Isotropization of scalar field Bianchi type-VII$_h$ models with an exponential potential

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(Received 16 October 1996)

We investigate the isotropization of the Bianchi type-VII$_h$ cosmological models possessing a scalar field with an exponential potential of the form $V(\phi)=\Lambda e^{k\phi}$. In the case $k^2>2$, we show that there is an open set of initial conditions in the set of anisotropic Bianchi type-VII$_h$ initial data such that the corresponding cosmological models isotropize asymptotically. Hence scalar field spatially homogeneous cosmological models having an exponential potential with $k^2>2$ can isotropize to the future. [S0556-2821(97)05606-3]

PACS number(s): 04.20.Jb, 98.80.Hw

I. INTRODUCTION

In the famous paper by Collins and Hawking [1] it was proven that, within the set of spatially homogeneous cosmological models (which satisfy reasonable energy conditions), those which approach isotropy at infinite times are of measure zero; that is, in general anisotropic models do not isotropize as they evolve to the future. Since we presently observe the universe to be highly isotropic, we therefore need an explanation of why our universe has evolved the way it has. This problem, known as the isotropy problem, can be easily solved with an idea popularized by Guth [2]. If the early universe experiences a period of inflation, then all anisotropies are essentially pushed out of our present observable light-cone and are therefore not presently observed. The cosmic no-hair conjecture asserts that under appropriate conditions, any universe model will undergo a period of inflation and will consequently isotropize.

A significant amount of work on the cosmic no-hair conjecture has already been done for spatially homogeneous (Bianchi) cosmologies [3–9]. For instance, Wald [3] has proven a version of the cosmic no-hair conjecture for spatially homogeneous spacetimes with a positive cosmological constant; namely, he has shown that all initially expanding Bianchi models asymptotically approach a spatially homogenous and isotropic model, except the subclass of Bianchi type-IX models which recollapse. It was shown by Grøn [9] that inflation is not prevented by any amount of initial anistropy in a class of generalized de Sitter models.

Anisotropic models with scalar fields and with particular forms of the scalar field potential have also been investigated. Heusler [5], has analyzed the case in which the potential function passes through the origin and is concave up and, like Collins and Hawking [1] has found that the only models that can possibly isotropize to the future are those of Bianchi types I, V, VII, and IX.

Recent work on scalar field models with an exponential potential of the form $V(\phi)=\Lambda e^{k\phi}$ (where Λ and k are positive constants and ϕ is the scalar field) has been done by Kitada and Maeda [6,7] and Ibáñez *et al.* [8]. In a series of two papers Kitada and Maeda [6,7] have proven that if $k<\sqrt{2}$, then all initially expanding Bianchi models except possibly those of type IX must isotropize. In the case of the Bianchi type-IX models there exists a subclass of models which isotropize and a subclass of models which recollapse. The aim here is to determine what happens in the case $k>\sqrt{2}$. In Ibáñez *et al.* [8] it was proven, using results from Heusler's paper [5], that the only models that can possibly isotropize when $k>\sqrt{2}$ are those of Bianchi types I, V, VII, or IX. Since the Bianchi type-I, type-V, and type-VII$_0$ models are restricted classes of models, the only general spatially homogeneous models that can possibly isotropize are consequently of types VII$_h$ or IX. Here we shall study the possible isotropization of the Bianchi type-VII$_h$ models when $k>\sqrt{2}$.

II. THE BIANCHI TYPE-VII$_h$ MODEL

A. The equations

The Bianchi type-VII$_h$ models belong to the Bianchi type-B models as classified by Ellis and MacCallum [10]. Hewitt and Wainwright [11] have derived the equations describing the evolution of the general Bianchi type-B models. We shall utilize these equations, adjusted so that they describe a model with a minimally coupled scalar field ϕ with an exponential potential $V(\phi)=\Lambda e^{k\phi}$. The energy-momentum tensor describing a minimally coupled scalar field is given by

$$T_{ab}=\phi_{;a}\phi_{;b}-g_{ab}\left[\frac{1}{2}\phi_{;c}\phi^{;c}+V(\phi)\right],$$

where, for a homogeneous scalar field, $\phi=\phi(t)$. In this case we can formally treat the energy-momentum tensor as a perfect fluid with velocity vector $u^a = \phi^{;a}/\sqrt{-\phi_{;b}\phi^{;b}}$, where the energy density, μ_ϕ, and the pressure, p_ϕ, are given by

$$\mu_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.1}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.2}$$

Our variables are the same as those used by Hewitt and Wainwright [11], with the addition of

$$\Phi \equiv \frac{\sqrt{3V}}{\theta}, \quad \Psi \equiv \sqrt{\frac{3}{2}}\frac{\dot{\phi}}{\theta} \tag{2.3}$$

to describe the scalar field. The quantity Ω, defined in [11] as $\Omega \equiv 3\mu/\theta^2$, becomes $\Omega = \Phi^2 + \Psi^2$ in the case under investigation here.

The dimensionless evolution equations are then (see Hewitt and Wainwright [11] for notation)

$$\Sigma_+' = (q-2)\Sigma_+ - 2\tilde{N}, \tag{2.4}$$

$$\tilde{\Sigma}' = 2(q-2)\tilde{\Sigma} - 4\Delta N_+ - 4\Sigma_+\tilde{A}, \tag{2.5}$$

$$\Delta' = 2(q+\Sigma_+ - 1)\Delta + 2(\tilde{\Sigma} - \tilde{N})N_+, \tag{2.6}$$

$$\tilde{A}' = 2(q+2\Sigma_+)\tilde{A}, \tag{2.7}$$

$$N_+' = (q+2\Sigma_+)N_+ + 6\Delta, \tag{2.8}$$

$$\Psi' = (q-2)\Psi - \sqrt{\frac{3}{2}}k\Phi^2, \tag{2.9}$$

where the prime denotes differentiation with respect to the new dimensionless time τ, where $dt/d\tau = 3/\theta$ [11], and where

$$\tilde{N} = \frac{1}{3}(N_+^2 - \ell\tilde{A}), \tag{2.10}$$

$$q = 2\Sigma_+^2 + 2\tilde{\Sigma} + 2\Psi^2 - \Phi^2. \tag{2.11}$$

There also exists the constraint

$$\tilde{\Sigma}\tilde{N} - \Delta^2 - \tilde{A}\Sigma_+^2 = 0, \tag{2.12}$$

and the equations are subject to the conditions

$$\tilde{A} \geqslant 0, \quad \tilde{\Sigma} \geqslant 0, \quad \tilde{N} \geqslant 0. \tag{2.13}$$

The generalized Friedmann equation, written in dimensionless variables, becomes

$$\Phi^2 = 1 - \Sigma_+^2 - \tilde{\Sigma} - \tilde{A} - \tilde{N} - \Psi^2, \tag{2.14}$$

which serves to define Φ, and the evolution of Φ is governed by

$$\Phi' = \left(q + 1 + \sqrt{\frac{3}{2}}k\Psi\right)\Phi. \tag{2.15}$$

Equations (2.9) and (2.15) are equivalent to the Klein-Gordon equation $\ddot{\phi} + \theta\dot{\phi} + V'(\phi) = 0$ written in dimensionless variables. The parameter $\ell = 1/h$ defines the group parameter h in the Bianchi type-VII$_h$ models.

The variables Σ_+ and $\tilde{\Sigma}$ describe the shear anisotropy. The variables $\tilde{A}$, N_+, and $\tilde{N}$ describe the spatial curvature of the models. The variable Δ describes the relative orientation of the shear and spatial curvature eigenframes.

In this short paper we are not interested in the complete qualitative behavior of the cosmological models but simply whether the Bianchi type-VII$_h$ models isotropize to the future when $k > \sqrt{2}$. This question can be easily answered by examining the stability of the isotropic equilibrium points of the above six-dimensional dynamical system (2.4)–(2.9).

B. Stability analysis

All of the isotropic equilibrium points lie in the invariant set defined by $\{\Sigma_+ = 0, \tilde{\Sigma} = 0, \tilde{N} = 0, \Delta = 0\}$. Therefore, we are able to find all of the isotropic equilibrium points and easily determine whether any are stable attractors or sinks. By definition an equilibrium point is a sink if the eigenvalues of the linearization of the dynamical system in a neighborhood of the equilibrium point have negative real parts.

The equilibrium point

$$\Sigma_+ = 0, \quad \tilde{\Sigma} = 0, \quad \Delta = 0, \quad A = 1, \quad N_+ = \sqrt{\ell}, \quad \Psi = 0 \tag{2.16}$$

implies $\Phi = 0$ and represents the negatively curved Milne vacuum model. The linearization of the dynamical system in the neighborhood of this equilibrium point has eigenvalues

$$0, 2, -2, -4, -2+4\sqrt{-\ell}, -2+4\sqrt{-\ell}. \tag{2.17}$$

Therefore, this equilibrium point is a saddle.

The equilibrium point(s)

$$\Sigma_+ = 0, \quad \tilde{\Sigma} = 0, \quad \Delta = 0, \quad A = 0, \quad N_+ = 0, \quad \Psi = 1 \text{ or } -1 \tag{2.18}$$

imply that $\Phi = 0$ and represent flat noninflationary Friedmann-Robertson-Walker (FRW) model(s). The eigenvalues in both cases are

$$0, 0, 2, 2, 4, 6 + \sqrt{6}k. \tag{2.19}$$

These equilibrium points are unstable with an unstable manifold of at least dimension 4.

The equilibrium point

$$\Sigma_+ = 0, \quad \tilde{\Sigma} = 0, \quad \Delta = 0, \quad A = 0,$$

$$N_+ = 0, \quad \Psi = -\frac{\sqrt{6}}{6}k \tag{2.20}$$

implies that $\Phi=\sqrt{1-k^2/6}$. The eigenvalues are

$$\frac{k^2-6}{2},\quad \frac{k^2-6}{2},\quad k^2-6,\quad k^2-4,$$

$$k^2-2,\ \frac{k^2-2}{2}. \tag{2.21}$$

For $k<\sqrt{2}$, this equilibrium point represents the usual power-law inflationary attractor. If $\sqrt{2}<k<2$, then the equilibrium point has an unstable manifold of dimension 4. If $2<k<\sqrt{6}$, then the equilibrium point has an unstable manifold of dimension 3. This equilibrium point does not exist if $k>\sqrt{6}$.

The equilibrium point

$$\Sigma_+=0,\quad \tilde{\Sigma}=0,\quad \Delta=0,\quad A=1-\frac{2}{k^2},$$

$$N_+=\sqrt{\ell\left(1-\frac{2}{k^2}\right)},\quad \Psi=-\frac{\sqrt{6}}{3k}, \tag{2.22}$$

denoted F, implies that $\Phi=2\sqrt{3}/3k$ and represents a noninflationary negatively curved FRW model. The eigenvalues are

$$-1+\sqrt{\frac{8}{k^2}-3},\quad -1-\sqrt{\frac{8}{k^2}-3},$$

$$-2+\frac{\sqrt{2}}{k}\sqrt{(k^2-4\ell k^2+8\ell)+E},$$

$$-2+\frac{\sqrt{2}}{k}\sqrt{(k^2-4\ell k^2+8\ell)-E}, \tag{2.23}$$

$$-2-\frac{\sqrt{2}}{k}\sqrt{(k^2-4\ell k^2+8\ell)+E},$$

$$-2-\frac{\sqrt{2}}{k}\sqrt{(k^2-4\ell k^2+8\ell)-E},$$

where

$$E\equiv\sqrt{(k^2+4\ell k^2-8\ell)^2+32\ell(2-k^2)}.$$

After some algebra it can be shown that if $k>\sqrt{2}$ (note that $\ell>0$ in the Bianchi type-VII$_h$ models) then all of the eigenvalues have negative real parts. Therefore, if $k>\sqrt{2}$, then this equilibrium point is a stable attractor. (Note that this equilibrium point does not exist if $k<\sqrt{2}$.) In other words, there exists an open set of initial conditions in the set of anisotropic Bianchi type-VII$_h$ (with a scalar field and exponential potential) initial data for which the corresponding cosmological models asymptotically approach an isotropic and negatively curved FRW model.

III. CONCLUSION

In this paper our goal has been to expand upon recent results concerning the isotropization of scalar field models with an exponential potential. In [8] it was proven that if $k>\sqrt{2}$, then the only Bianchi models that can possibly isotropize are those of Bianchi types I, V, VII, and IX, with I and V being very restricted classes of Bianchi cosmologies.

We have shown here that within the set of all spatially homogeneous initial data, there exists an open set of initial data describing the Bianchi type-VII$_h$ models (having a scalar field with an exponential potential and $k>\sqrt{2}$) such that the models approach isotropy at infinite times. This compliments the results of Kitada and Maeda [6,7], who showed that all such spatially homogeneous models (including the Bianchi type-VII$_h$ models) with $k<\sqrt{2}$ approach isotropy to the future. In other words, there exists a set of spatially homogeneous initial data of nonzero measure for which models will isotropize to the future for all positive values of k. Of course, there also exists a set of spatially homogeneous initial data of nonzero measure for which models will not isotropize to the future when $k>\sqrt{2}$ (e.g., the Bianchi type-VIII models).

If $k<\sqrt{2}$, then all models will inflate as they approach the power-law inflationary attractor represented by Eq. (2.20) [6,7]. For $k>\sqrt{2}$, the stable equilibrium point F, given by Eq. (2.22), which does not exist for $k<\sqrt{2}$, is isotropic and resides on the surface $q=0$. This means that the corresponding exact solution is marginally noninflationary. However, this does not mean that the corresponding cosmological models are not inflating as they asymptotically approach this equilibrium state. As orbits approach F they may have $q<0$ or $q>0$ (or even $q=0$) and consequently the models may or may not be inflating. If they are inflating, then the rate of inflation is decreasing as F is approached (i.e., $q\to 0$). When $\sqrt{2}<k<\sqrt{8/3}$, we find that F is nodelike, hence there is an open set of models that inflate as they approach F and an open set which do not. When $k>\sqrt{8/3}$, F is found to be spiral-like, and so it is expected that orbits experience regions of both $q<0$ and $q>0$ as they wind their way towards F [12].

As in Kitada and Maeda [6,7] the inclusion of matter in the form of a perfect fluid is not expected to change the results of our analysis provided the matter satisfies appropriate energy conditions. Finally, we note that the analysis of the Bianchi type-IX models (with $k>\sqrt{2}$) is rather more complicated. However, it is apparent that a subclass of the Bianchi type-IX models will isotropize and a subclass will recollapse.

ACKNOWLEDGMENTS

A.A.C. was supported by the Natural Sciences and Engineering Research Council of Canada and J.I. was supported by the CICYT (Spain) Grant No. PB93-0507. J.I. also thanks the Dept. of Mathematics, Statistics and Computing Science, Dalhousie University for their kind hospitality.

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