AIP | Journal of Mathematical Physics

Space-times admitting special affine conformal vectors

A. A. Coley and B. O. J. Tupper

Citation: Journal of Mathematical Physics **31**, 649 (1990); doi: 10.1063/1.528899
View online: http://dx.doi.org/10.1063/1.528899
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/31/3?ver=pdfcov
Published by the AIP Publishing

Space-times admitting special affine conformal vectors

A. A. Coley
Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia B3H 3J5, Canada

B. O. J. Tupper
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick E3B 5A3, Canada

(Received 21 August 1989; accepted for publication 18 October 1989)

Space-times admitting a special affine conformal vector (SACV) are shown to be precisely the space-times that admit a special conformal Killing vector. All possible SACV space-times are listed together with the corresponding SACV's and covariantly constant tensors.

I. INTRODUCTION

In general relativity the existence of certain symmetries in the space-time manifold is often assumed in the pursuit of exact solutions of Einstein's field equations, and such symmetries and their corresponding vector fields have been studied by many authors (see Ref. 1 for references). The Lie derivative of the metric tensor, g_{ab}, in the direction of the vector field ξ^a can be written in the general form

$$\mathop{\mathcal{L}}_{\xi} g_{ab} = 2\psi g_{ab} + K_{ab}, \tag{1}$$

where ψ is a scalar function of the coordinates and K_{ab} is a symmetric tensor. Among the various special symmetries generated by ξ^a are:

(i) Killing vector (KV): $\psi = 0$, $K_{ab} = 0$,
(ii) Homothetic vector (HV): $\psi = \text{const} \neq 0$, $K_{ab} = 0$,
(iii) Proper conformal Killing vector (CKV): $\psi_{,a} \neq 0$, $K_{ab} = 0$,
(iv) Special conformal Killing vector (SCKV): $\psi_{;ab} = 0$, $\psi_{,a} \neq 0$, $K_{ab} = 0$,
(v) Affine vector (AV): $\psi = 0$, $K_{ab} \neq 0$, $K_{ab;c} = 0$,
(vi) Proper affine conformal vector (ACV): $\psi_{,a} \neq 0$, $K_{ab} \neq 0$, $K_{ab;c} = 0$,
(vii) Special affine conformal vector (SACV): $\psi_{;ab} = 0$, $\psi_{,a} \neq 0$, $K_{ab} \neq 0$, $K_{ab;c} = 0$,

where, in (v), (vi), and (vii), K_{ab} is not proportional to g_{ab}. An ACV or a CKV [which can be regarded as a special case of (vi) in which K_{ab} is proportional to g_{ab}] generates a conformal collineation characterized by

$$\mathop{\mathcal{L}}_{\xi} \begin{Bmatrix} a \\ bc \end{Bmatrix} = \delta^a_b \psi_{,c} + \delta^a_c \psi_{,b} - g_{bc}\psi^{a}. \tag{2}$$

An SACV or an SCKV generates a Ricci collineation characterized by

$$\mathop{\mathcal{L}}_{\xi} R_{ab} = 0. \tag{3}$$

Recently, we have investigated SCKV's[1] and have found all space-times admitting SCKV's. The existence of the covariantly constant vector $\psi_{,a}$ and the SCKV equations given by (1) and (iv) imply that there are very few SCKV space-times, none of which can represent a perfect fluid, a non-null electromagnetic field, or a vacuum space-time other than a *pp*-wave space-time. However, such space-times can represent viscous fluids and a set of anisotropic fluids satisfying particularly restrictive equations of state. A thorough study of AV's has been made by Hall and da Costa,[2] while ACV's have been discussed by Duggal[3], and SACV's have been studied by Duggal and Sharma[4] and Mason and Maartens.[5]

In this paper we discuss SACV's and show that the only space-times admitting SACV's are precisely those which admit SCKV's. We display the form of the SACV for each space-time and give the associated covariantly constant tensor K_{ab}.

II. SACV SPACE-TIMES

We first note a result due to Hall and da Costa,[2,6] namely if a simply connected space-time admits a global, nowhere zero, covariantly constant, second-order symmetric tensor, K_{ab}, which is not a constant multiple of the metric tensor, then one of the following three possibilities occur:

(a) There exists locally a timelike or spacelike nowhere zero covariantly constant vector field $\eta_a \equiv \eta_{,a}$ such that $K_{ab} = \eta_{,a}\eta_{,b}$ and the space-time decomposes into a $1+3$ space-time, in the notation of Ref. 2.

(b) There exists locally a null nowhere zero covariantly constant vector field $\eta_a \equiv \eta_{,a}$ such that $K_{ab} = \eta_{,a}\eta_{,b}$ and the space-time is the generalized *pp*-wave space-time[1,7] which, in general, is not decomposable.

(c) The space-time is locally decomposable into a $2+2$ space-time and no covariantly constant vector exists unless the space-time decomposes further.

We note that, if a $2+2$ space-time does admit a covariantly constant vector, then it must locally decompose further into a $1+1+2$ space-time. This follows immediately from holonomy considerations[6] or is shown easily by writing the general $2+2$ space-time metric in the form

$$ds^2 = e^{2\mu}(-dt^2 + dx^2) + e^{2\nu}(dy^2 + dz^2), \tag{4}$$

where $\mu = \mu(t,x)$ and $\nu = \nu(y,z)$, and solving the equations, $\eta_{a;b} = 0$. It is found that $\eta_a = 0$ unless one (or both) of the following conditions hold:

$$\mu_{tt} - \mu_{xx} = 0, \quad \nu_{yy} + \nu_{zz} = 0. \tag{5}$$

The first of these implies that the first two-space is flat (i.e., decomposes into a $1+1$ space), while the second condition implies that the second two-space is flat. Since the SACV

 0022-2488/90/030649-04$03.00

space-times admit the covariantly constant vector $\psi_{,a}$, it follows that we need consider only cases (a) and (b), the $1+1+2$ decomposition being treated as a special case of the $1+3$ decomposition.

In cases (a) and (b) Eq. (1) becomes

$$\xi_{a;b} + \xi_{b;a} = 2\psi g_{ab} + \eta_{,a}\eta_{,b}, \qquad (6)$$

which can be rewritten in the form

$$(\xi_a - \tfrac{1}{2}\eta\eta_{,a})_{;b} + (\xi_b - \tfrac{1}{2}\eta\eta_{,b})_{;a} = 2\psi g_{ab}, \qquad (7)$$

so that $\zeta_a \equiv \xi_a - \frac{1}{2}\eta\eta_{,a}$ is a CKV (which will be an SCKV if $\psi_{;ab} = 0$ or an HV if $\psi_{,a} = 0$), and $\tau_a \equiv \xi_a - \zeta_a = \frac{1}{2}\eta\eta_{,a}$ is an AV. Thus, in cases (a) and (b), the ACV is necessarily the sum of a CKV and an AV, so that space-times admitting ACV's must also admit a CKV and an AV.

Applying this result to SACV's, we see that space-times admitting SACV's are precisely those which admit SCKV's and which were found in Ref. 1. On the other hand, given an SCKV space-time satisfying

$$\zeta_{a;b} + \zeta_{b;a} = 2\psi g_{ab} \qquad (8)$$

with $\psi_{;ab} = 0$, we see that the vector $\chi_a = \frac{1}{2}\psi\psi_{,a}$ is an AV since

$$\mathop{\mathscr{L}}_{\chi} g_{ab} \equiv \chi_{a;b} + \chi_{b;a} = \psi_{,a}\psi_{,b} \qquad (9)$$

so that $\xi_a = \zeta_a + \chi_a$ satisfies

$$\mathop{\mathscr{L}}_{\xi} g_{ab} = 2\psi g_{ab} + \psi_{,a}\psi_{,b} \qquad (10)$$

and ξ_a is an SACV. Thus we have the following theorem.[8]

Theorem: *A simply connected space-time will admit an SACV if and only if it admits an SCKV.* The theorem asserts the complete equivalence between SACV and SCKV space-times.

From Ref. 1, when $\psi_{,a}$ is timelike we can choose local coordinates such that $\psi_{,a} = (-1,0,0,0)$ and the space-time metric is

$$ds^2 = -dt^2 + dx^2 + x^2(dy^2 + f^2(y,z)dz^2). \qquad (11)$$

The SCKV is $\zeta^a = (-\frac{1}{2}t^2 - \frac{1}{2}x^2, -tx,0,0)$ and, since $\psi_{,a}$ is, up to constant scalings, the only covariantly constant vector, we have $K_{ab} = \psi_{,a}\psi_{,b}$ so that $K_{00} = 1$ is the only nonzero component. It follows that the AV, τ^a, and the SACV, ξ^a, have components

$$\tau^a = (\tfrac{1}{2}t,0,0,0), \qquad (12)$$

$$\xi^a = (-\tfrac{1}{2}t^2 + \tfrac{1}{2}t - \tfrac{1}{2}x^2, -tx,0,0). \qquad (13)$$

When $\psi_{,a}$ is spacelike, the space-time metric takes one of the following possible forms:

$$ds^2 = dx^2 - dt^2 + t^2(dy^2 + g^2(y,z)dz^2), \qquad (14)$$

$$ds^2 = dx^2 + dy^2 + y^2(-dt^2 + h^2(t,z)dz^2). \qquad (15)$$

In each case $\psi_{,a} = (0,1,0,0)$, which is the only covariantly constant vector, and $K_{ab} = \psi_{,a}\psi_{,b}$ has $K_{11} = 1$ as its only nonzero component, so that the AV is

$$\tau^a = (0,\tfrac{1}{2}x,0,0). \qquad (16)$$

For the metric (14) the SCKV and SACV are, respectively,

$$\zeta^a = (xt,\tfrac{1}{2}x^2 + \tfrac{1}{2}t^2,0,0), \qquad (17)$$

$$\xi^a = (xt,\tfrac{1}{2}x^2 + \tfrac{1}{2}x + \tfrac{1}{2}t^2,0,0), \qquad (18)$$

while for the metric (15), the corresponding quantities are

$$\zeta^a = (0,\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2,xy,0), \qquad (19)$$

$$\xi^a = (0,\tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{1}{2}y^2,xy,0). \qquad (20)$$

Note that the space-time metric (15) does not satisfy the dominant energy condition and so has no reasonable physical interpretation.

When $\psi_{,a}$ is null, the space-time must be the generalized *pp* wave space-time with metric

$$ds^2 = P^{-2}(dx^2 + dy^2) - 2du(dv - mdx + Hdu), \qquad (21)$$

where H, m, and P are arbitrary functions of u, x, and y only. Not all space-times of the form (21) admit an SCKV, but in the case when the Ricci scalar, R, is not zero, if an SCKV exists it is of the form

$$\begin{aligned}\zeta^a = [&-(u^2+\alpha u+\beta), \alpha v - D(u,x,y)\\ &+ (2H + m^2P^2)(u^2+\alpha u+\beta)\\ &+ mP^2B(u,x,y), mP^2(u^2+\alpha u+\beta)\\ &+ P^2B(u,x,y), P^2C(u,x,y)],\end{aligned} \qquad (22)$$

where α, β are arbitrary constants and B, C, and D satisfy a set of six first-order differential equations [Eqs. (4.23) of Ref. 1] which serves to delineate those members of the set of space-times (21) which admit an SCKV.

When $R = 0$, the space-time metric can be written in the form

$$ds^2 = dx^2 + dy^2 - 2du(dv + H\,du), \qquad (23)$$

and such a space-time will admit the SCKV

$$\begin{aligned}\zeta^a = [&-(u^2+\alpha u+\beta), \alpha v - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + J_u x\\ &+ K_u y + L(u),\\ &-ux + \gamma y + J(u), -uy - \gamma x + K(u)],\end{aligned} \qquad (24)$$

provided that the metric function H satisfies the equation

$$\begin{aligned}&H_u(u^2+\alpha u+\beta) + H_x(ux - \gamma y - J)\\ &+ H_y(uy + \gamma x - K)\\ &+ 2H(u+\alpha) - J_{uu}x - K_{uu}y + L_u = 0,\end{aligned} \qquad (25)$$

where α, β, and γ are arbitrary constants and J, K, and L are arbitrary functions of u only.

In each of the above cases, the null vector $\psi_{,a} = (-1,0,0,0)$ is the only covariantly constant vector (see the Appendix), and $K_{ab} = \psi_{,a}\psi_{,b}$ has only $K_{00} = 1$ as a nonzero component. The AV and SACV are, respectively,

$$\tau^a = (0, -\tfrac{1}{2}u,0,0), \qquad (26)$$

$$\xi^a = \zeta^a + \tau^a, \qquad (27)$$

with ζ^a given by either (22) or (24).

The equivalence of SACV and SCKV space-times and the form of K_{ab} can be demonstrated also by using a coordinate dependent approach similar to that used in Ref. 1 (see MacLean[9]). In addition, the various results obtained in Ref. 1 are applicable in the SACV case. In particular (Ref. 1, Theorem 7), the energy-momentum tensors for the space-times (11) and (14) are of Segré type {(1,1) (1 1)}, while the space-times (21) and (23) are either of this type or of type {2(1 1)}, where, in either case, the bracketed pair of space-like vectors have zero eigenvalue (see also Hall[10]). This implies that the existence of an SCKV or, equivalently, an SACV, eliminates all perfect fluid space-times and all non-null electrovac fields. The only vacuum space-times are the *pp*-wave solutions, and the only null electrovac fields are the conformally flat *pp*-wave type. On the other hand, the space-times can be interpreted as viscous fluid solutions or, if $R \neq 0$, as anisotropic fluid solutions subject to the restriction

$$\mu = -p_{\parallel} = \tfrac{1}{2}R, \quad p_{\perp} = 0, \tag{28}$$

where μ is the energy density, and $p_{\parallel}$ and $p_{\perp}$ denote the parallel and perpendicular pressures, respectively. Equation (28) indicates that the various subcases considered by Mason and Maartens[5] are not, in fact, possible.

III. CONCLUSION

We have found all space-times that admit an SACV [i.e., space-times (11), (14), (15), and the appropriate members of (21) and (23)]; these are identical to the set of space-times admitting an SCKV and, consequently, have only a limited number of possible physical interpretations.

Duggal and Sharma[4] have considered space-times admitting SACV's (which they refer to as "special conformal collineations"), particularly those representing anisotropic fluids, subject to the condition

$$K_{ab} = \gamma R_{ab}, \tag{29}$$

where γ is a scalar function. This condition implies that the space-time is Ricci recurrent. However, on calculating the Ricci tensor in the cases of the space-times (11), (14), (15), and (21), it is easy to see that $K_{ab} = \psi_{,a}\psi_{,b}$ is never proportional to R_{ab}. In fact, (29) cannot be satisfied even if we take K_{ab} to be of the form $K_{ab} = \psi_{,a}\psi_{,b} + Cg_{ab}$, for some constant C. In the case of the metric (23), the only nonzero component of K_{ab} is $K_{00} = 1$ and the only nonzero component of R_{ab} is R_{00}, so Eq. (29) does hold. Since $R = 0$, the metric (23) admits no anisotropic fluid solutions—only viscous fluid, null electrovac, and pure radiation solutions—and the energy-momentum tensor is of the form $T_{ab} = \lambda\psi_{,a}\psi_{,b}$. Thus many of the results concerning anisotropic and isotropic fluids presented in Ref. 4 are illusory since there exist virtually no SACV space-times satisfying the physical interpretations and mathematical conditions considered in that article.

ACKNOWLEDGMENTS

We are grateful to Graham Hall for preprints of his work and for discussions and communications which enabled us to better understand the geometrical aspects of the problem.

This research was supported in part by the Natural Sciences and Engineering Research Council of Canada through operating grants to each author.

APPENDIX

The statement that each of the SACV space-times admits only one covariantly constant vector, namely $\psi_{,a}$, is easily proved in the cases of the space-times (11), (14), (15), and (23) by solving the equations $m_{a;b} = 0$ and showing that $m_a = \psi_{,a}$ is the only solution. However, in the case of the space-time (21) the proof is rather less simple; here we present an outline of the proof.

We consider the metric (21) and specify that $R \neq 0$ [otherwise we have the metric (23)]. The covariantly constant vector $\psi_{,a} \equiv k_a$ is of the form $k^a = (0,1,0,0)$. Suppose there exists another vector m_a satisfying $m_{a;b} = 0$. From the integrability conditions we find that

$$R_{ab}m^b = 0, \tag{A1}$$

and since the only nonzero components of R_{ab} are R_{00}, R_{02}, R_{03}, and $R_{22} = R_{33} = \frac{1}{2}P^{-2}R$, these conditions are

$$\begin{aligned} R_{00}m^0 + R_{02}m^2 + R_{03}m^3 &= 0, \\ R_{20}m^0 + R_{22}m^2 \qquad\qquad &= 0, \\ R_{30}m^0 \qquad\qquad + R_{33}m^3 &= 0. \end{aligned} \tag{A2}$$

Now m^0, m^2, and m^3 cannot all be zero, otherwise m^a will be a constant multiple of k^a, so that the determinant of the system (A2) must vanish, i.e.,

$$RR_{00} = 2P^2(R_{02}^2 + R_{03}^2). \tag{A3}$$

But, from Ref. 1, this is precisely the condition for the metric (21) to admit a T_{ab} of Segré type {(1,1) (1 1)}, which implies that there are two null eigenvectors, k^a and l^a, such that $k^a l_a = 1$, each corresponding to the same eigenvalue, and T_{ab} is given by[1]

$$T_{ab} = -\tfrac{1}{2}R(k_a l_b + k_b l_a), \tag{A4}$$

so that $T_{ab}k^b = -\frac{1}{2}Rk_a$ and $T_{ab}l^b = -\frac{1}{2}Rl_a$. Now from (A1) we see that m_a also satisfies $T_{ab}m^b = -\frac{1}{2}Rm_a$, so that m_a lies in the two-space spanned by k_a and l_a and, without loss of generality, we may take $m_a \equiv l_a$. Defining a timelike unit vector, u_a, and a spacelike unit vector, n_a, by $u_a = (1/\sqrt{2})(k_a + l_a)$, $n_a = (1/\sqrt{2})(k_a - l_a)$, so that $u_a n^a = 0$, we see that u_a and n_a are each covariantly constant and coordinates can be chosen so that the space-time can be written in the form

$$ds^2 = -dt^2 + dv^2 + p_{AB}(x^C)dx^A dx^B,$$

where $(A,B) = (2,3)$, and a further coordinate transformation leads to the form

$$ds^2 = -du^2 - 2\,du\,dv + P^{-2}(x,y)(dx^2 + dy^2), \tag{A5}$$

which is the metric (21) with $m = 0$, $H = 1/2$, $P_u = 0$.

We now have to determine whether or not this metric admits an SACV or SCKV. Applying Eqs. (4.23) of Ref. 1 to the metric (A5), we find that these equations can be satisfied only if

$$P_x^2 + P_y^2 - PP_{xx} - PP_{yy} = 0,$$

which is precisely the condition $R = 0$, thus contradicting our initial premise. Hence, none of the space-times admitting SACV's or SCKV's admit any covariantly constant vector other than $\psi_{,a}$.

[1]A. A. Coley and B. O. J. Tupper, J. Math. Phys. **30**, 2616 (1989).
[2]G. S. Hall and J. da Costa, J. Math. Phys. **29**, 2465 (1988).
[3]K. L. Duggal, J. Math. Phys. **30**, 1316 (1989).
[4]K. L. Duggal and R. Sharma, J. Math. Phys. **27**, 2511 (1986).
[5]D. P. Mason and R. Maartens, J. Math. Phys. **28**, 2182 (1987).
[6]G. S. Hall, Gen. Relativ. Gravit., to appear.
[7]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge U.P., Cambridge, 1980).
[8]G. S. Hall (private communication).
[9]M. MacLean, in *Proceedings of the Third Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by A. Coley, F. Cooperstock, and B. Tupper (World Scientific, Singapore, 1989).
[10]G. S. Hall, Gen. Relativ. Gravit., to appear.