

**Averaging in spherically symmetric cosmology**

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The averaging problem in cosmology is of fundamental importance. When applied to study cosmological evolution, the theory of macroscopic gravity (MG) can be regarded as a long-distance modification of general relativity. In the MG approach to the averaging problem in cosmology, the Einstein field equations on cosmological scales are modified by appropriate gravitational correlation terms. We study the averaging problem within the class of spherically symmetric cosmological models. That is, we shall take the microscopic equations and effect the averaging procedure to determine the precise form of the correlation tensor in this case. In particular, by working in volume-preserving coordinates, we calculate the form of the correlation tensor under some reasonable assumptions on the form for the inhomogeneous gravitational field and matter distribution. We find that the correlation tensor in a Friedmann-Lemaître-Robertson-Walker (FLRW) background must be of the form of a spatial curvature. Inhomogeneities and spatial averaging, through this spatial curvature correction term, can have a very significant dynamical effect on the dynamics of the Universe and cosmological observations; in particular, we discuss whether spatial averaging might lead to a more conservative explanation of the observed acceleration of the Universe (without the introduction of exotic dark matter fields). We also find that the correlation tensor for a non-FLRW background can be interpreted as the sum of a spatial curvature and an anisotropic fluid. This may lead to interesting effects of averaging on astrophysical scales. We also discuss the results of averaging an inhomogeneous Lemaître-Tolman-Bondi solution as well as calculations of linear perturbations (that is, the backreaction) in an FLRW background, which support the main conclusions of the analysis.

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**I. INTRODUCTION**

The Universe is not isotropic or spatially homogeneous on local scales. The correct governing equations on cosmological scales are obtained by averaging the Einstein equations of general relativity (GR). An averaging of inhomogeneous spacetimes can lead to dynamical behavior different from the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model [1]; in particular, the expansion rate may be significantly affected [2]. Consequently, a solution of the averaging problem is of considerable importance for the correct interpretation of cosmological data. Unfortunately, this is a very difficult problem.

There are a number of theoretical approaches to the averaging problem [2–8]. In the approach of Buchert [6], a  $3 + 1$  cosmological spacetime splitting that depends on the spacetime foliation is utilized and, in addition, only scalar quantities are averaged (and hence, in general, the equations are not closed and consequently additional assumptions are necessary). The perturbative approach [2,8] involves averaging the perturbed Einstein equations; however, a perturbation analysis cannot provide detailed information about the averaged geometry.

In all of these approaches, an averaging of the Einstein equations is performed to obtain the averaged field equations. The macroscopic gravity (MG) approach to the averaging problem in cosmology is an attempt to [3] give an exact (and tensorial) prescription for the correlation functions which inevitably emerges in an averaging of

the field equations (without which the averaging simply amounts to definitions of the new averaged terms). In MG the Einstein equations on cosmological scales with a continuous distribution of cosmological matter are modified by appropriate gravitational correlation (correction) terms. The MG approach provides a covariant method of averaging tensors; consequently, it does not rely on assumptions regarding the nature of perturbations (e.g., in principle there are no approximations and no higher order terms are dropped). We shall adopt the MG averaging approach in this paper.

The spacetime averaging procedure adopted in MG is based on the concept of Lie dragging of averaging regions,<sup>1</sup> which makes it valid for any differentiable manifold with a volume  $n$ -form, and it has been proven to exist on an arbitrary Riemannian spacetime with well-defined local averaged properties [3]. Averaging of the structure equations for the geometry of GR leads to the structure equations for the averaged (macroscopic) geometry and the definitions and the properties of the correlation tensor. The averaged Einstein equations for the macroscopic metric tensor together with a set of algebraic and differential equations for the correlation tensors become a coupled system of the macroscopic field equations for the unknown macroscopic metric, correlation tensor, and other objects

<sup>1</sup>In this paper, averaging is performed in one coordinate patch of a volume-preserving coordinate system and so the issue of Lie dragging of averaging regions does not apply.

of the theory. The averaged Einstein equations can always be written in the form of the Einstein equations for the macroscopic metric tensor when the correlation terms are moved to the right-hand side of the averaged Einstein equations, and consequently can be regarded as a geometric modification to the averaged (macroscopic) matter energy-momentum tensor [3].

MG is a nonperturbative geometric field theory with a built-in scale related to the spatial scale over which averages are taken (we recall that, in principle, averaging is performed over a 4-volume region). The microscopic field to be averaged is supposed to have two essentially different variation scales,  $\lambda$  and  $L_H$ , satisfying  $\lambda \ll L_H$ , where  $L_H$  is the horizon size related to the inverse Hubble scale. An averaging region must be taken of an intermediate size  $L$  such as  $\lambda \ll L \ll L_H$ , so that the averaging effectively smooths out all the variations of the microscopic field of the scale  $\lambda$ . In cosmological applications  $\lambda$  is taken to be the scale on which astrophysical objects such as galaxies or clusters of galaxies have structure, and the size of the averaging space regions has been tacitly assumed to be  $\approx 100$  Mpc, or a fraction of the order of the inverse Hubble scale, and thus any terms (e.g., a cosmological constant or a curvature term) appearing in the correlation tensor might be expected to be related to the inverse Hubble scale. In principle the scale, given by the size of the spacetime averaging region, is a free parameter of the theory.

A procedure for solving the MG equations with one connection correlation tensor was discussed in [9]. The macroscopic field equations were written in the form of the Einstein equations of GR, with a “modified” stress-energy tensor consisting of the averaged microscopic stress-energy tensor  $\langle \mathbf{t}^{(\text{micro})} \rangle$  and an additional effective stress-energy tensor  $\mathbf{C}$  arising from the correlation tensor  $\mathbf{Z}$  [3]. In [9] it was found that the averaged Einstein equations for a spatially homogeneous, isotropic macroscopic spacetime geometry have the form of the Einstein equations of GR for a spatially homogeneous, isotropic spacetime geometry with an additional spatial curvature term (i.e., the correlation tensor  $\mathbf{C}$  is of the form of a spatial curvature term).

Therefore, assuming spatial homogeneity and isotropy on largest scales, then the inhomogeneities affect the dynamics through correction terms (the correlation tensor) of the form of a curvature term [9], which will dominate at late times and on largest scales. Thus even for FLRW backgrounds it is important to understand how these correction terms affect cosmological observations. For example, a spatially averaged metric is not a local physical observable: the averaged value of the expansion will not be the same as the expansion rate of the averaged geometry, because of the nonlinear nature of the expansion.

The spacetime averaging in MG utilizes bilocal averaging operators. The MG averaging scheme is especially simple in a proper coordinate system [4], in which the bilocal operators take on the simplest possible forms. In

particular, any proper coordinate system is necessarily a volume-preserving (system of) coordinates (VPC), and in a pseudo-Riemannian spacetime the spacetime averages defined in proper coordinates are Lorentz tensors exactly like the averages in Minkowski spacetime; that is, VPC on an arbitrary differentiable metric manifold is a natural counterpart of the Cartesian coordinate system on a Minkowski manifold. A brief review of the spacetime averaging scheme adopted in macroscopic gravity and the role of proper systems of coordinates is presented in Appendix A.

Spherically symmetric cosmological models are of special cosmological importance, partially motivated by the observed isotropy of the CMB. Therefore, it is important to study the averaging problem comprehensively within the class of spherically symmetric cosmological models (i.e., to determine the form of the MG equations in the case of spherical symmetry). We shall take the microscopic equations and effect the averaging procedure to determine the precise form of the correlation tensor in this case.

In the next section we shall calculate the form of the MG equations in the case of spherical symmetry. The first step is to choose an appropriate spherically symmetric VPC system. It is also instructive to investigate the FLRW metric in VPC. We then make some reasonable assumptions on the form of the inhomogeneous gravitational field and matter distribution, and in Sec. III we calculate the resulting form of the correlation tensor in both a FLRW and non-FLRW background.

In Sec. IV we average an inhomogeneous Lemaître-Tolman-Bondi (LTB) solution. The first step in this calculation is to rewrite the LTB dust solution in VPC (which is done in Appendix B). This solution is presented as a perturbation about a flat FLRW model. In Sec. V we then assume a spatially homogeneous and isotropic background and discuss the effect of perturbations (that is, the back-reaction) on this FLRW background.

In Sec. VI we discuss the results obtained in light of recent observations, with particular emphasis on the effect of inhomogeneities on the local expansion rate. We discuss whether inhomogeneities and spatial averaging might lead to a more conservative explanation of the observed acceleration of the Universe (without the introduction of exotic dark matter fields). The conclusions are given in the final section. In Appendix C we briefly discuss the relationship between our work and the work of Buchert [6] in the case of spherical symmetry.

## II. SPHERICAL SYMMETRY

We shall calculate the form of the MG equations in the case of spherical symmetry. That is, we shall take the microscopic equations and effect the averaging procedure to determine the precise form of the correlation tensor,  $C^a_b$ , in this case.

We begin by choosing an appropriate coordinate system. Starting from the general form of the spherically symmetric metric, we first choose a new angular coordinate,  $u = \cos(\theta)$ , to eliminate any angular dependence in  $\sqrt{-g}$ , where  $g = \det(g_{ab})$ . Next, we use the remaining coordinate freedom to set  $\sqrt{-g} = 1$ ; this is done by choosing an appropriate form for the ‘‘radial’’ metric function that multiplies the spherical line element  $ds^2(u, \phi)$ . The line element is thus<sup>2</sup>

$$ds^2 = -Bdt^2 + Adr^2 + \frac{du^2}{\sqrt{AB}(1-u^2)} + \frac{1-u^2}{\sqrt{AB}}d\phi^2, \quad (1)$$

where the functions  $A$  and  $B$  depend on  $t$  and  $r$ . These are volume-preserving coordinates (VPC) for the spherically symmetric metric [4]. It is the adoption of VPC that enables us to calculate the averaged quantities in a relatively straightforward manner.

We next calculate [10] the form of the Einstein tensor  $G^a_b$  (note the position of indices)

$$G^t_t = \frac{5}{8} \frac{B_r A_r}{A^2 B} - \frac{1}{2} \frac{B_{rr}}{AB} + \frac{3}{16} \frac{A_t^2}{A^2 B} + \frac{1}{8} \frac{A_t B_t}{AB^2} + \frac{11}{16} \frac{B_r^2}{AB^2} - \frac{1}{16} \frac{B_t^2}{B^3} - \frac{1}{2} \frac{A_{rr}}{A^2} - \sqrt{AB} + \frac{15}{16} \frac{A_r^2}{A^3} \quad (2)$$

$$G^r_t = -\frac{5}{8} \frac{B_r A_t}{A^2 B} - \frac{7}{8} \frac{B_r B_t}{AB^2} - \frac{7}{8} \frac{A_r A_t}{A^3} - \frac{1}{8} \frac{A_r B_t}{A^2 B} + \frac{1}{2} \frac{A_{rt}}{A^2} + \frac{1}{2} \frac{B_{rt}}{AB} \quad (3)$$

$$G^r_r = -\frac{5}{8} \frac{A_t B_t}{AB^2} + \frac{1}{2} \frac{A_{tt}}{AB} - \frac{3}{16} \frac{B_r^2}{AB^2} - \frac{1}{8} \frac{B_r A_r}{A^2 B} - \frac{11}{16} \frac{A_t^2}{A^2 B} + \frac{1}{16} \frac{A_r^2}{A^3} - \sqrt{AB} - \frac{15}{16} \frac{B_t^2}{B^3} + \frac{1}{2} \frac{B_{tt}}{B^2} \quad (4)$$

$$G^u_u = \frac{1}{8} \frac{A_t B_t}{AB^2} - \frac{7}{16} \frac{B_t^2}{B^3} - \frac{1}{4} \frac{A_{tt}}{AB} + \frac{1}{4} \frac{B_{tt}}{B^2} + \frac{1}{16} \frac{A_t^2}{A^2 B} - \frac{1}{8} \frac{B_r A_r}{A^2 B} - \frac{1}{16} \frac{B_r^2}{AB^2} + \frac{7}{16} \frac{A_r^2}{A^3} - \frac{1}{4} \frac{A_{rr}}{A^2} + \frac{1}{4} \frac{B_{rr}}{AB} \quad (5)$$

and  $G^t_r = -AB^{-1}G^r_t$ ,  $G^\phi_\phi = G^u_u$ . The subscripts  $r$  and  $t$  on the metric functions  $A$  and  $B$  denote partial differentiation with respect to  $r$  and  $t$ , respectively.

We note that all terms in the expressions for  $G^r_t$  and  $G^u_u$  originate from the basic metric functions  $g_{rr}$  and  $g_{tt}$  ( $A$  and

$B$ ). The terms of the form  $\sqrt{AB}$  in  $G^t_t$  and  $G^r_r$  arise from derived ‘‘radial’’ metric functions [e.g., the term  $\sqrt{AB}$  in Eq. (2) arises as a product of the metric components  $g_{tt}$ ,  $g_{rr}$ ,  $g_{uu}$  and  $g_{\phi\phi}$ ; also see the comment in Sec. III A 3]. In order to avoid unnecessary complications in the averaging procedure, we shall not deal with derived quantities directly.<sup>3</sup> Therefore, we eliminate the  $\sqrt{AB}$  terms by consid-

$$G^t_t - G^r_r = \frac{3}{4} \frac{B_r A_r}{A^2 B} + \frac{7}{8} \frac{B_t^2}{B^3} + \frac{7}{8} \frac{A_r^2}{A^3} + \frac{3}{4} \frac{A_t B_t}{AB^2} - \frac{1}{2} \frac{A_{rr}}{A^2} + \frac{7}{8} \frac{B_r^2}{AB^2} + \frac{7}{8} \frac{A_t^2}{A^2 B} - \frac{1}{2} \frac{B_{rr}}{AB} - \frac{1}{2} \frac{A_{tt}}{AB} - \frac{1}{2} \frac{B_{tt}}{B^2} \quad (6)$$

and use the contracted Bianchi identity to determine the remaining component of the Einstein tensor. Taking averages, we now obtain the appropriate form for the MG equations and hence the correlation tensor  $C^a_b$ . For example, we have that<sup>4</sup>

$$C^r_t \equiv G^r_t \langle g \rangle - \langle G^r_t \rangle = -\frac{5}{8} \left[ \frac{\langle B_r \rangle \langle A_t \rangle}{\langle A \rangle^2 \langle B \rangle} - \left\langle \frac{B_r A_t}{A^2 B} \right\rangle \right] - \frac{7}{8} \left[ \frac{\langle B_r \rangle \langle B_t \rangle}{\langle A \rangle \langle B \rangle^2} - \left\langle \frac{B_r B_t}{AB^2} \right\rangle \right] - \frac{7}{8} \left[ \frac{\langle A_r \rangle \langle A_t \rangle}{\langle A \rangle^3} - \left\langle \frac{A_r A_t}{A^3} \right\rangle \right] - \frac{1}{8} \left[ \frac{\langle A_r \rangle \langle B_t \rangle}{\langle A \rangle^2 \langle B \rangle} - \left\langle \frac{A_r B_t}{A^2 B} \right\rangle \right] + \frac{1}{2} \left[ \frac{\langle A_{rt} \rangle}{\langle A \rangle^2} - \left\langle \frac{A_{rt}}{A^2} \right\rangle \right] + \frac{1}{2} \left[ \frac{\langle B_{rt} \rangle}{\langle A \rangle \langle B \rangle} - \left\langle \frac{B_{rt}}{AB} \right\rangle \right]. \quad (7)$$

In the above, by virtue of VPC, the average is now a simple average defined by

$$\langle f(r, t) \rangle \equiv \frac{1}{TL} \int_{t'=-T/2}^{T/2} dt' \int_{r'=-L/2}^{L/2} dr' f(r+r', t+t'), \quad (8)$$

which, for smooth functions with a sufficiently slowly varying dependence on cosmological time, essentially reduces to a spatial average in terms of the averaging scale  $L$  (see the next section).

It is instructive to consider the FLRW metric in VPC. The metric is given by (1), with

$$A = \frac{R^2}{F^4}; \quad B = \frac{1}{R^6}, \quad (9)$$

where  $R = R(t)$  and  $F = F(r)$ , subject to  $\frac{dF}{dr} = \sqrt{1-kF^2}F^{-2}$ , and  $k = -1, 0$  or  $1$ . The Einstein tensor in these coordinates is given by

<sup>3</sup>This was pointed out to us by Zalaletdinov.

<sup>4</sup> $C^a_b$  is implicitly defined by this expression. The formal definition of the correlation tensor is given in [3].

<sup>2</sup>In general, these are not comoving coordinates.

$$G^a_b = \begin{bmatrix} -3R^4\dot{R}^2 - 3kR^{-2} & 0 & 0 & 0 \\ 0 & -7R^4\dot{R}^2 - 2R^5\ddot{R} - kR^{-2} & 0 & 0 \\ 0 & 0 & -7R^4\dot{R}^2 - 2R^5\ddot{R} - kR^{-2} & 0 \\ 0 & 0 & 0 & -7R^4\dot{R}^2 - 2R^5\ddot{R} - kR^{-2} \end{bmatrix}. \quad (10)$$

The spatial curvature term is given by  $G^a_b = -kR_0^{-2}\text{diag}[3, 1, 1, 1]$  (whereby spatial curvature term we mean the Einstein tensor corresponding to a spacetime with constant spatial curvature), with effective equation of state  $p_{\text{eff}} = -\frac{1}{3}\rho_{\text{eff}}$ .

### III. INHOMOGENEOUS SPACETIMES

The form of the correlation tensor now depends on the assumed form for the inhomogeneous gravitational field and matter distribution. Let us assume that

$$A(r, t) = \langle A(r, t) \rangle \left[ 1 + \sum_{n=1}^{\infty} a_n(t)L^n \sin\left(\frac{2n\pi}{L}r\right) + \sum_{n=1}^{\infty} \bar{a}_n(t)L^n \cos\left(\frac{2n\pi}{L}r\right) \right], \quad (11)$$

$$B(r, t) = \langle B(r, t) \rangle \left[ 1 + \sum_{n=1}^{\infty} b_n(t)L^n \sin\left(\frac{2n\pi}{L}r\right) + \sum_{n=1}^{\infty} \bar{b}_n(t)L^n \cos\left(\frac{2n\pi}{L}r\right) \right], \quad (12)$$

where  $r$  is a radial variable (and, strictly speaking, we are assuming that  $r \leq L$ ). The assumptions (11) and (12) constitute a spatial Fourier decomposition of the metric functions in which the variation in the timelike direction is assumed small and the dominant source of inhomogeneity arises from a spatial variation of the gravitational field (thus the 4-volume average effectively reduces, in this case, to a smoothing on a spatial domain). Note that the coordinates  $t$  and  $r$  appearing in (1) are not the usual “time” and “radial” coordinates; however, the unit magnitude timelike coordinate basis vector has zero vorticity, which implies the existence of a foliation of spacetime (where the  $r$  coordinate parameterizes the spatial hypersurfaces). Since the coordinate basis vectors  $\partial_t$  and  $\partial_r$  are independent (i.e., the metric is diagonal), it follows that variation along timelike and spatial directions is not coupled. Although other forms for the inhomogeneous gravitational field are possible [i.e., different assumptions to (11) and (12)], it is not expected that the main conclusions obtained in this paper will be qualitatively affected. In Eqs. (11) and (12),  $L \equiv L_0$  is treated as a parameter and in the calculations that follow  $L_0$  is effectively taken to be a small dimensionless parameter after a renormalization of the variables using the speed of light (set to unity) and the

present value of the Hubble parameter,  $H_0$ , and a redefinition of the functions  $a_n, b_n$ .

With these assumptions<sup>5</sup> we have essentially assumed that averages effectively become space averages with

$$\langle f(r) \rangle = \frac{1}{L} \int_{-L/2}^{L/2} f(r+r')dr'. \quad (13)$$

We note that integrating the left-hand side of the first of these equations (i.e., taking spatial averages) yields  $\langle A(r, t) \rangle$ . We also note that

$$\frac{\partial}{\partial r} A(r, t) = \frac{\partial}{\partial r} \langle A(r, t) \rangle \left( 1 + \sum_{n=1}^{\infty} a_n L^n \sin\left(\frac{2n\pi}{L}r\right) + \sum_{n=1}^{\infty} \bar{a}_n(t)L^n \cos\left(\frac{2n\pi}{L}r\right) \right) \quad (14)$$

$$+ \langle A \rangle \left\langle \sum_{n=1}^{n=\infty} 2n\pi a_n L^{n-1} \cos\left(\frac{2n\pi}{L}r\right) - \sum_{n=1}^{n=\infty} 2n\pi \bar{a}_n L^{n-1} \sin\left(\frac{2n\pi}{L}r\right) \right\rangle, \quad (15)$$

and

$$\left\langle \frac{\partial}{\partial r} \langle A(r, t) \rangle \right\rangle = \left\langle \frac{\partial}{\partial r} A(r, t) \right\rangle, \quad (16)$$

$$\left\langle \frac{\partial}{\partial t} \langle A(r, t) \rangle \right\rangle = \left\langle \frac{\partial}{\partial t} A(r, t) \right\rangle.$$

Thus the assumed forms for the inhomogeneous functions  $A$  and  $B$  satisfy a set of appropriate and self-consistent conditions.

We shall expand in powers of  $L \equiv L_0 < 1$ . Since the coefficients ( $a_1, a_2$ , for example) are evolving functions of time, these expansions may only be valid for a transient period of time. We calculate the correlation tensor  $C^a_b$  up to  $\mathcal{O}(L^2)$ :

<sup>5</sup>The calculations will depend on the assumed form for the inhomogeneities. The assumptions (11) and (12) are self-consistent and physically reasonable (at least at lowest orders); see Sec. III B 2 for further discussion.

$$\begin{aligned}
 C^r_t = 0 + \frac{\pi}{8\langle A \rangle} & \left[ 3(b_1 - a_1)\dot{\bar{b}}_1 + 3(\bar{a}_1 - \bar{b}_1)\dot{b}_1 \right. \\
 & + (5b_1 - a_1)\dot{a}_1 + (\bar{a}_1 - 5\bar{b}_1)\dot{\bar{a}}_1 \\
 & \left. + \left\{ (\bar{a}_1 b_1 - a_1 \bar{b}_1) \left( 3 \frac{\langle B \rangle_t}{\langle B \rangle} + 5 \frac{\langle A \rangle_t}{\langle A \rangle} \right) \right\} \right] L + \mathcal{O}(L^2)
 \end{aligned} \quad (17)$$

$$\begin{aligned}
 C^u_u = \frac{\pi^2}{8\langle A \rangle} & [(\bar{a}_1 - \bar{b}_1)^2 + (a_1 - b_1)^2 - 4(b_1^2 + \bar{b}_1^2)] \\
 & + \frac{\pi}{8\langle A \rangle} (\bar{a}_1 b_1 - a_1 \bar{b}_1) \left[ 3 \frac{\langle B \rangle_r}{\langle B \rangle} - \frac{\langle A \rangle_r}{\langle A \rangle} \right] L + \mathcal{O}(L^2)
 \end{aligned} \quad (18)$$

$$\begin{aligned}
 C^t_t - C^r_r = \frac{\pi^2}{4\langle A \rangle} & [(\bar{a}_1 - \bar{b}_1)^2 + (a_1 - b_1)^2 - 4(b_1^2 + \bar{b}_1^2)] \\
 & + \frac{3\pi}{4\langle A \rangle} (\bar{a}_1 b_1 - a_1 \bar{b}_1) \left[ \frac{\langle B \rangle_r}{\langle B \rangle} + \frac{\langle A \rangle_r}{\langle A \rangle} \right] L \\
 & + \mathcal{O}(L^2).
 \end{aligned} \quad (19)$$

The  $\mathcal{O}(L^2)$  terms have been calculated, but we have not explicitly displayed them here.

### A. Lowest order calculation

From Eqs. (7) and (14) we obtain, for example,  $C^r_t = 0 + \mathcal{O}(L)$ , and from Eqs. (5) and (6)

$$C^a_b = \begin{bmatrix} C + \frac{2\ell}{\langle A \rangle} & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & \frac{\ell}{\langle A \rangle} & 0 \\ 0 & 0 & 0 & \frac{\ell}{\langle A \rangle} \end{bmatrix} + \mathcal{O}(L), \quad (20)$$

where  $C \equiv C^r_r$  and

$$\ell(t) \equiv \frac{\pi^2}{8} [(a_1 - 3b_1)(a_1 + b_1) + (\bar{a}_1 - 3\bar{b}_1)(\bar{a}_1 + \bar{b}_1)].$$

We calculate  $C$  from the contracted Bianchi identities. We note that if  $C^a_b$  is isotropic (i.e., of the form of a perfect fluid) then  $C = \frac{\ell}{\langle A \rangle}$  and  $C^a_b$  is of the form of a spatial curvature term.

### 1. Bianchi identities

For the metric (1) and correlation tensor (20), to  $\mathcal{O}(L^0)$  the contracted Bianchi identities yield

$$C_r - \frac{1}{2} C \left( \frac{A_r}{A} + \frac{B_r}{B} \right) + \frac{1}{2} \frac{\ell}{\langle A \rangle} \left( \frac{A_r}{A} - \frac{B_r}{B} \right) = 0 \quad (21)$$

$$C_t + \frac{2\dot{\ell}}{\langle A \rangle} - \frac{1}{2} C \left( \frac{A_t}{A} + \frac{B_t}{B} \right) - \frac{1}{2} \frac{\ell}{\langle A \rangle} \left( \frac{A_t}{A} + \frac{B_t}{B} \right) = 0, \quad (22)$$

where  $C_r \equiv \frac{\partial C}{\partial r}$  and  $C_t \equiv \frac{\partial C}{\partial t}$ . The solution of these equations (for  $C$ ) depends on whether  $B_r$  is zero or not.

## 2. FLRW background

In the case that  $B_r = 0$ , as in the case of a FLRW background, Eq. (21) immediately yields  $C \equiv \frac{\ell}{\langle A \rangle}$  and  $\langle A \rangle_{,r} = 0$ , and Eq. (22) then yields  $\frac{\dot{\ell}}{\langle A \rangle} = \ell_0 R^{-2}$  (where  $\ell_0$  is a constant that depends on the spatial averaging scale). Therefore, in this case we obtain

$$C^a_b = \ell_0 R^{-2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (23)$$

and  $C^a_b$  is necessarily of the form of a spatial curvature term.

In order to fully reconcile (20) and (23), we note that since  $A = R^2 F^{-4}$  in the FLRW case, we obtain

$$\frac{\ell}{\langle A \rangle} = \frac{\ell \langle F^4 \rangle}{R^2} = \frac{\ell_0}{R^2} = \frac{-k}{R^2}. \quad (24)$$

Here  $\langle A \rangle$  must be interpreted as an averaged spatial curvature, so that  $\langle F^4(r) \rangle$  is replaced by  $\langle 2F^2 F_r^2 + F^3 F_{rr} \rangle$ , which is constant. That is, the term  $\langle A \rangle$  in the above equation must be interpreted correctly.

## 3. Non-FLRW background

If  $B_r \neq 0$ , then Eq. (21) can be integrated to obtain

$$C = -\frac{\ell}{\langle A \rangle} + f(t) \frac{(AB)^{1/2}}{A^{2\ell}} \quad (25)$$

[we note that, in general, this expression is different to what we would have obtained if we had averaged Eqs. (1), (11), and (12), with the  $\sqrt{AB}$  term, directly]. Equations (22) and (25) then yield

$$\dot{\ell} = - \left[ \frac{2f}{A^{2\ell}} \right]_{,t} A^{3/2} B^{1/2}.$$

Since  $A = A(r, t)$  ( $B_r \neq 0$ ), in general the solution of this equation yields  $\ell = \ell_0$  (constant) and  $f(t) = g(r)A^{2\ell}$ , so that if  $f \neq 0$ ,  $A(r, t)$  is separable. In the latter case, in general  $B(r, t)$  is also separable, and the two separate terms in (25) can be of a comparable form.

We note that  $C^a_b$  is necessarily anisotropic (and cannot be formally equivalent to a perfect fluid). We can always write

$$C^a_b = \frac{\ell_0}{\langle A \rangle} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \Pi \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (26)$$

where  $\Pi \equiv -\{g(r)\langle AB \rangle^{1/2} - \frac{2\ell_0}{\langle A \rangle}\}$ . The correlation tensor  $C^a_b$  automatically satisfies the contracted Bianchi identities (21) and (22). It can be interpreted as the sum of a perfect fluid and an anisotropic fluid (when  $B_r \neq 0$ ). For an anisotropic fluid in spherically symmetric coordinates the



energy-momentum tensor is of the form  $\text{diag}[-\mu, p_{\parallel}, p_{\perp}, p_{\perp}]$ , where  $p_{\parallel} = p + \frac{2}{3}\pi$  and  $p_{\perp} = p - \frac{1}{3}\pi$ , and  $\pi$  is the anisotropic pressure. From above, we see that, if the (total) correlation tensor  $C^a_b$  is interpreted as an anisotropic fluid, it follows that  $\Pi = -\pi$  and  $\mu = 3p$ .

If both terms separately satisfy the contracted Bianchi identity, then the first term can be interpreted as a spatial curvature term. The second term can be interpreted as an anisotropic fluid with  $p_{\perp} = 0$  and  $p_{\parallel} = -\rho_{\text{eff}}$  (which is similar to the equation of state for a cosmological constant). For  $f(t) = g(r)A^{2\ell}$  [ $g(r) \neq 0$ ] and  $A \equiv \bar{F}^{-4}(r)\bar{R}^2(t)$ , we obtain  $\rho_{\text{eff}} = 2\ell_0\bar{F}^4(r)R^{-2} = g(r)\bar{F}^{-2}\bar{R}(t)B^{1/2}$ , so that  $B$  is separable and of the form  $B = b(r)\bar{R}^{-6}(t)$  [compare with Eq. (9)]. One solution gives  $\rho_{\text{eff}} = 0$ , so that in this case  $C^a_b$  is of the form of a perfect fluid and is thus necessarily of the form of a spatial curvature term.

Although the correlation tensor  $C^a_b$  satisfies the contracted Bianchi identities, when interpreted as the sum of a spatial curvature perfect fluid and an anisotropic fluid through (26), the two separate fluid do not in general satisfy separate conservation equations. However, Eq. (22) can be rewritten in the form of a conservation law for the anisotropic pressure  $\Pi$ ,

$$\Pi_t - \frac{1}{2}\Pi\left(\frac{A_t}{A} + \frac{B_t}{B}\right) + \frac{\ell_0}{\langle A \rangle}\left(2\frac{A_t}{A} + \frac{B_t}{B}\right) = 0, \quad (27)$$

in VPC where the metric is given by Eq. (1) and  $u^a$  is comoving to order  $\mathcal{O}(L^2)$  [note that the expansion and shear are given by

$$\theta = -\frac{1}{2B^{1/2}}\left(\frac{B_t}{B}\right), \quad \sigma = \frac{1}{2\sqrt{6}B^{1/2}}\left(3\frac{A_t}{A} + \frac{B_t}{B}\right), \quad (28)$$

respectively; compare this with Eq. (C9) in Appendix C].

#### 4. Anisotropic fluid

The second term in the correlation tensor is of the form of an energy-momentum tensor for an anisotropic fluid, with energy density  $\mu$ , a pressure  $p_{\parallel}$  parallel to the radial unit normal, and a perpendicular pressure  $p_{\perp}$ .<sup>6</sup> Fluids with an anisotropic pressure have been studied in the cosmological context for a number of reasons: an energy-momentum tensor of this form arises formally if the source consists of two perfect fluids with distinct four-velocities, a heat conducting viscous fluid under some circumstances, a perfect fluid and a magnetic field, and in the presence of particle production [11–13]. In particular, the energy-

momentum tensor of a cosmic string [14] is of the form of an anisotropic fluid with  $\mu = -p_{\parallel}$ ,  $p_{\perp} = 0$  (such an equation of state also arises in other early universe applications). Anisotropic fluids in spherically symmetric cosmological models have been studied in [11,12,15]. In addition, the energy-momentum arising from the gravitational field of a global monopole is formally an anisotropic fluid which is static and spherically symmetric [14]. We also note that, in an investigation of the consequences of an imperfect dark energy component on the large scale structure, the effect of anisotropic perturbations (due to the dark energy) on the cosmic microwave background radiation was studied. It was found that an anisotropic stress is not excluded by the present day cosmological observations [16].

Let us comment on the astrophysical applications of an anisotropic fluid. It is known that dark matter is a major constituent of the halos of galaxies [17]. By an analysis of observed rotation curves, under reasonable assumptions (e.g., that galaxies can be modeled as spherically symmetric) it has been found that the dark matter is of the form of an anisotropic fluid [18]. This has been taken up in [19], in which the consequences of anisotropic dark matter stresses are discussed in the weak field gravitational lensing (where it was noted that in any attempt to model dark matter in galactic halos with classical fields will lead to anisotropic stresses comparable in magnitude with the energy density).

Finally, we note that for a 4-dimensional spacetime with a metric of the form  $g_{ab} = \text{diag}[-1, 1, (1 + cu^2)^{-1}, (1 + cu^2)]$  (in VPC), where  $c$  is a constant, we have that  $G^a_b = \text{diag}[c, c, 0, 0]$ . The metric is of the product form  $R^2 \times S^2$ , and is therefore the tensor product of a 2-dimensional flat space and (for  $c < 0$ ) a 2-dimensional sphere (which are two 2-dimensional spaces of constant curvature). Hence, the second term can also be interpreted in terms of spatial curvature (although we again note that each of the two terms, namely, the spatial curvature term and the anisotropic term, do not separately satisfy the contracted Bianchi identity).

## B. Further calculations

### 1. Higher order terms

We can consider the contribution of the higher order  $\mathcal{O}(L)$  and  $\mathcal{O}(L^2)$  terms. In the case of a FLRW background (with  $B_r = 0$ ,  $\frac{\ell}{\langle A \rangle} = \ell_0 R^{-2}$ , where  $\ell$  is constant), the  $\mathcal{O}(L)$  terms all vanish. We can see from Appendix B that this occurs trivially when  $(\bar{a}_1 b_1 - a_1 \bar{b}_1) = 0$ , which occurs for a sine series expansion (only) or a cosine series (only), or for a single trigonometric series with a composite argument (using the double angle formulas). The  $\mathcal{O}(L^2)$  contributions to  $C^r_t$  then vanish, and the  $\mathcal{O}(L^2)$  contributions to the remaining components of the correlation tensor are of the form of a spatial curvature term [the  $\mathcal{O}(L^2)$  terms are not displayed explicitly in the Appendix]. That is, the

<sup>6</sup>We emphasize that, although the correlation tensor has the form of an anisotropic fluid (with certain properties), it is not a material source but arises from the averaging procedure. For example, the conservation law (27) is simply a formal expression for the contracted Bianchi identities.

effect of the higher order terms is simply to renormalize the spatial curvature term.

In the non-FLRW background case, in general we must have  $(\bar{a}_1 b_1 - a_1 \bar{b}_1) = 0$ , and the higher order terms do not play any significant role (as above).

## 2. Discussion

In Eqs. (11) and (12),  $L$  is essentially treated as a dimensionless parameter, which is sufficient to the lowest order of approximation. In principle, in the cosmological setting  $L$  depends on the Hubble scale  $H^{-1}$  and might also be related to a scale dependent on structure formation, both of which vary with cosmological time. Therefore, in general,  $L$  will be time dependent.

To lowest order we assume that  $L = L_0$  is fixed and integration is taken over a comoving domain (and presently  $L_0 \sim 10^{-1}$ ). Assuming  $L$  is time dependent,  $L = L(t)$ , we have that a typical correction term is of the form

$$C_T \equiv \left\{ 1 - \frac{1}{c_n(t)L^n} \frac{1}{T} \int_{-T/2}^{T/2} c_n(t)L^n dt \right\}. \quad (29)$$

Assuming time evolution is of the order of the Hubble scale, we have that  $C_T \sim \mathcal{O}(L_0 \times H_0 T)$ , where  $H_0$  is the current value of the Hubble parameter. Clearly such corrections are of order  $\mathcal{O}(L_0)$  compared to the contributions calculated above. Moreover, these corrections are negligible over small time averaging scales  $T$  (compared to  $H_0$ ; i.e.,  $H_0 T$  small). In addition, we have that

$$\left\langle \frac{\partial}{\partial t} \langle f \rangle \right\rangle \equiv \left\langle \frac{\partial}{\partial t} \left[ \frac{1}{L} \int_{-L/2}^{L/2} f dr \right] \right\rangle \equiv \left\langle \frac{\partial f}{\partial t} \right\rangle + C_{\partial T}, \quad (30)$$

where

$$C_{\partial T} \equiv \left\langle \frac{1}{L} \frac{dL}{dt} \{f - \langle f \rangle\} \right\rangle \sim L_0 H_0 \bar{C}, \quad (31)$$

where the term  $\bar{C}$  in the particular case of inhomogeneities of the form (11)/(12) is negligible. Therefore,  $T C_{\partial T} \sim \mathcal{O}(L_0) \times H_0 T$ . These corrections are consequently of higher order and generally will only renormalize the spatial curvature term.

## 3. Other inhomogeneous models

The form of the correlation tensor depends on the assumed form for the inhomogeneous gravitational field and matter distribution. We could consider alternatives to the form of the inhomogeneous metric (11)/(12). We shall consider two alternative approaches here. First, we shall average an exact inhomogeneous Lemaître-Tolman-Bondi solution. Second, we shall discuss a linear inhomogeneous perturbation of an exact FLRW model.

However, the main conclusions of this section will not be affected; namely, in most applications of interest the correlation tensor is of the form of a spatial curvature, but in general it is not even of the form of a perfect fluid.

Moreover, higher order corrections are not expected to lead to significant effects; e.g., they alone cannot account for a current acceleration.

## IV. LEMAÎTRE-TOLMAN-BONDI MODEL

Let us consider averaging an exact solution. The spherically symmetric dust solution is the exact Lemaître-Tolman-Bondi (LTB) model [11,20], which can be regarded as an exact inhomogeneous generalization of the FLRW solution. In the dust LTB model, from the Gauss-Codazzi equations the Einstein tensor has the form of a spatial curvature tensor on spacelike hypersurfaces (which we recall is not the same as the projected Einstein tensor). Various aspects of the averaging problem in LTB spacetimes have been studied [21,22].

The first step is to take the LTB solution [11,20] and rewrite it in VPC. This is done in Appendix B. We note that taking averages using VPC is of interest in its own right, and is an advantage in that averaging can now be done in both space and time (this will be discussed further in [23]). From the Appendix, we obtain

$$ds^2 = - \left( 1 - \frac{U^2}{R^4} \right) dt^2 - 2 \frac{U}{R^4} dt dx + \frac{dx^2}{R^4} + R^2 \left[ \frac{du^2}{1-u^2} + (1-u^2) d\phi^2 \right]. \quad (32)$$

The constraints of the original LTB metric become a defining equation for  $U(t, x)$  (B11) and a differential equation for  $R(t, x)$  (B12), which then ensures an exact dust solution with density  $G^t{}^t(t, x)$ . Since in VPC the velocity of the dust flow is  $u^a = (1, U(t, x), 0, 0)$ , the Einstein tensor components satisfy  $G^{tx} = U G^{tt}$ ,  $G^{xx} = U^2 G^{tt}$ ,  $G^{uu} = G^{\phi\phi} = 0$  [see Eqs. (B13)].

In Appendix B we explicitly construct the FLRW dust models in VPC. The spatially flat ( $E_0 = 0$ ) FLRW model in VPC is given by (B14). The spatially closed ( $E_0 < 0$ ) FLRW model in VPC is given in Eq. (B18) in terms of an expansion (of  $R$ ,  $U$  and  $G^t{}^t$ ) about the spatially flat FLRW model with  $E_0 = 0$  (a similar expression exists for the  $E_0 > 0$  FLRW model).

## A Perturbative Solution

We shall assume that  $t_B(r)$  is zero, which implies that the bang time is uniform and we are consequently restricting our choice of LTB models to those with no decaying modes. Such models are of interest at later times, and particularly in the study of structure formation [24], and are suitable for our purposes here. We shall also consider solutions of the LTB metric in VPC as perturbations about the spatially flat FLRW model given in (B14). In this respect our approximate solution will be an expansion with respect to  $E_0$  and we require the Einstein tensor to have the form of (B13) [i.e., the form of dust, after truncation of terms of  $\mathcal{O}(E_0^2)$  or higher]. We begin by making the

following ansatz on the form of  $R$

$$R(t, x) = R_0 + \alpha_1 x^a t^b E_0 + \alpha_2 x^c t^d E_0^2, \quad (33)$$

where  $\alpha_1, \alpha_2, a, b, c$ , and  $d$  are constants to be determined from requiring the Einstein tensor has the form of a dust solution up to order  $E_0$ . Substituting (B11) into (B12) gives a partial differential equation (PDE) involving only  $R$ , a subsequent substitution by (33) then shows that the first nontrivial term in the PDE occurs at  $\mathcal{O}(E_0^3)$ . At this stage we are only interested in a perturbative solution of the PDE, therefore to obtain necessary conditions for a dust solution we simply require that the coefficient of  $E_0^3$  vanish, which yields three cases ( $a = 1/3, -2/3$ , or  $1/3 - b/2$ ). Choosing  $a$ , we can use Eqs. (33) and (B11) to obtain  $U(t, x)$ . Calculating the Einstein tensor and requiring it have the form of (B13) allows us to determine the remaining constants (the details are discussed in [23]).

If  $a = 1/3$ , a number of subcases occur. A typical solution (e.g.,  $b = 0, c = 5/6$ , and  $d = -1$ ) gives rise to  $G^t = 4/(3t^2) - 2\alpha_1 x^{-2/3} E_0 + \mathcal{O}(E_0^2)$  and  $G^{\phi\phi} = \mathcal{O}(E_0^2)$ , so that the truncation of  $E_0^2$  and higher terms results in an Einstein tensor of the form of dust (B13). Other solutions give rise to a  $G^t$  with no  $E_0$  terms (but containing higher orders of  $E_0$ , with  $G^{\phi\phi}$  beginning at these higher orders) and  $G^t$  and  $G^{xx}$  components with no  $\mathcal{O}(1)$  terms (and beginning with  $E_0$  terms, whereas the other components begin at higher orders of  $E_0$ ). After averaging the ( $-2\alpha_1 x^{-2/3} E_0$ ) contribution, we obtain a correction term to the density which is independent of  $t$ . This constant correction in the dust model may be related to a cosmological constant or to an anisotropic source (this will be further investigated in [23]).

If  $a = -2/3$ , then the resulting dust solutions are of the form of a flat or curved FLRW model [the closed model is given by (B16)]. That is, the solution in this case is typically of the form of a flat FLRW model with spatial curvature corrections.

If  $a = 1/3 - b/2$  (where  $b \neq 0, 2$ , these cases are discussed above), setting  $c = -1/3 - b, d = 2(b + 1)$ , and  $\alpha_2 = -81\alpha_1^2 b(b - 2)^2 / (80(3b - 1))$  gives a dust solution up to order  $E_0^2$ . Unlike previous cases, here we have an arbitrary power of  $x$ , and there are two free parameters  $b$  and  $\alpha_1$ ; however, these solutions are not more general because other free parameters are constrained.

We can obtain more general perturbative dust solutions containing more free parameters by a superposition of the solutions discussed above. For example, one such solution is found by adding a solution with  $a = 1/3, b = 0, c = 5/6$ , and  $d = -1$  and a solution with  $b = -16/3, a = 3, c = 5, d = -26/3$ . and  $\alpha_2 = -1452\alpha_1^2/85$ , to obtain a solution of the form (to order  $E_0^2$ )

$$R(t, x) = \alpha_0 R_0 + [\alpha_1 x^{1/3} + \beta_1 x^3 t^{-16/3}] E_0 + [\alpha_2 x^{5/6} t^{-1} - (1452/85)\beta_1^2 x^5 t^{-26/3}] E_0^2, \quad (34)$$

with corresponding forms for  $U(t, x)$  and  $G^t$ , thus giving rise to another (more general) perturbative dust solution. Therefore, we can construct perturbative LTB solutions which can be interpreted as having both spatial curvature and constant correction terms.

## V. COSMOLOGICAL PERTURBATIONS

### A. Backreaction

The theoretical approach is to solve the full problem to obtain the equations satisfied by the averaged quantities, without assuming a given background. An alternative but more practical approach is to assume a spatially homogeneous and isotropic background and study the effect of perturbations (that is, the backreaction) on this FLRW background [2,8,25,26]. The starting point is the Einstein equations in an appropriately defined background [25,27]. The Einstein and energy-momentum tensors are then expanded in metric and matter perturbations up to second order. The linear equations are assumed to be satisfied, and the spatially averaged remnants provide the new background metric which takes into account the backreaction effect of linear fluctuations computed up to quadratic order. The backreaction has been studied for scalar gravitational perturbations [25], and it was found that the equation of state of the dominant infrared contribution to the energy-momentum tensor which describes backreaction can take the form of a negative cosmological constant. This has led to the speculation that gravitational backreaction may lead to a dynamical cancellation mechanism for a bare cosmological constant [28]. Since, in the perturbative approach, the averaged Einstein tensor will rapidly come to dominate over the correlation tensor, it has been argued that this might also explain the presence of a source of late-time acceleration [28].

The main aim is to investigate the effect of these perturbations on the local expansion rate and to see how it might differ from the background expansion rate. In an important study, using a 3 + 1 split and the Zeldovich approach and assuming inhomogeneous perturbations about a dust Einstein de Sitter (FLRW) background, the equations governing the time dependence of the scale factor due to backreaction were obtained by spatial averaging [2]. The metric perturbations were assumed small, even when the density contrast is large (much larger than unity). It was found that the scale factor dependence on the correlation terms acts like a (negative) spatial curvature term (and, curiously, that the age is greater than in the exact flat background FLRW model) [2]. Typically perturbations are small corrections, but since they are time dependent they can become larger, although likely vanishing asymptotically to the future [26].

Therefore, the resulting correlation terms are simply of the form of a spatial curvature term in the linear perturbation analysis. This is true in general, and is certainly true in spherically symmetric cosmological models. There are, as



mentioned earlier, problems with the perturbative approach. First, the perturbation scheme breaks down when perturbations become significant and affect the background. Second, there are potentially gauge effects arising from the choice of hypersurface on which to do spatial averaging.

### B. Discussion

Recent observations are usually interpreted as implying that the Universe is very nearly flat, currently accelerating [29], and indicating the existence of dark matter and dark energy [30]. A cosmological constant or a negative pressure fluid (or quintessence field) are candidates for the dark energy. However, as noted earlier, inhomogeneities can affect the dynamics and may significantly affect the expansion rate of the spatially averaged “background” FLRW universe (the effect depending on the scale of the initial inhomogeneity) [2]. Therefore, a more conservative approach to explain the acceleration of the Universe without introduction of exotic fields is to utilize a backreaction effect due to inhomogeneities of the Universe.

It has been suggested that backreactions from inhomogeneities smaller than the Hubble scale could explain the apparently observed accelerated expansion of the Universe today. This has been investigated by studying the effective Friedmann equation describing an inhomogeneous Universe after smoothing out of the subhorizon cosmological perturbations, and it has been suggested that the acceleration in our Hubble volume might be possible even if local fluid elements do not individually undergo accelerated expansion [31,32]. However, in [33] it was claimed that the perturbative effect proposed amounts to a simple renormalization of the spatial curvature, and in other work it has been argued that the acceleration cannot be explained by the effects of inhomogeneities [34,35]. However, after density fluctuations in the Universe grow to be nonlinear and begin to recollapse, the perturbative expansion breaks down and reliable results cannot be obtained beyond this based on perturbative calculations. More recently, a solution using the gravitational backreaction of long wavelength (super-Hubble) fluctuation modes on the background metric was presented [28], and it was shown that in the presence of entropy fluctuations backreaction of the nongradient terms is physically measurable (compare with [35]).

In further work [26], the relationship between backreaction and spatial curvature using exact equations which do not rely on perturbation theory was studied in more detail, and it was argued that even though the effect does not simply reduce to spatial curvature, the acceleration that results is accompanied by a growth of spatial curvature to an extent that it is unlikely to be compatible with the CMB data. On the other hand, an explicit example of an inhomogeneous Universe has been presented that leads to accelerated expansion after taking spatial averaging [22].

The model universe is the LTB solution and contains both a region with positive spatial curvature and a region with negative spatial curvature. It was found that, after the region with positive spatial curvature begins to recollapse, the deceleration parameter of the spatially averaged universe becomes negative and the averaged universe starts accelerated expansion. Further examples, in which the assumption of spherical symmetry is relaxed, are discussed in [36]. In addition, inhomogeneities can lead to a reinterpretation of the luminosity distance of cosmological sources in terms of its redshift, which may account for the observed acceleration [37]. However, it should be reiterated that there are subtleties when dealing with spatially averaged quantities, even if the spatial averaging is over a limited domain, and that the results discussed above may not apply to the quantities of physical interest [36]. This point has also been further emphasized in [35] (also see [28,32]).

## VI. CONCLUSIONS

We have calculated the form of the MG equations in the case of spherical symmetry. By working in VPC, we calculated the form of the correlation tensor under some reasonable assumptions on the form for the inhomogeneous gravitational field and matter distribution. *The main result of this paper is that the correlation tensor in a FLRW background must be of the form of a spatial curvature, while for a non-FLRW background (with  $B_r \neq 0$ ) the correlation tensor can be interpreted as the sum of a spatial curvature and an anisotropic fluid.* We note that working in VPC (in which the determinant of the metric and hence the volume element is constant) is useful in its own right, particularly in the context of averaging.

The cosmological result that in the spherically symmetric case the averaged Einstein equations in an FLRW background have the form of the Einstein equations of GR for a spatially homogeneous, isotropic macroscopic spacetime geometry with an additional spatial curvature term, confirms the results in previous work in which we were able to explicitly solve the MG equations to find a correction term in the form of a spatial curvature [9]. The results of the calculations regarding averaging of an exact inhomogeneous LTB solution (presented above), as well as calculations of linear perturbations (that is, the backreaction) in a spatially homogeneous and isotropic background and the results of Buchert [6] also confirm and support this result.

The MG method adopted here is an exact approach in which inhomogeneities affect the dynamics on large scales through correction terms.<sup>7</sup> Averaging can have a very significant dynamical effect on the evolution of the

<sup>7</sup>Hence, the main criticisms of the backreaction approach to studying the possible contributions to an accelerated expansion [26,28,31,33–35] do not apply here.

Universe; the correction terms change the interpretation of observations so that they need to be accounted for carefully to determine if the models may be consistent with an accelerating Universe. Averaging may or may not explain the observed acceleration. However, it is clear that it cannot be neglected, and a proper analysis will not be possible without a comprehensive understanding of the effects of averaging.

On cosmological scales (of the order of the inverse Hubble scale), in which  $B_r = 0$  and we have a FLRW background, averaging only gives rise to a spatial curvature term. However, the effects of averaging on astrophysical scales, such as galactic scales, are also of interest. Under the assumption that a galaxy can be approximated as spherically symmetric, where the background has  $B_r \neq 0$ , averaging is found to give rise to a correlation tensor of the form of an anisotropic fluid. This is of particular interest since, as noted earlier, dark matter in galactic halos is more accurately described by an anisotropic fluid [18].

In this paper we have also discussed averaging in an inhomogeneous LTB solution. Writing the LTB solution in volume-preserving coordinates, we found a perturbative solution in which the correlation term can be interpreted as the sum of a spatial curvature term and a constant correction term. We also discussed linear inhomogeneous perturbations (that is, the backreaction) on an exact FLRW background. It was noted that the resulting correlation terms are simply of the form of a spatial curvature term.

If the underlying microscopic spacetime has positive spatial curvature (as perhaps suggested by recent observations [29,30,38]), then we could obtain a cosmological model which is “closed” on local scales, but as a result of the MG correlations behaves dynamically on macroscopically large scales as a flat model, which might have considerable physical implications. Indeed, cosmological models which act like an Einstein static model on the largest scales are possible even for models with zero or negative curvature on small scales; thus at late times (and on the largest scale) a spatial curvature term will dominate the dynamics and the correlations might stabilize the Einstein static model [39] (also see Sec. 3.3 of [36]).

The MG analysis presented here is a self-consistent analysis. However, the MG approach is quite complicated and can be difficult to implement in practice. Therefore, it is of interest to compare our results to the work of Buchert [36]. In this latter approach a  $3 + 1$  split of the equations is effected (which introduces some gauge issues that presumably can be appropriately dealt with in the cosmological setting). More importantly, only scalar quantities appear in the averaged equations. This implies some sort of “truncation” of the Einstein equations in order for the equations to reduce to scalar equations. As a result, in general the Buchert equations are not closed. In the approach taken here, the actual averages are constructed and therefore, in principle, the forms for the averaged quantities take on a

specific form. Consequently, our approach is more restrictive in the sense that the system of governing equations is closed and so no further assumptions to close the system, which may or may not be physical, are necessary. However, it is anticipated that it is possible to derive the averaged scalar equations of Buchert [6] as some appropriate limit of MG. Whether any significant effects are neglected in the Buchert approach could then be determined.

In Appendix C we show that in the spherically symmetric case the governing Einstein equations can reduce to equations in terms of scalar quantities in some circumstances. Therefore, we may be able to compare our results with the work of Buchert [6] in this case. Certainly, a spatial curvature term appears in Buchert’s scheme when averaging in a FLRW background, and it is expected that our analysis is consistent with the work of Buchert in a more fundamental sense. Unfortunately, the perfect fluid case is very complicated for comparisons, and so it is more sensible to first try to make the comparison in the dust case (i.e., the LTB model). Some brief comments are made in Appendix C and we shall pursue this further in [23]. Indeed, we note that for the models studied in Sec. III, if the correlation tensor is of the form of a perfect fluid then necessarily it is of the form of a spatial curvature (and this result is thus trivially consistent with the work of Buchert). On the other hand, if the correlation tensor is of the form of an anisotropic fluid, then a comparison with the work of Buchert is not possible.

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## APPENDIX A: MACROSCOPIC GRAVITY

Let us review the spacetime averaging scheme adopted in macroscopic gravity (MG) [3]. It is based on the concept of Lie dragging of averaging regions and is valid for any differentiable manifold. Choosing a compact region  $\Sigma \subset \mathcal{M}$  in an  $n$ -dimensional differentiable metric manifold  $(\mathcal{M}, g_{\alpha\beta})$  with a volume  $n$ -form and a supporting point  $x \in \Sigma$  to which the average value will be prescribed, the average value of a geometric object,  $p_{\beta}^{\alpha}(x)$ ,  $x \in \mathcal{M}$ , over a region  $\Sigma$  at the supporting point  $x \in \Sigma$  is defined as

$$\bar{p}_{\beta}^{\alpha}(x) = \frac{1}{V_{\Sigma}} \int_{\Sigma} \mathbf{p}_{\beta}^{\alpha}(x, x') \sqrt{-g'} d^n x' \equiv \langle \mathbf{p}_{\beta}^{\alpha} \rangle, \quad (\text{A1})$$

where  $V_{\Sigma}$  is the volume of the region  $\Sigma$ ,

$$V_{\Sigma} = \int_{\Sigma} \sqrt{-g} d^n x, \quad (\text{A2})$$

the integration is carried out over all points  $x' \in \Sigma$ ,  $g' = \det(g_{\alpha\beta}(x'))$  and the boldface object  $\mathbf{p}_{\beta}^{\alpha}(x, x')$  is a bilocal extension of the object  $p_{\beta}^{\alpha}(x)$ ,

$$\mathbf{p}_{\beta}^{\alpha}(x, x') = \mathcal{W}_{\mu'}^{\alpha}(x, x') p_{\nu'}^{\mu'}(x') \mathcal{W}_{\beta}^{\nu'}(x', x), \quad (\text{A3})$$

by means of the bilocal averaging operator  $\mathcal{W}_{\beta}^{\alpha}(x, x')$  and its inverse  $\mathcal{W}_{\beta}^{\alpha'}(x', x)$ . The averaging scheme is covariant and linear by construction, and the averaged object  $\bar{p}_{\beta}^{\alpha}$  has the same tensorial character as  $p_{\beta}^{\alpha}$ . As a result of the coincidence limit ( $\lim_{x' \rightarrow x} \mathcal{W}_{\beta}^{\alpha}(x, x') = \delta_{\beta}^{\alpha}$ ) and the idempotency condition, the average tensor  $\bar{p}_{\beta}^{\alpha}(x)$  takes the same value as the original tensor  $p_{\beta}^{\alpha}(x)$  when the integrating region  $\Sigma$  tends to zero, which implies that the averaging procedure commutes with the operation of index contraction.

In order to obtain the averaged fields of geometric objects on  $\mathcal{M}$ , it is necessary to assign an averaging region  $\Sigma_x$  to each point  $x$  of  $\mathcal{U} \subset \mathcal{M}$ , where the averaging integral is to be evaluated. To calculate directional, partial, and covariant derivatives of the averaged fields, regions are related by Lie dragging by means of a second bilocal operator, which can also be taken to be  $\mathcal{W}_{\beta}^{\alpha'}(x', x)$  (which satisfies a divergence-free condition in order for Lie dragging of a region to be a volume-preserving diffeomorphism) [3]. The commutation relations simplify, and the differential constraint for the idempotent bilocal reduces to

$$\mathcal{W}_{[\beta, \gamma]}^{\alpha'} + \mathcal{W}_{[\beta, \delta']}^{\alpha'} \mathcal{W}_{\gamma]}^{\delta'} = 0, \quad (\text{A4})$$

which has the general solution

$$\mathcal{W}_{\beta}^{\alpha'}(x', x) = f_i^{\alpha'}(x') f_{\beta}^{-1i}(x), \quad (\text{A5})$$

where  $f_i^{\alpha}(x) \partial_{\alpha} = \mathbf{f}_i$  is any vector basis satisfying the commutation relations  $[\mathbf{f}_i, \mathbf{f}_j] = C_{ij}^k \mathbf{f}_k$ , with constant structure functions  $C_{ij}^k$ . In any  $n$ -dimensional differentiable metric manifold  $(\mathcal{M}, g_{\alpha\beta})$  with a volume  $n$ -form there always exist locally volume-preserving divergence-free operators  $\mathcal{W}_{\beta}^{\alpha'}(x', x)$  of the form (A5) [3].

### 1. Proper systems of coordinates

We can consider the MG averaging scheme for a particular subclass of operators in which the averages and their properties are especially simple. Such a coordinate system is the analogue for MG of the Cartesian coordinates in Minkowski spacetime [4]. Let us hereby restrict the class of solutions of the equations (A4) to the subclass satisfying  $[\mathbf{f}_i, \mathbf{f}_j] = 0$ ; that is,  $C_{ij}^k \equiv 0$ . In this case the vector fields  $f_i^{\alpha}$  constitute a coordinate system and there always exist  $n$  functionally independent scalar functions  $\phi^i(x)$  such that the vector and corresponding dual 1-form bases are of the

form

$$f_i^{\alpha}(x(\phi^k)) = \frac{\partial x^{\alpha}}{\partial \phi^i}, \quad f^{-1i}(\phi(x^{\mu})) = \frac{\partial \phi^i}{\partial x^{\alpha}}. \quad (\text{A6})$$

Thus, the bilocal operator  $\mathcal{W}_{\beta}^{\alpha'}(x', x)$  becomes

$$\mathcal{W}_{\beta}^{\alpha'}(x', x) = \frac{\partial x^{\alpha'}}{\partial \phi^i} \frac{\partial \phi^i}{\partial x^{\beta}}. \quad (\text{A7})$$

Since they are functionally independent, the set of  $n$  functions  $\phi^i(x)$  can be taken as a system of local coordinates on the manifold  $\mathcal{M}$ , which will be called a proper coordinate system [4]. Therefore, in a proper coordinate system the bilocal operator  $\mathcal{W}_{\beta}^{\alpha'}(x', x)$  takes the simplest possible form:

$$\mathcal{W}_j^i(\phi', \phi) \equiv \mathcal{W}_{\beta}^{\alpha'}(x', x)|_{x^{\alpha}=\phi^i} = \delta_{\beta}^{\alpha'} \equiv \delta_j^i, \quad (\text{A8})$$

where the bilocal Kronecker symbol  $\delta_{\beta}^{\alpha'}$  is defined as  $\delta_{\beta}^{\alpha'} = \delta_i^{\alpha'} \delta_{\beta}^i$ .

The definition of an average consequently takes on a particularly simple form when written using a proper coordinate system. The existence of volume-preserving bilocal operators  $\mathcal{W}_{\beta}^{\alpha'}$  of this form was proven in [4]. Moreover, any proper coordinate system with a corresponding divergence-free bivector is necessarily a system of VPC. In the case of a pseudo-Riemannian manifold the Christoffel symbols,  $\Gamma_{\beta\alpha}^{\alpha}$ , vanish and partial differentiation and averaging commute in VPC.

Consequently, if the manifold  $(\mathcal{M}, g_{\alpha\beta})$  is a pseudo-Riemannian spacetime, the spacetime averages defined in proper coordinates are Lorentz tensors, precisely like the averages in Minkowski spacetime. The average value of a tensor field  $p_{\beta}^{\alpha}(x)$ ,  $x \in \mathcal{E}$ , over a compact space region  $S$  and a finite time interval  $\Delta t$  at a supporting point  $(t, x^a) \in \Delta t \times S$  is thus

$$\langle p_{\beta}^{\alpha}(t, x^a) \rangle_{\mathcal{E}} = \frac{1}{\Delta t V_S} \int_{\Delta t} \int_S p_{\beta}^{\alpha}(t + t', x^a + x^{a'}) dt' d^3 x', \quad (\text{A9})$$

where  $V_S$  is the 3-volume of the region  $S$ , which is usually taken as a 3-sphere of radius  $R$  around the point  $x^a$  at the instant of time  $t$ ,  $V_S = \int_S d^3 x'$ .

One issue of concern in the MG approach is the question of uniqueness; to what extent do spacetime averages depend on the choice of the bilocal operator. In the context of the present analysis, this raises the question of whether the results obtained in this paper could depend on the choice of VPC (1). It is clearly of interest to study this question, and we hope to return to this in future work. However, it is strongly anticipated that the main conclusions of this paper will not be qualitatively affected by the choice of VPC; namely, that in physical applications the correlation tensor is of the form of a spatial curvature (while in general the correlation tensor is of the form of an anisotropic fluid).

## APPENDIX B: LEMAÎTRE-TOLMAN-BONDI MODELS

A spherically symmetric solution of the Einstein equation with dust field is given by the Lemaître-Tolman-Bondi (LTB) solution [11] with metric

$$ds^2 = -d\tau^2 + \frac{(R,r)^2}{1+2E(r)} dr^2 + R^2 d\Omega_2^2, \quad (\text{B1})$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2E(r)}{R^2} + \frac{2M(r)}{R^3}, \quad (\text{B2})$$

where  $E(r)$  and  $M(r)$  are arbitrary functions of  $r$ . The solution of Eq. (B2) can be written parametrically by using the variable  $\eta = \int d\tau/R$ ,

$$R(\eta, r) = \frac{M(r)}{-2E(r)} [1 - \cos(\sqrt{-2E(r)}\eta)], \quad (\text{B3})$$

$$t(\eta, r) = \frac{M(r)}{-2E(r)} \left[ \eta - \frac{1}{\sqrt{-2E(r)}} \sin(\sqrt{-2E(r)}\eta) \right].$$

By introducing the following variables

$$a(\tau, r) = \frac{R(\tau, r)}{r}, \quad k(r) = -\frac{2E(r)}{r^2}, \quad (\text{B4})$$

$$\rho_0(r) = \frac{6M(r)}{r^3},$$

the metric and the evolution equation for the scale factor  $a(\tau, r)$  become

$$ds^2 = -d\tau^2 + a^2 \left[ \left(1 + \frac{a,r r}{a}\right)^2 \frac{dr^2}{1 - k(r)r^2} + r^2 d\Omega_2^2 \right], \quad (\text{B5})$$

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k(r)}{a^2} + \frac{\rho_0(r)}{3a^3}. \quad (\text{B6})$$

Equation (B6) is the same as the Friedmann equation with dust, and we can regard the LTB solution as a model of an inhomogeneous universe whose local behavior is equivalent to a FLRW universe with a spatial curvature  $k(r)$ .

As a specific case, Nambu and Tanimoto [22] assumed the following spatial distribution of spatial curvature:

$$k(r) = \frac{1}{L^2} [2\theta(r - r_0) - 1], \quad 0 \leq r \leq L, \quad (\text{B7})$$

$$0 \leq r_0 \leq L$$

and assumed that  $\rho_0(r) = \rho_0 = \text{constant}$ . For  $0 \leq r < r_0$ , the solution is that of a spatially open FLRW universe and for  $r_0 < r \leq L$ , the solution is that of a spatially closed FLRW universe.

### 1. Volume-preserving coordinate system

Starting with the LTB metric in the standard coordinate system  $(\tau, r, \theta, \phi)$  above that is aligned with the fluid flow,

we obtain a volume-preserving coordinate system (VPC),  $(t, x, u, \phi)$ , by making the following coordinate transformation:

$$t = \tau, \quad x = \int \frac{R(\tau, r)^2 R_r}{\sqrt{1+2E(r)}} dr, \quad u = \cos\theta. \quad (\text{B8})$$

Defining  $U(t, x) := x_\tau$ , and regarding  $R = R(t, x)$ , the line element becomes

$$ds^2 = -\left(1 - \frac{U^2}{R^4}\right) dt^2 - 2\frac{U}{R^4} dt dx + \frac{dx^2}{R^4} + R^2 \left[ \frac{du^2}{1-u^2} + (1-u^2) d\phi^2 \right], \quad (\text{B9})$$

which has  $g = -1$  as desired. The constraints on the original LTB metric ensuring a dust solution with density

$$G^{\tau\tau} = 2\frac{M_r}{R(\tau, r)^2 R_r}, \quad (\text{B10})$$

now become

$$U(t, x) = -\frac{2R_t R_x + R R_{tx}}{2R_x^2 + R R_{xx}}, \quad (\text{B11})$$

and

$$2(3R_x R_t - R R_x U_x) U - 2R R_x U_t + (7R_x^2 + 2R R_{xx}) U^2 - R_t^2 - 2R R_{tt} + R_x^2 R^4 = 1. \quad (\text{B12})$$

Using Eq. (B11), we can view Eq. (B12) as a differential constraint for  $R(t, x)$  [where  $U(t, x)$  is then derived once  $R(t, x)$  is known]. As a result of Eqs. (B8), (B11), and (B12), the Einstein tensor has the following form:

$$G^{tt} = G^{\tau\tau}(t, x), \quad G^{tx} = U G^{tt}, \quad (\text{B13})$$

$$G^{xx} = U^2 G^{tt}, \quad G^{uu} = G^{\phi\phi} = 0.$$

These equations describe the corresponding LTB dust solution in volume-preserving coordinates. We note that the fluid was comoving in the original coordinate system whereas in VPC the velocity of the dust flow is  $u^a = (1, U(t, x), 0, 0)$ , and hence is no longer comoving but is normalized,  $u_a u^a = -1$ .

### 2. Friedmann-Lemaître-Robertson-Walker cosmologies

In order to compare with calculations we present the FLRW dust models in VPC. In the original coordinate system the FLRW models have  $E(r) = E_0 r^2$ ,  $M(r) = M_0 r^3$  and a constant bang time  $t_B$ . Setting  $L_0 = |E_0|^{3/2}/M_0$  throughout, the spatially flat ( $E_0 = 0$ ) FLRW model in VPC is given by

$$R(t, x) = (3x)^{1/3}, \quad U(t, x) = \frac{2x}{t - t_B}, \quad (\text{B14})$$

giving an Einstein tensor component [the other compo-



nents are determined from (B13)

$$G'' = \frac{4}{3(t - t_B)^2}. \quad (\text{B15})$$

The exact spatially closed ( $E_0 < 0$ ) FLRW model in VPC is given by

$$\begin{aligned} R(t, x) &= \frac{1}{2\sqrt{2}L_0} \sin\theta(1 - \cos\eta), \\ U(t, x) &= \frac{6\sqrt{2}L_0 x \sin\eta}{(1 - \cos\eta)^2}, \\ \eta - \sin\eta &= 2\sqrt{2}L_0(t - t_B), \\ \theta - \sin\theta \cos\theta &= \frac{32\sqrt{2}L_0^3 x}{(1 - \cos\eta)^3}, \end{aligned} \quad (\text{B16})$$

where  $\eta = \eta(t)$  and  $\theta = \theta(t, x)$ . The resulting Einstein tensor component is

$$G'' = \frac{48L_0^2}{(1 - \cos\eta)^3}. \quad (\text{B17})$$

It is of interest to consider the form of  $R$ ,  $U$ , and  $G''$  in the closed FLRW model written as a perturbation about the spatially flat FLRW model. Expanding about  $E_0 = 0$  and defining  $R_0 = (3x)^{1/3}$ , we obtain

$$\begin{aligned} R(t, x) &= R_0 \left( 1 + \frac{12^{1/3} x^{2/3}}{15M_0^{2/3}(t - t_B)^{4/3}} E_0 \right. \\ &\quad \left. - \frac{2^{1/3} x^{2/3} (19 \cdot 3^{2/3} x^{2/3} + 126(t - t_B)^2)}{1575M_0^{4/3}(t - t_B)^{8/3}} E_0^2 \right) \\ &\quad + \mathcal{O}(E_0^3), \end{aligned} \quad (\text{B18})$$

$$\begin{aligned} U(t, x) &= \frac{2x}{t - t_B} + \frac{6^{2/3} x}{5M_0^{2/3}(t - t_B)^{1/3}} E_0 \\ &\quad - \frac{39 \cdot 6^{1/3} x (t - t_B)^{1/3}}{175M_0^{4/3}} E_0^2 + \mathcal{O}(E_0^3), \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} G'' &= \frac{4}{3(t - t_B)^2} - \frac{2 \cdot 6^{2/3}}{5M_0^{2/3}(t - t_B)^{4/3}} E_0 \\ &\quad + \frac{102 \cdot 6^{1/3}}{175M_0^{4/3}(t - t_B)^{2/3}} E_0^2 + \mathcal{O}(E_0^3). \end{aligned} \quad (\text{B20})$$

In the above perturbation scheme we have assumed that  $E_0$  (and  $x$  and  $t$ ) are small. The spatially open ( $E_0 > 0$ ) FLRW model in VPC can also be displayed (it is similar to the closed case but with hyperbolic trigonometric functions replacing trigonometric functions and appropriate sign changes—see [23]).

### APPENDIX C: SPHERICALLY SYMMETRIC MODELS IN THE 1 + 3 FORMALISM

Using the Uggla and van Elst 1 + 3 formalism [40] for a perfect fluid energy-momentum tensor, in the case of spherical symmetry we have the evolution equations:

$$\dot{\theta} = -\frac{1}{3}\theta^2 + (\mathbf{e}_1 + \dot{u} - 2a)(\dot{u}) - \frac{2}{3}(\sigma_+)^2 - \frac{1}{2}(\mu + 3p) + \Lambda \quad (\text{C1})$$

$$\dot{\sigma}_+ = -\theta\sigma_+ - (\mathbf{e}_1 + \dot{u} + a)(\dot{u}) - *S_+ \quad (\text{C2})$$

$$\dot{a} = -\frac{1}{3}(\theta + \sigma_+)(a + \dot{u}) \quad (\text{C3})$$

$${}^2\dot{K} = -\frac{2}{3}(\theta + \sigma_+)^2 K \quad (\text{C4})$$

$$\dot{\mu} = -(\mu + p)\theta \quad (\text{C5})$$

and the Friedmann constraint

$$0 = \frac{1}{3}\theta^2 + \frac{1}{2}R - \frac{1}{3}\sigma_+^2 - \mu - \Lambda * \quad (\text{C6})$$

(and the spatial constraint equations), where  ${}^2K(t, x)$  is the 2-curvature of the spheres.  $\dot{u}$  is specified by choosing a temporal gauge, and  $p$  is specified by the fluid model.

There is only a single shear component,  $\sigma \equiv \sigma_+$ , so that in the spherically symmetric case the shear can be described by a single scalar. The curvature, in some cases, can also be described by a scalar. Therefore, in the spherically symmetric case we may be able to use scalar equations in some appropriate limit to describe the model. In particular, choosing a gauge in which  $\dot{u} = 0$  and defining

$$R_d = {}^2K \quad Q_d = -\frac{2}{3}\sigma^2,$$

we obtain from the Friedmann constraint

$$0 = \frac{1}{3}\theta^2 + \frac{1}{2}R_d + Q_d - \mu - \Lambda + 2e_1(a) - 6a^2.$$

The evolution equations (C2) and (C4) yield

$$2\theta Q_d + \dot{\theta}_d = \sqrt{\frac{2}{3}} Q_d^{1/2} R_d \quad \dot{R}_d + \frac{2}{3}\theta R_d = -\sqrt{\frac{2}{3}} Q_d^{1/2} R_d$$

which yields

$$a_d^{-2}[R_d a^2]^* + a_d^{-6}[Q_d a^6]^* = 0,$$

where  $\theta \equiv 3\dot{a}_d/a_d$ . These equations are valid for both averaged and nonaveraged scalar quantities.

We can see that, in the case of spherical symmetry, the resulting governing equations are equations for scalar quantities under some circumstances. Therefore, we may be able to compare our results with the work of Buchert [6] in this case. Certainly Buchert [6] can obtain a spatial curvature term in an FLRW background, and it is expected that our analysis is consistent with the work of Buchert in a deeper sense. Unfortunately, the perfect fluid case is very complicated for comparisons, and so it is more sensible to

try to make the comparison in the dust case (i.e., the LTB model).

### 1. The dust case

Let us consider an inhomogeneous universe with irrotational dust. Following Buchert [6], one then obtains from Einstein's equations the following equations of motion for the effective scale factor,  $a_{\mathcal{D}}$ :

$$3 \frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -\frac{\kappa^2}{2} \langle \rho \rangle_{\mathcal{D}} + Q_{\mathcal{D}}, \quad (\text{C7})$$

$$3 \left( \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 = \kappa^2 \langle \rho \rangle_{\mathcal{D}} - \frac{1}{2} \mathcal{R}_{\mathcal{D}} - \frac{1}{2} Q_{\mathcal{D}}, \quad (\text{C8})$$

together with

$$\frac{\partial}{\partial t} (a_{\mathcal{D}}^6 Q_{\mathcal{D}}) + a_{\mathcal{D}}^4 \frac{\partial}{\partial t} (a_{\mathcal{D}}^2 \mathcal{R}_{\mathcal{D}}) = 0. \quad (\text{C9})$$

Here  $\mathcal{R}$  is the spatial scalar curvature (not necessarily isotropic) and

$$Q_{\mathcal{D}} \equiv \frac{2}{3} (\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2) - \langle \sigma_{ij} \sigma^{ij} \rangle_{\mathcal{D}}. \quad (\text{C10})$$

From the analysis of the spherically symmetric models we see that, in principle, we can obtain precise evolution equations for  $\mathcal{R}_{\mathcal{D}}$ ,  $Q_{\mathcal{D}}$  (and, more generally, for the additional terms in the averaged equations for perfect fluid models). Indeed, since we actually take averages in our analysis we will obtain specific forms for the averaged quantities (i.e., in principle we will obtain explicit expressions for  $\mathcal{R}$ ,  $Q_{\mathcal{D}}$ ). We intend to study this in the LTB models in more detail elsewhere [23].

However, in an analysis of the spherically symmetric collapse of dust in the Newtonian regime [assuming an

irrotational velocity field of the form  $\mathbf{v} = v(r)\mathbf{e}_r$  and other physical restrictions] [41], it was found that the backreaction,  $Q_{\mathcal{D}}$ , vanishes (as might be expected in the Newtonian approximation). Therefore, in this case the only effect of averaging in the Buchert approach is through a spatial curvature term. Unfortunately, the Newtonian limit is not expected to capture the relativistic backreaction; a GR treatment is needed to discuss the global effects of averaging.

In our relativistic MG approach let us assume a Newtonian-like 4-velocity field of the form

$$u^a = \frac{1}{\sqrt{B - Av^2}} [1, v, 0, 0], \quad (\text{C11})$$

where  $A$  and  $B$  are the metric functions and  $v = v(r)$  is assumed to be small (i.e.,  $v \ll 1$ ). The corresponding 4-acceleration is then given by

$$A^a = -\frac{F}{2(B - Av^2)^2} \left[ \frac{v}{B}, \frac{1}{A}, 0, 0 \right], \quad (\text{C12})$$

where

$$F \equiv AA_t v^3 + (2AB_r - A_r B) v^2 + (AB_t - 2A_t B - 2AB_v) v - BB_r. \quad (\text{C13})$$

If we assume that the fluid is pressure-free (i.e., dust), then the acceleration is zero. To lowest order in  $v$ , this implies that  $B_r = 0$ , and from earlier we conclude that the correlation tensor is of the form of a spatial curvature. Consequently, in this approximation the Buchert approach [41] and the MG approach are consistent.

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