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A family of cosmological solutions in higher-dimensional Einstein gravity

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A two-parameter family of solutions to Einstein's vacuum field equations in six dimensions is presented. The solutions are a natural extension of a one-parameter family of five-dimensional vacuum solutions found by Ponce de Leon [Gen. Relativ. Gravit. **20**, 539 (1988)] and are closely related to the generalized Kasner metrics. The solution is generalized to $4+n$ dimensions and is briefly discussed. © *1995 American Institute of Physics.*

I. INTRODUCTION

In a recent article,[1] several solutions to Einstein's vacuum field equations in five dimensions were presented. A primary focus of this earlier work was to investigate whether the four-dimensional properties of matter could be contemplated as being purely geometrical in origin.[2,3] The embedding of the four-dimensional space–time in the five-dimensional vacuum space–time was interpreted as producing an effective four-dimensional stress-energy tensor.[4–7] In particular, McManus observed that a one-parameter family of five-dimensional vacuum solutions previously discovered by Ponce de Leon[8] was in fact completely flat in five dimensions, that is, the five-dimensional Riemann tensor was identically zero. In this article, we are mainly concerned with determining if generalizations of these solutions exist in higher dimensions.

Before proceeding with our general analysis, we first present some background on the problem of embedding pseudo-Riemannian manifolds in flat spaces of higher dimensions. Of course, it is well known that every analytic four-dimensional space–time can be interpreted, at least locally, as a subspace of a flat pseudo-Euclidean space of no more than ten dimensions;[9] that is, every space–time can be (locally) isometrically embedded into a flat space with (minimum) embedding class $p\leqslant 6$. Normally space–times with high symmetry induce the existence of subspaces of low embedding class.[10] For example, spherically symmetric space–times are of class $p\leqslant 2$. (However, not all such space–times will admit coordinates such that the metric components will be independent of the extra higher-dimensional coordinates.) In addition, all conformally flat perfect fluid solutions are of embedding class 1 (Theorem 32.15 in Ref. 10), and consequently all Friedman–Robertson–Walker (FRW) space–times can be regarded as subspaces of flat, five-dimensional pseudo-Euclidean space.[1] Curiously, we note that there are no class 1 vacuum solutions (Theorem 32.11 in Ref. 10).

Related to the question of isometric embedding, within the context of this work, is the question of the reduction of the higher-dimensional flat space to the "underlying" four-dimensional space–time, that is, the question of how to choose the appropriate subspace (or slicing) of the flat space. In the simplest of cases it may be possible to make a natural selection of the four-dimensional subspace. Wesson *et al.*[6,7] examined a class of metrics of the form $ds^2={}^4g_{ab}(x^c)dx^a\,dx^b+\phi^2(x^c,y)dy^2$ where the natural choice was to take $y=$const. However, these models always lead to four-dimensional space–times whose Ricci scalar is zero. In general no such natural choice exists, and the physical consequences of this scenario is not clear.

In Sec. II, a power-law ansatz is presented for a class of six-dimensional solutions and all the vacuum solutions (up to symmetries) are found. In Sec. III, the solution is generalized to $4+n$ dimensions and is discussed in the context of induced matter theory.[4–7]

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II. GENERAL ANALYSIS

Ponce de Leon[8] found a one-parameter family of five-dimensional vacuum solutions

$$ds^2 = -\psi^2\, dt^2 + t^{2/\alpha}\psi^{2/(1-\alpha)}(dx^2+dy^2+dz^2) + (\alpha/(\alpha-1))^2 t^2\, d\psi^2, \tag{1}$$

where α is a constant. In these solutions the intrinsic geometry of the four-dimensional hypersurfaces ψ=const is Robertson–Walker. In terms of Wesson's[4,7,1] induced matter theory, these solutions can be interpreted as perfect-fluid, spatially flat, Friedman–Robertson–Walker cosmologies with an associated density

$$\mu = \frac{3}{\alpha^2}\frac{1}{\psi^2 t^2} \tag{2}$$

and pressure

$$p = \frac{2\alpha-3}{\alpha^2}\frac{1}{\psi^2 t^2}. \tag{3}$$

The pressure and the energy density satisfy a barotropic equation of state $p=\frac{1}{3}(2\alpha-3)\mu$. Furthermore, the above solution is Riemann flat,[1] that is, ${}^5R_{ijkl}=0$.

Is it possible to generalize the above solution (1) to higher dimensions? In particular, we will concentrate on six dimensions for the remainder of this section. Thus, can we find a two-parameter family of six-dimensional vacuum solutions similar to the solutions above, and if so, under what conditions will these solutions also be Riemann flat? We start by considering the following power-law ansatz for the metric:

$$ds^2 = -u^{2a}v^{2b}\, dt^2 + t^{2c}u^{2d}v^{2f}(dx^2+dy^2+dz^2) + A^2t^{2g}v^{2h}\, du^2 + B^2t^{2j}u^{2k}\, dv^2, \tag{4}$$

where $a\cdots k$, A and B are constants. (We make no *a priori* assumption about whether the higher-dimensional coordinates u and v are compact or not.) The six-dimensional vacuum field equations, ${}^6R_{ij}=0$, produces a system of seven independent nontrivial equations (see Appendix A). Finding all of the solutions to these equations is simple but tedious. Basically, one works through all the equations in a systematic exhaustive fashion breaking the solutions into mutually exclusive classes. There are essentially ten independent solutions (not counting solutions obtained by interchanging u and v). Four of these solutions induce four-dimensional vacuum solutions and thus they do not concern us here. For completeness, we list these solutions in Appendix B. The remaining six solutions are as follows:

Solution (i):

$$ds^2 = -v^2\, dt^2 + t^{2c}v^{2f}(dx^2+dy^2+dz^2) + t^{2g}v^{2h}\, du^2 + B^2t^2\, dv^2, \tag{5}$$

with $B=(3c+g)/[3c(c-1)+g(g-1)]$, $f=cB$ and $h=gB$;

Solution (ii):

$$ds^2 = -dt^2 + t^{2c}(dx^2+dy^2+dz^2) + t^{2g}\, du^2 + t^{2j}\, dv^2, \tag{6}$$

with $3c^2+g^2+j^2=3c+g+j=1$;

Solution (iii):

$$ds^2 = -dt^2 + t^{2c}v(dx^2+dy^2+dz^2) + t^{2g}v^{-1}\, du^2 + t^{2j}dv^2, \tag{7}$$

with $3c^2+g^2+j^2=3c+g+j=1=2g+3j$;

Solution (iv):

$$ds^2=-v^{2b}\,dt^2+t^{2c}v^{2f}(dx^2+dy^2+dz^2)+t^{2g}v^{2h}\,du^2+t^{2j}\,dv^2, \tag{8}$$

with $3c^2+g^2+j^2=3c+g+j=1$, $b^2+3f^2+h^2=b+3f+h=1$, and $3cf-3bc-3jf+gh-bg-jh=0$;

Solution (v):

$$ds^2=-u^{2a}v^2\,dt^2+t^{2c}u^{2d}v^{2f}(dx^2+dy^2+dz^2)+t^{2g}v^{2h}\,du^2+B^2t^2u^{2a}\,dv^2, \tag{9}$$

with $2a^2+3d^2=2a+3d=1$, $B=d(1+3d)/[c(3d^2+1)-d(1+3d)]$, $f=cB$, $h=3f[(5d-1)/(1+3d)]$, and $g=3c[(5d-1)/(1+3d)]$;

Solution (vi):

$$ds^2=-u^2v^2\,dt^2+(tuv)^{2c}(dx^2+dy^2+dz^2)+At^2v^2\,du^2+[A/(A-1)]t^2u^2\,dv^2, \tag{10}$$

with $c=1\pm 2/\sqrt{3}$ and $A>1$.

Most of the solutions are variants of the Kasner solution.[11] Solution (6) is a version of the generalized Kasner solution.[12–14] All the solutions form one-parameter families with the exceptions of (7) which is the only solution in its class and (5) which is the only two-parameter family of solutions. Indeed, solution (5) is the generalization of the Ponce de Leon solution as can easily be seen by setting $g=0$.

The only Riemann flat solutions with $c\neq 0$ (if $c=0$ then Wesson's induced matter theory implies that the four-dimensional space given by $u, v=\text{const}$ is simply Minkowski space–time and hence completely flat) are (5) with $g=\epsilon c$ where either $\epsilon=1$ or $\epsilon=0$. Both of these solutions form a one-parameter family and are essentially the same as the five-dimensional Ponce de Leon solution (1).

III. DISCUSSION

As we saw in the preceding section, the only six-dimensional metrics of the form (4) that are Riemann flat are simple extensions of the five-dimensional Ponce de Leon solutions (1), namely, solution (6) with $g=0$ which constitutes a one-parameter family of solutions. In addition, solution (6) is Ricci flat and is a natural generalization of the Ponce de Leon metric (1).

Indeed, it is not too difficult to conjecture what the form of the *generalized metric* is in $4+n$ dimensions

$$ds^2=-v^2\,dt^2+t^{2c}v^{2f}(dx^2+dy^2+dz^2)+B^2t^2\,dv^2+t^{2g_1}v^{2h_1}\,du_1^2+\cdots+t^{2g_{n-1}}v^{2h_{n-1}}\,du_{n-1}^2, \tag{11}$$

with $h_i=g_iB$ and $f=cB$ where

$$B=\frac{3c+\Sigma_{i=1}^{n-1}\,g_i}{3c(c-1)+\Sigma_{i=1}^{n-1}\,g_i(g_i-1)}. \tag{12}$$

The proof of the above conjecture is as follows: consider a $5+(n-1)$ orthogonal splitting of space–time[15] where the metric is decomposed as

$$ds^2=\gamma_{ab}(\xi^a)d\xi^a\,d\xi^b+\gamma_{AB}(\xi^a)d\zeta^A\,d\zeta^B, \tag{13}$$

where

$$\gamma_{ab}(\xi^a)d\xi^a\,d\xi^b\equiv-v^2\,dt^2+t^{2c}v^{2f}(dx^2+dy^2+dz^2)+B^2t^2\,dv^2, \tag{14}$$

$$\gamma_{AB}(\xi^a)d\zeta^A\, d\zeta^B \equiv t^{2g_1}v^{2Bg_1}\, du_1^2 + \cdots + t^{2g_{n-1}}v^{2Bg_{n-1}}\, du_{n-1}^2 \tag{15}$$

and B is an unspecified arbitrary constant $[\{\xi^a\}=(t,x,y,z,v)$ and $\zeta^A \equiv u_A]$. Following Ref. 15, the $5+(n-1)$-dimensional vacuum field equations may be written as

$$R_{ab} = {}^5R_{ab} - P_{a;b} - P_{aAB}P_b{}^{AB} = 0, \tag{16}$$

$$R_{AB} = -P^aP_{aAB} + 2P_{aA}{}^CP^a{}_{BC} - P^a{}_{AB;a} = 0, \tag{17}$$

where $P_{aAB} = \frac{1}{2}(\partial/\partial\xi^a)\gamma_{AB}$, $P_a = \gamma^{AB}P_{aAB}$, ";" indicates covariant differentiation with respect to the intrinsic metric γ_{ab}, ${}^5R_{ab}$ is the Ricci tensor corresponding to the five-metric γ_{ab}, and lower and upper case indices are raised and lowered with the intrinsic metrics γ_{ab} and γ_{AB}, respectively. Equation (17) is trivially satisfied, and Eq. (16) implies that B must have precisely the form as given in Eq. (12).

According to induced matter theory,[4–7] all of the solution of the form (1) and (11) can be interpreted as inducing four-dimensional spatially flat perfect-fluid FRW models (v, u=const). The densities and pressures associated with these solutions are given by

$$\mu = -\frac{3}{4g_{tt}}\left(\frac{\partial}{\partial t}\ln g_{xx}\right)^2 = \frac{3c^2}{T^2}, \tag{18}$$

$$p = \frac{1}{g_{tt}}\left[\frac{\partial^2}{\partial t^2}\ln g_{xx} + \frac{3}{4}\left(\frac{\partial}{\partial t}\ln g_{xx}\right)^2\right] = \frac{c(2-3c)}{T^2}, \tag{19}$$

where $T(=\int dt\sqrt{-g_{tt}})$ is the proper time.[1] Furthermore, the density and pressure satisfy the barotropic equation of state

$$p = \frac{2-3c}{3c}\,\mu. \tag{20}$$

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APPENDIX A: VACUUM FIELD EQUATIONS

The vacuum field equation for the metric (4) reduce to the following seven equations:

$$R_{tt}=0 \Rightarrow AB[3c(c-1)+g(g-1)+j(j-1)] + B[-3d-a+1-k]at^{2(1-g)}u^{2(a-1)}v^{2(b-h)}$$
$$+A[-3f-h-b+1]bt^{2(1-j)}u^{2(a-k)}v^{2(b-1)}=0,$$

$$R_{xx}=0 \Rightarrow AB[-3c^2+c-gc-cj] + B[a+3d-1+k]dt^{2(1-g)}u^{2(a-1)}v^{2(b-h)}$$
$$+A[3f+h+b-1]ft^{2(1-j)}u^{2(a-k)}v^{2(b-1)}=0,$$

$$R_{uu}=0 \Rightarrow AB[-g+1-3c-j]g + B[a(a-1)+3d(d-1)+k(k-1)]t^{2(1-g)}u^{2(a-1)}v^{2(b-h)}$$
$$+A[3f+h+b-1]ht^{2(1-j)}u^{2(a-k)}v^{2(b-1)}=0,$$

$$R_{vv}=0 \Rightarrow AB[-j+1-3c-g]j+B[3d+a+k-1]kt^{2(1-g)}u^{2(a-1)}v^{2(b-h)}$$

$$+A[b(b-1)+3f(f-1)+h(h-1)]t^{2(1-j)}u^{2(a-k)}v^{2(b-1)}=0,$$

$$R_{tu}=0 \Rightarrow 3cd-3ac-3gd+jk-ja-gk=0,$$

$$R_{tv}=0 \Rightarrow 3cf-3bc-3jf+gh-bg-jh=0,$$

$$R_{uv}=0 \Rightarrow ab-ah-bk+3df-3dh-3fk=0.$$

APPENDIX B: ADDITIONAL VACUUM SOLUTIONS

There are ten independent solutions to the vacuum field equation for the metric (1). Thus in addition to the six solutions listed in Sec. II there are the following four solutions:

Solution (vii):

$$ds^2=-v^{2b}\,dt^2+v^{2f}(dx^2+dy^2+dz^2)+v^{2h}\,du^2+dv^2,$$

with $b^2+3f^2+h^2=b+3f+h=1$;

Solution (viii):

$$ds^2=-v^{2b}\,dt^2+v^{2f}(dx^2+dy^2+dz^2)+t^2v^{2b}\,du^2+dv^2,$$

with $2b^2+3f^2=2b+3f=1$;

Solution (ix):

$$ds^2=-v^{2b}\,dt^2+uv^{2f}(dx^2+dy^2+dz^2)+v^{2h}\,du^2+u^{-1}\,dv^2,$$

with $b^2+3f^2+h^2=b+3f+h=1$;

Solution (x):

$$ds^2=-v^{2b}u^2\,dt^2+v^{2f}(dx^2+dy^2+dz^2)+t^2v^{2b}\,du^2+dv^2,$$

with $2b^2+3f^2=2b+3f=1$.

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