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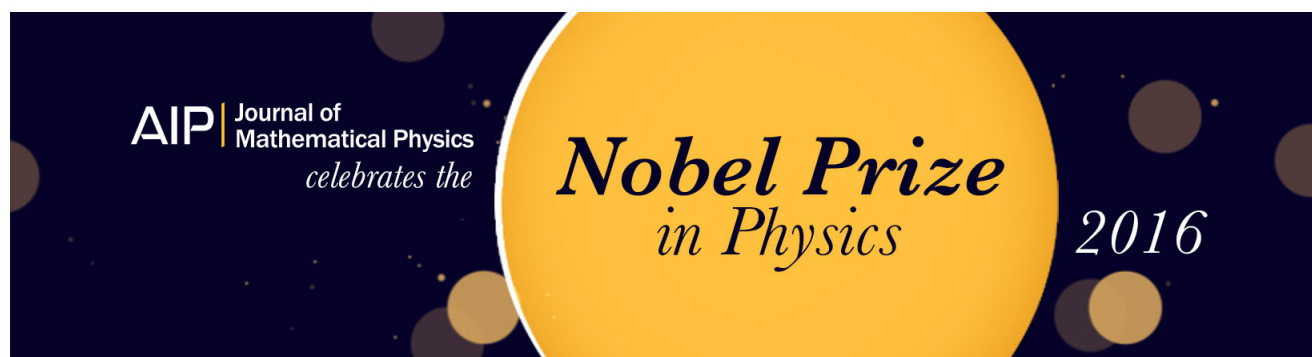
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Hamilton–Jacobi theory in three-dimensional Minkowski space via Cartan geometry

Joshua T. Horwood,^{1,a)} Raymond G. McLenaghan,^{2,b)} and Roman G. Smirnov^{3,c)}

¹Numerica Corporation, 4850 Hahns Peak Drive, Loveland, Colorado 80538, USA

²Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

³Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia B3H 3J5, Canada

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A complete solution to the problem of orthogonal separation of variables of the Hamilton–Jacobi equation in three-dimensional Minkowski space is obtained. The solution is based on the underlying ideas of Cartan geometry and ultimately developed into a general new algorithm that can be employed in the study of Hamiltonian systems defined by natural Hamiltonians within the framework of Hamilton–Jacobi theory. To demonstrate its effectiveness, we investigate from this viewpoint the Morosi–Tondo integrable system derived as a stationary reduction of the seventh-order Korteweg–de Vries flow to show explicitly that the system in question is an orthogonally separable Hamiltonian system. The latter result is a new characterization of the Morosi–Tondo system. © 2009 American Institute of Physics. [DOI: [10.1063/1.3094719](https://doi.org/10.1063/1.3094719)]

I. INTRODUCTION

The main purpose of this work is to solve the problem of orthogonal separation of variables for the Hamilton–Jacobi (HJ) equation in three-dimensional Minkowski space \mathbb{M}^3 by making *explicit* use of the underlying ideas of Cartan’s geometry.^{6,7,34} In his classic 1934 paper, Eisenhart⁹ employed Cartan’s approach *implicitly* to solve the corresponding problem in three-dimensional Euclidean space \mathbb{E}^3 for the special case of a geodesic Hamiltonian. Using similar techniques, Olevsky continued the program in 1950 by studying orthogonal separation of variables in three-dimensional Riemannian spaces of constant (nonvanishing) curvature.²⁹ The results presented in these influential articles laid the groundwork for the development of the theory in the decades that followed. For more details, we refer to Refs. 2, 18, 26, and 31, as well as the references therein.

Many aspects of Cartan’s approach to geometry have inevitably resurfaced in the related literature. In 1965, Winternitz and Friš³⁸ computed the *isometry group invariants* of second-order symmetries of the Laplace equation defined on the Euclidean plane \mathbb{E}^2 . These invariants were then used to classify the (orthogonal) *separable webs* of \mathbb{E}^2 . In 2002, the results of Ref. 38 were independently reproduced and extended by McLenaghan *et al.*²⁵ in the language of *Killing two-tensors* which appear naturally in the study of classical Hamiltonian systems. The latter paper also saw the implicit use of the *moving frame map*, a fundamental concept of Cartan geometry (see Refs. 10, 11, and 30 for more details). Another notable work is the classical 1976 paper by Boyer *et al.*⁴ The authors explicitly described the *problem of equivalence*³⁰ for the separable coordinate systems for the Helmholtz equation in \mathbb{E}^3 as the problem of classification of the orbits of the isometry group acting in the algebra of second-order symmetry operators of the equation. Not-

^{a)}Electronic mail: joshua.horwood@numerica.us.

^{b)}Electronic mail: rgmclena@uwaterloo.ca.

^{c)}Electronic mail: smirnov@mathstat.dal.ca.

withstanding this description, in the present paper as well as in many others written on the subject, the orbits of the corresponding Killing two-tensors (or the orbits of the isometry group acting in the algebra of second-order symmetry operators) are represented by the simplest canonical elements only, namely, those obtained by intersecting the orbits with the cross section corresponding to the unity element of the isometry group. We conclude therefore that from the viewpoint of the orbit analysis described in Ref. 4, Eisenhart⁹ solved the corresponding *canonical form problem*.³⁰

The application of Cartan's geometry makes perfect sense when one studies, for example, the orthogonal separation of variables of the HJ equation for a geodesic Hamiltonian (as in Ref. 9). In such a situation, one can choose *any* orthogonal coordinate system corresponding to *any* element of *any* orbit to perform orthogonal separation of variables of the associated HJ equation and ultimately find exact solutions of the Hamiltonian system. Clearly, the obvious choice in this case would be to use the simplest canonical elements in the orbit space. However, the situation changes drastically when the same problem is considered for natural Hamiltonians. The presence of a potential acts as an obstruction as far as orthogonal separation of variables of the HJ equation is concerned. The Calogero–Moser potential in \mathbb{E}^3 is a prime example in which this obstruction is manifested. As is well known, the corresponding HJ equation is multiseparable; however, the separable webs no longer correspond to the simplest canonical elements in the orbit space.^{3,16,31} This observation validates the development of a more general theory than the one introduced by Eisenhart.⁹ In brief, given a separable web generated by a Killing two-tensor, it is essential to know (i) the orbit to which it belongs and (ii) its location in the orbit in terms of the corresponding moving frames map. Recently, the authors solved a more general problem for \mathbb{E}^3 involving natural Hamiltonian systems and exposed the usefulness of various techniques from Cartan's geometry in the process.¹⁶ Indeed, in the language of orbit analysis, the main result presented in Ref. 16 is a solution to the corresponding equivalence problem. The results of Ref. 16 were later revisited by Horwood in Ref. 14 who employed various tools from algebra and geometry, including the fiber bundle approach to the study of the orbit problem described above, first introduced by Adlam *et al.*¹

In the present paper, we extend our study to the case of three-dimensional Minkowski space \mathbb{M}^3 and make the application of Cartan's geometry more transparent. Our paper is a continuation of the projects initiated by Kalnins,¹⁷ Kalnins and Miller, Jr.,^{20–23} Boyer *et al.*,⁵ Hinterleitner,¹³ and Horwood and McLenaghan.¹⁵ In the latter, the authors solved the corresponding canonical forms problem for orthogonal separation of variables of the HJ equation in \mathbb{M}^3 , in analogy with the classical solution in \mathbb{E}^3 presented by Eisenhart.⁹ The outcome of the present paper is a complete solution to the corresponding problem of equivalence in the framework of the invariant theory of Killing tensors. The main result is a general procedure to *systematically* solve the HJ equation in \mathbb{M}^3 via orthogonal separation of variables. To illustrate the effectiveness of our results, we apply our procedure to an integrable Hamiltonian system introduced by Morosi and Tondo in Ref. 27 describing a stationary reduction of a higher-order flow of the Korteweg–de Vries (KdV) equation.

The article is organized as follows. In Sec. II, we describe the HJ theory of orthogonal separation of variables from the viewpoint of Cartan's geometry. In Sec. III, we begin to apply the general theory to spaces of Killing tensors defined in \mathbb{M}^3 , culminating in the derivation of sets of fundamental isometry group invariants. We then employ the concept of a web symmetry in Sec. IV to compute additional sets of invariants for several invariant subspaces of Killing two-tensors in \mathbb{M}^3 . In Sec. V, we apply the invariants computed in the previous two sections to build an invariant classification scheme for the separable webs in \mathbb{M}^3 , thereby completely solving the equivalence problem. In Sec. VI, we give the explicit construction of the moving frame map for the space of Killing vectors and Killing two-tensors in \mathbb{M}^3 . Our algorithm for computing orthogonally separable coordinates for the HJ equation in \mathbb{M}^3 is then summarized in Sec. VII. Finally, in Sec. VIII, we apply our results to the study of the integrability of the Hamiltonian system presented by Morosi and Tondo.²⁷ Concluding remarks are made in Sec. IX. There are two appendixes. Appendix A lists the canonical forms for the Killing two-tensors characterizing the separable webs in \mathbb{M}^3 previously derived in Ref. 15. In Appendix B, we present some useful formulas and tools for computing moving frame maps for spaces of Killing tensors.

II. HAMILTON–JACOBI THEORY VIA CARTAN GEOMETRY

Let $(\mathcal{M}, \mathbf{g})$ be an n -dimensional pseudo-Riemannian manifold of constant curvature. Referring to Ref. 18 for specific details and references, we recall that the geodesics on $(\mathcal{M}, \mathbf{g})$ can be defined as the integral curves of the Hamiltonian vector field X_H given by

$$X_H = [\mathbf{P}_0, H], \quad (2.1)$$

where $\mathbf{P}_0 = \sum_{i=1}^n \partial / \partial q^i \wedge \partial / \partial p_i$ is the canonical Poisson bivector and H is the *geodesic Hamiltonian*,

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j. \quad (2.2)$$

The usual summation convention applies in (2.2) and throughout the paper. Note also that both \mathbf{P}_0 and H are given in terms of the position-momenta coordinates $(\mathbf{q}, \mathbf{p}) = (q^1, \dots, q^n, p_1, \dots, p_n)$ of the cotangent bundle $T^*(\mathcal{M})$, while $[\cdot, \cdot]$ in (2.1) and g^{ij} in (2.2) as well as throughout this paper denote the Schouten bracket³³ and the contravariant components of the metric tensor \mathbf{g} , respectively. To find the geodesic equation, one may consider [in terms of the Hamiltonian given by (2.2)] either the corresponding second-order Euler–Lagrange equations or the first-order canonical Hamilton equations. In the latter case, one can employ the associated HJ equation to determine the equations satisfied by the geodesics of $(\mathcal{M}, \mathbf{g})$. More specifically, the geodesic flow splits the cotangent bundle $T^*(\mathcal{M})$ into the level sets of constant energy,

$$\mathcal{M}_E \equiv \{(\mathbf{q}, \mathbf{p}) \in T^*(\mathcal{M}) \mid H(\mathbf{q}, \mathbf{p}) = E\}.$$

The problem may be solved by finding a complete integral W for the associated HJ equation,

$$H(\mathbf{q}, \mathbf{p}) = E, \quad p_i = \frac{\partial W}{\partial q^i}, \quad i = 1, \dots, n. \quad (2.3)$$

A function W is a *complete integral* of (2.3) if and only if (iff) the Lagrangian submanifold $S \subset T^*(\mathcal{M})$ defined by the equations $p_i = \partial W / \partial q^i$, $i = 1, \dots, n$, belongs to the level sets \mathcal{M}_E defined above. Finding such a function W is what is understood by “solving” in this context. As is well known from classical mechanics, once W is found, one can in principle solve the original canonical Hamilton equations for \mathbf{q} and \mathbf{p} defined by the Hamiltonian vector field (2.1). However, solving the problem in terms of the original position-momenta coordinates (\mathbf{q}, \mathbf{p}) without making any assumptions on W is a rare occurrence. Instead, the problem normally extends to finding a canonical transformation to separable coordinates, viz., $(\mathbf{q}, \mathbf{p}) \rightarrow (\mathbf{u}, \mathbf{v})$ in which the equation can be solved under the *additive separation ansatz*

$$W(\mathbf{u}; \mathbf{c}) = \sum_{i=1}^n W_i(u^i; \mathbf{c})$$

and the *nondegeneracy condition*

$$\det \left(\frac{\partial^2 W}{\partial u^i \partial c_j} \right)_{n \times n} \neq 0,$$

where $\mathbf{c} = (c_1, \dots, c_n)$ is a constant vector. *Orthogonal separation of variables* occurs in the case when the transformation to separable coordinates is a point transformation and the metric tensor \mathbf{g} is diagonal with respect to the coordinates (\mathbf{u}, \mathbf{v}) of separation (see, for example, Refs. 2, 9, and 18 for more details and references). The following definition is vital for further considerations.

Definition 2.1: A *Killing tensor field* \mathbf{K} of *valence* p defined in $(\mathcal{M}, \mathbf{g})$ is a symmetric $(p, 0)$ tensor field satisfying the Killing tensor equation

$$[\mathbf{K}, \mathbf{g}] = 0. \quad (2.4)$$

When $p=1$, \mathbf{K} is said to be a Killing vector field (infinitesimal isometry) and (2.4) reduces to

$$\mathcal{L}_{\mathbf{K}}\mathbf{g} = 0, \quad (2.5)$$

where \mathcal{L} denotes the Lie derivative operator.

Since the Schouten bracket $[\cdot, \cdot]$ is \mathbb{R} -bilinear, the set of solutions to the system of overdetermined partial differential equations (PDEs) given by (2.4) form a vector space over \mathbb{R} . Furthermore, since $(\mathcal{M}, \mathbf{g})$ is of constant curvature, the dimension of such a vector space is maximal (see the relevant references in Ref. 16 for more details). In what follows, we shall use the notation $\mathcal{K}^p(\mathcal{M})$ to denote the vector space of valence p Killing tensor fields defined on \mathcal{M} .

The geometric and algebraic properties of Killing tensors of valence two are essential for the considerations that follow.^{2,9,16,18} Firstly, a function $F \in T^*(\mathcal{M})$ which is quadratic in the momenta according to

$$F(\mathbf{q}, \mathbf{p}) = K^{ij}(\mathbf{q})p_i p_j \quad (2.6)$$

is a *first integral* of (2.1) iff the functions K^{ij} above are the components of a Killing tensor field $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$. Secondly, the following version of the Eisenhart theorem on orthogonal separation of variables⁹ gives a set of necessary and sufficient conditions for the Hamiltonian system (2.1) defined by the geodesic Hamiltonian (2.2) to be orthogonally separable in the sense described above.

Theorem 2.2. (Eisenhart): *The Hamiltonian system (2.1) defined by the geodesic Hamiltonian (2.2) is orthogonally separable iff it admits $n-1$ functionally independent first integrals of motion of the form (2.6), such that (i) all of the corresponding Killing tensors of valence two have real and pointwise simple (almost everywhere) eigenvalues, (ii) the eigenvectors (or eigenforms) of these Killing two-tensors are normal, and (iii) the Killing two-tensors defined by the $n-1$ first integrals have the same eigenvectors (eigenforms).*

Remark 2.3: Let $\mathbf{K}_1, \dots, \mathbf{K}_{n-1}$ be the Killing two-tensors of Theorem 2.2. Then $\{\mathbf{g}, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$ generates an n -dimensional vector subspace of $\mathcal{K}^2(\mathcal{M})$. The generic Killing tensor

$$\mathbf{K} = \mathbf{g} + \sum_{i=1}^{n-1} \mathbf{K}_i \quad (2.7)$$

has pointwise distinct eigenvalues and the same eigenvectors as any of the \mathbf{K}_i , $i=1, \dots, n-1$. The normality of the eigenvectors of each of the $n-1$ Killing tensors means that the eigenvectors generate n foliations that consist of $(n-1)$ -dimensional hypersurfaces orthogonal to the eigenvectors of the Killing tensor. Such a geometric construction is called an (*orthogonal*) *separable web* which defines the coordinates of separation for the HJ equation (2.3). Thus, one can associate either $\{\mathbf{g}, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$ or any linear combination thereof having distinct eigenvalues [such as \mathbf{K} given by (2.7)] with the corresponding separable web.

Remark 2.4: A Hamiltonian system (2.1) can admit more than one vector subspace of $\mathcal{K}^2(\mathcal{M})$ consisting of the metric in addition to $n-1$ Killing two-tensors with the properties prescribed by Theorem 2.2. In turn, this entails orthogonal separation of variables for the associated HJ equation in *more than one coordinate system*. For example, Eisenhart⁹ showed that in the case of $\mathcal{M} = \mathbb{E}^3$ the associated HJ equation for the geodesic Hamiltonian (2.2) is orthogonally separable in 11 coordinate systems.

The HJ theory of orthogonal separation of variables has been extended to the study of Hamiltonian systems defined by a *natural Hamiltonian*

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}g^{ij}(\mathbf{q})p_i p_j + V(\mathbf{q}). \quad (2.8)$$

The analogue of the Eisenhart Theorem 2.2 for such natural Hamiltonian systems is due to Benenti.²

Theorem 2.5 (Benenti): *The natural Hamiltonian system defined by (2.8) is orthogonally separable iff there exists a valence-two Killing tensor \mathbf{K} with (i) pointwise simple and real eigenvalues, (ii) normal eigenvectors (eigenforms), and (iii) such that*

$$d(\hat{K}dV) = 0, \quad (2.9)$$

where the $(1,1)$ -tensor $\hat{K} = K\mathbf{g}^{-1}$.

A Killing tensor satisfying conditions (i) and (ii) of Theorem 2.2 or 2.5 is called a *characteristic Killing tensor (CKT)*.²

Remark 2.6: When $V=0$ in (2.8) the Killing two-tensor K of Theorem 2.5 is simply the Killing tensor (2.7) of Theorem 2.2 (see Remark 2.3). Hence, the latter is a special case of the former.

Remark 2.7: The HJ theory of orthogonal separation of variables applied to a natural Hamiltonian (2.8) is more restrictive than the application of the theory to a geodesic Hamiltonian (2.2). Thus, if the natural Hamiltonian system (2.1) is orthogonally separable (as per the associated HJ equation) then the corresponding Hamiltonian system obtained by letting $V=0$ in (2.8) is also orthogonally separable with respect to the same coordinate systems. However, the converse statement is *not* true.

Remark 2.8: The left-hand side of the compatibility condition (2.9) written in local coordinates yields the Bertrand–Darboux PDEs of classical and quantum mechanics (see, for example, Ref. 19 and the references therein). The condition (2.9) implies that there exists a first integral F of the Hamiltonian flow defined by (2.8) of the form

$$F(\mathbf{q}, \mathbf{p}) = \frac{1}{2}K^{ij}(\mathbf{q})p_i p_j + U(\mathbf{q}), \quad (2.10)$$

where $K^{ij}(\mathbf{q})$ are the components of the CKT K and $dU = \hat{K}dV$.

In light of Theorem 2.5, given a natural Hamiltonian (2.8) defined on the cotangent bundle $T^*(\mathcal{M})$ of a pseudo-Riemannian manifold $(\mathcal{M}, \mathbf{g})$ of constant curvature, the following problems are essential for a satisfactory HJ theory of orthogonal separation of variables:

- (i) How many “inequivalent” coordinate systems afford orthogonal separation of variables in the corresponding HJ equation?
- (ii) If the answer to (i) is nonzero, how can one characterize intrinsically the coordinate systems that afford separation of variables in the HJ equation?
- (iii) What are the canonical coordinate transformations

$$(q^1, q^2, \dots, q^n) \rightarrow (u^1, u^2, \dots, u^n)$$

from the given position coordinates of (2.8) to the coordinate systems that afford orthogonal separation of variables of the HJ equation?

A careful look at the above problems suggests that one can naturally link the algebraic and geometric properties of CKTs. More specifically, the analysis we shall review strongly suggests that one can naturally employ and exploit geometric ramifications of the properties of CKTs in the more general framework of Cartan geometry. The resulting *invariant theory of killing tensors (ITKT)* that naturally follows from it is precisely the link between the algebraic and geometric properties of CKTs. In fact, as we shall demonstrate, the HJ theory of orthogonal separation of variables is one of the areas of mathematics where Cartan’s approach to geometry via moving frames manifests itself most strikingly. Indeed, because $(\mathcal{M}, \mathbf{g})$ is a space of constant curvature, we can make the identification $\mathcal{M} \simeq G/H$, where G is its isometry group $I(\mathcal{M})$ and H is a closed Lie subgroup of G . As a homogeneous space, $\mathcal{M} \simeq G/H$ is the base manifold in the (tautological) principal bundle projection

$$\pi_1: G \rightarrow G/H \simeq \mathcal{M}$$

(i.e., G is the principal H -bundle over G/H). Furthermore, the considerations above give rise to the (trivial) vector bundle projection

$$\pi_2: \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M} \simeq G/H.$$

The problem (i) is thus a problem of equivalence involving the study of the orbit space $\mathcal{K}^2(\mathcal{M})/G$ of the group action $G \curvearrowright \mathcal{K}^2(\mathcal{M})$.^{4,16,25,38} Indeed, the isometry group G that acts as an automorphism in $\mathcal{K}^2(\mathcal{M})$ maps CKTs to CKTs, since the algebraic and geometric properties of CKTs are preserved under the action of G . Therefore, two CKTs \mathbf{K}_1 and \mathbf{K}_2 that are connected by an action of G via the corresponding push-forward map (pull-back map if the CKTs are covariant) belong to the same orbit in the orbit space $\mathcal{K}^2(\mathcal{M})/G$. As far as orthogonal separation of variables is concerned, this fact entails that the normal eigenvectors of \mathbf{K}_1 and \mathbf{K}_2 respectively generate the same separable web. The problem of equivalence of the 11 separable webs generated in \mathbb{E}^3 by CKTs is in essence the orbit problem $\mathcal{K}^2(\mathbb{E}^3)/\text{SE}(3)$ explicitly stated in Ref. 4 (but not solved). Here and below, $\text{SE}(3)$ is the Lie group of (orientation-preserving) isometries of \mathbb{E}^3 . Note that in his classical 1934 work, Eisenhart⁹ showed (implicitly) that there were exactly 11 orbits in this setting and derived a set of canonical forms representing each of them. A complete solution for the orbit problem $\mathcal{K}^2(\mathbb{E}^3)/\text{SE}(3)$ for CKTs is presented in Ref. 16 and later in Ref. 14 where a different method is used.

To continue our discussion of the general problem in the setting of Cartan geometry, we proceed following the ideas developed by Adlam *et al.*¹ and Horwood.¹⁴ The orbit space $\mathcal{K}^2(\mathcal{M})/G$ can also be studied from the larger orbit space $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$ of the group action $G \curvearrowright (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})$, since the former is contained in the latter. Thus, we have the structure of a principal G -bundle with the total space $\mathcal{K}^2(\mathcal{M}) \times \mathcal{M}$ and a projection π_3 into the base space $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$, such that

$$\pi_3: \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} \rightarrow (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G.$$

The orbit $\pi_3^{-1}(b)$ is called the fiber over $b \in (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$. The group G acts freely and transitively on the orbits. Additionally, since $(\mathcal{M}, \mathbf{g})$ is a pseudo-Riemannian manifold, it is naturally equipped with the oriented quasiorthonormal frame bundle. Here and in what follows we shall only consider *oriented* frame bundles, since the mathematical consequences of this arrangement are sufficient for the development of a HJ theory from this perspective. Furthermore, consider a CKT $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ at a nonsingular point $\mathbf{x} \in \mathcal{M}$ (i.e., the eigenvalues of \mathbf{K} are all real and distinct at \mathbf{x}). Indeed, \mathbf{K} gives rise to a quasiorthonormal frame $E_{\mathbf{K},\mathbf{x}}(\mathcal{M})$ of eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of \mathbf{K} at $\mathbf{x} \in \mathcal{M}$, which is also a quasiorthonormal basis for $T_{\mathbf{x}}(\mathcal{M})$. Denoting $E(\mathcal{M})$ as the corresponding bundle of frames generated by CKTs in \mathcal{M} , it follows that $(E(\mathcal{M}), \mathcal{M}, \tilde{\pi}_2)$ defines a quasiorthonormal (oriented) frame bundle, where

$$\tilde{\pi}_2: E(\mathcal{M}) \rightarrow \mathcal{M}.$$

The fibers $\tilde{\pi}_2^{-1}(\mathbf{x})$ correspond to sets of all possible quasiorthonormal frames at $\mathbf{x} \in \mathcal{M}$ generated by the eigenvectors of CKTs. Finally, our construction leads to the fiber bundle projection

$$\tilde{\pi}_3: \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} \rightarrow E(\mathcal{M}).$$

For example, if $\mathbf{K} \in \mathcal{K}^2(\mathbb{E}^3)$, then $G = \text{SE}(3)$, $H = \text{SO}(3)$, and the set of all oriented orthonormal frames at a nonsingular point $\mathbf{x} \in \mathbb{E}^3$ generated by \mathbf{K} and its images under the action of $\text{SE}(3)$ is an $\text{SO}(3)$ -torsor.

Finally, to bring our arrangements in line with the postulates of Cartan's geometry, we introduce a *lift* $f: (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{\pi_1} & G/H \simeq \mathcal{M} \\
 \uparrow f & & \uparrow \pi_2 \swarrow \tilde{\pi}_2 \\
 (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G & \xleftarrow{\pi_3} & \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} \xrightarrow{\tilde{\pi}_3} E(\mathcal{M})
 \end{array} \tag{2.11}$$

Let \mathfrak{g} be the Lie algebra of G and let ω denote the \mathfrak{g} -valued left-invariant Maurer–Cartan form on G satisfying the Maurer–Cartan equation

$$d\omega = \omega \wedge \omega.$$

The existence of such a lift f is assured by the fact that G acts transitively on the bundle of frames $E(\mathcal{M})$ for a given CKT $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ and the classical Cartan lemma:¹²

Lemma 2.9 Cartan: *Suppose that φ is a \mathfrak{g} -valued one-form on a connected (or simply connected) manifold \mathcal{M} . Then there exists a C^∞ map $F: \mathcal{M} \rightarrow G$ with $F^*\omega = \varphi$ iff*

$$d\varphi = \varphi \wedge \varphi.$$

Moreover, the resulting map is unique up to left translation.

In view of Lemma 2.9, we define the map $F: E(\mathcal{M}) \rightarrow G$ to be $F = f \circ \pi_3 \circ \tilde{\pi}_3^{-1}$ [see the diagram (2.11)]. One can now solve the equivalence problem for the orbit space $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$ (or $\mathcal{K}^2(\mathcal{M})/G$) for CKTs using the classical calculus of differential forms. More specifically, the problem of invariant classification of the orbit(s) generated by a CKT $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ reduces to fixing a quasiorthonormal frame of eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and considering *in the frame* the corresponding Cartan structure equations

$$d\mathbf{e}^a + \boldsymbol{\omega}^a_b \wedge \mathbf{e}^b = \mathbf{T}^a, \tag{2.12}$$

$$d\boldsymbol{\omega}^a_b + \boldsymbol{\omega}^a_c \wedge \boldsymbol{\omega}^c_b = \boldsymbol{\Theta}^a_b, \tag{2.13}$$

together with the Killing tensor equations for the components K_{ab} of \mathbf{K} ,

$$K_{(ab;c)} = 0,$$

and the integrability conditions

$$\mathbf{e}^a \wedge d\mathbf{e}^a = 0 \quad (\text{no sum}).$$

In these equations, $\boldsymbol{\omega}^a_b = \Gamma_{cb}^a \mathbf{e}^c$ are the connection one-forms, $\mathbf{T}^a = \frac{1}{2} T^a_{bc} \mathbf{e}^b \wedge \mathbf{e}^c$ are the torsion two-forms, $\boldsymbol{\Theta}^a_b = \frac{1}{2} R^a_{bcd} \mathbf{e}^c \wedge \mathbf{e}^d$ are the curvature two-forms, $\{\mathbf{e}^1, \dots, \mathbf{e}^n\}$ is the dual basis of one-forms, the connection coefficients Γ_{cb}^a correspond to the Levi–Civita connection ∇ [and hence $\mathbf{T}^a = 0$ in (2.12)], and R^a_{bcd} are the components of the curvature tensor. We also note that, with respect to this frame, the components of the metric \mathbf{g} and CKT \mathbf{K} are given by

$$g_{ab} = \text{diag}(\epsilon_1, \dots, \epsilon_n), \quad K_{ab} = \text{diag}(\epsilon_1 \lambda_1, \dots, \epsilon_n \lambda_n),$$

respectively, where $\epsilon_a = \pm 1$, $a = 1, \dots, n$, and λ_a , $a = 1, \dots, n$, are the eigenvalues of \mathbf{K} . The *differential invariants* characterizing the orbits in question are determined by the connection one-forms $\boldsymbol{\omega}^a_b$ that are found from a fixed quasiorthonormal frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. We emphasize that such a problem may be extremely complicated computationally. To alleviate such computational difficulties, we can proceed by making use of the fact that the group G acts transitively in the bundle of frames $E(\mathcal{M})$. Thus, one can essentially solve the problem “in the group” using an alternative version of the moving frames method.^{10,11,30} In this setting, we first determine explicitly the corresponding action of the group G in the parameter space of $\mathcal{K}^2(\mathcal{M})$ and then find *algebraic invariants* to be used in solving the equivalence problem. The algebraic invariants are the coordinates of the canonical forms obtained by intersecting the orbits with an appropriate cross section. Indeed, choosing a frame via the above considerations corresponds to choosing a cross section through the orbits of the group G acting in the parameter space of $\mathcal{K}^2(\mathcal{M})$.

In order to obtain a complete solution to the equivalence problem, in many instances one also has to employ algebraic *covariants* of Killing tensors, introduced in Ref. 35. More specifically, the extended action of the isometry group $G \circledast (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})$ yields, in general, n additional fundamental covariants which are functions of both the parameters of the vector space $\mathcal{K}^2(\mathcal{M})$ and the base manifold \mathcal{M} . From the viewpoint of the diagram (2.11), covariants are the natural invariant functions which arise in the ITKT. The covariants also possess a remarkably simple geometric meaning: they are functions of the eigenvalues of Killing two-tensors. It is easy to see that the n fundamental covariants are functions of the eigenvalues of the generic element of $\mathcal{K}^2(\mathcal{M})$. The covariants (eigenvalues) play a pivotal role in the general procedure for solving the equivalence problem. Indeed, the dimension of the orbits generated by CKTs in the orbit space $\mathcal{K}^2(\mathcal{M})/G$ is determined by the number of eigenvalues of CKTs which are functionally independent. For example, the orbits in $\mathcal{K}^2(\mathbb{E}^3)/SE(3)$ corresponding to the asymmetric separable webs are generated by those CKTs having three functionally independent eigenvalues. Those corresponding to the translational and rotational webs are generated by the CKTs having two functionally independent eigenvalues, in exception to the circular cylindrical and Cartesian webs. The former is generated by the CKTs possessing only one functionally independent eigenvalue, while the CKTs associated with the latter have all three eigenvalues constant (see Ref. 16 for more details).

The concept of a *web symmetry*, which signifies that the web is invariant under at least a one-parameter group of isometries, is another interesting geometric consequence stemming from the properties of covariants (eigenvalues) of CKTs. Suppose, for example, that \mathbf{K} is a CKT in $\mathcal{K}^2(\mathcal{M})$ having $n-1$ functionally independent eigenvalues λ_i , $i=1, \dots, n-1$. It follows that there exists a vector field $\mathbf{V} \in \mathcal{K}^1(\mathcal{M})$ (\mathbf{V} is necessarily a Killing vector), such that $\mathcal{L}_{\mathbf{V}}(\mathbf{K})=0$. Indeed, the integral curves of \mathbf{V} are given by the common level sets

$$\bigcap_{i=1}^{n-1} \{\lambda_i = \text{const}\}.$$

The relationship between web symmetries and eigenvalues of CKTs will be explored in a subsequent paper.

We also note that the concept of a web symmetry is used to determine the dimension of the orbits of $\mathcal{K}^2(\mathcal{M})/G$. More specifically, recall that³⁷

$$\dim G = \dim O_x + \dim G_x,$$

where G is a group acting on \mathcal{M} , $\mathbf{x} \in \mathcal{M}$, O_x is the orbit through \mathbf{x} , and G_x is the isotropy subgroup of G through \mathbf{x} . The existence of nontrivial web symmetries indicates, the existence of isotropy subgroups and their number is equal to the dimension of the corresponding isotropy subgroup (see Sec. IV for more details).

In view of the above, solving the equivalence problem for the CKTs of a given vector space $\mathcal{K}^2(\mathcal{M})$ consists of the following steps. Given a Killing tensor $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$, the first step is to verify if it is characteristic. The computational details of this procedure are outlined in the next section. If \mathbf{K} is a CKT, we proceed next to verify whether the eigenvalues of \mathbf{K} admit any functional dependencies (or, equivalently, web symmetries), which determine the dimension of the corresponding orbit(s). Finally, we employ any relevant invariants to determine the type of orbit to which \mathbf{K} belongs to. Finally, when the equivalence problem is solved with the aid of differential or algebraic invariants, one can proceed to determine the *moving frame map*. Such an equivariant map sends any given point on an orbit (or any given CKT $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$) to the corresponding canonical form obtained by intersecting the orbit with a cross section. Finding such a map corresponds to solving the problem (iii) outlined earlier, namely, the determination of the canonical coordinate transformation $(q^1, \dots, q^n) \rightarrow (u^1, \dots, u^n)$ from the given position coordinates of (2.8) to the coordinates permitting orthogonal separation of variables of the HJ equation.

In what follows we shall employ these ideas to completely solve the problem of orthogonal separation of variables of the HJ equation of a natural Hamiltonian (2.8) defined in three-dimensional Minkowski space.

III. INVARIANT THEORY OF KILLING TENSORS IN MINKOWSKI SPACE

A representation of the group acting on a vector space of Killing tensors is the central object used in the generation of group invariants. As elaborated in Sec. II, the orientation-preserving isometry group $I(\mathcal{M})$ is a natural choice of group to employ in ITKT; it acts transitively on \mathcal{M} and induces a nontransitive action on any space of Killing tensors. As proved in Ref. 24, the map

$$\rho: I(\mathcal{M}) \rightarrow GL(\mathcal{K}^p(\mathcal{M}))$$

defines a representation of $I(\mathcal{M})$. The general Killing tensor field \mathbf{K} of $\mathcal{K}^p(\mathcal{M})$ may be represented by d arbitrary parameters a_1, \dots, a_d , with respect to a given basis. As each element $h \in I(\mathcal{M})$ induces a (nonsingular) transformation $\rho(h)$ of $\mathcal{K}^p(\mathcal{M})$, one may compute the group action $I(\mathcal{M}) \circ \mathcal{K}^p(\mathcal{M})$ in terms of the parameters a_1, \dots, a_d , yielding the explicit form of the transformation $h \cdot \mathbf{K} \equiv \rho(h)\mathbf{K}$. An algebraic invariant is thus any smooth real-valued function of the parameters a_i which is invariant under the action of $I(\mathcal{M})$. More precisely, the definition of such invariant functions is as follows.

Definition 3.1: Let $(\mathcal{M}, \mathbf{g})$ be a pseudo-Riemannian manifold of constant curvature and $p \geq 1$ be fixed. A smooth function $\mathcal{I}: \mathcal{K}^p(\mathcal{M}) \rightarrow \mathbb{R}$ is said to be an $I(\mathcal{M})$ -invariant of $\mathcal{K}^p(\mathcal{M})$ iff it satisfies

$$\mathcal{I}(h \cdot \mathbf{K}) = \mathcal{I}(\mathbf{K}) \quad (3.1)$$

for all $\mathbf{K} \in \mathcal{K}^p(\mathcal{M})$ and for all $h \in I(\mathcal{M})$.

The study of group *covariants* (first introduced in ITKT by Smirnov and Yue³⁵) naturally fits into the framework of this section. An $I(\mathcal{M})$ -covariant of $\mathcal{K}^p(\mathcal{M})$ is an $I(\mathcal{M})$ -invariant of the product space $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$. Thus, unlike an invariant, a covariant may also depend on the coordinates of the base manifold \mathcal{M} . We have also seen in Sec. II that covariants arise naturally in the Cartan approach to ITKT.

The description of the entire space of group invariants of a vector space, the fundamental problem of any invariant theory, is known explicitly for $\mathcal{K}^p(\mathcal{M})$, for any valence p , and for any flat pseudo-Euclidean space \mathcal{M} of any dimension or signature. This general result was recently developed by Horwood.¹⁴ For the remainder of this section, we shall specialize the results of Ref. 14 to three-dimensional Minkowski space \mathbb{M}^3 , culminating in a set of fundamental $I(\mathcal{M})$ -covariants of $\mathcal{K}^p(\mathbb{M}^3)$ for the $p=1$ and $p=2$ cases. In what follows, we work exclusively with the Minkowski metric

$$g_{ij} = \text{diag}(-1, 1, 1),$$

defined with respect to canonical pseudo-Cartesian coordinates $x^i = (t, x, y)$. Throughout this paper, we employ the summation convention: any repeated upper and lower Latin indices is summed from 0 to 2. Moreover, we use the covariant metric g_{ij} and its inverse g^{ij} to lower and raise tensor indices.

Any Killing tensor of valence p defined on a manifold of constant curvature is expressible as a sum of symmetrized tensor products of Killing vectors (see, for example, Ref. 24). In \mathbb{M}^3 , a basis for the space of Killing vectors may be written in pseudo-Cartesian coordinates x^i according to

$$\mathbf{X}_i = \frac{\partial}{\partial x^i}, \quad \mathbf{R}_i = \epsilon^k_{ji} x^j \mathbf{X}_k \quad (3.2)$$

for $i=0, 1, 2$, where ϵ_{ijk} is the Levi–Civita tensor with $\epsilon_{012}=1$. An arbitrary linear combination of these basis elements leads to the general Killing vector in $\mathcal{K}^1(\mathbb{M}^3)$ given by

$$\mathbf{K} = A^i \mathbf{X}_i + B^i \mathbf{R}_i, \quad (3.3)$$

where A^i and B^i are constants referred to as the *Killing vector parameters*. The general Killing tensor in $\mathcal{K}^2(\mathbb{M}^3)$ may be expressed as

$$\mathbf{K} = A^{ij}X_i \odot X_j + 2B^{ij}X_i \odot R_j + C^{ij}R_i \odot R_j, \quad (3.4)$$

where A^{ij} , B^{ij} , and C^{ij} , the *Killing tensor parameters*, are constants and satisfy the symmetry properties $A^{ij} = A^{(ij)}$ and $C^{ij} = C^{(ij)}$. On account of the syzygy $g^{ij}X_i \odot R_j = 0$, only 20 of the 21 Killing tensor parameters are independent. As (3.4) is unaffected by the addition of an arbitrary multiple of this syzygy, one can always fix the “trace” of B^{ij} . Without loss of generality, we shall assume that

$$B^i_i = B^{ij}g_{ji} = 0. \quad (3.5)$$

In the HJ theory of orthogonal separation of variables, it is the CKTs which play the primary role, as they characterize the associated separable webs. An arbitrary Killing tensor in $\mathcal{K}^2(\mathbb{M}^3)$ need not be a CKT; not every Killing tensor has real and distinct eigenvalues and normal eigenvectors. It is straightforward to verify if a given Killing tensor $\mathbf{K} \in \mathcal{K}^2(\mathbb{M}^3)$ is characteristic. If the discriminant of the characteristic polynomial of \mathbf{K} is not identically nonpositive, then it generally has real and distinct eigenvalues (at least on an open subset of \mathbb{M}^3). The verification of the normality of the eigenvectors is facilitated by the celebrated *Tonolo–Schouten–Nijenhuis (TSN) conditions*.^{28,32,36} Indeed, a symmetric tensor field K^{ij} has normal eigenvectors iff

$$\begin{aligned} N^{\ell}_{[jk} g_{i]\ell} &= 0, \\ N^{\ell}_{[jk} K_{i]\ell} &= 0, \\ N^{\ell}_{[jk} K_{i]m} K^m_{\ell} &= 0, \end{aligned} \quad (3.6)$$

where N^i_{jk} are the components of the Nijenhuis tensor²⁸ of K^{ij} defined by

$$N^i_{jk} = K^i_{\ell} K^{\ell}_{[j,k]} + K^{\ell}_{[j} K^i_{k],\ell}. \quad (3.7)$$

The special pseudo-Euclidean group $\text{SE}(2,1) = \text{SO}(2,1) \ltimes \mathbb{R}^n$, where $\text{SO}(2,1)$ is the Lie group of Lorentz transformations with positive unit determinant, is the isometry group of \mathbb{M}^3 . The group $\text{SE}(2,1)$ acts transitively on \mathbb{M}^3 according to

$$x^i = \Lambda^i_j \tilde{x}^j + \delta^i, \quad (3.8)$$

where $\Lambda^i_j \in \text{SO}(2,1)$, $\delta^i \in \mathbb{R}^3$, and x^i and \tilde{x}^i denote the original and transformed sets of pseudo-Cartesian coordinates, respectively. Acting on the basis (3.2) of Killing vectors, the group $\text{SE}(2,1)$ induces the transformation rules

$$X_i = \Lambda^j_i \tilde{X}_j, \quad R_i = \Lambda^j_i \tilde{R}_j + \mu^j_i \tilde{X}_j, \quad (3.9)$$

where

$$\mu^j_i = \epsilon^k_{\ell i} \Lambda_k^j \delta^{\ell}. \quad (3.10)$$

The nontransitive action of $\text{SE}(2,1)$ on $\mathcal{K}^1(\mathbb{M}^3)$ or $\mathcal{K}^2(\mathbb{M}^3)$ can now be derived using the transformation rules (3.9) in conjunction with (3.3) or (3.4). Explicitly, it follows that the action $\text{SE}(2,1) \odot \mathcal{K}^1(\mathbb{M}^3)$ is given by

$$\tilde{A}^i = \Lambda^j_i A^j + \mu^j_i B^j, \quad \tilde{B}^i = \Lambda^j_i B^j, \quad (3.11)$$

while the action $\text{SE}(2,1) \odot \mathcal{K}^2(\mathbb{M}^3)$ is

$$\tilde{A}^{ij} = \Lambda_k^i \Lambda_{\ell}^j A^{k\ell} + 2\Lambda_k^{(i} \mu_{\ell}^{j)} B^{k\ell} + \mu_k^i \mu_{\ell}^j C^{k\ell},$$

$$\begin{aligned}\tilde{B}^{ij} &= \Lambda_k^i \Lambda_\ell^j B^{k\ell} + \mu_k^i \Lambda_\ell^j C^{k\ell}, \\ \tilde{C}^{ij} &= \Lambda_k^i \Lambda_\ell^j C^{k\ell}.\end{aligned}\quad (3.12)$$

Having explicit forms of the representation of SE(2,1) on the two spaces of Killing tensors, we can now proceed to compute group invariants and covariants. For the case of $\mathcal{K}^1(\mathbb{M}^3)$, let us define

$$K^i = A^i + \epsilon_{kj}^i B^j x^k. \quad (3.13)$$

We note that the K^i are the components of the general Killing vector in $\mathcal{K}^1(\mathbb{M}^3)$ with respect to the natural basis. Moreover, the components (3.13) are consistent with the group action (3.11) since $K^i = \tilde{A}^i|_{\Lambda_j^i = \delta_j^i, \delta^i = x^i}$. Consequently, the general theory from Ref. 14 is applicable concerning the generation of SE(2,1)-covariants. Indeed, any scalar formed from contractions of g_{ij} , ϵ_{ijk} , K^i , and B^i is an SE(2,1)-covariant of $\mathcal{K}^1(\mathbb{M}^3)$. A set of functionally independent covariants is given by

$$\mathcal{C}_1 = B^i B_i, \quad \mathcal{C}_2 = K^i B_i = A^i B_i, \quad \mathcal{C}_3 = K^i K_i. \quad (3.14)$$

To generate SE(2,1)-covariants of $\mathcal{K}^2(\mathbb{M}^3)$, we define

$$K^{ij} = A^{ij} + 2\epsilon_{\ell k}^{(i} B^{j)k} x^\ell + \epsilon_{mk}^i \epsilon_{n\ell}^j C^{k\ell} x^m x^n, \quad (3.15)$$

$$L^{ij} = B^{ij} + \epsilon_{\ell k}^i C^{jk} x^\ell, \quad (3.16)$$

noting that they are obtained from the \tilde{A}^{ij} and \tilde{B}^{ij} transformation rules in (3.12), restricted to $\Lambda_j^i = \delta_j^i$ and $\delta^i = x^i$. Further, the K^{ij} defined in (3.15) are the components of the general Killing tensor in $\mathcal{K}^2(\mathbb{M}^3)$ with respect to the natural basis. By the general theory of Ref. 14, we therefore conclude that any scalar formed from contractions of g_{ij} , ϵ_{ijk} , K^{ij} , L^{ij} , and C^{ij} is an SE(2,1)-covariant of $\mathcal{K}^2(\mathbb{M}^3)$. Although the space $\mathcal{K}^2(\mathbb{M}^3)$ admits 17 functionally independent SE(2,1)-covariants, it turns out that only 12 fundamental covariants are required in the construction of our invariant classification scheme for the associated separable webs. In terms of the Killing tensor parameters C^{ij} and the K^{ij} and L^{ij} defined in (3.15) and (3.16), respectively, these 12 covariants are given by

$$\begin{aligned}\mathcal{C}_1 &= \text{Tr}(\mathbf{C}), \quad \mathcal{C}_2 = \text{Tr}(\mathbf{C}^2), \quad \mathcal{C}_3 = \text{Tr}(\mathbf{C}^3), \\ \mathcal{C}_4 &= \text{Tr}(\mathbf{L}^2), \quad \mathcal{C}_5 = \text{Tr}(\mathbf{L}\mathbf{L}^t), \quad \mathcal{C}_6 = \text{Tr}(\mathbf{L}\mathbf{C}\mathbf{L}^t), \\ \mathcal{C}_7 &= \text{Tr}(\mathbf{L}\mathbf{C}^2\mathbf{L}), \quad \mathcal{C}_8 = \text{Tr}(\mathbf{K}\mathbf{C}), \quad \mathcal{C}_9 = \text{Tr}(\mathbf{K}\mathbf{C}\mathbf{K}), \\ \mathcal{C}_{10} &= \text{Tr}(\mathbf{K}\mathbf{L}\mathbf{K}\mathbf{L}^t), \quad \mathcal{C}_{11} = \text{Tr}(\mathbf{K}\mathbf{L}^2), \quad \mathcal{C}_{12} = \text{Tr}(\mathbf{K}^2\mathbf{L}^2).\end{aligned}\quad (3.17)$$

The covariants $\mathcal{C}_1, \dots, \mathcal{C}_{12}$ in (3.17) are defined in terms of traces of the matrices \mathbf{C} , \mathbf{K} , and \mathbf{L} with components $(\mathbf{L}^t)^i_j = L^i_j = L^{ik} g_{kj}$, $(\mathbf{L}^t)^i_j = L^i_j = g_{jk} L^{ki}$, etc. We remind the reader that trace operator is defined with respect to the Minkowski metric, e.g., $\text{Tr}(\mathbf{C}) = C^i_i = C^{ij} g_{ji}$. The application of the covariants (3.14) and (3.17) to the problem of classifying the group orbits of the spaces $\mathcal{K}^1(\mathbb{M}^3)$ and $\mathcal{K}^2(\mathbb{M}^3)$ is treated in the next two sections.

IV. WEB SYMMETRIES, INVARIANT SUBSPACES, AND REDUCED INVARIANTS

The problems of *equivalence* and *canonical forms* are pivotal in the study of an invariant theory.³⁰ For the space $\mathcal{K}^2(\mathbb{M}^3)$, the former involves establishing if a group element $h \in \text{SE}(2,1)$ exists such that $h \cdot \mathbf{K}_1 = \mathbf{K}_2$, given $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{K}^2(\mathbb{M}^3)$. The solution to the equivalence problem therefore amounts to studying the orbit space $\mathcal{K}^2(\mathbb{M}^3)/\text{SE}(2,1)$. The concept of an orbit space evidently

defines an equivalence relation on $\mathcal{K}^2(\mathbb{M}^3)$. Thus, the canonical form problem seeks a suitably “simple” representative in each equivalence class. The selection of representatives is often determined by choosing an appropriate cross section through the orbits or fixing the frame. Eisenhart⁹ employed precisely this idea to obtain canonical forms for the CKTs in Euclidean space \mathbb{E}^3 . By solving the Killing tensor equations in the rigid frame of (normal) eigenvectors of Killing tensors, Eisenhart implicitly chose a cross section. Eisenhart’s method was recently extended to \mathbb{M}^3 in Ref. 15 culminating in 39 classes of canonical CKTs each characterizing a separable web. We shall now invariantly characterize each of the 39 classes, thereby proving their inequivalence.

The 39 canonical CKTs of \mathbb{M}^3 derived in Ref. 15 are tabulated in Appendix A. One could attempt to classify the orbits of these CKTs using $SE(2,1)$ -invariants or covariants alone. This approach was successfully employed in Ref. 14 for the analogous problem in \mathbb{E}^3 , resulting in a purely algebraic based invariant characterization of the 11 separable webs. However, several of the invariants used in the classification were rather lengthy and resource intensive Gröbner basis-type calculations were required to find them. These computational issues present in the \mathbb{E}^3 case suggest that it might not be efficient to seek a purely invariant based classification of the 39 separable webs in \mathbb{M}^3 . To compound matters, the noncompactness of the isotropy subgroup $SO(2,1)$ of $SE(2,1)$ does not even guarantee that any two inequivalent group orbits can be separated by invariants! We are thus led to find an alternate method for classifying the separable webs of \mathbb{M}^3 . Indeed, we shall adopt the strategy of Ref. 16 in which the 11 separable webs in \mathbb{E}^3 were classified using the concepts of *web symmetries* and *reduced invariants*. To apply the program of Ref. 16 to \mathbb{M}^3 , we shall first group the 39 CKTs into invariant subspaces according to the type of web symmetry they admit. Then, on each invariant subspace we will compute a new set of invariants and use these reduced invariants to classify the respective CKTs. For the sake of computation, it is more efficient to use the definition of a web symmetry first provided by Chanachowicz *et al.*⁸ rather than the definition formulated in Sec. II in terms of eigenvalues of CKTs.

Definition 4.1: A separable web defined by a CKT \mathbf{K} on a pseudo-Riemannian manifold $(\mathcal{M}, \mathbf{g})$ admits a web symmetry iff there exists a one-parameter subgroup ϕ_t of $I(\mathcal{M})$ such that

$$\phi_{t*}(\mathbf{K}) = \mathbf{K},$$

where ϕ_{t*} denotes the corresponding push-forward map.

A separable web satisfying the above definition is said to be ϕ_t -symmetric. We shall also say that any CKT defining a ϕ_t -symmetric web admits a web symmetry. The infinitesimal version of Definition 4.1 is useful for computations and is given by the following proposition.⁸

Proposition 4.2: Let V be an infinitesimal generator of a one-parameter subgroup ϕ_t of $I(\mathcal{M})$. A separable web defined by a CKT \mathbf{K} is ϕ_t -symmetric iff

$$\mathcal{L}_V \mathbf{K} = 0. \quad (4.1)$$

Remark 4.3: Definition 4.1 and Proposition 4.2 can be extended to the case when a CKT \mathbf{K} admits a homothetic web symmetry.⁸ For the case of CKTs defined in \mathbb{M}^3 , the generators of homothetic (dilatational) web symmetries are of the form $\mathbf{D} = (x^i + a^i)X_i$ where the a^i are constant. The Lie derivative condition (4.1) generalizes to

$$\mathcal{L}_D \mathbf{K} = c\mathbf{K}, \quad (4.2)$$

where c is a nonzero constant.

It follows from Proposition 4.2 and Remark 4.3 that 29 of the 39 CKTs listed in Appendix A admit a web symmetry. Of these 29 separable webs, 24 admit an isometry group symmetry, while the other 5 admit a conformal dilatational symmetry. For the 24 webs admitting an $SE(2,1)$ web symmetry, we can further divide them into five groups according to the type of Killing vector they admit using (4.1). This classification is described in Table I.

Remark 4.4: The construction of Table I is not unique as some CKTs in \mathbb{M}^3 admit more than one web symmetry. Indeed, the spacelike translational web I also admits timelike and null trans-

TABLE I. Web symmetries of the separable webs in Minkowski space.

Symmetry	Canonical vector field	Number of webs
Spacelike translational	X_2	10
Timelike translational	X_0	3
Spacelike rotational	R_2	4
Timelike rotational	R_0	4
Null rotational	R_0+R_2	3
Dilatational	$x^i X_i$	5
Asymmetric	\dots	10

lational symmetries, the spacelike translational web II admits a spacelike rotational symmetry, the timelike translational web I admits a timelike rotational symmetry, and both the spacelike, timelike, and null rotational web I admits dilatational symmetries.

It is still necessary to prove the invariance of Table I. In other words, we must show that a CKT contained in one of the seven groups in Table I remains in that same group under the action of SE(2,1). Clearly, a dilatational symmetry is preserved under the isometry group. For the remaining five groups of CKTs admitting a symmetry, it suffices to show that the orbits of the associated Killing vectors can be separated by invariants. This procedure can be accomplished by first classifying the space of Killing vectors using the fundamental SE(2,1)-covariants (3.14) of $\mathcal{K}^1(\mathbb{M}^3)$. An invariant classification scheme for $\mathcal{K}^1(\mathbb{M}^3)$ is depicted in the flowchart of Fig. 1. The figure thus demonstrates that the Killing vectors associated with the web symmetries in Table I are inequivalent. Without loss of generality, we may assume henceforth that if a given CKT admits a web symmetry, then its associated generator is one of the canonical vector fields in Table I.

The equivalence problem for the 39 separable webs in \mathbb{M}^3 can thus be reduced to seven simpler problems. Indeed, we can classify the CKTs in each of the seven groups in Table I using an appropriate set of invariants. For the ten asymmetric classes of CKTs, we shall employ the full SE(2,1)-covariants (3.17) and shall present this classification in the next section. For the six other types of web symmetries in Table I, we can determine the most general CKT which admits the given web symmetry by imposing either (4.1) or (4.2). Such CKTs define *invariant subspaces* of $\mathcal{K}^2(\mathbb{M}^3)$ under the action of the subgroup which preserves the CKT (i.e., the action which maps the subspace to itself). Once these subgroups are known, the corresponding group action on the

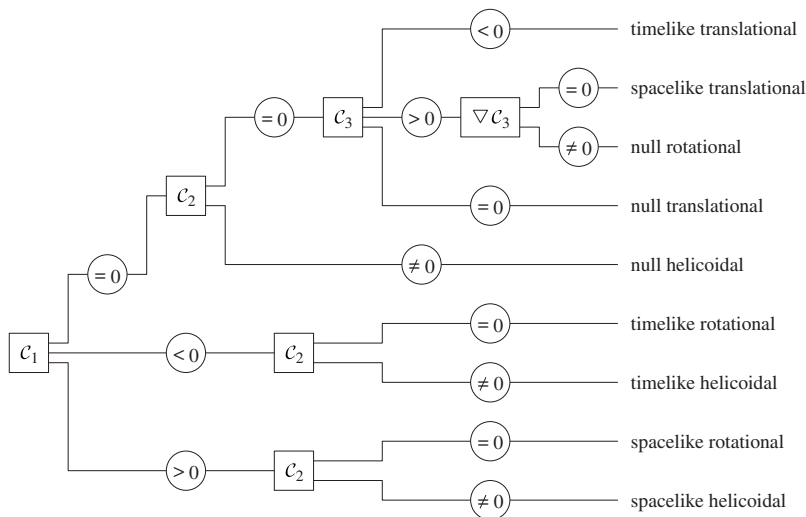


FIG. 1. Invariant classification of Killing vectors in Minkowski space. The quantities C_1 , C_2 , and C_3 refer to the SE(2,1)-covariants (3.14).

invariant subspace can be derived and *reduced invariants* can be computed. For the remainder of this section, we present the invariant subspaces for each of the six types of web symmetries and then give a set of reduced invariants. The classification schemes for these invariant subspaces are relegated to the next section.

Remark 4.5: It is straightforward to determine the subgroup H of $I(\mathcal{M})$ which maps an invariant subspace \mathcal{S} of CKTs to itself. Let V denote a generator corresponding to the web symmetry of \mathcal{S} . Clearly, U is a generator of H iff

$$\mathcal{L}_V(\mathcal{L}_U K) = 0 \tag{4.3}$$

for all $K \in \mathcal{S}$. By the Jacobi identity, (4.3) is equivalent to

$$[U, V] = cV \tag{4.4}$$

for some constant $c \in \mathbb{R}$. Therefore, H is generated by those Killing vectors of \mathcal{M} whose commutator with V is proportional to V .

A. The spacelike translational invariant subspace

The components of the most general CKT K^{ij} satisfying $\mathcal{L}_{X_2} K^{ij} = 0$ are given by

$$K^{00} = a_0 - 2b_{02}x + c_2x^2, \quad K^{11} = a_1 - 2b_{12}t + c_2t^2, \quad K^{22} = a_2,$$

$$K^{01} = \alpha_2 - b_{12}x - b_{02}t + c_2tx, \quad K^{12} = K^{20} = 0. \tag{4.5}$$

To derive (4.5), we impose the Lie derivative condition (4.1) using the components (3.15) of the general Killing tensor in \mathbb{M}^3 . This condition yields a set of linear equations in the Killing tensor parameters which are straightforward to solve. We then impose the TSN conditions (3.6) which guarantee the normality of the eigenvectors of the Killing tensor. Although the TSN conditions lead to nonlinear equations in the Killing tensor parameters, their solution is nevertheless tractable and yields (4.5).

The subgroup of $SE(2,1)$ which maps (4.5) to itself is spanned by the Killing vectors X_i , $i=0, 1, 2$, and R_2 , as follows from the commutation relation (4.4). Therefore, the restricted group action on \mathbb{M}^3 is of the form (3.8) where

$$\Lambda^i_j = \begin{pmatrix} \lambda^\alpha_\beta & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \delta^i = \begin{pmatrix} \delta^\alpha \\ 0 \end{pmatrix},$$

$\lambda^\alpha_\beta \in SO(1,1)$, and $\delta^\alpha \in \mathbb{R}^2$. Here and in what follows, all Greek indices take on the values of either 0 or 1 and the usual summation convention applies.

The study of the restricted group action on the spacelike translational invariant subspace is isomorphic to the study of the action $SE(1,1) \circ \mathcal{K}^2(\mathbb{M}^2)$. Indeed, the general valence-two Killing tensor on the Minkowski plane \mathbb{M}^2 is of the form

$$k = a^{\alpha\beta} X_\alpha \odot X_\beta + 2b^\alpha X_\alpha \odot R + cR \odot R,$$

where $a^{\alpha\beta} = a^{(\alpha\beta)}$, b^α and c are constants, $X_\alpha = \partial / \partial x^\alpha$, and $R = \epsilon^\beta_\alpha x^\alpha X_\beta$. From these definitions, it follows that the components of the general Killing tensor are

$$k^{\alpha\beta} = a^{\alpha\beta} + 2b^{(\alpha} \epsilon^{\beta)}_{\gamma} x^\gamma + c \epsilon^\alpha_\gamma \epsilon^\beta_\delta x^\gamma x^\delta. \tag{4.6}$$

Making the identification

$$a^{\alpha\beta} = \begin{pmatrix} a_0 & \alpha_2 \\ \alpha_2 & a_1 \end{pmatrix}, \quad b^\alpha = \begin{pmatrix} b_{02} \\ b_{12} \end{pmatrix}, \quad c = (c_2),$$

we observe that the upper 2×2 block of (4.5) coincides with (4.6). The group action $SE(1,1) \circlearrowleft \mathcal{K}^2(\mathbb{M}^2)$ can be derived analogously to the \mathbb{M}^3 case leading to

$$\begin{aligned} \tilde{a}^{\alpha\beta} &= \lambda_\gamma^\alpha \lambda_\delta^\beta a^{\gamma\delta} + 2\lambda_\gamma^{(\alpha} \mu^{\beta)} b^\gamma + c \mu^\alpha \mu^\beta, \\ \tilde{b}^\alpha &= \lambda_\beta^\alpha b^\beta + c \mu^\alpha, \\ \tilde{c} &= c, \end{aligned} \tag{4.7}$$

where $\mu^\alpha = \epsilon^{\beta\gamma} \lambda_\beta^\alpha \delta^\gamma$. The group action (4.7) is in the form amenable to the general theory of Ref. 14. Defining

$$\ell^\alpha = b^\alpha + c \epsilon^\alpha_\beta x^\beta, \tag{4.8}$$

it follows that any scalar formed from contractions of $g_{\alpha\beta}$, $\epsilon_{\alpha\beta}$, $k^{\alpha\beta}$, ℓ^α , and c is an $SE(1,1)$ -covariant of $\mathcal{K}^2(\mathbb{M}^2)$ or, equivalently, a set of reduced covariants of the spacelike translational invariant subspace of $\mathcal{K}^2(\mathbb{M}^3)$. A set of fundamental covariants is given by

$$C_1 = c, \quad C_2 = \ell^\alpha \ell_\alpha, \quad C_3 = k^\alpha_\alpha, \quad C_4 = k_{\alpha\beta} \ell^\alpha \ell^\beta, \quad C_5 = k^{\alpha\beta} k_{\alpha\beta}. \tag{4.9}$$

B. The timelike translational invariant subspace

The components of the most general CKT K^{ij} satisfying $\mathcal{L}_{X_0} K^{ij} = 0$ are given by

$$\begin{aligned} K^{00} &= a_0, \quad K^{11} = a_1 + 2b_{10}y + c_0y^2, \quad K^{22} = a_2 - 2b_{20}x + c_0x^2, \\ K^{12} &= \alpha_0 + b_{20}y - b_{10}x - c_0xy, \quad K^{20} = K^{01} = 0. \end{aligned} \tag{4.10}$$

The subspace of $SE(2,1)$ which leaves (4.10) invariant is spanned by the vectors X_i , $i=0, 1, 2$, and R_0 . Thus, the restricted group action on \mathbb{M}^3 is of the form (3.8) where

$$\Lambda^i_j = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^A_B \end{pmatrix}, \quad \delta^i = \begin{pmatrix} 0 \\ \delta^A \end{pmatrix},$$

$\lambda^A_B \in SO(2)$, and $\delta^A \in \mathbb{R}^2$. Here and in what follows, all uppercase Latin indices take on the values of either 1 or 2 and the usual summation convention applies.

Like the spacelike translational case, the study of the restricted group action on the timelike translational invariant subspace is also isomorphic to the study of an action on space of Killing tensors defined on a two-dimensional manifold, namely, the action $SE(2) \circlearrowleft \mathcal{K}^2(\mathbb{E}^2)$. Indeed, the general valence-two Killing tensor on the Euclidean plane \mathbb{E}^2 is of the form

$$k = a^{AB} X_A \circlearrowleft X_B + 2b^A X_A \circlearrowleft R + c R \circlearrowleft R,$$

where $a^{AB} = a^{(AB)}$, b^A and c are constants, $X_A = \partial / \partial x^A$, and $R = \epsilon^B_A x^A X_B$. From these definitions, it follows that the components of the general Killing tensor are

$$k^{AB} = a^{AB} + 2b^{(A} \epsilon^{B)}_C x^C + c \epsilon^A_C \epsilon^B_D x^C x^D. \tag{4.11}$$

Making the identification

$$a^{AB} = \begin{pmatrix} a_1 & \alpha_0 \\ \alpha_0 & a_2 \end{pmatrix}, \quad b^A = \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix}, \quad c = (c_0),$$

we observe that the lower 2×2 block of (4.10) coincides with (4.11). It follows that the group action $SE(2) \circlearrowleft \mathcal{K}^2(\mathbb{E}^2)$ is

$$\begin{aligned} \tilde{a}^{AB} &= \lambda_C^A \lambda_D^B a^{CD} + 2\lambda_C^{(A} \mu^{B)} b^C + c \mu^A \mu^B, \\ \tilde{b}^A &= \lambda_B^A b^B + c \mu^A, \\ \tilde{c} &= c, \end{aligned} \tag{4.12}$$

where $\mu^A = \epsilon^B_C \lambda_B^A \delta^C$. It is straightforward to show from (4.12) that the following quantities define $SE(2)$ -invariants of $\mathcal{K}^2(\mathbb{E}^2)$ or, equivalently, a set of reduced invariants of the timelike translational invariant subspace of $\mathcal{K}^2(\mathbb{M}^3)$:

$$\mathcal{I}_1 = c, \quad \mathcal{I}_2 = b^A b_A - c a^A_A, \quad \mathcal{I}_3 = (b^A b^B - c a^{AB})(b_A b_B - c a_{AB}). \tag{4.13}$$

C. The spacelike rotational invariant subspace

The components of the most general CKT K^{ij} satisfying $\mathcal{L}_{R_2} K^{ij} = 0$ are given by

$$\begin{aligned} K^{00} &= -a_1 + 2b_{01} + c_1 y^2 + c_2 x^2, & K^{11} &= a_1 - 2b_{01} + c_2 t^2 - c_1 y^2, \\ K^{22} &= a_2 - c_1 x^2 + c_1 t^2, & K^{01} &= c_2 t x, \\ K^{12} &= b_{01} x + c_1 x y, & K^{20} &= b_{01} t + c_1 y t. \end{aligned} \tag{4.14}$$

The only nontrivial Killing vector satisfying the commutation relation (4.4) is X_2 ; hence the restricted group action on \mathbb{M}^3 is simply

$$t = \tilde{t}, \quad x = \tilde{x}, \quad y = \tilde{y} + k,$$

where $k \in \mathbb{R}$. It follows that the restricted group action on the spacelike rotational subspace is

$$\tilde{a}_1 = a_1 - 2b_{01}k - c_1 k^2, \quad \tilde{b}_{01} = b_{01} + c_1 k, \quad \tilde{a}_2 = a_2, \quad \tilde{c}_1 = c_1, \quad \tilde{c}_2 = c_2. \tag{4.15}$$

Using the action (4.15), it is straightforward to derive reduced invariants of the spacelike rotational subspace. Two such invariants used in the classification scheme detailed in the next section are given by

$$\mathcal{I}_1 = c_1, \quad \mathcal{I}_2 = -b_{01}^2 + c_1(a_2 - a_1). \tag{4.16}$$

D. The timelike rotational invariant subspace

The components of the most general CKT K^{ij} satisfying $\mathcal{L}_{R_0} K^{ij} = 0$ are given by

$$\begin{aligned} K^{11} &= a_2 - 2b_{12}t + c_2 t^2 + c_0 y^2, & K^{22} &= a_2 - 2b_{12}t + c_0 x^2 + c_2 t^2, \\ K^{00} &= a_0 + c_2 y^2 + c_2 x^2, & K^{12} &= -c_0 x y, \end{aligned}$$

$$K^{20} = -b_{12}y + c_2yt, \quad K^{01} = -b_{12}x + c_2tx. \quad (4.17)$$

As in the spacelike rotational case, the subgroup of $SE(2,1)$ which leaves (4.17) invariant is spanned by a single translational Killing vector, namely, \mathbf{X}_0 ; hence the restricted group action on \mathbb{M}^3 is

$$t = \tilde{t} + k, \quad x = \tilde{x}, \quad y = \tilde{y},$$

where $k \in \mathbb{R}$. It follows that the restricted group action on the timelike rotational subspace is

$$\tilde{a}_2 = a_2 - 2b_{12}k + c_2k^2, \quad \tilde{b}_{12} = b_{12} - c_2k, \quad \tilde{a}_0 = a_0, \quad \tilde{c}_0 = c_0, \quad \tilde{c}_2 = c_2. \quad (4.18)$$

We can apply the action (4.18) to obtain reduced invariants of the timelike rotational subspace. Two such invariants are given by

$$\mathcal{I}_1 = c_2, \quad \mathcal{I}_2 = -b_{12}^2 + c_2(a_0 + a_2). \quad (4.19)$$

E. The null rotational invariant subspace

The components of the most general CKT K^{ij} satisfying $\mathcal{L}_{\mathbf{R}_0+\mathbf{R}_2}K^{ij}=0$ are given by

$$\begin{aligned} K^{00} &= -a_0 + 2\alpha_1 + 2b_{01}y + c_1y^2 + (\gamma_1 + c_1)x^2, \\ K^{11} &= a_0 - \alpha_1 + 2b_{01}(t - y) + (\gamma_1 + c_1)t^2 + (\gamma_1 - c_1)y^2 - 2\gamma_1yt, \\ K^{22} &= a_0 + 2b_{01}t + (\gamma_1 - c_1)x^2 + c_1t^2, \\ K^{12} &= b_{01}x + \gamma_1tx - (\gamma_1 - c_1)xy, \\ K^{20} &= \alpha_1 + b_{01}(y + t) + \gamma_1x^2 + c_1yt, \\ K^{01} &= b_{01}x - \gamma_1xy + (\gamma_1 + c_1)tx. \end{aligned} \quad (4.20)$$

The subspace of $SE(2,1)$ which leaves (4.20) invariant is spanned by the vectors $\mathbf{X}_0+\mathbf{X}_2$, \mathbf{R}_1 , and $\mathbf{R}_0+\mathbf{R}_2$; the restricted group action on \mathbb{M}^3 can be parametrized in the form

$$\begin{pmatrix} t \\ x \\ y \end{pmatrix} = \begin{pmatrix} \epsilon \cosh \psi & 0 & \sinh \psi \\ 0 & 1 & 0 \\ \sinh \psi & 0 & \epsilon \cosh \psi \end{pmatrix} \begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} k \\ 0 \\ k \end{pmatrix},$$

where $\epsilon = \pm 1$ and $\psi, k \in \mathbb{R}$. The restricted group action on the null rotational subspace can be computed as in the previous subsections. It can be shown that the function

$$\mathcal{C}_1 = (b_{01}^2 - \alpha_1c_1)(t - y)^2 \quad (4.21)$$

is a reduced covariant. In the next section, we shall use the covariant (4.21) to characterize the three classes of null rotational webs in \mathbb{M}^3 .

F. The dilatational invariant subspace

The most general CKT \mathbf{K} satisfying $\mathcal{L}_D\mathbf{K}=c\mathbf{K}$, where $D=x^i\mathbf{X}_i$ and c is a constant, can be written in the form

$$K = a_0 g^{ij} X_i \odot X_j + C^{ij} R_i \odot R_j, \quad (4.22)$$

where a_0 and $C^{ij} = C^{(ij)}$ are constant and the X_i and R_i , $i=0, 1, 2$, are the six basis Killing vectors of \mathbb{M}^3 defined in (3.2). Clearly, the group $\text{SO}(2,1)$ leaves (4.22) invariant and its action on \mathbb{M}^3 is

$$x^i = \Lambda^i_j \tilde{x}^j, \quad (4.23)$$

where $\Lambda^i_j \in \text{SO}(2,1)$. Moreover, it follows that the action of $\text{SO}(2,1)$ on the dilatational invariant subspace is

$$\tilde{a}_0 = a_0, \quad \tilde{C}^{ij} = \Lambda_k^i \Lambda_\ell^j C^{k\ell}. \quad (4.24)$$

By the elementary properties of the group $\text{SO}(2,1)$, we conclude from (4.23) and (4.24) that any scalar formed from contractions of the Minkowski metric g_{ij} , the Levi-Civita tensor ϵ_{ijk} , x^i and C^{ij} is an $\text{SO}(2,1)$ -covariant of the space of dilatational Killing tensors. A set of fundamental covariants used in the classification of the dilatational webs is given by

$$\begin{aligned} \mathcal{C}_1 &= C^i_i, & \mathcal{C}_2 &= C^i_j C^j_i, & \mathcal{C}_3 &= C^i_j C^j_k C^k_i, \\ \mathcal{C}_4 &= C_{ij} x^i x^j, & \mathcal{C}_5 &= C_{ik} C^k_j x^i x^j, & \mathcal{C}_6 &= g_{ij} x^i x^j. \end{aligned} \quad (4.25)$$

Remark 4.6: The study of the dilatational Killing tensors in \mathbb{M}^3 is equivalent to the study of the space of valence-two Killing tensors defined on two-dimensional *hyperbolic space* or two-dimensional *de Sitter space* under the action of $\text{SO}(2,1)$. Both of these manifolds are examples of homogeneous spaces of constant curvature embedded in \mathbb{M}^3 . Two-dimensional hyperbolic space \mathbb{H}^2 is a Riemannian manifold defined by one sheet of the two-sheet hyperboloid $-t^2 + x^2 + y^2 = -r^2$, where $r > 0$ is the radius of the ‘‘pseudosphere,’’ its curvature is $-r^{-2}$ everywhere. Two-dimensional de Sitter space dS_2 is a Lorentzian manifold of constant positive curvature defined by the hyperboloid of one sheet $-t^2 + x^2 + y^2 = r^2$, where $r > 0$ is constant. Viewed as homogeneous spaces,

$$\mathbb{H}^2 \simeq \text{SO}(2,1)/\text{SO}(2), \quad \text{dS}_2 \simeq \text{SO}(2,1)/\text{SO}(1,1).$$

Equation (4.22) represents the most general Killing tensor in $\mathcal{K}^2(\mathbb{H}^2)$ and $\mathcal{K}^2(\text{dS}_2)$, as seen in the ambient three-dimensional Minkowski space. In this section, we have also encountered Killing tensors defined in two-dimensional Euclidean and Minkowski space, as characterized by the translational invariant subspaces. Thus, the study of the orbits of all homogeneous subspaces of $\mathcal{K}^2(\mathbb{M}^3)$, namely, valence-two Killing tensors defined on lower dimensional manifolds of constant curvature (i.e., \mathbb{M}^2 , \mathbb{E}^2 , \mathbb{H}^2 , and dS_2), is paramount to the study of the full orbit space $\mathcal{K}^2(\mathbb{M}^3)/\text{SE}(2,1)$.

V. INVARIANT CLASSIFICATION OF SEPARABLE WEBS IN MINKOWSKI SPACE

In this section, we use the invariants derived in the previous two sections to study the orbits of the CKTs in \mathbb{M}^3 under the action of $\text{SE}(2,1)$ leading to the promised classification of the associated 39 separable webs. As demonstrated in Sec. IV, one can simplify the study of the orbit space and hence the derivation of a classification scheme by considering separately each of the six invariant subspaces of CKTs. We now give the invariant characterization of the associated separable webs in each of the six subspaces using the appropriate set of reduced invariants and classify the ten asymmetric webs using the full group covariants (3.17).

A. The spacelike translational Killing tensors

The study of the spacelike translational invariant subspace is isomorphic to the study of $\mathcal{K}^2(\mathbb{M}^2)$ under the action of $\text{SE}(1,1)$. A classification for the associated ten separable webs of \mathbb{M}^2 using group covariants was first developed by Smirnov and Yue.³⁵ For the sake of brevity, we shall

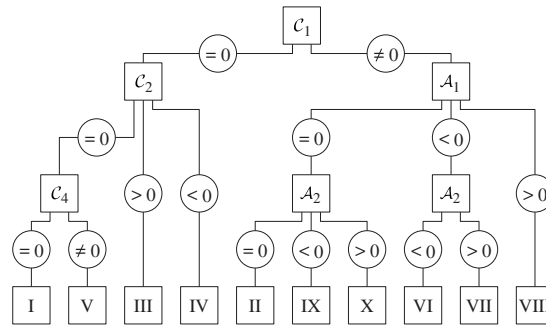


FIG. 2. Invariant classification of the spacelike translational Killing tensors in Minkowski space. The quantities \mathcal{C}_1 , \mathcal{C}_2 , and \mathcal{C}_4 refer to the reduced covariants (4.9). The auxiliary covariants \mathcal{A}_1 and \mathcal{A}_2 are defined by (5.1) and (5.2).

not give full details of this classification. Nevertheless, these ten webs can be characterized in terms of the reduced covariants (4.9) and two auxiliary covariants defined by

$$\mathcal{A}_1 = \mathcal{C}_1^2 \mathcal{C}_3^2 + 4\mathcal{C}_1 \mathcal{C}_4 - 2\mathcal{C}_1^2 \mathcal{C}_5 - \mathcal{C}_2^2 - 2\mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3, \tag{5.1}$$

$$\mathcal{A}_2 = \mathcal{C}_2^2 - 2\mathcal{C}_1 \mathcal{C}_4 + \mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3. \tag{5.2}$$

An invariant classification of the ten spacelike translational separable webs is depicted in the flowchart of Figs. 2. It is straightforward to verify by evaluating the required covariants on the canonical CKTs tabulated in Appendix A 1.

B. The timelike translational Killing tensors

The study of the timelike translational invariant subspace is isomorphic to the study of $\mathcal{K}^2(\mathbb{E}^2)$ under the action of $SE(2)$. Invariant characterizations of the four separable webs of \mathbb{E}^2 are well known (see, for example, Ref. 25) and so it is not necessary to provide full details here. For the purpose of completeness, we remark that the timelike translational Killing tensors can be classified using the reduced invariants (4.13) and one additional auxiliary invariant given by

$$\mathcal{A}_1 = \mathcal{I}_2^2 - 2\mathcal{I}_3. \tag{5.3}$$

The classification of the three timelike translational separable webs is depicted in the flowchart of Fig. 3. As mentioned in Remark 4.4, the spacelike translational web I (i.e., the pseudo-Cartesian web) also admits a timelike translational symmetry. It satisfies $\mathcal{I}_1=0$ and $\mathcal{A}_1=0$ and hence can also be inserted into the flowchart.

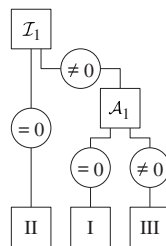


FIG. 3. Invariant classification of the timelike translational Killing tensors in Minkowski space. The reduced invariant \mathcal{I}_1 and the auxiliary invariant \mathcal{A}_1 are defined by (4.13) and (5.3), respectively.

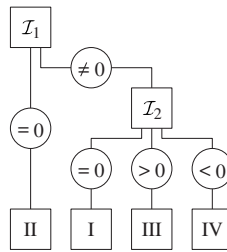


FIG. 4. Invariant classification of the spacelike or timelike rotational Killing tensors in Minkowski space. The reduced invariants \mathcal{I}_1 and \mathcal{I}_2 refer either to (4.16) or (4.19) depending on whether one is considering the spacelike or timelike case.

C. The spacelike rotational Killing tensors

Let $\mathbf{K}_1, \dots, \mathbf{K}_4$ denote the four canonical spacelike rotational Killing tensors listed in Appendix A 3. The reduced invariants (4.16) distinguish between the orbits of this invariant subspace. Indeed, $\mathcal{I}_1=0$ for \mathbf{K}_2 and $\mathcal{I}_1=c_1 \neq 0$ for the other three. For \mathbf{K}_1 , $\mathcal{I}_2=0$, while for \mathbf{K}_3 and \mathbf{K}_4 , $\mathcal{I}_2=c_1(a_2-a_1)$ which is positive and negative on these respective Killing tensors. This classification is summarized in Fig. 4.

D. The timelike rotational Killing tensors

Let $\mathbf{K}_1, \dots, \mathbf{K}_4$ denote the four canonical timelike rotational Killing tensors listed in Appendix A 4. The reduced invariants (4.19) distinguish between the orbits of this invariant subspace. Indeed, $\mathcal{I}_1=0$ for \mathbf{K}_2 and $\mathcal{I}_1=c_2 \neq 0$ for the other three. For \mathbf{K}_1 , $\mathcal{I}_2=0$, while for \mathbf{K}_3 and \mathbf{K}_4 , $\mathcal{I}_2=c_1(a_0+a_2)$ which is positive and negative on these respective Killing tensors. This classification is summarized in Fig. 4.

E. The null rotational Killing tensors

The reduced covariant \mathcal{C}_1 defined by (4.21) characterizes the three null rotational separable webs. Indeed, let $\mathbf{K}_1, \mathbf{K}_2$, and \mathbf{K}_3 denote the three canonical null rotational Killing tensors listed in Appendix A 5. It follows that $\mathcal{C}_1=0$ for \mathbf{K}_1 , $\mathcal{C}=-c_1^2(t-y)^2 < 0$ for \mathbf{K}_2 , and $\mathcal{C}_1=c_1^2(t-y)^2 > 0$ for \mathbf{K}_3 . We note that $c_1 \neq 0$ for the latter two cases; otherwise the CKTs would have a repeated eigenvalue. This classification is depicted in the flowchart of Fig. 5.

F. The dilatational Killing tensors

Let $\mathbf{K}_1, \dots, \mathbf{K}_5$ denote the five canonical dilatational CKTs tabulated in Appendix A 6. We can characterize the orbits of these Killing tensors using several auxiliary covariants defined in terms of the reduced covariants (4.25). Let

$$\mathcal{A}_1 = \mathcal{C}_1^2 - 3\mathcal{C}_2, \quad \mathcal{A}_2 = \mathcal{C}_1^3 - 9\mathcal{C}_3.$$

The pair $(\mathcal{A}_1, \mathcal{A}_2)$ vanishes identically for \mathbf{K}_1 and we claim that $(\mathcal{A}_1, \mathcal{A}_2) \neq 0$ for the other four. Indeed, for \mathbf{K}_2 and \mathbf{K}_3 , $\mathcal{A}_1 = -2\gamma_2^2$, while for \mathbf{K}_4 ,

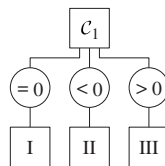


FIG. 5. Invariant classification of the null rotational Killing tensors in Minkowski space. The reduced covariants \mathcal{C}_1 is defined by (4.21).

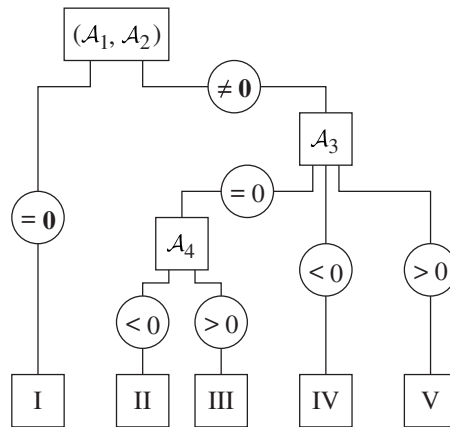


FIG. 6. Invariant classification of the dilatational Killing tensors in Minkowski space. The auxiliary reduced covariants $\mathcal{A}_1, \dots, \mathcal{A}_4$ are defined in Sec. V F.

$$\mathcal{A}_1 = -2\left(c_0 + \frac{1}{2}c_1 + \frac{1}{2}c_2\right)^2 - \frac{3}{2}(c_1 - c_2)^2.$$

Clearly, $\mathcal{A}_1 \neq 0$ for $\mathbf{K}_2, \dots, \mathbf{K}_4$; otherwise these Killing tensors would have a repeated eigenvalue. Finally, for \mathbf{K}_5 ,

$$\mathcal{A}_1 = 6\gamma_2^2 - 2(c_0 + c_2)^2.$$

If \mathcal{A}_1 were to vanish for \mathbf{K}_5 , then $\mathcal{A}_2 = -8(c_0 + c_2)^3$. Thus, $(\mathcal{A}_1, \mathcal{A}_2) \neq 0$ for \mathbf{K}_5 ; otherwise $c_0 = -c_2$ and $\gamma_2 = 0$ forcing the CKT to have repeated eigenvalues. To distinguish between $\mathbf{K}_2, \dots, \mathbf{K}_5$, we define

$$\mathcal{A}_3 = \mathcal{C}_1^6 - 9\mathcal{C}_1^4\mathcal{C}_2 + 8\mathcal{C}_1^3\mathcal{C}_3 + 21\mathcal{C}_1^2\mathcal{C}_2^2 - 36\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 - 3\mathcal{C}_2^3 + 18\mathcal{C}_3^2.$$

It follows that $\mathcal{A}_3 = 0$ for \mathbf{K}_2 and \mathbf{K}_3 , while for \mathbf{K}_4 and \mathbf{K}_5 ,

$$\mathcal{A}_3 = -6(c_0 + c_1)^2(c_0 + c_2)(c_1 - c_2)^2 < 0 \tag{5.4}$$

and

$$\mathcal{A}_3 = 24\gamma_2^2[\gamma_2^2 + (c_0 + c_2)^2]^2 > 0, \tag{5.5}$$

respectively. Finally, to distinguish between \mathbf{K}_2 and \mathbf{K}_3 , we define

$$\mathcal{A}_4 = (\mathcal{C}_1\mathcal{C}_2 - 3\mathcal{C}_3)\mathcal{C}_4 - (\mathcal{C}_1^2 - 3\mathcal{C}_2)\mathcal{C}_5 + (\mathcal{C}_1\mathcal{C}_3 - \mathcal{C}_2^2)\mathcal{C}_6.$$

Evaluating \mathcal{A}_4 on \mathbf{K}_2 and \mathbf{K}_3 , we find that $\mathcal{A}_4 = \mp 2\gamma_2^4(t+x)^2$, respectively. The invariant classification of the dilatational separable webs is summarized in the flowchart of Fig. 6.

Remark 5.1: As discussed in Remark 4.6, a separable web in \mathbb{M}^3 admitting a dilatational web symmetry also defines a separable web on \mathbb{H}^2 or dS_2 when appropriately restricted. With the inclusion of the classification of the dilatational CKTs in \mathbb{M}^3 , we essentially have a complete description of the orbit spaces $\mathcal{K}^2(\mathbb{H}^2)/SO(2,1)$ and $\mathcal{K}^2(dS_2)/SO(2,1)$. Recall that on each dilatational separable web, there exists an atlas of coordinate patches with corresponding three-metrics with respect to a system of local separable coordinates. These atlases are described explicitly in Ref. 15 for all 39 separable webs in \mathbb{M}^3 . For those atlases covering the dilatational webs, the separable metrics have one of two forms:

$$ds^2 = -du^2 + u^2(A(v) + B(w))(dv^2 + dw^2), \tag{5.6}$$

TABLE II. Dilatational separable webs in Minkowski space and their corresponding separable metrics. The metrics listed in the second column are given in Ref. 15. The third column of the table specifies whether the corresponding coordinate patch covers \mathbb{H}^2 or dS_2 .

Dilatational separable web	Separable metric	Manifold
Spacelike rotational I	Family 5.3	\mathbb{H}^2
	Family 6.2	dS_2
	Family 7.1	dS_2
Timelike rotational I	Family 5.2	\mathbb{H}^2
	Family 6.3	dS_2
Null rotational I	Family 5.1	\mathbb{H}^2
	Family 6.1	dS_2
Dilatational I	Family 10.3	\mathbb{H}^2
	Family 11.3	dS_2
Dilatational II	Family 10.2	\mathbb{H}^2
Dilatational III	Family 11.2	dS_2
Dilatational IV	Family 10.1	\mathbb{H}^2
	Family 11.1	dS_2
Dilatational V	Family 10.4	\mathbb{H}^2
	Family 11.4	dS_2

$$ds^2 = w^2(A(u) + B(v))(-du^2 + dv^2) + dw^2. \tag{5.7}$$

Here, u, v, w denote the separable coordinates and A and B are real analytic functions. The corresponding two-metrics obtained by setting $u=r=\text{const}$ in (5.6) and $w=r=\text{const}$ in (5.7) have constant curvature of $-r^{-2}$ and $+r^{-2}$, respectively, and hence define separable metrics on \mathbb{H}^2 and dS_2 , respectively. Thus, by examining the metrics for each of the dilatational separable webs, one can determine whether the corresponding coordinate patch covers \mathbb{H}^2 or dS_2 . This procedure is summarized in Table II. We conclude from this table that both \mathbb{H}^2 and dS_2 admit seven distinct classes of separable webs.

G. The asymmetric Killing tensors

We now complete our study of the orbit space $\mathcal{K}^2(\mathbb{M}^3)/\text{SE}(2,1)$ by invariantly classifying the ten asymmetric separable webs in \mathbb{M}^3 which admit no web symmetry. Let $\mathbf{K}_1, \dots, \mathbf{K}_{10}$ denote the ten canonical asymmetric CKTs listed in Appendix A 7. It is necessary to define “degenerate” versions of these classes of CKTs as the nondegenerate and degenerate cases will often need to be treated separately. That being said, we define

$$\begin{aligned} \hat{\mathbf{K}}_1 &= \mathbf{K}_1|_{c_1=0}, & \hat{\mathbf{K}}_2 &= \mathbf{K}_2|_{c_0=0}, & \hat{\mathbf{K}}_3 &= \mathbf{K}_3|_{c_2=0}, \\ \hat{\mathbf{K}}_4 &= \mathbf{K}_4|_{c_0=0, b_{12}+b_{21}=0}, & \hat{\mathbf{K}}_5 &= \mathbf{K}_5|_{c_2=0, b_{10}+b_{01}=0}, & \hat{\mathbf{K}}_6 &= \mathbf{K}_6|_{c_2=0, \beta_1=0}, \\ \hat{\mathbf{K}}_7 &= \mathbf{K}_7|_{c_0+2\gamma_0=0}, & \hat{\mathbf{K}}_{8a} &= \mathbf{K}_8|_{\gamma_2=0}, & \hat{\mathbf{K}}_{8b} &= \mathbf{K}_8|_{c_0+\gamma_2=0, a_0+a_2+\alpha_2=0}, \\ \hat{\mathbf{K}}_9 &= \mathbf{K}_9|_{c_0+c_2=0, c_1-c_2=0}, & \hat{\mathbf{K}}_{10} &= \mathbf{K}_{10}|_{c_0-c_2=0, \gamma_2=0}. \end{aligned}$$

Note that the “unhatted” Killing tensors defined in Appendix A 7 are just the complement of the set of “hatted” tensors defined above (e.g., for the CKT $\mathbf{K}_1, c_1 \neq 0$). An unhatted CKT and its hatted counterpart still define the same type of separable web. In what follows, the quantities $\mathcal{C}_1, \dots, \mathcal{C}_{12}$ will refer to the full $\text{SE}(2,1)$ -covariants of $\mathcal{K}^2(\mathbb{M}^3)$ defined in (3.17).

To begin, we define an auxiliary invariant

$$\mathcal{A}_1 = \mathcal{C}_1^6 - 9\mathcal{C}_1^4\mathcal{C}_2 + 8\mathcal{C}_1^3\mathcal{C}_3 + 21\mathcal{C}_1^2\mathcal{C}_2^2 - 36\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 - 3\mathcal{C}_2^3 + 18\mathcal{C}_3^2.$$

This invariant vanishes for all of the asymmetric CKTs in exception to \mathbf{K}_9 and \mathbf{K}_{10} . For these CKTs, \mathcal{A}_1 evaluates to the right-hand sides of (5.4) and (5.5), respectively. Therefore, \mathcal{A}_1 is negative for \mathbf{K}_9 and positive for \mathbf{K}_{10} . Next, we evaluate \mathcal{C}_2 and note that it vanishes identically for $\mathbf{K}_1, \hat{\mathbf{K}}_1, \mathbf{K}_2, \hat{\mathbf{K}}_2, \hat{\mathbf{K}}_3, \hat{\mathbf{K}}_4, \hat{\mathbf{K}}_5, \hat{\mathbf{K}}_6$, and $\hat{\mathbf{K}}_7$. It is straightforward to show that $\mathcal{C}_2 \neq 0$ for the remaining CKTs. We now split the analysis into two cases: $\mathcal{C}_2=0$ and $\mathcal{C}_2 \neq 0$.

Case I: Suppose $\mathcal{C}_2=0$. We define another auxiliary covariant given by

$$\mathcal{A}_2 = \mathcal{C}_4 + \mathcal{C}_5.$$

It follows that $\mathcal{A}_2=0$ for $\hat{\mathbf{K}}_1, \dots, \hat{\mathbf{K}}_6$, while for $\mathbf{K}_1, \mathbf{K}_2$, and $\hat{\mathbf{K}}_7$, \mathcal{A}_2 evaluates to $192c_1^2$, $-16c_0^2(t-x)$, and $3\gamma_0^2(t+x)^2$, respectively, all of which are nonvanishing. There are now two subcases to consider.

Case I.1: Suppose $\mathcal{A}_2=0$. The covariant \mathcal{C}_4 vanishes for $\hat{\mathbf{K}}_1$ and $\hat{\mathbf{K}}_2$, while for $\hat{\mathbf{K}}_3, \hat{\mathbf{K}}_5$, and $\hat{\mathbf{K}}_6$, $\mathcal{C}_4=2b_{01}^2 > 0$, and for $\hat{\mathbf{K}}_4$, $\mathcal{C}_4=-2b_{01}^2 < 0$. To distinguish between $\hat{\mathbf{K}}_1$ and $\hat{\mathbf{K}}_2$, we observe that $\mathcal{C}_{11}=0$ for the former and $\mathcal{C}_{11}=-8(a_0+a_2)^3 \neq 0$ for the latter. Finally, to distinguish between $\hat{\mathbf{K}}_3, \hat{\mathbf{K}}_5$, and $\hat{\mathbf{K}}_6$, we define

$$\mathcal{A}_3 = \mathcal{C}_{11}^2 + \mathcal{C}_4\mathcal{C}_{10}.$$

It follows that $\mathcal{A}_3=0$ for $\hat{\mathbf{K}}_3$, while for $\hat{\mathbf{K}}_5$, and $\hat{\mathbf{K}}_6$, $\mathcal{A}_3=b_{01}^4(a_0+a_1)^2 > 0$ and $\mathcal{A}_3=-4\alpha_2^2b_{01}^4 < 0$, respectively.

Case I.2: Suppose $\mathcal{A}_2 \neq 0$. We define the auxiliary covariant

$$\mathcal{A}_4 = \mathcal{C}_4 - 2\mathcal{C}_8.$$

For \mathbf{K}_1 and \mathbf{K}_2 , $\mathcal{A}_4=0$, while for $\hat{\mathbf{K}}_7$, $\mathcal{A}_4=6\gamma_0^2(t+x)^2 > 0$. Finally, to distinguish between \mathbf{K}_1 and \mathbf{K}_2 , we note that $\mathcal{C}_6=0$ for the former and $\mathcal{C}_6=64c_0^3 \neq 0$ for the latter.

Case II: Suppose $\mathcal{C}_2 \neq 0$ and define

$$\mathcal{A}_5 = \mathcal{C}_1^2 - 3\mathcal{C}_2.$$

It can be verified that $\mathcal{A}_5=0$ for $\mathbf{K}_7, \hat{\mathbf{K}}_{8a}, \hat{\mathbf{K}}_9$, and $\hat{\mathbf{K}}_{10}$, and is nonvanishing for $\mathbf{K}_3, \dots, \mathbf{K}_6, \mathbf{K}_8$ and $\hat{\mathbf{K}}_{8a}$. We now split the analysis into two subcases.

Case II.1: Suppose $\mathcal{A}_5=0$. Evaluating \mathcal{A}_2 (as defined in Case I) on the CKTs contained in this case, we find that it vanishes for $\hat{\mathbf{K}}_{8a}, \hat{\mathbf{K}}_9$, and $\hat{\mathbf{K}}_{10}$, while $\mathcal{A}_2=3\gamma_0^2(t+x)^2 \neq 0$ for \mathbf{K}_7 . To distinguish between $\hat{\mathbf{K}}_{8a}, \hat{\mathbf{K}}_9$, and $\hat{\mathbf{K}}_{10}$, a rather complicated auxiliary covariant is required which is defined in terms of other covariants and transvectants. We let

$$\mathcal{A}_6 = \mathcal{C}_4^2 - 2\mathcal{C}_9, \quad \mathcal{A}_7 = \square \mathcal{A}_6, \quad \mathcal{A}_8 = \square(g^{ij}(\partial_i \mathcal{A}_6)(\partial_j \mathcal{A}_6)),$$

$$\mathcal{A}_9 = \text{Hess } \mathcal{A}_6, \quad \mathcal{A}_{10} = 3\mathcal{C}_4^3 - 4\mathcal{C}_2\mathcal{C}_{12},$$

$$\mathcal{A}_{11} = 128\mathcal{C}_1^4(\mathcal{A}_{10} - 3\mathcal{C}_4\mathcal{A}_6) + 243\mathcal{C}_4(\mathcal{A}_7^2 - 2\mathcal{A}_8), \quad \mathcal{A}_{12} = \square \mathcal{A}_{11},$$

$$\mathcal{A}_{13} = \mathcal{A}_7\mathcal{A}_{12}(\mathcal{A}_7^3 + 20\mathcal{A}_9) - 54\mathcal{C}_1^2(\mathcal{A}_8^3 - 5\mathcal{A}_7^2\mathcal{A}_8^2 - 88\mathcal{A}_7\mathcal{A}_8\mathcal{A}_9 - 432\mathcal{A}_9^2).$$

In these definitions, \square and Hess refer to the d'Alembertian and Hessian determinant invariant differential operators, defined by

$$\square C = g^{ij} \partial_i \partial_j C, \quad \text{Hess } C = \frac{1}{3!} \epsilon^{ikm} \epsilon^{jln} (\partial_i \partial_j C) (\partial_k \partial_l C) (\partial_m \partial_n C).$$

The auxiliary covariants or “transvectants” \mathcal{A}_7 , \mathcal{A}_8 , \mathcal{A}_9 , and \mathcal{A}_{12} are indeed SE(2,1)-covariants because the partial derivative operator ∂_i transforms like a tensor, i.e., $\tilde{\partial}_i = \Lambda_i^j \partial_j$. It follows that $\mathcal{A}_{13} = 0$ for $\hat{\mathbf{K}}_{8a}$, while for $\hat{\mathbf{K}}_9$ and $\hat{\mathbf{K}}_{10}$,

$$\mathcal{A}_{13} = -2^{23} \cdot 3^5 c_2^{20} (a_0 + a_1)^2 (a_0 + a_2)^2 (a_1 - a_2)^2 < 0$$

and

$$\mathcal{A}_{13} = 2^{25} \cdot 3^5 c_2^{20} \alpha_2^2 [\alpha_2^2 + (a_0 - a_2)^2] > 0,$$

respectively.

Case II.2: Suppose $\mathcal{A}_5 \neq 0$. We define two additional auxiliary invariants given by

$$\mathcal{A}_{14} = \mathcal{C}_1^2 - \mathcal{C}_2, \quad \mathcal{A}_{15} = \mathcal{C}_1^3 - \mathcal{C}_3.$$

The pair $(\mathcal{A}_{14}, \mathcal{A}_{15})$ vanishes identically for $\mathbf{K}_3, \dots, \mathbf{K}_6$ and $\hat{\mathbf{K}}_{8a}$, while for \mathbf{K}_8

$$\mathcal{A}_{14} = 2(c_0 + \gamma_2)(3c_0 + 5\gamma_2), \quad \mathcal{A}_{15} = -6(c_0 + \gamma_2)(2c_0 + 3\gamma_2)^2.$$

Clearly, $(\mathcal{A}_{14}, \mathcal{A}_{15}) \neq 0$ for \mathbf{K}_8 ; otherwise it would reduce to $\hat{\mathbf{K}}_{8a}$. We now let

$$\mathcal{A}_{16} = \mathcal{C}_1 \mathcal{C}_6 - \mathcal{C}_2 (\mathcal{C}_4 + \mathcal{C}_5) + 2\mathcal{C}_7.$$

This auxiliary covariant vanishes for \mathbf{K}_3 and $\hat{\mathbf{K}}_{8a}$, while for \mathbf{K}_5 , $\mathcal{A}_{16} = c_2^2 (b_{01} + b_{10})^2 > 0$, for \mathbf{K}_4 , $\mathcal{A}_{16} = -c_0^2 (b_{12} + b_{21})^2 < 0$, and for \mathbf{K}_6 , $\mathcal{A}_{16} = -4c_2^2 \beta_1^2 < 0$. It remains to distinguish between the pair \mathbf{K}_3 and $\hat{\mathbf{K}}_{8a}$ and the pair \mathbf{K}_4 and \mathbf{K}_6 . For the former, we note that $\mathcal{C}_7 = 0$ for \mathbf{K}_3 and $\mathcal{C}_7 = -\gamma_2^4 (t + x)^2 \neq 0$ for $\hat{\mathbf{K}}_{8a}$. It does not appear possible to distinguish between the orbits of \mathbf{K}_4 and \mathbf{K}_6 using covariants alone. The lack of discriminating power in the covariants is manifested by the fact that no SE(2,1)-covariant can distinguish between the quadratics $x^2 + y^2$ and $-t^2 + x^2$. Indeed, the d'Alembertian and Hessian determinant of these quadratics are identical and all other transvectants are proportional to the original quadratics. To complete the classification of the asymmetric webs, we can extend the definition of a web symmetry to a covariant. If we evaluate the covariant \mathcal{C}_6 on the canonical CKTs \mathbf{K}_4 and \mathbf{K}_6 , then it follows that $\mathcal{L}_{R_0} \mathcal{C}_6|_{\mathbf{K}_4} = 0$ and $\mathcal{L}_{R_2} \mathcal{C}_6|_{\mathbf{K}_6} = 0$. Therefore, the covariant \mathcal{C}_6 admits a timelike rotational symmetry for \mathbf{K}_4 and a spacelike rotational symmetry for \mathbf{K}_6 ; this is a manifestly invariant statement.

We have now invariantly characterized the orbits of the asymmetric separable webs in \mathbb{M}^3 . A flowchart depicting the classification scheme is given in Fig. 7.

VI. THE MOVING FRAME MAP

At the heart of Cartan geometry and the problem of equivalence is the construction of the moving frame map. In the previous section, we addressed the problem of how to determine if two given CKTs $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{K}^2(\mathbb{M}^3)$ are equivalent by constructing a purely algebraic invariant classification scheme for the associated 39 separable webs in \mathbb{M}^3 . If \mathbf{K}_1 and \mathbf{K}_2 are equivalent in the sense that they characterize the *same* separable web, then it is desirable to determine the explicit group action $h \in \text{SE}(2,1)$ which maps \mathbf{K}_1 to \mathbf{K}_2 . In particular, if one of the two CKTs is one of the 39 classes of canonical forms, then the group element h is the moving frame map, i.e., the map which sends any point on a orbit to its canonical form. The moving frame map is also of key importance in the solution of the HJ equation by orthogonal separation of variables. As a separable web is defined up to an isometry, there is no guarantee that its associated CKT is in one of the canonical forms. The transformation of the CKT to canonical form leads directly to the transformation to separable coordinates for the HJ equation.

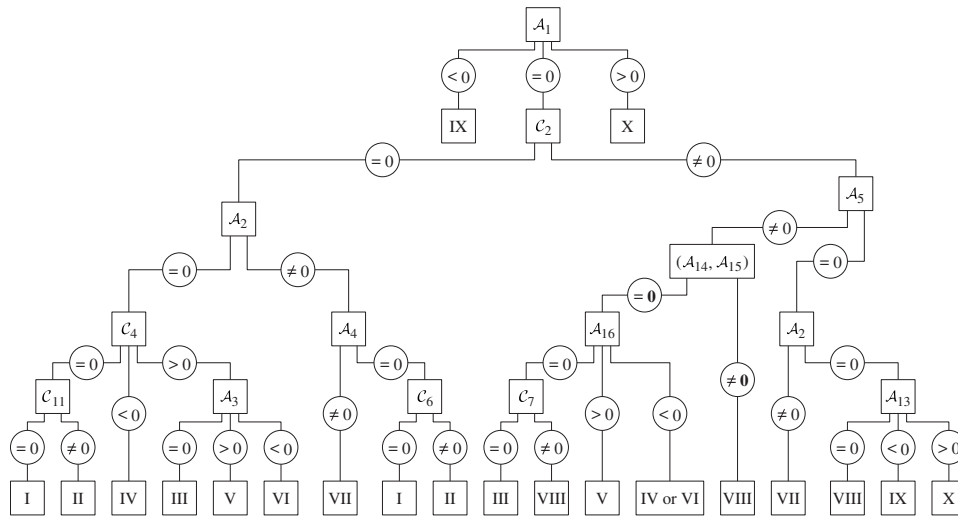


FIG. 7. Invariant classification of the asymmetric Killing tensors in Minkowski space. The auxiliary covariants $\mathcal{A}_1, \dots, \mathcal{A}_{16}$ are defined in Sec. V G.

The moving frame map is found by determining the explicit Lorentz transformation $\Lambda_j^i \in \text{SO}(2,1)$ and translation $\delta^i \in \mathbb{R}^3$ which casts a given CKT K^{ij} to its appropriate canonical form \tilde{K}^{ij} . The group action $\text{SE}(2,1) \circ \mathcal{K}^2(\mathbb{M}^3)$ given by (3.12) relates the Killing tensor parameters A^{ij} , B^{ij} , and C^{ij} of K^{ij} to the transformed parameters \tilde{A}^{ij} , \tilde{B}^{ij} , and \tilde{C}^{ij} of the canonical form \tilde{K}^{ij} through the group parameters. In order to efficiently solve the resulting system of algebraic equations for the group parameters Λ_j^i and δ^i , an appropriate parametrization of $\text{SO}(2,1)$ is required. Such a parametrization is trivial when the Killing tensor admits an isometry group web symmetry in which case the determination of the moving frame map amounts to simple algebra. In what follows, we shall only present the results for these cases and not give full details of the derivation. For those CKTs admitting a dilatational web symmetry or no symmetry at all, an appropriate parametrization for the Lorentz transformation Λ_j^i can usually be deduced from the eigenvectors of the C^{ij} parameter matrix. Moreover, it is usually necessary to transform one of the eigenvectors into a canonical spacelike, timelike, or null vector. Upon substituting this parametrization of $\text{SO}(2,1)$ back into the group action, the solution to the resulting system of nonlinear equations for the parameters of the Lorentz transformation, the translation components, and the Killing tensor parameters of the canonical form is tractable in all cases. We review the solution to the generalized eigenproblem and the problem of transforming a Lorentz three-vector to canonical form in Appendix B.

The section is structured as follows. In Sec. VI A, we discuss the moving frame map for the space $\mathcal{K}^1(\mathbb{M}^3)$ in view of Table I describing the types of web symmetries. For the five invariant subspaces of CKTs admitting $\text{SE}(2,1)$ web symmetries, we give the explicit moving frame map in Secs. VI B–VI F. Finally, the moving frame map is derived for the dilatational and asymmetric cases in Secs. VI G and VI H, respectively.

A. The space of Killing vectors

Although one could give the moving frame map corresponding to all nine classes of Killing vectors in \mathbb{M}^3 (see Fig. 1), for the sake of brevity, we shall only consider those Killing vectors which characterize web symmetries of CKTs. We have already seen that if a CKT admits an isometry group web symmetry, then its associated generator (Killing vector) could be spacelike or timelike translational or spacelike, timelike, or null rotational. Canonical forms for each of these five classes are given in Table I. We now give the moving frame map for each of these cases using

the group action $SE(2,1) \circlearrowleft \mathcal{K}^1(\mathbb{M}^3)$ specified by (3.11). For all cases, let A^i and B^i denote the parameters of the given Killing vector, as defined in (3.3), and \tilde{A}^i and \tilde{B}^i denote the parameters of the canonical form.

- (1) For a spacelike translational Killing vector, $\tilde{A}^i=(0,0,\tilde{a}_2)$ and $\tilde{B}^i=0$, where $\tilde{a}_2=(g_{ij}A^iA^j)^{1/2}$. The components of Λ^i_j are given by (B3) with $v^i=A^i/\tilde{a}_2$; the parameters ϵ and ψ are arbitrary. The components of the translation δ^i are also arbitrary.
- (2) For a timelike translational Killing vector, $\tilde{A}^i=(\tilde{a}_0,0,0)$ and $\tilde{B}^i=0$, where $\tilde{a}_0=(-g_{ij}A^iA^j)^{1/2}$. The components of Λ^i_j are given by (B4) with $v^i=A^i/\tilde{a}_0$; the parameter ψ is arbitrary. The components of δ^i are also arbitrary.
- (3) For a spacelike rotational Killing vector, $\tilde{A}^i=0$ and $\tilde{B}^i=(0,0,\tilde{b}_2)$, where $\tilde{b}_2=(g_{ij}B^iB^j)^{1/2}$. The components of Λ^i_j are given by (B3) with $v^i=B^i/\tilde{b}_2$; the parameters ϵ and ψ are arbitrary. The transformation rule for the A^i in (3.11) is a system of linear equations for the components of the translation δ^i which can be solved.
- (4) For a timelike rotational Killing vector, $\tilde{A}^i=0$ and $\tilde{B}^i=(\tilde{b}_0,0,0)$, where $\tilde{b}_0=(-g_{ij}B^iB^j)^{1/2}$. The components of Λ^i_j are given by (B4) with $v^i=B^i/\tilde{b}_0$; the parameter ψ is arbitrary. As in the spacelike rotational case, the components of δ^i are determined from the transformation rule for the A^i in (3.11).
- (5) For a null rotational Killing vector, $\tilde{A}^i=0$ and $\tilde{B}^i=(1,0,1)$. The components of Λ^i_j are formed from the composition of two Lorentz transformations, viz. $\Lambda=\Lambda_2\Lambda_1$. Here, Λ_1 is the elementary rotation

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and Λ_2 is formed from (B5⁺) with $v^i=B^i/B^0$. The parameter κ in Λ_2 may be arbitrarily chosen while $\epsilon e^{\epsilon\psi}=B^0$. Finally, the components of the translation δ^i are determined from the transformation rule for the A^i in (3.11).

B. The spacelike translational Killing tensors

The restricted group action on the spacelike translational invariant subspace is given by (4.7). We parametrize the subgroup $SO(2,1)$ according to

$$\Lambda^i_j = \begin{pmatrix} \epsilon \cosh \psi & \sinh \psi & 0 \\ \sinh \psi & \epsilon \cosh \psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\psi \in \mathbb{R}$ and $\epsilon = \pm 1$. We now give the boost parameter ψ , the sign ϵ , and the translation components δ^0 and δ^1 for each of the ten classes of spacelike translational CKTs.

- (I) The components of such a CKT are constant (it characterizes the pseudo-Cartesian web) and the group action (3.12) reduces to $\tilde{A}^{ij}=\Lambda_k^i\Lambda_\ell^jA^{k\ell}$, where \tilde{A}^{ij} is diagonal. Therefore, the moving frame map can be computed by solving an eigenproblem, as detailed in Appendix B 1. The translation δ^i can be arbitrarily chosen.
- (II) ψ and ϵ are arbitrary, $\delta^0=b_{12}/c_2$, and $\delta^1=b_{02}/c_2$.
- (III)

$$\tanh \epsilon\psi = \frac{b_{02}}{b_{12}}, \quad \delta^0 = \frac{b_{12}(a_0 + a_1) - 2\alpha_2 b_{02}}{2(b_{12}^2 - b_{02}^2)}, \quad \text{and} \quad \delta^1 = \frac{b_{02}(a_0 + a_1) - 2\alpha_2 b_{12}}{2(b_{02}^2 - b_{12}^2)}.$$

- (IV) As in (III) but with $\tanh \epsilon\psi=b_{12}/b_{02}$.
- (V)

$$e^{\epsilon\psi} = 2\epsilon/b_{02}, \quad \delta^0 - \delta^1 = \frac{(a_0 + a_1 + 2\alpha_2)b_{02}^4 - 16b_{02}^2 - 16(a_0 + a_1 - 2\alpha_2)}{64b_{02}}, \quad \text{and}$$

$$\delta^0 + \delta^1 = \frac{a_0 - a_1}{2b_{02}}.$$

(VI) $\delta^0 = b_{12}/c_2$, $\delta^1 = b_{02}/c_2$, and $e^{4\epsilon\psi} = \sigma_1/\sigma_2$ where

$$\sigma_1 = b_{12}^2 + b_{02}^2 - c_2(a_0 + a_1) + 2(b_{12}b_{02} - \alpha_2c_2),$$

$$\sigma_2 = b_{12}^2 + b_{02}^2 - c_2(a_0 + a_1) - 2(b_{12}b_{02} - \alpha_2c_2).$$

(VII) As in (VI).

(VIII) As in (VI) and (VII) but with $e^{4\epsilon\psi} = -\sigma_1/\sigma_2$.

(IX) As in (VI)-(VIII) but with $e^{2\epsilon\psi} = -\sigma_1/c_2^2$.

(X) As in (VI)-(IX) but with $e^{2\epsilon\psi} = \sigma_1/c_2^2$.

C. The timelike translational Killing tensors

The restricted group action on the timelike translational invariant subspace is given by (4.12). We parametrize the subgroup $SO(2,1)$ according to

$$\Lambda^i_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{pmatrix},$$

where $\psi \in \mathbb{R}$. We now give the rotation parameter ψ and the translation components δ^1 and δ^2 for each of the three classes of spacelike translational CKTs.

(I) ψ is arbitrary, $\delta^1 = b_{20}/c_0$, and $\delta^2 = -b_{10}/c_0$.

(II) $\tan \psi = -b_{10}/b_{20}$ for $b_{20} \neq 0$, $\psi = \pi/2$ (unique mod π) for $b_{20} = 0$,

$$\delta^1 = \frac{b_{20}(a_2 - a_1) + 2\alpha_0 b_{10}}{2(b_{10}^2 + b_{20}^2)} \quad \text{and} \quad \delta^2 = \frac{b_{10}(a_2 - a_1) - 2\alpha_0 b_{20}}{2(b_{10}^2 + b_{20}^2)}.$$

(III) As in (I) but with $\tan \psi = (\sigma_1 - \sqrt{\sigma_1^2 + 4\sigma_2^2})/(2\sigma_2)$, where

$$\sigma_1 = b_{20}^2 - b_{10}^2 + c_0(a_1 - a_2), \quad \sigma_2 = b_{20}b_{10} - \alpha_0c_0.$$

If $\sigma_2 = 0$, then $\psi = 0$ for $\sigma_1 > 0$; otherwise $\psi = \pi/2$ (unique mod π).

D. The spacelike rotational Killing tensors

The restricted group action on the spacelike rotational invariant subspace is given by (4.15). Since the invariant subgroup acting in this case involves only a single translation in terms of a parameter k , the computation of the moving frame map becomes a trivial calculation. For the spacelike rotational CKTs (I), (III), and (IV), it follows that $k = -b_{01}/c_1$, while for the CKT (II), $k = (a_1 - a_2)/(2b_{01})$.

E. The timelike rotational Killing tensors

The restricted group action on the timelike rotational invariant subspace is given by (4.18). As in the spacelike rotational case, the action is parametrized in terms of a single parameter k . For the timelike rotational CKTs (I), (III), and (IV), the moving frame map is found to be $k = b_{12}/c_2$, while for the CKT (II), $k = (a_0 + a_2)/(2b_{12})$.

F. The null rotational Killing tensors

Although we did not provide explicit details of the restricted group action on the null rotational invariant subspace in Sec. IV E, the calculation of the moving frame map is nevertheless straightforward to compute for all three classes of null rotational CKTs. We parametrize the subgroup $SO(2,1)$ according to

$$\Lambda_j^i = \begin{pmatrix} \epsilon \cosh \psi & 0 & \sinh \psi \\ 0 & 1 & 0 \\ \sinh \psi & 0 & \epsilon \cosh \psi \end{pmatrix},$$

where $\psi \in \mathbb{R}$ and $\epsilon = \pm 1$ and the translation, viz. $\delta^i = (k, 0, k)$, where $k \in \mathbb{R}$. It follows that $k = -b_{01}/c_1$ in all three cases. For the null rotational CKT (I), ψ and ϵ can be arbitrarily chosen, while for the CKTs (II) and (III), $e^{\epsilon\psi} = (\alpha_1 c_1 - b_{01}^2)/c_1^2$ and $e^{\epsilon\psi} = (b_{01}^2 - \alpha_1 c_1)/c_1^2$, respectively.

G. The dilatational Killing tensors

As developed in Sec. IV F, the restricted group action on the dilatational invariant subspace is given by

$$\tilde{C}^{ij} = \Lambda_k^i \Lambda_\ell^j C^{k\ell}. \quad (6.1)$$

Note that only the subgroup $SO(2,1)$ acts on this subspace; thus $\delta^i = 0$ in the moving frame map. Moreover, the transformation rules (6.1) for the C^{ij} Killing tensor parameter matrix are in the form of a generalized eigenproblem, the solution of which is reviewed in Appendix B 1. Indeed, if the matrix product \mathbf{gCg} (whose components are C_{ij} , i.e., the given C^{ij} with both indices lowered with the Minkowski metric) is diagonalizable, then its eigenvectors (when suitably normalized) uniquely determine the Lorentz transformation Λ_j^i . If \mathbf{gCg} is not diagonalizable, it will still admit one nontrivial eigenvector which can be used to suitably parametrize Λ_j^i . These parametrizations are derived in Appendix B and are given by one of (B3)–(B5). Upon substituting the compatible parametrization back into (6.1), one can solve for the unknown group parameters as well as the Killing tensor parameters appearing in the canonical form parameter matrix \tilde{C}^{ij} . Although the resulting equations are nonlinear, they nevertheless reduce to polynomial equations and can always be solved in closed form. We shall not give the explicit solution to these equations. In what follows, for each of the five classes of dilatational CKTs, we give the correct parametrization for the Lorentz transformation Λ_j^i , which can be deduced from the eigenvectors of \mathbf{gCg} .

- (I) \mathbf{gCg} admits one null eigenvector $\tilde{v}^i = (1, -1, 0)$; Λ_j^i is parametrized by (B5⁻), where v^i is the null eigenvector of \mathbf{gCg} normalized such that $v^0 = 1$.
- (II) \mathbf{gCg} admits one spacelike eigenvector $\tilde{v}^i = (0, 0, 1)$; Λ_j^i is parametrized by (B3), where v^i is the spacelike eigenvector of \mathbf{gCg} normalized such that $g_{ij}v^i v^j = 1$.
- (III) As in (II)
- (IV) The eigenvalues of \mathbf{gCg} are necessarily real and distinct; Λ_j^i is uniquely determined by solving the associated eigenproblem for \mathbf{gCg} , as detailed in Appendix B 1. Note that it might be necessary to relabel the parameters \tilde{c}_1 and \tilde{c}_2 in the canonical form, so that the constraints $-\tilde{c}_0 < \tilde{c}_2 < \tilde{c}_1$ or $-\tilde{c}_0 > \tilde{c}_2 > \tilde{c}_1$ are satisfied.
- (V) As in (II) and (III)

H. The asymmetric Killing tensors

The Killing tensor parameter matrices associated with an asymmetric CKT are generally arbitrary and the full group action (3.12) is required in order to compute the moving frame map. As in the dilatational case, we begin with the transformation rules for the C^{ij} parameter matrix [see (6.1)] by computing the generalized eigenvectors of the matrix \mathbf{gCg} . This calculation imposes restrictions on the components of the Lorentz transformation Λ_j^i and, in many cases, determines

Λ^i_j up to a single parameter. Once this parametrization is substituted back into the remaining transformation rules in (3.12), one can solve for the Lorentz transformation parameter(s), the components of the translation δ^i , and the Killing tensor parameters appearing in the canonical form. This final step usually requires solving a system of nonlinear equations, but the solution is always tractable for the ten classes of asymmetric CKTs. We now show how an appropriate parametrization for Λ^i_j can be found for each of the ten cases. In what follows, the functions C_i refer to the full SE(2,1)-covariants defined in (3.17) and the \mathcal{A}_i refer to the auxiliary covariants defined in Sec. V G.

- (I) If $\mathcal{A}_2=0$, then $\tilde{C}^{ij}=0$ in the canonical form. In this case, the transformation rules for the B^{ij} collapse to an eigenproblem. It follows that $\mathbf{g}\tilde{\mathbf{B}}\mathbf{g}$ has a zero eigenvalue with a one-dimensional eigenspace spanned by the null eigenvector $\tilde{v}^i=(1,-1,0)$. Therefore, Λ^i_j is parametrized by (B5⁻), where v^i is the null eigenvector of $\mathbf{g}\mathbf{B}\mathbf{g}$ normalized such that $v^0=1$. Similarly, for the case $\mathcal{A}_2\neq 0$, we have $\tilde{C}^{ij}\neq 0$. Indeed, $\mathbf{g}\tilde{\mathbf{C}}\mathbf{g}$ also admits the null eigenvector $\tilde{v}^i=(1,-1,0)$; Λ^i_j is parametrized by (B5⁻), where v^i is the null eigenvector of $\mathbf{g}\mathbf{C}\mathbf{g}$ normalized such that $v^0=1$.
- (II) As in (I), except that the null eigenvector of $\mathbf{g}\tilde{\mathbf{B}}\mathbf{g}$ (for the case $\mathcal{A}_2=0$) and $\mathbf{g}\tilde{\mathbf{C}}\mathbf{g}$ (for the case $\mathcal{A}_2\neq 0$) is $\tilde{v}^i=(1,1,0)$; Λ^i_j is parametrized by (B5⁺).
- (III) If $\mathcal{C}_2=0$, then $\tilde{c}_2=0$ in the canonical form and the transformation rules for the B^{ij} collapse to an eigenproblem. It follows that $\mathbf{g}\tilde{\mathbf{B}}\mathbf{g}$ admits one zero eigenvalue with a spacelike eigenspace spanned by $\tilde{v}^i=(0,0,1)$; Λ^i_j is parametrized by (B3), where v^i is the spacelike eigenvector corresponding to the zero eigenvalue of $\mathbf{g}\mathbf{B}\mathbf{g}$ normalized such that $g_{ij}v^iv^j=1$. If $\mathcal{C}_2\neq 0$, then $\mathbf{g}\tilde{\mathbf{C}}\mathbf{g}$ admits one nonzero eigenvalue with a spacelike eigenspace spanned by $\tilde{v}^i=(0,0,1)$; Λ^i_j is parametrized by (B3), where v^i is the spacelike eigenvector corresponding to the nonzero eigenvalue of $\mathbf{g}\mathbf{C}\mathbf{g}$, normalized such that $g_{ij}v^iv^j=1$.
- (IV) As in (III), except that the corresponding one-dimensional eigenspaces are now timelike; Λ^i_j is parametrized by (B4), where v^i is the timelike eigenvector corresponding to the zero eigenvalue of $\mathbf{g}\mathbf{B}\mathbf{g}$ (when $\mathcal{C}_2=0$) or the timelike eigenvector corresponding to the nonzero eigenvalue of $\mathbf{g}\mathbf{C}\mathbf{g}$ (when $\mathcal{C}_2\neq 0$). In both cases, v^i is normalized such that $g_{ij}v^iv^j=-1$.
- (V) As in (III).
- (VI) As in (III) and (V).
- (VII) The canonical form and moving frame map depend on the covariant \mathcal{C}_2 . If \mathcal{C}_2 is zero (nonzero), then $\tilde{c}_0+2\tilde{\gamma}_0$ is zero (nonzero) in the canonical form. In both cases, $\mathbf{g}\tilde{\mathbf{C}}\mathbf{g}$ only admits one nontrivial null eigenvector $\tilde{v}^i=(1,-1,0)$. Thus, Λ^i_j is parametrized by (B5⁻), where v^i is the null eigenvector of $\mathbf{g}\mathbf{C}\mathbf{g}$ normalized such that $v^0=1$.
- (VIII) There are three cases to consider, namely,
 - (A) $\mathcal{A}_5=0$ ($\tilde{\gamma}_2=0$),
 - (B) $\mathcal{A}_5\neq 0$, $(\mathcal{A}_{14},\mathcal{A}_{15})=\mathbf{0}$ ($\tilde{c}_0+\tilde{\gamma}_2=0$, $\tilde{\alpha}_0+\tilde{\alpha}_2+\tilde{\alpha}_2=0$),
 - (C) $\mathcal{A}_5\neq 0$, $(\mathcal{A}_{14},\mathcal{A}_{15})\neq \mathbf{0}$ ($\tilde{c}_0+\tilde{\gamma}_2\neq 0$, $\tilde{\gamma}_2\neq 0$).

For Case (A), \tilde{C}^{ij} is a multiple of the metric. The group action (3.12) implies that $C^{ij}=-\tilde{c}_0g^{ij}$ and hence $\tilde{c}_0=C^{00}$. Using this simplification in the transformation rules (3.12) for the B^{ij} , we can obtain the components of the translation explicitly, viz.,

$$\delta^i = \frac{1}{2}\tilde{c}_0^{-1}\epsilon^i_{jk}B^{jk}. \quad (6.2)$$

Substituting everything back into the transformation rules for the A^{ij} and simplifying yields

$$A^{ij} + \tilde{c}_0(\delta^i\delta^j - g^{ij}g_{kl}\delta^k\delta^l) = \Lambda^i_k\Lambda^j_\ell\tilde{A}^{k\ell}, \quad (6.3)$$

which is in the form of a generalized eigenproblem. It follows that $\mathbf{g}\tilde{\mathbf{A}}\mathbf{g}$ admits one spacelike eigenvector $\tilde{v}^i=(0,0,1)$. Thus, Λ^i_j is parametrized by (B3), where v^i is the spacelike eigenvector of the left-hand side of (6.3) (with both indices lowered), normalized such that $g_{ij}v^iv^j=1$. For both

Cases (B) and (C), $\mathbf{g}\tilde{\mathbf{C}}\mathbf{g}$ admits the spacelike eigenvector $\tilde{v}^i=(0,0,1)$; Λ_j^i is parametrized by (B3), where v^i is the spacelike eigenvector of $\mathbf{g}\mathbf{C}\mathbf{g}$, normalized such that $g_{ij}v^i v^j=1$.

- (IX) If $\mathcal{A}_1=0$, then $\tilde{C}^{ij}=-\tilde{c}_0 g^{ij}$. As in Case (A) of (VIII), $\tilde{c}_0=C^{00}$, the components of the translation δ^i are given by (6.2), and (6.3) is also satisfied. Moreover, as the eigenvalues of \tilde{A}^{ij} are necessarily real and distinct (otherwise the CKT would admit a web symmetry), the associated eigenvectors of the left-hand side of (6.3) (with both indices lowered) uniquely determine the components of Λ_j^i . If $\mathcal{A}_1 \neq 0$, then the eigenvalues of $\mathbf{g}\tilde{\mathbf{C}}\mathbf{g}$ are necessarily real and distinct; Λ_j^i is uniquely determined by solving the associated eigenproblem for $\mathbf{g}\mathbf{C}\mathbf{g}$.
- (X) If $\mathcal{A}_1=0$, then $\tilde{C}^{ij}=\tilde{c}_0 g^{ij}$ and hence $\tilde{c}_0=-C^{00}$. Moreover, $\mathbf{g}\tilde{\mathbf{A}}\mathbf{g}$ admits one spacelike eigenvector given by $\tilde{v}^i=(0,0,1)$. Therefore, we may proceed in analogy to Case (A) of (VIII). If $\mathcal{A}_1 \neq 0$, it follows that $\mathbf{g}\tilde{\mathbf{C}}\mathbf{g}$ admits a complex conjugate pair of eigenvalues and one real eigenvalue with a spacelike eigenspace spanned by $\tilde{v}^i=(0,0,1)$. Therefore, Λ_j^i is parametrized by (B3), where v^i is the spacelike eigenvector of $\mathbf{g}\mathbf{C}\mathbf{g}$, normalized such that $g_{ij}v^i v^j=1$.

VII. MAIN ALGORITHM

The results of the previous sections lead to a systematic method for determining orthogonally separable coordinates and first integrals of motion quadratic in the momenta for natural Hamiltonian systems in \mathbb{M}^3 . Given that \mathbb{M}^3 admits a total of 39 separable webs with 58 corresponding separable coordinate systems (contrast this to \mathbb{E}^3 where the number of webs and coordinate systems is only 11), it is not surprising that our algorithm is rather involved. Computationally, however, our algorithm is *purely algebraic* and hence is straightforward to implement in a computer algebra system. We now summarize the main steps of the algorithm.

- (1) *Impose the compatibility condition.* Using the components K^{ij} of the general Killing tensor in $\mathcal{K}^2(\mathbb{M}^3)$ specified in (3.15) and the given potential V with respect to pseudo-Cartesian coordinates x^i , impose the compatibility condition (2.9). In components, this condition reads

$$\partial_i (K^k_{\ j} \partial_k V) = 0. \quad (7.1)$$

The compatibility condition (7.1) places linear constraints on the Killing tensor parameters A^{ij} , B^{ij} , and C^{ij} which are readily solved in any computer algebra system.

- (2) *Extract the CKTs.* Using the compatible Killing tensor obtained in step (1), impose the condition that the Killing tensor be characteristic, as described in the paragraph preceding Eq. (3.5). These conditions give rise to nonlinear (polynomial) constraints in the Killing tensor parameters, in particular, the TSN conditions (3.6) governing the normality of the eigenvectors. In some cases, a general solution to these conditions might not always be tractable. Nevertheless, one can always attempt to solve the conditions for special cases thereby enabling one to proceed in the algorithm.
- (3) *Search for web symmetries.* Determine if the CKT obtained in step (2) admits an isometry group web symmetry by imposing the Lie derivative condition (4.1) using the components V^i of the general Killing vector in $\mathcal{K}^1(\mathbb{M}^3)$ specified in (3.13). This condition leads to a system of linear equations in the Killing vector parameters A^i and B^i . If a nontrivial solution exists, classify the resulting Killing vector using Fig. 1. Then, determine the group element $h_1 \in \text{SE}(2,1)$ which casts the Killing vector to canonical form using the moving frame map construction detailed in Sec. VI A. If a group web symmetry does not exist, determine if the CKT admits a dilatational web symmetry by imposing the Lie derivative condition (4.2), where $V=(x^i+a^i)X_i$ is the general dilatational vector field. This condition generates a system of linear equations in the parameters a^i . If a solution for the a^i exists, then the CKT admits a dilatational web symmetry. In this case, construct $h_1 \in \text{SE}(2,1)$, viz., $(\Lambda_j^i, \delta^i)=(\delta_j^i, a^i)$, otherwise set $h_1=\text{identity}$. Finally, compute the transformed CKT $h_1 \cdot \mathbf{K}$ using the standard tensor transformation rules or, alternatively, the explicit group action (3.12). It follows that

the resulting transformed Killing tensor is contained in one of the six invariant subspaces listed in Table I (in the case when the CKT admits a web symmetry) or is asymmetric.

- (4) *Classify the CKT and determine the moving frame map.* For the transformed CKT $h_1 \cdot \mathbf{K}$ computed in step (3), classify it using the scheme detailed in Sec. V. This calculation determines the type of separable web in \mathbb{M}^3 which is characterized by the CKT. Then, using the moving frame map construction detailed in Sec. VI, determine the group element $h_2 \in \text{SE}(2,1)$ which casts the classified CKT to canonical form.
- (5) *Construct the transformation to separable coordinates and the first integrals of motion.* Let Λ_j^i and δ^j be the corresponding Lorentz transformation and translation of the composition $h_2 \circ h_1$. The transformation from pseudo-Cartesian coordinates x^i to separable coordinates u^i is

$$x^i = \Lambda_j^i T^j(u^k) + \delta^i, \quad (7.2)$$

where $x^i = T^i(u^j)$ is (one of) the canonical coordinate transformations associated with the separable web. These coordinate patches and transformations to separable coordinates are given explicitly in Ref. 15 for each of the 39 separable webs in \mathbb{M}^3 . Finally, the most general first integral quadratic in the momenta compatible with the potential V is given by (2.10), where K^{ij} is the general solution obtained from step (1) and the potential function U is obtained by integrating the system of PDEs $\partial_i U = K^i_j \delta^j V$.

VIII. APPLICATION: A STATIONARY FLOW OF THE KORTEWEG–de VRIES HIERARCHY

Hamiltonian systems often arise naturally from the stationary flows of soliton equations, such as the celebrated KdV equation. Morosi and Tondo²⁷ considered the integrability of the natural Hamiltonian system defined by

$$H = \frac{1}{2}(2p_u p_v + p_y^2) - \frac{5}{8}u^4 + \frac{5}{2}u^2 v + \frac{1}{2}u y^2 - \frac{1}{2}v^2, \quad (8.1)$$

which is obtained as a stationary reduction of the seventh-order KdV flow. The Hamiltonian (8.1) is defined on the base manifold \mathbb{M}^3 with respect to the position-momenta coordinates $q^i = (u, v, y)$ and $p_i = (p_u, p_v, p_y)$. Before we can apply the algorithm of Sec. VII to this Hamiltonian system, we must first convert the null coordinates u and v to pseudo-Cartesian coordinates t and x because we have assumed throughout that the \mathbb{M}^3 metric is the diagonal Minkowski metric $g_{ij} = \text{diag}(-1, 1, 1)$. Upon defining

$$u = -\frac{1}{\sqrt{2}}(t+x), \quad v = \frac{1}{\sqrt{2}}(t-x),$$

it follows that potential in (8.1) is transformed to

$$V = -\frac{5}{32}(t+x)^4 + \frac{5\sqrt{2}}{8}(t+x)^2(t-x) - \frac{\sqrt{2}}{4}(t+x)y^2 - \frac{1}{4}(t-x)^2. \quad (8.2)$$

We now apply our method to the potential (8.2).

The general solution of the compatibility condition (7.1) subject to the potential (8.2) is the Killing tensor

$$K^{ij} = a_1 K_1^{ij} + a_2 K_2^{ij} + a_3 K_3^{ij}, \quad (8.3)$$

where a_1 , a_2 , and a_3 are arbitrary constants and

$$K_1^{ij} = \begin{pmatrix} 1 + 2\sqrt{2}x & 1 + \sqrt{2}(t-x) & -\sqrt{2}y \\ 1 + \sqrt{2}(t-x) & 1 - 2\sqrt{2}t & \sqrt{2}y \\ -\sqrt{2}y & \sqrt{2}y & -2\sqrt{2}(t+x) \end{pmatrix},$$

$$K_2^{ij} = \begin{pmatrix} -2 - 2\sqrt{2}x & -1 - \sqrt{2}(t-x) & \sqrt{2}y \\ -1 - \sqrt{2}(t-x) & 2\sqrt{2}t & -\sqrt{2}y \\ \sqrt{2}y & -\sqrt{2}y & 1 + 2\sqrt{2}(t+x) \end{pmatrix},$$

$$K_3^{ij} = \begin{pmatrix} -y^2 & y^2 & -y(t+x+\sqrt{2}) \\ y^2 & -y^2 & y(t+x-\sqrt{2}) \\ -y(t+x+\sqrt{2}) & y(t+x-\sqrt{2}) & -(t+x)^2 - 2\sqrt{2}(t-x) \end{pmatrix}.$$

The TSN conditions (3.6) are identically satisfied for the Killing tensor (8.3) and hence it has normal eigenvectors for all a_1 , a_2 , and a_3 . The discriminant of the characteristic polynomial of (8.3) is a lengthy polynomial in the constants a_i and the pseudo-Cartesian coordinates; nevertheless it is generally nonzero and vanishes only if $a_1=a_2$ and $a_3=0$, in which case (8.3) reduces to a multiple of the metric. Therefore, we conclude that (8.3) generally has normal eigenvectors and real and distinct eigenvalues, thereby defining a CKT.

Imposing the Lie derivative condition (4.1) on the CKT (8.3), as described in step (2) of the algorithm in Sec. VII, we conclude that (8.3) admits no web symmetry for any values of the constants a_i . The search for a dilatational web symmetry proves equally unsuccessful. Therefore, the CKT (8.3) characterizes one of the ten asymmetric separable webs in \mathbb{M}^3 .

We now proceed to classify the asymmetric Killing tensor (8.3) using the scheme detailed in Sec. VI. It follows that $\mathcal{A}_1=0$, $\mathcal{C}_2=0$ and

$$\mathcal{A}_2 = 4\sqrt{2}a_3^2(t+x).$$

There are two cases to consider, namely, $a_3=0$ and $a_3 \neq 0$. Firstly, if $a_3=0$, then $\mathcal{A}_4=0$ and $\mathcal{A}_{11} = -8(a_1-a_2)^3 \neq 0$ (otherwise the CKT would reduce to the metric). Therefore, the CKT (8.3) with $a_3=0$ characterizes the asymmetric web II. Secondly, if $a_3 \neq 0$, then $\mathcal{A}_4=0$ and $\mathcal{A}_6 = -8a_3^3$; thus in this case the CKT also characterizes the asymmetric web II.

We now compute the moving frame map for (8.3) which transforms it to the canonical form tabulated in Appendix A 7. Following the procedure detailed in Sec. VI H, it follows that for the case when $a_3=0$, the parameter matrix \mathbf{gBg} admits a null eigenvector $v^i=(1,-1,0)$. [Note that if (8.3) was in canonical form to begin with, this null vector would be $(1, 1, 0)$.] Therefore, Λ_j^i is parametrized by (B5⁺), where v^i is the aforementioned null vector. Solving the system of equations induced by the group action (3.12) for the parameters appearing in Λ_j^i , the components of the translation δ^i and the Killing tensor parameters appearing in the canonical form, we find that

$$\Lambda_j^i = \frac{1}{2\sqrt{2}} \begin{pmatrix} -3 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & -2\sqrt{2} \end{pmatrix}, \quad \delta^i = 0, \quad (8.4)$$

defines the moving frame map. Similarly, for the case $a_3 \neq 0$, we also find that the parameter matrix \mathbf{gCg} admits the null eigenvector $v^i=(1,-1,0)$. It follows that the moving frame map for this case is also given by (8.4).

We conclude that the CKT (8.3) compatible with the potential (8.2) characterizes only one of the 39 separable webs in \mathbb{M}^3 , namely, the asymmetric web II, for any (nontrivial) values of the constants a_1 , a_2 , and a_3 . Referring to Ref. 15 the coordinate system (F12.9) is the only such system of coordinates which covers this web. The transformation from canonical pseudo-Cartesian coordinates (t,x,y) to separable coordinates (μ, ν, ω) is given by

$$t+x = \frac{1}{8}(\omega^2 + (\mu + \nu)^2)(\omega^2 + (\mu - \nu)^2), \quad t-x = \mu^2 + \nu^2 - \omega^2, \quad y = \mu\nu\omega. \quad (8.5)$$

Equation (7.2) in conjunction with (8.4) and (8.5) implies that the transformation to separable coordinates for the Hamiltonian (8.1) is

TABLE III. Solutions to the canonical forms and equivalence problems.

	Euclidean space \mathbb{E}^3	Minkowski space \mathbb{M}^3
Canonical forms problem	Eisenhart (Ref. 9) Boyer <i>et al.</i> (Ref. 4)	Horwood and McLenaghan (Ref. 16)
Equivalence problem	Horwood <i>et al.</i> (Ref. 16) Horwood (Ref. 14)	Present article

$$u = -\frac{1}{\sqrt{2}}(t+x) = \frac{1}{2}(\mu^2 + \nu^2 - \omega^2),$$

$$v = \frac{1}{\sqrt{2}}(t-x) = -\frac{1}{8}(\omega^2 + (\mu + \nu)^2)(\omega^2 + (\mu - \nu)^2),$$

$$y = -\mu\nu\omega. \quad (8.6)$$

In closing, we remark that it is possible to construct the most general first integral quadratic in the momenta for the Hamiltonian (8.1) using the Killing tensor (8.3), as described in step (5) of the algorithm in Sec. VII. This computation yields a three-parameter family of first integrals in the constants a_1 , a_2 , and a_3 . It is possible to extract two additional functionally independent first integrals [e.g., with $(a_1, a_2, a_3) = (1, 0, 0)$ and $(a_1, a_2, a_3) = (0, 0, 1)$], thereby showing that the Hamiltonian system (8.1) is *completely integrable*, in agreement with the results of Morosi and Tondo.²⁷

IX. CONCLUSION

This work concludes the program of studying orthogonal separation of variables for natural Hamiltonians defined in three-dimensional Euclidean and Minkowski spaces pursued by different authors over the years. Geometrically, the program in each case consists of solutions to the corresponding canonical forms and equivalence problems. Table III encapsulates the presentation of the solutions to each problem in the corresponding papers.

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APPENDIX A: CANONICAL CHARACTERISTIC KILLING TENSORS

In this appendix, we give 39 classes of canonical forms for the CKTs in \mathbb{M}^3 , each of which characterizes one of the 39 separable webs in \mathbb{M}^3 . These canonical forms were first derived in Ref. 15 by Eisenhart's method. We shall express these canonical forms by giving the components of the Killing tensor parameter matrices A^{ij} , B^{ij} and C^{ij} , as defined in Eq. (3.4). The components K^{ij} of the corresponding Killing tensor can then be reconstructed from (3.15). Unless indicated otherwise, any component not given explicitly is zero and any parameters appearing in the components are unconstrained.

1. Spacelike translational CKTs

$$(I) \quad A^{ij} = \text{diag}(a_0, a_1, a_2), \quad B^{ij} = 0, \quad C^{ij} = 0.$$

- (II) $A^{ij} = \text{diag}(-a_0, a_0, a_2)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(0, 0, c_2)$.
 (III) $A^{ij} = \text{diag}(-a_0, a_0, a_2)$, $B^{12} = b_{12}$, $C^{ij} = 0$.
 (IV) $A^{ij} = \text{diag}(-a_0, a_0, a_2)$, $B^{02} = b_{02}$, $C^{ij} = 0$.
 (V)

$$A^{ij} = \begin{pmatrix} -a_0 & \alpha_2 & 0 \\ \alpha_2 & a_0 + 2\alpha_2 & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \quad B^{ij} = \begin{pmatrix} 0 & 0 & 2\alpha_2 \\ 0 & 0 & -2\alpha_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^{ij} = 0.$$

- (VI) $A^{ij} = \text{diag}(a_0, a_1, a_2)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(0, 0, c_2)$, $(a_0 + a_1)c_2 > 0$.
 (VII) As in (VI) but with $(a_0 + a_1)c_2 < 0$.
 (VIII)

$$A^{ij} = \begin{pmatrix} -a_0 & \alpha_2 & 0 \\ \alpha_2 & a_0 & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \quad B^{ij} = 0, \quad C^{ij} = \text{diag}(0, 0, c_2), \quad \alpha_2 c_2 < 0.$$

(IX)

$$A^{ij} = \begin{pmatrix} -a_0 & \alpha_2 & 0 \\ \alpha_2 & a_0 + 2\alpha_2 & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \quad B^{ij} = 0, \quad C^{ij} = \text{diag}(0, 0, 4\alpha_2).$$

(X)

$$A^{ij} = \begin{pmatrix} -a_0 & \alpha_2 & 0 \\ \alpha_2 & a_0 + 2\alpha_2 & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \quad B^{ij} = 0, \quad C^{ij} = \text{diag}(0, 0, -4\alpha_2).$$

2. Timelike translational CKTs

- (I) $A^{ij} = \text{diag}(a_0, a_1, a_1)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(c_0, 0, 0)$.
 (II) $A^{ij} = \text{diag}(a_0, a_1, a_1)$, $B^{20} = b_{20}$, $C^{ij} = 0$.
 (III) $A^{ij} = \text{diag}(a_0, a_1, a_2)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(c_0, 0, 0)$, $(a_1 - a_2)c_0 > 0$.

3. Spacelike rotational CKTs

- (I) $A^{ij} = \text{diag}(-a_1, a_1, a_1)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(-c_1, c_1, c_2)$.
 (II) $A^{ij} = \text{diag}(-a_1, a_1, a_1)$, $B^{01} = -B^{10} = b_{01}$, $C^{ij} = \text{diag}(0, 0, c_2)$.
 (III) $A^{ij} = \text{diag}(-a_1, a_1, a_2)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(-c_1, c_1, c_2)$, $(a_2 - a_1)c_1 > 0$.
 (IV) As in (III) but with $(a_2 - a_1)c_1 < 0$.

4. Timelike rotational CKTs

- (I) $A^{ij} = \text{diag}(-a_0, a_0, a_0)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(c_0, c_2, c_2)$.
 (II) $A^{ij} = \text{diag}(-a_0, a_0, a_0)$, $B^{12} = -B^{21} = b_{12}$, $C^{ij} = \text{diag}(c_0, 0, 0)$.
 (III) $A^{ij} = \text{diag}(a_0, a_2, a_2)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(c_0, c_2, c_2)$, $(a_0 + a_2)c_2 > 0$.
 (IV) As in (III) but with $(a_0 + a_2)c_2 < 0$.

5. Null rotational CKTs

(I)

$$A^{ij} = \text{diag}(-a_0, a_0, a_0), \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} \gamma_1 - c_1 & 0 & \gamma_1 \\ 0 & c_1 & 0 \\ \gamma_1 & 0 & \gamma_1 + c_1 \end{pmatrix}.$$

(II)

$$A^{ij} = \begin{pmatrix} -a_0 + c_1 & 0 & c_1 \\ 0 & a_0 & 0 \\ c_1 & 0 & a_0 + c_1 \end{pmatrix}, \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} \gamma_1 - c_1 & 0 & \gamma_1 \\ 0 & c_1 & 0 \\ \gamma_1 & 0 & \gamma_1 + c_1 \end{pmatrix}.$$

(III)

$$A^{ij} = \begin{pmatrix} -a_0 - c_1 & 0 & -c_1 \\ 0 & a_0 & 0 \\ -c_1 & 0 & a_0 - c_1 \end{pmatrix}, \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} \gamma_1 - c_1 & 0 & \gamma_1 \\ 0 & c_1 & 0 \\ \gamma_1 & 0 & \gamma_1 + c_1 \end{pmatrix}.$$

6. Dilatational CKTs

(I)

$$A^{ij} = \text{diag}(-a_0, a_0, a_0), \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} c_0 & 0 & -\gamma_0 \\ 0 & -c_0 & \gamma_0 \\ -\gamma_0 & \gamma_0 & -c_0 \end{pmatrix}.$$

(II)

$$A^{ij} = \text{diag}(-a_0, a_0, a_0), \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} -c_2 - 2\gamma_2 & \gamma_2 & 0 \\ \gamma_2 & c_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix}.$$

(III)

$$A^{ij} = \text{diag}(-a_0, a_0, a_0), \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} -c_2 & \gamma_2 & 0 \\ \gamma_2 & c_2 - 2\gamma_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix}.$$

(IV) $A^{ij} = \text{diag}(-a_0, a_0, a_0), B^{ij} = 0, C^{ij} = \text{diag}(c_0, c_1, c_2), -c_0 < c_2 < c_1$ or $-c_0 > c_2 > c_1$.

(V)

$$A^{ij} = \text{diag}(-a_0, a_0, a_0), \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} c_0 & \gamma_2 & 0 \\ \gamma_2 & -c_0 & 0 \\ 0 & 0 & c_2 \end{pmatrix}.$$

7. Asymmetric CKTs

(I)

$$\begin{aligned}
 A^{00} &= -a_0, & B^{00} &= -B^{11} = 4c_1, \\
 A^{11} &= a_0 + 2c_1, & B^{22} &= 8c_1, & C^{00} &= 16c_1, \\
 A^{22} &= a_0 + c_1, & B^{01} &= -B^{10} = 4c_1, & C^{11} &= 16c_1, \\
 A^{01} &= c_1, & B^{12} &= -B^{21} = -4b_{02}, & C^{01} &= -16c_1. \\
 A^{02} &= A^{12} = -b_{02}, & B^{20} &= -B^{02} = -4b_{02},
 \end{aligned}$$

(II)

$$\begin{aligned}
 A^{00} &= 2a_0, & B^{02} &= -a_0 - a_2, & C^{00} &= c_0, \\
 A^{11} &= 2a_0 + 4a_2, & B^{12} &= -a_0 - a_2, & C^{11} &= c_0, \\
 A^{22} &= 2a_2, & B^{20} &= a_0 + a_2 + 4c_0, & C^{01} &= c_0. \\
 A^{01} &= -2a_0 - 2a_2, & B^{21} &= a_0 + a_2 - 4c_0,
 \end{aligned}$$

(III)

$$\begin{aligned}
 A^{00} &= -a_0, & B^{00} &= -c_2, \\
 A^{11} &= a_0 + 2b_{01} + 2c_2, & B^{11} &= -c_2, & C^{22} &= c_2. \\
 A^{22} &= a_0 + b_{01} + c_2, & B^{01} &= b_{01}, \\
 A^{01} &= b_{01} + c_2, & B^{10} &= -b_{01} - 2c_2,
 \end{aligned}$$

(IV)

$$A^{ij} = \text{diag}(a_0, a_1, a_2), \quad B^{12} = b_{12}, \quad C^{ij} = \text{diag}(c_0, 0, 0), \\
 B^{21} = b_{21},$$

$$b_{12}(b_{12} + b_{21}) + c_0(a_0 + a_1) = 0 \quad \text{and} \quad b_{12}(a_0 + a_2) + 2b_{21}(a_0 + a_1) = 0.$$

(V)

$$A^{ij} = \text{diag}(a_0, a_1, a_2), \quad B^{01} = b_{01}, \quad C^{ij} = \text{diag}(0, 0, c_2), \\
 B^{10} = b_{10},$$

$$b_{01}(b_{01} + b_{10}) - c_2(a_0 + a_2) = 0 \quad \text{and} \quad b_{01}(a_2 - a_1) + 2b_{10}(a_0 + a_2) = 0.$$

(VI)

$$A^{ij} = \begin{pmatrix} -a_0 & \alpha_2 & 0 \\ \alpha_2 & a_0 & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \quad B^{ij} = \begin{pmatrix} -\beta_1 & b_{01} & 0 \\ -b_{01} & -\beta_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^{ij} = \text{diag}(0, 0, c_2),$$

$$b_{01}\beta_1 - \alpha_2 c_2 = 0 \quad \text{and} \quad \beta_1^2 + c_2(a_0 - a_2) = 0 \quad \text{if} \quad \beta_1^2 + c_2^2 > 0,$$

$$a_0 - a_2 = 0 \quad \text{if} \quad \beta_1^2 + c_2^2 = 0.$$

(VII)

$$\begin{aligned}
A^{00} &= a_0, & C^{00} &= c_0, \\
A^{11} &= -a_0 + 16c_0 + 24\gamma_0, & C^{11} &= -c_0 - 4\gamma_0, \\
A^{22} &= -a_0 + 8c_0 + 12\gamma_0, & B^{ij} &= 0, & C^{22} &= -c_0 - 2\gamma_0, \\
A^{01} &= -8c_0 - 12\gamma_0, & C^{01} &= 2\gamma_0, \\
A^{02} &= -A^{12} = 4c_0 + 8\gamma_0, & C^{02} &= -C^{12} = -\gamma_0.
\end{aligned}$$

(VIII)

$$\begin{aligned}
A^{00} &= a_0, & C^{00} &= c_0, \\
A^{11} &= -a_0 - 2\alpha_2, & C^{11} &= -c_0 - 2\gamma_2, \\
& & B^{ij} &= 0, \\
A^{22} &= a_2, & C^{22} &= -c_0 - 2\gamma_2, \\
A^{01} &= \alpha_2, & C^{01} &= \gamma_2,
\end{aligned}$$

$$(a_0 + a_2)(c_0 + 2\gamma_2)\gamma_2 + \alpha_2(2c_0 + 3\gamma_2)\gamma_2 = 0.$$

(IX) $A^{ij} = \text{diag}(a_0, a_1, a_2)$, $B^{ij} = 0$, $C^{ij} = \text{diag}(c_0, c_1, c_2)$, $(a_0 + a_1)c_0c_1 + (a_1 - a_2)c_1c_2 - (a_2 + a_0)c_2c_0 = 0$.

(X)

$$A^{ij} = \begin{pmatrix} -a_0 & \alpha_2 & 0 \\ \alpha_2 & a_0 & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \quad B^{ij} = 0, \quad C^{ij} = \begin{pmatrix} -c_0 & \gamma_2 & 0 \\ \gamma_2 & c_0 & 0 \\ 0 & 0 & c_2 \end{pmatrix},$$

$$\alpha_2\gamma_2^2 + c_2\gamma_2(a_2 - a_0) - c_0\alpha_2(c_2 - c_0) = 0.$$

APPENDIX B: THE GENERALIZED EIGENVALUE-EIGENVECTOR PROBLEM

The majority of the calculations required to compute the moving frame map for a space of Killing tensors in \mathbb{M}^3 amounts to the use of elementary linear algebra and properties of the Lorentz group $\text{SO}(2,1)$. In particular, we frequently require the generalized eigenvalues and eigenvectors of a 3×3 matrix with respect to the Minkowski metric. We review the steps required for this computation in Appendix B 1. In the case when the matrix fails to be diagonalizable or has a repeated eigenvalue, one can find a natural parametrization of $\text{SO}(2,1)$ from one of the (nonzero) eigenvectors by determining the Lorentz transformation which maps the eigenvector to either the spacelike vector $(0,0,1)$, the timelike vector $(1,0,0)$, or the null vectors $(1, \pm 1, 0)$. We derive the explicit transformation for each of these four cases in Appendixes B 2–B 4.

1. Solution to the eigenproblem in Minkowski space

For motivation, let us consider the group action $\text{SE}(2,1) \circlearrowleft \mathcal{K}^2(\mathbb{M}^3)$ given in (3.12) and its effect on the C^{ij} Killing tensor parameter matrix. It follows that the transformation rule for the C^{ij} can be written in the form

$$\Lambda^i_j \tilde{C}^{jk} g_{k\ell} = C^{ij} g_{jk} \Lambda^k_\ell,$$

where $\Lambda^i_j \in \text{SO}(2,1)$. In matrix form, this equation reads

$$\Lambda \tilde{C} g = C g \Lambda \Leftrightarrow \Lambda(\tilde{C} g) = g^{-1}(g C g) \Lambda, \quad (\text{B1})$$

where $(\Lambda)_{ij} = \Lambda^i_j$, $(\tilde{C})_{ij} = \tilde{C}^{ij}$, etc.

Let us now recall the generalized eigenproblem from linear algebra. Let A and B be real (or complex) $n \times n$ matrices, where the latter is also assumed to be invertible. The matrix A is said to admit a *generalized eigenvalue* k with respect to the matrix B if there exists a nonzero vector \mathbf{v} such that $A\mathbf{v} = kB\mathbf{v}$. The vector \mathbf{v} is called a generalized eigenvector of A with respect to B . If A admits n linearly independent generalized eigenvectors, then A is said to be *diagonalizable* with respect to B . In this case, there exists an invertible matrix S , whose columns are the n generalized eigenvectors, and a diagonal matrix D , whose elements are the corresponding generalized eigenvalues, such that

$$AS = BSD \Leftrightarrow SD = B^{-1}AS. \quad (\text{B2})$$

Comparing Eqs. (B1) and (B2), we see that the transformation rule for the C^{ij} parameter matrix is in the form of a generalized eigenproblem with

$$A = gCg, \quad B = g, \quad D = \tilde{C}g, \quad S = \Lambda.$$

Moreover, if C is diagonalizable (e.g., it has distinct eigenvalues), then S can be formed so that it is a proper Lorentz transformation. Let us review the steps required to determine the matrices \tilde{C} and Λ .

- (1) Solve the characteristic equation $\det(gCg - k_i g) = 0$ to obtain a set of eigenvalues k_i , $i = 0, 1, 2$.
- (2) For each eigenvalue k_i , compute a nonzero generalized eigenvector \mathbf{v}_i satisfying $gCg\mathbf{v}_i = k_i g\mathbf{v}_i$ (no sum).
- (3) If C (or equivalently, gCg) is diagonalizable, then one of the three \mathbf{v}_i is necessarily timelike while the other two are spacelike. That being said, reorder the \mathbf{v}_i so that \mathbf{v}_0 is timelike and \mathbf{v}_1 and \mathbf{v}_2 are spacelike. Finally, rescale the \mathbf{v}_i so that $g_{k\ell} \mathbf{v}_i^k \mathbf{v}_i^\ell = \pm 1$.
- (4) Form the matrix Λ whose i th column is \mathbf{v}_i .
- (5) Compute the determinant of Λ and verify that it is equal to ± 1 . If the determinant is negative, multiply one of the columns of Λ by an overall sign.
- (6) Construct $\tilde{C} = \text{diag}(-k_0, k_1, k_2)$.

If C fails to be diagonalizable, then it is possible to use one of its generalized eigenvectors to write Λ in terms of one or more arbitrary parameters. For example, if C admits a spacelike generalized eigenvector \mathbf{v} , then one can parameterize Λ in terms of a boost parameter ψ in the plane whose normal is \mathbf{v} . The construction of such a one-parameter family of Lorentz transformations requires one to transform such an eigenvector \mathbf{v} to some canonical form. We discuss these computations in the remaining subsections of this appendix.

For the sake of brevity, we shall drop the use of the word ‘‘generalized’’ when discussing generalized eigenvalues and eigenvectors. Throughout this paper, an eigenvalue (eigenvector) of some matrix will always mean a generalized eigenvalue (generalized eigenvector) of the said matrix with respect to the Minkowski metric $g_{ij} = \text{diag}(-1, 1, 1)$.

2. Transformation of a constant vector to (0,0,1)

Let v^i be the components a constant spacelike vector in \mathbb{M}^3 , normalized so that $g_{ij} v^i v^j = 1$. We seek the most general Lorentz transformation $\Lambda_j^i \in \text{SO}(2,1)$ which maps the vector $\tilde{v}^i = (0, 0, 1)$ to v^i or, equivalently, a general Lorentz transformation characterizing a boost in the plane whose normal is v^i . To begin, let us first determine a rotation which maps $v^i \rightarrow v'^i = (v'^0, 0, v'^2)$ and then a boost which maps $v'^i \rightarrow \tilde{v}^i = (0, 0, 1)$. Indeed, for the former, $S_1 \mathbf{v} = \mathbf{v}'$, where

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix},$$

and

$$\cos \theta_1 = \frac{v_2}{\sqrt{(v^1)^2 + (v^2)^2}}, \quad \sin \theta_1 = \frac{v_1}{\sqrt{(v^1)^2 + (v^2)^2}},$$

so that $v'^i = (v^0, 0, \sqrt{(v^1)^2 + (v^2)^2})$. For the latter, $S_2 v' = \tilde{v}$, where

$$S_2 = \begin{pmatrix} \cosh \theta_2 & 0 & \sinh \theta_2 \\ 0 & 1 & 0 \\ \sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix},$$

and

$$\cosh \theta_2 = \sqrt{(v^1)^2 + (v^2)^2}, \quad \sinh \theta_2 = -v^0.$$

Finally, we note that the Lorentz transformation

$$S_0 = \begin{pmatrix} \epsilon \cosh \psi & \sinh \psi & 0 \\ \sinh \psi & \epsilon \cosh \psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

a boost in the tx -plane, where $\epsilon = \pm 1$, maps \tilde{v} to itself. Therefore, the required Lorentz transformation is $\Lambda = (S_2 S_1)^{-1} S_0$ or, explicitly,

$$\Lambda^i_j = \begin{pmatrix} \frac{\epsilon \sqrt{(v^1)^2 + (v^2)^2} \cosh \psi}{\sqrt{(v^1)^2 + (v^2)^2}} & \frac{\sqrt{(v^1)^2 + (v^2)^2} \sinh \psi}{\sqrt{(v^1)^2 + (v^2)^2}} & v^0 \\ \frac{\epsilon v^0 v^1 \cosh \psi + v^2 \sinh \psi}{\sqrt{(v^1)^2 + (v^2)^2}} & \frac{v^0 v^1 \sinh \psi + \epsilon v^2 \cosh \psi}{\sqrt{(v^1)^2 + (v^2)^2}} & v^1 \\ \frac{\epsilon v^0 v^2 \cosh \psi - v^1 \sinh \psi}{\sqrt{(v^1)^2 + (v^2)^2}} & \frac{v^0 v^2 \sinh \psi - \epsilon v^1 \cosh \psi}{\sqrt{(v^1)^2 + (v^2)^2}} & v^2 \end{pmatrix}. \quad (\text{B3})$$

3. Transformation of a constant vector to $(1,0,0)$

Let v^i be the components of a constant timelike vector in \mathbb{M}^3 , normalized so that $g_{ij}v^i v^j = -1$. We seek the most general Lorentz transformation $\Lambda^i_j \in \text{SO}(2,1)$ which maps the vector $\tilde{v}^i = (1, 0, 0)$ to v^i or, equivalently, a general Lorentz transformation characterizing a rotation whose axis of rotation is v^i . The derivation of such a Lorentz transformation is analogous to the spacelike case in the previous subsection. It follows that the required Lorentz transformation is

$$\Lambda^i_j = \begin{pmatrix} v^0 & \frac{v^0 v^2 \sin \psi + v^1 \cos \psi}{\sqrt{1 + (v^2)^2}} & \frac{v^0 v^2 \cos \psi - v^1 \sin \psi}{\sqrt{1 + (v^2)^2}} \\ v^1 & \frac{v^1 v^2 \sin \psi + v^0 \cos \psi}{\sqrt{1 + (v^2)^2}} & \frac{v^1 v^2 \cos \psi - v^0 \sin \psi}{\sqrt{1 + (v^2)^2}} \\ v^2 & \sqrt{1 + (v^2)^2} \sin \psi & \sqrt{1 + (v^2)^2} \cos \psi \end{pmatrix}. \quad (\text{B4})$$

4. Transformation of a constant vector to $(1, \pm 1, 0)$

Let v^i be the components of a constant null vector in \mathbb{M}^3 , rescaled so that $v^0 = 1$. We seek the most general Lorentz transformation $\Lambda^i_j \in \text{SO}(2,1)$ which maps the vector $\tilde{v}^i = (1, \pm 1, 0)$ to v^i . To begin, let us first find the rotation which maps $v \rightarrow \tilde{v}$. Indeed, $S_1 v = \tilde{v}$, where

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix},$$

and

$$\cos \theta_1 = \pm v^1, \quad \sin \theta_1 = \mp v^2.$$

Finally, let S_0 be the most general proper Lorentz transformation which maps $\tilde{\mathbf{v}}$ to itself. To determine S_0 , we first impose the condition $S_0 \tilde{\mathbf{v}} = \tilde{\mathbf{v}}$, which places restrictions on the components of S_0 . Next, we impose the condition that S_0 be a proper Lorentz transformation. Although these conditions are quadratic, they can nevertheless be solved systematically. One finds that

$$S_0 = \begin{pmatrix} 1 + \frac{1}{2}\kappa^2 & \mp \frac{1}{2}\kappa^2 & \kappa \\ \pm \frac{1}{2}\kappa^2 & 1 - \frac{1}{2}\kappa^2 & \pm \kappa \\ \kappa & \mp \kappa & 1 \end{pmatrix}$$

for any $\kappa \in \mathbb{R}$. Therefore, the required Lorentz transformation is $\Lambda = S_1^{-1} S_0$.

We remark that there is a further degree of freedom one could impose on the derived Lorentz transformation. The null vectors $\tilde{\mathbf{v}}$ and $k\tilde{\mathbf{v}}$ are equivalent for any $k > 0$. We observe that any boost in the tx -plane given by

$$\bar{S} = \begin{pmatrix} \epsilon \cosh \psi & \sinh \psi & 0 \\ \sinh \psi & \epsilon \cosh \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies $\bar{S}\tilde{\mathbf{v}} = k\tilde{\mathbf{v}}$, where $k = \epsilon e^{\pm \epsilon \psi}$ and $\epsilon = \pm 1$. Therefore, the Lorentz transformation $\Lambda = S_1^{-1} \bar{S} S_0$ or, explicitly,

$$\Lambda^0_0 = \epsilon \left(1 + \frac{1}{2}\kappa^2 \right) \cosh \psi \pm \frac{1}{2}\kappa^2 \sinh \psi,$$

$$\Lambda^0_1 = \mp \frac{1}{2}\epsilon \kappa^2 \cosh \psi + \left(1 - \frac{1}{2}\kappa^2 \right) \sinh \psi,$$

$$\Lambda^0_2 = \epsilon \kappa \cosh \psi \pm \kappa \sinh \psi,$$

$$\Lambda^1_0 = \frac{1}{2}\epsilon \kappa^2 v^1 \cosh \psi \pm \left(1 + \frac{1}{2}\kappa^2 \right) v^1 \sinh \psi \mp \kappa v^2,$$

$$\Lambda^1_1 = \pm \epsilon \left(1 - \frac{1}{2}\kappa^2 \right) v^1 \cosh \psi - \frac{1}{2}\kappa^2 v^1 \sinh \psi + \kappa v^2,$$

$$\Lambda^1_2 = \epsilon \kappa v^1 \cosh \psi \pm \kappa v^1 \sinh \psi \mp v^2,$$

$$\Lambda^2_0 = \frac{1}{2}\epsilon \kappa^2 v^2 \cosh \psi \pm \left(1 + \frac{1}{2}\kappa^2 \right) v^2 \sinh \psi \pm \kappa v^1,$$

$$\Lambda^2_1 = \pm \epsilon \left(1 - \frac{1}{2}\kappa^2 \right) v^2 \cosh \psi - \frac{1}{2}\kappa^2 v^2 \sinh \psi - \kappa v^1,$$

$$\Lambda^2_2 = \epsilon \kappa v^2 \cosh \psi \pm \kappa v^2 \sinh \psi \pm v^1 \tag{B5}$$

maps the vector $(1, \pm 1, 0)$ to the null vector $\epsilon e^{\pm \epsilon \psi} \mathbf{v}$ with $v^0 = 1$.

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