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Coefficients to $O(\epsilon^3)$ for the mixed fixed point of the nm-component field model

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The nm-component field model has a number of applications in the description of critical phenomena. The coefficients in the ϵ expansion for the stability matrix eigenvalues and correlation function exponent η for the mixed fixed point are presented to $O(\epsilon^3)$ and $O(\epsilon^4)$, respectively.

In this Brief Report we present the coefficients in the $\epsilon(=4-d)$ expansion to order ϵ^3 of the eigenvalues which determine the stability of the mixed fixed point of the general nm-component field model[1]

$$-\mathscr{H}=\int_k (r+k^2)\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\phi_i^\alpha\phi_i^\alpha + u\left[\sum_i\sum_\alpha \phi_i^\alpha\phi_i^\alpha\right]^2+\Delta\sum_i\left[\sum_\alpha\phi_i^\alpha\phi_i^\alpha\right]^2 \tag{1}$$

(where, as usual, there is momentum conservation for all products of ϕ's in $\mathscr{H}$).

The eigenvalues were determined by the method of minimal subtraction[2] in which the spin field renormalization constant Z and renormalized coupling constants u_R and Δ_R are chosen so that after renormalization of the N point functions,

$$\Gamma_R^{(N)}(u_R,\Delta_R,\mathbf{p},K)=Z^{N/2}\Gamma^{(N)}(u,\Delta,\mathbf{p}) \ , \tag{2a}$$

$$u=K^\epsilon u_R Z_u(u_R,\Delta_R) \ , \tag{2b}$$

$$\Delta=K^\epsilon \Delta_R Z_\Delta(u_R,\Delta_R) \ , \tag{2c}$$

$\partial\Gamma_R^{(2)}/\partial p^2$ and the two distinct four-point functions are finite when $\epsilon\to 0$. [In (2) $\mathbf{p}$ represents the external momenta and K is an arbitrary rescaling factor.]

The fixed points (u^*,Δ^*) of the renormalization-group flow are determined by the vanishing of the beta functions

$$\beta_u=K\frac{\partial u_R}{\partial K} \ , \tag{3a}$$

and

$$\beta_\Delta=K\frac{\partial \Delta_R}{\partial K} \ . \tag{3b}$$

Writing

$$Z_u=Z_u^{(1)}\frac{1}{\epsilon}+Z_u^{(2)}\frac{1}{\epsilon^2}+Z_u^{(3)}\frac{1}{\epsilon^3}+\cdots \ , \tag{4a}$$

$$Z_\Delta=Z_\Delta^{(1)}\frac{1}{\epsilon}+Z_\Delta^{(2)}\frac{1}{\epsilon^2}+Z_\Delta^{(3)}\frac{1}{\epsilon^3}+\cdots \ , \tag{4b}$$

and using the finiteness of the beta functions as $\epsilon\to 0$, we obtain

$$\beta_u=-\epsilon u_R+u_R\left[u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial\Delta_R}\right]Z_u^{(1)} \ , \tag{5}$$

and the identities

$$0=-u_R^2\left[u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial\Delta_R}\right]Z_u^{(1)}\frac{\partial Z_u^{(1)}}{\partial u_R}-u_R\Delta_R\left[u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial\Delta_R}\right]Z_\Delta^{(1)}\frac{\partial Z_u^{(1)}}{\partial\Delta_R}+u_R\left[u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial\Delta_R}\right]\left[Z_u^{(2)}-\frac{Z_u^{(1)^2}}{2}\right], \tag{6a}$$

and

$$0=-u_R^2\left[u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial\Delta_R}\right]Z_u^{(1)}\frac{\partial}{\partial u_R}\left[Z_u^{(2)}-\frac{Z_u^{(1)^2}}{2}\right]-u_R\Delta_R\left[u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial\Delta_R}\right]Z_\Delta^{(1)}\frac{\partial}{\partial\Delta_R}\times\left[Z_u^2-\frac{Z_u^{(1)^2}}{2}\right]+u_R\left[u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial\Delta_R}\right]\left[Z_u^{(3)}-Z_u^{(1)}Z_u^{(2)}+\frac{Z_u^{(1)^3}}{3}\right] \ , \tag{6b}$$

and similar expressions obtained from the above by interchanging u and Δ. This is the generalization to two coupling constants of Brézin's results.[2]

It is therefore only necessary to obtain the coefficients in Z_u and Z_Δ from the Feynman graph expansions for the two- and four-point functions. The beta functions are then determined immediately from (5) and the corresponding identity for β_Δ. The integrals associated with the Feynman graphs are known from the calculation for the isotropic model and therefore it was necessary only to determine the multiplicities of the graphs and to perform the lengthy algebra to derive Z_u and Z_Δ. The multiplicities of the graphs were determined by a brute force counting of component configurations and checked by an algebraic method. Identities like (6) provide a useful check on the self-consistency of the calculations.

The beta functions were determined for general n and m in the way described above. A computer program was then written to determine the fixed points for particular values of n and m. It was previously shown by Brézin, Le Guillou, and Zinn-Justin[1] that this model has only three fixed points; these are the nm-component and m-component isotropic fixed points and the "mixed" fixed point. These authors found that to $O(\epsilon)$ only one fixed point is stable.

Rather than present the fixed points themselves (which have no direct physical meaning) we tabulate the eigenvalues λ_1 and λ_2 of the stability matrix,

$$\begin{pmatrix} \dfrac{\partial \beta_u}{\partial u_R} & \dfrac{\partial \beta_u}{\partial \Delta_R} \\ \dfrac{\partial \beta_\Delta}{\partial u_R} & \dfrac{\partial \beta_\Delta}{\partial \Delta_R} \end{pmatrix},$$

when the derivatives are evaluated at the mixed fixed points (Tables I and II). If both eigenvalues are positive, the associated fixed point is stable and the critical behavior of experimental systems undergoing second-order transitions is determined only by such stable fixed points. Tables I and II exhibit these eigenvalues for values of n and m of interest for the study of dilute systems $(n \rightarrow 0)$,[3] magnetic systems with cubic anisotropy[4] $(m=1)$ and systems which undergo a change in the unit cell at the transition temperature $(m=2)$.[5]

In the course of this calculation we also determined the eigenvalues of the above stability matrix at the isotropic fixed points. However, in the case of the nm-component isotropic fixed point, the stability is dependent on the product nm and not on the particular value of m, and so has previously been determined by Ketley and Wallace[6] to $O(\epsilon^3)$. We therefore refer the reader to the comments of these authors on the considerable difficulty of determining the stability of this fixed point from truncated ϵ expansions. The stability of the m-component isotropic fixed point is expected to depend only on the sign of the specific-heat exponent α_m of the m-component isotropic model.[7] In the limit $n \rightarrow \infty$ this fixed point is expected to be stable only if $\alpha_m < 0$.[8] However, it may be shown that to all orders in ϵ the eigenvalues are independent of n.

In addition to the eigenvalues, we have determined the correlation function exponent

$$\eta = K \frac{\partial}{\partial K} \log Z \tag{7}$$

to order ϵ^4. The results are listed in Table III. Writing

$$Z = Z^{(1)} \frac{1}{\epsilon} + Z^{(2)} \frac{1}{\epsilon^2} + Z^{(3)} \frac{1}{\epsilon^3} + \cdots , \tag{8}$$

the finiteness of η as $\epsilon \rightarrow 0$ leads to

$$\eta = -\left(u_R \frac{\partial}{\partial u_R} + \Delta_R \frac{\partial}{\partial \Delta_R} \right) Z^{(1)} , \tag{9}$$

TABLE I. Coefficients of ϵ, ϵ^2, and ϵ^3 in the ϵ expansion of the eigenvalue λ_1 for the mixed fixed point (see text).

n	m	1	2	3
0	ϵ	a	0.500	0.125
	ϵ^2		−0.547	−0.569
	ϵ^3		−13.36	0.865
1	ϵ	1.000	1.000	1.000
	ϵ^2	0.000	2.000	16.000
	ϵ^3	−4.983	−51.849	−1055.000
2	ϵ	1.000	1.000	1.000
	ϵ^2	−0.630	−0.542	−0.542
	ϵ^3	1.618	1.153	1.020
3	ϵ	1.000	1.000	1.000
	ϵ^2	−0.581	−0.527	−0.552
	ϵ^3	1.362	0.993	1.093
4	ϵ	1.000	1.000	1.000
	ϵ^2	−0.542	−0.535	−0.557
	ϵ^3	1.153	1.007	1.141

[a]See Ref. 15.

TABLE II. Coefficients of ϵ, ϵ^2, and ϵ^3 in the ϵ expansion of the eigenvalue λ_2 for the mixed fixed point (see text).

n	m	1	2	3
0	ϵ	a	1.000	1.000
	ϵ^2		−2.656	−0.638
	ϵ^3		32.553	2.306
1	ϵ	−1.000	−1.000	−1.000
	ϵ^2	0.000	0.000	0.000
	ϵ^3	0.000	0.000	0.000
2	ϵ	−0.333	0.000	0.058
	ϵ^2	0.234	0.167	−0.107
	ϵ^3	−0.249	−0.320	−0.195
3	ϵ	−0.111	0.091	0.075
	ϵ^2	0.232	−0.002	0.203
	ϵ^3	−0.328	−0.122	−0.011
4	ϵ	0.000	0.125	0.080
	ϵ^2	0.167	−0.082	−0.238
	ϵ^3	−0.264	−0.008	0.063

[a]See Ref. 15.

TABLE III. Coefficients of ϵ^2, ϵ^3, and ϵ^4 in the ϵ expansion of the exponent η for the mixed fixed point (see text).

n	m	1	2	3
0	ϵ^2	a	0.0156	0.0205
	ϵ^3		0.0791	0.0210
	ϵ^4		−0.5452	−0.0277
1	ϵ^2	0.0	0.0	0.0
	ϵ^3	0.0	0.0	0.0
	ϵ^4	0.0	0.0	0.0
2	ϵ	0.0185	0.0208	0.0208
	ϵ^2	0.0187	0.0174	0.0173
	ϵ^3	−0.0083	−0.0071	−0.0035
3	ϵ	0.0206	0.0207	0.0207
	ϵ^2	0.0188	0.0166	0.0177
	ϵ^3	−0.0083	−0.0041	−0.0043
4	ϵ^2	0.0208	0.0205	0.0207
	ϵ^3	0.0174	0.0168	0.0179
	ϵ^4	−0.0071	−0.0035	−0.0050

[a]See Ref. 15.

and

$$-\left(u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial \Delta_R}\right)\left(Z^{(2)}-\frac{Z^{(1)^2}}{2}\right)+\left(u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial \Delta_R}\right)Z_u^{(1)}u_R\frac{\partial Z^{(1)}}{\partial u_R}+\left(u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial \Delta_R}\right)Z_\Delta^{(1)}\Delta_R\frac{\partial Z^{(1)}}{\partial \Delta_R}=0 \ , \qquad (10a)$$

and

$$-\left(u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial \Delta_R}\right)\left(Z^{(3)}-Z^{(1)}Z^{(2)}+\frac{Z^{(1)^3}}{3}\right)+\left(u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial \Delta_R}\right)Z_u^{(1)}u_R\frac{\partial}{\partial u_R}\left(Z^{(2)}-\frac{Z^{(1)^2}}{2}\right)$$
$$+\left(u_R\frac{\partial}{\partial u_R}+\Delta_R\frac{\partial}{\partial \Delta_R}\right)Z_\Delta^{(1)}\Delta_R\frac{\partial}{\partial \Delta_R}\left(Z^{(2)}-\frac{Z^{(1)^2}}{2}\right)=0 \ . \qquad (10b)$$

Thus, the determination of Z from the Feynman graph expansion for $\Gamma^{(2)}$ determines η and identities such as (10) provide checks on the self-consistency of the calculation.

Many of the previous predictions for experimental systems from the nm-component model have been based on truncation of the ϵ expansions and setting ϵ to one. Our own interest in the model was stimulated by the applicability of the cases $m=2$, $n=2$, and $m=2$, $n=3$ to systems which undergo a change in the unit cell of the transition temperature.[5] Both experimental[9] and theoretical[10–13] results indicate that α_m is negative when $m=2$ for three-dimensional systems, and hence the isotropic m-component fixed point is expected to be stable in three dimensions in this case. Truncation of the eigenvalues for the mixed fixed point of $O(\epsilon^2)$ and setting ϵ to one for the cases $m=2$, $n=2$, and $m=2$, $n=3$ leads to positive values for both eigenvalues.[5] If the mixed fixed point is stable in three dimensions for these cases, we would have the interesting case of a region of n and m values for which two fixed points are simultaneously stable. However, our present results show that, for the cases $m=2$, $n=2$, and $m=2$, $n=3$ and for the case $m=1$, $n=3$ (of interest in the study of systems with cubic anisotropy), the second eigenvalue λ_2 (Table II) changes in sign depending on the order at which the expansion is truncated, and the relative magnitude of the coefficients indicates that truncation of the expansion is probably an unreliable device (c.f. Ref. 6) for these eigenvalues.

It might be hoped that comparison of experimentally determined values of the critical exponents with the values determined for different fixed points would indicate which fixed points are stable in three dimensions. It is therefore highly desirable to extend the known ϵ expansions for critical exponents at the mixed fixed point in order that proper resummation methods may be employed. Our present results for η are a first step in this direction.

A particular difficulty arises in trying to identify the appropriate fixed point from experimentally determined exponents for systems with $n=2$, $m=2$. Previously,[5] it has been noted that to order ϵ^3 the coefficients in the expansion of η at the mixed fixed point are the same as those for a four-component isotropic system. [A similar result holds to $O(\epsilon^2)$ for othe exponents.] We find that this remains true for the $O(\epsilon^4)$ coefficient (and is also true for the other four-component system listed in Table III, $n=4$, $m=1$).

In summary, reliable determination of the stability of the mixed fixed point in the nm model seems to require the calculation of further terms in the ϵ expansion as well as an appropriate resummation procedure.[10–13]

Note added. Since the submission of this manuscript, a detailed study of the most general four-component ϕ^4 field model to $O(\epsilon^2)$ has appeared.[14]

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