1\}, F_3 = \{x_5, x_6, a_2\},$$
$$F_4 = \{x_2, x_3, a_2\}, F_5 = \{x_1, x_2, a_1\}, F_6 = \{x_6, x_7, a_1\}.$$

If we consider $\alpha = (1, 3, 5)$ and $\beta = (2, 4, 6)$ then $deg_\alpha(a_1) = 1$ and $deg_\beta(a_1) = 2$.

Suppose $I = (f_1, \ldots, f_q)$ is a squarefree monomial ideal in R with $\Delta = \langle F_1, \ldots, F_q \rangle$ its facet complex and let $\alpha, \beta \in \mathcal{I}_s$ where $s \geq 2$ is an integer. We set $\alpha = (i_1, \ldots, i_s)$ and $\beta = (j_1, \ldots, j_s)$ and consider the following sequence of not necessarily distinct facets of

Δ

$$\mathcal{C}_{\alpha,\beta} = F_{i_1}, F_{j_1}, \ldots, F_{i_s}, F_{j_s}.$$

Then (5.1.1) becomes

$$T_{\alpha,\beta}(I) = \left(\prod_{deg_\alpha(x) < deg_\beta(x)} x^{deg_\beta(x) - deg_\alpha(x)} \right) T_\alpha - \left(\prod_{deg_\alpha(x) > deg_\beta(x)} x^{deg_\alpha(x) - deg_\beta(x)} \right) T_\beta \tag{5.2.1}$$

where the products vary over the vertices x of $\mathcal{C}_{\alpha,\beta}$.

Definition 5.2.3 (simplicial even walk)**.** Let $\Delta = \langle F_1, \ldots, F_q \rangle$ be a simplicial complex and let $\alpha = (i_1, \ldots, i_s), \beta = (j_1, \ldots, j_s) \in \mathcal{I}_s$, where $s \geq 2$. The following sequence of not necessarily distinct facets of Δ

$$\mathcal{C}_{\alpha,\beta} = F_{i_1}, F_{j_1}, \ldots, F_{i_s}, F_{j_s}$$

is called a **simplicial even walk**, or simply "even walk," if the following conditions hold

- For every $i \in\ Supp\ (\alpha)$ and $j \in\ Supp\ (\beta)$ we have

$$F_i \backslash F_j \not\subset \{x \in \mathrm{V}(\Delta) : deg_\alpha(x) > deg_\beta(x)\} \quad \text{and} \quad F_j \backslash F_i \not\subset \{x \in \mathrm{V}(\Delta) : deg_\alpha(x) < deg_\beta(x)\}.$$

If $C_{\alpha,\beta}$ is connected, we call the even walk $C_{\alpha,\beta}$ is a **connected** even walk.

Remark 5.2.4. It follows from the definition, if $\mathcal{C}_{\alpha,\beta}$ is an even walk then $\ Supp\ (\alpha) \cap$ $Supp\ (\beta) = \emptyset$.

Remark 5.2.5. Even walk are a generalization of monomial walks in hypergraph theory which is introduced by Petrović and Stasi [37]. They have used monomial walks to contribute a combinatorial description of the toric ideal of a hypergraph. In fact they have generalized a Villarreal's work of the graph case [47]. (Villarreal's description of toric ideals of a graph have been proven with different techniques by Ohsugi and Hibi [36] and Reyes, Tatakis and Thoma [38]).

Example 5.2.6. In Figures [5.1a] and [5.1b] by setting $\alpha = (1, 3, 5), \beta = (2, 4, 6)$ we have $\mathcal{C}_{\alpha,\beta} = F_1, \ldots, F_6$ is an even walk in [5.1a] but in [5.1b] $\mathcal{C}_{\alpha,\beta} = F_1, \ldots, F_6$ is not an even walk because

$$F_1 \backslash F_2 = \{x_1, a_1\} = \{x : deg_\alpha(x) > deg_\beta(x)\}.$$

Remark 5.2.7. A question which naturally arises here is if a minimal even walk (an even walk that does not properly contain an other even walk) can have repeated facets. The answer is positive since for instance, the bicycle graph in Figure 5.2 is a minimal even walk, because of Theorem 5.2.18 below, but it has a pair of repeated edges.

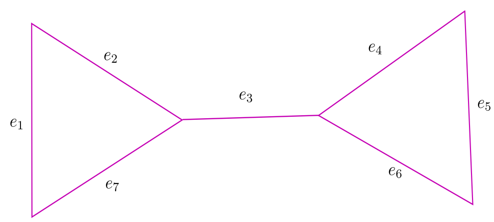


Figure 5.2: A minimal even walk with repeated facets

5.2.1 The Structure of Even Walks

Proposition 5.2.8 (structure of even walks)**.** *Let $\mathcal{C}_{\alpha,\beta} = F_1, F_2, \ldots, F_{2s}$ be an even walk. Then we have*

(i) If $i \in$ Supp (α) (or $i \in$ Supp (β)) there exist distinct $j, k \in$ Supp (β) (or $j, k \in$ Supp (α)) such that

$$F_i \cap F_j \neq \emptyset \quad \textit{and} \quad F_i \cap F_k \neq \emptyset. \tag{5.2.2}$$

(ii) The simplicial complex $\langle \mathcal{C}_{\alpha,\beta} \rangle$ contains an extended trail of even length labeled $F_{v_1}, F_{v_2}, \ldots, F_{v_{2l}}$ where $v_1, \ldots, v_{2l-1} \in$ Supp (α) and $v_2, \ldots, v_{2l} \in$ Supp (β).

Proof. To prove (i) let $i \in$ Supp (α), and consider the following set

$$\mathcal{A}_i = \{j \in \text{ Supp } (\beta) : F_i \cap F_j \neq \emptyset\}.$$

We only need to prove that $|\mathcal{A}_i| \geq 2$.

Suppose $|\mathcal{A}_i| = 0$ then for all $j \in$ Supp (β) we have

$$F_i \backslash F_j = F_i \subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) > deg_\beta(x)\}$$

because for each $x \in F_i \backslash F_j$ we have $deg_\beta(x) = 0$ and $deg_\alpha(x) > 0$; a contradiction.

Suppose $|\mathcal{A}_i| = 1$ so that there is one $j \in$ *Supp* (β) such that $F_i \cap F_j \neq \emptyset$. So for every $x \in F_i \backslash F_j$ we have $deg_\beta(x) = 0$. Therefore, we have

$$F_i \backslash F_j \subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) > deg_\beta(x)\},$$

again a contradiction and so we must have $|\mathcal{A}_i| \geq 2$ and we are done.

To prove (ii) pick $u_1 \in$ *Supp* (α). By using the previous part we can say there are $u_0, u_2 \in$ *Supp* (β), $u_0 \neq u_2$, such that

$$F_{u_0} \cap F_{u_1} \neq \emptyset \quad \text{and} \quad F_{u_1} \cap F_{u_2} \neq \emptyset.$$

By a similar argument there is $u_3 \in$ *Supp* (α) such that $u_1 \neq u_3$ and $F_{u_2} \cap F_{u_3} \neq \emptyset$. We continue this process. Pick $u_4 \in$ *Supp* (β) such that

$$F_{u_4} \cap F_{u_3} \neq \emptyset \quad \text{and} \quad u_4 \neq u_2.$$

If $u_4 = u_0$, then $F_{u_0}, F_{u_1}, F_{u_2}, F_{u_3}$ is an even length extended trail. If not, we continue this process each time taking

$$F_{u_0}, \ldots, F_{u_n}$$

and picking $u_{n+1} \in$ *Supp* (α) (or u_{n+1} *Supp* (β)) if $u_n \in$ *Supp* (β) (or $u_n \in$ *Supp* (α)) such that

$$F_{u_{n+1}} \cap F_{u_n} \neq \emptyset \quad \text{and} \quad u_{n+1} \neq u_{n-1}.$$

If $u_{n+1} \in \{u_0, \ldots, u_{n-2}\}$, say $u_{n+1} = u_m$, then the process stops and we have

$$F_{u_m}, F_{u_{m+1}}, \ldots, F_{u_n}$$

is an extended trail. The length of this cycle is even since the indices $u_m, u_{m+1}, \ldots, u_n$ alternately belong to *Supp* (α) and *Supp* (β) (which are disjoint by our assumption), and if $u_m \in$ *Supp* (α), then by construction $u_n \in$ *Supp* (β) and vice-versa. So there are an even length of such indices and we are done.

If $u_{n+1} \notin \{u_0, \ldots, u_{n-2}\}$ we add it to the end of the sequence and repeat the same process for $F_{u_0}, F_{u_1}, \ldots, F_{u_{n+1}}$. Since $\mathcal{C}_{\alpha,\beta}$ has a finite number of facets, this process has to stop. $\square$

Corollary 5.2.9. *An even walk has at least* 4 *distinct facets.*

In Corollary 5.2.11, we will see that every even walk must contain a simplicial cycle. The following theorem, which we prove directly can also be deduced from combining Theorem 1.14 in Soleyman Jahan and Zheng [29] and Theorem 2.4 in Conca and De Negri [14].

Theorem 5.2.10. *A simplicial forest contains no simplicial even walk.*

Proof. Assume the forest Δ contains an even walk $\mathcal{C}_{\alpha,\beta}$ where $\alpha, \beta, \in \mathcal{I}_s$ and $s \geq 2$ is an integer. Since Δ is a simplicial forest so is its subcollection $\langle \mathcal{C}_{\alpha,\beta} \rangle$, so by Theorem 2.6.6 $\langle \mathcal{C}_{\alpha,\beta} \rangle$ contains a good leaf F_0. So we can consider the following order on the facets $F_0, \ldots, F_q$ of $\langle \mathcal{C}_{\alpha,\beta} \rangle$

$$F_q \cap F_0 \subseteq \cdots \subseteq F_2 \cap F_0 \subseteq F_1 \cap F_0. \tag{5.2.3}$$

Without loss of generality we suppose $0 \in \ Supp\ (\alpha)$. Since $\ Supp\ (\beta) \neq \emptyset$, we can pick $j \in \{1, \ldots, q\}$ to be the smallest index with $F_j \in \ Supp\ (\beta)$. Now if $x \in F_0 \backslash F_j$, by (5.2.3) we will have $deg_\alpha(x) \geq 1$ and $deg_\beta(x) = 0$, which shows that

$$F_0 \backslash F_j \subset \{x \in V(\mathcal{C}_{\alpha,\beta}); deg_\alpha(x) > deg_\beta(x)\},$$

a contradiction. $\square$

Corollary 5.2.11. *Every simplicial even walk contains a simplicial cycle.*

An even walk is not necessarily an extended trail. For instance see the following example.

Example 5.2.12. Let $\alpha = (1, 3, 5, 7), \beta = (2, 4, 6, 8)$ and $\mathcal{C}_{\alpha,\beta} = F_1, \ldots, F_8$ as in Figure 5.3. It can easily be seen that $\mathcal{C}_{\alpha,\beta}$ is an even walk of distinct facets but $\mathcal{C}_{\alpha,\beta}$ is not an extended trail.

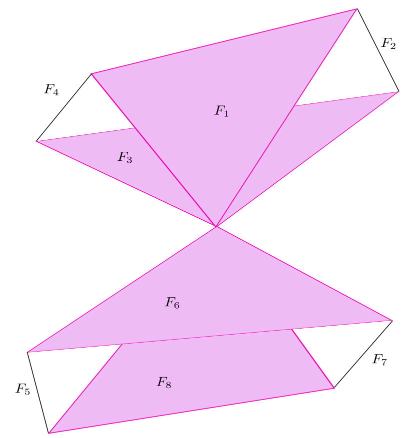


Figure 5.3: An even walk which is not an extended trail

The main point here is that we do not require that $F_i \cap F_{i+1} \neq \emptyset$ in an even walk which is necessary condition for extended trails. For example $F_4 \cap F_5 \neq \emptyset$ in this case.

On the other hand, every even-length special cycle is an even walk.

Proposition 5.2.13 (even special cycles are even walks)**.** *If* $F_1, \ldots, F_{2s}$ *is a special cycle (under the written order) then it is an even walk under the same order.*

Proof. Let $\alpha = (1, 3, \ldots, 2s-1)$ and $\beta = (2, 4, \ldots, 2s)$, and set $\mathcal{C}_{\alpha,\beta} = F_1, \ldots, F_{2s}$. Suppose $\mathcal{C}_{\alpha,\beta}$ is not an even walk, so there is $i \in \textit{ Supp }(\alpha)$ and $j \in \textit{ Supp }(\beta)$ such that at least one of the following conditions holds

$$\begin{aligned} F_i \backslash F_j &\subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) > deg_\beta(x)\} \qquad (5.2.4)\\ F_j \backslash F_i &\subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) < deg_\beta(x)\}. \end{aligned}$$

Without loss of generality we can assume that the first condition holds. Pick $h \in \{i-1, i+1\}$ such that $h \neq j$. Then by definition of special cycle there is a vertex $z \in F_i \cap F_h$ and $z \notin F_l$ for $l \notin \{i, h\}$. In particular, $z \in F_i \backslash F_j$, but $deg_\alpha(z) = deg_\beta(z) = 1$ which contradicts (5.2.4). $\square$

The converse of Proposition 5.2.13 is not true: not every even walk is a special cycle. (see Figure [5.1a] or Figure [5.3] which are not even extended trails). But one can show that it is true for even walks with four facets.

Lemma 5.2.14. *Let* $\alpha = (1, 3), \beta = (2, 4)$ *and* $\mathcal{C}_{\alpha,\beta} = (F_1, F_2, F_3, F_4)$ *be an even walk with distinct facets. Then* $\mathcal{C}_{\alpha,\beta}$ *is a special cycle.*

Proof. By Proposition 5.2.8 we know F_1, F_2, F_3, F_4 is a extended trail. If it is not a special cycle then, without loss of generality, suppose

$$F_1 \cap F_2 \subseteq F_3 \cup F_4. \tag{5.2.5}$$

Since $\mathcal{C}_{\alpha,\beta}$ is an even walk there is $x \in F_1 \cap F_4$ such that

$$1 \leq deg_\alpha(x) \leq deg_\beta(x) \leq 2.$$

Since there are only 4 facets, it follows that $deg_\alpha(x) = deg_\beta(x) = 1$. So $x \in F_1 \cap F_2$ which, by (5.2.5), implies that $x \in F_3 \cap F_4$. So $deg_\alpha(x) > 1$ or $deg_\beta(x) > 1$, a contradiction. □

5.2.2 The Case of Graphs

We demonstrate that simplicial closed even walks in dimension 1 restricts to closed even walks in graph.

Lemma 5.2.15. *Let G be a simple graph and let $\mathcal{C} = e_{i_1}, \ldots, e_{i_{2s}}$ be a sequence of not necessarily distinct edges of G where $s \geq 2$ and $e_i = \{x_i, x_{i+1}\}$ and $f_i = x_i x_{i+1}$ for $1 \leq i \leq 2s$. Let $\alpha = (i_1, i_3, \ldots, i_{2s-1})$, $\beta = (i_2, i_4, \ldots, i_{2s})$. Then $\mathcal{C}$ is a closed even walk if and only if $f_\alpha = f_\beta$.*

Proof. ($\Longrightarrow$) This direction follows from the definition of closed even walks.

($\Longleftarrow$) We can give to each repeated edge in $\mathcal{C}$ a new label and consider $\mathcal{C}$ as a multigraph (a graph with multiple edges). The condition $f_\alpha = f_\beta$ implies that every $x \in \mathrm{V}(\mathcal{C})$ has even degree, as a vertex of the multigraph $\mathcal{C}$ (a graph containing edges that are incident to the same two vertices). Theorem 2.4.5 implies that $\mathcal{C}$ is a closed even walk with no repeated edges. Now we revert back to the original labeling of the edges of $\mathcal{C}$ (so that repeated edges appear again) and then since $\mathcal{C}$ has even length we are done. □

To prove the main theorem of this section (Theorem 5.2.18) we need the following lemma.

Lemma 5.2.16. *Let $\mathcal{C} = \mathcal{C}_{\alpha,\beta}$ be a 1-dimensional simplicial even walk and $\alpha, \beta \in \mathcal{I}_s$. If there is $x \in V(\mathcal{C})$ for which $deg_\beta(x) = 0$ (or $deg_\alpha(x) = 0$), then we have $deg_\beta(v) = 0$ ($deg_\alpha(v) = 0$) for all $v \in V(\mathcal{C})$.*

Proof. First we show the following statement.

$$e_i = \{w_i, w_{i+1}\} \in E(\mathcal{C}) \quad \text{and} \quad deg_\beta(w_i) = 0 \Longrightarrow deg_\beta(w_{i+1}) = 0. \tag{5.2.6}$$

where $E(\mathcal{C})$ is the edge set of $\mathcal{C}$.

Suppose $deg_\beta(w_{i+1}) \neq 0$. Then there is $e_j \in E(\mathcal{C})$ such that $j \in$ $Supp\ (\beta)$ and $w_{i+1} \in e_j$. On the other hand since $w_i \in e_i$ and $deg_\beta(w_i) = 0$ we can conclude $i \in$ $Supp\ (\alpha)$ and thus $deg_\alpha(w_i) > 0$. Therefore, we have

$$e_i \backslash e_j = \{w_i\} \subseteq \{z : deg_\alpha(z) > deg_\beta(z)\}$$

and it is a contradiction. So we must have $deg_\beta(w_{i+1}) = 0$.

Now we proceed to the proof of our statement. Pick $y \in$ V$(\mathcal{C})$ such that $y \neq x$. Since $\mathcal{C}$ is connected we can conclude there is a path $\gamma = e_{i_1}, \ldots, e_{i_t}$ in $\mathcal{C}$ in which we have

- $e_{i_j} = \{x_{i_j}, x_{i_{j+1}}\}$ for $j = 1, \ldots, t$;
- $x_{i_1} = x$ and $x_{i_{t+1}} = y$.

Since γ is a path it has neither repeated vertices nor repeated edges. Now note that since $deg_\beta(x) = deg_\beta(x_{i_1}) = 0$ and $\{x_{i_1}, x_{i_2}\} \in E(\mathcal{C})$ from (5.2.6) we have $deg_\beta(x_{i_2}) = 0$. By repeating a similar argument we have

$$deg_\beta(x_{i_j}) = 0 \quad \text{for } j = 1, 2, \ldots, t+1.$$

In particular we have $deg_\beta(x_{i_{t+1}}) = deg_\beta(y) = 0$ and we are done. $\square$

We now show that a simplicial even walk in a graph (considering a graph as a 1-dimensional simplicial complex) is a closed even walk in that graph as defined in Definition 2.4.1.

Theorem 5.2.17. *Let G be a simple graph with edges $e_1, \ldots, e_q$. Let $e_{i_1}, \ldots, e_{i_{2s}}$ be a sequence of edges of G such that $\langle e_{i_1}, \ldots, e_{i_{2s}} \rangle$ is a connected subgraph of G and $\{i_1, i_3, \ldots, i_{2s-1}\} \cap \{i_2, i_4, \ldots, i_{2s}\} = \emptyset$. Then $e_{i_1}, \ldots, e_{i_{2s}}$ is a simplicial even walk if and only if*

$$\{x \in V(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) > deg_\beta(x)\} = \{x \in V(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) < deg_\beta(x)\} = \emptyset$$

Proof. ($\Longleftarrow$) is straightforward. To prove the converse we assume $\alpha = (i_1, i_3, \ldots, i_{2s-1})$, $\beta = (i_2, i_4, \ldots, i_{2s})$ and $\mathcal{C}_{\alpha,\beta}$ is a simplicial even walk. We only need to show

$$deg_\alpha(x) = deg_\beta(x) \qquad \text{for all } x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}).$$

Assume without loss of generality $deg_\alpha(x) > deg_\beta(x) \geq 0$, so there exists $i \in Supp\ (\alpha)$ such that $x \in e_i$. We set $e_i = \{x, w_1\}$.

Suppose $deg_\beta(x) \neq 0$. We can choose an edge e_k in $\mathcal{C}_{\alpha,\beta}$ where $k \in\ Supp\ (\beta)$ such that $x \in e_i \cap e_k$. We consider two cases.

(1) If $deg_\beta(w_1) = 0$, then since $deg_\alpha(w_1) \geq 1$ we have

$$e_i \backslash e_k = \{w_1\} \subseteq \{z \in \mathrm{V}(G) : deg_\alpha(z) > deg_\beta(z)\},$$

a contradiction.

(2) If $deg_\beta(w_1) \geq 1$, then there exists $h \in\ Supp\ (\beta)$ with $w_1 \in e_h$. So we have

$$e_i \backslash e_h = \{x\} \subseteq \{z \in \mathrm{V}(G) : deg_\alpha(z) > deg_\beta(z)\},$$

again a contradiction.

So we must have $deg_\beta(x) = 0$. By Lemma 5.2.16 this implies that $deg_\beta(v) = 0$ for every $v \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})$, a contradiction, since $Supp\ (\beta) \neq \emptyset$. $\square$

Corollary 5.2.18 (1-dimensional simplicial even walks)**.** *Let G be a simple graph with edges $e_1, \ldots, e_q$. Let $e_{i_1}, \ldots, e_{i_{2s}}$ be a sequence of edges of G such that $\langle e_{i_1}, \ldots, e_{i_{2s}} \rangle$ is a connected subgraph of G and $\{i_1, i_3, \ldots, i_{2s-1}\} \cap \{i_2, i_4, \ldots, i_{2s}\} = \emptyset$. Then $e_{i_1}, \ldots, e_{i_{2s}}$ is a simplicial even walk if and only if $e_{i_1}, \ldots, e_{i_{2s}}$ is a closed even walk in G.*

Proof. Let $I(G) = (f_1, \ldots, f_q)$ be the edge ideal of G and $\alpha = (i_1, i_3, \ldots, i_{2s-1})$ and $\beta = (i_2, i_4, \ldots, i_{2s})$ so that $\mathcal{C}_{\alpha,\beta} = e_{i_1}, \ldots, e_{i_{2s}}$. Assume $\mathcal{C}_{\alpha,\beta}$ is a closed even walk in G. Then we have

$$f_\alpha = \prod_{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})} x^{deg_\alpha(x)} = \prod_{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})} x^{deg_\beta(x)} = f_\beta,$$

where the second equality follows from Lemma 5.2.15.

So for every $x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta})$ we have $deg_\alpha(x) = deg_\beta(x)$. In other words we have

$$\{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) > deg_\beta(x)\} = \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : deg_\alpha(x) < deg_\beta(x)\} = \emptyset,$$

and therefore we can say $\mathcal{C}_{\alpha,\beta}$ is a simplicial even walk. The converse follows directly from Theorem 5.2.17 and Lemma 5.2.15. $\square$

We will need the following proposition in the next sections.

Proposition 5.2.19. *Let $\mathcal{C}_{\alpha,\beta}$ be a 1-dimensional even walk, and $\langle \mathcal{C}_{\alpha,\beta} \rangle = G$. Then every vertex of G has degree > 1. In particular, G is either an even cycle or contains at least two cycles.*

Proof. Suppose G contains a vertex of degree 1 namely v. Without loss of generality we can assume $v \in e_i$ where $i \in$ $Supp\ (\alpha)$. So $deg_\alpha(v) = 1$ and from Theorem 5.2.17 we have $deg_\beta(v) = 1$. Therefore, there is $j \in$ $Supp\ (\beta)$ such that $v \in e_j$. Since $deg(v) = 1$ we have $i = j$ and it is a contradiction since $Supp\ (\alpha)$ and $Supp\ (\beta)$ are disjoint.

Note that from Corollary 5.2.11 G contains a cycle. Now we show that G contains at least two distinct cycles or it is an even cycle.

Suppose G contains only one cycle C_n. Then removing the edges of C_n leaves a forest of n components. Since every vertex of G has degree > 1, each of the components must be singleton graphs (a null graph with only one vertex). So $G = C_n$. Therefore, by Corollary 5.2.18 and the fact that $Supp\ (\alpha)$ and $Supp\ (\beta)$ are disjoint, n must be even. $\square$

5.3 A Necessary Criterion for a Squarefree Monomial Ideal to be of Linear Type

We are ready to state one of the main results of this thesis which is a combinatorial method to detect irredundant Rees equations of squarefree monomial ideals. We first show that these Rees equations come from even walks.

Lemma 5.3.1. *Let $I = (f_1, \ldots, f_q)$ be a squarefree monomial ideal in the polynomial ring R. Suppose s, t, h are integers with $s \geq 2$, $1 \leq h \leq q$ and $1 \leq t \leq s$. Let $0 \neq \gamma \in R$, $\alpha = (i_1, \ldots, i_s), \beta = (j_1, \ldots, j_s) \in \mathcal{I}_s$. Then* *(i)* $\operatorname{lcm}(f_\alpha, f_\beta) = \gamma f_h \widehat{f}_{\alpha_t} \iff T_{\alpha,\beta} = \lambda \widehat{T}_{\alpha_t} T_{(i_t),(h)} + \mu T_{\alpha_t(h),\beta}$ *for some monomials* $\lambda, \mu \in R$, $\lambda \neq 0$.

(ii) $\operatorname{lcm}(f_\alpha, f_\beta) = \gamma f_h \widehat{f}_{\beta_t} \iff T_{\alpha,\beta} = \lambda \widehat{T}_{\beta_t} T_{(h),(j_t)} + \mu T_{\alpha,\beta_t(h)}$ *for some monomials* $\lambda, \mu \in R$, $\lambda \neq 0$.

Proof. We only prove (i); the proof of (ii) is similar.

First note that if $h = i_t$ then we have

$$\operatorname{lcm}(f_\alpha, f_\beta) = \gamma f_\alpha \iff T_{\alpha,\beta} = T_{\alpha,\beta} \quad \text{(Setting } \mu = 1)$$

and we have nothing to prove, then without loss of generality we can assume that $h \neq i_t$.

If we have $\operatorname{lcm}(f_\alpha, f_\beta) = \gamma f_h \widehat{f}_{\alpha_t}$, then the monomial γf_h is divisible by f_{i_t}, so there exists a nonzero exists a monomial $\lambda \in R$ such that

$$\lambda \operatorname{lcm}(f_{i_t}, f_h) = \gamma f_h. \tag{5.3.1}$$

By using (5.3.1) we have

$$\begin{aligned} T_{\alpha,\beta} &= \left(\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha}\right) T_\alpha - \left(\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\beta}\right) T_\beta = \left(\frac{\gamma f_h}{f_{i_t}}\right) T_\alpha - \left(\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\beta}\right) T_\beta \\ T_{\alpha,\beta} &= \lambda \widehat{T}_{\alpha_t} T_{(i_t),(h)} + \left(\frac{\lambda \operatorname{lcm}(f_{i_t}, f_h)}{f_h}\right) \widehat{T}_{\alpha_t} T_h - \left(\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\beta}\right) T_\beta. \end{aligned} \tag{5.3.2}$$

On the other hand note that since we have

$$\operatorname{lcm}(f_\alpha, f_\beta) = \gamma f_h \widehat{f}_{\alpha_t} = \gamma f_{\alpha_t(h)}, \tag{5.3.3}$$

we can conclude $\operatorname{lcm}(f_{\alpha_t(h)}, f_\beta)$ divides $\operatorname{lcm}(f_\alpha, f_\beta)$. Thus there exists a monomial $\mu \in R$ such that

$$\operatorname{lcm}(f_\alpha, f_\beta) = \mu \operatorname{lcm}(f_{\alpha_t(h)}, f_\beta). \tag{5.3.4}$$

By using (5.3.1), (5.3.3) and (5.3.4) we have

$$\frac{\lambda \operatorname{lcm}(f_{i_t}, f_h)}{f_h} = \frac{\lambda \operatorname{lcm}(f_{i_t}, f_h) \widehat{f}_{\alpha_t}}{f_{\alpha_t(h)}} = \frac{\gamma f_h \widehat{f}_{\alpha_t}}{f_{\alpha_t(h)}} = \frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_{\alpha_t(h)}} = \frac{\mu \operatorname{lcm}(f_{\alpha_t(h)}, f_\beta)}{f_{\alpha_t(h)}}. \tag{5.3.5}$$

Substituting (5.3.4) and (5.3.5) in (5.3.2) we get

$$T_{\alpha,\beta} = \lambda \widehat{T}_{\alpha_t} T_{(i_t),(h)} + \mu T_{\alpha_t(h),\beta}.$$

For the converse since $h \neq i_t$, by comparing coefficients we have

$$\begin{aligned} \frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha} = \lambda \left(\frac{\operatorname{lcm}(f_{i_t}, f_h)}{f_{i_t}} \right) = \lambda \prod_{x \in F_h \setminus F_{i_t}} x \implies & \operatorname{lcm}(f_\alpha, f_\beta) = \lambda \left(\prod_{x \in F_h \setminus F_{i_t}} x \right) f_\alpha \\ \implies & \operatorname{lcm}(f_\alpha, f_\beta) = \lambda_0 f_h \hat{f}_{\alpha_t} \end{aligned}$$

where $0 \neq \lambda_0 \in R$. This concludes our proof. □

Now we show that there is a direct connection between redundant Rees equations and the above lemma.

Theorem 5.3.2. *Let $\Delta = \langle F_1, \ldots, F_q \rangle$ be a simplicial complex , $\alpha, \beta \in \mathcal{I}_s$ and $s \geq 2$ an integer. If $\mathcal{C}_{\alpha,\beta}$ is not an even walk then*

$$T_{\alpha,\beta} \in J_1 S + J_{s-1} S.$$

Proof. Let $I = (f_1, \ldots, f_q)$ be the facet ideal of Δ and let $\alpha = (i_1, \ldots, i_s)$ and $\beta = (j_1, \ldots, j_s) \in \mathcal{I}_s$. If $C_{\alpha,\beta}$ is not an even walk, then by Definition 5.2.3 there exist $i_t \in Supp\,(\alpha)$ and $j_l \in\ Supp\,(\beta)$ such that one of the following cases is true

(1) $F_{j_l} \setminus F_{i_t} \subseteq \{x \in \mathrm{V}(\Delta) : deg_\alpha(x) < deg_\beta(x)\}$;

(2) $F_{i_t} \setminus F_{j_l} \subseteq \{x \in \mathrm{V}(\Delta) : deg_\alpha(x) > deg_\beta(x)\}$.

Suppose (1) is true. Then there exists a monomial $m \in R$ such that

$$\frac{\operatorname{lcm}(f_\alpha, f_\beta)}{f_\alpha} = \prod_{deg_\beta(x) > deg_\alpha(x)} x^{deg_\beta(x) - deg_\alpha(x)} = m \prod_{x \in F_{j_l} \setminus F_{i_t}} x. \qquad (5.3.6)$$

So we have

$$\operatorname{lcm}(f_\alpha, f_\beta) = m f_\alpha \prod_{x \in F_{j_l} \setminus F_{i_t}} x = m_0 f_{j_l} \widehat{f}_{\alpha_t}$$

where $m_0 \in R$. On the other hand by using Lemma 5.3.1 we can say there exist monomials $0 \neq \lambda, \mu \in R$ such that

$$\begin{aligned} T_{\alpha,\beta} &= \lambda \widehat{T}_{\alpha_t} T_{(i_t),(j_l)} + \mu T_{\alpha_t(j_l),\beta} \\ &= \lambda \widehat{T}_{\alpha_t} T_{(i_t),(j_l)} + \mu T_{j_l} T_{\widehat{\alpha}_t,\widehat{\beta}_l} \in J_1 S + J_{s-1} S \quad \text{(since } j_l \in \ Supp\,(\beta)). \end{aligned}$$

If case (2) holds, a similar argument settles our claim. $\square$

Corollary 5.3.3. *Let $\Delta = \langle F_1, \ldots, F_q \rangle$ be a simplicial complex and $s \geq 2$ be an integer. Then we have*

$$J = J_1 S + \left(\bigcup_{i=2}^{\infty} P_i \right) S$$

where $P_i = \{T_{\alpha,\beta} : \alpha, \beta \in \mathcal{I}_i$ and $\mathcal{C}_{\alpha,\beta}$ is an even walk$\}$.

Theorem 5.3.4 (main theorem)**.** *Let I be a squarefree monomial ideal in R and suppose the facet complex $\mathcal{F}(I)$ has no even walk. Then I is of linear type.*

The following corollary, can also be deduced from combining Theorem 1.14 in Conca, De Negri [29] and Theorem 2.4 in Soleyman Jahan, Zheng [14].

More precisely Conca and De Negri have showed that M-sequence ideals (ideals which are generated by M-sequences) are of linear type. More recently Soleyman Jahan and Zheng proved that M-sequence ideals can be viewed as the facet ideal of a simplicial tree. Thus from these two facts one can conclude that the facet ideal of trees are of linear type. In our case, it follows directly from Theorem 5.3.4 and Theorem 5.2.10.

Corollary 5.3.5. *The facet ideal of a simplicial forest is of linear type.*

The converse of Theorem 5.3.2 is not in general true. For example see the following.

Example 5.3.6. Let $\alpha = (1,3), \beta = (2,4)$. In Figure [5.4] we see $\mathcal{C}_{\alpha,\beta} = F_1, F_2, F_3, F_4$ is an even walk but we have

$$T_{\alpha,\beta} = x_4 x_8 T_1 T_3 - x_1 x_6 T_2 T_4 = x_8 T_3 (x_4 T_1 - x_2 T_5) + T_5 (x_2 x_8 T_3 - x_5 x_6 T_4) + x_6 T_4 (x_5 T_5 - x_1 T_2) \in J_1 S.$$

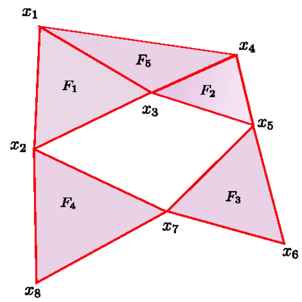


Figure 5.4: A simplicail complex with 5 facets containing an even walk

By Theorem 5.3.2, all irredundant generators of J of deg > 1 correspond to even walks. However irredundant generators of J do not correspond to minimal even walks in Δ (even walks that do not properly contain other even walks). For instance $\mathcal{C}_{(1,3,5),(2,4,6)}$ as displayed in Figure [5.1a] is an even walk which is not minimal (since $\mathcal{C}_{(3,5),(2,4)}$ and $\mathcal{C}_{(1,5),(2,6)}$ are even walks which contain properly in $\mathcal{C}_{(1,3,5),(2,4,6)}$). But $T_{(1,3,5),(2,4,6)} \in J$ is an irredundant generator of J.

We can now state a simple necessary condition for a simplicial complex to be of linear type in terms of its line graph.

Definition 5.3.7. Let $\Delta = \langle F_1, \ldots, F_n \rangle$ be a simplicial complex. The **line graph** $L(\Delta)$ of Δ is a graph whose vertices are labeled with the facets of Δ, and two vertices labeled F_i and F_j are adjacent if and only if $F_i \cap F_j \neq \emptyset$.

Theorem 5.3.8 (a simple test for linear type)**.** *Let Δ be a simplicial complex and suppose $L(\Delta)$ contains no even cycle. Then $\mathcal{F}(\Delta)$ is of linear type.*

Proof. We show that Δ contains no even walk $\mathcal{C}_{\alpha,\beta}$. Otherwise by Proposition 5.2.8 $\mathcal{C}_{\alpha,\beta}$ contains an even extended trail B, and $L(B)$ is then an even cycle contained in $L(\Delta)$ which is a contradiction. Theorem 5.3.4 settles our claim. $\square$

Theorem 5.3.8 generalizes results of Fouli and Lin [18], where they showed if $L(\Delta)$ is a tree or is an odd cycle then I is of linear type.

It must be noted, however, that our test is far from being criterion. Below is an example of an ideal of linear type whose path graph contains an even cycle.

Example 5.3.9. In the following simplicial complex Δ, $L(\Delta)$ contains an even cycle but its facet ideal $\mathcal{F}(\Delta)$ is of linear type.

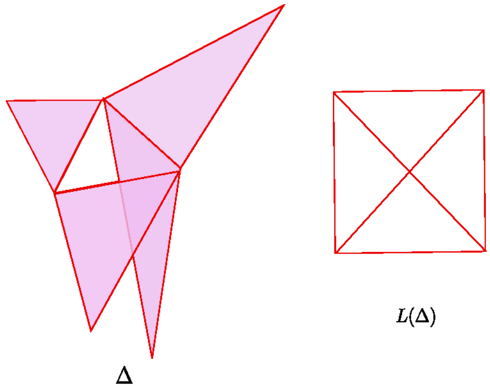


Figure 5.5: A simplicial complex and its line graph

By applying Corollary 5.3.3 and Proposition 5.2.19 we conclude the following statement, which was originally proved by Villarreal in [47].

Corollary 5.3.10. *Let G be a graph which is either tree or contains a unique cycle which is odd. Then the edge ideal $\mathcal{F}(G)$ is of linear type.*

Chapter 6

Conclusion

In this chapter we discuss some open problems and further work related to the topics covered in this thesis. These can be divided into two groups, questions related to path ideals of cycles and paths and questions related to Rees algebras of squarefree monomial ideals. For path ideals we would mainly like to investigate the arithmetic rank of these ideals for cycles. For Rees algebras of monomial ideals, the equations of Rees algebra of quadratic squarefree monomial ideals are known. The focus of my work is to find a combinatorial interpretation for equations of Rees algebras of squarefree monomial ideals of higher degrees.

6.1 Path Ideals

Definition 6.1.1. Let I be an ideal in R, then the minimum number of elements of R that generate I up to radical is called the **arithmetic rank** of I. More precisely the arithmetic rank of I is

$$ara(I) = \min \left\{ r : \exists a_1, \ldots, a_r \in R; \sqrt{I} = \sqrt{(a_1, \ldots, a_r)} \right\}.$$

Finding the arithmetic rank of squarefree monomial ideals is inspired by the work of Lyubeznik [31] in 80's, who used Taylor's resolution of monomial ideals to study questions about cohomological dimensions.

The computation of the arithmetic rank of an ideal is a challenging problem. For this reason researchers are developing bounds for the arithmetic rank of classes of ideals. The simplest upper bound for the arithmetic rank of an ideal I is $\mu(I)$, the minimal number of generators of I. By using Krull's Principal Ideal Theorem [46], one can find a lower bound; it is the height of the ideal I. Thus

$$ht(I) \leq ara(I) \leq \mu(I).$$

According to Lyubeznik's result in [31], one can find a less obvious lower bound for the arithmetic rank of squarefree monomial ideals. For a squarefree monomial ideal I we have the following

$$ht(I) \leq pd(R/I) \leq ara(I) \leq \mu(I).$$

In 2000, Zhao Yan [49] showed that the inequality is sharp in some cases. The following question arises immediately.

Question 6.1.2. For which classes of squarefree monomial ideals does $pd(R/I) = ara(I)$?

This is a broad question. Some classes of squarefree monomial ideals in which the equality holds have been investigated (cf. [6], [4], [5], [8], [7]). The edge ideals of cycles have also been investigated. Barile, Kiani, Mohammadi and Yassemi [8] showed that

$$ara(I_2(C_n)) = pd(R/I_2(C_n)),$$

where C_n is a graph cycle with n vertices. Now we can ask what happens for $t \neq 2$? Regarding the arithmetic rank of path ideals of a cycle, Macchia [32] in his unpublished work showed the following.

Lemma 6.1.3. *Let C_n be a cycle graph over n vertices and let $2 \leq t \leq n$. Then we have*

$$ara(I_t(C_n)) \in \{pd(R/I_t(C_n)), pd(R/I_t(C_n)) + 1\}.$$

Since it is proved for $t = 2$ in [8] that $ara(I_2(C_n)) = pd(R/I_2(C_n))$, we have the following open question.

Question 6.1.4. Let C_n be a graph cycle over n vertices and $2 < t \leq n$. Is it true to say $ara(I_t(C_n)) = pd(R/I_t(C_n))$?

I have been able to show that for some values of t and n this equality holds. However, there are still lots of unexamined cases.

6.2 Rees Algebras of Squarefree Monomial Ideals

Villarreal gave a combinatorial characterization of irredundant generators for edge ideals of graphs [47] by attributing irredundant generators of J_s to closed even walks. Motivated by this work I defined **simplicial closed even walks** in this thesis for simplicial complexes to generalize graph case.

Earlier in this thesis I showed that if $T_{\alpha,\beta}(I)$ is an irredundant generator of J_s, then the generators of I involved in $T_{\alpha,\beta}(I)$ form a simplicial even walk. Our class of simplicial even walks includes *even special cycles* (cf. Herzog et al. [23]) as they are known in hypergraph theory.

Also in this thesis I showed that every simplicial closed even walk contains a simplicial cycle (not necessarily of the same length). By using this new result we can conclude every simplicial tree is of linear type.

Generally the converse of Theorem 5.3.2 is not correct. My goal is to further investigate the structure of simplicial closed even walk so that I can give a more effective criterion to replace Theorem 5.3.2.

Question 6.2.1. Find a more effective criterion to characterize squarefree monomial ideals of linear type.

Fouli and Lin [18] used the line graphs of squarefree monomial ideals to give a simple test for ideals of linear type. Here I improved their test as demonstrated in Theorem 5.3.8.

One can ask what happens for line graphs which contain an even cycle. In general we can say nothing about linearity of I by using these line graphs. I am currently working on improving Theorem 5.3.8.

Bibliography

[1] A. Alilooee and S. Faridi. “Graded Betti numbers of the path ideals of cycles and lines”. In: *arXiv:11106653[math.AC], Submitted* (2013).

[2] A. Alilooee and S. Faridi. “On the resolution of path ideals of cycles”. In: *arXiv:1305.1651v1 [math.AC], accepted for publication in Communications in Algebra* (2013).

[3] A. Alilooee and S. Faridi. “When is a Squarefree Monomial Ideal of Linear Type?” In: *arXiv:1309.1771v2 [math.AC], accepted for publication in Commutative Algebra and Noncommutative Algebraic Geometry and Representation Theory* (2014).

[4] M. Barile. “Arithmetical ranks of Stanley-Reisner ideals via linear algebra”. In: *Comm. Algebra* 36.12 (2008), p. 4540. ISSN: 0092-7872.

[5] M. Barile. “On the arithmetical rank of the edge ideals of forests”. In: *Comm. Algebra* 36.12 (2008), p. 4678. ISSN: 0092-7872.

[6] M. Barile. “On the number of equations defining certain varieties”. In: *Manuscripta Math.* 91.4 (1996), p. 483. ISSN: 0025-2611.

[7] M. Barile and N. Terai. “Arithmetical ranks of Stanley-Reisner ideals of simplicial complexes with a cone”. In: *Comm. Algebra* 38.10 (2010), p. 3686. ISSN: 0092-7872.

[8] M. Barile et al. “Arithmetical rank of the cyclic and bicyclic graphs”. In: *J. Algebra Appl.* 11.2 (2012), pp. 1250039, 14. ISSN: 0219-4988.

[9] C. Berge. *Hypergraphs*. Vol. 45. North-Holland Mathematical Library. Combinatorics of finite sets, Translated from the French. North-Holland Publishing Co., Amsterdam, 1989, pp. x+255. ISBN: 0-444-87489-5.

[10] C. Berge. *Hypergraphs, Combinatorics of finite sets*. Vol. 45. North Holland Publishing Co., Amsterdam, 1989.

[11] R. R. Bouchat, H. T. Hà, and A. O’Keefe. “Path ideals of rooted trees and their graded Betti numbers”. In: *J. Combin. Theory Ser. A* 118.8 (2011), p. 2411. ISSN: 0097-3165.

[12] W. Bruns and J. Herzog. *Cohen-Macaulay rings*. Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993, pp. xii+403. ISBN: 0-521-41068-1.

[13] M. Caboara, S. Faridi, and P. Selinger. "Simplicial cycles and the computation of simplicial trees". In: *J. Symbolic Comput* 42 (1-2 2007), p. 74.

[14] A. Conca and E. De Negri. "M-sequences, graph ideals, and ladder ideals of linear type". In: *J. Algebra* 211.2 (1999), p. 599. ISSN: 0021-8693.

[15] D. L. Costa. "Sequences of linear type". In: *J. Algebra* 94.1 (1985), p. 256. ISSN: 0021-8693.

[16] S. Faridi. "Simplicial trees are sequentially Cohen-Macaulay". In: *J. Pure Appl. Algebra* 190.1-3 (2004), p. 121. ISSN: 0022-4049.

[17] M. Fiorentini. "On relative regular sesquences". In: *J. Algebra* 18 (1971), p. 384. ISSN: 0021-8693.

[18] L. Fouli and K-N Lin. "Rees algebras of square-free monomial ideals". In: *J. Commut. Algebra* 7.1 (2015), p. 25. ISSN: 1939-0807.

[19] R. P. Grimaldi. *Discreet and Combinatorial Mathematics, An applied Introduction*. 2004.

[20] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[21] J. He and A. Van Tuyl. "Algebraic Properties of Path Ideal of a Tree". In: *Comm. Algebra* 38 (2010), p. 1725.

[22] J. Herzog, A. Simis, and W. V. Vasconcelos. "Koszul homology and blowing-up rings". In: *Commutative algebra (Trento, 1981)*. Vol. 84. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1983, p. 79.

[23] J. Herzog et al. "Standard graded vertex cover algebras, cycles and leaves". In: *Trans. Amer. Math. Soc.* 360.12 (2008), p. 6231. ISSN: 0002-9947.

[24] M. Hochster. "Cohen-Macaulay rings, combinatorics, and simplicial complexes". In: *Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975)*. Dekker, New York, 1977, 171, Vol. 26.

[25] C. Huneke. "Determinantal ideals of linear type". In: *Arch. Math. (Basel)* 47.4 (1986), p. 324. ISSN: 0003-889X.

[26] C. Huneke. “The theory of d-sequences and powers of ideals”. In: *Adv. in Math.* 46.3 (1982), p. 249. ISSN: 0001-8708.

[27] C. Huneke and I. Swanson. *Integral closure of ideals, rings, and modules*. Vol. 336. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006, pp. xiv+431. ISBN: 978-0-521-68860-4; 0-521-68860-4.

[28] S. Jacques. “Betti numbers of graph ideals”. PhD thesis. University of Sheffield, arXiv.math.AC/0410107, 2004.

[29] A. Soleyman Jahan and X. Zheng. “Monomial ideals of forest type”. In: *Comm. Algebra* 40.8 (2012), p. 2786. ISSN: 0092-7872.

[30] L. Lovász. *Combinatorial Problems and Exercises*. North-Holland,Amsterdam, 1979.

[31] G. Lyubeznik. “On the local cohomology modules $H^i_a(R)$ for ideals a generated by monomials in an R-sequence”. In: *Complete intersections (Acireale, 1983)*. Vol. 1092. Lecture Notes in Math. Berlin: Springer, 1984, p. 214.

[32] A. Macchia. “Private communication”. In: (2013).

[33] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*. Vol. 227. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xiv+417. ISBN: 0-387-22356-8.

[34] S. E. Morey. *The equations of Rees algebra of ideals of low codimension*. Thesis (Ph.D.)–Rutgers The State University of New Jersey - New Brunswick. ProQuest LLC, Ann Arbor, MI, 1995, p. 66.

[35] F. Muiños and F. Planas-Vilanova. “The equations of Rees algebras of equimultiple ideals of deviation one”. In: *Proc. Amer. Math. Soc.* 141.4 (2013), p. 1241. ISSN: 0002-9939.

[36] H. Ohsugi and T. Hibi. “Toric ideals generated by quadratic binomials”. In: *J. Algebra* 218.2 (1999), p. 509.

[37] S. Petrović and D. Stasi. “Toric algebra of hypergraphs”. In: *J. Algebraic Combin.* 39.1 (2014), p. 187. ISSN: 0925-9899.

[38] E. Reyes, C. Tatakis, and A. Thoma. “Minimal generators of toric ideals of graphs”. In: *Adv. Appl. Math.* 48.1 (2012), p. 64.

[39] R. P. Stanley. *Enumerative combinatorics. Volume 1*. Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2012.

[40] D. Taylor. *Ideals generated by monomials in an R-sequences*. Thesis (Ph.D.)–The University of Chicago. ProQuest LLC, Ann Arbor, MI, 1966, (no paging).

[41] B. Ulrich and W. V. Vasconcelos. "The equations of Rees algebras of ideals with linear presentation". In: *Math. Z.* 214.1 (1993), p. 79. ISSN: 0025-5874.

[42] W. V. Vasconcelos. *Arithmetic of blowup algebras*. Vol. 195. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1994, pp. viii+329. ISBN: 0-521-45484-0.

[43] W. V. Vasconcelos. *Integral Closure: Rees Algebras, Multiplicity, Algorithms*. Springer Monographs in Mathematics, 2005.

[44] W. V. Vasconcelos. "On the equations of Rees algebras". In: *J. Reine Angew. Math.* 418 (1991), p. 189. ISSN: 0075-4102.

[45] R. H. Villarreal. "Cohen-Macaulay graphs". In: *Manuscripta Math.* 66.3 (1990), p. 277. ISSN: 0025-2611.

[46] R. H. Villarreal. *Monomial algebras*. Vol. 238. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 2001, pp. x+455. ISBN: 0-8247-0524-6.

[47] R. H. Villarreal. "Rees algebras of edge ideals". In: *Comm. Algebra* 23.9 (1995), p. 3513. ISSN: 0092-7872.

[48] D. B. West. *Introduction to graph theory*. Prentice Hall, Inc., Upper Saddle River, NJ, 1996, pp. xvi+512. ISBN: 0-13-227828-6.

[49] Z. Yan. "An étale analog of the Goresky-MacPherson formula for subspace arrangements". In: *J. Pure Appl. Algebra* 146.3 (2000), p. 305. ISSN: 0022-4049.

[50] X. Zheng. "Homological Properties of Monomial Ideals Associated to Quasi-Trees and Lattices". PhD thesis. University of Duisburg-Essen, 2004.