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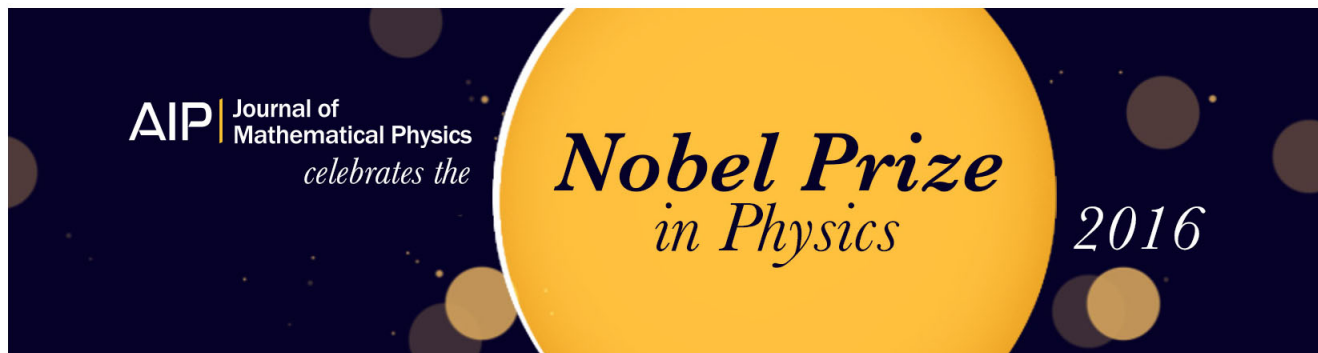
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Equivalence problem for the orthogonal webs on the 3-sphere

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We solve the equivalence problem for the orthogonally separable webs on the 3-sphere under the action of the isometry group. This continues a classical project initiated by Olevsky in which he solved the corresponding canonical forms problem. The solution to the equivalence problem together with the results by Olevsky forms a complete solution to the problem of orthogonal separation of variables for the Hamilton–Jacobi equation defined on the 3-sphere via orthogonal separation of variables. It is based on invariant properties of characteristic Killing tensors in addition to properties of the corresponding algebraic curvature tensor and the associated Ricci tensor. The result is illustrated by a nontrivial application to a natural Hamiltonian defined on the 3-sphere. © 2011 American Institute of Physics. [doi:10.1063/1.3578773]

I. INTRODUCTION

This paper presents a complete solution to the *equivalence problem for orthogonally separable webs* (OSWs) generated by the *characteristic Killing tensors* (CKTs) defined on the three-dimensional sphere \mathbb{S}^3 . In addition, we lay the ground work for solving the analogous problem for three-dimensional hyperbolic space \mathbb{H}^3 as well as the spaces of constant nonzero curvature of higher dimensions. This work continues and generalizes our studies of the OSWs defined in flat (pseudo-)Riemannian spaces (see, for instance Refs. 1, 14, and 16 and the relevant references therein).

Reportedly, the study of OSWs generated by Killing two-tensors was conceived in various classical articles published throughout the 19th century. A list of such works should include, among many others, Liouville,²¹ Neumann,²⁶ Bertrand,³ Morera,²⁵ and Stäckel.³⁴ The research in the area continued throughout the last century to make this study more systematic and develop new results. Of note was the celebrated 1934 paper by Eisenhart.⁸ In this paper, the author completely solved the *canonical forms problem* for the OSWs generated by CKTs defined in \mathbb{E}^3 by showing that there were exactly eleven inequivalent (in some sense) such OSWs and represented each of them with the corresponding canonical form. Furthermore, he gave a general criterion linking orthogonal separability with geometric and algebraic properties of Killing two-tensors. In addition to \mathbb{E}^3 , he also partially solved the canonical forms problem for \mathbb{S}^3 . This problem, as well as the case of hyperbolic space \mathbb{H}^3 , was later completed by Olevsky in 1950. In his paper,²⁸ Olevsky determined the number of orbits corresponding to the orthogonally separable coordinates on \mathbb{S}^3 and \mathbb{H}^3 , listing the metrics in these coordinates and the transformations to separable coordinates in each case. Kalnins *et al.*²⁰ completed Olevsky's work on \mathbb{S}^3 by also listing the CKTs (using the language of differential operators) in their respective canonical forms representing the OSWs. In recent years, the

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research in the area has been successfully continued yielding many important results in connection with the study of integrable and superintegrable classical and quantum Hamiltonian systems (see, for example, Refs. 4, 14, 16, 19, 30, and 35 and the relevant references therein).

It must be noted, however, that many of the aforementioned investigations concerned the canonical forms problems for the CKTs in question, while the applications arising in the field of classical and quantum Hamiltonian systems often involve the properties of the CKTs in their general form as far as the action of the corresponding isometry groups are concerned (see Ref. 14, for example). From this perspective, the main goal of this paper is to extend the results by Olevsky and others and solve the equivalence problem for the CKTs defined on \mathbb{S}^3 .

The paper is organized as follows. Section II contains a brief review of the required mathematical tools defined in the framework of the invariant theory of Killing tensors (ITKTs) that will be used in solving the main problem. In Sec. III we determine the form of the general Killing tensor on \mathbb{S}^n by considering it as an imbedded hypersurface in \mathbb{E}^{n+1} . We also obtain algebraic conditions for the orthogonal integrability of the eigendirections of the CKTs. In Sec. IV we formulate and solve the equivalence problems for CKTs defined on \mathbb{S}^3 . Section V is devoted to applications to problems of classical mechanics that subsume the results of the previous sections. Section VI contains the conclusion.

II. INVARIANT THEORY OF KILLING TENSORS

In what follows we formulate and solve the equivalence problem for the OSWs generated by CKTs defined on \mathbb{S}^3 within the framework of the invariant theory of Killing tensors, which is a natural extension of the classical invariant theory of homogeneous polynomials (see Ref. 18 for more details and relevant references). Thus, the study of OSWs is based on algebraic and geometric properties of Killing two-tensors in which invariant theory comes into play as the natural link between algebra and geometry. Moreover, recently it has been shown *explicitly* (see Ref. 16 for more details and references) that in the case of Killing two-tensors generating OSWs the study can be naturally cast into the general setting of Cartan's geometry.^{5,6,12}

Let $(\mathcal{M}, \mathbf{g})$ be an n -dimensional (pseudo-)Riemannian manifold of constant curvature.

Definition II.1: A (contravariant) *Killing tensor of valence p* defined in $(\mathcal{M}, \mathbf{g})$ is a symmetric $(p, 0)$ tensor field satisfying the Killing tensor equation

$$[\mathbf{K}, \mathbf{g}] = 0, \quad (2.1)$$

where $[\ , \]$ denotes the Schouten bracket.³² When $p = 1$, \mathbf{K} is said to be a Killing vector field (infinitesimal isometry) and (2.1) reduces to

$$\mathcal{L}_{\mathbf{K}} \mathbf{g} = 0, \quad (2.2)$$

where \mathcal{L} denotes the Lie derivative operator.

Since the Schouten bracket $[\ , \]$ is \mathbb{R} -bilinear, the set of solutions to the system of overdetermined partial differential equations given by (2.1) forms a vector space over \mathbb{R} . Furthermore, since $(\mathcal{M}, \mathbf{g})$ is of constant curvature, the dimension of such a vector space is maximal (see the relevant references in Ref. 14 for more details). In what follows, we shall use the notation $\mathcal{K}^p(\mathcal{M})$ to denote the vector space of valence p Killing tensor fields defined on \mathcal{M} .

Remark II.2: Note that Eq. (2.1) can be equivalently rewritten in the more familiar (covariant) form as follows:

$$K_{(i_1 \dots i_p i_{p+1})} = 0, \quad (2.3)$$

where $K_{i_1 \dots i_p}$ denotes the covariant components of the Killing tensor \mathbf{K} ; ∇_{i_p} the covariant derivative with respect to the Levi-Civita connection defined by \mathbf{g} , and (\dots) symmetrization over the enclosed indices.

As is well-known the Killing tensors defined on spaces of constant curvature are sums of symmetrized tensor products of Killing vectors forming finite-dimensional vector spaces. Denote by d the dimension of the vector space $\mathcal{K}^p(\mathcal{M})$ and recall that d is given by (see reference 16 for details)

$$d = \dim \mathcal{K}^p(\mathcal{M}) = \frac{1}{n} \binom{n+p}{p+1} \binom{n+p-1}{p}, \quad p \geq 1. \quad (2.4)$$

Therefore the general element of $\mathcal{K}^p(\mathcal{M})$ may be represented by d arbitrary parameters a^1, \dots, a^d with respect to a given basis. Let G denote the isometry group of $(\mathcal{M}, \mathbf{g})$. Our next observation is that the group G acting on \mathcal{M} induces, via the pushforward map, a linear transformation of $\mathcal{K}^p(\mathcal{M})$ which defines a representation of G .²³ Moreover, the action $G \circ \mathcal{K}^2(\mathcal{M})$ is not transitive. These observations are the crux of the invariant theory of Killing tensors allowing us to formulate the canonical forms and equivalence problems for the Killing tensors defined on spaces of constant curvature.

Of particular importance for applications are the elements of $\mathcal{K}^2(\mathcal{M})$ enjoying additional geometric and algebraic properties. More specifically, let $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ be such that its eigenvalues are pointwise simple and real, and the eigenvector fields are normal (orthogonally integrable). Such a Killing tensor \mathbf{K} is called a *characteristic Killing tensor*.^{2,8} The remarkable property of CKTs is that their eigenvalues and eigenvectors generate OSWs, which are n foliations of the space \mathcal{M} that consist of $(n-1)$ -dimensional hypersurfaces orthogonal to the eigenvectors of the CKT in question.

The action $G \circ \mathcal{K}^2(\mathcal{M})$, thus defined, foliates the vector space $\mathcal{K}^2(\mathcal{M})$ into the orbit space $\mathcal{K}^2(\mathcal{M})/G$. These orbits represent Killing tensors of valence 2 that share the geometric (and algebraic) properties that are equivalent modulo the group action $G \circ \mathcal{K}^2(\mathcal{M})$. Of particular importance in applications are the orbits that correspond to the CKTs belonging to the vector space $\mathcal{K}^2(\mathcal{M})$. It must be noted at this point that, in general, the topology of the orbit space $\mathcal{K}^2(\mathcal{M})/G$ (or $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$) is far from being trivial which makes the problem of *invariant classification* of the orbits fairly complicated. The problem consists of two “subproblems,” namely, the *canonical forms problem* and the *equivalence problem* which can briefly be formulated in this setting as follows:

1. **Canonical forms problem:** Consider the action $G \circ \mathcal{K}^2(\mathcal{M})$. The problem is to determine the number of *inequivalent* orbits corresponding to the CKTs defined on $(\mathcal{M}, \mathbf{g})$ as well as the canonical forms representing each of them.
2. **Equivalence problem:** Consider again the action $G \circ \mathcal{K}^2(\mathcal{M})$. Let $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$. First, the problem is to determine whether or not \mathbf{K} is a CKT. If the answer is “yes,” the main problem is to determine the corresponding orbit in the quotient space $\mathcal{K}^2(\mathcal{M})/G$ to which \mathbf{K} belongs. Finally, we also want to determine the *moving frame map*^{9,10} that maps \mathbf{K} to its respective canonical form.

Recall that Eisenhart⁸ outlined the solution to the canonical forms problem for CKTs formulated above and solved it for the case $\mathcal{M} = \mathbb{E}^3$. In Ref. 16 (see also the relevant references therein) Horwood *et al.* reformulated Eisenhart’s approach in the language of the Cartan geometry (which Eisenhart employed implicitly). Note that the canonical forms problems for the cases $\mathcal{M} = \mathbb{S}^3$ and $\mathcal{M} = \mathbb{H}^3$ were solved by Olevsky in Ref. 28, while the case $\mathcal{M} = \mathbb{M}^3$ was treated by Horwood and McLenaghan in Ref. 15. Moreover, these solutions have been used to solve the corresponding equivalence problems. Thus, Horwood *et al.*¹⁴ and Horwood¹⁷ employed two different methods to solve the equivalence problem for the CKTs defined on \mathbb{E}^3 , while Horwood *et al.*¹⁶ solved the equivalence problem for the case $\mathcal{M} = \mathbb{M}^3$.

In order to set the stage for our theory and in what follows solve the equivalence problem for $\mathcal{K}^2(\mathbb{S}^3)$, we now employ the fundamental ideas from Cartan’s approach to geometry. Observe first that $\mathcal{M} \simeq G/H$, where G is the isometry group of \mathcal{M} and H is a closed subgroup of G . Thus, for example, $\mathbb{S}^3 \simeq SO(4)/SO(3)$. Next, we observe that in view of this identification the homogeneous space G/H can be treated as the base manifold in the tautological principal bundle projection $\pi_1 : G \rightarrow G/H \simeq \mathcal{M}$ (i.e., G is the principal H -bundle over G/H). Similarly, consider the vector bundle

$\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$; then $\pi_2 : \mathcal{K}^p(\mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M} \simeq G/H$ is the vector bundle projection with the same base manifold. Upon noticing that the transitive action of the isometry group $I(\mathcal{M}) = G$ in \mathcal{M} yields the nontransitive action of G in $\mathcal{K}^p(\mathcal{M})$, where G acts as an automorphism, we consider next the orbit space $(\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G$ leading to the third projection $\pi_3 : \mathcal{K}^p(\mathcal{M}) \times \mathcal{M} \rightarrow (\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G$, having the structure of a principal G -bundle with $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$ as the total space. Following Ref. 16, we introduce the lift $f : (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G \rightarrow G$, so that the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{\pi_1} & G/H \simeq \mathcal{M} \\
 f \uparrow & & \uparrow \pi_2 \\
 (\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G & \xleftarrow{\pi_3} & \mathcal{K}^p(\mathcal{M}) \times \mathcal{M}
 \end{array} \tag{2.5}$$

Geometrically, the existence of f is equivalent to the existence of a cross section through the orbits of the orbit-space $(\mathcal{K}^p(\mathcal{M}) \times \mathcal{M})/G$. The intersections of such a cross section with the orbits are the corresponding *canonical forms*, and their coordinates are *covariants (invariants)* which play an important role in the considerations that follow.

Consider now the case of $p = 2$.

Let $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ be a CKT at a nonsingular point $\mathbf{x} \in \mathcal{M}$ (i.e., the eigenvalues of \mathbf{K} are all real and distinct at \mathbf{x}). Indeed, \mathbf{K} gives rise to a quasi-orthonormal frame $E_{\mathbf{K},\mathbf{x}}(\mathcal{M})$ of eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of \mathbf{K} at $\mathbf{x} \in \mathcal{M}$, which is also a quasi-orthonormal basis for $T_{\mathbf{x}}(\mathcal{M})$. Denoting $E(\mathcal{M})$ as the corresponding bundle of frames generated by CKTs in \mathcal{M} , it follows that $(E(\mathcal{M}), \mathcal{M}, \tilde{\pi}_2)$ defines an (oriented) quasi-orthonormal frame bundle, where $\tilde{\pi}_2 : E(\mathcal{M}) \rightarrow \mathcal{M}$. The fibres $\tilde{\pi}_2^{-1}(\mathbf{x})$ correspond to sets of all possible quasi-orthonormal frames at $\mathbf{x} \in \mathcal{M}$ generated by the eigenvectors of CKTs. Finally, this arrangement leads to the fibre bundle projection $\tilde{\pi}_3 : \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} \rightarrow E(\mathcal{M})$. Accordingly, our diagram (2.5) now assumes the following form:

$$\begin{array}{ccccc}
 G & \xrightarrow{\pi_1} & G/H \simeq \mathcal{M} & & \\
 f \uparrow & & \uparrow \pi_2 & \swarrow \tilde{\pi}_2 & \\
 (\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G & \xleftarrow{\pi_3} & \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} & \xrightarrow{\tilde{\pi}_3} & E(\mathcal{M})
 \end{array} \tag{2.6}$$

The existence of such a lift f is assured by the fact that G acts transitively on the bundle of frames $E(\mathcal{M})$ for a given CKT $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ and the classical Cartan lemma:¹²

Lemma II.3 (Cartan): Suppose that φ is a \mathfrak{g} -valued one-form on a connected (or simply connected) manifold \mathcal{M} . Then there exists a C^∞ map $F : \mathcal{M} \rightarrow G$ with $F^*\omega = \varphi$ iff

$$d\varphi = \varphi \wedge \varphi,$$

where ω is the Maurer–Cartan form on G . Moreover, the resulting map is unique up to left translation.

Furthermore, we define the map $F : E(\mathcal{M}) \rightarrow G$ to be $F = f \circ \pi_3 \circ \tilde{\pi}_3^{-1}$. Clearly, in this case G may be identified with the set of frames associated with the CKT \mathbf{K} in question, or alternatively, cross sections of the fibration $G \rightarrow G/H$ over \mathbf{K} . In view of Lemma II.3 the Maurer–Cartan form on G can be restricted to this choice of frames and thus produces a complete set of invariants (covariants) to solve the equivalence problem, which one can now solve for the orbit space $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$ (or $\mathcal{K}^2(\mathcal{M})/G$) for CKTs using the classical calculus of differential forms. More specifically, the problem of invariant classification of the orbit(s) generated by a CKT $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ reduces to fixing a quasi-orthonormal frame of eigenvectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and considering *in the frame* the corresponding Cartan structure equations

$$d\mathbf{e}^a + \omega^a_b \wedge \mathbf{e}^b = \mathbf{T}^a, \tag{2.7}$$

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = \Theta^a_b, \tag{2.8}$$

together with the Killing tensor equations for the components K_{ab} of \mathbf{K} (2.3) and the integrability conditions

$$e^a \wedge de^a = 0 \quad (\text{no sum}). \tag{2.9}$$

In these equations, $\omega^a{}_b = \Gamma_{cb}{}^a e^c$ are the connection one-forms, $T^a = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c$ are the torsion two-forms, $\Theta^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d$ are the curvature two-forms, $\{e^1, \dots, e^n\}$ is the dual basis of one-forms, the connection coefficients $\Gamma_{cb}{}^a$ correspond to the Levi-Civita connection ∇ [and hence $T^a = 0$ in (2.7)] and $R^a{}_{bcd}$ are the components of the curvature tensor. Note that (2.9) are the integrability conditions for the normality of the eigenvectors e_a . We also note that, with respect to this frame, the components of the metric g and CKT \mathbf{K} are given by

$$g_{ab} = \text{diag}(\epsilon_1, \dots, \epsilon_n) \quad \text{and} \quad K_{ab} = \text{diag}(\epsilon_1 \lambda_1, \dots, \epsilon_n \lambda_n), \tag{2.10}$$

respectively, where $\epsilon_a = \pm 1$, $a = 1, \dots, n$, and λ_a , $a = 1, \dots, n$, are the eigenvalues of \mathbf{K} . The differential invariants characterizing the orbits in question are determined by the connection one-forms $\omega^a{}_b$ that are found from a fixed quasi-orthonormal frame $\{e_1, \dots, e_n\}$. More specifically, solving the Killing tensor Eq. (2.3) for K_{ab} in this case modulo the integrability conditions (2.9) will produce a set of canonical forms (thus solving the canonical forms problem) corresponding to the orbits, while finding the connection one-forms $\omega^a{}_b$ will provide a means to distinguish between the orbits, thus solving the equivalence problem (or, rather, most of it). However, at this point we would like to point out that solving the equivalence problem in this way (i.e., by finding a complete set of differential invariants) may be extremely challenging computationally. Instead, one can make use of the fact that the group G acts transitively on the bundle of frames and try to solve it “in the group,” employing an algebraic approach to Cartan’s method of moving frames.^{9,10,29} More specifically, via the tensor transformation laws, we can determine the action of G on the parameters a_1, \dots, a_d (and the local coordinates of \mathcal{M} if necessary) that determine the vector space $\mathcal{K}^2(\mathcal{M})$ (or the product space $\mathcal{K}^2(\mathcal{M}) \times \mathcal{M}$) and then find a set of complete invariants (covariants) as algebraic functions of the parameters (parameters and local coordinates). This approach (analogous to the classical invariant theory of homogeneous polynomials) is also more preferable from the applications point of view, because the Killing two-tensors in Classical Mechanics (for example) normally appear in terms of local coordinates on \mathcal{M} , rather than in the frame of its eigenvectors as in (2.10).

The concept of a *web symmetry*, which signifies that the web is invariant under at least a one-parameter group of isometries, is another interesting geometric consequence stemming from the properties of covariants (eigenvalues) of CKTs. Suppose, for example, \mathbf{K} is a CKT in $\mathcal{K}^2(\mathcal{M})$ having $n - 1$ functionally independent eigenvalues λ_i , $i = 1, \dots, n - 1$. It follows that there exists a vector field $V \in \mathcal{K}^1(\mathcal{M})$ (V is necessarily a Killing vector), such that $\mathcal{L}_V(\mathbf{K}) = 0$. Indeed, the integral curves of V are given by the common level sets

$$\bigcap_{i=1}^{n-1} \{\lambda_i = \text{const.}\}.$$

Now we can be more specific about solving the equivalence problem outlined above. Let $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$. We first verify whether or not the Killing two-tensor \mathbf{K} in question is a CKT. This requires verifying that the eigenvalues of \mathbf{K} are real and distinct and that the eigenvector fields of \mathbf{K} are normal. The latter step can be accomplished by verifying the vanishing of the *Haantjes tensor* (H condition)¹³ (see Ref. 18 for more details):

$$H_{\hat{K}}(X, Y) := \hat{K}^2 N_{\hat{K}}(X, Y) + N_{\hat{K}}(\hat{K}X, \hat{K}Y) - \hat{K}(N_{\hat{K}}(X, \hat{K}Y) + N_{\hat{K}}(\hat{K}X, Y)) = 0, \tag{2.11}$$

where the (1, 1)-tensor $\hat{K} = \mathbf{K} g^{-1}$, $N_{\hat{K}}$ is the Nijenhuis tensor²⁷ and X, Y are arbitrary vector fields on \mathcal{M} . Thus, the vanishing of the Haantjes tensor defined by (2.11) is equivalent to the Killing tensor \mathbf{K} being a CKT. Alternatively, the formula (2.11) can be given in index form as follows:

$$H^i{}_{jk} = N^i{}_{\ell m} K^\ell{}_j K^m{}_k + 2N^\ell{}_{m[j} K^m{}_{k]} K^i{}_\ell + N^\ell{}_{jk} K^m{}_\ell K^i{}_m = 0, \tag{2.12}$$

where $K^i{}_j$ denotes the components of the (1, 1)-tensor \hat{K} and $[\dots]$ denotes skew symmetrization over the enclosed indices.

One can also employ the Tonolo–Schouten–Nijenhuis (TSN) conditions to verify the normality of the eigenvectors of a symmetric tensor field \mathbf{K} of valence (0,2). Indeed, \mathbf{K} with real distinct eigenvalues has normal eigenvectors iff the following conditions are satisfied:

$$\begin{aligned} N_{[ijgk]\ell}^\ell &= 0, \\ N_{[ijk]\ell}^\ell &= 0, \\ N_{[ijk]m}^\ell K^m_\ell &= 0, \end{aligned} \tag{2.13}$$

where N^i_{jk} are the components of the Nijenhuis tensor²⁷ of K^i_j defined by

$$N^i_{jk} = K^i_\ell K^\ell_{[j,k]} + K^\ell_{[j} K^i_{k],\ell}. \tag{2.14}$$

In the index-free form, the formula (2.14) can be obtained from the formula (2.11) by replacing the operator $N_{\hat{K}}(\cdot, \cdot)$ with the Lie bracket $[\cdot, \cdot]$.

Remark II.4: If the eigenvalues of \mathbf{K} are real and distinct, the H condition and the TSN conditions are equivalent. If the multiplicity of some of the eigenvalues of \mathbf{K} is greater than one, the TSN conditions no longer apply. However, in this case the following more general result¹³ needed in Sec. IV is true: *the distributions defined by the eigenspaces of \mathbf{K} in a Riemannian space are orthogonally integrable if and only if the H condition is satisfied.* Recall that *orthogonal integrability* signifies that the distribution defined by the orthogonal complement of the eigenspace is integrable.

If $\mathbf{K} \in \mathcal{K}^2(\mathcal{M})$ is a CKT, our next step is to determine which orbit in $\mathcal{K}^2(\mathcal{M})/G$ it belongs to, noting that the corresponding canonical form for the orbit in question is known from solving the canonical forms problem. This problem can in principle be solved by employing a number of mathematical tricks and tools, although it is hard in general due to the fact that the group action $G \circlearrowleft \mathcal{K}^2(\mathcal{M})$ is not regular and thus the orbits have different dimensions.

To solve the problem, one can employ *algebraic invariants* of the group action $G \circlearrowleft \mathcal{K}^2(\mathcal{M})$. These are the invariants of the group G acting in the *parameter space* of the vector space $\mathcal{K}^2(\mathcal{M})$, or more specifically, functions of the parameters that remain unchanged under the induced action of the group G (see, for example, Refs. 14 and 35 for more details). It must be noted, however, that the problem can normally be solved in relatively simple cases using only algebraic invariants, that is, when one has to deal with few types of orbits (e.g., $\dim \mathcal{M}$ is small). Recall that algebraic invariants of the CKTs were first introduced by Winernitz and Friš³⁵ and later rediscovered in McLenaghan *et al.*²² in a different context. More generally, one has to recover more information about the orbits and their degeneracies. One way to deal with the problem is to “improve” the group action by considering it in the extended space $\mathcal{K}^2(\mathcal{M}) \times \mathcal{M}$. After appropriate computations, this yields *algebraic covariants* of Killing tensors, which are functions of both the parameters of the vector space $\mathcal{K}^2(\mathcal{M})$ and coordinates of \mathcal{M} . The covariants of Killing tensors were introduced by Smirnov and Yue in Ref. 33 and have been successfully employed to solve a number of equivalence problems (see, for example, Horwood¹⁷). The degeneracies of the orbits in $\mathcal{K}^2(\mathcal{M})/G$ (or $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$) manifest themselves in the orbits having different dimensions—a fact which for our problem has appropriate group theoretical, geometric, and algebraic interpretations. Thus recall that in general the dimension of an orbit is determined by the formula $\dim G = \dim O_x + \dim G_x$, where G is a group acting on \mathcal{M} , $x \in \mathcal{M}$, O_x is the orbit through x , and G_x is the isotropy subgroup of G through x . The existence of isotropy subgroups indicates the degeneracies of the orbits (i.e., their dimensions drop), which in our case of $\mathcal{M} = \mathcal{K}^2(\mathcal{M})/G$ from the geometric perspective is equivalent to the CKTs corresponding to the degenerate orbits admitting *web symmetries* (see below). Algebraically this means that the CKTs corresponding to the degenerate orbits have functionally *dependent* eigenvalues. Note that if all the eigenvalues of a CKT \mathbf{K} are functionally *independent* they can be taken as the orthogonal coordinates (see Ref. 16 for more details). Degeneracies of the orbits can also be characterized by *singular points* of the CKTs in question, that is the points where the eigenvalues coincide. Unfortunately, most of the orthogonal coordinate webs used in applications come from the CKTs with degeneracies.

Solving the equivalence and canonical forms problems described above forms the mathematical foundation for applications arising in Classical Mechanics, in particular in the Hamilton–Jacobi theory of orthogonal separation of variables for the natural Hamiltonians defined in spaces of constant curvature (see Ref. 16 for relevant references and details). Let a Hamiltonian system be defined by a natural Hamiltonian of the form

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q}). \quad (2.15)$$

The following theorem (Benenti²), which is a generalization of the corresponding result due to Eisenhart concerning geodesic Hamiltonians,⁸ establishes an important link between orthogonal separation of variables in the associated Hamilton–Jacobi equation for the Hamiltonian system defined by (2.15).

Theorem II.5: *A Hamiltonian system defined by (2.15) is orthogonal separable iff there exists a CKT \mathbf{K} such that*

$$d(\hat{\mathbf{K}}dV) = 0, \quad (2.16)$$

where the (1, 1)-tensor $\hat{\mathbf{K}} = \mathbf{K}g$.

In the light of Theorem II.5 the canonical and equivalence problems formulated above can be reformulated in the language of orthogonal separation of variables for natural Hamiltonian systems defined by (2.15) in spaces of constant curvature as follows:

- (i) How many “inequivalent” coordinate systems afford orthogonal separation of variables in the corresponding HJ equation?
- (ii) If the answer to (i) is nonzero, how can one characterize intrinsically the coordinate systems that afford separation of variables in the HJ equation?
- (iii) What are the canonical coordinate transformations

$$(q^1, q^2, \dots, q^n) \rightarrow (u^1, u^2, \dots, u^n)$$

from the given position coordinates of (2.15) to the coordinate systems that afford orthogonal separation of variables of the HJ equation?

In what follows we employ some of the tools and techniques described above to solve the equivalence problem for the CKTs defined on \mathbb{S}^3 employing the results obtained by Olevsky²⁸ and Kalnins *et al.*²⁰ who solved the corresponding canonical forms problem.

III. KILLING TWO-TENSORS ON $\mathbb{S}^n \subset \mathbb{E}^{n+1}$

To fully explore the algebraic and geometric properties of Killing two-tensors defined on a sphere, we employ the (Cartesian) coordinates of the corresponding ambient Euclidean space. In this view the vector space of Killing two-tensors $\mathcal{K}^2(\mathbb{S}^n)$ is viewed as a subspace of $\mathcal{K}^2(\mathbb{E}^{n+1})$ whose elements are defined in terms of the Cartesian coordinates of \mathbb{E}^{n+1} .

Indeed, since every Killing tensor defined in a space of constant curvature is expressible as a sum of symmetrized products of Killing vectors, let us begin by defining basic Killing vectors of a Euclidean space \mathbb{E}^m in terms of the corresponding Cartesian coordinates. For our purposes, we also need to define the dilatational vector field \mathbf{D} (Euler vector field). Thus, the translational \mathbf{X}_i and rotational \mathbf{R}_{ij} Killing vectors and the vector \mathbf{D} in terms of a system of Cartesian coordinates x^i can be defined as follows:

$$\mathbf{X}_i = \frac{\partial}{\partial x^i}, \quad \mathbf{R}_{ij} = 2\delta_{ij}^{k\ell} g_{\ell m} x^m \mathbf{X}_k, \quad \mathbf{D} = x^i \mathbf{X}_i, \quad (3.1)$$

where $g_{ij} = \text{diag}(1, \dots, 1)$, denotes the Euclidean metric of the Euclidean space in question and $\delta_{ij}^{k\ell} = \delta_i^k \delta_j^\ell$, the generalized Kronecker delta. Note that the translational and rotational Killing vectors defined above form a basis for the Lie algebra $\mathcal{K}^1(\mathbb{E}^m)$ of the group of rigid motions of \mathbb{E}^m . Next,

the commutation relations among these vectors are given by

$$\begin{aligned} [X_i, X_j] &= 0, & [X_i, R_{jk}] &= 2\delta_{jk}^{\ell m} g_{mi} X_\ell, & [X_i, D] &= X_i, \\ [R_{ij}, R_{k\ell}] &= 4\delta_{ij}^{mn} \delta_{k\ell}^{pr} g_{mp} R_{nr}, & [D, R_{ij}] &= 0. \end{aligned} \quad (3.2)$$

The Killing vectors also satisfy the following algebraic identities (*syzygies*):

$$X_{[i} \odot R_{jk]} = 0, \quad R_{i[j} \odot R_{kl]} = 0, \quad (3.3)$$

where \odot denotes the symmetric tensor product. The general Killing vector thus has the form

$$K = A^i X_i + B^{ij} R_{ij}, \quad (3.4)$$

where the constants A^i and B^{ij} , called the *Killing vector parameters*, satisfy the symmetry relation

$$B^{(ij)} = 0. \quad (3.5)$$

From (3.1) and (3.4) we obtain the covariant components of K with respect to the natural basis dx^i :

$$K_i = A_i + 2B_{ij}x^j. \quad (3.6)$$

Similarly, the general valence two Killing tensor thus has the form

$$K = A^{ij} X_i \odot X_j + B^{ijk} X_i \odot R_{jk} + C^{ijkl} R_{ij} \odot R_{kl}, \quad (3.7)$$

where the constants A^{ij} , B^{ijk} , and C^{ijkl} called the *Killing tensor parameters* satisfy the following symmetry relations:

$$\begin{aligned} A^{[ij]} &= 0, \\ B^{i(jk)} &= 0, & B^{[ijk]} &= 0, \\ C^{(ij)k\ell} &= 0, & C^{ijkl} &= C^{klij}, & C^{i[jk\ell]} &= 0. \end{aligned} \quad (3.8)$$

A key observation is that the parameter set C^{ijkl} has the same symmetries as the Riemann curvature tensor. For this reason it is sometimes called an *algebraic curvature tensor*. From (3.1) and (3.7) we find that the covariant components of K with respect to the natural basis are given by

$$K_{ij} = A_{ij} + 2B_{(ij)k}x^k + 4C_{ikj\ell}x^kx^\ell. \quad (3.9)$$

This result is consistent with that obtained in Ref. 23 by the use of representation theory.

Now we may determine the form of the general Killing tensor on \mathbb{S}^n by considering it as an imbedded hypersurface in \mathbb{E}^m with $m = n + 1$ defined implicitly by the equation

$$\mathbf{x} \cdot \mathbf{x} = 1, \quad (3.10)$$

where $\mathbf{x} = (x^1, \dots, x^{n+1})$, and \cdot denotes the Euclidean inner product. Let

$$\mathbf{x} = \mathbf{f}(u^1, \dots, u^n), \quad (3.11)$$

be a local parametrization of \mathbb{S}^n . Then from (3.10) and (3.11) we obtain by differentiation

$$\mathbf{x} \cdot \mathbf{x}_\alpha = 0, \quad \mathbf{x}_{\alpha\beta} \cdot \mathbf{x} = -g_{\alpha\beta}, \quad (3.12)$$

where $\mathbf{x}_\alpha = \frac{\partial \mathbf{x}}{\partial u^\alpha}$, $(x_\alpha^i = \frac{\partial x^i}{\partial u^\alpha})$, and

$$g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta, \quad (3.13)$$

denotes the pullback of the Euclidean metric to \mathbb{S}^n . The characterization is given by the following proposition:

Proposition III.1: The general Killing vector and general valence two Killing tensor fields on \mathbb{S}^n are given by

$$K_\alpha = x_\alpha^i K_i, \quad K_{\alpha\beta} = x_\alpha^i x_\beta^j K_{ij}, \quad (3.14)$$

where

$$K_i = 2B_{ij}x^j, \quad K_{ij} = 4C_{ikj\ell}x^kx^\ell. \quad (3.15)$$

Proof: The pullback to \mathbb{S}^n of the covariant derivative of any smooth tensor field $K_{i_1\dots i_p}$ defined on \mathbb{E}^{n+1} is given by

$$x_{\alpha_1}^{i_1} \dots x_{\alpha_p}^{i_p} x_{\alpha_{p+1}}^{i_{p+1}} K_{i_1\dots i_p, i_{p+1}} = K_{\alpha_1\dots\alpha_p, \alpha_{p+1}} + px^i K_{i(\alpha_2\dots\alpha_{p-1} g_{\alpha_p}\alpha_{p+1})}. \quad (3.16)$$

If $K_{i_1\dots i_p}$ is any solution of the KT Eq. (2.3), it follows that

$$K_{(\alpha_1\dots\alpha_p, \alpha_{p+1})} + px^i K_{i(\alpha_1\dots\alpha_{p-1} g_{\alpha_p}\alpha_{p+1})} = 0. \quad (3.17)$$

We conclude that \mathbf{K} will define a Killing tensor field on \mathbb{S}^n if and only if

$$x^i K_{i(\alpha_1\dots\alpha_{p-1} g_{\alpha_p}\alpha_{p+1})} = 0. \quad (3.18)$$

This condition is clearly satisfied if

$$x^j K_{ji_1\dots i_{p-1}} = 0, \quad (3.19)$$

on \mathbb{E}^{n+1} . For the cases $p = 1, 2$ this condition takes the form

$$x^i K_i = 0, \quad x^j K_{ij} = 0. \quad (3.20)$$

It follows from (3.5), (3.6), (3.8), and (3.9) that

$$A_i = 0, \quad A_{ij} = 0, \quad B_{(ij)k=0}. \quad (3.21)$$

These conditions imply that K_i and K_{ij} have form given by (3.15). We note that, with the use of (3.2), the conditions (3.21) are equivalent to the invariant condition

$$[\mathbf{D}, \mathbf{K}] = 0. \quad (3.22)$$

It remains to demonstrate that (3.14) and (3.15) indeed define the general Killing vector and general valence two Killing tensor on \mathbb{S}^n . This may be achieved by a dimensional argument. From (2.4) it follows, respectively, for $p = 1, 2$ that

$$d = \frac{1}{2}n(n+1), \quad \frac{1}{12}n(n+1)^2(n+2). \quad (3.23)$$

For the case $p = 1$, the number d is identical to the number of independent components of the Killing vector parameter B_{ij} , while for the case $p = 2$, d is identical to the number of independent components of the algebraic curvature tensor C_{ijkl} in a $(n+1)$ -dimensional space. \square

Remark III.2: Proposition III.1 is closely related to a result proved by Delong⁷ which states that the valence p Killing tensors (considered as functions on the cotangent bundle of \mathbb{E}^{n+1}) that are in involution with the Euclidean distance function and the function $x^i p_i$, where p_i denotes the canonical momenta, are Killing tensors on \mathbb{S}^n . Delong's conditions are equivalent to (3.20) and (3.22), respectively, only one of which seems to be required to obtain the result. Furthermore, our proposition goes beyond his by giving the explicit form of the general valence one and two KTs on \mathbb{S}^n in terms of the appropriate KTs in the ambient space \mathbb{E}^{n+1} . Our proposition may be easily generalized to valence p Killing tensors by the use of the results of Horwood.¹⁷

Remark III.3: Note that the image of the contravariant metric $g^{\alpha\beta}$ on \mathbb{S}^n with respect to spherical coordinates under the pushforward induced by the parametrization gives rise to a Killing tensor \mathbf{C} called the *Casimir tensor* with the property that it commutes with every element of the vector subspace of $\mathcal{K}^2(\mathbb{E}^{n+1})$ given by (3.15). In view of the above $\mathbf{C} \in \mathcal{K}^2(\mathbb{E}^{n+1})$ is degenerate as an element of the vector subspace determined by $4C_{ikj\ell}x^kx^\ell$. We also note that \mathbf{C} is a CKT for the

spherical web on S^n . The explicit form of the Casimir tensor is given by

$$C = kg^{ik}g^{\ell j}R_{ij} \odot R_{k\ell}, \tag{3.24}$$

or with respect to the natural basis by

$$C_{ij} = kg_{i[j}g_{\ell]k}x^kx^\ell, \tag{3.25}$$

where k is some constant. It may be shown by direct calculation that $[C, K] = 0$, which verifies the above mentioned commutation property.

The next step is to determine the KT's among those given by (3.14) and (3.15) that are CKTs, that is which have pointwise distinct eigenvalues and normal eigenvectors. To effect this determination we use the fact that the eigenvalues of $K_{\alpha\beta}$ are also eigenvalues of K_{ij} and that the pushforward of the eigenvectors of $K_{\alpha\beta}$ are eigenvectors of K_{ij} . These results follow by contracting both sides of the eigenvalue equation

$$K_{ij}X^j = \lambda g_{ij}X^j, \tag{3.26}$$

with x_α^i , which yields

$$K_{\alpha\beta}X^\beta = \lambda g_{\alpha\beta}X^\beta, \tag{3.27}$$

where X^j in (3.26) is the pushforward of X^β :

$$X^j = x_\beta^jX^\beta. \tag{3.28}$$

We conclude that if K_{ij} has distinct eigenvalues then so does $K_{\alpha\beta}$. The converse is not true since if $K_{\alpha\beta}$ has a zero eigenvalue then by (3.20) K_{ij} will have zero as a repeated eigenvalue. However, the remaining eigenvalues will be nonzero and distinct. We also observe by (3.12) that the eigenvector x^i corresponding to the zero eigenvalue of K_{ij} pulls back to the zero vector on S^n .

We next show that the pullback of a normal covector field on \mathbb{E}^{n+1} is a normal vector field on S^n . To see this consider any covector field E_i on \mathbb{E}^{n+1} which satisfies the normality condition

$$E_{[i,j}E_{k]} = 0. \tag{3.29}$$

We note that (3.29) is the component form of the integrability condition (2.9). An easy computation shows that

$$E_{i,j}E_kx_\alpha^ix_\beta^jx_\gamma^k = E_{\alpha,\beta}E_\gamma - x_{\alpha\beta}^iE_iE_\gamma. \tag{3.30}$$

It follows from (3.29) and (3.30) that

$$E_{[\alpha,\beta}E_\gamma] = 0, \tag{3.31}$$

which implies that the pullback E_α is a normal vector field. The above results imply that the pullback of a characteristic KT on \mathbb{E}^{n+1} is a characteristic KT on S^n . In view of the above results we are able to study CKTs on S^n as pullbacks of CKTs on \mathbb{E}^{n+1} .

We observe that the use of (2.9) to determine whether the eigenvector fields of K are normal is impractical since it requires the explicit determination of the eigenvector fields, an intractable problem. However, alternatively we may utilize either the TSN conditions (2.13) or the H condition defined by (2.12). We first consider the TSN conditions for a KT of the form (3.15). These conditions impose the corresponding algebraic conditions on the algebraic curvature tensor C :

$$C^\ell{}_{(pq[i}C_{jk]r)\ell} = 0, \tag{3.32}$$

$$C_{\ell(pq}{}^m C^\ell{}_{r[ij}C_{k]st)m} - 2C_{\ell(pq[i}C_{j}{}^\ell{}_{|r|}{}^m C_{k]st)m} = 0, \tag{3.33}$$

$$3C_{\ell(pq}{}^m C^\ell{}_{r|n|s}C^n{}_{t[ij}C_{k]uv)m} + 2C_{\ell(pq}{}^m C_{|n|rs[i}C_{j]t|}{}^n C_{k]uv)m} + 2C_{\ell(pq}{}^m C_{|n|rs[i}C_{j}{}^n{}_{|t|}{}^\ell C_{k]uv)m} = 0, \tag{3.34}$$

where $|\dots|$ denotes exclusion of the enclosed indices from the symmetrization process. The first TSN condition (3.32) implies

$$C_{\ell(pq}{}^m C_{r[ij}{}^\ell C_{k]st)m} + 2C_{\ell(pq[i} C_{j}{}^\ell |r|}{}^m C_{k]st)m} = 0. \quad (3.35)$$

This equation together with the second TSN condition (3.33) yields the two equations

$$C_{\ell(pq}{}^m C_{r[ij}{}^\ell C_{k]st)m} = 0, \quad (3.36)$$

$$C_{\ell(pq[i} C_{j}{}^\ell |r|}{}^m C_{k]st)m} = 0. \quad (3.37)$$

These equations replace the second TSN condition (3.33). We now show that the third TSN condition is a consequence of the first and second conditions. The cyclical identity implies

$$C_{\ell(pq}{}^m C_{|n|rs[i} C_{j}{}^\ell |t|}{}^n C_{k]uv)m} = -C_{\ell(pq}{}^m C_{|n|rs[i} C_{j|t|}{}^{n\ell} C_{k]uv)m} + C_{\ell(pq}{}^m C_{|n|rs[i} C_{j}{}^n |t|}{}^\ell C_{k]uv)m}. \quad (3.38)$$

From the above equation and (3.37) we obtain

$$C_{\ell(pq}{}^m C_{|n|rs[i} C_{j|t|}{}^{n\ell} C_{k]uv)m} = 0, \quad (3.39)$$

and

$$C_{\ell(pq}{}^m C_{|n|rs[i} C_{j}{}^\ell |t|}{}^n C_{k]uv)m} = C_{\ell(pq}{}^m C_{|n|rs[i} C_{j}{}^n |t|}{}^\ell C_{k]uv)m}. \quad (3.40)$$

Now from the first TSN condition and (3.40) we find

$$C_{\ell(pq}{}^m C_{r|n|s}{}^\ell C_{t[ij}{}^n C_{k]uv)m} - 2C_{\ell(pq}{}^m C_{|n|rs[i} C_{j}{}^n |t|}{}^\ell C_{k]uv)m} = 0. \quad (3.41)$$

On the other hand (3.36) implies

$$C_{\ell(pq}{}^m C_{r|n|s}{}^\ell C_{t[ij}{}^n C_{k]uv)m} + 2C_{\ell(pq}{}^m C_{|n|rs[i} C_{j}{}^n |t|}{}^\ell C_{k]uv)m} = 0. \quad (3.42)$$

These equations together imply

$$C_{\ell(pq}{}^m C_{r|n|s}{}^\ell C_{t[ij}{}^n C_{k]uv)m} = 0, \quad (3.43)$$

$$C_{\ell(pq}{}^m C_{|n|rs[i} C_{j}{}^n |t|}{}^\ell C_{k]uv)m} = 0. \quad (3.44)$$

Finally (3.39) and (3.43), and (3.44) imply that the third TNS condition (3.34) is identically satisfied.

Remark III.4: This result was first proven in \mathbb{E}^3 by Czapor¹⁴ using Gröbner basis theory and computer algebra. Schöbel³¹ extended the result to n -dimensional spaces of nonzero constant curvature using representation theory of the symmetric group. However, it seems that our proof, based on standard indicial tensor algebra, is simpler and more concise.

Substituting the Killing tensor (3.15) into the H condition (2.12) we obtain the following condition on the coefficients:

$$\begin{aligned} & 4C_{\ell(pq}{}^k C_{rs[i}{}^m C_{j]tu}{}^n C_{v]mn}{}^\ell + 2C_{\ell(p|m|}{}^k C_{qr[i}{}^n C_{j]st}{}^m C_{uv)n}{}^\ell \\ & - 5C_{\ell(pq}{}^k C_{rs[i}{}^m C_{j]m|t}{}^n C_{uv)n}{}^\ell + C_{\ell(pq}{}^k C_{rs[i}{}^m C_{j]t}{}^n C_{n|uv)m}{}^\ell \\ & + C_{\ell(pq}{}^k C_{rs[i}{}^m C_{j]tu}{}^n C_{n|v]m}{}^\ell - 3C_{\ell(pq}{}^k C_{r|ij}{}^m C_{st|m|}{}^n C_{uv)n}{}^\ell \\ & - 2C_{\ell(pq}{}^k C_{rs[i}{}^m C_{j]t|m|}{}^n C_{uv)n}{}^\ell = 0. \end{aligned} \quad (3.45)$$

Using this condition we can verify whether or not a given Killing tensor defined on \mathbb{S}^n is characteristic. It may be shown that the first and second TSN conditions (3.32) and (3.33) imply that the H condition (3.45) is satisfied.

We next examine the pullback of the quadratic first integral

$$K = K^{ij} p_i p_j + U. \quad (3.46)$$

The vanishing of the Poisson bracket implies that K_{ij} satisfies (2.3) for $p = 2$, and that

$$\frac{\partial U}{\partial x^i} = g_{ij} K^{j\ell} \frac{\partial V}{\partial x^\ell}, \quad (3.47)$$

where V is the potential function. The pullback to \mathbb{S}^n is given by

$$\frac{\partial U}{\partial x^i} x_\alpha^i = g_{ij} K^{\beta\gamma} x_\beta^j x_\gamma^\ell \frac{\partial V}{\partial x^\ell} x_\alpha^i, \quad (3.48)$$

which may be written as

$$\frac{\partial U}{\partial u^\alpha} = g_{\alpha\beta} K^{\beta\gamma} \frac{\partial V}{\partial u^\gamma}. \quad (3.49)$$

We conclude this section by studying the action of $SE(m)$ on the Killing vector and Killing tensor parameters.¹⁷ The action $SE(m) \circlearrowleft \mathbb{E}^m$ is given by

$$x^i = \Lambda^i_j \tilde{x}^j + \delta^i, \quad (3.50)$$

where $\Lambda^i_j \in SO(m)$, $\delta^i \in \mathbb{E}^m$, and \tilde{x}^i denote the transformed Cartesian coordinates. This transformation induces by (3.1) the following transformation of the Killing vectors:

$$\mathbf{X}_i = \Lambda^j_i \tilde{\mathbf{X}}_j, \quad \mathbf{R}_{ij} = \Lambda^k_i \Lambda^\ell_j \tilde{\mathbf{R}}_{k\ell} + \mu_{ij}{}^k \tilde{\mathbf{X}}_k, \quad (3.51)$$

where

$$\mu_{ij}{}^k = 2\delta_{ij}^{\ell m} g_{mn} \Lambda^k_\ell \delta^n. \quad (3.52)$$

It follows that the Killing vector and parameters transform as

$$\tilde{A}^i = \Lambda^i_j A^j + \mu_{jk}{}^i B^{jk}, \quad \tilde{B}^{ij} = \Lambda^i_k \Lambda^j_\ell B^{k\ell}, \quad (3.53)$$

while the valence two Killing tensor parameters transform as

$$\begin{aligned} \tilde{A}^{ij} &= \Lambda^i_k \Lambda^j_\ell A^{k\ell} + 2\Lambda^{(i}_k \mu_{\ell m}{}^j) B^{klm} + \mu_{k\ell}{}^i \mu_{mn}{}^j C^{klmn}, \\ \tilde{B}^{ijk} &= \Lambda^i_\ell \Lambda^j_m \Lambda^k_n B^{\ell mn} + \mu_{mn}{}^i \Lambda^j_p \Lambda^k_q C^{mnpq}, \\ \tilde{C}^{ijkl} &= \Lambda^i_m \Lambda^j_n \Lambda^k_p \Lambda^\ell_q C^{mnpq}. \end{aligned} \quad (3.54)$$

In order to make our formulas more compact we introduce a multi-index notation where any upper-case index represents a pair of skew-symmetric lower-case indices. Thus \mathbf{R}_I will represent \mathbf{R}_{ij} . Using this notation we may rewrite (3.51) as

$$\mathbf{R}_I = \Lambda^J_I \tilde{\mathbf{R}}_J + \mu_I{}^k \tilde{\mathbf{X}}_k, \quad (3.55)$$

where $\Lambda^J_K = \Lambda^i_{[k} \Lambda^j_{\ell]}$, represents the second compound of Λ^i_j . With the same notation (3.53) reads

$$\tilde{A}^i = \Lambda^i_j A^j + \mu_J{}^i B^J, \quad \tilde{B}^I = \Lambda^I_J B^J, \quad (3.56)$$

whereas (3.54) reads

$$\begin{aligned} \tilde{A}^{ij} &= \Lambda^i_k \Lambda^j_\ell A^{k\ell} + 2\Lambda^{(i}_k \mu_{\ell L}{}^j) B^{kL} + \mu_K{}^i \mu_L{}^j C^{KL}, \\ \tilde{B}^{iJ} &= \Lambda^i_\ell \Lambda^J_M B^{\ell M} + \mu_L{}^i \Lambda^J_M C^{LM}, \\ \tilde{C}^{IJ} &= \Lambda^I_L \Lambda^J_M C^{LM}. \end{aligned} \quad (3.57)$$

We are now in a position to give an invariant classification of the translational Killing vectors on \mathbb{E}^m under the action of $SE(m)$. The invariant needed to effect this classification is $\mathcal{I}_1 = B_{ij} B^{ij}$.

A translational Killing vector in \mathbb{E}^m is invariantly defined by the condition $\mathcal{I}_1 = 0$, which implies $B^{ij} = 0$. Thus by (3.4) a translational KV has the form

$$\mathbf{K} = A^i \mathbf{X}_i, \quad (3.58)$$

where it is assumed that not all the A^i are zero so that \mathbf{K} is nontrivial. It follows from the transformation formula (3.53) that, by an appropriate choice of $\Lambda^i_j \in \text{SO}(m)$,¹⁴ we can transform (3.58) to the form

$$\mathbf{K} = A^1 \mathbf{X}_1, \quad (3.59)$$

for some $A^1 \neq 0$, where the tildes have been dropped.

The case when $\mathcal{I}_1 \neq 0$, characterizes the *nontranslational Killing vectors* the analysis of which will be left to a future paper.

However, we are in a position to classify the rotational Killing vectors in \mathbb{E}^4 given by (3.15) that define KVs on \mathbb{S}^3 . For this we need the additional invariant $\mathcal{I}_2 = \epsilon_{ijkl} B^{ij} B^{kl}$. We utilize the property¹¹ that the skew-symmetric matrix B^{ij} , assumed to be not the 0 matrix, may, by use of the second formula of (3.53), be transformed to the following canonical form by an element of $\text{O}(4)$:

$$B^{ij} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_2 \\ 0 & 0 & -\kappa_2 & 0 \end{pmatrix}, \quad (3.60)$$

where $\kappa_1, \kappa_2 \in \mathbb{R}$. If $\mathcal{I}_1 \neq 0$ and $\mathcal{I}_2 = 0$, then $\kappa_1^2 + \kappa_2^2 \neq 0$ and $\kappa_1 \kappa_2 = 0$. It follows by use of the first formula of (3.53), under the assumption $\kappa_2 = 0$, that there exists $\delta^i \in \mathbb{E}^m$ and $\Lambda^i_j \in \text{SO}(m)$ such that the KV may be written as

$$\mathbf{K} = A^3 \mathbf{X}_3 + b_{12} \mathbf{R}_{12}, \quad (3.61)$$

where A^3 and $b_{12} = 2\kappa_1$ are constant. We now assume $\mathcal{I}_2 \neq 0$, which implies that $\kappa_1 \kappa_2 \neq 0$. In this case the matrix B^{ij} is rank four from which it follows by use of (3.53) that the KV may be expressed as

$$\mathbf{K} = b_{12} \mathbf{R}_{12} + b_{34} \mathbf{R}_{34}, \quad (3.62)$$

where $b_{12} = 2\kappa_1$ and $b_{34} = 2\kappa_2$ are constant. The Eqs. (3.61) and (3.62) via Proposition III.1 give the two possible canonical forms for the rotations on \mathbb{S}^3 .

IV. THE EQUIVALENCE PROBLEM FOR KILLING TENSORS ON \mathbb{S}^3

To solve the equivalence problem outlined in Sec. III for the CKTs on \mathbb{S}^3 we employ the Cartesian coordinates of the ambient Euclidean space $\mathbb{E}^4 \supset \mathbb{S}^3$. In terms of these coordinates, the general Killing tensor of \mathbb{S}^3 is given by

$$\mathbf{K} = 4C^{ijkl} \mathbf{R}_{ij} \odot \mathbf{R}_{kl}, \quad (4.1)$$

where

$$\mathbf{R}_{ij} = \delta^m_{ijk} x^k \mathbf{X}_m, \quad \delta^m_{ijk} = \delta_i^m g_{jk} - \delta_j^m g_{ik}, \quad (4.2)$$

$$i, j, k, \ell, m = 1, \dots, 4,$$

x^1, \dots, x^4 are Cartesian coordinates of \mathbb{E}^4 , $\mathbf{X}_i = \partial_i$, g_{ij} are the components of the metric and \odot is the symmetric tensor product. Note that (4.2) defines the six generators of $\mathfrak{so}(4)$, the Lie algebra of the isometry group $\text{SO}(4)$ of \mathbb{S}^3 (no reflections). Therefore the RHS of (4.1) represents a vector space of Killing tensors on \mathbb{S}^3 . Furthermore, the dimension d of this space is determined by the tensor C^{ijkl} which has the same symmetries as that of the Riemann curvature tensor, hence $d = 20$.

The special orthogonal group $\text{SO}(4)$ is a Lie subgroup of the orthogonal group $\text{O}(4)$ consisting of all orthogonal matrices Λ with positive unit determinant. The transitive action of $\text{SO}(4)$ on \mathbb{E}^4 can be specified by

$$x^i = \Lambda^i_j \tilde{x}^j,$$

where $\Lambda^i_j \in \text{SO}(4)$ and x^i denote Cartesian coordinates. This, in turn, induces the following transformation

$$\mathbf{R}_{ij} = \Lambda^k_i \Lambda^\ell_j \tilde{\mathbf{R}}_{k\ell}, \quad K^{ij} = \Lambda^i_k \Lambda^j_\ell \tilde{K}^{k\ell}, \tag{4.3}$$

on the Killing vectors (4.2) and Killing tensors (4.1) of \mathbb{E}^4 . At the same time, this action induces the following transformations:

$$\tilde{B}^{ij} = \Lambda^i_k \Lambda^j_\ell B^{k\ell}, \quad \tilde{C}^{ijkl} = \Lambda^i_p \Lambda^j_q \Lambda^k_r \Lambda^\ell_s C^{pqrs}, \tag{4.4}$$

on the Killing vector and Killing tensor parameters.

To make our formulas more compact, we will once again adopt a multi-index notation (i.e., \mathbf{R}_I will represent \mathbf{R}_{ij}). Using this notation we may rewrite the first equation of (4.3) as

$$\mathbf{R}_I = \Lambda^J_I \tilde{\mathbf{R}}_J, \tag{4.5}$$

where $\Lambda^I_K = \Lambda^i_{[k} \Lambda^j_{\ell]}$, represents the second compound of Λ^i_j . With the same notation (4.4) reads

$$\tilde{B}^I = \Lambda^I_J B^J, \quad \tilde{C}^{IJ} = \Lambda^I_K \Lambda^J_L C^{KL}. \tag{4.6}$$

To obtain the invariants and covariants of the group action $\text{SO}(4) \curvearrowright \mathcal{K}^2(\mathbb{S}^3)$, we apply the theory developed in Ref. 17. In particular, by taking contractions of products of the general Killing tensor, the Euclidean metric, and the coefficient tensor, we can obtain a complete set of $\text{SO}(4)$ -covariants and invariants. Using matrices \mathbf{K} and \mathbf{C} , where $(\mathbf{K})^i_j = K^i_j$, and $(\mathbf{C})^I_J = C^{ij}_{k\ell}$, as well as the trace operator Tr , we can express these covariants \mathcal{C}_i and invariants \mathcal{I}_i as follows:

$$\mathcal{C}_1 = \text{Tr}(\mathbf{K}), \quad \mathcal{C}_2 = \text{Tr}(\mathbf{K}^2), \quad \mathcal{C}_3 = \text{Tr}(\mathbf{K}^3), \quad \mathcal{C}_4 = g_{ij} x^i x^j,$$

$$\mathcal{I}_1 = \text{Tr}(\mathbf{C}), \quad \mathcal{I}_2 = \text{Tr}(\mathbf{C}^2), \quad \mathcal{I}_3 = \text{Tr}(\mathbf{C}^3), \quad \dots, \quad \mathcal{I}_{14} = \text{Tr}(\mathbf{C}^{14}),$$

where, for example, $\text{Tr}(\mathbf{K}) = K^i_i$ and $\text{Tr}(\mathbf{C}) = C^I_I$.

As we stated in Sec. II, the action $\text{SO}(4) \curvearrowright \mathcal{K}^2(\mathbb{S}^3)$ foliates the vector space $\mathcal{K}^2(\mathbb{S}^3)$ into the orbit space $\mathcal{K}^2(\mathbb{S}^3)/\text{SO}(4)$. Each orbit is represented by a canonical form, and the solution to the equivalence problem requires determining such canonical forms as well as a classification scheme for finding which orbit a given CKT belongs to. As we mentioned in the introduction, the canonical forms problem has been solved. Please refer to the appendix for a list of six canonical forms for the orbit space $\mathcal{K}^2(\mathbb{S}^3)/\text{SO}(4)$.

Let us now develop a classification scheme for the orbit space $\mathcal{K}^2(\mathbb{S}^3)/\text{SO}(4)$. The set of invariants and covariants listed above could be used to try and classify the orbits of these CKTs. This approach was successfully implemented in the solution to the equivalence problem of Killing tensors defined on \mathbb{E}^3 ,¹⁷ although the calculations were quite cumbersome. Indeed, the difficulty in this approach lies in finding certain combinations of the above invariants and covariants which distinguish between the orbits. To reduce many of these difficulties, the *method of web symmetries* can be used as an initial step in developing a classification scheme. The central idea of this method is to use the symmetry properties of the associated orthogonal separable web of a canonical CKT to characterize its orbit. As a result, webs are first categorized according to their symmetry properties and then in each symmetry subspace a new set of “reduced invariants” is determined to classify the webs of the category. In Sec. II we stated that the web symmetries of a Killing tensor \mathbf{K} are generated by Killing vectors on the manifold. Thus, to determine all of the symmetry generators of a given web we impose the following condition:

$$\mathcal{L}_V \mathbf{K} = 0 \tag{4.7}$$

on the Killing tensor \mathbf{K} defining the web using the general Killing vector V of our manifold.

In what follows we demonstrate the surprising result that the six CKTs of \mathbb{S}^3 can be classified based purely on the symmetry properties of their associated webs. To this end, we are interested in obtaining the symmetry properties of a web *before* we impose the spherical constraint, which will yield additional web symmetries for the CKTs. Visually, this corresponds to capturing all of the symmetry properties of a web before it is intersected with the surface of \mathbb{S}^3 .

TABLE I. Symmetry classification of the six webs of \mathbb{S}^3 under the action of $\text{SO}(4)$.

Category	Symmetry	Separable webs	Generators
1	2 rotations	cylindrical	$\mathbf{R}_{12}, \mathbf{R}_{34}$
2	1 translation & 1 rotation	spherical	$\mathbf{R}_{12}, \mathbf{X}_4$
3	1 translation	spheroelliptic	\mathbf{X}_4
4	1 rotation	elliptic-cylindrical I elliptic-cylindrical II	\mathbf{R}_{12}
5	none	ellipsoidal	

To achieve this, we impose condition (4.7) on each of the six CKTs using the general Killing vector of the ambient space \mathbb{E}^4 . This will enable us to determine if a web is rotationally and/or translationally symmetric before it intersects the surface of \mathbb{S}^3 . Upon applying this method, we find that four of the six webs admit at least one rotational web symmetry. We can go even further by noting the number of rotational symmetries a CKT admits, which effectively divides the six canonical forms into three categories. Lastly, we find that two of the six webs admit translational symmetry, which provides the final distinguishing feature between each of the six webs. Please refer to Table I for a summary of these results.

Remark IV.1: In an application problem, it is possible that a given CKT \mathbf{K} may have the Casimir tensor present. Specifically,

$$\mathbf{K} = \alpha \mathcal{C} + \mathbf{K}_1,$$

where \mathcal{C} is the Casimir tensor (3.24) and \mathbf{K}_1 is a CKT. If \mathbf{K}_1 is translationally symmetric, then the addition of the rotationally symmetric Casimir tensor destroys this translational symmetry. Thus in order to determine all of the symmetries of \mathbf{K} with or without the presence of the Casimir tensor, it is necessary to check the more general condition

$$\mathcal{L}_{\mathbf{V}}(\mathbf{K} + \alpha \mathcal{C}) = 0,$$

for arbitrary $\alpha \in \mathbb{R}$.

It is necessary to prove that the aforementioned symmetry properties of a Killing tensor are invariant under the action of $\text{SO}(4)$. To do so, it suffices to solve the equivalence problem of Killing vectors of $\mathcal{K}^1(\mathbb{E}^4)$ under the action of the group $\text{SO}(4)$. To begin, we note that the general Killing vector of \mathbb{E}^4 is given by

$$\mathbf{K} = A^i \mathbf{X}_i + B^I \mathbf{R}_I,$$

where A^i and B^I denote the Killing vector parameters, and \mathbf{X}_i and \mathbf{R}_I are the Killing vector fields defined previously. The action of $\text{SO}(4)$ on \mathbf{K} induces the following transformations:

$$\tilde{A}^i = \Lambda^i_j A^j, \quad \tilde{B}^I = \Lambda^I_K B^K,$$

on the Killing vector parameters. Therefore, it follows that

$$\mathcal{I}_1 = B^I B_I, \quad \mathcal{I}_2 = A^i A_i$$

are invariants in the orbit space $\mathcal{K}^1(\mathbb{S}^3)/\text{SO}(4)$. Using either of these two invariants it is possible to distinguish between two different types of symmetry generators. Please refer to Table II for a summary of these results. We can conclude that the translational and rotational web symmetries as defined by the Killing vectors of $\mathcal{K}^1(\mathbb{E}^4)$ are inequivalent under the action of $\text{SO}(4)$.

In addition to a classification scheme, a solution to the equivalence problem also requires a method for determining the *moving frames map* which identifies the group action required to return a given CKT to the canonical form of its orbit. On the two-dimensional manifolds $\mathbb{E}^2, \mathbb{M}^2$, and \mathbb{S}^2 algebraic formulas have been derived^{18,22,24} for determining the moving frame map of a given CKT. On \mathbb{E}^3 and \mathbb{M}^3 , a combination of web symmetry and eigenvalues and eigenvectors of the

TABLE II. Invariant classification of Killing vectors on \mathbb{E}^4 under the action of $SO(4)$.

Category	Canonical form	\mathcal{I}_1
1	\mathcal{R}_{12}	$\neq 0$
2	X_1	0

parameter matrices has been used to determine such a map.^{14,16} In our case, however, the situation is complicated by the fact that our coefficient tensor, C^{ijkl} , has order 6 when regarded as a matrix. As such, we will need to devise a different strategy for determining the moving frame map of a CKT on S^3 .

It has been noted in Sec. III [see formula (3.8)] that the coefficient tensor C^{ijkl} has the same symmetries as the curvature tensor, and thus can be called an *algebraic curvature tensor*. In light of this property, let us lower the last three indices of C^{ijkl} and contract on the first and third indices

$$Ric = C^i_{jil} = \mathcal{R}_{j\ell},$$

to obtain an *algebraic Ricci tensor*. The coefficient tensor for each of the six canonical forms listed in the Appendix can be contracted to define a *canonical Ricci tensor* in each case. The following proposition demonstrates that the Ricci tensor can be used to define the moving frame map for a given CKT.

Proposition IV.2: A Killing tensor (4.1) is in canonical form if and only if its Ricci tensor is in canonical form.

Proof 2: Since the canonical form of the Ricci tensor is defined by the canonical form of \mathbf{K} , the “if” part of the proof is trivial. To prove the “only if” part, we argue by contradiction. Suppose the Ricci tensor of a Killing tensor \mathbf{K} is in canonical form, but \mathbf{K} is not. Since $SO(4)$ acts transitively on the orbits of $\mathcal{K}^2(S^3)/SO(4)$, we can find a group action $\Lambda \in SO(4)$ which sends \mathbf{K} to its canonical form $\tilde{\mathbf{K}}$. In particular, the components of \mathbf{K} transform according to (4.3) which induces the following transformation:

$$\tilde{C}^i_{jkl} = \Lambda^i_m \Lambda^n_j \Lambda^p_k \Lambda^q_\ell C^m_{npq},$$

on the coefficient tensor C^i_{jkl} . At the same time, this action on C induces the following transformation:

$$\tilde{\mathcal{R}}_{j\ell} = \Lambda^m_j \Lambda^n_\ell \mathcal{R}_{mn},$$

on its Ricci tensor \mathcal{R} . Since the Ricci tensor of a canonical Killing tensor is necessarily canonical, we must have $\tilde{\mathcal{R}} = \mathcal{R}$, which is a contradiction. \square

According to Proposition IV.2, the moving frame map of a CKT can be constructed by determining the moving frame map of the corresponding Ricci tensor. Note that each canonical Ricci tensor can be represented by a diagonal matrix of order 4. Therefore, the determination of the moving frame map for the Ricci tensor is an eigenvalue–eigenvector problem for matrices of order 4. Before we illustrate this technique with the application in Sec. V, we summarize our results in the following algorithm:

1. *Compatibility condition.* Begin by substituting the potential into the compatibility condition (2.16) to determine the most general Killing tensor compatible with the potential. Using this Killing tensor determine the subspace of CKTs.
2. *Classification.* Next, we classify a CKT \mathbf{K} by determining whether it admits any symmetry. Namely, impose the constraint

$$\mathcal{L}_{\mathbf{V}}(\mathbf{K} + \alpha C) = 0,$$

where α is an arbitrary parameter, \mathcal{C} is the Casimir tensor, and \mathbf{V} is the general Killing vector of \mathbb{E}^4 ,

$$\mathbf{V} = A^i \mathbf{X}_i + B^I \mathbf{R}_I.$$

If \mathbf{K} does admit symmetry, determine which type and the number of generators for each type. Consult Table I to classify the CKT.

3. *Moving frame map.* To determine the moving frame map for \mathbf{K} find the Ricci tensor \mathcal{R} of the coefficient tensor. Diagonalize \mathcal{R} by solving the corresponding eigenvalue–eigenvector problem. The matrix Λ , which diagonalizes \mathcal{R} defines the moving frame map.
4. *Orthogonally separable coordinates.* Finally, define the orthogonally separable set of coordinates corresponding to \mathbf{K} by substituting Λ found in the previous step into the equation

$$x^i = \Lambda^i_j T^j(u^k),$$

where $x^i = T^j(u^k)$ denote the canonical orthogonally separable coordinates corresponding to \mathbf{K} .

V. APPLICATION

Consider the following Hamiltonian

$$H = \frac{1}{2} C^{ij} p_i p_j + \frac{1}{(x-y)^2},$$

defined on \mathbb{S}^3 , where C^{ij} denotes the Casimir tensor and x, y, z, w are the Cartesian coordinates of the ambient space \mathbb{E}^4 . Using this Hamiltonian, we will now demonstrate how to apply the theory outlined in this paper.

First, we impose the compatibility condition (2.16) to obtain a family of Killing tensors which are compatible with the potential. Of this family, the following restrictions on the parameters yield a subfamily of Killing tensors which satisfy the Haantjes condition (2.11) and generally admit three distinct eigenvalues:

$$\begin{aligned} C_{1212} &= C_{3434}, \quad C_{1313} = C_{2323}, \quad C_{1414} = C_{2424}, \\ C_{1323} &= C_{1313} - C_{1212}, \quad C_{1424} = C_{1212} - C_{1414}. \end{aligned}$$

Therefore we conclude that \mathbf{K} must characterize at least one of the six orthogonally separable webs of \mathbb{S}^3 . After a direct calculation, we find that \mathbf{K} admits the following family of rotational Killing vectors:

$$\mathbf{V} = (c_3 z + c_6 w) \frac{\partial}{\partial x} + (c_6 w - c_3 z) \frac{\partial}{\partial y} + (c_3 y - c_6 x) \frac{\partial}{\partial z} - (c_6 y + c_3 x) \frac{\partial}{\partial w},$$

for arbitrary constants c_3 and c_6 . Using the classification scheme outlined in Table I we conclude that \mathbf{K} characterizes a noncanonical cylindrical web.

In order to determine the orthogonally separable coordinates for this Killing tensor, we need to determine the transformation which maps \mathbf{K} to its canonical form. As discussed in Sec. IV, such a map can be constructed by diagonalizing the Ricci tensor of the coefficient tensor. Contracting indices, we obtain the following noncanonical Ricci tensor for this family of characteristic Killing tensors

$$\mathcal{R}_{jk} = \begin{pmatrix} C_{1212} + C_{1313} + C_{1414} & C_{1414} - C_{1313} & 0 & 0 \\ C_{1414} - C_{1313} & C_{1212} + C_{1313} + C_{1414} & 0 & 0 \\ 0 & 0 & 2C_{1313} + C_{1212} & 0 \\ 0 & 0 & 0 & 2C_{1414} + C_{1212} \end{pmatrix}.$$

TABLE III. Solutions to the canonical forms and equivalence problems.

	Canonical forms problem	Equivalence problem
Euclidean space \mathbb{E}^3	Eisenhart, (Ref. 8) Boyer <i>et al.</i> , (Ref. 4)	Horwood <i>et al.</i> (Ref. 14) Horwood, (Ref. 17)
Minkowski space \mathbb{M}^3	Horwood and McLanaghan, (Ref. 15)	Horwood <i>et al.</i> (Ref. 16)
Sphere \mathbb{S}^3	Olevsky, (Ref. 28)	The present paper

After calculating the eigenvalues and corresponding eigenvectors of \mathcal{R}_{jk} and applying the Gram–Schmidt orthonormalization procedure, we obtain an orthogonal matrix

$$\Lambda^i_j = \begin{pmatrix} \frac{-\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which brings the Ricci tensor into canonical form

$$\tilde{\mathcal{R}}_{jk} = \text{diag}(2C_{1313} + C_{1212}, 2C_{1313} + C_{1212}, 2C_{1414} + C_{1212}, 2C_{1414} + C_{1212}).$$

Therefore, we conclude that

$$\begin{aligned} x &= -\frac{\sqrt{2}}{2} \cos t \cos u + \frac{\sqrt{2}}{2} \sin t \sin v, \\ y &= \frac{\sqrt{2}}{2} \cos t \cos u + \frac{\sqrt{2}}{2} \sin t \sin v, \\ z &= -\cos t \sin u, \\ w &= \sin t \cos v \end{aligned}$$

is a system of orthogonally separable coordinates for this Hamiltonian.

VI. CONCLUSION

The results presented in this paper conclude an important project within the framework of a more general program on the development of the Hamilton–Jacobi theory of orthogonal separation of variables for natural Hamiltonians defined in spaces of constant curvature (see Table III). Having solved the equivalence problem, thus extending the classical result of Olevsky,²⁸ we have developed a general algorithm for solving natural Hamiltonians defined on the 3-sphere via orthogonal separation of variables. In addition we give a simple and concise proof of the fact that the validity of the first and second TSN conditions imply the validity of the third. We have also derived a set of analogous algebraic conditions following from the vanishing of the Haanjes tensor which can be used to study and characterize algebraic and geometric properties of Killing two-tensors defined in spaces of nonzero constant curvature. These conditions provide an alternative characterization of CKTs to the one derived in Ref. 31 and this paper based on the TSN criterion.

The results presented here lay the groundwork for a project that concerns orthogonal separation of variables afforded by characteristic Killing two-tensors defined in three-dimensional hyperbolic space which is the subject of a forthcoming paper.

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APPENDIX. SEPARABLE METRICS AND KILLING TENSOR CANONICAL FORMS

1. Orthogonally separable metrics and coordinate systems

The following is a list of the orthogonally separable metrics and associated coordinate systems for \mathbb{S}^3 . In each case, the coordinate systems define a transformation from the separable coordinates (t, u, v) to Cartesian coordinates (x, y, z, w) .

I. Spherical coordinates

$$\begin{cases} ds^2 = dt^2 + \sin^2 t (du^2 + \sin^2 u dv^2) \\ x = \sin t \sin u \cos v, \quad y = \sin t \sin u \sin v, \\ z = \sin t \cos u, \quad w = \cos t \\ 0 \leq t \leq \pi, \quad 0 \leq u \leq \pi, \quad 0 \leq v < 2\pi \end{cases}$$

II. Cylindrical coordinates

$$\begin{cases} ds^2 = dt^2 + \cos^2 t du^2 + \sin^2 t dv^2 \\ x = \cos t \cos u, \quad y = \cos t \sin u, \\ z = \sin t \cos v, \quad w = \sin t \sin v \\ 0 \leq t \leq \pi, \quad 0 < u \leq 2\pi, \quad 0 < v \leq 2\pi \end{cases}$$

III. Elliptic-cylindrical coordinates of type 1

$$\begin{cases} ds^2 = (\operatorname{dn}^2(u; \tilde{k}) - k^2 \operatorname{sn}^2(t; k))(dt^2 + du^2) + \operatorname{sn}^2(t; k) \operatorname{dn}^2(u; \tilde{k}) dv^2 \\ x = \operatorname{sn}(t; k) \operatorname{dn}(u; \tilde{k}) \cos v, \quad y = \operatorname{sn}(t; k) \operatorname{dn}(u; \tilde{k}) \sin v, \\ z = \operatorname{dn}(t; k) \operatorname{sn}(u; \tilde{k}), \quad w = \operatorname{cn}(t; k) \operatorname{cn}(u; \tilde{k}) \\ 0 \leq t \leq 2K(k), \quad -\tilde{K}(\tilde{k}) \leq u \leq \tilde{K}(\tilde{k}), \quad 0 \leq v < 2\pi, \\ 0 < k^2 < 1, \quad k^2 + \tilde{k}^2 = 1 \end{cases}$$

IV. Elliptic-cylindrical coordinates of type 2

$$\begin{cases} ds^2 = (\operatorname{dn}^2(u; \tilde{k}) - k^2 \operatorname{sn}^2(t; k))(dt^2 + du^2) + \operatorname{cn}^2(t; k) \operatorname{cn}^2(u; \tilde{k}) dv^2 \\ x = \operatorname{cn}(t; k) \operatorname{cn}(u; \tilde{k}) \cos v, \quad y = \operatorname{cn}(t; k) \operatorname{cn}(u; \tilde{k}) \sin v, \\ z = \operatorname{sn}(t; k) \operatorname{dn}(v; \tilde{k}), \quad w = \operatorname{dn}(t; k) \operatorname{sn}(u; \tilde{k}) \\ 0 \leq t \leq 2K(k), \quad -\tilde{K}(\tilde{k}) \leq u \leq \tilde{K}(\tilde{k}), \quad 0 \leq v < 2\pi, \\ 0 < k^2 < 1, \quad k^2 + \tilde{k}^2 = 1 \end{cases}$$

V. Spheroelliptic coordinates

$$\begin{cases} ds^2 = dt^2 + \sin^2 t (\operatorname{dn}^2(u; k) - \tilde{k}^2 \operatorname{sn}^2(v; \tilde{k}))(du^2 + dv^2) \\ x = \sin t \operatorname{sn}(u; k) \operatorname{dn}(v; \tilde{k}), \quad y = \sin t \operatorname{cn}(u; k) \operatorname{cn}(v; \tilde{k}), \\ z = \sin t \operatorname{dn}(u; k) \operatorname{sn}(v; \tilde{k}), \quad w = \cos t \\ 0 \leq t \leq \pi, \quad -K(k) \leq u < K(k), \quad -2\tilde{K}(\tilde{k}) \leq v \leq 2\tilde{K}(\tilde{k}), \\ 0 < k^2 < 1, \quad k^2 + \tilde{k}^2 = 1 \end{cases}$$

VI. Ellipsoidal

$$\begin{cases} ds^2 = \frac{(t-u)(t-v)}{4(t-a)(t-b)(t-1)} dt^2 + \frac{(u-v)(u-t)}{4(u-a)(u-b)(u-1)} du^2 + \\ \frac{(v-t)(v-u)}{4(v-a)(v-b)(v-1)} dv^2 \\ x^2 = \frac{(t-1)(u-1)(v-1)}{(a-1)(b-1)(-1)}, \quad y^2 = \frac{(t-a)(u-a)(v-a)}{-a(1-a)(b-a)}, \\ z^2 = \frac{(t-b)(u-b)(v-b)}{-b(1-b)(a-b)}, \quad w^2 = \frac{tuv}{ab} \\ 0 < v < 1 < u < b < t < a \end{cases}$$

2. Canonical forms

The following is a list of canonical forms for the six orthogonally separable coordinate systems of S^3 . The cylindrical, elliptic-cylindrical types 1 and 2, and ellipsoidal webs are defined by $\mathbf{K} = \alpha\mathbf{K}_1 + \beta\mathbf{K}_2 + \gamma\mathbf{C}$, where $\mathbf{K}_1, \mathbf{K}_2$ come from Eisenhart's equations, \mathbf{C} is the Casimir tensor, and $\alpha, \beta, \gamma \in \mathbb{R}$. Since it is possible to determine the presence of the Casimir tensor for the spherical and spheroelliptic webs, and thus subtract it, these webs are defined by $\mathbf{K} = \alpha\mathbf{K}_1 + \beta\mathbf{K}_2$, where $\mathbf{K}_1, \mathbf{K}_2$ come from Eisenhart's equations and $\alpha, \beta \in \mathbb{R}$.

3. Category 1

Cylindrical web

$$\mathbf{K} = c_1\mathbf{R}_{12} \odot \mathbf{R}_{12} + c_2(\mathbf{R}_{13} \odot \mathbf{R}_{13} + \mathbf{R}_{14} \odot \mathbf{R}_{14} + \mathbf{R}_{23} \odot \mathbf{R}_{23} + \mathbf{R}_{24} \odot \mathbf{R}_{24}) + c_3\mathbf{R}_{34} \odot \mathbf{R}_{34}.$$

$$R_{ij} = \text{diag}(c_1 + 2c_2, c_1 + 2c_2, 2c_2 + c_3, 2c_2 + c_3).$$

4. Category 2

Spherical web

$$\mathbf{K} = c_1\mathbf{R}_{12} \odot \mathbf{R}_{12} + c_2(\mathbf{R}_{13} \odot \mathbf{R}_{13} + \mathbf{R}_{23} \odot \mathbf{R}_{23}),$$

$$R_{ij} = \text{diag}(c_1 + c_2, c_1 + c_2, 2c_2, 0)$$

5. Category 3

Spheroelliptic web

$$\mathbf{K} = c_1\mathbf{R}_{12} \odot \mathbf{R}_{12} + c_2\mathbf{R}_{13} \odot \mathbf{R}_{13} + c_3\mathbf{R}_{23} \odot \mathbf{R}_{23}.$$

$$\text{Essential parameter: } k^2 = \frac{c_2 - c_3}{c_1 - c_3}$$

$$R_{ij} = \text{diag}(c_1 + c_2, c_1 + c_3, c_2 + c_3, 0)$$

6. Category 4

Elliptic-cylindrical webs of type 1 and type 2

$$\mathbf{K} = c_1\mathbf{R}_{12} \odot \mathbf{R}_{12} + c_2(\mathbf{R}_{13} \odot \mathbf{R}_{13} + \mathbf{R}_{23} \odot \mathbf{R}_{23}) + c_3(\mathbf{R}_{14} \odot \mathbf{R}_{14} + \mathbf{R}_{24} \odot \mathbf{R}_{24}) + c_4\mathbf{R}_{34} \odot \mathbf{R}_{34}.$$

$$\text{Discriminating parameter: } \delta = \frac{c_4 - c_2}{c_4 - c_3}$$

$$\text{Essential parameters: } k^2, \tilde{k}^2 = 1 - k^2$$

$$\text{Type 1: } \delta > 0, k^2 = \frac{c_4 - c_2}{c_4 - c_3}$$

$$\text{Type 2: } \delta < 0, k^2 = \frac{c_4 - c_3}{c_2 - c_3}$$

$$R_{ij} = \text{diag}(c_1 + c_2 + c_3, c_1 + c_2 + c_3, 2c_2 + c_4, 2c_3 + c_4).$$

7. Category 5

Ellipsoidal web

$$\mathbf{K} = c_1\mathbf{R}_{12} \odot \mathbf{R}_{12} + c_2\mathbf{R}_{13} \odot \mathbf{R}_{13} + c_3\mathbf{R}_{14} \odot \mathbf{R}_{14} + c_4\mathbf{R}_{23} \odot \mathbf{R}_{23} + c_5\mathbf{R}_{24} \odot \mathbf{R}_{24} + c_6\mathbf{R}_{34} \odot \mathbf{R}_{34},$$

where the c_i satisfy the constraint

$$(c_3 + c_4)(c_1 c_6 - c_2 c_5) + (c_2 + c_5)(c_3 c_4 - c_1 c_6) + (c_1 + c_6)(c_2 c_5 - c_3 c_4) = 0.$$

Essential parameters:

$$a = \frac{c_1(c_2 - c_4) + c_6(c_2 - c_3) - c_2(c_3 + c_4) + 2c_3 c_4}{c_1(c_2 - c_4) + c_4(c_6 - c_2) + c_5(c_4 - c_6)},$$

$$b = \frac{c_2(c_1 - c_4) + c_1(c_5 - c_4) - c_3(c_1 + c_5) + 2c_3 c_4}{c_1(c_2 - c_4) + c_4(c_6 - c_2) + c_5(c_4 - c_6)}$$

$$R_{ij} = \text{diag}(c_1 + c_2 + c_3, c_1 + c_4 + c_5, c_2 + c_4 + c_6, c_3 + c_5 + c_6)$$

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