Analysis of nonmetric theories of gravity. I. Electromagnetism

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In this article we are concerned with an analysis of nonmetric theories of gravity, and, in particular, with an analysis of a class of theories called metric-affine theories of gravity (MATG's). The purely gravitational laws of these theories are written down. We then establish a suitable set of laws representing electromagnetism in the presence of a gravitational field in metric-affine theories of gravity. We find that these laws simplify if we assume the gravitational field to be spherically symmetric and static. Consequently we define the concepts of spherical symmetry and staticity in the context of MATG's, and we calculate the form of the connection Γ and the $\binom{0}{2}$ tensor g explicitly in such a gravitational field generated by a central mass. Finally, the laws that are established are investigated.

I. INTRODUCTION

This paper is the first in a series of articles involved in a systematic analysis of nonmetric theories of gravity. The class of metric theories of gravity (MTG's) is well defined in the literature.[1,2] A nonmetric theory of gravity is a theory not belonging to the class of MTG's. Although the techniques and ideas to be used in this analysis are quite general, they are primarily applied to a subclass of the class of all nonmetric theories of gravity, called metric-affine theories of gravity (MATG's). The precise definition of an MATG is given in Ref. 1, but such a theory is essentially characterized by the following:

(a) It is a geometric theory of gravity; that is, spacetime is characterized by a four-dimensional, Hausdorff, differentiable manifold of signature -2.

(b) It is an affine theory of gravity (ATG). Essentially an ATG is a theory in which the spacetime manifold is endowed with a connection Γ, and the gravitational field is represented (at least in part) by Γ. The unique curves of freely falling test bodies are then associated with the natural geometric curves in the spacetime manifold, called paths. That is, the motion of freely falling particles is governed by the (path) equation given by

$$\frac{d^2x^a}{d\lambda^2}+\Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}=0\,, \tag{1.1}$$

where λ is an affine parameter.

(c) In addition, the spacetime manifold is endowed with a $\binom{0}{2}$ tensor field g. The gravitational field is represented (completely) by Γ and g. (Note that an MTG is a special case of an MATG with $\Gamma=\{\ \}$, where $\{\ \}$ denotes the metric connection. We are specifically interested in the case where $\Gamma\neq\{\ \}$.)

These conditions represent the purely gravitational laws of the theories, an analysis of which would include a theoretical investigation into the structure of these laws (see Ref. 1) and an investigation concerning their experimental verification or nonverification. In the case of the latter a parametrized post-Newtonian (PPN)-type analysis might be appropriate; unfortunately, for a general MATG, the PPN-type expansion would include far too many independent terms for the analysis to be useful.[1] However, the number of independent terms reduces considerably if we restrict attention to spherically symmetric and static (SSS) gravitational fields. We shall find that a useful analysis of the class of theories of gravity can be made within the SSS idealization. In Sec. II we shall discuss the concepts of spherical symmetry and staticity in the context of MATG's, and we shall calculate the general forms of Γ and g in an SSS gravitational field due to a central, spherically symmetric mass. The question of solar-system experiments within this framework will be dealt with in a later paper.

In actual fact, an analysis of the purely gravitational laws of an MATG turns out to be rather limited. However, in order for an MATG to be complete it must also specify how other physical fields act in a gravitational field. When other fields are included in the analysis the results that are obtained are far more interesting.

In this paper we shall discuss the laws of electromagnetism in a gravitational field. In particular, we wish to establish a set of such laws in a general (or generalized) form in order to include the possible laws of electromagnetism in a gravitational field for nonmetric theories of gravity. [We shall call these the gravitationally generalized laws of electromagnetism (laws of GGEM), and these laws consist of the gravitationally generalized Maxwell equations (GGM equations) and the gravitationally generalized Lorentz equations (GGL equations).] We require that these laws of GGEM must be general enough to include all possible laws of GGEM for MATG's. (We also require that the laws of GGEM should reduce to the special relativistic laws of electromagnetism in the appropriate limit, and that the laws of electromagnetism in MTG's are a special case.)

We shall find that the laws of GGEM will be written in terms of a $\binom{0}{2}$ tensor field g, and arbitrary functions of the gravitational field. If we do not assume that g is the metric (which is not necessary here; g could simply be a tensor occurring in the laws of GGEM), and we assume that the arbitrary functions are completely general, the class of nonmetric theories under investigation is very general. If we assume that g is the metric tensor, and the arbitrary functions are in fact functions of g and Γ, we are specifically investigating MATG's.

It is instructive to consider the laws of electromagnetism in an MTG. The Lorentz force law is given by

$$\frac{d^2x^a}{d\tau^2}+\left\{{a \atop bc}\right\}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau}=\frac{e}{m}F_n{}^a\frac{dx^n}{d\tau}\,, \tag{1.2}$$

where F_{ab} is the electromagnetic tensor (indices are raised using the metric), m is the mass of the test particle and e its electromagnetic charge.

The Maxwell equations in an MTG are

$$F_{ab,c}+F_{bc,a}+F_{ca,b}=0\,, \tag{1.3}$$

and

$$F^{ab}{}_{;b}=4\pi J^a\,, \tag{1.4}$$

where a semicolon denotes covariant differentiation with respect to the Christoffel symbol and J^a is the electromagnetic four-current given by $J^a=(\rho,\vec{j}\,)-\rho$ is the charge density and $\vec{j}$ the electromagnetic three-current.

Equation (1.3) guarantees the existence of a four-vector A_a, called the electromagnetic four-potential, such that

$$F_{ab}=A_{b,a}-A_{a,b}\,. \tag{1.5}$$

We obtain the equation for the conservation of charge by covariantly differentiating (1.4), viz.,

$$J^a{}_{;a}=0\,. \tag{1.6}$$

First let us consider the gravitationally generalized Lorentz (GGL) law. Equation (1.2) suggests that we take this law in the form

$$\frac{d^2x^a}{d\lambda^2}+\Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}=\bar{L}\left[\text{grav},\frac{dx^c}{d\lambda}\right]\frac{e}{m}F_n{}^a\frac{dx^n}{d\lambda}\,, \tag{1.7}$$

where $\bar{L}$ is some function of the gravitational field and the four-velocity $dx^c/d\lambda$. Equation (1.7) will reduce to the special-relativistic Lorentz law in the "absence" of gravity providing that in this limit $\bar{L}(\text{grav},dx^i/d\lambda)$ is unity. This equation has the following points in its favor: (1) it reduces to the path equation in the "absence" of electromagnetism; (2) the "metrically" modified Lorentz law (1.2) is a special case; and (3) all the possible forms for the specific GGL equations investigated by the author are special cases of (1.7).[1]

Next we consider the first of the gravitationally generalized Maxwell (GGM) equations. Since we require that (a) the appropriate equation should agree with special relativity as gravity is "turned off", and (b) F_{ab} can be written in terms of the electromagnetic four-potential so that the laws of GGEM are gauge invariant (ensuring the photon can be interpreted as a massless, spin-1, elementary particle[3]), we again take Eq. (1.3) to be valid. That is, we assume that the first of the GGM equations does not couple to gravity, as is the case in MTG's. (This is precisely the assumption that Hehl *et al.*[4] make in their theory with nonzero torsion—they refer to it as the "principle of minimal coupling.")

Finally, we consider the second of the GGM equations. Using the "Lorentz" gauge condition defined by

$$g^{bc}A_{b,c}=0\,, \tag{1.8}$$

we can rewrite Eq. (1.4) as

$$-g^{bc}A_{a,bc}+\Lambda^{bc}{}_a(g,g_{,d})A_{b,c}=4\pi J_a\,, \tag{1.9}$$

where $\Lambda^{bc}{}_a$ is defined by

$$\Lambda^{bc}{}_a = -g^{bc}{}_{,a} - g^{mn}\{{}^{b}_{mn}\}\delta^c_a + g^{mn}\{{}^{c}_{mn}\}\delta^b_a - g^{bm}\{{}^{c}_{am}\} + g^{cm}\{{}^{b}_{am}\} \ . \tag{1.10}$$

Working with the electromagnetic four-potential and again demanding the correct special-relativistic limit, the appropriate form for the GGM equations is

$$-g^{bc}A_{a,bc} + \Pi^{bc}{}_a(\eta, g, g_{,d}, \Gamma)A_{b,c} = 4\pi J_a \ , \tag{1.11}$$

where $\Pi^{bc}{}_a$ is a general function of the gravitational field, and depends on the Minkowski tensor η,g and its first derivatives and Γ [and must be such that (1.11) reduces to the special relativistic Maxwell equations in the appropriate limit]. Equation (1.11) has the following favorable features: (1) the "metrically" modified Maxwell equations are a special case; (2) in the optical limit, the equation is conformally invariant and electromagnetic waves are null; and (3) all the possible forms for the specific GGM equations investigated by the author are special cases of (1.11).

It will sometimes be convenient to write the GGM equations in terms of F_{ab}. Using the "Lorentz" gauge condition, (1.11) becomes

$$g^{bc}F_{ab,c} + \Sigma^{bc}{}_a F_{bc} = 4\pi J_a \ , \tag{1.12}$$

where

$$\Pi^{bc}{}_a = \Sigma^{cb}{}_a - \Sigma^{bc}{}_a - g^{bc}{}_{,a} \ . \tag{1.13}$$

In the remainder of this paper we shall be concerned with an analysis of these laws of GGEM, and, in particular, with an analysis of the laws of GGEM in an SSS gravitational field (whose form will be calculated explicitly in Sec. III).

Finally, a few brief comments on notation. We shall use indices (a,b,c) to range from 0 to 3, and greek indices (μ,ν,σ) to range from 1 to 3 (alternatively, we shall use three-vector notation). When considering an electromagnetic source consisting of charged point particles, we shall use the subscript k to denote the kth particle.

II. SPHERICAL SYMMETRY AND STATICITY

There are numerous problems in gravitational physics in which the simplification to a spherically symmetric and static (SSS) idealization is useful. First we need to define the concepts of spherical symmetry and staticity in the context of a metric-affine theory of gravity. We can then calculate the general form of the connection Γ and the metric g in an SSS gravitational field. The forms of Γ and g simplify further when we consider an SSS gravitational field generated by a spherically symmetric central mass.

The concepts of spherical symmetry and staticity are discussed in some detail in Ref. 1. The outcome of this discussion is that the following definitions of spherical symmetry and staticity appear as the most appropriate.

If a spacetime is stationary, then

$$\mathscr{L}_X g = 0 \tag{2.1}$$

and

$$\mathscr{L}_X \Gamma = 0 \tag{2.2}$$

for some timelike vector field X. ($\mathscr{L}$ denotes the Lie derivative.) The spacetime is static if, in addition, the timelike vector field is orthogonal to a family of spacelike hypersurfaces.

A spacetime is spherically symmetric if both g and Γ are Lie invariant with respect to the three spherically symmetric vector fields (denoted ξ_A, $A=1,2,3$), that is, if

$$\mathscr{L}_{\xi_A} g = 0 \tag{2.3}$$

and

$$\mathscr{L}_{\xi_A} \Gamma = 0 \ , \tag{2.4}$$

where [in a spherical polar coordinate system (t,r,θ,ϕ)]

$$\xi_1 = \frac{\partial}{\partial\phi} \ ,$$
$$\xi_2 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \ , \tag{2.5}$$
$$\xi_3 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \ .$$

In components, $\mathscr{L}_\xi g = 0$ yields

$$\xi^c\frac{\partial g_{ab}}{\partial x^c} + g_{ac}\frac{\partial\xi^c}{\partial x^b} + g_{bc}\frac{\partial\xi^c}{\partial x^a} = 0 \ . \tag{2.6}$$

The proof of this is given in most modern textbooks on general relativity. In components, $\mathscr{L}_\xi\Gamma = 0$ yields

$$(\mathscr{L}_\xi\Gamma)^a{}_{bc} = \frac{\partial\Gamma^a{}_{bc}}{\partial x^d}\xi^d + \Gamma^a{}_{bd}\frac{\partial\xi^d}{\partial x^c} + \Gamma^a{}_{dc}\frac{\partial\xi^d}{\partial x^b} - \Gamma^d{}_{bc}\frac{\partial\xi^a}{\partial x^d} + \frac{\partial^2\xi^a}{\partial x^b\partial x^c} = 0 \ . \tag{2.7}$$

The proof of this is straightforward and can be found in Ref. 1.

In order to determine the required form of g and Γ, we invoke the conditions of spherical symmetry and staticity, as defined above. However, since the coordinates in which we work are not completely specified, g and Γ will retain a degree of arbitrariness. Consequently, we invoke not only the SSS conditions, but also four coordinate conditions (thus completely fixing the coordinate system).

The conditions imposed on g and Γ by spherical symmetry are found by solving Eqs. (2.3) and (2.4) [or, rather, Eqs. (2.6) and (2.7)] for the three spherically symmetric vector fields given by (2.5); these conditions are given by [in a (t,x,y,z) coordinate system related to the (t,r,θ,ϕ) coordinate system by $t=t$, $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$]

$$
\begin{aligned}
&g_{00}=g_{tt}=f(r,t)\;,\\
&g_{0\alpha}=l(r,t)\frac{x^\alpha}{r}\;,\\
&g_{\alpha\beta}=g(r,t)\delta_{\alpha\beta}+h(r,t)\frac{x^\alpha x^\beta}{r^2}
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
\Gamma^\delta{}_{\alpha\beta}&=Ax^\alpha x^\beta x^\delta+(B\delta^\delta_\alpha x^\beta+C\delta^\delta_\beta x^\alpha+D\delta_{\alpha\beta}x^\delta)\\
&\quad+(E\epsilon_{\delta\alpha\mu}x^\mu x^\beta+F\epsilon_{\delta\beta\mu}x^\mu x^\alpha+G\epsilon_{\beta\alpha\mu}x^\mu x^\delta)\;,\\
\Gamma^\delta{}_{\alpha 0}&=Hx^\alpha x^\delta+I\epsilon_{\delta\alpha\mu}x^\mu+J\delta^\delta_\alpha\;,\\
\Gamma^\delta{}_{0\alpha}&=Kx^\alpha x^\delta+L\epsilon_{\delta\alpha\mu}x^\mu+M\delta^\delta_\alpha\;,\\
\Gamma^\delta{}_{00}&=Nx^\delta\;,\\
\Gamma^0{}_{\alpha\beta}&=Px^\alpha x^\beta+Q\epsilon_{\beta\alpha\mu}x^\mu+R\delta_{\alpha\beta}\;,\\
\Gamma^0{}_{0\delta}&=Sx^\delta\;,\\
\Gamma^0{}_{\delta 0}&=Tx^\delta\;,\\
\Gamma^0{}_{00}&=W\;,
\end{aligned}
\tag{2.9}
$$

where f, g, h, l, and the 20 functions $A,B,\ldots,W$ (omitting O and U) are arbitrary functions of r and t. The proof of (2.8) is well known, and can be found in many modern textbooks. The outline of the proof of (2.9) is as follows. Solving Eq. (2.7) for ξ_1 [in (2.5)] yields

$$
\frac{\partial\Gamma^a{}_{bc}}{\partial\phi}=0\;.
\tag{2.10}
$$

Solving (2.7) for ξ_2 and ξ_3 then consists of solving two (corresponding to ξ_2 and ξ_3) simultaneous sets of 64 ($a,b,c=1,2,3,4$) simultaneous, first-order differential equations (this solution is given in Ref. 1). The result is then obtained by transforming these results, given in the (t,r,θ,ϕ) coordinate system, to the (t,x,y,z) coordinate system, using the equation of transformation of a connection

$$
\Gamma^a{}_{bc}\frac{\partial x^b}{\partial x'^j}\frac{\partial x^c}{\partial x'^k}+\frac{\partial^2 x^a}{\partial x'^j\partial x'^k}=\Gamma'^{i'}{}_{j'k'}\frac{\partial x^a}{\partial x'^i}\;.
\tag{2.11}
$$

We note that in obtaining the results given by Eqs. (2.8) and (2.9) we have used up two of our coordinate conditions (essentially specifying θ and ϕ).

Next we impose the conditions due to staticity. Choosing a coordinate system so that the timelike vector field has components

$$
X^a=(1,0,0,0)\;,
\tag{2.12}
$$

the conditions that $\mathscr{L}_X g=0$ and that the time-like vector field X is orthogonal to a family of spacelike hypersurfaces are sufficient to prove that there is an appropriate coordinate system in which

$$
g_{0\alpha}=0\;.
\tag{2.13}
$$

[Essentially the "proof" is as follows. The form of the vector field given by (2.12) is invariant under a change of coordinates represented by $x^\alpha\to x'^\alpha=x^\alpha$, $x^0\to x'^0=x^0+f(x^\alpha)$, where f is some function of the spatial coordinates; we can then choose f, which is equivalent to completely specifying the t coordinate, such that conditions (2.13) holds.[1]]

Finally, the conditions imposed by Eqs. (2.1) and (2.2) [with X^a given by (2.12)] are

$$
\frac{\partial g_{ab}}{\partial t}=0
\tag{2.14}
$$

and

$$
\frac{\partial\Gamma^a{}_{bc}}{\partial t}=0\;.
\tag{2.15}
$$

Applying these conditions to the general form of g and Γ, we find that Γ is given by (2.9), where the arbitrary functions appearing [in (2.9)] are now functions of r only, and g is given by

$$
\begin{aligned}
&g_{00}=f(r)\;,\\
&g_{0\alpha}=0\;,\\
&g_{\alpha\beta}=g(r)\delta_{\alpha\beta}+h(r)\frac{x^\alpha x^\beta}{r^2}\;.
\end{aligned}
\tag{2.16}
$$

To summarize, two conditions of staticity were used to obtain Eqs. (2.14) and (2.15). The third condition of staticity, plus one coordinate condition (the fixing of the t coordinate) is used to obtain result (2.13).

We have one more coordinate condition to use, essentially the fixing of the r coordinate. Assuming that $g(r)$ in (2.16) is nonzero, it can then be shown that this last condition can be used to specify a coordinate system in which $g_{\alpha\beta}$ takes on the form $g_{\alpha\beta}=g(r)\delta_{\alpha\beta}$, for some arbitrary function g. [Moreover, it can be shown that the form of Γ given by (2.9) is the same in this "new" coordinate system.[1]]

We have now completed the calculation, and we have found the general form of Γ and g in an SSS spacetime (in a completely fixed coordinate system). The general form of Γ is given by (2.9), where $A,B,\ldots,W$ (omitting O and U) are arbitrary functions of r only. The general form of g is given by

$$g_{00}=f(r)\,,\qquad g_{\alpha\beta}=g(r)\delta_{\alpha\beta}\,. \tag{2.17}$$

Next we shall argue that for the physically significant situation of an SSS gravitational field generated by a spherically symmetric central mass (at rest with respect to the coordinate system), the form of Γ simplifies further.

In the special case of a spherically symmetric, central mass, we assume that Γ is a function of a single dimensionless variable U and its first derivatives only. (Note that we are using "gravitational" units in which $c=G=1$, so that M/R is dimensionless—M denotes mass and R length.) This assumption is justified on physical grounds. Also, the following dimensional analysis argument supports this. From the field equations, crudely speaking, we expect the first derivatives of Γ to be related to the energy density ρ, viz.,

$$\frac{\Gamma}{R}\sim\rho\sim\frac{M}{R^3}\,; \tag{2.18}$$

consequently,

$$\Gamma\sim\frac{M}{R}\frac{1}{R}\sim\frac{U}{R}\sim\frac{f(U)}{R}\,. \tag{2.19}$$

Therefore, we assume that Γ is a function of U and its first derivatives only. In addition, we also assume that the form of Γ is such that there are no contributions to the acceleration d^2x^α/dt^2 of a test particle in the gravitational field of the spherically symmetric central mass [calculated from Eq. (1.1)] of the form $\epsilon_{\alpha\beta\gamma}x^\beta\,dx^\gamma/dt$ (since there are no mechanisms that could realistically generate contributions to the acceleration in the $\vec{x}\otimes\vec{v}$ direction). Consequently, Γ takes on the following simplified form:

$$\begin{aligned}
&\Gamma^\delta{}_{\alpha\beta}=(\alpha)_{,\beta}\delta_{\alpha\delta}+(\bar\alpha)_{,\alpha}\delta_{\beta\delta}+(\beta)_{,\delta}\delta_{\alpha\beta}\,,\\
&\Gamma^\delta{}_{00}=(\gamma)_{,\delta}\,,\\
&\Gamma^0{}_{0\alpha}=(\delta)_{,\alpha}\,,\\
&\Gamma^0{}_{\alpha 0}=(\bar\delta)_{,\alpha}
\end{aligned} \tag{2.20}$$

(all other components of Γ are zero) ,

where α, $\bar\alpha$, β, γ, δ, and $\bar\delta$ are arbitrary functions of U (rather than r), and replace the arbitrary functions given in (2.9).[5]

We also note that in this situation g takes on the form

$$g_{ij}=\begin{bmatrix} f(U) & 0\\ 0 & -g(U)\delta_{\alpha\beta}\end{bmatrix}\,. \tag{2.21}$$

It is always possible to decompose Γ according to

$$\Gamma^a{}_{bc}=\left\{{a \atop bc}\right\}+A^a{}_{bc}\,, \tag{2.22}$$

where $\left\{{a \atop bc}\right\}$ denotes the Christoffel symbol constructed from g_{mn}, and $A^a{}_{bc}$ is a tensor (sometimes called the difference tensor). In an SSS gravitational field $\{\ \}$ is constructed from f and g, and the decomposition takes on the form [see (2.20)]

$$\begin{aligned}
&\alpha_{,\mu}=\frac{1}{2g}g_{,\mu}+\hat\alpha_{,\mu},\quad \bar\alpha_{,\mu}=\frac{1}{2g}g_{,\mu}+\hat{\bar\alpha}_{,\mu}\,,\\
&\beta_{,\mu}=-\frac{1}{2g}g_{,\mu}+\hat\beta_{,\mu},\quad \gamma_{,\mu}=\frac{1}{2g}f_{,\mu}+\hat\gamma_{,\mu}\,,\\
&\delta_{,\mu}=\frac{1}{2f}f_{,\mu}+\hat\delta_{,\mu},\quad \bar\delta_{,\mu}=\frac{1}{2f}f_{,\mu}+\hat{\bar\delta}_{,\mu}
\end{aligned} \tag{2.23}$$

where the caret denotes the part of Γ corresponding to A in (2.22).

III. THE GRAVITATIONALLY GENERALIZED EQUATIONS OF ELECTROMAGNETISM IN AN SSS GRAVITATIONAL FIELD

First we consider the GGL equations. In terms of the three-position x^σ and the coordinate time $t\equiv x^0$, equation (1.7) becomes (for $a=\sigma$)

$$\frac{d^2x^\sigma}{dt^2}+\Gamma^\sigma{}_{np}\frac{dx^n}{dt}\frac{dx^p}{dt}-\Gamma^0{}_{np}\frac{dx^\sigma}{dt}\frac{dx^n}{dt}\frac{dx^p}{dt}
=\frac{e}{m}\left[L\frac{d\lambda}{dt}\right]\left[F_n{}^\sigma\frac{dx^n}{dt}-F_n{}^0\frac{dx^\sigma}{dt}\frac{dx^n}{dt}\right], \tag{3.1}$$

where $\bar{L}\,d\lambda/dt$ now takes on the form of the arbitrary "nonmetric" multiplying factor representing possible gravitational coupling. For an SSS gravitational field, with Γ and g taking on the forms given by (2.20) and (2.21) this equation becomes

$$\frac{d^2x^\sigma}{dt^2}+(\gamma)_{,\sigma}+(\alpha+\bar{\alpha}-\delta-\bar{\delta})_{,\mu}\frac{dx^\mu}{dt}\frac{dx^\sigma}{dt}+(\beta)_{,\sigma}\left[\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\delta_{\mu\nu}\right]$$
$$=\frac{e}{m}\left[L\left[U,U_{,\rho},\frac{dx^\rho}{dt}\right]\right]\left[-\frac{1}{g}F_{n\sigma}\frac{dx^n}{dt}-\frac{1}{f}F_{n0}\frac{dx^n}{dt}\frac{dx^\sigma}{dt}\right], \quad (3.2)$$

where

$$L\left[U,U_{,\rho},\frac{dx^\rho}{dt}\right]\equiv\bar{L}\frac{d\lambda}{dt}\;. \quad (3.3)$$

Writing this in terms of the electromagnetic four-potential [using (1.5) and $A_a\equiv(-\phi,\vec{A})$] we obtain the simplified GGL equations (in vector notation)

$$\frac{d^2\vec{x}}{dt^2}+\vec{\nabla}(\gamma)+[\vec{\nabla}(\alpha+\bar{\alpha}-\delta-\bar{\delta})\cdot\vec{v}]\vec{v}+\vec{\nabla}(\beta)\vec{v}^2$$
$$=\frac{e}{m}L(U,\vec{\nabla}U,\vec{v})\left[-\frac{1}{g}\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi-\vec{\nabla}(\vec{A}\cdot\vec{v})+(\vec{v}\cdot\vec{\nabla})\vec{A}\right]+\frac{1}{f}\left\{\left[\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi\right]\cdot\vec{v}\right]\vec{v}\right\}\right], \quad (3.4)$$

where $\vec{v}=d\vec{x}/dt$ is the coordinate three-velocity and $\vec{\nabla}$ is the usual gradient operator. The two terms in bold parentheses on the right-hand side of Eq. (3.4) are generalizations of the Lorentz force.

The GGM equations simplify a great deal in an SSS gravitational field. Immediately we can write (1.12) as

$$\frac{1}{f}F_{a0,0}-\frac{1}{g}\delta^{\mu\nu}F_{a\mu,\nu}+\Sigma^{bc}{}_a(f,g,\alpha,\bar{\alpha},\beta,\gamma,\delta,\bar{\delta}\text{ and their spatial derivatives})F_{bc}=4\pi J_a\;. \quad (3.5)$$

Only the antisymmetric part of $\Sigma^{bc}{}_a$ occurs in the above equation. In the SSS idealization $\Sigma^{[bc]}{}_a$ will take on the form

$$\Sigma^{[0\nu]}{}_0=\frac{1}{2}\frac{\mathscr{A}}{g}g^\nu\,,$$
$$\Sigma^{[00]}{}_\sigma=0\,, \quad (3.6)$$
$$\Sigma^{[\mu\nu]}{}_\sigma=\frac{\mathscr{B}}{g}(g^\nu\delta^\mu_\sigma-g^\mu\delta^\nu_\sigma)$$

in terms of just two arbitrary functions (of the gravitational field) $\mathscr{A}$ and $\mathscr{B}$. (g^ρ represents the ρth component of $\vec{g}$, i.e., $g^\rho=\partial U/\partial x^\rho$—do not confuse this with g defined by $g_{\mu\nu}=-g\delta_{\mu\nu}$.) The "proof" of this is as follows. $\Sigma^{bc}{}_a$ is constructed from terms such as $g^{bc}{}_{,a}$ and $\Gamma^{bc}{}_a$; consequently $\Sigma^{bc}{}_a$ will take on the form represented by (2.20) and (2.23) in an SSS gravitational field. Antisymmetrization then gives the result.

Using (3.6), Eq. (3.5) becomes for $a=0$

$$-\frac{1}{g}\delta^{\mu\nu}F_{0\mu,\nu}=4\pi J_0-\frac{\mathscr{A}}{g}F_{0\rho}g^\rho\,, \quad (3.7a)$$

and for $a=\sigma$

$$\frac{1}{f}F_{\sigma0,0}-\frac{1}{g}\delta^{\mu\nu}F_{\sigma\mu,\nu}=4\pi J_\sigma-\frac{\mathscr{B}}{g}F_{\sigma\rho}g^\rho\;. \quad (3.7b)$$

We can write (3.7) in terms of the electromagnetic four-potential. Using (1.5), $A_a\equiv(-\phi,\vec{A})$ and the "Lorentz" gauge condition

$$\frac{g}{f}\frac{\partial\phi}{\partial t}+\vec{\nabla}\cdot\vec{A}=0\,, \quad (3.8)$$

in vector notation (3.7a) becomes

$$\nabla^2\phi=\frac{g}{f}\frac{\partial^2\phi}{\partial t^2}+\mathscr{A}\vec{g}\cdot\left[\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\phi\right]-4\pi gJ_0\,, \quad (3.9a)$$

and (3.7b) becomes

$$\nabla^2\vec{A}=\frac{g}{f}\frac{\partial^2\vec{A}}{\partial t^2}+\frac{f}{g}(\vec{\nabla}\cdot\vec{A})\vec{\nabla}\left[\frac{g}{f}\right]$$
$$+\mathscr{B}(\vec{\nabla}\times\vec{A})\times\vec{g}+4\pi g\vec{J}\,, \quad (3.9b)$$

where $\vec{J} \equiv J_\sigma$.

As mentioned above, $\mathscr{A}$ and $\mathscr{B}$ are general functions of the gravitational field. In particular, $\mathscr{A}$ and $\mathscr{B}$ are functions of f, g, α, $\bar{\alpha}$, β, γ, δ, and $\bar{\delta}$ and their derivatives with respect to U. For each different type of GGM equations $\mathscr{A}$ and $\mathscr{B}$ would take on (different) explicit forms. For example, suppose we take the GGM equations in an MATG to be

$$g^{bc}F_{ab\,|\,c}=4\pi J_a \; . \tag{3.10}$$

Using the definition of the covariant derivative with respect to Γ (denoted by the bar), this becomes

$$g^{bc}(F_{ab,c}-\Gamma^m{}_{ac}F_{mb}-\Gamma^m{}_{bc}F_{am})=4\pi J_a \; . \tag{3.11}$$

Expanding this equation using (2.20) and (2.21), and writing it in terms of the electromagnetic four-potential [using (3.8)], in the SSS limit Eq. (3.11) then takes on the form of (3.9) with $\mathscr{A}$ and $\mathscr{B}$ defined by

$$\begin{aligned}\mathscr{A}&=\delta'+\alpha'+\bar{\alpha}'+3\beta' \; ,\\ \mathscr{B}&=2\alpha'+2\beta'+\bar{\alpha}'-\frac{g}{f}\gamma' \; ,\end{aligned} \tag{3.12a}$$

where the prime denotes differentiation with respect to U (i.e., $df/dU=f'$).

Other possible GGM equations would yield the following values for $\mathscr{A}$ and $\mathscr{B}$:

$$F_a{}^c{}_{|c}=(g^{bc}F_{ab})_{|c}=4\pi J_a : \begin{cases}\mathscr{A}=\dfrac{g'}{g}+\dfrac{g\gamma'}{f}-3\alpha'-\bar{\alpha}'-\beta' \; ,\\ \mathscr{B}=\dfrac{g'}{g}-\delta'-2\alpha'-\bar{\alpha}'-2\beta' \; ;\end{cases} \tag{3.12b}$$

$$F^{ac}{}_{|c}=4\pi J^a : \begin{cases}\mathscr{A}=\dfrac{f'}{f}+\dfrac{g'}{g}-\delta'-3\alpha'-\bar{\alpha}'-\beta' \; ,\\ \mathscr{B}=2\dfrac{g'}{g}-\delta'-4\bar{\alpha}'-2\beta' \; ;\end{cases} \tag{3.12c}$$

$$g^{bc}F^a{}_{b\,|\,c}=4\pi J^a : \begin{cases}\mathscr{A}=\dfrac{f'}{f}-\dfrac{g\gamma'}{f}+2\delta'-2\bar{\delta}'+\alpha'+\bar{\alpha}'+3\beta' \; ,\\ \mathscr{B}=\dfrac{g'}{g}-\dfrac{g\gamma'}{f}+4\alpha'-2\bar{\alpha}'+2\beta' \; ;\end{cases} \tag{3.12d}$$

$$g^{bc}F_{ab;c}=4\pi J_a : \begin{cases}\mathscr{A}=-\dfrac{1}{2}\left[\dfrac{g'}{g}-\dfrac{f'}{f}\right] ,\\ \mathscr{B}=\dfrac{1}{2}\left[\dfrac{g'}{g}-\dfrac{f'}{f}\right] .\end{cases} \tag{3.12e}$$

This last example is the "metrically" modified Maxwell equations. That is, the modified Maxwell equations for an MTG in an SSS gravitational field take on the form of (3.9) with $\mathscr{A}$ and $\mathscr{B}$ given by (3.12e).

Finally, an electromagnetic source consisting of electromagnetic point particles is defined by

$$J^m(\vec{x})=\sum_k e_k[-\det(g_{ab})]^{-1/2}\delta^3(\vec{x}-\vec{x}_k)\frac{dx_k{}^m}{dt} \; , \tag{3.13}$$

where k denotes the kth particle, e_k its charge, $\vec{x}_k$ its three-position and $dx_k{}^m/dt$ its four-velocity. Consequently, using $J_a=g_{am}J^m$, in an SSS gravitational field (3.13) becomes for $a=0$

$$J_0=\frac{f^{1/2}}{g^{3/2}}\sum_k e_k\delta^3(\vec{x}-\vec{x}_k) \; , \tag{3.14a}$$

and for $a=\sigma$

$$\vec{J}\equiv J_\sigma=-\frac{1}{f^{1/2}g^{1/2}}\sum_k e_k\delta^3(\vec{x}-\vec{x}_k)\vec{v}_k \; . \tag{3.14b}$$

IV. CHARGE CONSERVATION

Since we have insisted that the GGM equations reduce to the special-relativistic equations in the appropriate limit, charge will be conserved in the "absence" of gravity. Indeed, charge conservation in special relativity is a very-well-tested law of physics. However, it is possible to put forward theories in which charge is not conserved in the presence of a gravitational field. In general, charge is not conserved with respect to the GGM equations as given by (1.11) or (1.12).

But it may be felt that charge should be conserved. This would then lead to constraints on the form of the GGM equations. In this section we shall investigate the constraints obtained by imposing charge conservation.

It is convenient to take the GGM equations in the form given by (1.12), viz.,

$$g^{bc}F_{ab,c}+\Sigma^{bc}{}_aF_{bc}=4\pi J_a \ . \tag{4.1}$$

What form should the equation for charge conservation take in the presence of gravity? We certainly want the equation to be covariant. Also the equation should reduce to the special-relativistic equation in the correct limit. Hence, we take the following equation as appropriate:

$$g^{am}J_{a:m}=g^{am}(J_{a,m}-L^n{}_{am}J_n)=0 \ , \tag{4.2}$$

where the colon denotes covariant differentiation with respect to some "connection" L (L is assumed to depend on the gravitational field). L could be the metric connection $\{\ \}$, or the affine connection Γ, but for the moment we shall assume that L is any "general connection."

Therefore, taking the covariant derivative of (4.1) with respect to L, and using (4.2), we obtain (after some simplification)

$$(\Sigma^{ab}{}_{c,m}g^{cm}-L^n{}_{cm}\Sigma^{ab}{}_ng^{cm})F_{ab} +(g^{bc}{}_{,m}g^{am}+\Sigma^{ab}{}_mg^{mc}-L^a{}_{nm}g^{bc}g^{nm})F_{ab,c}=0 \ . \tag{4.3}$$

Since this equation must be satisfied for all possible electromagnetic fields, each term must be zero independently, viz.,

$$(\Sigma^{[ab]}{}_{c,m}-L^n{}_{cm}\Sigma^{[ab]}{}_n)g^{cm}F_{ab}=0 \tag{4.4a}$$

and

$$(g^{bc}{}_{,m}g^{am}+\Sigma^{[ab]}{}_mg^{mc}-L^a{}_{nm}g^{bc}g^{nm})F_{ab,c}=0 \ . \tag{4.4b}$$

These two equations represent constraints on Σ (in terms of g and L) in order for charge to be conserved. For general L and Σ no more information can be obtained. For a particular theory, where L and Σ are given, we could work out explicitly the conditions imposed by (4.4).

In the case of an SSS gravitational field the forms of g, Γ, L, and Σ simplify considerably, and Eqs. (4.4) reduce to a very simple form. Indeed, in such a gravitational field the forms of g and Σ are given by (2.21) and (3.6), respectively, and the form of L is given by

$$\begin{aligned} &L^\sigma{}_{\mu\nu}=(l_1)_{,\nu}\delta^\sigma_\mu+(l_2)_{,\mu}\delta^\sigma_\nu+(l_3)_{,\sigma}\delta_{\mu\nu} \ , \\ &L^\sigma{}_{00}=(l_4)_{,\sigma} \ , \\ &L^0{}_{0\nu}=(l_5)_{,\nu}, \quad L^0{}_{\mu 0}=(l_6)_{,\mu} \ , \end{aligned} \tag{4.5}$$

where the l_i's are functions of U. Consequently, equations (4.4a) and (4.4b) become, after some tedious algebra (using the symmetries of g and F),

$$0=0 \ , \tag{4.6a}$$

and

$$\frac{1}{fg}\left[-\frac{f'}{f}+\mathscr{A}+l\right]F_{0\mu,0}g^\mu +\frac{1}{g^2}\left[\frac{g'}{g}-\mathscr{B}-l\right]F_{\mu\nu,\rho}\delta^{\rho\mu}g^\nu=0, \tag{4.6b}$$

where l is defined by

$$l=\frac{g}{f}l_4'-l_1'-l_2'-3l_3' \ , \tag{4.7}$$

and (for example)

$$(f)_{,\sigma}=\frac{df}{dU}\frac{\partial U}{\partial x^\sigma}=f'g^\sigma \ , \tag{4.8}$$

where the prime denotes differentiation with respect to U.

Equation (4.6a) indicates that (4.4a) is redundant in an SSS gravitational field. If charge is to be conserved, it must be so for all possible charge distributions, and hence, for all possible electromagnetic fields F_{ab}. Therefore, the constraints imposed on Σ by (4.6b) are

$$\frac{f'}{f}-\mathscr{A}-l=0 \ , \tag{4.9a}$$

and

$$\frac{g'}{g}-\mathscr{B}-l=0 \ . \tag{4.9b}$$

If we demand that charge should be conserved "with respect to" the metric connection [so that from (2.23) $l_4'=\frac{1}{2}f'/g$ and $l_1'=l_2'=-l_3'=\frac{1}{2}g'/g$], then

$$l=\frac{1}{2}\frac{f'}{f}+\frac{1}{2}\frac{g'}{g}\ , \tag{4.10}$$

and the constraints on $\mathscr{A}$ and $\mathscr{B}$ become

$$\mathscr{A}=\frac{1}{2}\frac{f'}{f}-\frac{1}{2}\frac{g'}{g}\ ,$$
$$\mathscr{B}=\frac{1}{2}\frac{g'}{g}-\frac{1}{2}\frac{f'}{f} \tag{4.11}$$

(that is, the GGM equations take on their metric form).

If, on the other hand, we demand that charge should be conserved "with respect to" the affine connection Γ [so that from (2.20) $l'_4=\gamma'$, $l'_1=\alpha'$, $l'_2=\bar{\alpha}'$, and $l'_3=\beta'$], then

$$l=\frac{g}{f}\gamma'-\alpha'-\bar{\alpha}'-3\beta'\ , \tag{4.12}$$

and (4.9) becomes

$$\mathscr{A}=\frac{f'}{f}-\frac{g}{f}\gamma'+\alpha'+\bar{\alpha}'+3\beta'\ ,$$
$$\mathscr{B}=\frac{g'}{g}+\frac{g}{f}\gamma'-\alpha'-\bar{\alpha}'-3\beta'\ . \tag{4.13}$$

We note from (4.9) that we have one constraint that is independent of the form of L (by eliminating l from the equations). That is, in order for charge to be conserved for any arbitrary L (the most general form of charge conservation), the following condition must be satisfied:

$$\mathscr{A}-\mathscr{B}-\frac{f'}{f}+\frac{g'}{g}=0\ . \tag{4.14}$$

In Sec. III we considered some possible forms of the GGM equations, calculating the explicit values of $\mathscr{A}$ and $\mathscr{B}$ in each case [see Eq. (3.12)]. Equations (4.13) and (4.14) could then be used to constrain the form of Γ in each of these particular sets of GGM equations.

V. THE OPTICAL LIMIT

In this section we shall discuss the geometric optics approximation, or the optical limit, of the laws of electromagnetism. In this appproximation we are essentially looking for wavelike solutions for the electromagnetic potential A_a, by splitting up the potential into a very slowly changing complex-amplitude part, and a rapidly changing real-phase part, viz.,

$$A_a=\mathrm{Re}(A_{a0}e^{i\theta}) \tag{5.1}$$

(θ is sometimes called the eikonal, and the approximation scheme referred to as the eikonal approximation).

Therefore, we are looking for a high-frequency (or short-wavelength) approximation. More precisely, geometric optics is valid whenever the wavelength λ is very short compared with the typical length l over which the amplitude (and polarization) vary, and compared with the typical radius of curvature R of the spacetime through which the waves propagate.

We wish to expand (5.1) in powers of λ. We introduce the dummy parameter $\epsilon\sim\lambda$ to qualify the approximation scheme. Expanding the amplitude in powers of ϵ, and putting $\theta=\theta/\epsilon$ (since the phase is proportional to λ^{-1}), A_a therefore expands according to

$$A_a=\mathrm{Re}[(a_a+\epsilon b_a+\epsilon^2c_a+\cdots)e^{i\theta/\epsilon}]\ . \tag{5.2}$$

The key results of geometric optics are then obtained by substituting A_a [given by Eq. (5.2)] into the source-free GGM equations [given by Eq. (1.11)]. In particular, collecting together the highest-order or ϵ^{-2}-order terms of the "expanded" GGM equations, we find that

$$\mathrm{Re}(\epsilon^{-2}g^{bc}\theta_{,b}\theta_{,c}a_ae^{i\theta/\epsilon})=0\ . \tag{5.3}$$

Defining the "wave vector" k_a by $k_a=\theta_{,a}$, (5.3) becomes

$$g^{bc}k_bk_c=0\ . \tag{5.4}$$

Light rays are then defined to be curves normal to surfaces of constant phase θ. Since $k_a=\theta_{,a}$ is the normal to these surfaces, the differential equation for a light ray is

$$\frac{dx^a}{d\sigma}=k^a=g^{ab}k_b\ . \tag{5.5}$$

Therefore, Eq. (5.4) tells us that light rays are null. This is the main result of geometric optics (and the only result that we shall use here). For a discussion of the other laws of geometric optics see Ref. 6.

Taking the covariant derivative of (5.4) with respect to the metric connection, and using the fact that $k_{b;c}=\theta_{,b;c}=k_{c;b}$, we find that

$$k^a{}_{;b}k^b=\frac{d^2x^a}{d\sigma^2}+\left\{{a \atop bc}\right\}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}=0\ , \tag{5.6}$$

so that light rays are null geodesics (σ is an affine parameter with respect to the geodesic).

We recall that in MATG's the motion of timelike test particles is governed by the path equation

(1.1) (do not confuse the affine parameter λ in (1.1) with wavelength). In general, Eqs. (1.1) and (5.6) are not equivalent. However, it may be felt that the equations of motion governing photons should be the limit (as mass $\rightarrow 0$ or speed $\rightarrow 1$) of the equations of motion governing timelike particles. (Note that if this is not the case the equivalence principle is broken.) Consequently, it may be assumed that for particles with $dx^a/d\sigma$ satisfying (5.4), Eqs. (1.1) and (5.6) are identical. In order to make a comparison of these two equations we rewrite Eq. (1.1) as

$$\frac{d^2x^a}{d\sigma^2}+\left\{ {a \atop bc} \right\}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}+A^a{}_{bc}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}+\Lambda\frac{dx^a}{d\sigma}=0 , \qquad (5.7)$$

where Γ has been decomposed according to (2.22), and $\Lambda=(d^2\sigma/d\lambda^2)(d\lambda/d\sigma)^2$ is a scalar associated with the change in parametrization. Therefore, the equivalence of (5.6) and (5.7) yields

$$A^a{}_{bc}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}+\Lambda\frac{dx^a}{d\sigma}=0 . \qquad (5.8)$$

For general connections (i.e., general $A^a{}_{bc}$), Eq. (5.8) is in its simplest form. For special connections (5.8) reduces to a more simple expression. In particular, it reduces to a very simple form in the case of an SSS gravitational field. That is, for $A^a{}_{bc}$ given by (2.23), (5.8) becomes for $a=0$

$$(\hat{\delta}+\hat{\hat{\delta}})_{,\mu}\frac{dx^\mu}{d\sigma}\frac{dt}{d\sigma}+\Lambda\frac{dt}{d\sigma}=0 . \qquad (5.9a)$$

and for $a=\mu$

$$(\hat{\alpha}+\hat{\hat{\alpha}})_{,\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}+(\hat{\beta})_{,\mu}\delta_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma}+(\hat{\gamma})_{,\mu}\left[\frac{dt}{d\sigma}\right]^2+\Lambda\frac{dx^\mu}{d\sigma}=0 . \qquad (5.9b)$$

Equations (5.9) are subject to Eq. (5.4), which can be written as [using (2.21)]

$$f\left[\frac{dt}{d\sigma}\right]^2-g\delta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}=0 . \qquad (5.10)$$

Using (5.9a) and (5.10), Eq. (5.9b) becomes

$$(\hat{\alpha}+\hat{\hat{\alpha}}-\hat{\delta}-\hat{\hat{\delta}})_{,\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}+\left[\hat{\gamma}_{,\mu}+\frac{f}{g}\hat{\beta}_{,\mu}\right]\left[\frac{dt}{d\sigma}\right]^2=0 . \qquad (5.11)$$

Therefore, in an SSS gravitational field, the constraints on Γ, imposed by demanding that Eq. (1.1) is equivalent to equation (5.6) for particles satisfying (5.4), take the form

$$(\hat{\alpha}_{,\mu}+\hat{\hat{\alpha}}_{,\mu}-\hat{\delta}_{,\mu}-\hat{\hat{\delta}}_{,\mu})=0 \qquad (5.12a)$$

and

$$\left[\hat{\gamma}_{,\mu}+\frac{f}{g}\hat{\beta}_{,\mu}\right]=0 . \qquad (5.12b)$$

The question as to whether Eq. (5.11) is realized is open to experiment. In particular, experiments in the solar system that test, on the one hand, the motion of planets and satellites, and, on the other, the motion of electromagnetic radiation, do verify Eq. (5.11), at least up to first order within the PPN approximation scheme (we shall discuss this in more detail in a later paper).

VI. THE EQUIVALENCE PRINCIPLE

As we have already stated, we require the laws of electromagnetism to take on their special-relativistic form in the "absence" of gravity. But the principle of equivalence says rather more than this. It states that the laws of physics should take on their special relativistic form (approximately) in freely falling (or locally inertial) coordinate frames.

For MATG's, the local inertial frames are the Riemann-normal (RN) frames,[7] in which

$$g_{bc}=\eta_{bc}+o(y), \Gamma^a{}_{(bc)}=0+o(y) ,$$
$$\left\{ {a \atop bc} \right\}\sim g^{ab}{}_{,c}\sim\Gamma^a{}_{[bc]}\sim o(y^0) , \qquad (6.1)$$

where $o(y)$ denotes a "small" correction, the order of magnitude of the coordinate axes y^a, and hence the size of the region under consideration. As this region becomes smaller, the accuracy of the approximation therefore increases. In an RN coordinate system the equation of motion of a freely falling test particle in a gravitational field becomes $d^2x^a/d\lambda^2=0$, and so the gravitational laws of physics approximately take on their special-relativistic form.

Here we shall consider the principle of equivalence with respect to laws of GGEM, and in particular, the GGM equations. That is, we shall examine the form of the GGM equations in a local inertial frame and see whether (approximately) they take on their special-relativistic form. Indeed, if we contrast (6.1) with the corresponding conditions in MTG's, where the local inertial frames are represented by geodesic-normal coordinate frames in which[7]

$$g_{bc}=\eta_{bc}+o(y^2),\ \ \{{}^{a}_{bc}\}\sim g^{ab}{}_{,c}\sim o(y)\ , \qquad (6.2)$$

we see that g_{bc}, $\Gamma^a{}_{[bc]}$, and $g^{ab}{}_{,c}$ are an order of magnitude larger than their metric counterparts. Therefore, we might anticipate that in an RN coordinate system the GGM equations will not take on the correct form, and the equivalence principle will consequently be broken.

We wish to rewrite the GGM equations. We put $\Pi^{bc}{}_a=\Pi_1{}^{bc}{}_a+\Pi_2{}^{bc}{}_a+\Pi_3{}^{bc}{}_a$, where the first term depends on the symmetric part of the connection Γ (and g^{bc} and η^{bc} only), the second on the first derivatives of the metric, and the third on the antisymmetric part of Γ. We also expand g^{bc} by a Taylor series about the center of the RN coordinate system, so that $g^{bc}=\eta^{bc}+a^{bc}{}_dy^d+o(y^2)$, where the $a^{bc}{}_d$'s are constants. Therefore, we can write (1.11) as

$$-\eta^{bc}A_{a,bc}-a^{bc}{}_dA_{a,bc}y^d+o(y^2)A_{a,bc}$$
$$+b^{bc}{}_aA_{b,c}+o(y)A_{b,c}=4\pi J_a\ , \qquad (6.3)$$

where the $b^{bc}{}_a$'s are the constant values of $\Pi_2{}^{bc}{}_a+\Pi_3{}^{bc}{}_a$ (evaluated at the center of the coordinate system). Furthermore, we do not expect all the constants $a^{bc}{}_d$ and $b^{bc}{}_d$ to be zero [since they depend on $g^{bc}{}_{,d}$, and we could not have $g^{bc}{}_{,d}\sim o(y)$ and $\Gamma^a{}_{(bc)}\sim o(y)$ simultaneously].

The analogous equation [to (6.3)] in an MTG would not contain the terms $a^{bc}{}_dA_{a,bc}y^d$ and $b^{bc}{}_aA_{b,c}$. Since the first term contains the y^d factor, in general it will not contribute to any violation of the equivalence principle. It is the second term which gives rise to such violations.

As an illustration we look for a wavelike solution (propagating in the z direction) to a source-free equation (6.3) of the form $A_a\sim\exp[i(kz-\omega z)]$. For simplicity, we take as the only nonzero components of the constants $b^{ab}{}_c$, $b^{0\nu}{}_0=\xi_\nu$ and $b^{\mu\nu}{}_\rho=\xi_\mu\delta_{\nu\rho}+\xi_\nu\delta_{\mu\rho}$ [$\xi_\nu\equiv(\xi_x,\xi_y,\xi_z)$]. From (6.3) we obtain the dispersion relation $k^2-\omega^2=\pm\xi_zki$. Since this relation does not approach the special-relativistic relation as we restrict attention to smaller regions of spacetime, it represents a theoretical violation of the equivalence principle. (Note, however, that the relations only differ significantly for extremely small wavelengths and therefore such a violation is probably not observable.) If we define the electromagnetic energy in the usual way, the above violation of the equivalence principle is accompanied by a breakdown in the conservation of energy.

VII. GRAVITATIONAL RED-SHIFT

The gravitational red-shift is usually calculated using the equation governing ideal clocks in a gravitational field. However, such an equation is a (theoretical) idealization. Real clocks, made up of actual atomic systems, are subject to the various laws of physics. With sufficient knowledge of such systems and the corresponding physics, we could determine the gravitational red-shift directly from the underlying physics. In particular, we might expect that the GGM equations would give us information on the frequency shift of radiation when regarded as a wavelike phenomenon.

The wavelike properties of radiation are obtained by considering the optical limit of the source-free GGM equations (see Sec. V). The essential part of these equations are the terms involving second derivatives of A_a, viz.,

$$g^{bc}A_{a,bc}=0\ . \qquad (7.1)$$

If we look for a wavelike solution of Eq. (7.1) (propagating in the z direction) of the form

$$A\sim A_0\exp[i(kz-\omega t)]\ , \qquad (7.2)$$

then using the result that radiation travels on null geodesics, in an SSS gravitational field we find that

$$A\sim A_0\exp[i(g^{1/2}z-f^{1/2}t)]\ . \qquad (7.3)$$

Here A_0, a function of the gravitational field, is a slowly varying amplitude part. The wave vector k_a is given by

$$k_a=(f^{1/2},0,0,g^{1/2})\ . \qquad (7.4)$$

Due to the static nature of the coordinate time, any stationary observer will have a four-velocity U^a given by

$$U^a=(1,0,0,0)\ . \qquad (7.5)$$

Consequently, a wave will undergo a frequency shift with respect to an emitter and a receiver at rest in an SSS gravitational field, according to

$$\frac{U^ak_a(\mathrm{em})-U^ak_a(\mathrm{rec})}{U^ak_a}=\frac{f^{1/2}(\mathrm{em})-f^{1/2}(\mathrm{rec})}{f^{1/2}}\ . \qquad (7.6)$$

In nonmetric theories of gravity there is no universal equation for the gravitational red-shift; a study of the phenomenon can only be made with a complete theory of measurement and a knowledge of the actual measuring instruments involved (which is beyond the scope of the present investigation). However, one would expect that the results of any such study would have to be consistent with the above analysis concerning the frequency shift calculated from the GGM equations.

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