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# POLYNOMIALS RELATED TO EXPANSIONS OF CERTAIN RATIONAL FUNCTIONS IN TWO VARIABLES*

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**Abstract.** A difference equation corresponding to a certain partial differential equation leads to a "Pascal type" triangle. The entries of a row of this triangle can be regarded as coefficients of a polynomial; the sequence of these polynomials is studied, together with its generating function and related polynomials. The entries of a more general class of number triangles are explicitly determined, as well as asymptotic expressions for the columns of the triangles. Chebyshev and Gegenhauer polynomials, as well as hypergeometric functions are used in the proofs.

**Key words.** sequences of polynomials, recursions, zeros of polynomials, Chebyshev polynomials, ultraspherical polynomials, hypergeometric functions, asymptotics, Darboux's method

**AMS(MOS) subject classifications.** 11B37, 33A30, 33A50, 41A60

**1. Introduction.** Let $u=u(x,t)$ be a function in two variables, and consider the (hyperbolic) partial differential equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}.$$

If we change this into a difference equation, we get

$$2u(x,t+1)=u(x-1,t)+u(x,t)+u(x+1,t)-u(x,t-1).$$

This suggests the "Pascal type" triangle (after normalizing)

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & 1 & 1 & & & & \\
 & & & 1 & 2 & 1 & 2 & 1 & & & \\
 & & 1 & 3 & 2 & 3 & 2 & 3 & 1 & & \\
 & 1 & 4 & 4 & 4 & 5 & 4 & 4 & 4 & 1 & \\
1 & 5 & 7 & 6 & 9 & 7 & 9 & 6 & 7 & 5 & 1 \\
 & & & & & \vdots & & & & &
\end{array} \tag{1.1}$$

where each element in the $n$th row is the sum of the three closest elements in the $(n-1)$th row, minus twice the closest element in the $(n-2)$th row.

Now we expand

$$G(z,t):=\frac{t}{1-t(1+z+z^2)+2z^2t^2}=\sum_{n=1}^{\infty} f_n(z)t^n;$$

it is clear that the $f_n(z)$ are polynomials of degree $2n$, and their coefficients are the rows of the triangle (1.1).

More generally, let $\nu>\frac{1}{2}$ and $\lambda$ be real parameters. We expand

$$G^{\lambda,\nu}(z,t):=(1-(1+z+z^2)t+\lambda z^2t^2)^{-\nu}=\sum_{n=0}^{\infty} f_n^{\lambda,\nu}(z)t^n.$$

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If we compare this with the generating function

$$(1-2zt+t^2)^{-\nu}=\sum_{n=0}^{\infty} C_n^{\nu}(z)t^n$$

for the ultraspherical (Gegenhauer) polynomials $C_n^{\nu}(z)$, we find

$$f_n^{\lambda,\nu}(z)=\lambda^{n/2}z^n C_n^{\nu}\left(\frac{1+z+z^2}{2\sqrt{\lambda}\,z}\right). \tag{1.2}$$

Using the recurrence relation for the ultraspherical polynomials (see, e.g., [1, p. 782]), we get $f_0^{\lambda,\nu}(z)=1$, $f_1^{\lambda,\nu}(z)=\nu(1+z+z^2)$, and

$$f_n^{\lambda,\nu}(z)=\left(1+\frac{\nu-1}{n}\right)(1+z+z^2)f_{n-1}^{\lambda,\nu}(z)-\left(1+2\frac{\nu-1}{n}\right)\lambda z^2 f_{n-2}^{\lambda,\nu}(z). \tag{1.3}$$

The polynomials $f_n^{\lambda,\nu}(z)$ are self-inverse, i.e., $f_n^{\lambda,\nu}(z)=z^{2n}f_n^{\lambda,\nu}(1/z)$. If we denote

$$f_n^{\lambda,\nu}(z)=C_{n,n}^{\lambda,\nu}+C_{n,n-1}^{\lambda,\nu}z+\cdots+C_{n,0}^{\lambda,\nu}z^n+C_{n,1}^{\lambda,\nu}z^{n+1}+\cdots+C_{n,n}^{\lambda,\nu}z^{2n}, \tag{1.4}$$

we get the triangle

$$\begin{matrix} & & C_{0,0}^{\lambda,\nu} & & \\ & C_{1,1}^{\lambda,\nu} & C_{1,0}^{\lambda,\nu} & C_{1,1}^{\lambda,\nu} & \\ C_{2,2}^{\lambda,\nu} & C_{2,1}^{\lambda,\nu} & C_{2,0}^{\lambda,\nu} & C_{2,1}^{\lambda,\nu} & C_{2,2}^{\lambda,\nu} \\ & & \vdots & & \end{matrix} \tag{1.5}$$

where

$$C_{n,k}^{\lambda,\nu}=\left(1+\frac{\nu-1}{n}\right)(C_{n-1,k-1}^{\lambda,\nu}+C_{n-1,k}^{\lambda,\nu}+C_{n-1,k+1}^{\lambda,\nu})-\left(1+2\frac{\nu-1}{n}\right)\lambda C_{n-2,k}^{\lambda,\nu}, \tag{1.6}$$

with $C_{n,k}^{\lambda,\nu}=C_{n,-k}^{\lambda,\nu}$. For $\lambda=2$ and $\nu=1$, the triangle (1.5) has the form (1.1).

The main purpose of this paper is to study the coefficients $C_{n,k}^{\lambda,\nu}$. We derive the following explicit and asymptotic expressions.

THEOREM 1.

$$C_{n,k}^{\lambda,\nu}=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{[(n-k)/2]}(-\lambda)^s\frac{\Gamma(\nu+n-s)}{s!(n-2s)!}\sum_{j=0}^{[(n-k-2s)/2]}\binom{2j+k}{j}\binom{n-2s}{2j+k}.$$

*Here $[x]$ denotes, as usual, the greatest integer function.*

THEOREM 2. *For fixed real $\nu>\frac{1}{2}$ and $\lambda$, and integer $k\geqq 0$, we have asymptotically as $n\to\infty$,*

(a) *if $\lambda<\frac{9}{4}$,*

$$C_{n,k}^{\lambda,\nu}\sim\frac{1}{2n\Gamma(\nu)\sqrt{\pi}}\left(\frac{n}{\sqrt{9-4\lambda}}\right)^{\nu-1/2}\left(\frac{3+\sqrt{9-4\lambda}}{2}\right)^{n+\nu};$$

(b) *if $\lambda>\frac{9}{4}$,*

$$C_{n,k}^{\lambda,\nu}\sim\frac{(\sqrt{\lambda})^{n+\nu}n^{\nu-3/2}}{\Gamma(\nu)\sqrt{\pi}}\left\{(\sqrt{4\lambda-1})^{1/2-\nu}\cos\left[(\alpha+\pi)n+\nu\alpha+\left(k+\frac{1}{4}-\frac{3\nu}{2}\right)\pi\right]\right.$$
$$\left.+(\sqrt{4\lambda-9})^{1/2-\nu}\cos\left[\beta n+\nu\beta+\left(\frac{1}{4}-\frac{\nu}{2}\right)\pi\right]\right\},$$

*where*

$$\alpha = \mathrm{Cos}^{-1}(1/2\sqrt{\lambda}), \qquad \beta = \mathrm{Cos}^{-1}(3/2\sqrt{\lambda});$$

(c) *if* $\lambda = \frac{9}{4}$,

$$C_{n,k}^{\lambda,\nu} \sim \frac{(3/2)^n}{\Gamma(\nu)}\left\{\frac{\sqrt{2/3}}{\Gamma(\nu)}\left(\frac{n}{2}\right)^{2\nu-2} + \frac{(\sqrt{8})^{1/2-\nu}}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\nu} n^{\nu-3/2}\cos\left[(\alpha+\pi)n+\nu\alpha+\left(k+\frac{1}{4}-\frac{3\nu}{2}\right)\pi\right]\right\},$$

*where*

$$\alpha = \mathrm{Cos}^{-1}\left(\tfrac{1}{3}\right).$$

Next we fix $\nu = 1$ and expand $G^{\lambda,1}(z, t)$ according to powers of $z$. If we set $t = 1-\alpha^{-1}$, we get

$$G^{\lambda,1}(z,t) = \frac{\alpha}{1-(\alpha-1)z+\{(\lambda-1)\alpha+(1-2\lambda)+\lambda\alpha^{-1}\}z^2} = \alpha\sum_{n=0}^{\infty} g_n^{\lambda}(\alpha)z^n.$$

We have $g_0^{\lambda}(\alpha) = 1$, $g_1^{\lambda}(\alpha) = \alpha - 1$, and

$$g_{n+1}^{\lambda}(\alpha) = (\alpha-1)\{g_n^{\lambda}(\alpha)+(\lambda\alpha^{-1}+1-\lambda)g_{n-1}^{\lambda}(\alpha)\}. \tag{1.7}$$

In § 2 we find explicit expressions for the zeros of the $f_n^{\lambda,1}(z)$ and the $g_n^{\lambda,1}(\alpha)$ for all values of $\lambda$. In §§ 3–5, Theorems 1 and 2 are proved, and § 6 contains some further remarks and generalizations.

**2. The zeros.** The Chebyshev polynomials of the second kind $U_n(z)$ can be defined by the recursion $U_0(z) = 1$, $U_1(z) = 2z$, and

$$U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z). \tag{2.1}$$

By taking $z = p(x)/2\sqrt{q(x)}$ we get the following lemma.

LEMMA 1. *Let* $p(x)$ *and* $q(x)$ *be arbitrary functions, and define the sequence* $V_n(x)$ *recursively by* $V_0(x) = 1$, $V_1(x) = p(x)$, *and*

$$V_{n+1}(x) = p(x)V_n(x) - q(x)V_{n-1}(x).$$

*Then, if* $q(x) \neq 0$,

$$V_n(x) = q(x)^{n/2}U_n(p(x)/2\sqrt{q(x)}).$$

To find the zeros of $g_n^{\lambda}(\alpha)$, we take $p(\alpha) := \alpha - 1$ and $q(\alpha) := -(\alpha-1)(\lambda\alpha^{-1}+1-\lambda)$. With (1.7) and Lemma 1 we find that

$$g_n^{\lambda}(\alpha) = \{(\alpha-1)(\lambda\alpha^{-1}+1-\lambda)\}^{n/2}U_n\left(\frac{i}{2}\left(\frac{\alpha-1}{\lambda\alpha^{-1}+1-\lambda}\right)^{1/2}\right).$$

Hence $g_n^{\lambda}(\alpha)$ has zeros when

$$\frac{\alpha-1}{\lambda\alpha^{-1}+1-\lambda} = -4\cos^2\frac{k\pi}{n+1},$$

i.e., the zeros are given by

$$\alpha = \frac{1}{2} - 2(1-\lambda)\cos^2\frac{k\pi}{n+1} \pm \left\{4(1-\lambda^2)\cos^4\left(\frac{k\pi}{n+1}\right) - 2(1+\lambda)\cos^2\left(\frac{k\pi}{n+1}\right)+\frac{1}{4}\right\}^{1/2}$$

$(k=1,2,\cdots,n)$. It is easy to see that these zeros are real unless

$$\frac{1+\lambda-2\sqrt{\lambda}}{4(\lambda-1)^2}<\sin^2\left(\frac{k\pi}{n+1}\right)<\frac{1+\lambda+2\sqrt{\lambda}}{4(\lambda-1)^2},$$

in which case they lie on the circle

$$y^2+\left(x-\frac{\lambda}{\lambda-1}\right)^2=\left(\frac{\sqrt{\lambda}}{\lambda-1}\right)^2 \qquad (\alpha=x+iy).$$

To find the zeros of $f_n^{\lambda,1}(z)$, we use the facts that $C_n^1(z)=U_n(z)$, and that the zeros of $U_n(z)$ are given by $\cos(k\pi/(n+1))$, $k=1,2,\cdots,n$. Hence with (1.2) we find that the $2n$ zeros of $f_n^{\lambda,1}(z)$ are

$$z=-\sqrt{\lambda}\cos\frac{k\pi}{n+1}-\frac{1}{2}\pm\left(\lambda\cos^2\frac{k\pi}{n+1}+\sqrt{\lambda}\cos\frac{k\pi}{n+1}-\frac{3}{4}\right)^{1/2}$$

for $k=1,2,\cdots,n$. We note that these zeros are real except when

$$\frac{-3}{2\sqrt{\lambda}}<\cos\frac{k\pi}{n+1}<\frac{1}{2\sqrt{\lambda}},$$

in which case they lie on the unit circle.

**3. Proof of Theorem 1.** Using the well-known explicit expression for the ultraspherical polynomials (see, e.g., [1, p. 775]) and (1.2), we get

$$f_n^{\lambda,\nu}(z)=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{[n/2]}(-\lambda)^s\frac{\Gamma(\nu+n-s)}{s!(n-2s)!}z^{2s}(1+z+z^2)^{n-2s}. \tag{3.1}$$

If $r$ is a positive integer, the binomial theorem, applied twice, gives

$$\begin{aligned}(1+z+z^2)^r&=\sum_{j=0}^{r}\sum_{i=0}^{j}\binom{r}{j}\binom{j}{i}z^{2j-i}\\&=\sum_{m=0}^{2r}z^m\sum_{j=0}^{[m/2]}\binom{r}{m-j}\binom{m-j}{m-2j},\end{aligned}$$

and with (3.1) we obtain

$$\begin{aligned}f_n^{\lambda,\nu}(z)&=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{[n/2]}(-\lambda)^s\frac{\Gamma(\nu+n-s)}{s!(n-2s)!}z^{2s}\sum_{m=0}^{2n-4s}z^m\sum_{j=0}^{[m/2]}\binom{n-2s}{m-j}\binom{m-j}{m-2j}\\&=\frac{1}{\Gamma(\nu)}\sum_{k=-n}^{n}z^{n-k}\sum_{s=0}^{[(n-k)/2]}(-\lambda)^s\frac{\Gamma(\nu+n-s)}{s!(n-2s)!}\sum_{j=0}^{[(n-k-2s)/2]}\binom{n-2s}{n-k-j-2s}\\&\quad\cdot\binom{n-k-j-2s}{n-k-2j-2s}.\end{aligned}$$

The theorem now follows if we compare the last equation with (1.4) and note that the product of the two binomial coefficients in the last line is equal to that in Theorem 1.

**4. Lemmas.** We can rewrite Theorem 1 in the form

$$C_{n,k}^{\lambda,\nu}=\frac{1}{\Gamma(\nu)}\sum_{s=0}^{[(n-k)/2]}(-\lambda)^s\binom{n-k-s}{s}\frac{\Gamma(\nu+n-s)}{k!(n-k-s)!}B_k^{(n-k-2s)}, \tag{4.1}$$

where

$$B_k^{(m)} := \sum_{j=0}^{[m/2]} \binom{2j}{j}\binom{m}{2j} \Big/ \binom{k+j}{j}.$$

LEMMA 2.

$$B_k^{(m)} = (-i\sqrt{3})^m \frac{m!(2k)!}{(m+2k)!} C_m^{k+1/2}\left(\frac{i}{\sqrt{3}}\right).$$

*Proof.* We have

$$\begin{aligned} B_k^{(m)} &= \sum_{j=0}^{[m/2]} \frac{(2j)!\,m!\,j!\,k!}{j!\,j!\,(m-2j)!\,(2j)!\,(k+j)!} \\ &= \sum_{j=0}^{[m/2]} \frac{(-m)_{2j}}{(k+1)_j j!} = \sum_{j=0}^{[m/2]} \frac{(-m/2)_j((1-m)/2)_j}{(k+1)_j} \frac{2^{2j}}{j!} \\ &= F\left(-\frac{m}{2}, \frac{1-m}{2}; k+1; 4\right), \end{aligned}$$

where $(a)_j$ is the Pochhammer symbol $(a)_j = a(a+1)\cdot\cdots\cdot(a+j-1)$ and $(a)_0 = 1$ and $F(a, b; c; x) = {}_2F_1(a, b; c; x)$ is the Gauss hypergeometric series (see, e.g., [1, p. 556]). The ultraspherical polynomials can be expressed as

$$C_m^{\nu}(x) = \frac{(2\nu)_m}{m!} x^m F\left(-\frac{m}{2}, \frac{1-m}{2}; \nu+\frac{1}{2}; 1-x^{-2}\right);$$

this gives the lemma, with $\nu = k+\frac{1}{2}$, $x = i/\sqrt{3}$.

If we combine (4.1) with Lemma 2, we get

LEMMA 3.

$$C_{n,k}^{\lambda,\nu} = \frac{(2k)!}{\Gamma(\nu)k!}(-i\sqrt{3})^{n-k} \sum_{s=0}^{[(n-k)/2]} \left(\frac{\lambda}{3}\right)^s \frac{\Gamma(\nu+n-s)}{s!(n+k-2s)!} C_{n-k-2s}^{k+1/2}\left(\frac{i}{\sqrt{3}}\right).$$

**5. Proof of Theorem 2.** First we determine the generating functions for the $C_{n,k}^{\lambda,\nu}$, where $k$, $\lambda$, and $\nu$ are fixed. To simplify notation, we write $C_n := C_{n,k}^{\lambda,\nu}$. We denote

$$d_n := \frac{\Gamma(\nu)k!}{\Gamma(k+\nu)}(\sqrt{\lambda})^{k-n} C_n \tag{5.1}$$

and

$$t_1 := \frac{-1}{2\sqrt{\lambda}}(1-\sqrt{1-4\lambda}), \qquad t_2 := \frac{-1}{2\sqrt{\lambda}}(1+\sqrt{1-4\lambda}),$$

$$t_3 := \frac{1}{2\sqrt{\lambda}}(3+\sqrt{9-4\lambda}), \qquad t_4 := \frac{1}{2\sqrt{\lambda}}(3-\sqrt{9-4\lambda});$$

note that $t_1 t_2 = 1 = t_3 t_4$.

LEMMA 4. *For real $\nu > 1/2$ and $\lambda$, and for $k = 0, 1, \cdots$, we have*

$$F_k(t) := t^k[(1+tt_1)(1+tt_2)]^{\nu-k-1}[(1-tt_1)(1-tt_2)(1-tt_3)(1-tt_4)]^{1/2-\nu} \cdot F\left(\frac{k-\nu+2}{2}, \frac{k-\nu+1}{2}; k+1; \frac{4t^2/\lambda}{(t^2-(t/\sqrt{\lambda})+1)^2}\right) = \sum_{n=k}^{\infty} d_n t^n.$$

*Proof.* We use Lemma 3, change the order of summation, and apply the binomial theorem

$$\begin{aligned}
\sum_{n=k}^{\infty}\left(\frac{i}{\sqrt{3}}\right)^{n-k}\frac{\Gamma(\nu)k!}{(2k)!}C_n t^n &= \sum_{n=k}^{\infty}\left\{\sum_{s=0}^{[(n-k)/2]}\left(\frac{\lambda}{3}\right)^s\frac{\Gamma(\nu+n-s)}{s!(n+k-2s)!}C_{n-k-2s}^{k+1/2}\left(\frac{i}{\sqrt{3}}\right)\right\}t^n \\
&= \sum_{m=k}^{\infty} C_{m-k}^{k+1/2}\left(\frac{i}{\sqrt{3}}\right)t^m\sum_{s=0}^{\infty}\frac{\Gamma(\nu+m+s)}{s!(m+k)!}\left(\frac{\lambda}{3}t^2\right)^s \\
&= \sum_{m=k}^{\infty} C_{m-k}^{k+1/2}\left(\frac{i}{\sqrt{3}}\right)t^m\frac{\Gamma(\nu+m)}{(m+k)!}\sum_{s=0}^{\infty}\frac{\Gamma(\nu+m+s)}{s!\Gamma(\nu+m)}\left(\frac{\lambda}{3}t^2\right)^s \\
&= \sum_{m=k}^{\infty} C_{m-k}^{k+1/2}\left(\frac{i}{\sqrt{3}}\right)\frac{\Gamma(\nu+m)}{(m+k)!}t^m\left(1-\frac{\lambda}{3}t^2\right)^{-m-\nu} \\
&= \frac{t^k}{(1-(\lambda/3)t^2)^{k+\nu}}\frac{\Gamma(k+\nu)}{(2k)!}\sum_{m=0}^{\infty}C_m^{k+1/2}\left(\frac{i}{\sqrt{3}}\right) \\
&\quad\cdot\frac{(k+\nu)_m}{(2k+1)_m}\left(\frac{t}{1-(\lambda/3)t^2}\right)^m,
\end{aligned}$$

where we have used $(x)_n=\Gamma(x+n)/\Gamma(x)$. After changing the variable $t$ to $-it\sqrt{3/\lambda}$, we get

$$\sum_{n=k}^{\infty} d_n t^n = \frac{t^k}{(1+t^2)^{k+\nu}}\sum_{m=0}^{\infty}C_m^{k+1/2}\left(\frac{i}{\sqrt{3}}\right)\frac{(k+\nu)_m}{(2k+1)_m}\left(\frac{-it\sqrt{3/\lambda}}{1+t^2}\right)^m. \tag{5.2}$$

Now we are going to use the following generating function for ultraspherical polynomials (see, e.g., [3, p. 279]),

$$\sum_{m=0}^{\infty}\frac{(\gamma)_m}{(2\alpha)_m}C_m^{\alpha}(x)z^m=(1-xz)^{-\gamma}F\left(\frac{\gamma}{2},\frac{\gamma+1}{2};\alpha+\frac{1}{2};\frac{z^2(x^2-1)}{(1-xz)^2}\right). \tag{5.3}$$

With $\gamma=k+\nu$, $\alpha=k+\frac{1}{2}$, $x=i\sqrt{3}$, and $z=-it\sqrt{3/\lambda}/(1+t^2)$, (5.2) becomes

$$\sum_{n=k}^{\infty} d_n t^n=\frac{t^k}{(t^2-(t/\sqrt{\lambda})+1)^{k+\nu}}F\left(\frac{k+\nu}{2},\frac{k+\nu+1}{2};k+1;y\right), \tag{5.4}$$

where

$$y=\frac{4t^2/\lambda}{(t^2-(t/\sqrt{\lambda})+1)^2}.$$

Using Euler's identity

$$F(a,b;c;y)=(1-y)^{c-a-b}F(c-a,c-b;c;y)$$

(for $|y|<1$; see [3, p. 60]), we get

$$F\left(\frac{k+\nu}{2},\frac{k+\nu+1}{2};k+1;y\right)=(1-y)^{1/2-\nu}F\left(\frac{k-\nu+2}{2},\frac{k-\nu+1}{2};k+1;y\right). \tag{5.5}$$

Now it is easy to verify that

$$1-y=[(1+tt_1)(1+tt_2)]^{-2}(1-tt_1)(1-tt_2)(1-tt_3)(1-tt_4);$$

the lemma now follows from (5.4) and (5.5).

*Proof of Theorem* 2. To find asymptotics for $d_n$, we apply Darboux's method (see, e.g., [2, p. 310]) on the generating function $F_k(t)$. Possible singularities of $F_k(t)$ are at $t=t_j$, $j=1,\cdots,4$, and at $t=-t_1$, $t=-t_2$. To examine the behaviour of $F_k(t)$ in the neighbourhood of $-t_1$ and $-t_2$, we apply the identity (see, e.g., [1, p. 559])

$$F(a,b;c;z)=d_1(-z)^{-a}F\left(a,1-c+a;1-b+a;\frac{1}{z}\right)$$
$$+d_2(-z)^{-b}F\left(b,1-c+b;1-a+b;\frac{1}{z}\right),$$

where $d_1$ and $d_2$ are constants depending on $a$, $b$, $c$, to the right-hand side of (5.4). We find

$$F_k(t)=d_1\left(\frac{-\lambda}{4}\right)^{(k+\nu)/2}t^{-\nu}F\left(\frac{k+\nu}{2},\frac{-k+\nu}{2};\frac{1}{2};\frac{1}{y}\right)$$
$$+d_2\left(\frac{-\lambda}{4}\right)^{(k+\nu+1)/2}t^{-\nu-1}\left(t^2-\frac{t}{\sqrt{\lambda}}+1\right)F\left(\frac{k+\nu+1}{2},\frac{-k+\nu+1}{2};\frac{3}{2};\frac{1}{y}\right)$$

which shows that the singularities at $t=-t_1$ and $-t_2$ are removable.

Now we denote

$$F_k^{(j)}(t):=(1-tt_j)^{\nu-1/2}F_k(t)\qquad (j=1,\cdots,4)$$

and

$$F_k^{(5)}(t):=\{(1-tt_3)(1-tt_4)\}^{\nu-1/2}F_k(t).$$

Using the Gauss summation formula (see, e.g., [3, p. 49])

$$F\left(\frac{k-\nu+2}{2},\frac{k-\nu+1}{2};k+1;1\right)=\frac{\Gamma(k+1)\Gamma(\nu-1/2)}{\Gamma\left(\frac{k+\nu}{2}\right)\Gamma\left(\frac{k+\nu+1}{2}\right)}:=\Gamma,$$

which holds for $\nu>\frac{1}{2}$, we find

$$F_k^{(1)}(t_2)=\left(\frac{-1}{2}\right)^{k+\nu}(-\sqrt{1-4\lambda})^{1/2-\nu}(\sqrt{\lambda})^{k+\nu}\left(\frac{-1+\sqrt{1-4\lambda}}{2\sqrt{\lambda}}\right)^{\nu}\Gamma,\tag{5.6}$$

$$F_k^{(2)}(t_1)=\left(\frac{-1}{2}\right)^{k+\nu}(\sqrt{1-4\lambda})^{1/2-\nu}(\sqrt{\lambda})^{k+\nu}\left(\frac{-1-\sqrt{1-4\lambda}}{2\sqrt{\lambda}}\right)^{\nu}\Gamma,\tag{5.7}$$

$$F_k^{(3)}(t_4)=\left(\frac{1}{2}\right)^{k+\nu}(\sqrt{9-4\lambda})^{1/2-\nu}(\sqrt{\lambda})^{k+\nu}\left(\frac{3+\sqrt{9-4\lambda}}{2\sqrt{\lambda}}\right)^{\nu}\Gamma,\tag{5.8}$$

$$F_k^{(4)}(t_3)=\left(\frac{1}{2}\right)^{k+\nu}(-\sqrt{9-4\lambda})^{1/2-\nu}(\sqrt{\lambda})^{k+\nu}\left(\frac{3-\sqrt{9-4\lambda}}{2\sqrt{\lambda}}\right)^{\nu}\Gamma;\tag{5.9}$$

the arguments of these (in general multi-valued) expressions will be determined later.

We note that

(a) if $\lambda<\frac{9}{4}$, then $|t_4|<|t_j|$ for $j=1,2,3$;

(b) if $\lambda>\frac{9}{4}$, then $|t_j|=1$ $(j=1,\cdots,4)$ and no two $t_j$ are equal;

(c) if $\lambda=\frac{9}{4}$, then $t_3=t_4=1$, $|t_1|=|t_2|=1$, $t_1\neq t_2$, $t_1\neq 1$, $t_2\neq 1$.

We prove Theorem 2 according to this distinction.

(a) Let $\lambda<\frac{9}{4}$. Then according to Darboux's method the coefficients in the MacLaurin expansion of

$$f(t):=F_k^{(3)}(t_4)(1-tt_3)^{1/2-\nu}$$

are asymptotics to the $d_n$, as $n\to\infty$. The binomial theorem gives

$$(1-tt_3)^{1/2-\nu}=\sum_{n=0}^{\infty}\frac{\Gamma(\nu-1/2+n)}{\Gamma(\nu-1/2)n!}\left(\frac{3+\sqrt{9-4\lambda}}{2\sqrt{\lambda}}\right)^n t^n,\tag{5.10}$$

and we obtain from (5.1), (5.8), and (5.10)

$$C_n \sim A2^{-k-\nu}(\sqrt{9-4\lambda})^{1/2-\nu}\left(\frac{3+\sqrt{9-4\lambda}}{2}\right)^{n+\nu}, \tag{5.11}$$

where all the gamma function and factorial terms from (5.1), (5.8) and (5.10) are collected in $A$. Using the duplication formula (see, e.g., [3, p. 24])

$$\frac{\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \frac{2^{2z-1}}{\sqrt{\pi}}, \tag{5.12}$$

we find

$$A = \frac{\Gamma(\nu-\frac{1}{2}+n)}{\Gamma(\nu)n!}\,\frac{2^{2k+\nu-1}}{\sqrt{\pi}}. \tag{5.13}$$

Stirling's formula now gives

$$\Gamma(\nu-\tfrac{1}{2}+n)/n! \sim n^{\nu-3/2} \qquad (\text{as } n\to\infty),$$

so finally we get with (5.11) and (5.13)

$$C_n \sim \frac{1}{\Gamma(\nu)\sqrt{\pi}} n^{\nu-3/2}(9-4\lambda)^{1/4-\nu/2}\left(\frac{3+\sqrt{9-4\lambda}}{2}\right)^{n+\nu},$$

which implies Theorem 2(a).

(b) Let $\lambda > \frac{9}{4}$. Asymptotics to the $d_n$ (as $n\to\infty$) are given by the coefficients of the expansion of

$$g(t) := \sum_{j=1}^{4} F_k^{(j)}(t_j^{-1})(1-tt_j)^{1/2-\nu}.$$

We note that, in general, the values in (5.6)-(5.9) are not uniquely determined. However, the powers $(1-xz)^{-\gamma}$ and $(1-y)^{1/2-\nu}$ in (5.3), resp. (5.5) are to be taken with their principal values. With this in mind, we find that we have to take (5.6) and (5.7) with arguments

$$\varepsilon_1 := k\pi - \nu\alpha - \frac{\pi}{2}\left(\frac{1}{2}-\nu\right), \qquad \varepsilon_2 := k\pi + \nu\alpha + \frac{\pi}{2}\left(\frac{1}{2}-\nu\right),$$

respectively, where $\alpha := \arg\left((1+i\sqrt{4\lambda-1})/2\sqrt{\lambda}\right)$. Using the equivalent of (5.10) for $t_1$ and $t_2$, and with (5.1), (5.6), and (5.7), we find that the combined contribution from the first and second term of $g(t)$ is

$$\begin{aligned} &A2^{-k-\nu}(\sqrt{4\lambda-1})^{1/2-\nu}(\sqrt{\lambda})^{n+\nu}\left\{e^{i\varepsilon_1}\left(\frac{-1+i\sqrt{4\lambda-1}}{2\sqrt{\lambda}}\right)^n + e^{i\varepsilon_2}\left(\frac{-1-i\sqrt{4\lambda-1}}{2\sqrt{\lambda}}\right)^n\right\} \\ &\sim \frac{n^{\nu-3/2}}{\Gamma(\nu)\sqrt{\pi}}(\sqrt{4\lambda-1})^{1/2-\nu}(\sqrt{\lambda})^{n+\nu}\cos\left\{(\alpha+\pi)n+\nu\alpha+\left(k+\frac{1}{4}-\frac{3\nu}{2}\right)\pi\right\}. \end{aligned} \tag{5.14}$$

Similarly, we find that we have to take (5.8) and (5.9) with arguments

$$\varepsilon_3 := \nu\beta + \frac{\pi}{2}\left(\frac{1}{2}-\nu\right), \qquad \varepsilon_4 := -\nu\beta - \frac{\pi}{2}\left(\frac{1}{2}-\nu\right),$$

respectively, where $\beta := \arg((3+i\sqrt{4\lambda-9})/2\sqrt{\lambda})$. With (5.10) and its equivalent for $t_4$, and with (5.1), (5.8), and (5.9) we find the combined contribution from the third and fourth term of $g(t)$ to be

$$A2^{-k-\nu}(\sqrt{4\lambda-9})^{1/2-\nu}(\sqrt{\lambda})^{n+\nu}\left\{e^{i\varepsilon_3}\left(\frac{3+i\sqrt{4\lambda-9}}{2\sqrt{\lambda}}\right)^n+e^{i\varepsilon_4}\left(\frac{3-i\sqrt{4\lambda-9}}{2\sqrt{\lambda}}\right)^n\right\}$$
$$\sim\frac{n^{\nu-3/2}}{\Gamma(\nu)\sqrt{\pi}}(\sqrt{4\lambda-9})^{1/2-\nu}(\sqrt{\lambda})^{n+\nu}\cos\left\{\beta n+\nu\beta+\left(\frac{1}{4}-\frac{\nu}{2}\right)\pi\right\}.$$

This and (5.14) lead to Theorem 2(b).

(c) Let $\lambda=\frac{9}{4}$. Since $t_3=t_4=1$, we have to find the coefficients of the expansion of

$$h(t):=F_k^{(1)}(t_2)(1-tt_1)^{1/2-\nu}+F_k^{(2)}(t_1)(1-tt_2)^{1/2-\nu}+F_k^{(5)}(1)(1-t)^{1-2\nu}.$$

The contribution from the first two terms of $h(t)$ is the same as in (5.14), with $\lambda=\frac{9}{4}$. Furthermore,

$$F_k^{(5)}(1)=\left(\frac{3}{4}\right)^{k-1/2}2^{1/2-\nu}\Gamma \tag{5.15}$$

for $\lambda=\frac{9}{4}$, and the binomial theorem gives

$$(1-t)^{1-2\nu}=\sum_{n=0}^{\infty}\frac{\Gamma(2\nu-1+n)}{\Gamma(2\nu-1)n!}t^n. \tag{5.16}$$

Hence the contribution to the asymptotics of $C_n$ from the third term of $h(t)$, with (5.16), (5.15), and (5.1) is

$$\frac{\Gamma(2\nu-1+n)}{n!\Gamma(2\nu-1)}\frac{\Gamma(k+\nu)\Gamma(\nu-1/2)}{\Gamma(\nu)\Gamma\left(\frac{k+\nu}{2}\right)\Gamma\left(\frac{k+\nu+1}{2}\right)}\left(\frac{3}{2}\right)^{n-1/2}2^{1-k-\nu}. \tag{5.17}$$

By applying the duplication formula (5.12) twice, we get

$$\frac{\Gamma(k+\nu)\Gamma(\nu-1/2)}{\Gamma(2\nu-1)\Gamma\left(\frac{k+\nu}{2}\right)\Gamma\left(\frac{k+\nu+1}{2}\right)}=\frac{1}{\Gamma(\nu)}2^{1+k-\nu},$$

and Stirling's formula gives

$$\Gamma(2\nu-1+n)/n!\sim n^{2\nu-2}\qquad(\text{as } n\to\infty);$$

hence (5.17) is asymptotically equal to

$$\frac{1}{(\Gamma(\nu))^2}\left(\frac{n}{2}\right)^{2\nu-2}\left(\frac{3}{2}\right)^{n-1/2}.$$

This and (5.14) for $\lambda=\frac{9}{4}$ finally gives Theorem 2(c).

**6. Further remarks.** (1) Darboux's method can actually be used to find a complete asymptotic expansion for the $C_{n,k}^{\lambda;\nu}$; this would be a stronger result than Theorem 2. See, e.g., [5, Thm. 8.4].

(2) The sum of the elements of the $n$th row in the triangle (1.5) is easy to determine. Since the $C_{n,k}^{\lambda;\nu}$ are the coefficients of $f_n^{\lambda;\nu}(z)$, this sum is in fact $f_n^{\lambda;\nu}(1)$. Now (1.2) implies

$$f_n^{\lambda;\nu}(1)=(\sqrt{\lambda})^nC_n^{\nu}(3/2\sqrt{\lambda}).$$

More can be said in the case $\nu=1$, since $C_n^1(x)=U_n(x)$. From (2.1) we get with a standard method (Binet's formula)

$$U_n(x)=\frac{1}{2\sqrt{x^2-1}}\{(x+\sqrt{x^2-1})^{n+1}-(x-\sqrt{x^2-1})^{n+1}\},$$

and therefore

$$f_n^{\lambda,1}(1)=\frac{(\sqrt{\lambda})^{n+1}}{\sqrt{9-4\lambda}}\left\{\left(\frac{3+\sqrt{9-4\lambda}}{2\sqrt{\lambda}}\right)^{n+1}-\left(\frac{3-\sqrt{9-4\lambda}}{2\sqrt{\lambda}}\right)^{n+1}\right\}. \tag{6.1}$$

As examples, we have

$$f_n^{0,1}(1)=3^n,$$

$$f_n^{1,1}(1)=\frac{1}{\sqrt{5}}\left\{\left(\frac{3+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\right\}$$

(the odd-index Fibonacci numbers 1, 3, 8, 21, $\cdots$);

$$f_n^{2,1}(1)=2^{n+1}-1$$

(see (1.1)), and

$$f_n^{9/4,1}(1)=(n+1)\left(\frac{3}{2}\right)^n.$$

If $\lambda>\frac{9}{4}$ then (6.1) can be rewritten as

$$f_n^{\lambda,1}(1)=2(4\lambda-9)^{-1/2}(\sqrt{\lambda})^{n+1}\sin\{(n+1)\theta\} \tag{6.2}$$

where $\theta$ is such that $\exp(i\theta)=(3+i\sqrt{4\lambda-9})/2\sqrt{\lambda}$, or $\theta=\operatorname{Cos}^{-1}(3/2\sqrt{\lambda})$. (6.2) gives easy explicit formulas for $\lambda=3(\theta=\pi/6)$, $\lambda=\frac{9}{2}$ $(\theta=\pi/4)$, $\lambda=9$ $(\theta=\pi/3)$.

(3) The generating function $G^{\lambda,1}(z,t)$ of § 1 can be generalized as follows. Let $p(z):=a_0+a_1z+\cdots+a_rz^r$ and $q(z)=b_0+b_1z+\cdots+b_sz^s$ and expand

$$G(z,t):=\frac{1}{1-tp(z)+t^2q(z)}=\sum_{n=0}^{\infty}Q_n(z)(z)t^n.$$

Then $Q_0(z)=1$, $Q_1(z)=p(z)$, and

$$Q_{n+1}(z)=p(z)Q_n(z)-q(z)Q_{n-1}(z), \tag{6.3}$$

and we see that $Q_n(z)$ is a polynomial of degree $\leqq nr$. If we denote

$$Q_n(z)=C_0^{(n)}+C_1^{(n)}z+\cdots+C_{nr}^{(n)}z^{nr},$$

we have the recursion

$$C_k^{(n+1)}=\sum_{j=0}^{k}a_jC_{k-j}^{(n)}-\sum_{j=0}^{k}b_jC_{k-j}^{(n-1)} \tag{6.4}$$

where $a_j:=0$ for $j<0, j>r$, and $b_j:=0$ for $j<0, j>s$. We note the following special cases.

(a) $p(z):=1+z$, $q(z):=0$ gives $Q_n(z)=(1-z)^n$, and the $C_k^{(n)}$ are the binomial coefficients.

(b) $p(z):=1+z+z^2$, $q(z)=\lambda z^2$; this is the case dealt with in this paper, with $\nu=1$.

(c) To generalize (b), we set $p(z):=1+z+\cdots+z^{2m}$, $q(z):=\lambda z^{2m}$. After reindexing (so that $Q_n(z)=C_{nm}^{(n)}+C_{nm-1}^{(n)}z+\cdots+C_0^{(n)}z^{nm}+\cdots+C_{nm-1}^{(n)}z^{2nm-1}+C_{nm}^{(n)}z^{2nm}$) (6.4) becomes

$$C_k^{(n+1)}=C_{k-m}^{(n)}+\cdots+C_k^{(n)}+\cdots+C_{k+m}^{(n)}-\lambda C_k^{(n-1)}; \tag{6.5}$$

this is the analogue to (1.6). No attempt has been made to determine the $C_k^{(n)}$ explicitly or asymptotically. However, it is easy to derive the sums of the elements in the rows of the triangle generated by (6.5). In analogy to and as a generalization of (6.1) we obtain

$$Q_n(1)=\frac{(\sqrt{\lambda})^{n+1}}{\sqrt{(2m+1)^2-4\lambda}} \cdot\left\{\left(\frac{2m+1+\sqrt{(2m+1)^2-4\lambda}}{2\sqrt{\lambda}}\right)^{n+1}-\left(\frac{2m+1-\sqrt{(2m+1)^2-4\lambda}}{2\sqrt{\lambda}}\right)^{n+1}\right\}.$$

For $\lambda \leqq (2m+1)^2/4$, $Q_n(1)$ is positive for all $n$, and for $\lambda > (2m+1)^2/4$ it is an alternating sequence.

(4) K. B. Stolarsky [4] recently studied the recurrence $p_0(x)=1$, $p_1(x)=x$, and

$$p_n(x)=x^n p_{n-1}(x^{-1})+p_{n-2}(x).$$

He showed that for $n \geqq 0$

$$p_{2n+1}(x)=xf_{2n+1}^{1,1}(x)$$

(in our notation), i.e., the $p_{2n+1}(x)$ are self-inverse polynomials, or in other words, the coefficients are "centrally symmetric." However, it is easy to see that the $p_{2n}(x)$ do not have this property; in fact, it is shown in [4] that the coefficients of $p_{2n}(x)$ are "strongly noncentrally symmetric."

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