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Qualitative analysis of diagonal Bianchi type V imperfect fluid cosmological models

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The Einstein field equations for diagonal Bianchi type V imperfect fluid cosmological models with both viscosity and heat conduction are set up as an autonomous system of differential equations using dimensionless variables and a set of dimensionless equations of state. Models with and without a cosmological constant, λ, are investigated using the techniques from dynamical systems theory. It is shown that all models that satisfy the weak energy conditions isotropize. The introduction of viscosity (in particular) allows for a variety of different qualitative behaviors (including, for example, models with a negative deceleration parameter). Exact solutions that correspond to the singular points of the dynamical system are found. It is shown that the past asymptotic states are represented by self-similar cosmological models and, if $\lambda=0$, the future asymptotic states are also, in general, represented by self-similar cosmological models; in the exceptional cases the late time asymptotic state is represented by a de Sitter model with constant expansion, as is the case for solutions with $\lambda \neq 0$.

I. INTRODUCTION

Perfect fluid cosmological models have been extensively studied.[1] Recently, spatially homogeneous perfect fluid models have been investigated using techniques from dynamical systems theory.[2,3] It is of interest to take into account dissipative processes such as viscosity and heat conduction in cosmological models; for example, viscous fluid models have been used in an attempt to explain the currently observed highly isotropic matter distribution and the high entropy per baryon in the present state of the universe.[4,5] Imperfect fluid Bianchi models, and particularly Bianchi type V models [which are simple generalizations of the negatively curved Friedmann–Robertson–Walker models (FRW)], have also been analyzed recently using techniques from dynamical systems theory.[6–9] The purpose of this paper is to further extend this analysis.

The paper is organized as follows. In Sec. II, we study diagonal Bianchi type V imperfect fluid cosmological models with both viscosity and heat conduction. The energy momentum tensor contains a general source term which can be used to include a variety of different physical fields; for example, the general source term can represent an electromagnetic field or a cosmological constant. The Einstein field equations and the energy-momentum conservation equations are utilized to set up a system of ordinary differential equations governing the models. When the extra source term represents a cosmological constant or is identically zero, following Collins[2] and using the dimensionless equations of state introduced by Coley,[5] this system becomes an autonomous system of ordinary differential equations, thus enabling us to use geometric techniques to determine the qualitative behavior of the system. In Sec. III, we investigate the case where the extra source term represents a cosmological constant. In Sec. IV, we investigate the case where the extra source term is identically zero; this generalizes the work of Collins,[2] who assumed a perfect fluid, and the work of Coley and Dunn,[8] who assumed, for simplicity, a locally rotationally symmetric (LRS) Bianchi type V metric and considered an imperfect fluid source with both viscosity and heat conduction. In Sec. V, we present a thorough discussion of the singular points and give the corresponding exact solutions. In Sec. VI, we conclude with a discussion and outline some avenues for future research.

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Motivation for the use of dimensionless equations of state for the bulk and shear viscosity coefficients (in addition to the pressure), the primary assumption in this work, was given in Ref. 5. In particular, such equations of state contain as a special case the physically important subclass (see Ref. 5 and references within) in which (in addition to, and in analogy with, the barotropic equation of state in which the pressure and the energy-density are linearly related) the bulk and shear viscosity coefficients are proportional to the square root of the energy-density [i.e., $m=n=1/2$ in Eqs. (2.22) below] and hence contain no explicit dependence on the expansion. Indeed, this special subclass of equations of state is found to play a central role in the exact solutions representing the asymptotic states of the models (see Sec. VI).

However, the main reason for employing dimensionless equations of state is that they are the most general equations of state for which the governing equations of the Bianchi type V models (under investigation here) in particular, and of the orthogonal spatially homogeneous imperfect fluid models in general, reduce to an autonomous system of differential equations, thereby enabling us to study the models qualitatively using standard geometrical techniques. Moreover, it has recently been shown that these are also the most general equations of state for which the asymptotic states of the resulting system of differential equations are represented by *self-similar* cosmological models (see the Appendix), whereby the present work is the natural generalization of recent work by Wainwright and his collaborators.[3]

II. ANALYSIS

The diagonal form of the Bianchi type V metric is given by

$$ds^2=-dt^2+a(t)^2\,dx^2+b(t)^2e^{2x}\,dy^2+c(t)^2e^{2x}\,dz^2. \tag{2.1}$$

The energy-momentum tensor considered in this work is due to an imperfect fluid that includes bulk viscosity, shear viscosity, and heat conduction, viz.,

$$T_{ab}=(\rho+\bar{p})u_au_b+\bar{p}g_{ab}-2\eta\sigma_{ab}+q_au_b+q_bu_a+\chi_{ab}, \tag{2.2}$$

where u^a is the fluid 4-velocity, ρ is the energy density, the quantity $\bar{p}$ is defined to be $\bar{p}=p-\zeta\theta$, where p is the thermodynamic pressure, ζ is the bulk viscosity coefficient, θ is the expansion scalar, η is the shear viscosity coefficient, σ_{ab} is the shear tensor, and q_a is the heat conduction vector such that $q_au^a=0$ (which implies for a comoving fluid, along with the field equations, that the only nonzero component of q_a is q_1). The term χ_{ab} represents any additional sources; for example, the additional source term in the energy momentum tensor can be used to include a cosmological constant.

For a comoving fluid $u_a=(-1,0,0,0)$. The expansion scalar, which determines the volume behavior of the fluid, is then given by

$$\theta=\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}. \tag{2.3}$$

The shear tensor, σ_{ab}, determines the distortion arising in the fluid flow leaving the volume invariant. The nonzero components of the shear tensor are

$$\sigma_{11}=\frac{a}{3}(3\dot{a}-a\theta),\quad \sigma_{22}=\frac{be^{2x}}{3}(3\dot{b}-b\theta),\quad \sigma_{33}=\frac{ce^{2x}}{3}(3\dot{c}-c\theta), \tag{2.4}$$

and the shear scalar, $\sigma^2=\frac{1}{2}\sigma^{ab}\sigma_{ab}$, is given by

$$\sigma^2=\frac{1}{2}\left[\left(\frac{\dot{a}}{a}\right)^2+\left(\frac{\dot{b}}{b}\right)^2+\left(\frac{\dot{c}}{c}\right)^2\right]-\frac{\theta^2}{6}. \tag{2.5}$$

In the case under consideration here, there is no rotation and no acceleration.

The Einstein field equations are:

$$\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{a}\dot{c}}{ac}+\frac{\dot{b}\dot{c}}{bc}-\frac{3}{a^2}=\rho+\chi_{00}, \tag{2.6}$$

$$2\frac{\dot{a}}{a}-\frac{\dot{b}}{b}-\frac{\dot{c}}{c}=-q_1+\chi_{01}, \tag{2.7}$$

$$-\left(\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{b}\dot{c}}{bc}\right)+\frac{1}{a^2}=\bar{p}+\frac{2}{3}\eta\left(2\frac{\dot{a}}{a}-\frac{\dot{b}}{b}-\frac{\dot{c}}{c}\right)+\chi_1^1, \tag{2.8}$$

$$-\left(\frac{\ddot{a}}{a}+\frac{\ddot{c}}{c}+\frac{\dot{a}\dot{c}}{ac}\right)+\frac{1}{a^2}=\bar{p}-\frac{2}{3}\eta\left(2\frac{\dot{b}}{b}-\frac{\dot{a}}{a}-\frac{\dot{c}}{c}\right)+\chi_2^2, \tag{2.9}$$

$$-\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\dot{a}\dot{b}}{ab}\right)+\frac{1}{a^2}=\bar{p}-\frac{2}{3}\eta\left(2\frac{\dot{c}}{c}-\frac{\dot{a}}{a}-\frac{\dot{b}}{b}\right)+\chi_3^3. \tag{2.10}$$

From these equations one obtains the Friedmann equation

$$\theta^2=3\sigma^2+3\rho+9/a^2+3\chi_{00}, \tag{2.11}$$

and the Raychaudhuri equation

$$\dot{\theta}=-2\sigma^2-\tfrac{1}{3}\theta^2-\tfrac{1}{2}(\rho+3\bar{p})-\tfrac{1}{2}(\chi_{00}+\chi_1^1+\chi_2^2+\chi_3^3), \tag{2.12}$$

and defining

$$\sigma_1=\left(\frac{\dot{a}}{a}-\frac{\dot{b}}{b}\right),\qquad \sigma_2=\left(\frac{\dot{a}}{a}-\frac{\dot{c}}{c}\right), \tag{2.13}$$

so that

$$\sigma^2=\tfrac{1}{3}(\sigma_1+\sigma_2)^2-\sigma_1\sigma_2, \tag{2.14}$$

it can be shown that

$$\dot{\sigma}_1=-2\eta\sigma_1-\theta\sigma_1+\chi_1^1-\chi_2^2, \tag{2.15}$$

$$\dot{\sigma}_2=-2\eta\sigma_2-\theta\sigma_2+\chi_1^1-\chi_3^3. \tag{2.16}$$

From the energy conservation law $T^{ab}{}_{;b}u_a=0$, we obtain

$$\dot{\rho}+\dot{\chi}_{00}=4\eta\sigma^2-\theta(\rho+\bar{p})-\frac{2}{a^2}(q_1-\chi_{01})-\theta\chi_{00}-\frac{\dot{a}}{a}\chi_1^1-\frac{\dot{b}}{b}\chi_2^2-\frac{\dot{c}}{c}\chi_3^3. \tag{2.17}$$

Equations (2.7) and (2.13) imply that

$$-q_1+\chi_{01}=\sigma_1+\sigma_2. \tag{2.18}$$

From the Friedmann equation (2.11), we also obtain the inequality

$$\theta^2-\sigma_1^2-\sigma_2^2+\sigma_1\sigma_2-3\rho-3\chi_{00}=9/a^2\geqslant 0. \tag{2.19}$$

We define new dimensionless variables x, β_1, β_2, and a new time variable Ω as follows (Collins[2]):

$$x \equiv 3\rho/\theta^2 \tag{2.20}$$

(x measures the dynamical importance of the matter content),

$$\beta_1 \equiv 2\sigma_1/\theta , \qquad \beta_2 \equiv 2\sigma_2/\theta \tag{2.21}$$

[β_1 and β_2 measure the rate of shear (anisotropy) in terms of the expansion], and we define $l \equiv e^{\Omega}$, where $\theta = 3\dot{l}/l$, so that $d\Omega/dt = -\frac{1}{3}\theta$.

In order to complete the system of equations we need to specify equations of state for the quantities p, ζ, and η. In principle these equations of state can be derived from kinetic theory, but in practice one must specify phenomenological equations of state which may or may not have any physical foundations. These phenomenological equations of state must satisfy constraints such as p, ζ, and η should tend to zero as ρ tends to zero and they must also satisfy the energy conditions. Following Coley,[5] we introduce dimensionless equations of state of the form

$$p/\theta^2 = p_o x^l, \tag{2.22a}$$

$$\zeta/\theta = \zeta_o x^m, \tag{2.22b}$$

$$\eta/\theta = \eta_o x^n, \tag{2.22c}$$

where p_o, ζ_o, and η_o are positive constants, and l, m, and n are constant parameters (x is the dimensionless density parameter defined earlier). In the models under consideration, θ is strictly positive, thus Eqs. (2.22) are well defined. The most commonly used equation of state for the pressure is the barotropic equation of state $p=(\gamma-1)\rho$, hence $p_o = \frac{1}{3}(\gamma - 1)$ and $l=1$ (where $1 \leqslant \gamma \leqslant 2$ is necessary for local mechanical stability and for the speed of sound in the fluid to be no greater than the speed of light). We use these dimensionless equations of state because they are scale invariant and because the system reduces to an autonomous system of differential equations for appropriate χ_{ab}.

Using the new variables and the equations of state defined above, we obtain the following system of differential equations:

$$\begin{aligned}\frac{dx}{d\Omega} &= x((3\gamma-2)(1-x)-\beta_1^2-\beta_2^2+\beta_1\beta_2)-9\zeta_o x^m(1-x)-3\eta_o x^n(\beta_1^2+\beta_2^2-\beta_1\beta_2)\\ &\quad -\frac{\beta_1+\beta_2}{4}(4-4x-\beta_1^2-\beta_2^2+\beta_1\beta_2)-\frac{9}{\theta^3}\dot{\chi}_{00}+3\frac{(\beta_1+\beta_2+2)}{\theta^2}\chi_{00}\\ &\quad +\frac{3}{\theta^2}\left((1-x)(\chi_{00}+\chi_1^1+\chi_2^2+\chi_3^3)+\frac{\beta_1}{2}(\chi_1^1-2\chi_2^2+\chi_3^3)+\frac{\beta_2}{2}(\chi_1^1+\chi_2^2-2\chi_3^3)\right),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\frac{d\beta_1}{d\Omega} &= -\frac{\beta_1}{2}((3\gamma-2)x-4-9\zeta_o x^m-12\eta_o x^n+\beta_1^2+\beta_2^2-\beta_1\beta_2)+\frac{6}{\theta^2}(\chi_2^2-\chi_1^1)\\ &\quad -\frac{3\beta_1}{2\theta^2}(\chi_{00}+\chi_1^1+\chi_2^2+\chi_3^3),\end{aligned} \tag{2.24}$$

$$\frac{d\beta_2}{d\Omega} = -\frac{\beta_2}{2}((3\gamma-2)x-4-9\zeta_o x^m - 12\eta_o x^n + \beta_1^2 + \beta_2^2 - \beta_1\beta_2) + \frac{6}{\theta^2}(\chi_3^3 - \chi_1^1)$$
$$-\frac{3\beta_2}{2\theta^2}(\chi_{00} + \chi_1^1 + \chi_2^2 + \chi_3^3), \tag{2.25}$$

where

$$4 \geqslant \beta_1^2 + \beta_2^2 - \beta_1\beta_2 + 4x + \frac{12}{\theta^2}\chi_{00}. \tag{2.26}$$

If $\chi_{ab}=0$, these equations are autonomous. The above system of equations will be autonomous depending on the form of χ_{ab} and $\dot{\chi}_{ab}$, (the equations that govern the evolution of χ_{ab} will be derived from the appropriate physical laws; for example, the Einstein–Maxwell equations).

A. The extra source term

There are a number of forms that the extra source term χ_{ab} may take.

1. Tilt

A possible choice for the χ_{ab} is to incorporate tilt into the cosmological model. Here the four velocity is noncomoving and is given by $u_a^{\text{non-comoving}} = (-\cosh\psi,\ a\sinh\psi,0,0)$, where ψ is the tilt angle. If we let

$$\chi_{ab} = T_{ab}^{\text{non-comoving}} - T_{ab}^{\text{comoving}}, \tag{2.27}$$

we obtain the desired result. Because of the extra degree of freedom, one further equation is needed to complete the system. This equation will determine $\dot{\chi}_{ab}$ and will come from the energy-momentum conservation equations. The analysis is complicated and it will not be considered here (the perfect fluid case is studied in Collins[10]).

2. Electromagnetic field

Another possible choice for χ_{ab} is for it to represent a uniform electric or magnetic field. In this case $\dot{\chi}_{ab}$ will be obtained from the Einstein–Maxwell equations. If we consider either a source free electric field or a source free magnetic field, we find that all components are identically zero in the Bianchi V case considered here (see also Jacobs and Hughston[11]). That is, we cannot include a nonzero uniform electric or magnetic field in the models presently under consideration.

3. Cosmological constant

It is also possible to include a cosmological constant, λ, into the equations by defining $\chi_{ab} \equiv -\lambda g_{ab}$. Because of the extra degree of freedom, we need one more equation to complete the system. By defining a new variable $z \equiv 9/\theta^2$ and calculating $dz/d\Omega$ using Raychaudhuri's equation, we complete the system (and hence we have a four-dimensional system to analyze).

4. Extra source term identically zero

In the simplest case χ_{ab} is identically zero. We will then have a three-dimensional system to analyze. This is the case when we have an imperfect fluid with viscosity and heat conduction and zero cosmological constant (for example, see Coley and Dunn[8]).

In the remainder of this work we will consider two cases; the first is when the extra source term represents a cosmological constant (that is, $\chi_{ab} \equiv -\lambda g_{ab}$), where the system of differential equations is four-dimensional (see Sec. III), and the second is when the extra source term is identically zero (that is, $\chi_{ab}\equiv 0$), where the system of differential equations is three-dimensional (see Sec. IV).

III. THE CASE $\chi_{ab}\equiv-\lambda g_{ab}$

In the case when $\chi_{ab}\equiv-\lambda g_{ab}$, we have the equivalent of the Einstein field equations with a positive cosmological constant λ. A fourth equation is needed to complete the system (2.23)–(2.25), because of the extra degree of freedom. We define a new variable z and calculate its derivative as follows:

$$z\equiv 9/\theta^2, \tag{3.1}$$

$$\frac{dz}{d\Omega}=-z\left((3\gamma-2)x+2-9\zeta_o x^m+\beta_1^2+\beta_2^2-\beta_1\beta_2-\frac{2}{3}\lambda z\right). \tag{3.2}$$

[Note z is just a new expansion variable and its derivative is basically the Raychaudhuri equation (2.12).] The remaining equations are

$$\frac{d\beta_1}{d\Omega}=-\frac{\beta_1}{2}\left((3\gamma-2)x-4-9\zeta_o x^m-12\eta_o x^n+\beta_1^2+\beta_2^2-\beta_1\beta_2-\frac{2}{3}\lambda z\right), \tag{3.3}$$

$$\frac{d\beta_2}{d\Omega}=-\frac{\beta_2}{2}\left((3\gamma-2)x-4-9\zeta_o x^m-12\eta_o x^n+\beta_1^2+\beta_2^2-\beta_1\beta_2-\frac{2}{3}\lambda z\right), \tag{3.4}$$

$$\begin{aligned}\frac{dx}{d\Omega}=&x\left((3\gamma-2)(1-x)-\beta_1^2-\beta_2^2+\beta_1\beta_2+\frac{2}{3}\lambda z\right)-9\zeta_o x^m(1-x)-3\eta_o x^n\\&\times(\beta_1^2+\beta_2^2-\beta_1\beta_2)-\frac{\beta_1+\beta_2}{4}\left(4-4x-\beta_1^2-\beta_2^2+\beta_1\beta_2-\frac{4}{3}\lambda z\right),\end{aligned} \tag{3.5}$$

where

$$4\geqslant\beta_1^2+\beta_2^2-\beta_1\beta_2+4x+\tfrac{4}{3}\lambda z. \tag{3.6}$$

Equations (3.3) and (3.4) can then be integrated to obtain $\beta_1=k\beta_2$ for some constant k. We define a new variable β,

$$\beta\equiv(\sqrt{k^2-k+1})^{-1}(k\beta_1+(1-k)\beta_2), \tag{3.7}$$

so that

$$\beta^2=\beta_1^2+\beta_2^2-\beta_1\beta_2, \tag{3.8}$$

a term which occurs frequently in the equations above. Each value of k represents a different surface in the 4-dimensional phase space. As k ranges from $-\infty$ to ∞, the one parameter family of 3-dimensional surfaces will cover the entire 4-dimensional phase space. Hence the 4-dimensional phase portrait is the union of all the 3-dimensional phase portraits. The 4-dimensional system of equations can be considered as a one parameter family of 3-dimensional systems as follows:

$$\frac{dx}{d\Omega}=x\left((3\gamma-2)(1-x)-\beta^2+\frac{2}{3}\lambda z\right)-9\zeta_o x^m(1-x)-3\eta_o x^n\beta^2$$
$$-\frac{k+1}{\sqrt{k^2-k+1}}\frac{\beta}{4}\left(4-4x-\beta^2-\frac{4}{3}\lambda z\right), \tag{3.9}$$

$$\frac{dz}{d\Omega}=-z\left((3\gamma-2)x+2-9\zeta_o x^m+\beta^2-\frac{2}{3}\lambda z\right), \tag{3.10}$$

$$\frac{d\beta}{d\Omega}=-\frac{\beta}{2}\left((3\gamma-2)x-4-9\zeta_o x^m-12\eta_o x^n+\beta^2-\frac{2}{3}\lambda z\right), \tag{3.11}$$

where

$$4\geqslant\beta^2+4x+\tfrac{4}{3}\lambda z. \tag{3.12a}$$

From the definition of z we have that

$$z\geqslant 0. \tag{3.12b}$$

Finally, assuming that the energy density is non-negative, we have

$$x\geqslant 0. \tag{3.12c}$$

Equations (3.12) define a compact set in R^3. This set is the physical region of interest and will be denoted by $\mathfrak{R}$. In this case, Eq. (3.10) defines two invariant sets, $z=0$ and $z>0$. The invariant set $z=0$ is equivalent to case when the cosmological constant is zero and will be studied in detail in the next section (see also van den Hoogen[7]). Thus in this section, we need only investigate what happens in the invariant set $z>0$.

In order to simplify the analysis, we define a new parameter

$$C=\frac{k+1}{\sqrt{k^2-k+1}}. \tag{3.13}$$

The parameter C (as a function of k) ranges between -1 and 2; as k increases from $-\infty$ to 1, C increases monotonically from -1 to a maximum of 2, then as k increases from 1 to ∞, C decreases monotonically from 2 to 1. When $C=0$, $k=-1$ and we have no heat conduction, and when $C=2$, $k=1$ and we have the system analyzed by Coley and Dunn[8] with a positive cosmological constant. Allowing this parameter C to range through all its possible values, the entire four-dimensional phase portrait will be obtained. By choosing various values of the constants ζ_o, η_o, m, n, and C, the complete qualitative structure of the system in question will be determined.

A. Qualitative analysis

In the following analysis, the order of the coordinates is (x, z,β).

1. $m=n>1$

In this case there are at most seven singular points in $\mathfrak{R}$. The point $(0,3\lambda^{-1},0)$ is a repelling node. The point $(\Sigma,3\lambda^{-1}(1-\Sigma),0)$ is a saddle point where $\Sigma=(3\zeta_o/\gamma)^{1/(1-m)}$. The remaining singular points lie in the $z=0$ plane and all have $\lambda_z<0$ except for the point $(1,0,0)$; where λ_z is the eigenvalue associated with the positive z-direction. The point $(1,0,0)$ may or may not have $\lambda_z<0$ depending on the sign of $\gamma-3\zeta_o$. If $\gamma-3\zeta_o>0$ then $\lambda_z<0$ and if $\gamma-3\zeta_o<0$ then $\lambda_z>0$.

2. $m=n=1$

In this case there are at most five singular points and under certain conditions there will be a line singularity (line of nonisolated singular points) in $\mathfrak{R}$. The point $(0,3\lambda^{-1},0)$ is a repelling node if $\gamma \neq 3\zeta_o$ and if $\gamma=3\zeta_o$, the point becomes part of the line of singular points $(x,3\lambda^{-1}(1-x),0)$, where $0\leq x\leq 1$. When $\gamma=3\zeta_o$, we find that for $z>0$, $dz/d\Omega<0$, hence all trajectories are directed toward the $z=0$ plane. The remaining singular points lie in the $z=0$ plane and all have $\lambda_z<0$ except for the point $(1,0,0)$ which behaves in the same manner as in the previous case (i.e., *case 1*). When $9\zeta_o-(3\gamma-2)=0$, there is a line singularity in the $z=0$, plane but for $z>0$, $dz/d\Omega<0$, thus, this line is an attractor.

3. $m=n=1/2$

In this case there are at most seven singular points in $\mathfrak{R}$. For any singular point with $x=0$, the system becomes nonanalytic. By transforming to the variable u and time coordinate τ, where $u^2=x$; $d\Omega/d\tau=u$, these points can be analyzed using analytic methods. The point $(0,3\lambda^{-1},0)$ is degenerate, but closer analysis reveals that the point is nodelike in nature. The point $(\Sigma,3\lambda^{-1}(1-\Sigma),0)$ is a repelling node where $\Sigma=(3\zeta_o/\gamma)^2$. The remaining singular points lie in the $z=0$ plane and all have $\lambda_z<0$ except for the point $(1,0,0)$ which behaves in the same manner as in the first case (i.e., *case 1*).

4. $m=n=0$

In this case there are two separate situations depending upon whether $\zeta_o=0$ or $\zeta_o\neq 0$. For $\zeta_o=0$, the point $(0,3\lambda^{-1},0)$ is a repelling node and for $\zeta_o\neq 0$ the point is no longer singular. If $\zeta_o\neq 0$ there exists a singular point $(\Sigma,3\lambda^{-1}(1-\Sigma),0)$ where $\Sigma=(3\zeta_o/\gamma)$; it is a repelling node. The remaining singular points lie in the $z=0$ plane and all have $\lambda_z<0$ except for the point $(1,0,0)$ which behaves in the same manner as in the first case (i.e., *case 1*).

5. $\zeta_o=\eta_o=0$

In this case there are at most five singular points in $\mathfrak{R}$. The point $(0,3\lambda^{-1},0)$ is a repelling node. The remaining singular points lie in the $z=0$ plane and for $\gamma\neq 2$ will have $\lambda_z<0$. However, if $\gamma=2$ then there is a line singularity on the boundary in the plane $z=0$. Calculating $dz/d\Omega$ on the boundary, we found it to be negative, hence the boundary in the $z=0$ plane is attracting trajectories from above the plane.

B. Energy conditions

The singular points that are not located in the $z=0$ plane are located in the $\beta=0$ plane and thus represent isotropic cosmological models. The dominant energy condition (DEC) and the strong energy condition (SEC) in the $\beta=0$ plane are:

$$\begin{aligned} \text{DEC:} &\quad 0\leq \gamma x-3\zeta_o x^m \leq 2x,\\ \text{SEC:} &\quad \gamma x-3\zeta_o x^m \geq \tfrac{2}{3}x. \end{aligned} \tag{3.14}$$

The DEC and the SEC imply that the only singular point with a nonzero z coordinate permitted is the point $(0,3\lambda^{-1},0)$. The remaining singular points with nonzero z coordinate do not satisfy the above energy conditions.

Linearizing the system, we find for the case $m=n>1$ the point $(0,3\lambda^{-1},0)$ is a repelling node. For the case $m=n=1$, the point may be either a repelling node or a saddle, but if the energy conditions are satisfied, the point is a repelling node. For $m=n=\frac{1}{2}$, we have a degenerate case which is also found to be nodelike. For the remaining two cases $m=n=0$ and $\zeta_o=\eta_o=0$, the point $(0,3\lambda^{-1},0)$ is a repelling node. Thus we have found that the point $(0,3\lambda^{-1},0)$ is a repelling node in the Bianchi V case considered here if the energy conditions are satisfied.

The remaining singular points lie in the $z=0$ plane. We again linearize the system and calculate the eigenvalue associated with the z direction. We find that $\lambda_z<0$ for all singular points except for the point $(1,0,0)$. At the point $(1,0,0)$, if $3\zeta_o<\gamma$ then $\lambda_z<0$, if $3\zeta_o>\gamma$ then $\lambda_z>0$ and if $3\zeta_o=\gamma$ then $\lambda_z=0$. However, if we again assume that the DEC and the SEC be satisfied, then the inequality $3\zeta_o<\gamma$ must be satisfied, and therefore $\lambda_z<0$. Hence all eigenvalues associated with the z direction λ_z, for all the singular points in the $z=0$ plane are negative which means (combined with the previous result), that the $z=0$ plane is an attractor.

In conclusion, assuming both the SEC and DEC are satisfied, as $t\to\infty$ or $(\Omega\to-\infty)$ the point $(0,3\lambda^{-1},0)$ represents the late time behavior for the Bianchi V model with positive cosmological constant. This point is characterized by the fact that $\beta=0$, which implies isotropy, $x=0$, which implies that the energy density is zero, and that $z=3\lambda^{-1}$, which implies that the expansion is constant. The point represents an empty, homogeneous and isotropic model with constant expansion $\theta=\sqrt{3\lambda}$ and the corresponding exact solution is the empty de Sitter model. As $t\to 0$ or $(\Omega\to\infty)$, the $z=0$ plane attracts all trajectories and thus the $z=0$ plane indicates the asymptotic early time behavior. Hence, for early times (that is, $t\to 0$ or $\Omega\to\infty$), the qualitative nature will be the same as if we have $\lambda=0$; this corresponds to the case when $\chi_{ab}=0$, and is analyzed in detail in the following sections.

The above is a special case of the cosmic no-hair theorem (Wald[12]) proven for all Bianchi types (except some Bianchi IX models). Wald's proof simply assumes the dominant and strong energy conditions are satisfied for a general energy-momentum tensor. The energy conditions are not always satisfied if the cosmological constant is included in the calculation of these energy conditions. If the energy conditions are not satisfied then $(0,3\lambda^{-1},0)$ is not the only possible attractor. Here in almost all cases the point $(1,0,0)$ will become a late time attractor if $3\zeta_o>\gamma$ and in some cases, and under certain conditions, the point $(\Sigma,3\lambda^{-1}(1-\Sigma),0)$, where $\Sigma=(3\zeta_o/\gamma)^{1/(1-m)}$, may either be an attracting node or a saddle point for late times. With these conditions, there will exist trajectories that do not tend to the de Sitter model at $(0,3\lambda^{-1},0)$ but are attracted to one of the other singular points. The models corresponding to the singular points $(1,0,0)$ and $(\Sigma,3\lambda^{-1}(1-\Sigma),0)$ are not discussed here.

IV. THE CASE $\chi_{ab}=0$

The system of equations when $\chi_{ab}\equiv 0$ is as follows:

$$\frac{dx}{d\Omega}=x((3\gamma-2)(1-x)-\beta_1^2-\beta_2^2+\beta_1\beta_2)-9\zeta_o x^m(1-x)-3\eta_o x^n(\beta_1^2+\beta_2^2-\beta_1\beta_2)$$
$$-\frac{\beta_1+\beta_2}{4}(4-4x-\beta_1^2-\beta_2^2+\beta_1\beta_2), \tag{4.1}$$

$$\frac{d\beta_1}{d\Omega}=-\frac{\beta_1}{2}((3\gamma-2)x-4-9\zeta_o x^m-12\eta_o x^n+\beta_1^2+\beta_2^2-\beta_1\beta_2), \tag{4.2}$$

$$\frac{d\beta_2}{d\Omega}=-\frac{\beta_2}{2}((3\gamma-2)x-4-9\zeta_o x^m-12\eta_o x^n+\beta_1^2+\beta_2^2-\beta_1\beta_2), \tag{4.3}$$

where

$$4\geqslant\beta_1^2+\beta_2^2-\beta_1\beta_2+4x. \tag{4.4}$$

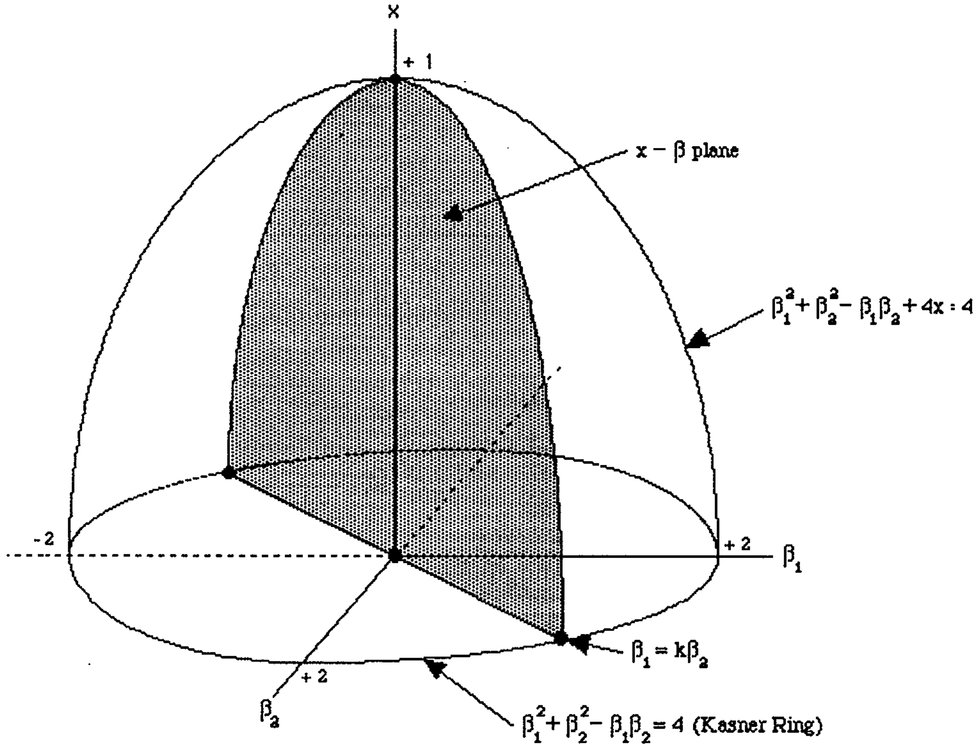


FIG. 1. The interior of the three-dimensional paraboloid represents the physical phase space. The two-dimensional shaded parabola represents a particular $x-\beta$ plane for a unique value of the parameter k. As we let k range through $-\infty$ to ∞, the two-dimensional $x-\beta$ plane will cover the entire three-dimensional phase space. For each value of k the plane intersects the Kasner ring of singular points at two points, thus we have effectively isolated the singular points.

From Eqs. (4.2) and (4.3) we again see that $\beta_1 = k\beta_2$, for some constant k. As in the previous case one defines a new variable β [see Eq. (3.7)]. Each value of k represents a different surface in the 3-dimensional phase space. As k ranges from $-\infty$ to ∞, the one parameter family of 2-dimensional surfaces will cover the entire 3-dimensional phase space. Hence the 3-dimensional phase portrait is the union of all the 2-dimensional phase portraits (i.e., all motion is "planar," in that it is restricted to a plane $\beta_1 = k\beta_2$, see Fig. 1).

The system of equations reduces to the following 1-parameter (k) family of 2-dimensional equations:

$$\frac{dx}{d\Omega} = x((3\gamma-2)(1-x)-\beta^2) - 9\zeta_o x^m(1-x) - 3\eta_o x^n\beta^2 - \frac{k+1}{\sqrt{k^2-k+1}}\frac{\beta}{4}(4-4x-\beta^2), \tag{4.5}$$

$$\frac{d\beta}{d\Omega} = -\frac{\beta}{2}((3\gamma-2)x - 4 - 9\zeta_o x^m - 12\eta_o x^n + \beta^2), \tag{4.6}$$

where

$$4 \geqslant \beta^2 + 4x, \tag{4.7a}$$

$$x \geqslant 0. \tag{4.7b}$$

We note that Eqs. (4.7) define a compact set in R^2. This set is the physical region of interest and will again be denoted by $\mathfrak{R}$. Equation (4.6), implies that $\beta=0$ defines an invariant set. Hence, the line, $\beta=0$, divides the phase space into three separate invariant sets, $\beta<0$, $\beta=0$, and $\beta>0$. In each set, $\beta<0$ and $\beta>0$, it can be shown that $d\beta/dx$ is never zero except at the singular points, hence there exists no closed orbits in $\mathfrak{R}$.

To simplify the analysis, the parameter C [see Eq. (3.13)] defined in the previous section is used. When $C=0$, $k=-1$ and we have no heat conduction thus we have an imperfect fluid with just viscosity (see Burd and Coley[10]), and when $C=2$, $k=1$ and we have the system analyzed by Coley and Dunn.[9] By allowing this new parameter C to range through all its possible values, the entire three dimensional phase portrait will be obtained. Choosing various values of the constants ζ_o, η_o, m, n, and C, the complete qualitative structure of the system in question will be determined.

We note in Abolghasem[6] an equation of state for the heat conduction vector of the form

$$-\frac{q_1}{\theta}=\kappa x^a \beta^b \tag{4.8}$$

was assumed; as a particular case Abolghasem[6] considered $a=0$ and $b=1$. In this paper, because of the fact that $\beta_1=k\beta_2$, Eq. (2.18) implies that

$$-\frac{q_1}{\theta}=\frac{(k+1)}{\sqrt{k^2-k+1}}\,\beta=C\beta. \tag{4.9}$$

Hence, an equation of state for the heat conduction vector cannot be independently made. [Note Eq. (4.9) was an assumption in the analysis of Abolghasem.[6]]

A. Qualitative analysis

Detailed analysis of the singular points (for example, their eigenvalues and eigendirections) can be found in van den Hoogen.[7] Information about the stability and other properties of the singular points is summarized in Tables I–III and some appropriate phase portraits are given in the figures. In the following analysis the order of the coordinates is (x,β).

1. $m=n>1$

If $\gamma\neq 2$, the point $(0,2)$ is generally a stable two-tangent node, unless $C=(3\gamma-2)/2$ in which case the point degenerates to a one-tangent node. When $C=2$ the point is degenerate but the single sector in $\mathfrak{R}$ is found to be hyperbolic in nature. Finally, if $\gamma=2$ the point behaves like a stable node. (See also Table A in Ref. 7.)

The point $(0,-2)$ (for $\gamma\neq 2$), is generally a stable two-tangent node, unless $C=-(3\gamma-2)/2$, in which case the point degenerates to a one-tangent node. There is also a degenerate case when $\gamma=2$ in which case the point behaves like a stable node.

The point $(0,0)$ is generally an unstable two-tangent node unless $\gamma=4/3$, in which case the point degenerates to a one-tangent node if $C\neq 0$ and a stellar node for $C=0$.

The point $(1,0)$ will take on a variety of different natures depending on the sign of $\Psi_1=9\zeta_o-(3\gamma-2)$. If $\Psi_1<0$ the point is a saddle point. When $\Psi_1=0$ the point $(1,0)$ is degenerate but with a change to polar coordinates we find that the point is saddlelike. When $\Psi_1>0$ the point is generally an unstable two-tangent node unless $\Psi_2=9\zeta_o-(3\gamma-2)-4-12\eta_o=0$, in which case the point degenerates to a stellar node.

When $\Psi_1>0$ we have a fifth singular point $(\Sigma,0)$ where $\Sigma=(9\zeta_o/(3\gamma-2))^{1/(1-m)}$. The point $(\Sigma,0)$ is found to be a saddle point.

TABLE I. Qualitative nature of the singular points for different values of the parameters (with respect to Ω time), in the case $\chi_{ab}\equiv 0$.

			(0,0)	$(\Sigma,0)$	(1,0)	(0,2)	(0,−2)
$m=n>1$	$\Psi_1>0$	$\gamma\neq 2$	$R-N_2{}^{\mathrm{a}}$	S	$R-N_2{}^{\mathrm{b}}$	$A-N_2{}^{\mathrm{c,d}}$	$A-N_2{}^{\mathrm{e}}$
		$\gamma=2$	$R-N_2$	S	$R-N_2{}^{\mathrm{b}}$	$A-N^{*\,\mathrm{d}}$	$A-N^*$
	$\Psi_1=0$	$\gamma\neq 2$	$R-N_2{}^{\mathrm{a}}$		S^*	$A-N_2{}^{\mathrm{c,d}}$	$A-N_2{}^{\mathrm{e}}$
		$\gamma=2$	$R-N_2$		S^*	$A-N^{*\,\mathrm{d}}$	$A-N^*$
	$\Psi_1<0$	$\gamma\neq 2$	$R-N_2{}^{\mathrm{a}}$		S	$A-N_2{}^{\mathrm{c,d}}$	$A-N_2{}^{\mathrm{e}}$
		$\gamma=2$	$R-N_2$		S	$A-N^{*\,\mathrm{d}}$	$A-N^*$
$m=n=1$	$\Psi_1>0$		S		$R-N_2{}^{\mathrm{b}}$	$A-N_2{}^{\mathrm{f,d}}$	$A-N_2{}^{\mathrm{g}}$
	$\Psi_1=0$		***	***	***	$A-N_2{}^{\mathrm{f,d}}$	$A-N_2{}^{\mathrm{g}}$
	$\Psi_1<0$		$R-N_2{}^{\mathrm{h}}$		S	$A-N_2{}^{\mathrm{f,d}}$	$A-N_2{}^{\mathrm{g}}$
$m=n=1/2$	$\Psi_1>0$		S^*		$R-N_2{}^{\mathrm{b}}$	$A-N^{*\mathrm{i}}$	$A-N^{*\mathrm{j}}$
	$\Psi_1=0$		S^*		$R-N^*$	$A-N^{*\mathrm{i}}$	$A-N^{*\mathrm{j}}$
	$\Psi_1<0$		S^*	$R-N_2$	S	$A-N^{*\mathrm{i}}$	$A-N^{*\mathrm{j}}$
$m=n=0$		$\zeta_o=0$	$R-N_2{}^{\mathrm{k,l}}$		S		
	$\Psi_1>0$	$\zeta_o\neq 0$			$R-N_2{}^{\mathrm{b}}$		
	$\Psi_1=0$	$\zeta_o\neq 0$			$R-N^*$		
	$\Psi_1<0$	$\zeta_o\neq 0$		$R-N_2{}^{\mathrm{m}}$	S		
$\zeta_o=\eta_o=0$		$\gamma\neq 2$	$R-N_2{}^{\mathrm{a}}$		S	$A-N_2{}^{\mathrm{c,d}}$	$A-N_2{}^{\mathrm{e}}$
		$\gamma=2$	$R-N_2$		***	***	***

[a] If $\gamma=4/3$ for $C\neq 0$ the point becomes a $R-N_1$, and for $C=0$ the point becomes a $R-SN$.
[b] If $\Psi_2=0$ the point becomes a $R-SN$.
[c] If $C=(3\gamma-2)/2$ the point becomes a $A-N_1$.
[d] If $C=2$ the point becomes a S^*.
[e] If $C=-(3\gamma-2)/2$ the point becomes a $A-N_1$.
[f] If $C=-\Psi_3$ the point becomes a $A-N_1$.
[g] If $C=\Psi_3$ the point becomes a $A-N_1$.
[h] If $\Psi_4=0$ for $C\neq 0$ the point becomes a $R-N_1$, and for $C=0$ the point becomes a $R-SN$.
[i] If $C>0$ the point becomes a S^*.
[j] If $C<0$ the point becomes a S^*.
[k] If $C\neq 0$ and $\Psi_5=0$ the point becomes a $R-N_1$.
[l] If $C=0$ and $\Psi_5\geqslant 0$ the point becomes a $R-SN$.
[m] If $\Psi_6=6$ for $C\neq 0$ the point becomes a $R-N_1$ and for $C=0$ the point becomes a $R-SN$.

2. $m=n=1$

The singular point $(0,2)$ is generally a stable two-tangent node, unless $\Psi_3+C=0$ whence the point degenerates to a one-tangent node [where $\Psi_3=\frac{1}{2}(9\zeta_o-(3\gamma-2)+12\eta_o)$]. There also exists a degenerate case when $C=2$ which is found to be hyperbolic in $\mathfrak{R}$. (See also Table B in Ref. 7.)

TABLE II. Definition of the quantities used in Table I.

	Conditional quantities
Ψ_1	$9\zeta_o-(3\gamma-2)$
Ψ_2	$9\zeta_o-(3\gamma-2)-4-12\eta_o$
Ψ_3	$\frac{1}{2}(9\zeta_o-(3\gamma-2)+12\eta_o)$
Ψ_4	$9\zeta_o-(3\gamma-2)+2$
Ψ_5	$(3\gamma-2)-2-6\eta_o$
Ψ_6	$9\zeta-(3\gamma-2)+2+6\eta_0$

TABLE III. Notation used to describe the singular points in Table I.

Notation	
R	Repelling
A	Attracting
S	Saddle point
$S*$	Saddlelike[a]
N_1	One-tangent node
N_2	Two-tangent node
$N*$	Nodelike[b]
SN	Stellar node
***	Line singularity
No entry	Not singular

[a]Degenerate point that has the qualitative nature of a saddle-point in the region of interest.
[a]Degenerate point that has the qualitative nature of a two-tangent node in the region of interest.

The singular point $(0,-2)$ is generally a stable two-tangent node, unless $\Psi_3 - C=0$ whence the point degenerates to a one-tangent node. There also exists a degenerate case when $C=2$ which is found to be hyperbolic in $\mathfrak{R}$.

The point (0,0) has a variety of different natures depending on the sign of Ψ_1. If $\Psi_1>0$, the point is a saddle point. When $\Psi_1=0$ the point $(0,0)$ actually becomes a singular point on a line singularity $\beta=0$, and will be discussed later. When $\Psi_1<0$, the point $(0,0)$ is generally an unstable two-tangent node unless $\Psi_4 = 9\zeta_o - (3\gamma - 2)+ 2=0$ in which case the point degenerates to a one-tangent node for $C \neq 0$ and is a stellar node for $C=0$.

The point $(1,0)$ in this case has the same qualitative behavior as in the case $m=n>1$ except in the degenerate case when $\Psi_1=0$ where it becomes part of a line singularity.

In the case when $\Psi_1=0$, we have a line singularity $(x,0)$, where $0\leqslant x\leqslant 1$. The line is an attracting singular line and the slope of the trajectories as $\beta \rightarrow 0$ is

$$\lim_{\beta\rightarrow 0} \frac{d\beta}{dx} = \frac{-2}{C}(1+ 3\eta_o)(1\text{-}x)^{-1}.$$

If $C<0$, the slope of the trajectories as $\beta \rightarrow 0$ is always positive, if $C=0$ the slope of the trajectories becomes infinite and the trajectories cross the line at right angles, and if $C>0$ the slope of the trajectories is negative.

3. $m=n=1/2$

In this case, there are at most five singular points in $\mathfrak{R}$. For the points (0,0), (0,2), and $(0,-2)$ the system becomes nonanalytic. By transforming to the variable u and time coordinate τ ($u^2=x;d\Omega/d\tau = u$), these points can be analyzed using analytic methods. All three points become degenerate, and by a change to polar coordinates, the qualitative behavior of the singular points is determined.

The point (0,0) has invariant rays $\theta=0$ and $\theta=\theta^*$ where tan $\theta^*=-9\zeta_o/C$. We find from the analysis that $dr/d\tau<0$ along the invariant ray $\theta=0$, and $dr/d\tau>0$ along the invariant ray $\theta=\theta^*$, thus each sector is hyperbolic. (See also Table C in Ref. 7.)

The point $(0,2)$ has invariant rays $\theta=0$ and $\theta=\theta^*$ where tan $\theta^*=(9\zeta_o+ 12\eta_o)/2C$. The region $\mathfrak{R}$ in the new coordinates is now bounded by $(\beta - 2)(\beta+ 2)+ 4u^2=4$, so the invariant ray $\theta=0$ corresponds to the trajectory along the boundary. If $C>0$, the single sector in $\mathfrak{R}$ is hyperbolic. If $C=0$, then $\theta_*=-\pi/2$, which corresponds to the $u=x=0$ boundary where $dr/d\tau$ <0; hence the trajectories are attracted to the point along the eigendirection $x=0$. If $C<0$, then

$\mathfrak{R}$ is divided into two sectors. One can show that $dr/dt<0$ along the invariant ray $\theta=\theta^*$. The trajectories are attracted to the point along the eigendirection corresponding to the invariant ray $\theta=\theta^*$.

The point $(0,-2)$ has invariant rays $\theta=0$ and $\theta=\theta^*$ where $\tan\theta^* = (9\zeta_o+12\eta_o)/2C$. If $C<0$, the single sector in $\mathfrak{R}$ is hyperbolic. If $C=0$, then $\theta_*=-\pi/2$ which corresponds to the $u=x=0$ boundary where $dr/d\tau<0$; hence the trajectories are attracted to the point along the eigendirection $x=0$. If $C>0$, then $\mathfrak{R}$ is divided into two sectors. One can show that $dr/d\tau<0$ along the invariant ray $\theta=\theta^*$. The trajectories are attracted to the point along the eigendirection corresponding to the invariant ray $\theta=\theta^*$.

The point $(1,0)$ has the same character as it did in the previous two cases except in the degenerate case when $\Psi_1=0$. In the degenerate case, changing to polar coordinates, the two sectors in $\mathfrak{R}$ are found to be parabolic in nature, and hence the point behaves like an unstable node.

When $\Psi_1<0$ there is a fifth singular point $(\Sigma,0)$, where $\Sigma=(9\zeta_o/(3\gamma-2))^2$. The point $(\Sigma,0)$ is an unstable two-tangent node, with the main eigendirection along the x axis.

4. $m=n=0$

In this case there are two separate situations depending upon whether $\zeta_o=0$ or $\zeta_o\neq 0$. If $\zeta_o=0$, there are two singular points $(0,0)$ and $(1,0)$. The point $(1,0)$ is a saddle point. The point $(0,0)$ is generally an unstable two-tangent node, unless $\Psi_5=(3\gamma-2)-2-6\eta_o=0$, whence the point degenerates to a one-tangent node for $C\neq 0$ and to a stellar node for $C=0$. (See also Table D in Ref. 7.)

If $\zeta_o\neq 0$, we have at most two singular points depending on the sign of Ψ_1. If $\Psi_1<0$, there are two singular points, $(1,0)$ and $(\Sigma,0)$, where $\Sigma=9\zeta_o/(3\gamma-2)$. In this case, the point $(1,0)$ is a saddle point. The point $(\Sigma,0)$ is generally an unstable two-tangent node, unless $\Psi_6=9\zeta_o-(3\gamma-2)+2+6\eta_o=0$, whence the point degenerates to an unstable one-tangent node for $C\neq 0$, and to a stellar node for $C=0$. If $\Psi_1=0$, we have only one singular point. The point $(1,0)$ becomes degenerate, but by changing to polar coordinates and using higher order terms in the variable r, we find that the point acts like an unstable node. If $\Psi_1>0$, the point $(1,0)$ is again the only singular point where the qualitative behavior is the same as in the previous cases for $\Psi_1>0$.

5. $\zeta_o=\eta_o=0$

For $\gamma\neq 2$ there are four singular points in $\mathfrak{R}$. The points $(0,2)$, $(0,-2)$, and $(0,0)$ behave in the same manner as the points in the case $m=n>1$ for $\gamma\neq 2$, and hence will not be summarized here. The point $(1,0)$ is a saddle when $\gamma\neq 2$. (See also Table E in Ref. 7.)

However, in the case $\gamma=2$ every point on the boundary $\beta^2+4x=4$ becomes singular. The system of equations can be solved explicitly when $\gamma=2$. The solution is given by the line $\beta=0$ and the family of parabolas $A\beta^2-C\beta+2x=0$ $(\beta\neq 0)$, where A is an arbitrary constant depending on initial conditions.

B. Energy conditions

We shall look at the conditions imposed on the parameters by the energy conditions (EC). A full and detailed analysis of the EC is difficult. For simplicity we shall look at the EC in a neighborhood of each individual singular point. All singular points in the model we are studying have either $x=0$ or $\beta=0$. The singular points with $x=0$ are $(0,2)$ and $(0,-2)$, in which case the EC are identically satisfied. The remaining singular points lie in the $\beta=0$ plane and the EC are given by Eq. (3.14). The point $(0,0)$ always satisfies the EC. The point $(\Sigma,0)$ satisfies the DEC and marginally satisfies the SEC. For the point $(1,0)$ two cases occur:

a. $\zeta_o=0$, in which case the EC are trivially satisfied.

b. $\zeta_o \neq 0$, in which case the DEC implies $\gamma \geq 3\zeta_o$ and the SEC implies $\Psi_1 \leq 0$.

Thus, the energy conditions put no direct constraints on the nature of the singular points $(0,0)$, $(\Sigma,0)$, $(0,2)$ and $(0,-2)$. However, the EC put constraints on the nature of the point $(1,0)$. From the previous analysis, $\Psi_1 < 0$ implies in all cases that the point is a saddle-point and $\Psi_1 = 0$ is precisely the degenerate case where the point may be saddlelike or nodelike depending on the values of m and n. If the requirement that the SEC be satisfied is dropped, then the point $(1,0)$ may become an attracting node.

V. DISCUSSION

In the case $\chi_{ab}=0$, Eq. (4.6) implies that there exists three invariant sets $\mathfrak{R}_- = \{(x,\beta)|\beta<0\}$, $\mathfrak{R}_o = \{(x,\beta)|\beta=0\}$, and $\mathfrak{R}_+ = \{(x,\beta)|\beta>0\}$. We shall discuss what happens in each invariant set and determine the exact solutions corresponding to each singular point. We will also show that most of these solutions represent space–times which are transitively self-similar (that is, there exists a homothetic vector field in addition to the three Killing vector fields).

In the set $\mathfrak{R}_-$ there exists only one isolated singular point, $(0,-2)$. It lies on the boundary $\beta^2 + 4x = 4$, where ${}^3R = 0$, and hence the solution is of Bianchi type I. The point is part of the Kasner ring of singularities with Kasner coefficients p_i:

$$p_1 = \frac{1}{3}\left(1 - \frac{(k+1)}{\sqrt{k^2-k+1}}\right), \quad p_2 = \frac{1}{3}\left(1 - \frac{(1-2k)}{\sqrt{k^2-k+1}}\right), \quad p_3 = \frac{1}{3}\left(1 - \frac{(k-2)}{\sqrt{k^2-k+1}}\right). \tag{5.1}$$

The solution is given by

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{5.2}$$

$$\theta = t^{-1}, \quad \zeta = 0, \quad \rho = 0, \quad \eta = 0, \quad \sigma = -(\sqrt{3}t)^{-1}, \quad q_1 = 0. \tag{5.3}$$

The space–time is transitively self-similar (Hsu and Wainwright[13]) with homothetic vector

$$\mathbf{X} = t\frac{\partial}{\partial t} + (1-p_1)x\frac{\partial}{\partial x} + (1-p_2)y\frac{\partial}{\partial y} + (1-p_3)z\frac{\partial}{\partial z}. \tag{5.4}$$

From a dynamical systems point of view, the singular point $(0,-2)$ is always a repellor in $t-time$ (even when the sector in $\mathfrak{R}_-$ is hyperbolic in nature, since trajectories are repelled in $t-time$ along an eigendirection that is not in $\mathfrak{R}_-$), which implies that this point represents an initial singularity. The singularity is generally of *cigar* type, but in the case when $k=0$ or $C=1$ the singularity is of *pancake* type.[14] For particular values of the parameters, there exist trajectories that start at $(0,-2)$ and leave $\mathfrak{R}_-$ at some finite time t_o. There also exist trajectories that start from the singular point $(0,-2)$ and remain in $\mathfrak{R}_-$ for all time; these models expand from the Kasner singularity towards one of the isotropic models located on the x axis (i.e., these models isotropize as $t \to \infty$). In some cases, there exist trajectories that enter $\mathfrak{R}_-$ at some finite time t_o; these models evolve towards one of the isotropic models and can only represent late time behavior.

In the case $m=n=0$, there are no singular points in the invariant set $\mathfrak{R}_-$. All trajectories enter $\mathfrak{R}_-$ at some finite time and evolve towards one of the isotropic models. Hence these models can only describe late time behavior.

In the case $\zeta_o = \eta_o = 0$ with $\gamma = 2$, there is a line singularity on the boundary $\beta^2 + 4x = 4$. The singular points represent stiff perfect fluid Bianchi I solutions (5.2) with coefficients p_i:

$$p_1 = \frac{1}{3}\left(1 - \frac{(k+1)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\right), \quad p_2 = \frac{1}{3}\left(1 - \frac{(1-2k)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\right), \tag{5.5}$$

$$p_3=\frac{1}{3}\left(1-\frac{(k-2)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\right),$$

$$\theta=t^{-1},\quad \zeta=0,\quad \rho=\frac{x_o}{3}t^{-2},\quad \eta=0,\quad \sigma=-\sqrt{\frac{(1-x_o)}{3}}t^{-1},\quad q_1=0, \tag{5.6}$$

where the parameter x_o is bounded by $0\leqslant x_o\leqslant 1$. The space-time is transitively self-similar[13] with homothetic vector (5.4) with the p_i now defined by (5.5). For $C\leqslant 0$ all trajectories remain in $\mathfrak{R}_-$ for all time and evolve towards the isotropic model at $(0,0)$. For $C>0$ all trajectories leave $\mathfrak{R}_-$ after some finite time and hence may only describe early time behavior.

In the invariant set $\mathfrak{R}_o$ there exists either 1, 2, 3, or a line of singular points. Points in this set represent negatively curved (i.e, $x<1$) or flat (i.e., $x=1$) FRW models with at most bulk viscosity.

The point $(1,0)$ is singular in all cases. The point lies on the boundary $\beta^2+4x=4$ and hence the point represents a flat FRW model. It is a saddle-point for $\Psi_1<0$. However, when $\Psi_1>0$ the point becomes an attracting node and hence represents a late-time attractor (but note that in this case the SEC is violated and the corresponding asymptotic solution may not be physically acceptable). In the case $\Psi_1=0$, the point is saddle-like for $m=n>1$ and nodelike in the remaining cases. The solution corresponding to this singular point depends upon whether $\gamma=3\zeta_o$ or not.

For $\gamma\neq 3\zeta_o$ the solution is

$$ds^2=-dt^2+(t)^{4/(3\gamma-3\zeta_o)}(dx^2+dy^2+dz^2), \tag{5.7}$$

$$\theta=\frac{2}{\gamma-3\zeta_o}t^{-1},\quad \zeta=\frac{2\zeta_o}{(\gamma-3\zeta_o)}t^{-1},\quad \rho=\frac{4}{3(\gamma-3\zeta_o)^2}t^{-2},\quad \eta=\frac{2\eta_o}{(\gamma-3\zeta_o)}t^{-1},\sigma=0,\quad q_1=0. \tag{5.8}$$

The space–time (5.7) admits the homothetic vector (5.4) with $p_1=p_2=p_3=2/(3\gamma-9\zeta_o)$, hence the space–time is transitively self-similar.[13]

For $\gamma=3\zeta_o$ the solution is

$$ds^2=-dt^2+e^{2Ht}(dx^2+dy^2+dz^2), \tag{5.9}$$

$$\theta=3H,\quad \zeta=9\zeta_oH^2,\quad \rho=3H^2,\quad \eta=9\eta_oH^2,\quad \sigma=0,\quad q_1=0. \tag{5.10}$$

The space–time (5.9) does not admit a homothetic vector; the space–time is not self-similar (Maartens and Maharaj[15]).

When the point $(1,0)$ is a saddle, models start from the matter dominated singularity at $(1,0)$ and evolve towards either the Milne model at $(0,0)$ or the FRW model at $(\Sigma,0)$. However, if the point is an attracting node, all models evolve towards the point $(1,0)$.

The point $(\Sigma,0)$ represents a negatively curved FRW model with bulk viscosity where $\Sigma = ((9\zeta_o/(3\gamma-2))^{1/(1-m)}$. The space–time is self-similar.[13] The corresponding exact solution is

$$ds^2=-dt^2+(1-\Sigma)^{-1}(t)^2(dx^2+e^{2x}dy^2+e^{2x}dz^2), \tag{5.11}$$

$$\theta=3t^{-1},\quad \zeta=3\zeta_o\Sigma^mt^{-1},\quad \rho=3\Sigma t^{-2},\quad \eta=3\eta_o\Sigma^mt^{-1},\quad \sigma=0,\quad q_1=0, \tag{5.12}$$

with homothetic vector

$$\mathbf{X}=t\frac{\partial}{\partial t}. \tag{5.13}$$

When the singular point $(\Sigma,0)$ is a saddle, models start from this matter dominated singularity at $(\Sigma,0)$ and evolve towards either the Milne model at $(0,0)$ or the FRW model at $(1,0)$ (however this latter model violates the SEC). However, if the point is a node, the solution is a late time asymptotic attractor (except in the degenerate case when there is a line singularity).

The point $(0,0)$ represents an empty cosmological model, commonly known as the Milne model. The space–time is transitively self-similar[13] with homothetic vector (5.13). The solution is

$$ds^2=-dt^2+(t)^2(dx^2+e^{2x}dy^2+e^{2x}dz^2), \tag{5.14}$$

$$\theta=3t^{-1}, \quad \zeta=0, \quad \rho=0, \quad \eta=0, \quad \sigma=0, \quad q_1=0. \tag{5.15}$$

(Note, there is one exception to the above solution; if $m=n=0$ and $\zeta_o=0$, then $\eta=3\eta_o t^{-1}$.) When point $(0,0)$ is an attracting node, the matter dominated singularities at $(\Sigma,0)$ or $(1,0)$ evolve towards the Milne model at $(0,0)$. However, when the point $(0,0)$ is a saddle, the Milne model evolves towards one of the other isotropic models.

In the set $\mathfrak{R}_+$ there exists only one isolated singular point, $(0,2)$. It lies on the boundary $\beta^2+4x=4$, where ${}^3R=0$, and hence the solution is of Bianchi type I. The point is part of the Kasner ring of singularities. The solution is given by Eqs. (5.2) and (5.3) [except that $\sigma=(\sqrt{3}t)^{-1}$], with the p_i defined as:

$$p_1=\frac{1}{3}\left(1+\frac{(k+1)}{\sqrt{k^2-k+1}}\right), \quad p_2=\frac{1}{3}\left(1+\frac{(1-2k)}{\sqrt{k^2-k+1}}\right), \quad p_3=\frac{1}{3}\left(1+\frac{(k-2)}{\sqrt{k^2-k+1}}\right). \tag{5.16}$$

The space–time is transitively self-similar with homothetic vector (5.4) with the p_i defined by (5.16). From a dynamical systems point of view, the singular point $(0,2)$ is always a repellor in $t-time$ (even when the sector in $\mathfrak{R}_+$ is hyperbolic in nature, since trajectories are repelled in $t-time$ along an eigendirection that is not in $\mathfrak{R}_+$), which implies that this point represents an initial singularity. The singularity is generally of *cigar* type, but in the case when $k=1$ or $C=2$ (the LRS case), the singularity is of *pancake* type.[14] For particular values of the parameters, there exist trajectories that start at $(0,2)$ and leave $\mathfrak{R}_+$ at some finite time t_o. There also exist trajectories that start from the singular point $(0,2)$ and remain in $\mathfrak{R}_+$ for all time; these models expand from the Kasner singularity towards one of the isotropic models located on the x axis (i.e., these models isotropize as $t\to\infty$). In some cases, there exist trajectories that enter $\mathfrak{R}_+$ at some finite time t_o; these models evolve towards one of the isotropic models and can only represent late time behavior.

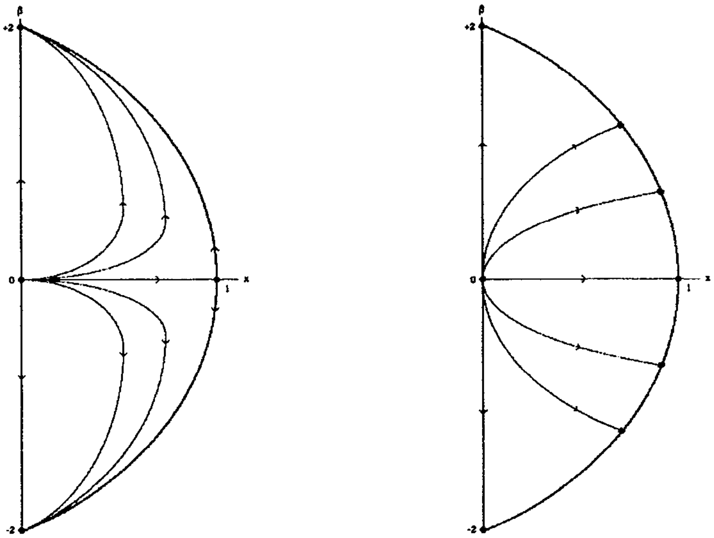
In the case $m=n=0$, there are no singular points in the invariant set $\mathfrak{R}_+$. All trajectories enter $\mathfrak{R}_+$ at some finite time and evolve towards one of the isotropic models. Hence these models can only describe late time behavior.

In the case $\zeta_o=\eta_o=0$ and $\gamma=2$ there is a line singularity on the boundary $\beta^2+4x=4$. The singular points represent stiff perfect fluid Bianchi I solutions (5.2) with coefficients p_i:

$$p_1=\frac{1}{3}\left(1+\frac{(k+1)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\right), \quad p_2=\frac{1}{3}\left(1+\frac{(1-2k)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\right), \tag{5.17}$$

$$p_3=\frac{1}{3}\left(1+\frac{(k-2)}{\sqrt{k^2-k+1}}\sqrt{1-x_o}\right),$$

$$\theta=t^{-1}, \quad \zeta=0, \quad \rho=\frac{x_o}{3}t^{-2}, \quad \eta=0, \quad \sigma=\sqrt{\frac{(1-x_o)}{3}}t^{-1}, \quad q_1=0, \tag{5.18}$$



(a) $C=0,\ \zeta_o=\eta_o=0,\ 1\leq\gamma<2$ (b) $C=0,\ \zeta_o=\eta_o=0,\ \gamma=2$

FIG. 2. The phase portraits describe the behavior of the perfect fluid Bianchi type V models with no heat conduction or viscosity in the case $\zeta_o=\eta_o=0$ and $C=0$. In all figures, the arrows refer to increasing Ω time or decreasing t time.

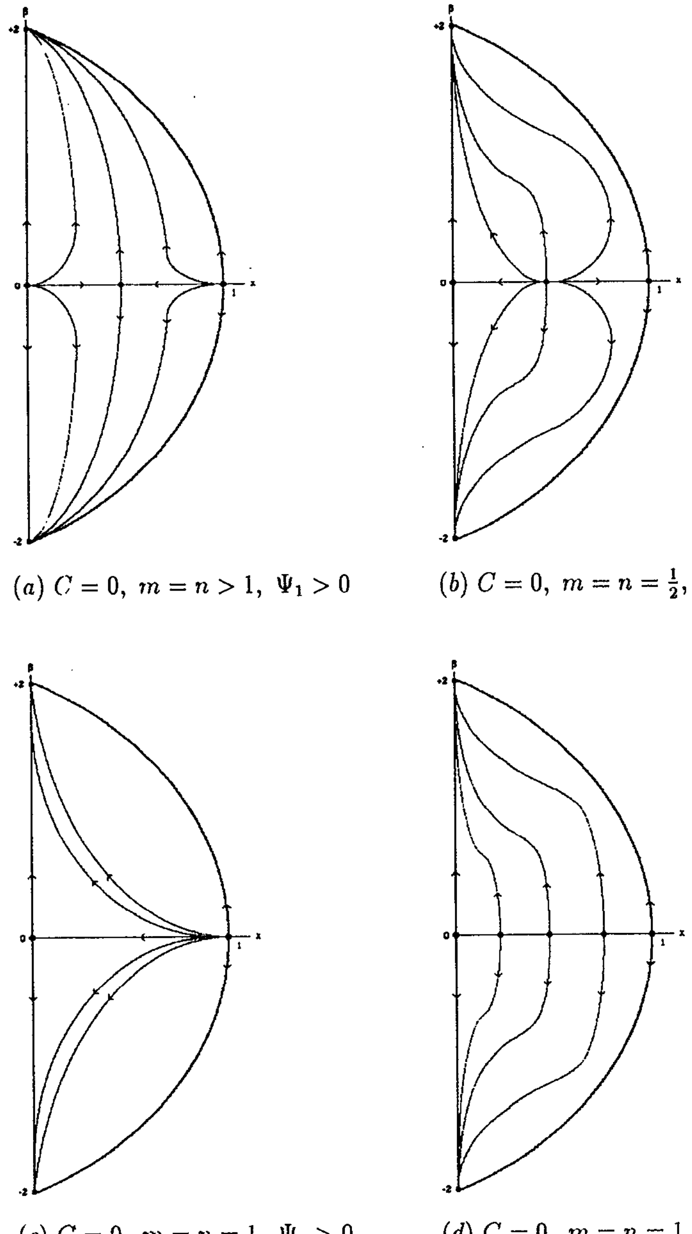
where the parameter x_o is bounded by $0\leqslant x_o\leqslant 1$. The space–time is transitively self-similar with homothetic vector (5.4) with the p_i defined by (5.17). For $C\geqslant 0$ all trajectories remain in $\mathfrak{R}_+$ for all time and evolve towards the isotropic model at $(0,0)$. For $C<0$ all trajectories leave $\mathfrak{R}_+$ after some finite time and hence may only describe early time behavior.

In the case of the perfect fluid Bianchi type V model with $C=0$ and $\zeta_o=\eta_o=0$, we note that all trajectories remain in $\mathfrak{R}$ for all time. The models evolve from the Kasner singularities at $(0,2)$ and $(0,-2)$ towards the Milne model at $(0,0)$. In this case there also exists exceptional trajectories; there are two trajectories along the boundary of $\mathfrak{R}$ that evolve from the Kasner points towards the FRW model at $(1,0)$ (these represent Bianchi I perfect fluid models), and one trajectory that evolves from the matter dominated FRW model at $(1,0)$ towards the Milne model at $(0,0)$. (See Fig. 2.)

In the case of the imperfect fluid Bianchi type V model with viscosity and zero heat conduction ($C=0$), there are two cases depending upon whether $m=n=0$ or not. If $m=n\neq 0$, Eq. (4.5) implies that $x=0$ is an invariant set and hence no trajectories can cross the β axis. Therefore, all trajectories remain in $\mathfrak{R}$ for all time. The behavior of the phase portraits depends critically on the sign of Ψ_1 as well as the parameters m and n. The models evolve from the Kasner singularities at $(0,2)$ and $(0,-2)$ towards one of the isotropic models either at $(0,0)$, $(\Sigma,0)$, or $(1,0)$ depending on the sign of Ψ_1. There exists exceptional trajectories emitting from $(0,2)$ and $(0,-2)$ towards the FRW models either at $(1,0)$ or $(\Sigma,0)$. There also exists exceptional trajectories on the x-axis that remain on the axis for all time [see Figs. 2(a) and 3].

However, in the case $m=n=0$, we find in all cases that the models start at some finite time t_o and evolve towards one of the isotropic models depending critically on the value of Ψ_1. Note that the initial Big Bang singularity is avoided in this case (see Figs. 4).

With the introduction of heat conduction, $x=0$ is no longer an invariant set, and hence trajectories may leave $\mathfrak{R}$, and consequently the weak energy condition (WEC) is violated. Equations (4.6) and (4.7) are invariant under the transformation $\beta\rightarrow-\beta$, $C\rightarrow -C$. Hence the phase



(a) $C=0,\ m=n>1,\ \Psi_1>0$ (b) $C=0,\ m=n=\frac{1}{2},\ \Psi_1<0$

(c) $C=0,\ m=n=1,\ \Psi_1>0$ (d) $C=0,\ m=n=1,\ \Psi_1=0$

FIG. 3. The phase portraits describe the behavior of the Bianchi type V models with viscosity and no heat conduction in the case $m=n\neq 0$ and $C=0$.

portraits for $C>0$ are reflections over the x-axis of the phase portraits for $C<0$. Hence, in the remainder of the analysis only the case $C<0$ will be considered. The specific case $C=2$ is done by Coley and Dunn[9] (see Figs. 2 in Coley and Dunn[9]).

Let us investigate what happens when we have a perfect fluid with heat conduction (i.e., no viscosity). Assuming that the WEC is satisfied for all time, for $1\leqslant\gamma<\frac{4}{3}$, the positive β quadrant has the same qualitative behavior as in the perfect fluid case. However when $\gamma\geqslant\frac{4}{3}$ the WEC is

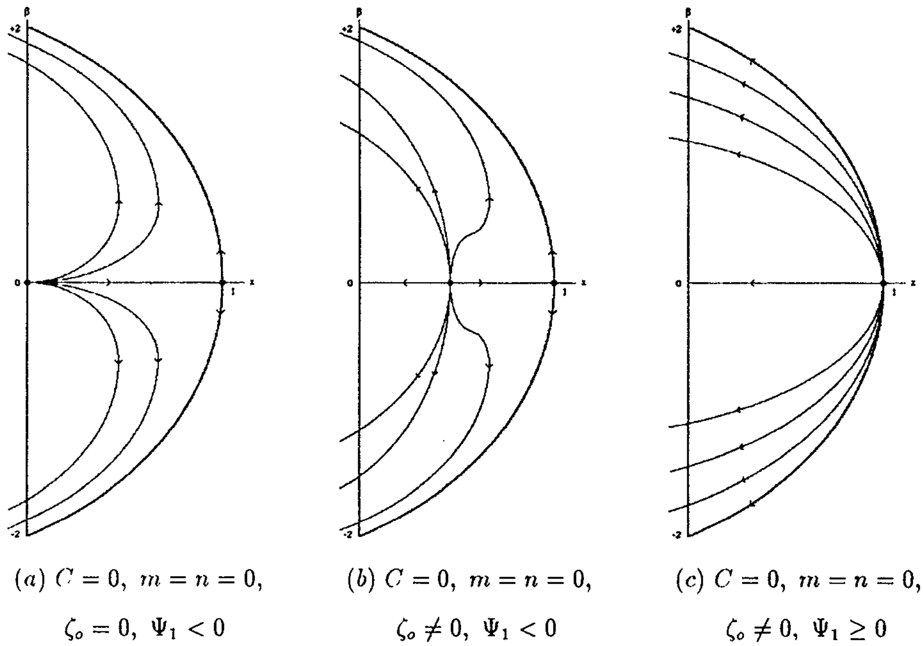


FIG. 4. The phase portraits describe the behavior of the Bianchi type V models with viscosity and no heat conduction in the case $m=n=0$ and $C=0$.

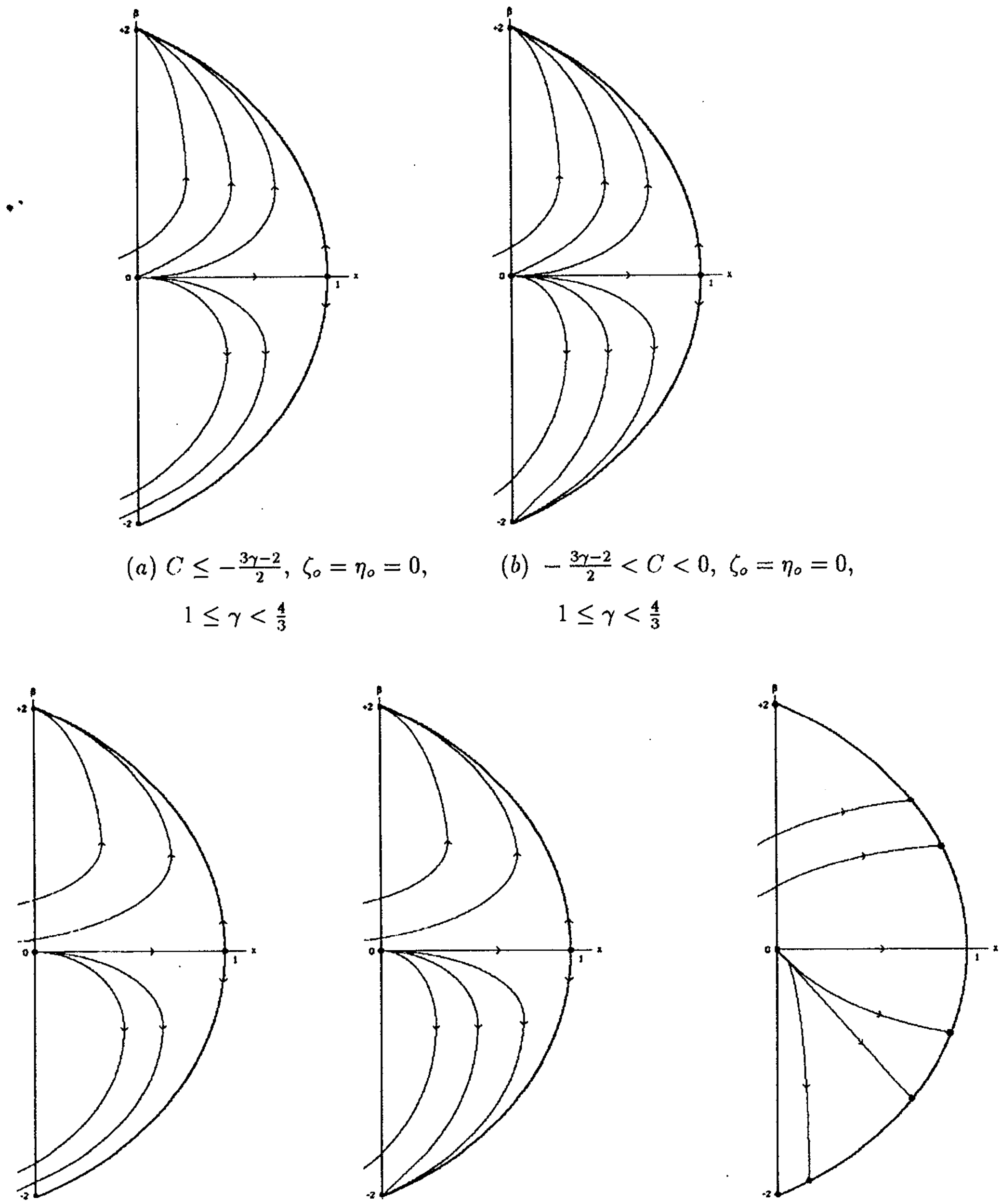
violated for all trajectories at some finite time (except for those exceptional trajectories which are qualitatively the same as in the perfect fluid case). In the negative β quadrant, for $C\leqslant-(3\gamma-2)/2$, all trajectories violate the WEC at some finite time (except again for those exceptional trajectories) and for $C>-(3\gamma-2)/2$, the qualitative behavior is the same as in the perfect fluid case (see Fig. 5).

Let us consider an imperfect fluid with both viscosity and heat conduction. In the case $m=n=0$, where all trajectories violate the WEC at some finite time and hence are only good for late time asymptotic behavior, we find in general that the qualitative behavior is the same as if we had viscosity and no heat (see Figs. 4). But when $\zeta_o=0$ and $\Psi_5\geqslant 0$ there is a slight difference in behavior; in the positive β quadrant all trajectories violate the WEC, while the negative quadrant is the same as if we had no heat (see Fig. 6).

When $m=n=1/2$ there are two different phase portraits depending on the sign of Ψ_1. If $\Psi_1<0$ and $C<0$ there is a fifth singular point at $(\Sigma,0)$. The positive β quadrant is the same as in the case where we had just viscosity (Fig. 4), but in the negative β quadrant all trajectories violate the WEC at some finite time [see Fig. 7(a)]. If $\Psi_1\geqslant 0$ and $C<0$, the positive β quadrant is the same as in the case where there was just viscosity [Fig. 3(c)], but in the negative β quadrant all trajectories violate the WEC at some finite time [see Fig. 7(b)].

In the case $m=n=1$, only the degenerate case when $\Psi_1=0$ is qualitatively different (to those already discussed). In this case, for $C<0$, the positive β quadrant is similar to that with just viscosity [Fig. 3(d)] but trajectories in the negative β quadrant will violate the WEC at some finite time (see Fig. 8).

In the case when $m=n>1$, $C<0$, and $\Psi_1\leqslant 0$ the qualitative behavior is the same as that for other cases [see Figs. 2(a), 5 (a)–(d)]. However in the case when $\Psi_1>0$ there are different possibilities. Again there exists a fifth singular point. When $1\leqslant\gamma<4/3$ and $C<0$, the positive β quadrant is similar to the case when no heat was present [Fig. (2a)], however in the negative β



(a) $C \leq -\frac{3\gamma-2}{2}$, $\zeta_o = \eta_o = 0$, $1 \leq \gamma < \frac{4}{3}$

(b) $-\frac{3\gamma-2}{2} < C < 0$, $\zeta_o = \eta_o = 0$, $1 \leq \gamma < \frac{4}{3}$

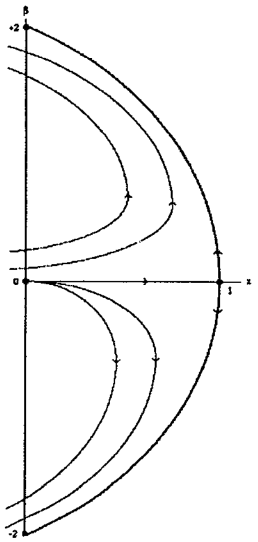
(c) $C = -1$, $\zeta_o = \eta_o = 0$, $\gamma = \frac{4}{3}$

(d) $-1 < C < 0$, $\zeta_o = \eta_o = 0$, $\frac{4}{3} \leq \gamma < 2$

(e) $C < 0$, $\zeta_o = \eta_o = 0$, $\gamma = 2$

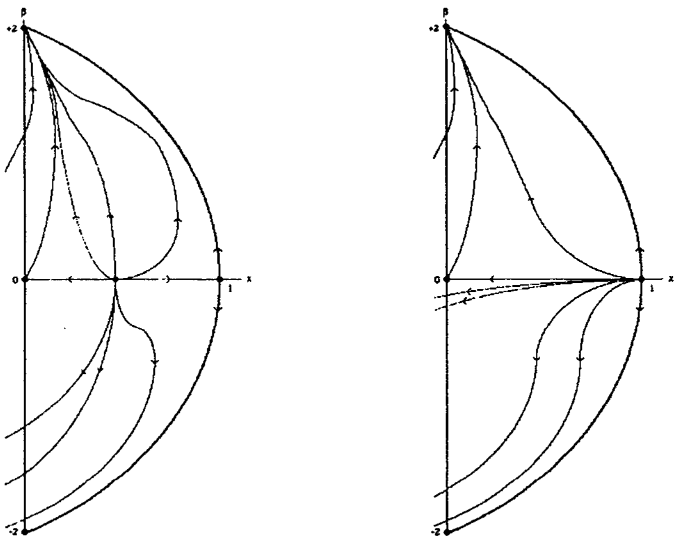
FIG. 5. The phase portraits describe the behavior of the Bianchi type V models with heat conduction and no viscosity in the case $\zeta_o = \eta_o = 0$ and $C \neq 0$.

quadrant some or all trajectories will violate the WEC at some finite time [see Figs. 9(a) and 9(b)]. When $\gamma < 4/3$ and $C < 0$ in the negative β quadrant some or all trajectories will violate the WEC at some finite time, while in the positive β quadrant the only physically realistic models evolve from the point (0,2) towards (1,0) whence the SEC is violated [see Figs. 9(c) and 9(d)].



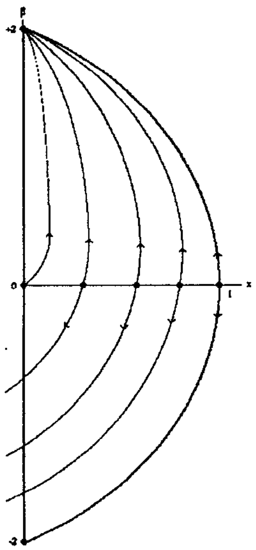
(a) $C<0,\ m=n=0,\ \zeta_o=0,\ \Psi_5\geq 0$

FIG. 6. The phase portrait describes the behavior of the Bianchi type V models with heat conduction and viscosity in the case $m=n=0$ and $C<0$.



(a) $C<0,\ m=n=\frac{1}{2},\ \Psi_1<0$ (b) $C<0,\ m=n=\frac{1}{2},\ \Psi_1\geq 0$

FIG. 7. The phase portraits describe the behavior of the Bianchi type V models with heat conduction and viscosity in the case $m=n=\frac{1}{2}$ and $C<0$.



(a) $C \leq \Psi_3,\ m = n = 1,\ \Psi_1 = 0$

FIG. 8. The phase portrait describes the behavior of the Bianchi type V models with heat conduction and viscosity for the degenerate case $m=n=1$ and $\Psi_1=0$.

VI. CONCLUSION

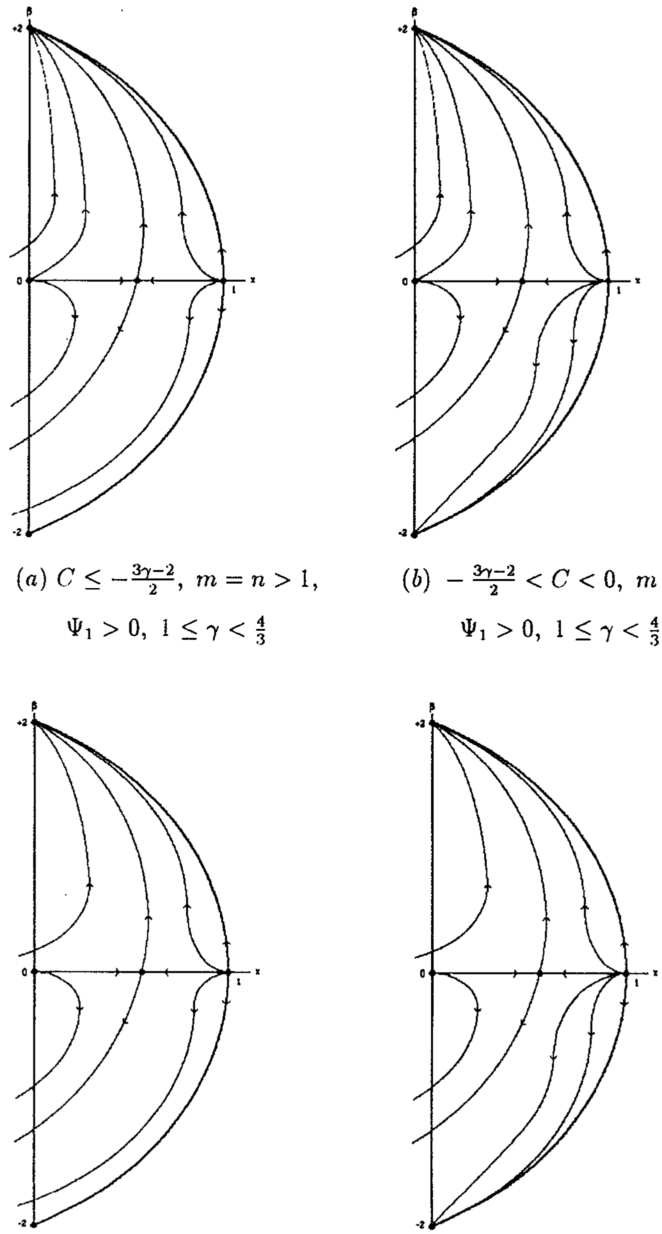
By using geometric techniques from dynamical systems theory we have been able to determine the qualitative behavior of a class of spatially homogeneous cosmological models that contain viscous matter and heat conduction.

With the introduction of viscosity into the fluid, the qualitative behavior of the models differ from that of the perfect fluid models (for example, in some instances an additional singular point is even created). In particular, it is the nature of Ψ_1 and hence ζ_o that affects the global behavior of the models, while the values of C and η_o change only the local behavior near a singular point. With the introduction of heat conduction, solutions that violate the WEC at some finite time t_o in the past (and, in some instances, in the future) arise.

With the introduction of bulk viscosity, the deceleration parameter q, defined by

$$q \equiv -\frac{\ddot{l}l}{\dot{l}^2} = \frac{1}{2}\,(\beta^2 + (3\gamma-2)x - 9\zeta_o x^m), \tag{6.1}$$

may become negative (where $\theta = \dot{l}/l$). A negative q indicates that there exists a region of phase space with an accelerated expansion; that is, inflation occurs. For $\Psi_1 > 0$, in all cases, there exists some region of phase space such that $q<0$, which implies that all models must inflate at some time in their evolution. For $m=n\leqslant 1$ and $\Psi_1>0$, the acceleration occurs as $t\to\infty$. For $m=n>1$, the models may inflate for all time t, or up to some finite time t_o. For $\Psi_1 \leqslant 0$ some models may inflate. In the perfect fluid case, inflation occurs, assuming an equation of state $p=(\gamma-1)\rho$, when $\gamma<\frac{2}{3}$.[16] With the addition of bulk viscosity, the fluid effectively acts like a perfect fluid with an equation of state $p=(\gamma_{\text{eff}} - 1)\rho$ where $\gamma_{\text{eff}} = \gamma - \zeta\theta\rho^{-1}$. Several authors[17] have investigated



(a) $C \leq -\frac{3\gamma-2}{2},\ m=n>1,$ $\Psi_1>0,\ 1\leq\gamma<\frac{4}{3}$

(b) $-\frac{3\gamma-2}{2}<C<0,\ m=n>1,$ $\Psi_1>0,\ 1\leq\gamma<\frac{4}{3}$

(c) $C=-1,\ m=n>1,$ $\Psi_1>0,\ \gamma=\frac{4}{3}$

(d) $-1<C<0,\ m=n>1,$ $\Psi_1>0,\ \gamma=\frac{4}{3}$

FIG. 9. The phase portraits describe the behavior of the Bianchi type V models with heat conduction and viscosity in the case $m=n>1$ and $C<0$.

whether a nonvanishing bulk viscosity could drive an inflationary phase in the early universe. Bulk viscosity can only act as a source for inflation if the SEC is violated. Models that include bulk viscosity and in which the SEC is violated have also been studied[18] since the initial singularity can, in a sense, be eliminated.

In both the cases $\lambda \neq 0$ and $\lambda = 0$, except for the exceptional trajectories located on the x-axis (as well as the stiff perfect fluid case, $\gamma = 2$, with $\zeta_o = \eta_o = 0$), all models that satisfy the WEC for all time start their evolution from the Kasner ring of singularities. For $\lambda \neq 0$, assuming all the EC are satisfied, all models evolve towards the de Sitter model. If the EC are not satisfied, then there exists other late time attractors. For the case $\lambda = 0$, assuming all EC are satisified, all models (except the exceptional trajectories) either evolve towards the Milne model at $(0,0)$ or the FRW model at the point $(\Sigma, 0)$. If the EC are not satisfied, then the FRW model at $(1,0)$ also becomes a late time attractor. In either case, models start from an anisotropic state and isotropize towards a flat or negatively curved FRW model.

In the case $\lambda = 0$, using the equations of state (2.22), all asymptotic states represent self-similar cosmological models [unless $\gamma = 3\zeta_o$ whence the point $(1,0)$ is no longer self-similar]. In the case $\lambda \neq 0$, the future asymptotic states are not self-similar, however the past asymptotic states are self-similar. This shows that the past asymptotic behavior of the imperfect fluid Bianchi V model with or without a cosmological constant is represented by self-similar solutions, and in the case with no cosmological constant if the EC are satisfied the future asymptotic states are also self-similar.

A number of properties of the asymptotic states described in the previous section are, in fact, general properties not only of the Bianchi type V models under investigation here but also of all orthogonal spatially homogeneous imperfect fluid models.

First, in general (in the case $\lambda = 0$ and when the energy conditions are satisfied) at the singular points (where x, β_1, and β_2 are constant) the right-hand side of the Raychaudhuri equation (2.12) will be (a negative definite) constant, whence on integration we obtain

$$\theta = \theta_s t^{-1} \tag{6.2}$$

(where the subscript 's' indicates a constant value; i.e., θ_s is a constant). Hence

$$\rho = \rho_s t^{-2}, \tag{6.3}$$

and Eqs. (2.22) then yield

$$p = p_s t^{-2}, \quad \zeta = \zeta_s t^{-1}, \quad \eta = \eta_s t^{-1}. \tag{6.4}$$

Therefore, all exact solutions corresponding to singular points will have $p \propto \rho$, $p - \zeta\theta \propto \rho$, and $\eta \propto \rho^{1/2}$.

Second, it has recently been shown[19] that dimensionless equations of state are necessary and sufficient for the asymptotic states of the governing system of differential equations to be represented by self-similar cosmological models. A precise statement of this result and an outline of its proof will be given in the Appendix.

We have shown in this paper that by using dimensionless variables (2.20), (2.21), and dimensionless equations of state (2.22), the Einstein field equations reduce to a system of autonomous ordinary differential equations. All models that satisfy the WEC for all time isotropize. Including viscosity (and heat conduction) in the models allow for processes such as inflation and the removal of the initial singularity. These models are sufficiently simple to allow us to analyze them qualitatively. By considering general orthogonal Bianchi models, non-orthogonal Bianchi V models, or more complex equations of state, more realistic models may be analyzed using similar techniques, which may lead to interesting and different qualitative behavior.

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APPENDIX: SELF-SIMILAR ASYMPTOTIC LIMITS OF THE EINSTEIN FIELD EQUATIONS

A space–time is defined to be self-similar if it admits a homothetic vector, and transitively self-similar if it admits an H_4 (Ref. 13) (that is, in addition to the homothetic vector, there exist three Killing vectors that act transitively on 3-dimensional hypersurfaces). In order to be consistent with previous work, we are using the term self-similarity to characterize the properties of the geometry, rather than, as is more conventional, to characterize the properties of the matter.[20] A space–time admits a simply transitive similarity group H_4 if and only if there exists an orthonormal frame $\{\mathbf{e}_a\}$ and a scalar field t such that $\gamma^c_{ab} = F^c_{ab}t^{-1}$ and $\mathbf{e}_a(t)=n_a$, where F^c_{ab} and n_a are constants that are not all zero (see theorem 4.1 in Hsu and Wainwright[13]). [**Note:** All quantities in this appendix are defined and given in either Ellis and MacCallum[21] or in MacCallum's Cargése lectures.[22] In particular, equations in these two papers will be referred to using EM or M, respectively; for example, M(113) refers to Eq. (113) in Ref. 22.]

Theorem. *Let there be a G_3 group of isometries acting transitively on a 3-dimensional hypersurface, and assume that the fluid is moving hypersurface orthogonal. Then, provided $\rho+3p\geqslant 0$, the asymptotic limits of the Einstein field equations are transitively self-similar if and only if the equations of state for the pressure p and anisotropic stress $\pi_{\alpha\beta}$ are homogeneous functions of degree two; that is,*

$$p(\lambda\theta,\ \lambda\sigma_{\alpha\beta},\lambda n_{\alpha\beta},\lambda a_\alpha)=\lambda^2 p(\theta,\ \sigma_{\alpha\beta},n_{\alpha\beta},a_\alpha),$$
$$\pi_{\alpha\beta}(\lambda\theta,\ \lambda\sigma_{\alpha\beta},\lambda n_{\alpha\beta},\lambda a_\alpha)=\lambda^2\pi_{\alpha\beta}(\theta,\ \sigma_{\alpha\beta},n_{\alpha\beta},a_\alpha) \tag{A1}$$

Indication of proof. (See Ref. 19 for details.) The Einstein field equations for the orthogonal spatially homogeneous models may be written in terms of an orthonormal tetrad $\{\mathbf{e}_a\}$. Let $\mathbf{e}_0=\mathbf{u}$ ($\mathbf{u}$ is the fluid four-velocity which is orthogonal to the spatial hypersurfaces), then the quantities γ^c_{ab} defined by the commutator relation $[\mathbf{e}_a,\ \mathbf{e}_b]=\gamma^c_{ab}\mathbf{e}_c$ are spatially independent and are functions of t only. The quantities γ^c_{ab} may be written in terms of θ, $\sigma_{\alpha\beta}$, and new variables $n_{\alpha\beta}$ and a_β (see Ref. 21). The generalized Friedmann equation, formally given by

$$\frac{\theta^2}{3}=\sigma^2+\rho+\frac{1}{2}\left(6a_\alpha a^\alpha+n^{\alpha\beta}n_{\alpha\beta}-\frac{(n_\alpha{}^\alpha)^2}{2}\right), \tag{A2}$$

is a first integral of the Bianchi field equations and serves to define ρ in terms of the remaining variables. In addition, the $[0\alpha]$ field equations serve to define q_α.

Let us assume the equations of state are given by (A1). The Einstein field equations M(113)–(115), the Jacobi identities M(116), (117), and the energy-momentum conservation equations M(118), (119) constitute a dynamical system in the remaining physical variables (which we shall denote as the DS). The DS is invariant under the transformation

$$\theta\to\lambda\theta,\quad a_\alpha\to\lambda a_\alpha,\quad p\to\lambda^2 p,\quad t\to\lambda^{-1}t,$$
$$\sigma_{\alpha\beta}\to\lambda\sigma_{\alpha\beta},\quad n_{\alpha\beta}\to\lambda n_{\alpha\beta},\quad \pi_{\alpha\beta}\to\lambda^2\pi_{\alpha\beta}, \tag{A3}$$

and this invariance implies that there exists a symmetry in the DS.[23] With the following change of variables

$$\Sigma_{\alpha\beta}=\frac{\sigma_{\alpha\beta}}{\theta},\quad A_\alpha=\frac{a_\alpha}{\theta},\quad N_{\alpha\beta}=\frac{n_{\alpha\beta}}{\theta},$$
$$\Theta=\ln\theta,\quad \frac{dt}{d\tau}=\frac{1}{\theta}, \tag{A4}$$

the new evolution equations for $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$, and A_α become independent of the variable Θ. Thus the DS can be considered as a reduced dynamical system for $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$, and A_α coupled to the evolution equation for Θ.

In order for the pressure p and anisotropic stress $\pi_{\alpha\beta}$ to satisfy the conditions that $p\rightarrow\lambda^2 p$ and $\pi_{\alpha\beta}\rightarrow\lambda^2\pi_{\alpha\beta}$ in Eq. (A3), the equations of state for p and $\pi_{\alpha\beta}$ must be homogeneous functions of degree two; that is,

$$p(\lambda\theta,\lambda\sigma_{\alpha\beta},\lambda n_{\alpha\beta},\lambda a_\alpha)=\lambda^2 p(\theta,\sigma_{\alpha\beta},n_{\alpha\beta},a_\alpha), \tag{A5}$$

$$\pi_{\alpha\beta}(\lambda\theta,\lambda\sigma_{\alpha\beta},\lambda n_{\alpha\beta},\lambda a_\alpha)=\lambda^2\pi_{\alpha\beta}(\theta,\sigma_{\alpha\beta},n_{\alpha\beta},a_\alpha). \tag{A6}$$

At the singular points of the reduced dynamical system $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$, and A_α are constant and consequently the equation

$$\frac{\dot{\theta}}{\theta^2}=-\frac{1}{3}-\Sigma_{\alpha\beta}\Sigma^{\alpha\beta}-\frac{1}{2}(\rho\theta^{-2}+3P) \tag{A7}$$

may be integrated to yield $\theta=\theta_o t^{-1}$ provided the right-hand side of Eq. (A7) is nonzero, which is guaranteed if $\rho+3p\geqslant 0$. The remaining physical variables may then be integrated to yield $\sigma_{\alpha\beta}=(\sigma_{\alpha\beta})_o t^{-1}$, $n_{\alpha\beta}=(n_{\alpha\beta})_o t^{-1}$, and $a_\alpha=(a_\alpha)_o t^{-1}$, where the subscript '$o$' denotes a constant. These solutions imply that the commutation coefficients γ^c_{ab} are inverse functions of t. Therefore, using Hsu and Wainwright's theorem, the singular points of the reduced system represent transitively self-similar cosmological models. However, the singular points of the reduced dynamical system represents the asymptotic limits to the Einstein field equations.

Conversely, if it is assumed that the asymptotic limit points are self-similar, Hsu and Wainwright's theorem implies that the commutation functions γ^c_{ab} are inverse functions of t. Therefore the physical variables θ, $\sigma_{\alpha\beta}$, $n_{\alpha\beta}$, and a_α are also inverse functions of t. The field equations then imply that the pressure p and anisotropic stress $\pi_{\alpha\beta}$ are inverse square functions of t; that is, $p(t)=p_o t^{-2}$ and $\pi_{\alpha\beta}(t)=(\pi_{\alpha\beta})_o t^{-2}$. If the pressure p has an equation of state of the form $p=p(\theta,\sigma_{\alpha\beta},n_{\alpha\beta},a_\alpha)$, then

$$\begin{aligned} p(t)&=p(\theta(t),\ \sigma_{\alpha\beta}(t),\ n_{\alpha\beta}(t),\ a_\alpha(t)),\\ &=p((\theta)_o t^{-1},\ (\sigma_{\alpha\beta})_o t^{-1},\ (n_{\alpha\beta})_o t^{-1},\ (a_\alpha)_o t^{-1}). \end{aligned} \tag{A8}$$

Thus, it follows that

$$\begin{aligned} p(\lambda^{-1}t)&=p(\lambda((\theta)_o t^{-1}),\ \lambda((\sigma_{\alpha\beta})_o t^{-1}),\ \lambda((n_{\alpha\beta})_o t^{-1}),\ \lambda((a_\alpha)_o t^{-1})),\\ &=p(\lambda\theta,\ \lambda\sigma_{\alpha\beta},\lambda n_{\alpha\beta},\lambda a_\alpha). \end{aligned} \tag{A9}$$

But $p(\lambda^{-1}t)=\lambda^2 p_o t^{-2}=\lambda^2 p(t)$, thus the equation of state for p is of the form $p(\lambda\theta,\lambda\sigma_{\alpha\beta},\lambda n_{\alpha\beta},\lambda a_\alpha)=\lambda^2 p(\theta,\sigma_{\alpha\beta},n_{\alpha\beta},a_\alpha)$. The result is similar for $\pi_{\alpha\beta}$. Therefore, assuming that the space–time is transitively self-similar, the equations of state for the pressure p and anisotropic stress $\pi_{\alpha\beta}$ must be homogeneous functions of degree two in their arguments. □

The equations of state in the new dimensionless variables (A4) are of the form

$$P\equiv\frac{p}{\theta^2}=\frac{p(\theta,\sigma_{\alpha\beta},n_{\alpha\beta},a_\alpha)}{\theta^2}=P\left(\frac{\theta}{\theta},\frac{\sigma_{\alpha\beta}}{\theta},\frac{n_{\alpha\beta}}{\theta},\frac{a_\alpha}{\theta}\right)=P(\Sigma_{\alpha\beta},N_{\alpha\beta},A_\alpha), \tag{A10}$$

$$\Pi_{\alpha\beta}\equiv\frac{\pi_{\alpha\beta}}{\theta^2}=\Pi_{\alpha\beta}(\Sigma_{\alpha\beta},N_{\alpha\beta},A_\alpha). \tag{A11}$$

We note that $P \equiv p\theta^{-2}$ and $\Pi_{\alpha\beta} \equiv \pi_{\alpha\beta}\theta^{-2}$ are dimensionless and we shall call the corresponding equations of state (A10), (A11) 'dimensionless' equations of state.[5] We also note from Eq. (A2) that $x \equiv 3\rho\theta^{-2}$ is a function of $\Sigma_{\alpha\beta}$, $N_{\alpha\beta}$, and A_α. A discussion of when the self-similarity of the asymptotic limits of the Einstein field equations is broken is given in Ref. 19.

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