

HILBERT FUNCTIONS IN MONOMIAL ALGEBRAS

by

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Abstract

In this thesis, we study Hilbert functions of monomial ideals in the polynomial ring and the Kruskal-Katona ring. In particular, we classify Gotzmann edge ideals and, more generally, Gotzmann squarefree monomial ideals. In addition, we discuss Betti numbers of Gotzmann ideals and measure how far certain edge ideals are from Gotzmann. This thesis also contains a thorough account the combinatorial relationship between lex segments and Macaulay representations of their dimensions and codimensions.

List of Abbreviations and Symbols Used

$a^{(d)}$	d -th upper Macaulay pseudopower of a , 31
$a^{(d)}$	d -th upper Kruskal-Katona pseudopower of a , 31
$a_{(d)}$	d -th lower Macaulay pseudopower of a , 31
$a_{(d)}$	d -th lower Kruskal-Katona pseudopower of a , 31
$\beta_i(I)$	Total Betti numbers of a homogeneous ideal I , 47
$\beta_{i,j}(I)$	Graded Betti numbers of a homogeneous ideal I , 47
Δ	Simplicial complex, 11
Δ^\vee	Alexander dual of a simplicial complex Δ , 14
I^\vee	Alexander dual of Stanley-Reisner ideal I , 14
Δ_I	Stanley-Reisner complex of a squarefree monomial ideal I , 12
ΔV	Lower shadow of a homogeneous vector space V , 8
$\Delta_R V$	Lower shadow of a homogeneous vector space V in ring R , 92
∇V	Upper shadow of a homogeneous vector space V , 8
$\nabla_R V$	Upper shadow of a homogeneous vector space V in ring R , 87
$E(V)$	Exterior algebra of a vector space V , 17
$\text{gens } I$	Minimal monomial generating set of a monomial ideal I , 8
$\text{gens } V$	Monomial basis of a monomial vector space V , 8
$\text{gin}_\sigma(I)$	Generic initial ideal of I with respect to term order σ , 62
$>_{\text{lex}}$	Lexicographically greater than, 19
HF_M	Hilbert function of a graded module M , 9
HS_M	Hilbert series of a graded module M , 9
I_Δ	Stanley-Reisner ideal of a simplicial complex Δ , 12
$I(G)$	Edge ideal of a simple graph G , 53
$\text{in}_\sigma(f)$	Leading term of a polynomial f , 11
$\text{in}_\sigma(I)$	Initial ideal of an ideal I , 11
I^{sf}	Image of $I \subseteq S$ in the Kruskal-Katona ring Q , 13
$\mathbb{k}[\Delta]$	Stanley-Reisner ring S/I_Δ of a simplicial complex Δ , 12
$\text{Lex}(d, m)$	Lex segment of degree d and dimension m , 19

$\text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}}$	Lex segment with last monomial $\mathbf{x}^{\mathbf{a}}$, 19
M_d	Homogeneous component of a graded module M , 6
$\text{mrep}_d(a)$	d -th Macaulay representation of the integer a , 22
\mathbb{N}	Natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$, 6
$\Phi_{\mathbf{A}, \omega}(t)$	Ordinary generating function of weighted set (\mathbf{A}, ω) , 81
$\Psi_{\mathbf{A}, \omega}(t)$	Exponential generating function of weighted set (\mathbf{A}, ω) , 81
Q	Kruskal-Katona ring $\mathbb{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$, 7
S	Polynomial ring $\mathbb{k}[x_1, \dots, x_n]$, 6
$T(V)$	Tensor algebra of a vector space V , 16
$\mathbf{x}^{\mathbf{a}}$	Monomial with exponent vector $\mathbf{a} = (a_1, \dots, a_n)$, 7

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Chapter 1

Introduction

1.1 Background

Every standard graded algebra R over a field \mathbb{k} decomposes into a sequence of vector spaces R_0, R_1, R_2 , and so on, and the dimensions of these vector spaces form an important invariant in commutative algebra and combinatorics.

There are a number of standard ways to encode this data: we can write this sequence as a function $\text{HF}_R(d) = \dim_{\mathbb{k}} R_d$, called the Hilbert function of R , or we can encode it in the coefficients of the Hilbert series $\text{HS}_R(t) = \sum_{d=0}^{\infty} (\dim_{\mathbb{k}} R_d) t^d$. It was first shown by Hilbert [Hil90] that this sequence of dimensions is eventually polynomial; there is a polynomial HP_R with $\text{HP}_R(d) = \text{HF}_R(d)$ for all sufficiently large integers d .

There are many different facets to the study of Hilbert functions and I will try to outline them here. First, there are many well known results on the classification of Hilbert functions in various settings that date back to Macaulay but also some which have more modern interpretations. Along with these are the well known connections between Hilbert functions and other numerical invariants of graded algebras. Second, there is the study of the growth of Hilbert functions and its interplay with Castelnuovo-Mumford regularity, Hilbert polynomials, and the Gotzmann property. This can be the entrance of a great deal of modern algebraic geometry, though this thesis will focus on combinatorial aspects of the Gotzmann property. Finally, there are strong connections between Hilbert functions and simplicial (and polyhedral) combinatorics. I will now touch on each of these aspects in a bit more depth.

Macaulay's theorem [Mac27] is a celebrated result that gives numerical bounds on the Hilbert function of a standard graded \mathbb{k} -algebra R . What it says is that, for all $d \geq 1$,

$$\text{HF}_R(d+1) \leq \text{HF}_R(d)^{\langle d \rangle}$$

for some numerical operation $(-)^{\langle d \rangle}$ that we will call a pseudopower. That is to say, the dimension of R_{d+1} is constrained by a function of the dimension of R_d , and for this reason we say that Macaulay's theorem bounds the growth of the Hilbert function. This condition is also sufficient for a Hilbert function; any function $H : \mathbb{N} \rightarrow \mathbb{N}$ is the Hilbert function of a standard graded \mathbb{k} -algebra if it satisfies Macaulay's inequalities.

Macaulay's theorem has been the inspiration for many similar classifications. Stanley first wrote Macaulay's theorem in its modern form in [Sta75a] and used it as the basis for the classifications of h -vectors of Cohen-Macaulay and Gorenstein domains, spawning a long line of study [Sta91, Sta78].

Ideals which achieve Macaulay's bound, at least componentwise, are called Gotzmann after Gerd Gotzmann who showed that this extremal growth persists once it occurs beyond the generators of the ideals and that this rate of growth is always achieved in the degree of the ideal's regularity for saturated ideals I [Got78]. Many modern presentations (e.g. [BH93] and [KR05]) of Macaulay's theorem and Gotzmann's persistence and regularity theorems are based on proofs by Mark Green [Gre89]. One should also refer to Green's lecture notes [Gre98] for detailed proofs of these results.

Murai and Hibi, whose work is important to this thesis in numerous ways, have used generic initial ideals to classify Gotzmann ideals with few generators [MH08, Mur07]. A generic initial ideal is a deformation of an ideal into one that is Borel-fixed and this is done while preserving its Hilbert function and increasing its Betti numbers. Generic initial ideals arose in Hartshorne's celebrated proof that the Hilbert scheme – the space of all saturated homogeneous ideals with the same Hilbert polynomial – is connected [Har66].

Finally, Kruskal proved a theorem on bounding the f -vectors of simplicial complexes in a way similar to Macaulay's theorem [Kru63]. Katona separately proved an equivalent result phrased in terms of Sperner families [Kat68]. Jeff Mermin has rephrased their theorem to exactly parallel Stanley's version of Macaulay's theorem except, in this case, for the Kruskal-Katona ring $\mathbb{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ [Mer08]. The Kruskal-Katona theorem holds equally true in the exterior algebra [AHH97]. The Kruskal-Katona theorem and combinatorial aspects of Macaulay's theorem have

found widespread application in Sperner theory, network reliability, and other graph problems [Eng97, BL05, Spe08, CL69].

In this thesis, we will be looking at Gotzmann ideals in both the polynomial ring and in the Kruskal-Katona ring. Since we will focus on edge ideals and Stanley-Reisner ideals (i.e., squarefree monomial ideals) there will be a lot of interplay between these two rings.

Stanley-Reisner ideals have their own rich history of study which connects simplicial homology to numerous algebraic properties. Probably the most celebrated of these results are Reisner's criterion for Cohen-Macaulayness, Hochster's formula for graded Betti numbers and Stanley's proof of the upper bound conjecture [Sta75b] (see also, [McM70] and [Sta96]). Stanley-Reisner ideals are still the subject of much activity.

1.2 Outline

The primary focus of this thesis is the study of Gotzmann ideals and monomial vector spaces in both the polynomial and Kruskal-Katona rings.

In Chapter 2, we begin with the basic definitions for standard graded algebras, monomial ideals, and Hilbert functions. Next, we introduce the notions required to work with homogeneous components of graded algebras; we define homogeneous and monomial vector spaces and their upper and lower shadows. In Section 2.2, we discuss the Stanley-Reisner correspondence between squarefree monomial ideals and simplicial complexes. Simplicial complexes give a combinatorial way to think about the basis of squarefree monomial quotients of the polynomial ring (and quotients of the Kruskal-Katona ring). In fact, the f -vector of a simplicial complex is essentially the Hilbert function of the Kruskal-Katona ring modulo the Stanley-Reisner ideal of the complex. The last section of Chapter 2 defines the exterior algebra and shows that Hilbert functions of ideals in the exterior algebra are also Hilbert functions of ideals in the Kruskal-Katona ring, and vice versa.

Chapter 3 gives a thorough account of Macaulay's theorem and the Kruskal-Katona theorem. The emphasis of this chapter is on connections between the numerics of Hilbert functions and the combinatorics of lexicographic ideals (which we call

lex ideals for convenience). In each homogeneous component of a lex ideal L , there is a lexicographically smallest monomial m and every monomial that is lexicographically larger than m is also in L . Since a homogeneous component of a lexicographic ideal, called a lex segment, is determined by knowing its smallest monomial m , the dimension and codimension of the component are determined by m .

In Section 3.1 we introduce lex ideals and segments, while in Section 3.2 we show how to explicitly compute the dimension and codimension of a lex segment as sum of binomials which are determined by the segment's last monomial. These sums of binomials are called Macaulay representations. Since upper and lower shadows of lex segments are also lex segments, the dimensions of shadows of lex segments can be described in terms of operations on Macaulay representations called pseudopowers. The various types of pseudopowers are described in Section 3.3. In Sections 3.4 and 3.5, we finally give Macaulay's theorem and the Kruskal-Katona theorem which explain the special role that lex ideals have in describing the Hilbert functions of all homogeneous ideals. We finish Chapter 3 with a discussion of the properties of Gotzmann ideals and a new result, Theorem 3.37, on the Betti numbers of Gotzmann ideals generated in a single degree. This proof relies on the Eliahou-Kervaire resolution of stable ideals and our combinatorial description of dimensions of lex segments (Proposition 3.8).

The results in Chapter 3 are widely known, though few references give comprehensive accounts of the connections between Macaulay representations and lex segments. Macaulay's original paper gave the dimensions of lex segments using Macaulay representations. The book "Computational Commutative Algebra 2" by Kreuzer and Robbiano [KR05] describes the Macaulay representations of both dimensions and codimensions of lex segments and has many useful lemmas for manipulating Macaulay representations. They do not, however, cover the Kruskal-Katona theorem, nor do they discuss lower shadows. Lower shadows and lower Kruskal-Katona pseudopowers are commonly seen in combinatorics since the original papers by Katona [Kat68] and Kruskal [Kru63] used lower pseudopowers (Kruskal also used upper pseudopowers). However, lower shadows and lower Macaulay pseudopowers in the polynomial ring do not get the same treatment. So the first half of Proposition 3.24 and part (4) of Theorem 3.26 (Macaulay's theorem) are an attempt to tell this part of the story. One

should note that the lower pseudopowers described in this thesis are different than those used by Green in his theorem on quotients of algebras by generic hyperplanes.

Proposition 3.8 on Macaulay representations of dimensions of lex segments appears in Macaulay's original paper [Mac27] and also in [BH93], while its counterpart, Proposition 3.11, does not appear in the literature despite being more applicable to Stanley's modern statement of Macaulay's theorem [Sta75a] which we give as Theorem 3.26.

In Chapter 4, we show that all quadratic squarefree Gotzmann ideals come from star graphs (Theorem 4.2). In the process, we show that Gotzmann edge ideals must have fewer variables than generators and hence fall into a classification of Murai and Hibi. This is explored further in Section 4.3 where we give an alternate proof of Theorem 4.2, suggested by an anonymous referee of [Hoe09], which takes advantage of generic initial ideals. Also, in the case of edge ideals with fewer generators than variables, we compute a lower bound on the distance an edge ideal is from Gotzmann. In certain cases, this bound is exact (see Example 4.8). This chapter is based on the paper [Hoe09] by the author.

Chapter 5 extends the previous result on edge ideals to all squarefree monomial ideals. Most of this is joint work with Jeff Mermin and based on the paper [HM10]. In particular, in Theorem 5.9, we characterize all Gotzmann squarefree monomial ideals of the polynomial ring and we enumerate them in Corollary 5.22. In Section 5.3, we give decomposition and reconstruction theorems (Theorem 5.32 and Theorem 5.34) for Gotzmann squarefree monomial ideals in the Kruskal-Katona ring. A full classification of these ideals is not currently known. Finally, in Section 5.4, we show that there are few Gotzmann ideals with Gotzmann Alexander duals.

Chapter 2

Monomial Algebras

2.1 Graded Algebras

Throughout this thesis $S = \mathbb{k}[x_1, \dots, x_n]$ will denote the **polynomial ring** in n variables over a field \mathbb{k} . The polynomial ring is a graded algebra in the sense that S decomposes as a direct sum of vector spaces,

$$S = \bigoplus_{d \in \mathbb{N}} S_d,$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$. Each S_d , called a **homogeneous component** of S , is a vector space spanned by monomials of degree d . If every term of a polynomial f has the same degree d (i.e., $f \in S_d$) then we say that f is a **homogeneous polynomial**. The product of two homogeneous polynomials f and g is another homogeneous polynomial whose degree is the sum of the degrees of f and g . An ideal is called a **homogeneous ideal** if it is generated by homogeneous polynomials.

A **commutative \mathbb{k} -algebra** is any commutative ring which contains \mathbb{k} as a subring. Consequently, commutative \mathbb{k} -algebras are also \mathbb{k} -vector spaces. We will only be concerned with commutative algebras, except for one exceptional case – namely, the exterior algebra. **Graded algebras**, R , like the polynomial ring, must decompose as $R = \bigoplus_{d \in \mathbb{Z}} R_d$ and have the property that the product of two **homogeneous elements** of R , namely $f \in R_d$ and $g \in R_e$, is again homogeneous of the appropriate degree; $fg \in R_{d+e}$. Naturally, the identity of R must have degree zero, but the generators of R as an algebra need not all have degree one. If all generators have degree one, we say R is **standard graded \mathbb{k} -algebra**. Throughout this thesis, we will only be concerned with standard graded algebras, though non-standard graded versions of some of the subject matter is known.

Despite this generality, standard graded \mathbb{k} -algebras are really concrete. Every standard graded (commutative) \mathbb{k} -algebra R is isomorphic to a quotient S/I of the

polynomial ring by a homogeneous ideal (and vice versa). Here the number of generators of S , which we always denote n , must be (at least) the number of algebra generators of R .

There are two graded algebras which are the focus of this thesis because of their nice combinatorial structure: the polynomial ring S and the **Kruskal-Katona ring**

$$Q = \mathbb{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$$

which will be denoted by Q throughout this thesis.

The polynomial ring can be thought of as the vector space spanned by all monomials. We will denote **monomials** by $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ or simply use m, m' , and so on when the **exponent vector** \mathbf{a} of $m = \mathbf{x}^{\mathbf{a}}$ is not needed. The **support** of a monomial m is the set variables dividing it. The Kruskal-Katona ring, as a vector space, is spanned by all **squarefree monomials** which we will denote by $x_{i_1} \cdots x_{i_d}$ for integers $1 \leq i_1 < \cdots < i_d \leq n$. Often we will refer to monomials in the Kruskal-Katona ring and it is implied that these are squarefree. Polynomials in the Kruskal-Katona ring multiply just as they do in the polynomial ring, except that any term containing the square of a variable becomes zero.

The ideals of any graded algebra R (and in particular, the ideals of S) are called **homogeneous ideals** if they are generated by homogeneous elements. Homogeneous ideals I also have a graded structure. That is to say, they can be decomposed as a direct sum of vector spaces

$$I = \bigoplus_{d \in \mathbb{N}} I_d$$

with $f \in R_d$ and $g \in I_e$ giving $fg \in I_{d+e}$. Again, we call each $I_d \subseteq R_d$ a **homogeneous component** (or graded component) of I .

Ideals of either S or Q which are generated by monomials are called **monomial ideals**. Monomial ideals are clearly homogeneous ideals and so, naturally decompose into homogeneous components. Of particular interest are **squarefree monomial ideals**, often called **Stanley-Reisner ideals**, which are ideals generated by square-free monomials. All monomial ideals of Q are squarefree monomial ideals. Each monomial ideal I has a uniquely determined set of **minimal monomial generators** which we denote $\text{gens } I$. Quotients of the polynomial ring by monomial ideals can

be referred to as **monomial algebras** and squarefree quotients are called **Stanley-Reisner rings** (we cover Stanley-Reisner rings in detail in Section 2.2).

One last homogeneous ideal of interest is the **homogeneous maximal ideal**

$$\mathfrak{m} = (x_1, \dots, x_n)$$

which can be thought of as an ideal in S or Q .

Though our primary interest is in homogeneous ideals and their quotient rings, we will often find it useful to consider the graded components of these ideals separate from the whole. Therefore, we define a **degree d homogeneous vector space** of S to be a subspace of S_d and a **degree d monomial vector space** to be a degree d homogeneous vector space spanned by monomials. We also use this terminology for subspaces of Q_d .

The homogeneous components of a monomial ideal are monomial vector spaces. Also, every monomial vector space $V \subseteq S_d$ generates a monomial ideal $I = (V)$ which is generated in degree d . We use $\text{gens } V$ to denote the **monomial basis** of a monomial vector space V .

Given a degree d homogeneous vector space V , we define its **upper shadow** to be the homogeneous vector space

$$\nabla V = \text{span}_{\mathbb{k}}\{x_i f \mid f \in V, 1 \leq i \leq n\}.$$

The upper shadow of a homogeneous component I_d of an ideal I is the space of forms lying above I_d . That is to say, $\nabla I_d = (\mathfrak{m} I)_{d+1}$ where $\mathfrak{m} = (x_1, \dots, x_n)$.

The **lower shadow** of a homogeneous vector space $V \subset S_d$ is defined to be

$$\Delta V = \text{span}_{\mathbb{k}}\{f \in S_{d-1} \mid \forall i, x_i f \in V\}.$$

In terms of ideals, we can equivalently define the lower shadow as

$$\Delta V = ((V) : \mathfrak{m})_{d-1}$$

where $(I : J) = \{f \in S \mid fJ \subseteq I\}$. Note that $V \subseteq \Delta \nabla V$ and $\nabla \Delta V \subseteq V$. For an example of both upper and lower shadows, see Example 2.3.

We will often drop the word upper when referring to upper shadows since they will be used more frequently than lower shadows.

We now define Hilbert functions which are numerical invariants of graded algebras and homogeneous ideals.

Definition 2.1 (Hilbert Function, Hilbert Series). Let R be a standard graded \mathbb{k} -algebra and let M be a positively graded R -module (e.g., $M = I$ or $M = R/I$ for a homogeneous ideal I). The **Hilbert function** of a M is the function $\text{HF}_M : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$\text{HF}_M(d) = \dim_{\mathbb{k}} M_d$$

while the **Hilbert series** of M is the generating series

$$\text{HS}_M(t) = \sum_{d \in \mathbb{N}} (\dim_{\mathbb{k}} M_d) t^d.$$

There is a simple relationship between the Hilbert function of a homogeneous ideal $I \subseteq R$ and the Hilbert function of its quotient ring R/I . The homogeneous components $(R/I)_d$ of the quotient ring are isomorphic to R_d/I_d and hence

$$\dim_{\mathbb{k}}(R/I)_d = \dim_{\mathbb{k}} R_d/I_d = \text{codim}_{\mathbb{k}} I_d,$$

where $\text{codim}_{\mathbb{k}} I_d = \dim_{\mathbb{k}} R_d - \dim_{\mathbb{k}} I_d$ is the **codimension** of I_d as a subspace of R_d . Thus, in every degree d ,

$$\text{HF}_{R/I}(d) = \text{HF}_R(d) - \text{HF}_I(d).$$

The following is an example of much of the terminology defined in this section. For examples in only $n = 3$ variables, we switch from variables x_1, x_2, x_3 to the variables x, y, z for readability.

Example 2.2. Let $S = \mathbb{k}[x, y, z]$. The ideal $I = (xy, xz^2, yz^3)$ is a monomial ideal with generators $\text{gens } I = \{xy, xz^2, yz^3\}$. The degree 0 and degree 1 components of I are trivial vector spaces. In degrees two and higher we have,

$$\begin{array}{ll} I_2 = \text{span}_{\mathbb{k}}\{xy\} & \text{HF}_I(2) = 1 \\ I_3 = \text{span}_{\mathbb{k}}\{x^2y, xy^2, xyz, xz^2\} & \text{HF}_I(3) = 4 \\ I_4 = \text{span}_{\mathbb{k}}\{x^3y, x^2y^2, xy^3, x^2yz, xy^2z, xyz^2, x^2z^2, xz^3, yz^3\} & \text{HF}_I(4) = 9 \\ \vdots & \vdots \end{array}$$

and so on. The homogeneous components $(S/I)_d$ of the quotient ring S/I are isomorphic to S_d/I_d . Thus, a basis for $(S/I)_d$ is given by equivalence classes of the monomials not in I_d :

$$\begin{array}{ll}
 (S/I)_0 \cong \text{span}_{\mathbb{k}}\{1\} & \text{HF}_{S/I}(0) = 1 \\
 (S/I)_1 \cong \text{span}_{\mathbb{k}}\{x, y, z\} & \text{HF}_{S/I}(1) = 3 \\
 (S/I)_2 \cong \text{span}_{\mathbb{k}}\{x^2, y^2, yz, z^2\} & \text{HF}_{S/I}(2) = 5 \\
 (S/I)_3 \cong \text{span}_{\mathbb{k}}\{x^3, x^2z, y^2z, yz^2, z^3\} & \text{HF}_{S/I}(3) = 6 \\
 (S/I)_4 \cong \text{span}_{\mathbb{k}}\{x^4, x^3z, y^4, y^3z, y^2z^2, z^4\} & \text{HF}_{S/I}(4) = 6 \\
 \vdots & \vdots
 \end{array}$$

and so on.

Example 2.3. Consider the degree 3 homogeneous component of $S = \mathbb{k}[x, y, z]$ which is depicted on the left of Figure 2.1. For each of the six monomials in S_2 we have drawn a triangle around the monomial's upper shadow in S_3 . For example, a triangle is drawn around y^3, xy^2 and y^2z as these are a basis for the upper shadow of $\text{span}_{\mathbb{k}}\{y^2\}$.

The monomial vector space $V = \text{span}_{\mathbb{k}}\{x^2y, xy^2, xyz, y^3, y^2z, x^3, xz^2, z^3\} \subset S_3$ is depicted on the middle of Figure 2.1. The lower shadow of V is $\Delta V = \text{span}_{\mathbb{k}}\{xy, y^2\}$ since these are the only monomials in S_2 which have their upper shadows entirely within V . Thus $\nabla \Delta V = \text{span}_{\mathbb{k}}\{x^2y, xy^2, xyz, y^3, y^2z\} \neq V$ as depicted on the right of Figure 2.1.

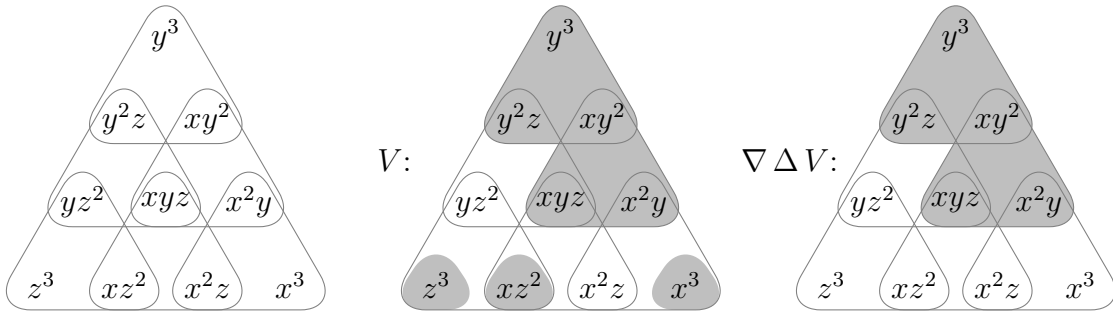


Figure 2.1: Upper and lower shadows.

In this thesis, our focus will primarily be on monomial and squarefree monomial ideals. In order to justify this focus, we now highlight the connection between monomial ideals and homogeneous ideals through initial ideals.

A **term order** σ on S or Q is a total order $>_\sigma$ on its monomial basis which has the property that $m''m >_\sigma m''m'$ whenever m, m' and m'' are three monomials with $m >_\sigma m'$. There are numerous term orders that one can choose for S or Q ; in Section 3.1 we will define the lexicographic term order, but for now an arbitrary term order will do.

If we are given a polynomial $f = \sum_{i=1}^k a_i m_i$ where $a_i \in \mathbb{k}$ and each m_i is a distinct monomial, we define the **initial term** of f to be $\text{in}_\sigma(f) = a_i m_i$ where m_i is the largest monomial with respect to σ with $a_i \neq 0$. Given an ideal I , we define its **initial ideal** to be

$$\text{in}_\sigma(I) = \text{span}_{\mathbb{k}}\{\text{in}_\sigma(f) \mid f \in I\}$$

which is an ideal in itself. Also, if $m \in \text{in}_\sigma(I)$ then there exists a polynomial $f \in I$ with $\text{in}_\sigma f = m$ (we can ignore coefficients because we are working over a field).

Taking the initial ideal of a homogeneous ideal I produces a monomial ideal. Initial ideals have many nice properties, but what we are most interested in is the following:

Theorem 2.4 (Macaulay’s Basis Theorem) [Mac27]. *Let I be a homogeneous ideal in S or Q and let σ be a term ordering. Then the initial ideal $\text{in}_\sigma(I)$ has the same Hilbert function as I .*

In [Mac27], Macaulay first described the reverse lexicographic order and noted that the span of the leading terms of an ideal, with respect to this order, “evidently satisfy the test for an ideal”. The Hilbert functions of I and $\text{in}_\sigma(I)$ are the same since the monomials in $\text{in}_\sigma(I)_d$ appear as the leading terms of the basis of I_d constructed by Gauss-Jordan elimination; in modern terms, we call this a reduced Gröbner basis. Macaulay’s basis theorem is named after these observations. Initial ideals have since been subsumed into Gröbner basis theory [Eis95, KR00].

2.2 Stanley-Reisner Rings

Throughout this thesis, we will be interested in squarefree monomial ideals in both S and Q . These ideals are best understood through the Stanley-Reisner correspondence between squarefree monomial ideals and simplicial complexes.

Definition 2.5 (Simplicial Complex). A **simplicial complex** Δ is a collection of subsets of the ground set $X = \{x_1, \dots, x_n\}$ with the property that $F \in \Delta$ and $E \subset F$ imply $E \in \Delta$.

Sets $F \in \Delta$ are called **faces** and their **dimensions** are given by $\dim F = |F| - 1$. The **dimension** of a non-empty simplicial complex is the largest dimension of its faces. Thus the simplicial complex $\{\emptyset\}$ has dimension -1 as its only face, the empty set, has dimension -1 . The dimension of the empty simplicial complex \emptyset is $-\infty$.

Definition 2.6 (f-vector). The **f-vector** of a d -dimensional simplicial complex is the vector $(f_{-1}, f_0, f_1, \dots, f_d) \in \mathbb{N}^{d+2}$ where f_i is the number of faces in Δ with dimension i .

Note that non-empty simplicial complexes contain the empty face, giving $f_{-1} = 1$.

A simplicial complex is often described by a list of its maximal faces with respect to inclusion, called **facets**. We use the notation $\Delta = \langle F_1, \dots, F_k \rangle$ for the simplicial complex with facets F_1, \dots, F_k . For brevity, we often write each facet $\{x_{i_1}, \dots, x_{i_d}\}$ as a squarefree monomial $x_{i_1} \cdots x_{i_d}$. A face of Δ is simply a subset of some facet F_i .

Every squarefree monomial ideal I corresponds to a simplicial complex Δ_I called the **Stanley-Reisner complex** of I which is defined as

$$\Delta_I = \left\{ F \subseteq \{x_1, \dots, x_n\} \mid \prod_{x_i \in F} x_i \notin I \right\}.$$

The ideal I can be recovered from Δ_I by taking the ideal generated by squarefree monomials $x_{i_1}x_{i_2} \cdots x_{i_k}$ for which $\{x_{i_1}, \dots, x_{i_k}\}$ is not a face Δ_I . It is easy to check that this gives a bijective correspondence between squarefree monomial ideals and simplicial complexes.

We use I_Δ to denote the squarefree monomial ideal whose Stanley-Reisner complex is Δ and we call I_Δ the **Stanley-Reisner ideal** of Δ . The **Stanley-Reisner ring** of a simplicial complex Δ , denoted $\mathbb{k}[\Delta]$, is the quotient ring S/I_Δ . Stanley-Reisner rings and their Hilbert functions are discussed at length in [BH93]. One should think of Δ_I as the simplicial complex formed by the squarefree monomials in the monomial basis of S/I .

Example 2.7. The Stanley-Reisner complex of the ideal $I = (x_1x_2x_3, x_1x_4)$ contained

in $\mathbb{k}[x_1, \dots, x_4]$ has faces $\{x_2, x_3, x_4\}$, $\{x_1, x_2\}$, $\{x_1, x_3\}$ and all subsets thereof. It is two dimensional and its f -vector is $(1, 4, 5, 1)$.

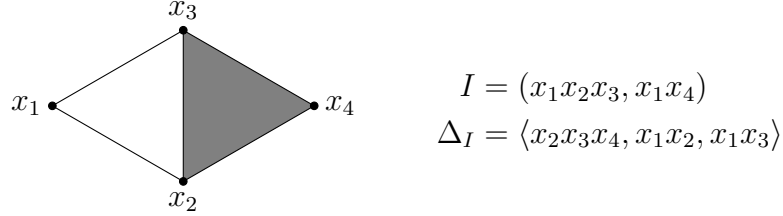


Figure 2.2: Stanley-Reisner ideal and complex.

Recall that monomial ideals in the Kruskal-Katona ring $Q = \mathbb{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ are squarefree monomial ideals in that every non-squarefree generator is zero and hence superfluous. Thus, every monomial ideal in Q can be written as the image of a squarefree monomial ideal I in the canonical map $S \rightarrow Q$. Given a squarefree monomial ideal $I \subseteq S$, we write $I^{\text{sf}} \subseteq Q$ for the image of I in Q . We call I^{sf} the **squarefree image** of I .

Consequently, simplicial complexes are in one-to-one correspondence with squarefree monomial ideals of S which, in turn, correspond bijectively to monomial ideals of Q . Given a simplicial complex Δ , its **Stanley-Reisner ideal in Q** is simply $I_{\Delta}^{\text{sf}} = (I_{\Delta})^{\text{sf}}$.

In particular, each face of $F = \{x_{i_1}, \dots, x_{i_d}\} \in \Delta$ corresponds to a squarefree monomial $\mathbf{x}_F = x_{i_1} \cdots x_{i_d}$ in the monomial basis of Q/I_{Δ}^{sf} . This leads to the following basic observation about f -vectors and Hilbert functions:

Proposition 2.8. *Let $I_{\Delta} \subseteq S$ be the Stanley-Reisner ideal of Δ and let I_{Δ}^{sf} be the image of I_{Δ} in the Kruskal-Katona ring Q . The Hilbert function of Q/I_{Δ}^{sf} is*

$$\text{HF}_{Q/I_{\Delta}^{\text{sf}}}(d) = \begin{cases} f_{d-1} & 0 \leq d \leq \dim \Delta + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $(f_{-1}, f_0, \dots, f_{\dim \Delta})$ is the f -vector of Δ .

One useful property of the Stanley-Reisner correspondence is that the Hilbert function of $\mathbb{k}[\Delta]$ can easily be described in terms of the f -vector of Δ . On the other

hand, it is often difficult to explicitly describe Δ_I when I is a complicated ideal. That is to say, it is computationally difficult to compute the generators of I from the facets of Δ_I or vice versa.

The following straightforward observation is Theorem 5.1.7 of [BH93].

Theorem 2.9. *Let $\mathbb{k}[\Delta]$ be a Stanley-Reisner ring and let $(f_{-1}, f_0, \dots, f_{\dim \Delta})$ be the f -vector of Δ . Then the Hilbert function of $\mathbb{k}[\Delta]$ is*

$$\mathrm{HF}_{\mathbb{k}[\Delta]}(d) = \begin{cases} 1 & d = 0 \\ \sum_{i=0}^{\dim \Delta} f_i \binom{d-1}{i} & d > 0. \end{cases}$$

Our convention here and throughout this thesis is that **binomial coefficients** are defined as

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(b-a)!} & a \geq 0, b \geq 0, b - a \geq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\binom{0}{0} = 1$.

Corollary 2.10. *Let $I \subseteq S$ be a squarefree monomial ideal in the polynomial ring and let I^{sf} be its image in the Kruskal-Katona ring Q . Then*

$$\mathrm{HF}_{S/I}(d) = \begin{cases} \sum_{i=1}^d \mathrm{HF}_{Q/I^{\mathrm{sf}}}(i) \binom{d-1}{d-i} & d > 0 \\ 1 & d = 0. \end{cases}$$

Consequently,

$$\mathrm{HS}_{S/I}(t) = \mathrm{HS}_{Q/I^{\mathrm{sf}}}\left(\frac{t}{1-t}\right).$$

Proof. The first statement about Hilbert functions follows from Proposition 2.8 and Theorem 2.9. The second part is simply a restatement of the first. \square

A useful aspect of Theorem 2.9 is that the dimension of I_d is determined by the Hilbert function of I^{sf} in degrees d and less. The converse is also true: the dimension of I_d^{sf} is determined by the Hilbert function of I in degrees d and less. This can be seen most easily from Corollary 2.10 by substituting $t = s/(1+s)$ to obtain

$$\mathrm{HS}_{Q/I^{\mathrm{sf}}}(s) = \mathrm{HS}_{S/I}\left(\frac{s}{1+s}\right).$$

A final important tool for working with Stanley-Reisner complexes is Alexander duality.

Definition 2.11 (Alexander Dual). Let Δ be a simplicial complex on vertices $V = \{x_1, \dots, x_n\}$. The **Alexander dual** of Δ is the simplicial complex

$$\Delta^\vee = \{F \subseteq V \mid V \setminus F \notin \Delta\}.$$

The **Alexander dual** of a Stanley-Reisner ideal $I = I_\Delta \subseteq S$ (or Q) is the ideal $I^\vee = I_{\Delta^\vee}$ in S (resp. Q).

Topologically, the Alexander dual of compact locally contractible subspace of the sphere is simply its complement in the sphere. Alexander duality of simplicial complexes, however, is best understood in terms of polarity [Bay96] or as a type of Koszul duality [BH93, BH97].

In terms of simplicial homology, the Alexander dual satisfies

$$H_i(\Delta; \mathbb{k}) \cong H_{n-i-3}(\Delta^\vee; \mathbb{k})$$

or more generally $H_i(\Delta; G) \cong H^{n-i-3}(\Delta^\vee; G)$ when G is a group [ER98, Bay96]. This homological relationship has been widely employed in the study of Stanley-Reisner rings. We will not have cause to use this formula, however, since our focus is on more direct combinatorial implications of Alexander duality.

Lemma 2.12. *Let Δ be a simplicial complex with n vertices and with Alexander dual Δ^\vee . If the f -vectors of Δ and Δ^\vee are $(f_i)_{i=-1}^{\dim \Delta}$ and $(f_i^\vee)_{i=-1}^{\dim \Delta}$ respectively, then*

$$f_i^\vee = \binom{n}{i+1} - f_{n-i-2}$$

for $i = -1, 0, \dots, n$ and where $f_i = 0$ for $i > \dim \Delta$ (and similarly for f_i^\vee).

Proof. The i -dimensional faces of Δ^\vee are of size $i+1$ and correspond to non-faces of Δ of size $n - (i+1)$, or in other words, non-faces of dimension $n - i - 2$. \square

Proposition 2.13. *Let Q be the Kruskal-Katona ring on n variables. A function $H : \mathbb{N} \rightarrow \mathbb{N}$ is the Hilbert function of a homogeneous ideal $I \subseteq Q$ if and only if $d \mapsto H(n-d)$ is the Hilbert function of a quotient Q/J for some homogeneous ideal J .*

Proof. We can assume that I is squarefree monomial by taking initial ideals.

Let Δ be the Stanley-Reisner complex of I and let $(f_i)_{i=-1}^{\dim \Delta}$ and $(f_i^\vee)_{i=-1}^{\dim \Delta^\vee}$ be the f -vectors of Δ and Δ^\vee respectively. By Theorem 2.8 and the previous lemma, the Hilbert function of Q/I_{Δ^\vee} is

$$\begin{aligned} \text{HF}_{Q/I_{\Delta^\vee}}(d) &= f_{d-1}^\vee \\ &= \binom{n}{d} - f_{n-d-1} \\ &= \binom{n}{n-d} - \text{HF}_{Q/I}(n-d) \\ &= \text{HF}_I(n-d). \end{aligned}$$

The same argument works in reverse if we start with the initial ideal of J and pick I to be the Alexander dual of the initial ideal of J . \square

2.3 Exterior Algebras

The exterior algebra, like the Kruskal-Katona ring, is a graded \mathbb{k} -algebra which has squarefree monomials for its \mathbb{k} -basis. The exterior algebra is used more frequently than the Kruskal-Katona ring to model the f -vectors of simplicial complexes. One advantage of exterior algebras is that they work nicely with changes of coordinates and generic initial ideals (which are defined later in Definition 4.10), while Kruskal-Katona rings do not. In fact, generic initial ideals in the exterior algebra have been used so widely that combinatorialists have renamed the operation of taking generic initial ideals to “algebraic shifting” [Kal02, Her02].

Since we will not use these techniques, we will be content to stick with the Kruskal-Katona ring which is otherwise easier to work with. We briefly describe the exterior algebra below and then prove, in Theorem 2.14, that any Hilbert function of a quotient of the exterior algebra is the Hilbert function of a quotient of the Kruskal-Katona ring.

Let V be a \mathbb{k} -vector space with basis e_1, \dots, e_n . The **tensor algebra** on V is

$$T(V) = \bigoplus_{d=0}^{\infty} V^{\otimes d}$$

where $V^{\otimes d} = \underbrace{V \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} V}_{d\text{-fold tensor}}$. The product of two **elementary tensors** $f_1 \otimes \cdots \otimes f_d$ and $g_1 \otimes \cdots \otimes g_e$ in $T(V)$ is simply $f_1 \otimes \cdots \otimes f_d \otimes g_1 \otimes \cdots \otimes g_e \in V^{\otimes(d+e)}$. This product extends linearly to all of $T(V)$. One can think of the tensor algebra as isomorphic to the non-commutative polynomial ring $\mathbb{k}\langle e_1, \dots, e_n \rangle$ where a word $e_{i_1} \cdots e_{i_d}$ is identified with the elementary tensor $e_{i_1} \otimes \cdots \otimes e_{i_d}$.

The exterior algebra $E = E(V)$ on a vector space V is the quotient $T(V)/I$ by the two-sided ideal

$$I = (f \otimes f \mid f \in T(V)).$$

The ideal I contains the relations $e_j \otimes e_i = -e_i \otimes e_j$ for any indices $i < j$ and, more generally, I contains the relations $f \otimes g = (-1)^{(\deg f)(\deg g)} g \otimes f$ for any homogeneous forms $f, g \in E(V)$. We call this **anticommutative multiplication**. In the exterior algebra, it is conventional to write tensors $f \otimes g$ as **wedge products**, $f \wedge g$, to suggest anticommutative multiplication.

From anticommutativity we can see that any elementary wedge product $e_{i_1} \wedge \cdots \wedge e_{i_d}$ can be written in ascending order (i.e., $i_1 < \cdots < i_d$) with an appropriate change of sign. Also, since $e_i \otimes e_i = 0$, we see that the squarefree wedge products of the form $e_{i_1} \otimes \cdots \otimes e_{i_d}$, where e_{i_j} is in our distinguished basis of V and where $i_1 < \cdots < i_d$, give a basis for E . A **monomial** in E is a wedge product of the form $e_{i_1} \wedge \cdots \wedge e_{i_d}$ where each e_{i_j} is an element of our fixed basis of V and where $i_1 < \cdots < i_d$. Note that the product of two monomials is another monomial up to a change of sign (which is needed to write the indices in ascending order).

A **term order** σ on E is a total order $>_{\sigma}$ on the monomials of E for which $f >_{\sigma} g$ implies $hf >_{\sigma} \pm hg$ for any monomials f, g and h in E . That is, term orders on monomials of E are defined much like term orders on monomials of S and Q , except that we need to ignore signs that occur from multiplying the monomials.

Likewise, we can define leading terms $\text{in}_{\sigma}(f)$ and initial ideals $\text{in}_{\sigma}(I) = \text{span}_{\mathbb{k}}\{\text{in}_{\sigma}(f) \mid f \in I\}$ for forms $f \in E$ and homogeneous ideals $I \subseteq E$. The ideals $I \subseteq E$ and $\text{in}_{\sigma}(I)$ share the same Hilbert function by Corollary 1.2 of [AHH97].

Theorem 2.14. *A function $h : \mathbb{N} \rightarrow \mathbb{N}$ is the Hilbert function of a quotient of the*

exterior algebra if and only if it is a Hilbert function of a quotient of the Kruskal-Katona ring.

Proof. Given an ideal $I \subseteq E$ we need to produce an ideal of Q with the same Hilbert function. The initial ideal $\text{in}_\sigma I$ has the same Hilbert function as I and its homogeneous components are spanned by monomials.

Thus we can form a graded vector subspace of Q defined as

$$J = \bigoplus_{d=0}^n \text{span}_{\mathbb{k}} \{x_{i_1} \cdots x_{i_d} \mid e_{i_1} \wedge \cdots \wedge e_{i_d} \in (\text{in}_\sigma I)_d\}$$

which has the same Hilbert function as I .

The graded vector space J is, in fact, an ideal. This follows from the observation that monomials in Q multiply in the same way as monomials of E , up to scalar multiplication. Thus, the Hilbert function of $I \subseteq E$ is the same as that of the ideal $J \subseteq Q$.

For the opposite direction, one can follow the same process of taking the initial ideal of an ideal in Q and then forming a graded vector space in E . \square

Chapter 3

Macaulay's Theorem

3.1 Lex Segments

Macaulay's theorem on Hilbert functions numerically describes all Hilbert functions of standard graded \mathbb{k} -algebras. The numerics of Macaulay's theorem are based on the combinatorics of lexicographic ideals which can be used to model the Hilbert function of any homogeneous ideal.

We say $\mathbf{x}^{\mathbf{a}}$ is **lexicographically greater than** (or occurs **lexicographically before**) $\mathbf{x}^{\mathbf{b}}$ and we write $\mathbf{x}^{\mathbf{a}} >_{\text{lex}} \mathbf{x}^{\mathbf{b}}$ if $a_i > b_i$ in the first index i where \mathbf{a} and \mathbf{b} differ. Often, we use **lex** as an abbreviation of lexicographic. The lexicographic order is a term order and hence a total order on the monomials of S .

We order the monomial basis of a homogeneous component S_d of the polynomial ring S lexicographically and we call monomial vector spaces $L \subseteq S_d$ spanned by initial intervals in this set **lex segments**. By an initial interval, we mean a set of monomials in S_d of the form $\{\mathbf{x}^{\mathbf{b}} \in S_d \mid \mathbf{x}^{\mathbf{b}} \geq_{\text{lex}} \mathbf{x}^{\mathbf{a}}\}$ for some monomial $\mathbf{x}^{\mathbf{a}} \in S_d$.

A lex segment can be specified by giving either the degree of the homogeneous component and the vector space dimension of the segment, or, for non-empty lex segments, by giving the lexicographically last monomial in the lex segment which implicitly contains both pieces of information. We will use the following two notations for lex segments:

$$\text{Lex}(d, m) = \text{span}_{\mathbb{k}}\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$$

where $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}$ are the first m monomials of degree d in the descending lexicographic order and

$$\text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}} = \text{span}_{\mathbb{k}}\{\mathbf{x}^{\mathbf{b}} \in \text{gens } S_d \mid \mathbf{x}^{\mathbf{b}} \geq_{\text{lex}} \mathbf{x}^{\mathbf{a}}\}$$

for some monomial $\mathbf{x}^{\mathbf{a}} \in S_d$.

In the Kruskal-Katona ring Q , lex segments are defined similarly with the caveat that $\text{Lex}^Q(d, m)$ is spanned by the first m squarefree monomials of degree d to ensure that $\text{Lex}^Q(d, m)$ has dimension m .

The **shadow of a lex segment** is also a lex segment. For the polynomial ring in particular, the shadow of the lex segment ending in $\mathbf{x}^{\mathbf{a}}$ is the lex segment ending in $\mathbf{x}^{\mathbf{a}}x_n$. In the Kruskal-Katona ring the shadow of $\text{Lex}_{\geq m}^Q$ is the lex segment ending in mx_i where i is the largest index of a variable not appearing in the support of m . If such an index does not exist, the shadow is the trivial vector space.

Example 3.1. In $S = \mathbb{k}[x, y, z]$ ordered lexicographically with $x > y > z$, the lex segment of degree 3 and dimension 7 is

$$\begin{aligned} \text{Lex}(3, 7) &= \text{span}_{\mathbb{k}}\{x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3\} \\ &= \text{Lex}_{\geq y^3}. \end{aligned}$$

The shadow of this lex segments is $\nabla \text{Lex}_{\geq y^3} = \text{Lex}_{\geq y^3z}$.

Example 3.2. In $Q = \mathbb{k}[x_1, \dots, x_5]/(x_1^2, \dots, x_5^2)$, the lex segment of degree 3 and dimension 7 is

$$\begin{aligned} \text{Lex}(3, 7) &= \text{span}_{\mathbb{k}}\{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4\} \\ &= \text{Lex}_{\geq x_2x_3x_4}. \end{aligned}$$

The shadow of this lex segments is $\nabla \text{Lex}_{\geq x_2x_3x_4} = \text{Lex}_{\geq x_2x_3x_4x_5} = Q_4$.

Definition 3.3 (Lex Ideal). Let $L \subseteq S$ or Q be a homogeneous ideal. If each homogeneous component of L is a lex segment, then L is called a lex ideal.

For any function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f(d) \leq \dim_{\mathbb{k}} S_d$, the graded vector space

$$L = \bigoplus_{d \geq 0} \text{Lex}(d, f(d))$$

has Hilbert function f by construction. An important consequence of Macaulay's theorem is that, when f is the Hilbert function of a homogeneous ideal, L is a lex ideal. In order for L to be a lex ideal, it suffices to show that L is an ideal; i.e., it is closed under multiplication by any variable. Indeed, we will see that $\nabla \text{Lex}(d, \text{HF}_I(d)) \subseteq \text{Lex}(d+1, \text{HF}_I(d+1))$ for any homogeneous ideal $I \subseteq S$. This allows us to lexify ideals:

Definition 3.4 (Lexification). Let $I \subseteq S$ be a homogeneous ideal. The lexification of I is the (unique) lex ideal with the same Hilbert function as I and is given by

$$L = \bigoplus_{d \geq 0} \text{Lex}(d, \text{HF}_I(d)).$$

Example 3.5. In Example 2.2, we showed that the ideal $I = (x^3y, yz, xz^3)$ in $\mathbb{k}[x, y, z]$ has Hilbert function $\text{HF}_I(2) = 1$, $\text{HF}_I(3) = 4$, $\text{HF}_I(4) = 9$, and so on. The lexification L of I has homogeneous components

$$L_2 = \text{span}_{\mathbb{k}}\{x^2\},$$

$$L_3 = \text{span}_{\mathbb{k}}\{x^3, x^2y, x^2z, xy^2\}, \text{ and}$$

$$L_4 = \text{span}_{\mathbb{k}}\{x^4, x^3y, x^3z, x^2y^2, x^2yz, x^2z^2, xy^3, xy^2z, xyz^2\}.$$

Even though I is generated in degrees 2, 3 and 4, the lex segments L_2, L_3 and L_4 are not sufficient to generate L as an ideal. The ideal J generated by L_2, L_3, L_4 has minimal generators x^2, xy^2, xyz^2 and Hilbert function

$$\text{HF}_J(2) = 1 = \text{HF}_I(2)$$

$$\text{HF}_J(3) = 4 = \text{HF}_I(3)$$

$$\text{HF}_J(4) = 9 = \text{HF}_I(4)$$

$$\text{HF}_J(5) = 14 < 15 = \text{HF}_I(5)$$

That is to say, the Hilbert function of J is smaller than that of I in degree 5. The full lexification of I is

$$L = (x^2, xy^2, xyz^2, xz^4, y^6).$$

It is natural to ask for a bound on the degrees of the generators of a lexification. It will follow from the Gotzmann persistence theorem (Theorem 3.36) that if I_d is Gotzmann and $\max\{\deg m \mid m \in \text{gens } I\} \leq d$ then all generators of the lexification of I have degree d or less. Determining the value of smallest value of d is a more complicated problem.

3.2 Macaulay Representations

The dimensions and codimensions of homogeneous vector spaces are best expressed as certain binomial sums which we call Macaulay representations. The value of these

representations will be seen in Section 3.3 where they are needed to determine the codimensions of shadows of lex segments. The codimensions of these shadows are crucial to Macaulay's theorem.

In this section, we make the combinatorial connection between the last monomial m in a lex segment L and the Macaulay representations of the dimension and codimension of L . This connection is made for lex segment in both the polynomial ring and the Kruskal-Katona ring. This will be explained further after the following definition and example.

Definition 3.6 (Macaulay Representation). Let $a \geq 0$ and $d > 0$ be integers. Then a can be expressed uniquely as

$$a = \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_1}{1}$$

where the b_i are integers satisfying $b_d > b_{d-1} > \cdots > b_1 \geq 0$. Note that each b_i depends on both a and d . This expression for a is called the **d-th Macaulay representation** of a and the coefficients b_i are called the **d-th Macaulay coefficients**. We will use

$$\text{mrep}_d(a) = (b_d, \dots, b_1)$$

to denote the **sequence of d-th Macaulay coefficients** of a .

The binomials $\binom{b_i}{i}$ are defined to be zero when $b_i = i - 1$, which is the smallest value the Macaulay coefficients can take on while still satisfying $b_d > \cdots > b_1 \geq 0$. When necessary, we will drop the zero terms in a Macaulay representation to write a as

$$a = \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_k}{k}$$

where $b_d > b_{d-1} > \cdots > b_k \geq k$. For convenience and clarity, we introduce the notation

$$(b_d, \dots, b_k; k-1) = (b_d, \dots, b_k, k-2, \dots, 0).$$

This notation is meant to indicate that the last $k-1$ terms of a Macaulay representation are zero.

Macaulay representations appeared in Macaulay's original paper [Mac27] in expressions for the dimensions of lex segments, but Kruskal [Kru63] gave the first

thorough treatment of Macaulay representations, calling them “canonical representations”.

Macaulay coefficients can be computed by the following greedy algorithm: assuming b_d, \dots, b_{i+1} have already been determined, take b_i to be the largest integer with $a - \sum_{j=i}^d \binom{b_j}{j} \geq 0$ (see, for example, [Gre98] or [KR05]). The uniqueness of Macaulay representations can also be seen from the greedy algorithm. If b_d is the largest integer with $\binom{b_d}{d} \leq a$, then any sum of binomials $\binom{c_d}{d} + \dots + \binom{c_1}{1}$ with $b_d > c_d > \dots > c_1 \geq 0$ cannot equal a as

$$\begin{aligned} \binom{c_d}{d} + \dots + \binom{c_1}{1} &\leq \binom{c_d}{d} + \binom{c_d - 1}{d - 1} + \dots + \binom{c_d - (d - 1)}{1} \\ &= \binom{c_d + 1}{d} - 1 \\ &< \binom{b_d}{d} \\ &\leq a. \end{aligned}$$

It may also be helpful to think of Macaulay representations as a type of numeral system where the i -th digit b_i must always be strictly less than the $(i + 1)$ -th digit.

Example 3.7. The third Macaulay representations of the numbers $1, \dots, 7$ are given in the following table.

a	1	2	3	4	5	6	7
$\text{mrep}_3(a)$	(3, 1, 0)	(3, 2, 0)	(3, 2, 1)	(4, 1, 0)	(4, 2, 0)	(4, 2, 1)	(4, 3, 0)

Figure 3.1: Macaulay representations.

An integer can be recovered from its Macaulay representation by evaluating the appropriate binomial sum. Therefore, if two integers have the same Macaulay representation, then they are in fact equal.

Macaulay’s original paper on Hilbert functions of polynomial ideals made a connection between Macaulay representations and the dimensions of lex segments which we give below as Proposition 3.8. What it states is that the $(n - 1)$ -st Macaulay representation of $\dim_{\mathbb{k}} \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}}$ can be given in terms of the exponent vector \mathbf{a} . In

Proposition 3.11, we make a similar connection between d -th Macaulay representations of codimensions of lex segments in degree d and their last monomials. A similar result appears as Proposition 5.5.13 of [KR05], but we give an alternate presentation and a different proof. We repeat this process in Propositions 3.16 and 3.14 for lex segments in the Kruskal-Katona ring. We have not been able to find references for these two results.

In later chapters, various proofs involve lex segments while others are based on the numerics of Macaulay representations. Often either technique could have been used. The thrust of this section is to make these connections explicit.

Proposition 3.8 [Mac27]. *Let $L \subset S_d$ be a non-empty lex segment and let $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} x_n^{a_n}$ be the last monomial in L where x_r is last variable dividing $\mathbf{x}^{\mathbf{a}}$ before x_n . Note that $a_n = 0$ is allowed. Then the $(n-1)$ -st Macaulay representation of $\dim_{\mathbb{k}} L$ is*

$$\text{mrep}_{n-1} \dim_{\mathbb{k}} L = (b_1 + n - 2, b_2 + n - 3, \dots, b_r + n - r - 1; n - r - 1).$$

where b_1, \dots, b_r are given by

$$b_i = \begin{cases} \sum_{j>i}^n a_j & i < r, \\ a_n + 1 & i = r. \end{cases}$$

Example 3.12 illustrates both Proposition 3.8 and Proposition 3.11, to come.

What may be interesting about this proposition is that it differs from more recent treatments (e.g., [Gre89, Gre98, BH93]) which deal only with Macaulay representations of $\text{codim}_{\mathbb{k}} L$ and $\text{codim}_{\mathbb{k}} \nabla L$. The main benefit of phrasing theorems in terms of codimension or, equivalently, in terms of the quotient ring rather than the ideal, is that the numerical theorems are independent of the number of variables n . For example, Proposition 3.8 relies on $(n-1)$ -st Macaulay representations while the next result, Proposition 3.11, uses d -th Macaulay representations. Phrasing a theorem in terms of codimension instead of dimension appears to be a superficial change, but it hides a meaningful change in the underlying combinatorics. It's not clear when it was in the years between Macaulay [Mac27] and say, for instance, Stanley [Sta75a] that this transition was made. Before we give Proposition 3.11 on Macaulay representations of codimensions of lex segments, we need some more notation.

Every degree d monomial in a polynomial ring on n variables can be represented by a string of d bullets and $n - 1$ bars by letting its variables be represented by bullets and by separating the different variables by bars.

Example 3.9. For $n = 3$ and $d = 5$ we have

$$x_1^3 x_2 x_3 \mapsto \bullet \bullet \bullet | \bullet | \bullet, \text{ and}$$

$$x_2^3 x_3^2 \mapsto | \bullet \bullet \bullet | \bullet \bullet$$

This allows one to count the number of monomials of degree d in n variables as $\binom{n-1+d}{n-1}$ since $n - 1$ positions need to be chosen for the bars among d balls. That is, we need to choose $n - 1$ of the $n - 1 + d$ locations in the string for the bars.

These diagrams of d bullets and $n - 1$ bars can also be described by a sequence of d integers (i_1, \dots, i_d) which express the positions of the bullets. We need to adopt the convention that positions in a string are indexed from right to left and start from zero.

Example 3.10. For $n = 3$ and $d = 5$ we have

$$\begin{array}{c} \bullet \bullet \bullet | \bullet | \bullet \\ 6 \ 5 \ 4 \ 2 \ 0 \end{array} \mapsto (6, 5, 4, 2, 0), \text{ and}$$

$$\begin{array}{c} | \bullet \bullet \bullet | \bullet \bullet \\ 5 \ 4 \ 3 \ 1 \ 0 \end{array} \mapsto (5, 4, 3, 1, 0).$$

The following proposition gives the combinatorial correspondence between lex segments and Macaulay representations. This result appears as part of Theorem 5.5.13 of [KR05], though it is stated differently. Also, their proof relies on Proposition 3.8 and properties of binomial sums, while the proof given here is a simple induction. Recall that the codimension of a vector subspace $W \subseteq V$ is simply $\text{codim}_{\mathbb{k}} W = \dim_{\mathbb{k}} V - \dim_{\mathbb{k}} W$.

Proposition 3.11. *Let $L \subset S_d$ be a non-empty lex segment and let $\mathbf{x}^{\mathbf{a}}$ be the last monomial in L . Let (i_1, \dots, i_d) be the sequence describing the string representation of $\mathbf{x}^{\mathbf{a}}$ as defined above. Then the Macaulay representation of the codimension of L in S_d is*

$$\text{mrep}_d \text{codim}_{\mathbb{k}} L = (i_1, \dots, i_d).$$

Proof. Proceed by induction on $n + d$, n being the number of variables in S , and consider the two cases when $\mathbf{x}^{\mathbf{a}}$ is divisible by x_1 and when $\mathbf{x}^{\mathbf{a}}$ is not divisible by x_1 .

First, when $\mathbf{x}^{\mathbf{a}}$ is divisible by x_1 we have $i_1 = n + d - 2$. Consider the lex segment $L' = \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}'}} \subset S_{d-1}$ where $\mathbf{x}^{\mathbf{a}'} = \mathbf{x}^{\mathbf{a}}/x_1$.

By induction, $\text{mrep}_{d-1} \text{codim}_{\mathbb{k}} L' = (i_2, \dots, i_d)$. Since every monomial in $\text{Lex}_{\geq \mathbf{x}^{\mathbf{a}'}}$ is in one to one correspondence with a monomial in $\text{Lex}_{\geq \mathbf{x}^{\mathbf{a}'}}$, by division by x_1 , we see that $\dim_{\mathbb{k}} \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}} = \dim_{\mathbb{k}} \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}'}}$. Therefore

$$\begin{aligned} \text{codim}_{\mathbb{k}} L &= \binom{n-1+d}{d} - \dim_{\mathbb{k}} L \\ &= \binom{n-1+d}{d} - \dim_{\mathbb{k}} L' \\ &= \binom{n-1+d}{d} - \binom{n+d-2}{d-1} + \text{codim}_{\mathbb{k}} L' \\ &= \binom{n+d-2}{d} + \text{codim}_{\mathbb{k}} L'. \end{aligned}$$

using Pascal's rule. We know that $i_2 < i_1 = n + d - 2$ by its definition. Therefore, the Macaulay representation of $\text{codim}_{\mathbb{k}} L$ is $(n + d - 2, i_2, \dots, i_d)$ as we expected.

In the second case, where $\mathbf{x}^{\mathbf{a}}$ is not divisible by x_1 , we have that all $\binom{n+d-2}{d-1}$ monomials of degree d and divisible by x_1 have already occurred in L . Therefore, $\text{codim}_{\mathbb{k}} L = \text{codim}_{\mathbb{k}} L'$ where $L' = \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}} \subseteq (S/(x_1))_d$.

The string representation of $\mathbf{x}^{\mathbf{a}}$ in $S/(x_1)$ is obtained by removing the first character, a bar, from the string representation of $\mathbf{x}^{\mathbf{a}}$ in S . Since this does not affect the indices of the bullets relative to the end of the string, we have $\text{mrep}_d \text{codim}_{\mathbb{k}} L' = (i_1, \dots, i_d)$ by induction. Since $\text{codim}_{\mathbb{k}} L = \text{codim}_{\mathbb{k}} L'$, we are done. \square

Example 3.12. Propositions 3.8 and 3.11 can be compared with Figure 3.2 in which we list dimensions and codimensions of each lex segment in the polynomial ring $S = \mathbb{k}[x, y, z]$ in $n = 3$ variables and degree $d = 4$.

Since Macaulay representations correspond to the last monomials in lex segments, it makes sense to order them lexicographically. For two d -th Macaulay representations, we say $(a_d, \dots, a_1) >_{\text{lex}} (b_d, \dots, b_1)$ if there exists an index i where $a_i > b_i$ and $a_j = b_j$ for all $j > i$. That is, we compare them lexicographically from left to right according to our convention of writing a_d first.

$\dim_{\mathbb{k}} L$	$\text{codim}_{\mathbb{k}} L$	last L	mrep_{n-1}	$\dim_{\mathbb{k}} L$	mrep_d	$\text{codim}_{\mathbb{k}} L$	string rep.
1	14	x^4	(2,0)	(5,4,3,2)	••••		
2	13	x^3y	(2,1)	(5,4,3,1)	••• •		
3	12	x^3z	(3,0)	(5,4,3,0)	••• •		
4	11	x^2y^2	(3,1)	(5,4,2,1)	•• ••		
5	10	x^2yz	(3,2)	(5,4,2,0)	•• • •		
6	9	x^2z^2	(4,0)	(5,4,1,0)	•• ••		
7	8	xy^3	(4,1)	(5,3,2,1)	• •••		
8	7	xy^2z	(4,2)	(5,3,2,0)	• •• •		
9	6	xyz^2	(4,3)	(5,3,1,0)	• • ••		
10	5	xz^3	(5,0)	(5,2,1,0)	• •••		
11	4	y^4	(5,1)	(4,3,2,1)	••••		
12	3	y^3z	(5,2)	(4,3,2,0)	••• •		
13	2	y^2z^2	(5,3)	(4,3,1,0)	•• ••		
14	1	yz^3	(5,4)	(4,2,1,0)	• •••		
15	0	z^4	(6,0)	(3,2,1,0)	••••		

Figure 3.2: Macaulay representations and lex segments in the polynomial ring.

Corollary 3.13. *Let d , s , and t be a positive integers. Then $s > t$ if and only if $\text{mrep}_d(s) >_{\text{lex}} \text{mrep}_d(t)$.*

Proof. Pick n sufficiently large that $\binom{n+d-1}{d} > \max(s, t)$ and let $S = \mathbb{k}[x_1, \dots, x_n]$. Let s and t represent the codimensions of degree d lex segments L and L' in S_d . If $s > t$ then last monomial of L comes lexicographically before the last monomial of L' . By comparing the string representations of these two monomials it follows that $\text{mrep}_d(s) > \text{mrep}_d(t)$. The same argument also works in reverse. \square

The previous corollary seems to be mentioned first in [Gre89] and is normally proved using properties of binomials (see [Gre98, Proposition 3.1] or [BH93, Lemma 4.2.7]).

In the Kruskal-Katona ring and exterior algebra there is a similar connection between the last monomial in a lex segment and the segment's dimension and codimension. The next proposition, for the codimension of squarefree lex segments, is the analogue of Proposition 3.11. The subsequent proposition, Proposition 3.16, is the analogue of Proposition 3.8 for dimensions of squarefree lex segments.

Proposition 3.14. *Let $L \subset Q_d$ be a non-empty lex segment and let $\mathbf{x}^{\mathbf{a}} = x_{i_1}x_{i_2} \cdots x_{i_d}$, with $i_1 < i_2 < \cdots < i_d$, be the last monomial in L . Then the Macaulay representation*

of the codimension of L in Q_d is

$$\text{mrep}_d \text{codim}_{\mathbb{k}} L = (n - i_1, \dots, n - i_d).$$

Proof. Consider the same two cases as before: $i_1 = 1$ and $i_1 > 1$.

If $i_1 = 1$ then every monomial in L is divisible by x_1 and hence $\dim_{\mathbb{k}} L = \dim_{\mathbb{k}} L'$ where $L' = \text{Lex}_{\geq x_{i_2} \cdots x_{i_d}}^{Q/(x_1)}$. Consequently,

$$\begin{aligned} \text{codim}_{\mathbb{k}} L &= \binom{n}{d} - \dim_{\mathbb{k}} L \\ &= \binom{n}{d} - \binom{n-1}{d-1} + \text{codim}_{\mathbb{k}} L' \\ &= \binom{n-1}{d} + \text{codim}_{\mathbb{k}} L'. \end{aligned}$$

By reindexing the variables of $Q/(x_1)$ we get $\dim_{\mathbb{k}} \text{Lex}_{\geq x_{i_2} \cdots x_{i_d}}^{Q/(x_1)} = \dim_{\mathbb{k}} \text{Lex}_{\geq x_{i_2-1} \cdots x_{i_d-1}}^{Q/(x_n)}$. So by induction on n ,

$$\begin{aligned} \text{mrep}_{d-1} \text{codim}_{\mathbb{k}} L' &= (n-1 - (i_2-1), \dots, n-1 - (i_d-1)) \\ &= (n - i_2, \dots, n - i_d). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{mrep}_d \text{codim}_{\mathbb{k}} L &= \text{mrep}_d \left(\binom{n-1}{d} + \text{codim}_{\mathbb{k}} L' \right) \\ &= (n-1, n - i_2, \dots, n - i_d) \end{aligned}$$

as desired.

If $i_1 \neq 1$ then $\text{codim}_{\mathbb{k}} L = \text{codim}_{\mathbb{k}} \text{Lex}_{\geq x_{i_1} \cdots x_{i_d}}^{Q/(x_1)}$ as all degree d monomials divisible by x_1 are in L . The result follows by induction using a reindexing of variables as above. \square

Lemma 3.15. *Take two monomials $m, m' \in Q_d$ and let $\bar{m} = \frac{x_1 \cdots x_n}{m}$ and $\bar{m}' = \frac{x_1 \cdots x_n}{m'}$ be their complementary monomials in Q_{n-d} . Then $m >_{\text{lex}} m'$ if and only if $\bar{m}' >_{\text{lex}} \bar{m}$.*

Proof. The first index i in which m and m' differ is the same as the index where their complements differ. Assuming $m >_{\text{lex}} m'$, $x_i | m$ but not m' and hence $x_i | \bar{m}'$ and not \bar{m} . \square

Proposition 3.16. *Let $L \subseteq Q_d$ be a non-empty lex segment and let m be the last monomial in L . For $1 \leq i \leq n - d$, let a_i be the number of variables in the support of m that occur after the i -th variable not in m (see Example 3.17). Then the $(n - d)$ -th Macaulay representation of $\dim_{\mathbb{k}} L - 1$ is*

$$\text{mrep}_{n-d}(\dim_{\mathbb{k}} L - 1) = (a_1 + (n - d - 1), \dots, a_i + (n - d - i), \dots, a_{n-d}).$$

Proof. For this proof we will use Alexander duality. Let $\Delta = \Delta_{(L)}$ be the Stanley-Reisner complex the ideal generated by L . By Proposition 2.13,

$$\dim_{\mathbb{k}} L = \text{HF}_{(L)}(d) = \text{HF}_{Q/I_{\Delta}^{\vee}}(n - d).$$

Let V be the degree $n - d$ component of Q/I_{Δ}^{\vee} which is spanned by the complements $\overline{m'}$ of each monomial m' in L . Since m is lexicographically last in L , by the previous lemma, \overline{m} is lexicographically first in V . That is,

$$\text{codim}_{\mathbb{k}} \text{Lex}_{\geq \overline{m}}^Q = \dim_{\mathbb{k}} V - 1 = \dim_{\mathbb{k}} L - 1.$$

If we write $\overline{m} = x_{i_1} \cdots x_{i_{n-d}}$ then, by Proposition 3.14,

$$\text{mrep}_{n-d}(\dim_{\mathbb{k}} L - 1) = (n - i_1, \dots, n - i_{n-d}).$$

All that remains is to show that a_j is equal to $(n - i_j) - (n - d - j) = d - (i_j - j)$. Notice that at the j -th variable not in m , that is, at variable x_{i_j} in $\overline{m} = x_{i_1} \cdots x_{i_{n-d}}$ we have already seen $i_j - j$ variables in the support of m that come before x_{i_j} . Thus, there are $d - (i_j - j)$ variables in the support of m coming after x_{i_j} . \square

The previous proposition can also be shown using an expansion for the lex segment similar to that in Macaulay's original proof of 3.8.

Example 3.17. Take $n = 7$ and let $L \subset Q_3$ be the lex segment ending in $m = x_2x_5x_7$. For the variables not in m , namely x_1, x_3, x_4 and x_6 there are 3, 2, 2 and 1 variables dividing m which come after each of them respectively. That is $a_1 = 3$, $a_2 = 2$, $a_3 = 2$ and $a_4 = 1$ in the previous theorem. Note that

$$\text{mrep}_{n-d}(\dim_{\mathbb{k}} L - 1) = \text{mrep}_4 23 = (6, 4, 3, 1).$$

Example 3.18. In Figure 3.3 we illustrate Propositions 3.14 and 3.16 in the Kruskal-Katona ring $Q = \mathbb{k}[x_1, \dots, x_5]/(x_1^2, \dots, x_5^2)$ on $n = 5$ variables and in degree $d = 3$.

$\dim_{\mathbb{k}} L$	$\text{codim}_{\mathbb{k}} L$	$\text{last } L$	$\text{mrep}_3 \text{codim}_{\mathbb{k}} L$	$\overline{\text{last } L}$	(a_1, a_2)	$\text{mrep}_2(\dim_{\mathbb{k}} L - 1)$
1	9	$x_1x_2x_3$	(4,3,2)	x_4x_5	(0,0)	(1,0)
2	8	$x_1x_2x_4$	(4,3,1)	x_3x_5	(1,0)	(2,0)
3	7	$x_1x_2x_5$	(4,3,0)	x_3x_4	(1,1)	(2,1)
4	6	$x_1x_3x_4$	(4,2,1)	x_2x_5	(2,0)	(3,0)
5	5	$x_1x_3x_5$	(4,2,0)	x_2x_4	(2,1)	(3,1)
6	4	$x_1x_4x_5$	(4,1,0)	x_2x_3	(2,2)	(3,2)
7	3	$x_2x_3x_4$	(3,2,1)	x_1x_5	(3,0)	(4,0)
8	2	$x_2x_3x_5$	(3,2,0)	x_1x_4	(3,1)	(4,1)
9	1	$x_2x_4x_5$	(3,1,0)	x_1x_3	(3,2)	(4,2)
10	0	$x_3x_4x_5$	(2,1,0)	x_1x_2	(3,3)	(4,3)

Figure 3.3: Macaulay representations and lex segments in the Kruskal-Katona ring.

3.3 Pseudopowers

Next we introduce pseudopowers which are operations on the Macaulay representation of an integer. These pseudopowers are paralleled by natural operations on lex segments.

Definition 3.19 (Pseudopowers). Let $a \geq 0$ be an integer with d -th Macaulay representation $\text{mrep}_d a = (b_d, \dots, b_k; k - 1)$ and with $b_k \geq k$. The **d-th upper Macaulay pseudopower** and the **d-th upper Kruskal-Katona pseudopower** of a are defined to be

$$a^{(d)} = \binom{b_d + 1}{d + 1} + \binom{b_{d-1} + 1}{d} + \dots + \binom{b_k + 1}{k + 1}$$

and

$$a^{(d)} = \binom{b_d}{d + 1} + \binom{b_{d-1}}{d} + \dots + \binom{b_k}{k + 1}$$

respectively. For convenience, we often drop the word upper when referring to upper pseudopowers.

The **lower d-th Macaulay and Kruskal-Katona pseudopowers** of a are defined to be

$$a_{(d)} = \binom{b_d - 1}{d - 1} + \binom{b_{d-1} - 1}{d - 2} + \dots + \binom{b_k - 1}{k - 1}$$

and

$$a_{(d)} = \binom{b_d}{d - 1} + \binom{b_{d-1}}{d - 2} + \dots + \binom{b_k}{k - 1}$$

respectively.

Recall that $\binom{p}{q} = 0$ for $p < 0$, $q < 0$ and $p < q$. Also, $\binom{0}{0} = 1$ which can appear when applying a lower pseudopowers.

Our notation for pseudopowers is a compromise with the many notations in use. Our notation for upper pseudopowers agrees with Stanley [Sta96], Bruns and Herzog [BH93], Green [Gre89, Gre98] and others. Kreuzer and Robbiano's book [KR05] prefers the suggestive notations a_+^\dagger and a_+ , while Kruskal [Kru63] used a notation $a^{(r'/r)}$ which is more general than the pseudopowers used here. Green uses a lower pseudopower for his theorem on Hilbert functions of rings modulo generic forms. His lower pseudopower, however, is different from the one we describe here, despite using the same notation. We have used this notation because it seems suggestive of our result in Proposition 3.24 and Proposition 3.21.

Pseudopowers act in a natural way on Macaulay representations. The upper pseudopowers act by padding the Macaulay representation with an extra zero term. That is, if $\text{mrep}_d a = (b_d, \dots, b_k; k - 1)$ then

$$\text{mrep}_{d+1} a^{(d)} = (b_d + 1, \dots, b_k + 1; k), \text{ and} \quad (3.3.1)$$

$$\text{mrep}_{d+1} a^{(d)} = (b_d, \dots, b_k; k). \quad (3.3.2)$$

The lower pseudopowers act in a slightly more complicated manner. If the last Macaulay coefficient, b_1 , is zero then the lower pseudopowers act by shifting the Macaulay coefficients to the right. That is, if $\text{mrep}_d a = (b_d, \dots, b_2, 0)$ then

$$\text{mrep}_{d-1} a_{(d)} = (b_d - 1, \dots, b_2 - 1), \text{ and} \quad (3.3.3)$$

$$\text{mrep}_{d-1} a_{(d)} = (b_d, \dots, b_2). \quad (3.3.4)$$

If $b_1 \neq 0$ then

$$\text{mrep}_{d-1}(a_{(d)} - 1) = (b_d - 1, \dots, b_2 - 1), \text{ and} \quad (3.3.5)$$

$$\text{mrep}_{d-1}(a_{(d)} - 1) = (b_d, \dots, b_2). \quad (3.3.6)$$

So, in these latter cases, applying a pseudopower to our Macaulay representations does not leave us with another Macaulay representation.

Note that the number of non-zero terms in the Macaulay representation of a does not change when the Macaulay upper pseudopower or lower Kruskal-Katona

pseudopower is applied (and $b_1 = 0$), while it may change in the other two cases (e.g., if $b_k = k$ for the Kruskal-Katona upper pseudopower or $k = 1$ for the lower Macaulay pseudopower).

Proposition 3.20. *The upper pseudopowers are increasing functions while the lower pseudopowers are non-decreasing functions.*

Proof. If $s > t$ then $\text{mrep}_d s >_{\text{lex}} \text{mrep}_d t$ by Corollary 3.13. After applying the upper Macaulay pseudopower and using equation 3.3.1, we obtain

$$\text{mrep}_{d+1} s^{(d)} >_{\text{lex}} \text{mrep}_{d+1} t^{(d)}$$

and so $s^{(d)} > t^{(d)}$ using Corollary 3.13 again.

For the lower pseudopowers, again $s > t$ implies $\text{mrep}_d s >_{\text{lex}} \text{mrep}_d t$. Let the Macaulay representations of s and t be

$$\text{mrep}_d s = (a_d, \dots, a_1) \quad \text{and} \quad \text{mrep}_d t = (b_d, \dots, b_1)$$

and let i be the first (largest) index where they differ. That is $a_i > b_i$ and $a_j = b_j$ for $j > i$. We have two cases: $b_1 = 0$ and $b_1 \neq 0$.

In the case where $b_1 = 0$, we have $\text{mrep}_{d-1} s_{(d)} = (a_d - 1, \dots, a_2 - 1)$ from equation 3.3.3 whenever $a_1 = 0$. If instead we have $a_1 \neq 0$ then

$$\begin{aligned} \text{mrep}_{d-1} s_{(d)} &>_{\text{lex}} \text{mrep}_{d-1}(s_{(d)} - 1) \\ &= (a_d - 1, \dots, a_2 - 1) \end{aligned}$$

by Corollary 3.13 and equation 3.3.5. Thus in both cases,

$$\begin{aligned} \text{mrep}_{d-1} s_{(d)} &\geq_{\text{lex}} (a_d - 1, \dots, a_2 - 1) \\ &\geq_{\text{lex}} (b_d - 1, \dots, b_2 - 1) \\ &= \text{mrep}_{d-1} t_{(d)} \end{aligned}$$

where the equality occurs if $i = 1$ and $a_1 = 0$. Thus, $s_{(d)} \geq t_{(d)}$.

Now consider the case where $b_1 \neq 0$. If $a_1 \neq 0$ then by the reusing the previous argument,

$$\text{mrep}_{d-1}(s_{(d)} - 1) \geq_{\text{lex}} \text{mrep}_{d-1}(t_{(d)} - 1)$$

and hence $s_{\langle d \rangle} - 1 \geq t_{\langle d \rangle} - 1$, as we wanted. If $a_1 = 0$ then $i > 1$ and

$$\begin{aligned} \text{mrep}_{d-1} s_{\langle d \rangle} &= (a_d - 1, \dots, a_2 - 1) \\ &>_{\text{lex}} (b_d - 1, \dots, b_2 - 1) \\ &= \text{mrep}_{d-1}(t_{\langle d \rangle} - 1). \end{aligned}$$

That is, $s_{\langle d \rangle} > t_{\langle d \rangle} - 1$ and therefore $s_{\langle d \rangle} \geq t_{\langle d \rangle}$.

The same arguments work for lower Kruskal-Katona pseudopowers. \square

The following proposition is essential for understanding the interplay between lower and upper pseudopowers. It also leads to various forms of Macaulay's theorem and the Kruskal-Katona theorem (in particular, see part (4) of both Theorem 3.26 and Theorem 3.32). While we have no direct reference for this result, it neatly parallels Theorem 5.5.11 of [KR05] for Green's lower pseudopower and could probably be stated in more generality.

Proposition 3.21. *Let s, t and d be positive integers.*

1. $s \leq t^{(d)}$ if and only if $s_{\langle d+1 \rangle} \leq t$.
2. $s \leq t^{(d)}$ if and only if $s_{\langle d+1 \rangle} \leq t$.

Also, if $\text{mrep}_{d+1} s = (a_{d+1}, \dots, a_1)$ with $a_1 = 0$ then all the inequalities above can be made strict. Finally, if $a_1 \neq 0$ and $s_{\langle d+1 \rangle} \leq t$ then $s < t^{(d)}$ and similarly for the Kruskal-Katona pseudopower.

Proof. Assume the Macaulay representations of s and t are

$$\begin{aligned} \text{mrep}_{d+1} s &= (a_{d+1}, \dots, a_1) \text{ and} \\ \text{mrep}_d t &= (b_d, \dots, b_1). \end{aligned}$$

We break the proof into the two cases where where $a_1 = 0$ and $a_1 \neq 0$.

Assuming $a_1 = 0$, we have

$$\begin{aligned} \text{mrep}_{d+1} t^{(d)} &= (b_d + 1, \dots, b_1 + 1, 0) \text{ and} \\ \text{mrep}_d s_{\langle d+1 \rangle} &= (a_{d+1} - 1, \dots, a_k - 1, \dots, a_2 - 1). \end{aligned}$$

We see, by Corollary 3.13, that

$$\begin{aligned}
s &\leq t^{(d)} \\
\iff (a_{d+1}, \dots, a_2, 0) &\leq_{\text{lex}} (b_d + 1, \dots, b_1 + 1, 0) \\
\iff (a_{d+1} - 1, \dots, a_2 - 1) &\leq_{\text{lex}} (b_d, \dots, b_1) \\
\iff s_{\langle d+1 \rangle} &\leq t
\end{aligned}$$

and also the inequalities above can be made strict.

Now we move to the case where $a_1 \neq 0$ and hence

$$\text{mrep}_d(s_{\langle d+1 \rangle} - 1) = (a_{d+1} - 1, \dots, a_2 - 1).$$

If we assume $s \leq t^{(d)}$ then we have

$$(a_{d+1}, \dots, a_2, a_1) \leq_{\text{lex}} (b_d + 1, \dots, b_1 + 1, 0)$$

as before. Since $a_1 \neq 0$, we know that this is in fact a strict inequality. Consequently,

$$(a_{d+1} - 1, \dots, a_2 - 1) <_{\text{lex}} (b_d, \dots, b_1)$$

which shows that $s_{\langle d+1 \rangle} - 1 < t$ or rather $s_{\langle d+1 \rangle} \leq t$ as desired.

In the other direction, we assume $s_{\langle d+1 \rangle} \leq t$ which gives $s_{\langle d+1 \rangle} - 1 < t$ and hence

$$(a_{d+1} - 1, \dots, a_2 - 1) <_{\text{lex}} (b_d, \dots, b_1)$$

Thus, $(a_{d+1}, \dots, a_2, a_1) <_{\text{lex}} (b_d + 1, \dots, b_1 + 1, 0)$ and so $s < t^{(d)}$. Note that for this direction of the $a_1 \neq 0$ case, we obtained a strict inequality.

The proof for the Kruskal-Katona pseudopowers follows similarly. \square

With the next proposition, we relate upper shadows with the upper pseudopowers. We have previously remarked that

$$\nabla \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}}^S = \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}} x_n}^S$$

and

$$\nabla \text{Lex}_{\geq x_{i_1} \cdots x_{i_d}}^Q = \text{Lex}_{\geq x_{i_1} \cdots x_{i_d} x_k}^Q$$

where k is largest index with x_k not dividing $x_{i_1} \cdots x_{i_d}$ (see Examples 3.1 and 3.2).

This will be used in the next proof.

Proposition 3.22. *Let $L \subset S_d$ and $L' \subset Q_d$ be degree d lex segments. Then the upper shadows of L and L' have codimensions given by*

$$\text{codim}_{\mathbb{k}}(\nabla L) = (\text{codim}_{\mathbb{k}} L)^{\langle d \rangle}$$

and

$$\text{codim}_{\mathbb{k}}(\nabla L') = (\text{codim}_{\mathbb{k}} L')^{\langle d \rangle}.$$

Proof. Let $\mathbf{x}^{\mathbf{a}}$ be the last monomial of L and let $x_{i_1} \cdots x_{i_d}$ be the last squarefree monomial of L' .

Let $\text{mrep}_d \text{codim}_{\mathbb{k}} L = (b_d, \dots, b_k; k-1)$ with $b_k \geq k$. The upper shadow of L , $\nabla L = (\mathbf{m}(L))_{d+1}$, is a lex segment with last monomial $\mathbf{x}^{\mathbf{a}}x_n$ and this monomial has the same string representation as $\mathbf{x}^{\mathbf{a}}$ except with an extra bullet added at the end. Therefore

$$\text{mrep}_{d+1} \text{codim}_{\mathbb{k}} \nabla L = (b_d + 1, \dots, b_k + 1; k) = \text{mrep}_{d+1} ((\text{codim}_{\mathbb{k}} L)^{\langle d \rangle})$$

using Proposition 3.11. Thus $\text{codim}_{\mathbb{k}} \nabla L = (\text{codim}_{\mathbb{k}} L)^{\langle d \rangle}$, proving the first part of the proposition.

Now consider L' . Assume for now that $i_d = n$. Let s be the first index for which i_s, i_{s+1}, \dots, i_d is a sequence of consecutive integers ending in n . That is,

$$(i_s, i_{s+1}, \dots, i_d) = (n-d+s, n-d+s+1, \dots, n)$$

and $i_{s-1} < n-d+s-1$. Using Proposition 3.14, this allows us to write the Macaulay representation of $\text{codim}_{\mathbb{k}} L'$ as

$$\begin{aligned} \text{mrep}_d \text{codim}_{\mathbb{k}} L' &= (n-i_1, \dots, n-i_{s-1}, d-s, d-s-1, \dots, 0) \\ &= (n-i_1, \dots, n-i_{s-1}; d-s+1). \end{aligned}$$

Note that if $i_d \neq n$ then $\text{mrep}_d \text{codim}_{\mathbb{k}} L' = (n-i_1, \dots, n-i_d)$. So the above still holds with $s = d+1$.

By the remark made before Example 3.1, the last monomial in the shadow of L' is $x_{i_1} \cdots x_{i_d} x_k$ where k is largest integer in $[n] \setminus \{i_1, \dots, i_d\}$ (and, in the $i_d \neq n$ case we have $x_k = x_n$). Thus, k is the last integer before the consecutive sequence of indices ending in n , namely $k = n-d+s-1$. And so, with indices in sorted order,

$$x_{i_1} \cdots x_{i_d} x_k = x_{i_1} \cdots x_{s-1} x_{n-d+s-1} x_{n-d-s} \cdots x_n$$

giving

$$\begin{aligned} \text{mrep}_{d+1} \text{codim}_{\mathbb{k}} \nabla L' &= (n - i_1, n - i_2, \dots, n - i_{s-1}, d - s + 1, d - s, \dots, 0) \\ &= (n - i_1, n - i_2, \dots, n - i_{s-1}; d - s + 2) \\ &= \text{mrep}_{d+1} ((\text{codim}_{\mathbb{k}} L')^{(d)}) \end{aligned}$$

by Proposition 3.14 and equation 3.3.2. Thus $\text{codim}_{\mathbb{k}} \nabla L' = (\text{codim}_{\mathbb{k}} L')^{(d)}$. \square

Before proving the analogous proposition for lower shadows, we will first describe the lower shadows of lex segments in S and Q .

Lemma 3.23. *The lower shadow of a lex segment in S or Q is also a lex segment.*

In particular, if $L = \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}}^S$ and $\mathbf{x}^{\mathbf{b}} \geq_{\text{lex}} \mathbf{x}^{\mathbf{a}}$ is the last degree d monomial before or equal to $\mathbf{x}^{\mathbf{a}}$ which is divisible by x_n then

$$\Delta L = \text{Lex}_{\geq \mathbf{x}^{\mathbf{b}}/x_n}^S.$$

If no such $\mathbf{x}^{\mathbf{b}}$ exists, then ΔL is the trivial vector space.

If $L' = \text{Lex}_{\geq m}^Q$ for a squarefree monomial $m \in Q_d$ and $m' \geq_{\text{lex}} m$ is the last degree d squarefree monomial before or equal to m which is divisible by x_n , then

$$\Delta L' = \text{Lex}_{\geq m'/x_n}^Q.$$

If no such monomial m' exists, then $\Delta L'$ is the trivial vector space.

Proof. Take $\mathbf{x}^{\mathbf{c}} \geq_{\text{lex}} \mathbf{x}^{\mathbf{b}}/x_n$ of degree $d - 1$. We want to show that $x_i \mathbf{x}^{\mathbf{c}} \in L$ for every variable x_i . As $>_{\text{lex}}$ is a term order,

$$x_i \mathbf{x}^{\mathbf{c}} \geq_{\text{lex}} x_i \mathbf{x}^{\mathbf{b}}/x_n \geq_{\text{lex}} x_n \mathbf{x}^{\mathbf{b}}/x_n = \mathbf{x}^{\mathbf{b}} \geq_{\text{lex}} \mathbf{x}^{\mathbf{a}}.$$

Thus $x_i \mathbf{x}^{\mathbf{c}} \in L$ and so $\text{Lex}_{\geq \mathbf{x}^{\mathbf{b}}/x_n}^S \subseteq \Delta L$.

If instead $\mathbf{x}^{\mathbf{b}}/x_n >_{\text{lex}} \mathbf{x}^{\mathbf{c}}$ then $\mathbf{x}^{\mathbf{b}} >_{\text{lex}} x_n \mathbf{x}^{\mathbf{c}}$. By the choice of $\mathbf{x}^{\mathbf{b}}$, $\mathbf{x}^{\mathbf{a}} >_{\text{lex}} x_n \mathbf{x}^{\mathbf{c}}$ and hence $x_n \mathbf{x}^{\mathbf{c}} \notin L$. Thus $\mathbf{x}^{\mathbf{c}} \notin \Delta L$, so $\text{Lex}_{\geq \mathbf{x}^{\mathbf{b}}/x_n}^S = \Delta L$.

The same argument works for lower shadows of lex segments in the Kruskal-Katona ring. \square

Proposition 3.24. *Let $L \subset S_d$ and $L' \subset Q_d$ be degree d lex segments. Then the lower shadows of L and L' have codimensions given by*

$$\text{codim}_{\mathbb{k}}(\Delta L) = (\text{codim}_{\mathbb{k}} L)_{\langle d \rangle}$$

and

$$\text{codim}_{\mathbb{k}}(\Delta L') = (\text{codim}_{\mathbb{k}} L')_{\langle d \rangle}.$$

Proof. Let $\mathbf{x}^{\mathbf{a}}$ be the last monomial in L . By Lemma 3.23, we know that ΔL is the lex segment ending in $\mathbf{x}^{\mathbf{b}}/x_n$ where $\mathbf{x}^{\mathbf{b}} \geq_{\text{lex}} \mathbf{x}^{\mathbf{a}}$ is the last monomial before $\mathbf{x}^{\mathbf{a}}$ divisible by x_n .

If x_n divides $\mathbf{x}^{\mathbf{a}}$ then $\mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{b}}$. Denote the d -th Macaulay representation of $\text{codim}_{\mathbb{k}} L$ by $\text{mrep}_d \text{codim}_{\mathbb{k}} L = (i_d, \dots, i_1)$. Since x_n divides $\mathbf{x}^{\mathbf{a}}$ we know $i_1 = 0$ by Proposition 3.11. Thus,

$$\text{mrep}_{d-1}(\text{codim}_{\mathbb{k}} L)_{\langle d \rangle} = (i_d - 1, \dots, i_2 - 1) = \text{mrep}_{d-1} \text{codim}_{\mathbb{k}} \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}/x_n}^S$$

again by Proposition 3.11. From Lemma 3.23, it follows that $(\text{codim}_{\mathbb{k}} L)_{\langle d \rangle} = \text{codim}_{\mathbb{k}} \Delta L$.

If x_n does not divide $\mathbf{x}^{\mathbf{a}}$ then $i_1 \neq 0$ and so, by Proposition 3.11,

$$\text{mrep}_{d-1}((\text{codim}_{\mathbb{k}} L)_{\langle d \rangle} - 1) = (i_d - 1, \dots, i_2 - 1) = \text{mrep}_{d-1} \text{codim}_{\mathbb{k}} \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}/x_r}^S$$

where x_r is the last variable dividing $\mathbf{x}^{\mathbf{a}}$. Consequently,

$$(\text{codim}_{\mathbb{k}} L)_{\langle d \rangle} - 1 = \text{codim}_{\mathbb{k}} \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}/x_r}^S.$$

It now suffices to show that $\mathbf{x}^{\mathbf{a}}/x_r$ appears immediately after $\mathbf{x}^{\mathbf{b}}/x_n$ in the lexicographic order. Clearly if $\mathbf{x}^{\mathbf{b}}/x_n \leq_{\text{lex}} \mathbf{x}^{\mathbf{a}}/x_r$ then $\mathbf{x}^{\mathbf{b}} \leq_{\text{lex}} x_n \mathbf{x}^{\mathbf{a}}/x_r <_{\text{lex}} \mathbf{x}^{\mathbf{a}}$ which contradicts $\mathbf{x}^{\mathbf{b}} \geq_{\text{lex}} \mathbf{x}^{\mathbf{a}}$. Thus $\mathbf{x}^{\mathbf{b}}/x_n >_{\text{lex}} \mathbf{x}^{\mathbf{a}}/x_r$. If we have some $\mathbf{x}^{\mathbf{c}}$ between $\mathbf{x}^{\mathbf{b}}/x_n$ and $\mathbf{x}^{\mathbf{a}}/x_r$ then, multiplying by x_n gives,

$$\mathbf{x}^{\mathbf{b}} >_{\text{lex}} x_n \mathbf{x}^{\mathbf{c}} >_{\text{lex}} x_n \mathbf{x}^{\mathbf{a}}/x_r >_{\text{lex}} \mathbf{x}^{\mathbf{a}}$$

which contradicts the definition of $\mathbf{x}^{\mathbf{b}}$. Thus, $\mathbf{x}^{\mathbf{a}}/x_r$ is the immediate successor of $\mathbf{x}^{\mathbf{b}}/x_n$ and so

$$\text{codim}_{\mathbb{k}} \Delta L = \text{codim}_{\mathbb{k}} \text{Lex}_{\geq \mathbf{x}^{\mathbf{a}}/x_r}^S + 1 = (\text{codim}_{\mathbb{k}} L)_{\langle d \rangle}.$$

A similar argument works from the lower Kruskal-Katona pseudopowers. □

Example 3.25. For an illustration of Proposition 3.24, consult Figure 3.4 which shows the lower shadows of degree $d = 4$ lex segments in $S = \mathbb{k}[x, y, z]$ and their codimensions.

$\text{codim}_{\mathbb{k}} L$	last L	mrep_d	$\text{codim}_{\mathbb{k}} L$	$(\text{codim}_{\mathbb{k}} L)_{\langle d \rangle}$	last ΔL	mrep_{d-1}	$\text{codim}_{\mathbb{k}} \Delta L$
14	x^4	(5, 4, 3, 2)	10	10	—	(5, 1, 0)	
13	x^3y	(5, 4, 3, 1)	10	10	—	(5, 1, 0)	
12	x^3z	(5, 4, 3, 0)	9	9	x^3	(4, 3, 2)	
11	x^2y^2	(5, 4, 2, 1)	9	9	x^3	(4, 3, 2)	
10	x^2yz	(5, 4, 2, 0)	8	8	x^2y	(4, 3, 1)	
9	x^2z^2	(5, 4, 1, 0)	7	7	x^2z	(4, 3, 0)	
8	xy^3	(5, 3, 2, 1)	7	7	x^2z	(4, 3, 0)	
7	xy^2z	(5, 3, 2, 0)	6	6	xy^2	(4, 2, 1)	
6	xyz^2	(5, 3, 1, 0)	5	5	xyz	(4, 2, 0)	
5	xz^3	(5, 2, 1, 0)	4	4	xz^2	(4, 1, 0)	
4	y^4	(4, 3, 2, 1)	4	4	xz^2	(4, 1, 0)	
3	y^3z	(4, 3, 2, 0)	3	3	y^3	(3, 2, 1)	
2	y^2z^2	(4, 3, 1, 0)	2	2	y^2z	(3, 2, 0)	
1	yz^3	(4, 2, 1, 0)	1	1	yz^2	(3, 1, 0)	
0	z^4	(3, 2, 1, 0)	0	0	z^3	(2, 1, 0)	

Figure 3.4: Lower shadows and pseudopowers.

3.4 Macaulay's Theorem

Theorem 3.26 (Macaulay's Theorem) [Mac27]. *Let $H : \mathbb{N} \rightarrow \mathbb{N}$ be a function and let \mathbb{k} be a field. The following conditions are equivalent:*

1. H is the Hilbert function of a standard graded \mathbb{k} -algebra;
2. H is the Hilbert function of a quotient of a polynomial ring over a \mathbb{k} by a proper lexicographic ideal;
3. H satisfies $H(0) = 1$ and

$$H(d+1) \leq H(d)^{\langle d \rangle} \quad \forall d \geq 1;$$

4. H satisfies $H(0) = 1$ and

$$H(d+1)_{\langle d+1 \rangle} \leq H(d) \quad \forall d \geq 1.$$

Proof. For every homogeneous ideal $I \subset S$ and degree d , $\nabla I_d \subseteq I_{d+1}$ which tells us that $\dim_{\mathbb{k}} \nabla I_d \leq \dim_{\mathbb{k}} I_{d+1}$. If I is a lex ideal, the shadow ∇I_d is a lex segment and, by Proposition 3.22, its codimension is known to be $\text{codim}_{\mathbb{k}} \nabla I_d = (\text{codim}_{\mathbb{k}} I_d)^{\langle d \rangle}$. So, our inequality on dimensions reverses to give

$$\text{codim}_{\mathbb{k}} I_{d+1} \leq \text{codim}_{\mathbb{k}} \nabla I_d = (\text{codim}_{\mathbb{k}} I_d)^{\langle d \rangle}.$$

Since $\text{HF}_{S/I}(d) = \text{codim}_{\mathbb{k}} I_d$, we have proven (2) implies (3).

Given a function H satisfying (3), we can construct the graded vector space

$$L = \bigoplus_{d=0}^{\infty} \text{Lex}(d, H(d)).$$

In order to show that L is an ideal, we need to show that $\nabla L_d \subseteq L_{d+1}$ for each degree d . Since the shadow of a lex segment is also lex segment, it follows that $\nabla L_d \subseteq L_{d+1}$ if and only if $\dim_{\mathbb{k}} L_d \leq \dim_{\mathbb{k}} L_{d+1}$. This is guaranteed by Proposition 3.22 and condition (3). So we have shown that (3) implies (2).

The equivalence of (3) and (4) follows from Proposition 3.21.

Clearly any lexicographic ideal, being a monomial ideal, is also a homogeneous ideal. Thus (2) implies (1).

The last step, and one we omit, is the proof that (1) implies (3). In most modern references, this is shown to follow from Green's theorem [Gre89] on the Hilbert function of standard graded algebra modulo a generic linear form. Macaulay's original proof [Mac27] is, in the words of Macaulay himself, "tedious reading", though it introduces a useful technique, called compression, which has been exploited by other authors [Mer08, CL69]. \square

In this thesis, when we casually say that lex ideals **model Hilbert functions**, implicitly we are referring to the equivalence of parts (1) and (2) of Macaulay's theorem. Similarly, the we refer to the inequality in part (3) of Macaulay's theorem as **Macaulay's inequality** on Hilbert functions.

A consequence of Macaulay's theorem is that lex segments have the smallest possible Hilbert function growth of all homogeneous vector spaces with the same degree and dimension. This idea is made precise in following definition:

Definition 3.27 (Gotzmann). Let R be a standard graded \mathbb{k} -algebra. A homogeneous vector space $V \subseteq R_d$ is called **Gotzmann** if for all other homogeneous vector spaces $W \subseteq R_d$ with $\dim_{\mathbb{k}} V = \dim_{\mathbb{k}} W$, we have

$$\dim_{\mathbb{k}} \nabla V \leq \dim_{\mathbb{k}} \nabla W.$$

A homogeneous ideal $I \subseteq R$ is called **Gotzmann** if each of its graded components are Gotzmann vector spaces.

As mentioned in the introduction to this thesis, Gotzmann ideals are named after Gerd Gotzmann who proved the regularity and persistence theorems which bear his name [Got78]. The definition of Gotzmann ideals given above first appears in the paper ‘‘Componentwise Linear Ideals’’ by Herzog and Hibi [HH99].

Corollary 3.28. *Lex segments and lex ideals in the polynomial ring are Gotzmann.*

Proof. Let L be a degree d lex segment and let V be a degree d homogeneous vector space with $\dim_{\mathbb{k}} V = \dim_{\mathbb{k}} L$. Also, let $I = (V)$. Macaulay’s theorem states that $\text{HF}_{S/I}(d+1) \leq \text{HF}_{S/I}(d)^{\langle d \rangle}$ or, in other words, $\text{codim}_{\mathbb{k}} \nabla V \leq (\text{codim}_{\mathbb{k}} V)^{\langle d \rangle}$. Therefore,

$$\begin{aligned} \dim_{\mathbb{k}} \nabla L &= \dim_{\mathbb{k}} S_{d+1} - \text{codim}_{\mathbb{k}} \nabla L \\ &= \dim_{\mathbb{k}} S_{d+1} - (\text{codim}_{\mathbb{k}} L)^{\langle d \rangle} \\ &= \dim_{\mathbb{k}} S_{d+1} - (\text{codim}_{\mathbb{k}} V)^{\langle d \rangle} \\ &\leq \dim_{\mathbb{k}} S_{d+1} - \text{codim}_{\mathbb{k}} \nabla V \\ &= \dim_{\mathbb{k}} \nabla V, \end{aligned}$$

using Proposition 3.22 on L . □

We can infer from Proposition 3.22 and the previous Corollary that Macaulay pseudopowers determine the growth of Gotzmann ideals. This gives the following numerical characterization of Gotzmann ideals.

Proposition 3.29. *Let V be a degree d homogeneous vector space in the polynomial ring S and let L be the degree d lex segment with $\dim_{\mathbb{k}} L = \dim_{\mathbb{k}} V$. The following are equivalent:*

1. V is Gotzmann.
2. $\dim_{\mathbb{k}} \nabla V = \dim_{\mathbb{k}} \nabla L$.
3. $\dim_{\mathbb{k}} S_{d+1}/\nabla V = (\dim_{\mathbb{k}} S_d/V)^{\langle d \rangle}$.

Also, if V is Gotzmann then

$$4. (\dim_{\mathbb{k}} S_{d+1}/\nabla V)_{\langle d+1 \rangle} = \dim_{\mathbb{k}} S_d/V$$

but not the converse (see Example 3.31).

Proof. By the definition of a Gotzmann homogeneous space, any two Gotzmann spaces $V, W \subseteq S_d$ of the same dimension have the same growth (i.e., $\dim_{\mathbb{k}} \nabla V = \dim_{\mathbb{k}} \nabla W$). Since lex segments are Gotzmann by Corollary 3.28, we have part (1) implies part (2). Similarly, if $\dim_{\mathbb{k}} \nabla V = \dim_{\mathbb{k}} \nabla L$ then

$$\dim_{\mathbb{k}} \nabla V = \dim_{\mathbb{k}} \nabla L \leq \dim_{\mathbb{k}} \nabla W$$

for any degree d homogeneous vector space W with $\dim_{\mathbb{k}} W = \dim_{\mathbb{k}} L = \dim_{\mathbb{k}} V$ since L is Gotzmann. Thus, part (2) implies part (1).

Parts (2) and (3) are equivalent by Proposition 3.22. This completes the equivalence of parts (1), (2) and (3).

Now, let V be a Gotzmann homogeneous vector space. From the equivalence of parts (1) and (3), we have $s = t^{\langle d \rangle}$ where $s = \dim_{\mathbb{k}} S_{d+1}/\nabla V$ and $t = \dim_{\mathbb{k}} S_d/V$. Let $\text{mrep}_d t = (a_d, \dots, a_1)$ so that $\text{mrep}_{d+1} t^{\langle d \rangle} = (a_d + 1, \dots, a_1 + 1, 0)$ and therefore $\text{mrep}_d (t^{\langle d \rangle})_{\langle d+1 \rangle} = (a_d, \dots, a_1)$, using equation 3.3.1 and equation 3.3.3. Thus $s_{\langle d+1 \rangle} = (t^{\langle d \rangle})_{\langle d+1 \rangle} = t$ which proves part (4). \square

Corollary 3.30. *A homogeneous ideal $I \subseteq S$ is Gotzmann if and only if for every degree d ,*

$$\text{HF}_{S/\mathfrak{m}I}(d+1) = \text{HF}_{S/I}(d)^{\langle d \rangle},$$

where $\mathfrak{m} = (x_1, \dots, x_n)$ is the homogeneous maximal ideal of S .

Proof. An ideal is Gotzmann if each component I_d is Gotzmann. Since $(S/\mathfrak{m}I)_d = S_d/\nabla I_d$, the result follows from part (3) of Proposition 3.29. \square

We often refer to this corollary by saying that an ideal I is Gotzmann if each graded component of I achieves Macaulay's bound.

Example 3.31. Let $I = (xy, yz, xz)$ in the polynomial ring $S = \mathbb{k}[x, y, z]$. We will show that this is not a Gotzmann monomial even though it satisfies part (4) of Proposition 3.29.

The degree 3 component I_3 of I is the shadow of I_2 and is spanned by

$$x^2y, xy^2, y^2z, yz^2, x^2z, xz^2, xyz.$$

So, $\text{HF}_I(2) = 3$ and $\text{HF}_I(3) = 7$ giving

$$\text{HF}_{S/I}(2) = \text{HF}_S(2) - \text{HF}_I(2) = 6 - 3 = 3 \quad \text{and}$$

$$\text{HF}_{S/I}(3) = \text{HF}_S(3) - \text{HF}_I(3) = 10 - 7 = 3.$$

Note that

$$\begin{aligned} \text{HF}_{S/I}(3)_{\langle 3 \rangle} &= \left(\binom{3}{3} + \binom{2}{2} + \binom{1}{1} \right)_{\langle 3 \rangle} \\ &= \binom{2}{2} + \binom{1}{1} + \binom{0}{0} \\ &= 3 \\ &= \text{HF}_{S/I}(2) \end{aligned}$$

showing that $V = I_2$ satisfies part (4) of Proposition 3.29. Since

$$\begin{aligned} \text{HF}_{S/I}(2)^{\langle 2 \rangle} &= \binom{3}{2}^{\langle 2 \rangle} \\ &= \binom{4}{3} \\ &= 4 \\ &\neq \text{HF}_{S/I}(3), \end{aligned}$$

we see that I is not Gotzmann.

3.5 The Kruskal-Katona Theorem

In this section we present the Kruskal-Katona theorem which is an analogue of Macaulay's theorem (Theorem 3.26) in the Kruskal-Katona ring or, equivalently by Theorem 2.14, in the exterior algebra.

The Kruskal-Katona theorem is most often stated as a bound on the f -vector of a simplicial complex. Rather than starting with the combinatorial version of the theorem, we will first give the algebraic version of the Kruskal-Katona theorem and then write the equivalent combinatorial version as a corollary. This algebraic presentation first appears for the exterior algebra in [AHH97], and for the Kruskal-Katona ring in Mermin's work [Mer08].

Theorem 3.32 (Kruskal-Katona) [Kru63, Kat68]. *Let $H : \mathbb{N} \rightarrow \mathbb{N}$ be a function and let \mathbb{k} be a field. The following conditions are equivalent:*

1. H is the Hilbert function of a quotient of the Kruskal-Katona ring over \mathbb{k} by a proper homogeneous ideal;
2. H is the Hilbert function of a quotient of the Kruskal-Katona ring over \mathbb{k} by a proper lexicographic ideal;
3. H satisfies $H(0) = 1$ and

$$H(d+1) \leq H(d)^{(d)} \quad \forall d \geq 1;$$

4. H satisfies $H(0) = 1$ and

$$H(d+1)_{(d+1)} \leq H(d) \quad \forall d \geq 1.$$

The Kruskal-Katona inequality applies to Hilbert functions of quotients of the exterior algebra by Theorem 2.14.

Recall Proposition 2.8 which states that the Hilbert function of a quotient Q/I of the Kruskal-Katona ring is precisely the f -vector of the Stanley-Reisner complex Δ_I with only a slight change of indices: $\text{HF}_{Q/I}(d) = f_{d-1}$. Thus, we have the following corollary:

Corollary 3.33. *A sequence of non-negative integers f_{-1}, \dots, f_D with $f_D \neq 0$ is the f -vector of a D -dimensional simplicial complex if and only if $f_{-1} = 1$ and*

$$f_{d+1} \leq f_d^{\langle d+1 \rangle}$$

for $0 \leq d < D$.

Proof. Given a simplicial complex Δ , we can construct the quotient Q/I_{Δ}^{sf} of the Kruskal-Katona ring Q by the squarefree image of the Stanley-Reisner ideal $I_{\Delta} \subseteq S$. By the Kruskal-Katona theorem and Proposition 2.8,

$$f_{d+1} = \text{HF}_{Q/I_{\Delta}^{\text{sf}}}(d+2) \leq \text{HF}_{Q/I_{\Delta}^{\text{sf}}}(d+1)^{\langle d+1 \rangle} = f_d^{\langle d+1 \rangle}.$$

In the other direction, if we are given a sequence f_{-1}, \dots, f_D satisfying $f_{-1} = 1$ and $f_{d+1} \leq f_d^{\langle d+1 \rangle}$ then the function

$$H(d) = \begin{cases} f_{d+1} & -1 \leq d < D \\ 0 & \text{otherwise} \end{cases}$$

satisfies part (3) of the Kruskal-Katona theorem. By part (2) of the Kruskal-Katona theorem, we know there exists a monomial quotient Q/I of Q with $\text{HF}_{Q/I}(d) = H(d)$. Consequently, the monomial basis of Q/I is a simplicial complex with the desired f -vector, again by Proposition 2.8. \square

Gotzmann ideals in the Kruskal-Katona ring, as defined in Definition 3.27, have many of the same properties as Gotzmann ideals of S . Lex ideals of Q are Gotzmann and the proof is the same as in Corollary 3.28. Proposition 3.29 also holds for Gotzmann homogeneous vector space in Q with the obvious changes to Kruskal-Katona pseudopowers:

Proposition 3.34. *Let V be a degree d homogeneous vector space in the Kruskal-Katona ring Q and let L be the degree d lex segment in Q with $\dim_{\mathbb{k}} L = \dim_{\mathbb{k}} V$. The following are equivalent:*

1. V is Gotzmann.
2. $\dim_{\mathbb{k}} \nabla V = \dim_{\mathbb{k}} \nabla L$.

$$3. \dim_{\mathbb{k}} Q_{d+1}/\nabla V = (\dim_{\mathbb{k}} Q_d/V)^{(d)}.$$

Also, if V is Gotzmann then

$$4. (\dim_{\mathbb{k}} Q_{d+1}/\nabla V)_{(d+1)} = \dim_{\mathbb{k}} Q_d/V$$

but not the converse.

Proof. The proof of this proposition parallels that of Proposition 3.29. Equally, this proposition can be thought of as a corollary to the Kruskal-Katona theorem and our results on shadows of lex segments.

From the definition of Gotzmann monomial vector spaces, Definition 3.27, any two Gotzmann vector spaces of the same dimension have the same growth. Since lex segments of Q are Gotzmann, parts (1) and (2) are equivalent. Parts (2) and (3) are equivalent by Proposition 3.22.

Now, let $V \subseteq Q_d$ be a Gotzmann homogeneous vector space. From the equivalence of parts (1) and (3), we have $s = t^{(d)}$ where $s = \dim_{\mathbb{k}} S_{d+1}/\nabla V$ and $t = \dim_{\mathbb{k}} S_d/V$. Let $\text{mrep}_d t = (a_d, \dots, a_1)$ so that $\text{mrep}_{d+1} t^{(d)} = (a_d, \dots, a_1, 0)$ and therefore $\text{mrep}_d(t^{(d)})_{(d+1)} = (a_d, \dots, a_1)$, using equation 3.3.2 and equation 3.3.4. Thus $s_{(d+1)} = (t^{(d)})_{(d+1)} = t$ which proves part (4). \square

We now show that any Gotzmann squarefree monomial ideal in S has a Gotzmann image in Q . This was first shown by Aramova, Avramov and Herzog [AAH00] for Gotzmann squarefree monomial ideals of S generated in a single degree. Their proof relied on their results relating Betti numbers of ideals in the exterior algebra to those in the polynomial ring. We provide the more general result below and prove it using a direct argument.

Proposition 3.35. *Let $I \subset S$ be a squarefree monomial ideal. If I is Gotzmann in S then $I^{\text{sf}} = IQ$ is Gotzmann in the Kruskal-Katona ring Q .*

Proof. Fix a degree d and let $I_{\leq d}$ be the squarefree monomial ideal of S generated by the homogeneous components of I that have degree d or less.

The benefit of looking at this truncation $I_{\leq d}$ of I is that their homogeneous components are equal in degrees less than or equal to d , while the $(d+1)$ -st component of $I_{\leq d}$ is ∇I_d while I_{d+1} is possibly larger.

The squarefree image of $I_{\leq d}$ also agrees with I^{sf} in degrees d and less. And similarly, the $(d+1)$ -st component of the squarefree image is $(I_{\leq d}^{\text{sf}})_{d+1} = \nabla I_d^{\text{sf}}$.

Let L be the lexification of I^{sf} in Q and let $L_{\leq d}$ be its truncation to degrees less than or equal to d . That is, $L_{\leq d}$ and I^{sf} have the same Hilbert function in degrees d and less. Also, $(L_{\leq d})_{d+1} = \nabla(L_{\leq d})_d = \nabla L_d$ as $L_{\leq d}$ has no minimal generators in degree $d+1$.

Let J be the ideal of S generated by the squarefree monomials in $L_{\leq d}$ so that $J^{\text{sf}} = L_{\leq d}$. Since $L_{\leq d}$ and $I_{\leq d}^{\text{sf}}$ have the same Hilbert functions in degrees d and less, by Corollary 2.10, $\dim_{\mathbb{k}} J_d = \dim_{\mathbb{k}} I_d$. Also, since J is generated in degrees d and less, $\nabla J_d = J_{d+1}$.

As lex segments are Gotzmann, we know $\dim_{\mathbb{k}} \nabla L_d \leq \dim_{\mathbb{k}} \nabla I_d^{\text{sf}}$ and so, by our characterization of Gotzmann spaces (Proposition 3.34), we only need to show the opposite inequality.

Using Corollary 2.10 and the assumption that I_d is Gotzmann, we have

$$\begin{aligned} \sum_{i=1}^{d+1} \text{HF}_{Q/L_{\leq d}}(i) \binom{d}{d+1-i} &= \text{HF}_{S/J}(d+1) \\ &\leq \text{HF}_{S/I_{\leq d}}(d+1) \\ &= \sum_{i=1}^{d+1} \text{HF}_{Q/I_{\leq d}^{\text{sf}}}(i) \binom{d}{d+1-i}. \end{aligned}$$

Since $\text{HF}_{Q/L_{\leq d}}(i) = \text{HF}_{Q/I_{\leq d}^{\text{sf}}}(i)$ for $i \leq d$, we get

$$\text{HF}_{Q/L_{\leq d}}(d+1) \leq \text{HF}_{Q/I_{\leq d}^{\text{sf}}}(d+1)$$

and since $(L_{\leq d})_{d+1} = \nabla L_d$ and $(I_{\leq d}^{\text{sf}})_{d+1} = \nabla I_d^{\text{sf}}$ we get $\dim_{\mathbb{k}} \nabla L_d \geq \dim_{\mathbb{k}} \nabla I_d^{\text{sf}}$ completing the proof. \square

3.6 Properties of Gotzmann Ideals

The Gotzmann persistence theorem states that the shadows of Gotzmann vector spaces are also Gotzmann.

Theorem 3.36 (Gotzmann Persistence) [Got78]. *If $V \subseteq S_d$ is a Gotzmann vector space then ∇V is also Gotzmann. Consequently, if I is a homogeneous ideal generated*

in degrees less than or equal to d and I_d is Gotzmann, then I_k is also Gotzmann for every degree $k > d$.

A simple corollary of this proposition is that an ideal generated by a Gotzmann homogeneous vector space is a Gotzmann ideal. This proposition also tells us that it is easy to check if an ideal I is Gotzmann – we simply need to know that its homogeneous components I_d are Gotzmann for every degree d up to and including the highest degree of a generator of I .

A **minimal graded free resolution** of a graded module M over a graded ring R is a long exact sequence,

$$\mathbb{F} : \quad \cdots \longrightarrow F_i \xrightarrow{d_i} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$$

where F_i is a free R -module

$$F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}}$$

and $\text{im } d_i \subseteq \mathfrak{m} F_{i-1}$ for each i . Here $R(-j)$, called the **j -th twist** of R , is a graded module isomorphic to R except with degrees shifted up by j (i.e., $R(-j)_d = R_{d-j}$). Twists are used to keep the maps d_i homogeneous. The exponents $\beta_{i,j} = \beta_{i,j}(M)$ are called the **graded Betti numbers** of M and they track the number of copies of $R(-j)$ that occur in stage i of the resolution. The numbers $\beta_i(M) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(M)$ are called the **(total) Betti numbers** of M .

Alternatively, graded Betti numbers can be defined homologically as

$$\beta_{i,j}(M) = \dim_{\mathbb{k}} \text{Tor}_i^R(M, \mathbb{k})_j.$$

This formula is proved in Proposition 1.7 of [Eis05]. Although Betti numbers are the main topic of [Eis05], this book does not discuss the basics of Tor. The reader may want to refer to Section 6.2 of [Eis95] or Section 7.1 of [Rot09] for a thorough account.

When $M = S/I$ for some homogeneous ideal I , the number $\beta_{1,d}$ counts the number of degree d forms in a minimal homogeneous generating set of I . The numbers $\beta_{1,d}$ can be given equivalently as

$$\beta_{1,d} = \dim_{\mathbb{k}} I_d - \dim_{\mathbb{k}} \nabla I_{d-1}.$$

Consequently, if we fix $\dim_{\mathbb{k}} I_{d-1}$ and $\dim_{\mathbb{k}} I_d$ then we can see that I_{d-1} is Gotzmann if and only if $\beta_{1,d}$ is as large as possible. That is to say, $\beta_{1,d}(S/I) \leq \beta_{1,d}(S/J)$

for any homogeneous ideals $I, J \subseteq S$ where J is Gotzmann and where I and J have the same Hilbert function (and equality holds if and only if I is Gotzmann as well).

Assuming once more that I and J are ideals with the same Hilbert function and that J is Gotzmann, if $\beta_1(S/I) = \beta_1(S/J)$ then $\sum_{j \in \mathbb{Z}} \beta_{1,j}(S/I) = \sum_{j \in \mathbb{Z}} \beta_{1,j}(S/J)$ and hence $\beta_{1,j}(S/I) = \beta_{1,j}(S/J)$ for each $j \in \mathbb{Z}$. That is, if two ideals have the same Hilbert function and the same total number of generators then if one ideal is Gotzmann, the other must be Gotzmann as well.

The next theorem, due to Herzog and Hibi, extends these observations past the first Betti numbers to show that Gotzmann ideals with the same Hilbert functions share all of their Betti numbers.

Theorem 3.37 [HH99, Corollary 1.4]. *Let I be a homogeneous ideal in S and let L be its lexification. The following are equivalent:*

1. I is Gotzmann.
2. $\beta_1(S/I) = \beta_1(S/L)$.
3. $\beta_{1,j}(S/I) = \beta_{1,j}(S/L)$ for all j .
4. $\beta_{i,j}(S/I) = \beta_{i,j}(S/L)$ for all i, j .

Parts (2) and (3) of the previous theorem, which are not explicitly stated in Herzog and Hibi's result [HH99, Corollary 1.4], do appear in a restatement of their result by Conca [Con04, Theorem 1.3].

The equivalence of parts (1) and (3) of Theorem 3.37 tells us that an ideal is Gotzmann if and only if it has the same number of generators in each degree as its lexification.

Theorem 3.38 (Bigatti, Hulett, Pardue) [Big93, Hul93, Par96]. *For any homogeneous ideal $I \subseteq S$ with lexification L ,*

$$\beta_{i,j}(S/I) \leq \beta_{i,j}(S/L).$$

Consequently, out of all ideals with a given Hilbert function, the lex ideal has the largest possible graded Betti numbers. These Betti numbers are shared by any Gotzmann ideal with that same Hilbert function.

We now explicitly compute the Betti numbers of a Gotzmann ideal generated in a single degree using a formula of Eliahou and Kervaire.

Let $\max(m)$ be the largest index of a variable dividing a monomial m . A monomial ideal I is called **stable** if for every monomial $m \in I$ and index $i < \max(m)$ we have $x_i m / x_{\max(m)} \in I$. Since $x_i m / x_{\max(m)} >_{\text{lex}} m$, lex ideals are stable.

The minimal resolutions of stable ideals were first described by Eliahou and Kervaire and their graded Betti numbers are easy to state.

Theorem 3.39 [EK90]. *If I is a stable monomial ideal in S and $\text{gens } I$ is its set of minimal monomial generators then*

$$\beta_{i,j}(I) = \sum_{\substack{m \in \text{gens } I \\ \deg m = j-i}} \binom{\max(m) - 1}{i}.$$

We use this formula to compute the graded Betti numbers of a Gotzmann ideal generated in a single degree. This is a new result although it has some overlap with [HH99, Theorem 2.1] by Herzog and Hibi. Herzog and Hibi give a formula for the Betti numbers of componentwise linear Stanley-Reisner ideals. All Gotzmann ideals are componentwise linear but “most” Gotzmann ideals are not Stanley-Reisner ideals. In Section 5.1, we will give a classification of Gotzmann Stanley-Reisner ideals which justifies that there is a small overlap between [HH99, Theorem 2.1] and our the result below.

Theorem 3.40. *Let I be a Gotzmann ideal generated in degree d and let*

$$\text{mrep}_{n-1} \dim_{\mathbb{k}} I_d = (b_1 + n - 2, b_2 + n - 3, \dots, b_r + n - r - 1; n - r - 1)$$

with $b_r \geq 1$ as in Proposition 3.8. Then the graded Betti numbers of I are

$$\beta_{i,i+d}(I) = \sum_{j=1}^n \binom{j-1}{i} \sum_{k=1}^j \binom{j-k+b_k-1}{j-k}.$$

and $\beta_{i,j}(I) = 0$ if $j \neq i + d$.

Proof. By Theorem 3.37, $\beta_{i,j}(S/I) = \beta_{i,j}(S/L)$ where $L = (\text{Lex}(d, \text{HF}_I(d)))$ is the lexification of I . Note that the Betti numbers any ideal and its quotient ring are related by $\beta_{i,j}(I) = \beta_{i+1,j}(S/I)$ so we know $\beta_{i,j}(I) = \beta_{i,j}(L)$ for each i and j .

If we let $\mathbf{x}^{\mathbf{a}}$ be the last monomial in L , then by Proposition 3.8, the exponents a_i and coefficients b_i are related by

$$\begin{aligned} b_i &= \begin{cases} \sum_{j=i+1}^n a_j & i < r, \\ a_n + 1 & i = r. \end{cases} \\ &= \begin{cases} d - \sum_{j=1}^i a_j & i < r, \\ d - \sum_{j=1}^r a_j + 1 & i = r. \end{cases} \end{aligned}$$

where $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_r^{a_r} x_n^{a_n}$ with $a_r > 0$.

We now decompose L as in Macaulay's original proof of Proposition 3.8:

$$\begin{aligned} L &= x_1^{a_1+1} (x_1, \dots, x_n)^{b_1-1} \\ &\quad + x_1^{a_1} x_2^{a_2+1} (x_2, \dots, x_n)^{b_2-1} \\ &\quad + \cdots \\ &\quad + x_1^{a_1} \cdots x_{r-1}^{a_{r-1}+1} (x_{r-1}, \dots, x_n)^{b_{r-1}-1} \\ &\quad + x_1^{a_1} \cdots x_r^{a_r} (x_r, \dots, x_n)^{b_r-1}. \end{aligned}$$

Note that the last summand differs from the rest in the exponent of x_r . Let $N_{j,d}(J)$ be the number of monomials m of degree d in the ideal J with maximum index $\max(m) = j$. Then

$$N_{j,b_k-1}((x_k, \dots, x_n)^{b_k-1}) = \binom{j-k+b_k-1}{j-k}$$

Note that $N_{j,b_k-1} = 0$ for $j < k$. Thus,

$$\begin{aligned} N_{j,d}(L) &= \sum_{k=1}^r N_{j,b_k-1}((x_k, \dots, x_n)^{b_k-1}) \\ &= \sum_{k=1}^j \binom{j-k+b_k-1}{j-k}. \end{aligned}$$

Now we can use Eliahou and Kervaire's formula for the graded Betti numbers of L .

$$\begin{aligned}\beta_{i,j}(I) &= \beta_{i,j}(L) \\ &= \sum_{\substack{m \in \text{gens } L \\ \deg m = j-i}} \binom{\max(m) - 1}{i} \\ &= \begin{cases} \sum_{m \in \text{gens } L} \binom{\max(m) - 1}{i} & j = i + d \\ 0 & j \neq i + d \end{cases}\end{aligned}$$

as all generators of L have degree d . So, we have

$$\begin{aligned}\beta_{i,i+d}(I) &= \sum_{m \in \text{gens } L} \binom{\max(m) - 1}{i} \\ &= \sum_{j=1}^n \sum_{\substack{m \in \text{gens } L \\ \max(m) = j}} \binom{j-1}{i} \\ &= \sum_{j=1}^n \binom{j-1}{i} N_{j,d}(L) \\ &= \sum_{j=1}^n \binom{j-1}{i} \sum_{k=1}^j \binom{j-k+b_k-1}{j-k}.\end{aligned}$$

□

Whether an ideal I is Gotzmann depends not only on its generators, but also on the number of variables in the ambient ring. Whether an ideal is lexicographic or not also depends on the ambient ring and, in fact, lex ideals may not even be Gotzmann if new variables are added.

Example 3.41. The ideal $I = (x_1x_2, x_2x_3, x_3x_4, x_1x_4)$ is Gotzmann in $\mathbb{k}[x_1, \dots, x_4]$ but not in $\mathbb{k}[x_1, \dots, x_5]$. Also, the ideal $L = (x_1^2, x_1x_2, x_1x_3, x_2^2)$ is lexicographic (and therefore Gotzmann) in $\mathbb{k}[x_1, x_2, x_3]$ but is neither lexicographic nor Gotzmann in $\mathbb{k}[x_1, \dots, x_4]$.

In Theorem 4.14, we will see Murai and Hibi's classification of Gotzmann ideals with n or fewer generators, where n is the number of variables. These Gotzmann ideals are interesting because they remain Gotzmann if variables are added to their rings.

This section has focused on the Betti numbers of Gotzmann ideals in the polynomial ring. In the squarefree case, the Betti numbers of the exterior algebra have been studied more than the Betti numbers of the Kruskal-Katona ring. For instance, in [AHH97], Aramova, Herzog and Hibi prove that lex ideals of the exterior algebra have the largest possible Betti numbers for their Hilbert function – the analogue of Theorem 3.38.

The following will be sufficient for our purposes:

Proposition 3.42. *Let $I \subseteq Q$ be a homogeneous ideal and let $L \subseteq Q$ be the lexification of I . Then I is Gotzmann if and only if I and L have the same number of minimal generators in each degree.*

Proof. The number of generators of I in degree d can be computed as

$$\beta_{1,d}(Q/I) = \dim_{\mathbb{k}} I_d - \dim_{\mathbb{k}} \nabla I_{d-1}$$

and similarly for L . Since $\dim_{\mathbb{k}} I_d = \dim_{\mathbb{k}} L_d$, we see that $\dim_{\mathbb{k}} \nabla I_{d-1} = \dim_{\mathbb{k}} \nabla L_{d-1}$ if and only if $\beta_{1,d}(Q/I) = \beta_{1,d}(Q/L)$. That is, by Proposition 3.34 part (2), I_{d-1} is Gotzmann if and only if $\beta_{1,d}(Q/I) = \beta_{1,d}(Q/L)$. \square

Chapter 4

Gotzmann Graphs

The first two sections of this chapter are based on [Hoe09].

4.1 Edge Ideals of Graphs

Definition 4.1 (Edge Ideal). Let $G = (V, E)$ be a simple graph on vertex set $V = \{x_1, \dots, x_n\}$ and edge set E . The **edge ideal** of G is defined to be

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E) \subset S.$$

Every quadratic squarefree monomial ideal is the edge ideal of some simple graph. Recently, “edge ideal” has become a broader term to encompass all squarefree monomial ideals by associating the minimal generators of the ideal with edges of a simple hypergraph [HVT08, Far02]. Simple hypergraphs are also known as clutters [HMV09]. We will only deal with edge ideals of graphs and we leave it to Section 5.1 to deal with all squarefree monomial ideals. Villarreal’s book [Vil01] gives a broad account of what is known about edge ideals of graphs.

A graph G is called a **star** if there exists a vertex $x_{i_0} \in G$ for which the degree of x_{i_0} is equal to the number of edges in G .

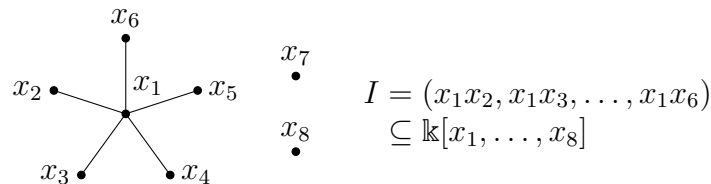


Figure 4.1: Star graph and its edge ideal.

The main result of this chapter is as follows:

Theorem 4.2. *Let G be a graph. The edge ideal $I(G)$ is Gotzmann if and only if G is a star.*

We will provide three separate proofs of this result.

The first proof, in Section 4.2, follows from elementary numerical arguments: In Theorem 4.3, it is first shown that Gotzmann edge ideals must have fewer than n edges. Subsequently, we bound the Hilbert function of edge ideals with fewer than n edges and use this to prove that Gotzmann edge ideals must come from stars. These results have also been published in [Hoe09].

The second proof, suggested by an anonymous reviewer of [Hoe09], follows by similar steps from results of Murai and Hibi [MH08]: One can show that Gotzmann edge ideals must have fewer than n edges by using Corollary 4.8 in [MH08]. The second step can be shown to follow from Murai and Hibi's characterization of Gotzmann ideals generated by at most n homogeneous polynomials (Theorem 4.14). This will be described in Section 4.3.

The final proof follows from more general arguments about Gotzmann squarefree ideals given by Hoefel and Mermin [HM10] and is reproduced with permission in Section 5.1.

4.2 Distance from Gotzmann

Before proving that only star graphs produce Gotzmann edge ideals, it is first shown in Theorem 4.3 that a Gotzmann edge ideal must have fewer edges than vertices in its graph. The next two lemmas give formulas for $\text{HF}_I(3)$ when I is either an edge ideal or Gotzmann with fewer than n generators. These lemmas are used together in Theorem 4.6 to bound how far an edge ideal is from being Gotzmann. The main result, Theorem 4.2, follows easily from this bound.

Theorem 4.3. *Let $I = I(G)$ be the edge ideal of a graph G on n vertices with e edges. If I is Gotzmann then $e < n$.*

Proof. Let $\Delta = \Delta_I$ be the Stanley-Reisner complex of I and let $(f_0, \dots, f_{\dim \Delta})$ be its f -vector. From Proposition 3.35 and Proposition 2.8 we know that $f_2 = f_1^{(2)}$ and so,

$$\begin{aligned} \text{HF}_{S/I}(2) &= f_0 + f_1 && \text{and} \\ \text{HF}_{S/I}(3) &= f_0 + 2f_1 + f_1^{(2)} \end{aligned}$$

from Theorem 2.9.

As I is generated in degree two, $\nabla I_2 = I_3$ and so, by Corollary 3.30,

$$\mathrm{HF}_{S/I}(3) = \mathrm{HF}_{S/I}(2)^{(2)}$$

giving,

$$f_0 + 2f_1 + f_1^{(2)} = (f_0 + f_1)^{(2)}. \quad (4.2.1)$$

Decompose f_1 and $f_0 + f_1$ into their second Macaulay representations as

$$f_1 = \binom{a_2}{2} + \binom{a_1}{1} \quad f_0 + f_1 = \binom{b_2}{2} + \binom{b_1}{1}$$

where $a_2 > a_1 \geq 0$ and $b_2 > b_1 \geq 0$. Substituting these Macaulay representations into equation 4.2.1 gives

$$\binom{b_2}{2} + \binom{b_1}{1} + \binom{a_2}{2} + \binom{a_1}{1} + \binom{a_2}{3} + \binom{a_1}{2} = \binom{b_2 + 1}{3} + \binom{b_1 + 1}{2}$$

which rearranges and simplifies to

$$\binom{b_2}{3} + \binom{b_1}{2} = \binom{a_2 + 1}{3} + \binom{a_1 + 1}{2}$$

using the binomial identity $\binom{i}{j} + \binom{i}{j+1} = \binom{i+1}{j+1}$.

These are third Macaulay representations and by the uniqueness of Macaulay representations we have

$$b_2 = a_2 + 1 \quad \text{and} \quad b_1 = a_1 + 1.$$

As I is generated in degree two, $f_0 = n$ and so

$$\begin{aligned} n &= (f_0 + f_1) - f_1 \\ &= \binom{a_2 + 1}{2} + \binom{a_1 + 1}{1} - \binom{a_2}{2} - \binom{a_1}{1} \\ &= a_2 + 1 \end{aligned}$$

again using the binomial identity mentioned earlier. Rearranging gives $a_2 = n - 1$ and so $f_1 = \binom{n-1}{2} + \binom{a_1}{1}$.

Recall from Proposition 2.8 that $f_1 = \mathrm{HF}_{Q/I^{\mathrm{sf}}}(2) = \binom{n}{2} - e$ is the number of non-edges of G . So,

$$e = \binom{n}{2} - f_1 = \binom{n}{2} - \binom{n-1}{2} - \binom{a_1}{1} = n - 1 - a_1$$

and since $a_1 \geq 0$ we have $e < n$. □

For the next few results, we will need to compare the Hilbert function of a homogeneous ideal $I \subseteq S$ with the Hilbert function of its quotient ring S/I by the following formula:

$$\mathrm{HF}_{S/I}(d) = \mathrm{HF}_S(d) - \mathrm{HF}_I(d) = \binom{n+d-1}{n-1} - \mathrm{HF}_I(d). \quad (4.2.2)$$

Lemma 4.4. *Let $I \subseteq S = \mathbb{k}[x_1, \dots, x_n]$ be a homogeneous ideal generated in degree two and let $m = \mathrm{HF}_I(2)$. If $m \leq n$ then I is Gotzmann if and only if*

$$\mathrm{HF}_I(3) = mn + \frac{1}{2}m - \frac{1}{2}m^2.$$

Proof. In the case where $\mathrm{HF}_I(2) = 0$, we have $I = (0)$ and the result clearly holds.

If $\mathrm{HF}_I(2) \neq 0$ then $\mathrm{HF}_{S/I}(2) < \mathrm{HF}_S(2) = \binom{n+1}{2}$. On the other hand, applying equation 4.2.2 to S/I in degree two gives a lower bound as follows:

$$\mathrm{HF}_{S/I}(2) = \mathrm{HF}_S(2) - \mathrm{HF}_I(2) \geq \binom{n+1}{2} - n = \binom{n}{2}.$$

Thus $\mathrm{HF}_{S/I}(2)$ can be written in its second Macaulay representation as

$$\mathrm{HF}_{S/I}(2) = \binom{n}{2} + \binom{a}{1} \quad (4.2.3)$$

for some integer a with $n > a \geq 0$.

We compute a by using equation 4.2.2 once more in degree two which gives

$$\begin{aligned} \mathrm{HF}_I(2) &= \mathrm{HF}_S(2) - \mathrm{HF}_{S/I}(2) \\ &= \binom{n+1}{2} - \binom{n}{2} - \binom{a}{1} \\ &= n - a \end{aligned}$$

and hence $a = n - H(I, 2) = n - m$. Replacing a with $n - m$ in equation 4.2.3 gives

$$\mathrm{HF}_{S/I}(2) = \binom{n}{2} + \binom{n-m}{1}.$$

By Proposition 3.29, I is Gotzmann if and only if

$$\mathrm{HF}_{S/I}(3) = \mathrm{HF}_{S/I}(2)^{\langle 2 \rangle} = \binom{n+1}{3} + \binom{n-m+1}{2}.$$

Applying 4.2.2 one last time yields an equivalent condition on $\text{HF}_I(3)$. Namely, I is Gotzmann if and only if

$$\begin{aligned} \text{HF}_I(3) &= \text{HF}_S(3) - \text{HF}_{S/I}(3) \\ &= \binom{n+2}{3} - \binom{n+1}{3} - \binom{n-m+1}{2} \\ &= mn + \frac{1}{2}m - \frac{1}{2}m^2. \end{aligned}$$

□

Given a graph $G = (V, E)$, a set $W \subseteq V$ is said to be **independent** if there are no edges $\{u, v\} \in E$ with $u, v \in W$. Subsets of V which are not independent are called **dependent**. The faces of the Stanley-Reisner complex Δ of an edge ideal $I(G)$ are simply the independent sets of G . Also note that the Stanley-Reisner ring $\mathbb{k}[\Delta]$ is equal to the quotient ring $S/I(G)$.

Consequently, the entries of the f -vector of Δ count the number of independent sets of G of a given size. For example, f_1 is the number of independent sets of size two or, put differently,

$$f_1 = \binom{n}{2} - |E(G)|$$

is the number of non-edges of G .

Lemma 4.5. *Let G be a graph with e edges and t dependent sets of size three. Then*

$$\text{HF}_{I(G)}(3) = 2e + t.$$

Proof. The monomial basis of $I(G)_3$, the degree three component of $I(G)$, can be partitioned into monomials of the form x_i^3 , $x_i^2x_j$ and $x_ix_jx_k$ where i, j and k are distinct. There are no monomials in $I(G)_3$ of the first type as $I(G)$ is generated by square-free monomials. There are two monomials of type $x_i^2x_j$ in $I(G)$ for each edge of G and there is one monomial of type $x_ix_jx_k$ in $I(G)$ for each dependent set of size three. □

The next theorem provides a bound on the Hilbert function of edge ideals for graphs with fewer edges than vertices. Implicit in its proof is an inductive procedure that could be carried out to compute the Hilbert function of an arbitrary edge ideal.

Theorem 4.6. *Let G be a graph with e edges and n vertices and where $e < n$. Let J be a Gotzmann ideal generated in degree 2 of $S = \mathbb{k}[x_1, \dots, x_n]$ with e generators. Given a vertex v of G with at least one neighbour, we have*

$$\mathrm{HF}_{I(G)}(3) \geq \mathrm{HF}_J(3) + (d-1)(e-d) + |E(G \setminus N(v))|$$

where $N(v)$ is the set of neighbours of v and $d = |N(v)|$ is the degree of v .

Proof. Consider the graph $H = G \setminus v$ obtained from G by deleting vertex v . Let $e_H = e - d$ denote the number of edges in H and define t_H to be the number of dependent sets in H of size three.

Let t_v be the number of dependent sets in G of size three that contain v . We can partition the dependent sets of G into those that contain v and those that are dependent sets of H . Thus, $t = t_v + t_H$ and using Lemma 4.5 we have

$$\begin{aligned} \mathrm{HF}_{I(G)}(3) &= 2e + t & (4.2.4) \\ &= 2d + 2e_H + t_H + t_v. \end{aligned}$$

Let L be the ideal generated by the degree 2 lex segment of dimension $e_H = \mathrm{HF}_{I(H)}(2)$ in the polynomial ring in $n - 1$ variables. As L is generated by a lex segment, it is Gotzmann. Thus,

$$\begin{aligned} 2e_H + t_H &= \mathrm{HF}_{I(H)}(3) & (4.2.5) \\ &\geq \mathrm{HF}_L(3) \\ &= (e-d)(n-1) + \frac{1}{2}(e-d)(1-e+d) \end{aligned}$$

using Lemma 4.5, that L is Gotzmann and Lemma 4.4.

We now compute t_v – the number of dependent sets in G of size three containing v . Partition these dependent sets into those that contain two neighbours, one neighbour and no neighbours of v . Every choice of two neighbours of v , along with v itself, is dependent. Every choice of a neighbour and a non-neighbour of v , along with v , is also dependent. Finally, every choice of two non-neighbours of v which have an edge between them gives a dependent set of size three when v is included. Thus,

$$t_v = \binom{d}{2} + d(n-d-1) + |E(G \setminus N(v))|. \quad (4.2.6)$$

Taking equation 4.2.4 and substituting in inequality 4.2.5 and equation 4.2.6 for $2e_H + t_H$ and t_v respectively gives the following:

$$\begin{aligned}
\text{HF}_{I(G)}(3) &= 2d + (2e_H + t_H) + t_v \\
&\geq 2d + (e - d)(n - 1) + \frac{1}{2}(e - d)(1 - e + d) \\
&\quad + \binom{d}{2} + d(n - d - 1) + |E(G \setminus N(v))| \\
&= ne - \frac{1}{2}e^2 + de - d^2 - \frac{1}{2}e + d + |E(G \setminus N(v))| \\
&= \left(ne + \frac{1}{2}e - \frac{1}{2}e^2 \right) + (d - 1)(e - d) + |E(G \setminus N(v))|.
\end{aligned}$$

Since we recognize $ne + \frac{1}{2}e - \frac{1}{2}e^2$ as $\text{HF}_J(3)$ from Lemma 4.4, this proves the theorem. \square

In the previous theorem, we gave a combinatorial bound on $\text{HF}_{I(G)}(3) - \text{HF}_J(3)$ for a Gotzmann ideal J generated in degree two by the same number of generators as $I(G)$. This difference is independent of J and when the difference is zero, $I(G)$ itself is Gotzmann. Thus, $\text{HF}_{I(G)}(3) - \text{HF}_J(3)$ is the **distance** that $I(G)$ is from Gotzmann.

Theorem 4.7. *Let G be a graph. The edge ideal $I(G)$ is Gotzmann if and only if G is a star.*

Proof. Let G be a graph on n vertices and e edges.

We begin by assuming that $I(G)$ is Gotzmann. We know from Theorem 4.3 that $e < n$.

The previous theorem applies to G for any choice of vertex v with at least one neighbour. As $I(G)$ is Gotzmann we must have $\text{HF}_{I(G)}(3) = \text{HF}_J(3)$ where J is a Gotzmann ideal generated in degree two with the same number of generators as $I(G)$. Thus,

$$\begin{aligned}
(d - 1)(e - d) &= 0 \quad \text{and} \\
|E(G \setminus N(v))| &= 0.
\end{aligned}$$

These equations hold for every choice of vertex v with degree $d \geq 1$ and so, the degree of each vertex in G is either 0, 1 or e . If every vertex v in G has degree 0 or

1 and if e is neither 0 nor 1 then G has more than one connected component and so $|E(G \setminus N(v))|$ cannot be zero for all v . Thus, G must have some vertex v with degree $d = e$ and hence G is a star.

Conversely, if G is a star then every dependent set of G must contain the common vertex v of all edges. That is, the number of size three dependent sets in G can be computed from equation 4.2.6 as

$$t = t_v = \binom{e}{2} + e(n - e - 1) + |E(G \setminus N(v))|.$$

However, $G \setminus N(v)$ contains no edges. Therefore,

$$\begin{aligned} \text{HF}_{I(G)}(3) &= 2e + t \\ &= 2e + \binom{e}{2} + e(n - e - 1) \\ &= ne + \frac{1}{2}e - \frac{1}{2}e^2 \end{aligned}$$

and hence $I(G)$ is Gotzmann by Lemma 4.4. \square

From the proof of Theorem 4.6, we can see that we have equality in the distance from Gotzmann,

$$\text{HF}_{I(G)}(3) = \text{HF}_J(3) + (d - 1)(e - d) + |E(G \setminus N(v))|$$

precisely when $e < n$ and $H = G \setminus v$ is a star.

Example 4.8. Let G be the graph of a square along with two extra vertices of degree zero. Then $I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1) \subset \mathbb{k}[x_1, \dots, x_6]$. Thus any Gotzmann ideal J with the same number of generators as $I(G)$ in degree two has Hilbert function,

$$\begin{aligned} \text{HF}_J(3) &= \dim_{\mathbb{k}} S_3 - (\dim_{\mathbb{k}} S_2 - 4)^{(2)} \\ &= 56 - (21 - 4)^{(2)} \\ &= 56 - \left(\binom{6}{2} + \binom{2}{1} \right)^{(2)} \\ &= 56 - \binom{7}{3} - \binom{3}{2} \\ &= 18. \end{aligned}$$

Now, to compute $\text{HF}_{I(G)}(3)$, we could either count dependent sets of size three in G and use Lemma 4.5 or simply note that, for any choice of vertex v , $I(G)$ is $(d-1)(e-d)+|E(G\setminus N(v))| = 2+0 = 2$ away from Gotzmann. That is, $\text{HF}_{I(G)}(3) = 20$.

4.3 Generic Initial Ideals of Edge Ideals

The purpose of this section is to provide a second proof of Theorem 4.2 which was outlined by an anonymous reviewer of [Hoe09]. This second proof uses machinery of generic initial ideals and their interplay with Betti numbers, Gotzmann ideals and critical ideals.

From now on, we assume that \mathbb{k} has characteristic zero since, in characteristic zero, generic initial ideals are strongly stable [Bay82]. For a discussion of generic initial ideals in positive characteristic, see [Eis95, Section 15.9].

Consider a linear transformation $\phi : S_1 \rightarrow S_1$ which acts on homogeneous polynomials of degree one. The map ϕ extends to a graded \mathbb{k} -algebra homomorphism $\Phi : S \rightarrow S$ which we call a **linear change of coordinates**. Thus, if $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial of S then $\Phi(\mathbf{x}^{\mathbf{a}}) = \phi(x_1)^{a_1} \cdots \phi(x_n)^{a_n}$.

If $\phi : S_1 \rightarrow S_1$ is an invertible linear transformation then $\Phi : S \rightarrow S$ is a graded \mathbb{k} -algebra automorphism. In particular, Φ sends ideals to ideals and preserves their Hilbert functions. Consequently, S/I and $S/\Phi(I)$ also have the same Hilbert functions.

For a term ordering σ , recall that the leading term $\text{in}_\sigma(f)$ of a polynomial f is simply the largest non-zero term of f with respect to σ . Also, remember that the initial ideal

$$\text{in}_\sigma(I) = \text{span}_{\mathbb{k}}\{\text{in}_\sigma(f) \mid f \in I\}$$

of a homogeneous ideal I always has the same Hilbert function as its initial ideal (Theorem 2.4).

An ideal I is called **Borel-fixed** if $\phi(I) = I$ for every invertible linear transformation ϕ of S_1 whose transformation matrix, with respect to the ordered basis x_1, \dots, x_n , is upper triangular. A **strongly stable** ideal is a monomial ideal I with the property that if $m \in I$ then $\frac{x_i}{x_j}m \in I$ for any $i < j \leq \max(m)$. In characteristic zero, as we have assumed, Borel-fixed ideals are strongly stable ideals [Bay82]. Recall

that **stable** ideals are monomial ideals for which $m \in I$ implies $\frac{x_i}{x_{\max(m)}}m \in I$ for any $i < \max(m)$, and clearly strongly stable ideals are stable.

In the following theorem, we use the concept of a generic invertible linear transformation. A property is **generic** if objects that do not have the property are contained within an algebraic set – a set defined by the intersection of zeros of polynomials. Algebraic sets are thought of as very small sets, and so a generic property holds “almost everywhere”.

Theorem 4.9 (Galligo-Bayer-Stillman) [BS87, Gal74]. *Let \mathbb{k} be an infinite field, let σ be a term ordering and let I be a homogeneous ideal. There exists a unique Borel-fixed monomial ideal J for which $\text{in}_\sigma \Phi(I) = J$ where $\Phi : S \rightarrow S$ is an automorphism induced by a generic invertible linear transformation $\phi : S_1 \rightarrow S_1$.*

The idea of the previous theorem is that if we change coordinates of our ideal I and then take an initial ideal, we almost always get the same Borel-fixed monomial ideal J .

Definition 4.10 (Generic Initial Ideal). Given a homogenous ideal I and term order σ , the generic initial ideal $\text{gin}_\sigma(I)$ is defined to be the unique ideal J given in the theorem of Galligo-Bayer-Stillman. We drop the subscript σ when an arbitrary term order suffices.

Since applying graded \mathbb{k} -algebra automorphisms and taking initial ideals both preserve Hilbert functions, we see that $\text{HF}_I = \text{HF}_{\text{gin } I}$.

Generic initial ideals are a powerful tool in the study of Hilbert functions, Hilbert polynomials and Betti numbers. One of their strongest properties is that they bound Betti numbers, as described below.

Theorem 4.11 (Cancellation Theorem) [Gre98]. *Let $I \subseteq S$ be a homogeneous ideal. Then for all i and j ,*

$$\beta_{i,j}(S/I) \leq \beta_{i,j}(S/\text{gin } I).$$

The following is a remark that appears in Theorem 4.6 of [Con04].

Corollary 4.12. *If I is a Gotzmann homogeneous ideal of S then $\text{gin } I$ is a Gotzmann monomial ideal.*

Proof. By Theorem 3.37 and Theorem 3.38, I has the largest possible Betti numbers for its Hilbert function. The generic initial ideal of I has the same Hilbert function as I but larger Betti numbers by the Cancellation theorem. Thus, I and $\text{gin } I$ must have the same Betti numbers. Using Theorem 3.37 a second time shows that $\text{gin } I$ is Gotzmann. \square

We now show the second proof of Theorem 4.2.

Lemma 4.13. *If $I \subset S$ is a strongly stable Gotzmann monomial ideal generated in degree two, then I is a lex ideal.*

Proof. Since the upper shadows of lex segments are lex segments, it suffices to show that I_2 is a lex segment. Let $m = x_i x_j$ be the lexicographically earliest monomial not in I and let $m' = x_k x_l$ be the lexicographically last monomial in I . Also, assume $i \leq j$ and $k \leq l$. Assume for a contradiction that I is not a lex ideal.

As I is not a lex ideal, we have $m >_{\text{lex}} m'$ and hence we either have $i < k$ or $i = k$ and $j < l$. In the second case, we must have $m = x_j m' / x_l \in I$ as I is strongly stable giving a contradiction. So we can assume that we are in the first case: $i < k$. If we can show that $j \leq l$, then again $m \in I$ and we will be done.

The Hilbert series of a stable ideal is given by Eliahou and Kervaire [EK90]:

$$\begin{aligned} \text{HS}(I, t) &= \sum_{d=0}^{\infty} \text{HF}_I(d) t^d \\ &= \sum_{u \in \text{gens } I} \frac{t^{\deg(u)}}{(1-t)^{n-\max(u)+1}}. \end{aligned}$$

Let $J = (u \in \text{gens } I \mid u \neq m') + (m)$. That is, J is obtained from I by replacing the generator m' with m . If we take a monomial m'' and replace a variable dividing m'' with one that occurs earlier, then the new monomial appears lexicographically before m'' . So, we see that $K = (u \in \text{gens } I \mid u \neq m')$ is strongly stable since m' was the lexicographically last monomial in I . And so, we can see that J is also strongly stable since it is obtained by adding m to K , and every monomial lexicographically before m is in K by our choice of m .

Thus the Hilbert series of J is also given by Eliahou and Kervaire. The difference between the Hilbert series of J and that of I is

$$\text{HS}(J, t) - \text{HS}(I, t) = \frac{t^2}{(1-t)^{n-j+1}} - \frac{t^2}{(1-t)^{n-l+1}}$$

and so, the difference between the Hilbert functions of I and J in degree three is

$$\begin{aligned} \text{HF}_J(3) - \text{HF}_I(3) &= [t^3](\text{HS}(J, t) - \text{HS}(I, t)) \\ &= [t] \left((1-t)^{-n+j-1} - (1-t)^{-n+l-1} \right) \\ &= n - j + 1 - (n - l + 1) \\ &= l - j. \end{aligned}$$

Here $[t^d]f(t) = d!f^{(d)}(0)$ is the coefficient of t^d occurring in the formal power series $f(t)$. As I is Gotzmann, $\text{HF}_I(3) \leq \text{HF}_J(3)$ and so $j \leq l$ which shows $m \in I$, a contradiction. \square

Murai and Hibi proved the following classification of Gotzmann ideals with n or fewer generators.

Theorem 4.14 [MH08]. *Let $I \subset S = \mathbb{k}[x_1, \dots, x_n]$ be an ideal with n or fewer generators. Then I is Gotzmann if and only if there exists a linear change of coordinates Φ such that*

$$\Phi(I) = (f_1x_1, f_1f_2x_2, \dots, f_1f_2 \cdots f_{s-1}x_{s-1}, f_1f_2 \cdots f_s)$$

where $f_i \in \mathbb{k}[x_i, \dots, x_n]$ are homogeneous polynomials the last of which, f_s , has degree at least one.

Murai and Hibi call Gotzmann ideals with fewer than n generators **critical ideals**. Critical ideals can also be characterized as ideals which remain Gotzmann if variables are added to their rings [MH08]. Ideals of the form

$$(f_1x_1, f_1f_2x_2, \dots, f_1f_2 \cdots f_{s-1}x_{s-1}, f_1f_2 \cdots f_s),$$

with the additional conditions mentioned in Theorem 4.14, are called **canonical critical ideals**.

The **reverse lexicographic order** is a term order defined by $\mathbf{x}^{\mathbf{a}} >_{\text{rlex}} \mathbf{x}^{\mathbf{b}}$ if either $\deg \mathbf{x}^{\mathbf{a}} > \deg \mathbf{x}^{\mathbf{b}}$ or $\deg \mathbf{x}^{\mathbf{a}} = \deg \mathbf{x}^{\mathbf{b}}$ and there exists an index i with $a_i = b_i$ for $i > k$

and $a_k < b_k$. The reverse lexicographic order and lexicographic order are the same on monomials of the same degree when n is 1 or 2, though one might have expected them to be the reverse of each other. The reverse lexicographic order is the reverse of the lexicographic order on monomials of the same degree if we also reverse the indices of the variables.

The following is Corollary 4.8 of [MH08].

Theorem 4.15 [MH08]. *Let I be an ideal generated in degree d . Then I is generated by the degree d component of a critical ideal if and only if I is Gotzmann and $\text{gin}_{\text{rlex}} I$ is a lex ideal. Here, gin_{rlex} is the generic initial ideal with respect to the reverse lexicographic order.*

Theorem 4.16. *Let G be a graph. The edge ideal $I(G)$ is Gotzmann if and only if G is a star.*

Proof. Let I be Gotzmann homogeneous vector space generated in degree two and let $J = \text{gin}_{\text{rlex}} I$.

We know that J is a strongly stable monomial ideal, has the same Hilbert function as I and, by Corollary 4.12, J is also Gotzmann. Therefore, I and J have the same Betti numbers and, in particular, J is also generated in degree two.

So, by Lemma 4.13, J is a lex ideal. By Theorem 4.15 of Murai and Hibi, I is generated by the degree two component of a critical ideal.

That is, there is a linear change of coordinates Φ with

$$\Phi(I)_2 = \Phi(I_2) = (f_1 x_1, f_1 f_2 x_2, \dots, f_1 \cdots f_{s-1} x_{s-1}, f_1 \cdots f_s)_2$$

for some f_i homogeneous in $\mathbb{k}[x_1, \dots, x_n]$ and $\deg f_s > 0$. Since generators of degree greater than two have no impact, we can assume that each f_i is either a scalar or linear. As f_1 divides each generator of our canonical critical ideal, and because Φ is a \mathbb{k} -algebra automorphism, $\Phi^{-1}(f_1)$ divides each element of I . So, $\Phi^{-1}(f_1)$ is either a scalar or a variable as I is squarefree.

If $\Phi^{-1}(f_1)$ is a variable, then every generator of I has a common variable and hence G is a star and we are done.

If $\Phi^{-1}(f_1)$ is a scalar then, as Φ is a graded map, f_1 is also a scalar. Therefore x_1 is an element of our canonical critical ideal, as is x_1^2 . Thus, $\Phi^{-1}(x_1)^2 = \Phi^{-1}(x_1^2) \in I$.

The leading term of $\Phi^{-1}(x_1)^2$ is not squarefree, contradicting that I is an edge ideal. Thus, $\Phi^{-1}(f_1)$ is not a scalar and G must be a star.

The other direction follows from Theorem 4.14 by letting Φ be the identity transformation, $f_1 = x_{s+1}$, $f_s = x_s$ and $f_i = 1$ for $1 < i < s$. \square

Chapter 5

Gotzmann Squarefree Monomial Ideals

5.1 Gotzmann Squarefree Monomial Ideals of the Polynomial Ring

In this section, we will classify the squarefree monomial ideals of the polynomial ring $S = \mathbb{k}[x_1, \dots, x_n]$ that are Gotzmann. To do this, we compare squarefree monomial ideals with their squarefree lexifications and exploit the interaction between S and the Kruskal-Katona ring $Q = \mathbb{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$.

This chapter is based on the paper [HM10] and is joint work with Jeff Mermin.

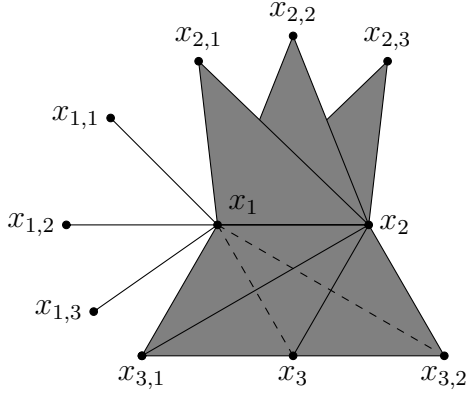
In the previous chapter, based on [Hoe09], we saw that a squarefree quadratic monomial ideal is Gotzmann if and only if it is the edge ideal of a **star-shaped** graph. We generalize this result as follows:

Definition 5.1. We say that a d -dimensional simplicial complex Δ is **star-shaped** if there exists a chain of faces $\emptyset \subset F_0 \subset F_1 \subset \dots \subset F_{d-1}$ of Δ such that every i -dimensional facet of Δ contains the $(i - 1)$ -dimensional face F_{i-1} .

The ideal we want to associate to a star-shaped complex is not the Stanley-Reisner ideal I_Δ , but instead the **facet ideal** $I(\Delta)$ which is generated by monomials formed from the facets of Δ . See Figure 5.1 for an example of a star-shaped complex, expressed by its facets, alongside the chain of faces which shows it is a star-shaped complex and a factorization of its facet ideal.

We show in Theorem 5.9 that a squarefree monomial ideal is Gotzmann if and only if it is the facet ideal of a star-shaped complex.

A consequence of Theorem 5.9 is that all Gotzmann squarefree monomial ideals of $S = \mathbb{k}[x_1, \dots, x_n]$ have at most n generators. We have already seen, in Theorem 4.14, Murai and Hibi's classification of Gotzmann ideals of S with at most n generators; it is clear from their classification that any Gotzmann squarefree monomial ideal with at most n generators must have the form prescribed by Theorem 5.9. Thus, if



$$F_0 = \{x_1\}, F_1 = \{x_1, x_2\}, F_2 = \{x_1, x_2, x_3\}$$

$$\Delta = \langle x_1x_{1,1}, x_1x_{1,2}, x_1x_{1,3}, \\ x_1x_2x_{2,1}, x_1x_2x_{2,2}, x_1x_2x_{2,3}, \\ x_1x_2x_3x_{3,1}, x_1x_2x_3x_{3,2} \rangle$$

$$I(\Delta) = x_1(x_{1,1}, x_{1,2}, x_{1,3}) \\ + x_1x_2(x_{2,1}, x_{2,2}, x_{2,3}) \\ + x_1x_2x_3(x_{3,1}, x_{3,2})$$

Figure 5.1: Star-shaped complex and its facet ideal.

this bound on the number of generators could be proved, Theorem 5.9 would be a corollary of [MH08, Theorem 1.1]. We have been unable to find a proof of this bound. Regardless, the smaller scope of our investigation allows a simpler proof than that given in [MH08].

Recall that we use I^{sf} as notation for the image of a squarefree monomial ideal $I \subseteq S$ in the Kruskal-Katona ring Q . We call an ideal L of S **squarefree lex** if L is squarefree monomial and its image $L^{\text{sf}} \subseteq Q$ is a lex ideal of Q .

Example 5.2. Let $S = \mathbb{k}[x_1, \dots, x_6]$ and let $L = (x_1x_2x_3, x_1x_2x_4, x_1x_2x_5x_6) \subset S$. Let L^{sf} be the image of L in $Q = S/(x_1^2, \dots, x_6^2)$. The degree three and degree four homogeneous components of L^{sf} are the lex segments

$$L_3^{\text{sf}} = \text{Lex}_{\geq x_1x_2x_3}^Q \quad \text{and} \quad L_4^{\text{sf}} = \text{Lex}_{\geq x_1x_2x_5x_6}^Q.$$

The upper shadow of a lex segment is a lex segment, so to tell if L^{sf} is a lex ideal, it suffices to check that every homogeneous component of L^{sf} is a lex segment up to the degree of the largest generator. Thus, we see that L^{sf} is lex and so L is squarefree lex.

We would like to define a squarefree version of lexification for homogeneous ideals in the polynomial ring. Clearly, not all Hilbert functions of ideals in S can be achieved by squarefree monomial ideals, let alone by squarefree lex ideals. If we start with a

squarefree monomial ideal, there is indeed a squarefree lex ideal with the same Hilbert function. We give the definition first and the existence proof second:

Definition 5.3. The **squarefree lexification** of a squarefree monomial ideal $I \subseteq S$ is the squarefree lex ideal L in S with the same Hilbert function as I .

The existence of the squarefree lexifications of a squarefree monomial ideal $I \subseteq S$ follows from the following construction: Let $J \subseteq Q$ be the lex ideal having the same Hilbert function as I^{sf} . Next, let L be the ideal of S with $L^{\text{sf}} = J$ (that is, L is generated by the monomial generators of J). Figure 5.2 depicts these relationships.

$$\begin{array}{ccc}
 S & \longrightarrow & Q \\
 I & \longrightarrow & I^{\text{sf}} \\
 & & \downarrow \text{lexification} \\
 L & \longrightarrow & J
 \end{array}$$

Figure 5.2: Construction of squarefree lexifications.

So, L is squarefree lex and has the same Hilbert function as I because, by Corollary 2.10,

$$\text{HS}_{S/I}(t) = \text{HS}_{Q/I^{\text{sf}}}\left(\frac{t}{1-t}\right) = \text{HS}_{Q/J}\left(\frac{t}{1-t}\right) = \text{HS}_{S/L}(t).$$

Recall Proposition 3.35 which states that if a squarefree monomial ideal of S is Gotzmann then its image in Q is Gotzmann.

Lemma 5.4. *If $I \subseteq S$ is a Gotzmann squarefree monomial ideal then its squarefree lexification L is Gotzmann.*

Proof. By Proposition 3.35, I^{sf} is Gotzmann in Q . Thus, applying Theorem 3.42, I^{sf} and L^{sf} have the same number of minimal generators in every degree. Now I and I^{sf} have the same generating set, as do L and L^{sf} , so I and L have the same number of generators in every degree. Applying Theorem 3.42 again, L must be Gotzmann in S . \square

Lemma 5.5. *Let $I \subseteq S$ be a squarefree monomial ideal and let L be its squarefree lexification. Then $L \subseteq (x_1)$ if and only if $I \subseteq (x_i)$ for some variable x_i .*

Proof. If $I \subseteq (x_i)$ then $I^{\text{sf}} \subseteq (x_i)^{\text{sf}}$ and hence

$$\dim_{\mathbb{k}}(L_d^{\text{sf}}) = \dim_{\mathbb{k}}(I_d^{\text{sf}}) \leq \dim_{\mathbb{k}}((x_i)_d^{\text{sf}}) = \dim_{\mathbb{k}}((x_1)_d^{\text{sf}}).$$

As $(x_1)_d^{\text{sf}}$ is a lex segment in Q , we have $L_d^{\text{sf}} \subseteq (x_1)_d^{\text{sf}}$ and hence every generator of L is divisible by x_1 .

Conversely, assume that $L \subseteq (x_1)$. We have $\dim_{\mathbb{k}}(L_{n-1}^{\text{sf}}) \leq n - 1$, so there is at least one squarefree monomial m of degree $n - 1$ which is not in L . Since L^{sf} and I^{sf} have the same Hilbert function, there must also be a squarefree monomial $m = \frac{x_1 \cdots x_n}{x_i}$ of degree $n - 1$ which is not in I . We claim that $I \subseteq (x_i)$.

Let $m' \in \text{gens } I$ be a minimal monomial generator of I . As I is a squarefree monomial ideal, we know m' is a squarefree monomial. If we assume, for a contradiction, that $m' \notin (x_i)$ then x_i does not divide m' . Thus, m' must divide $m = \frac{x_1 \cdots x_n}{x_i}$ and hence $m \in I$, which is a contradiction. Therefore, every generator of I is in (x_i) and so $I \subseteq (x_i)$. \square

Lemma 5.6. *If $I \subseteq S$ is a Gotzmann squarefree monomial ideal then either $I \subseteq (x_i)$ for some variable x_i or $(x_i) \subseteq I$ for some variable x_i .*

Proof. Suppose to the contrary that I is Gotzmann but, for all i , $I \not\subseteq (x_i)$ and $(x_i) \not\subseteq I$. We will show that L , the squarefree lexification of I , is not Gotzmann, contradicting Lemma 5.4.

It follows from Lemma 5.5 that $L \not\subseteq (x_1)$. Therefore we may choose a generator m of L which is not divisible by x_1 . Let d be the degree of m .

Since I contains no variable, L cannot contain x_1 . This allows us to choose a squarefree monomial $m' \in (x_1) \setminus L$ of maximal degree d' . As L is squarefree lex and m is not divisible by x_1 , L contains all squarefree monomials that are divisible by x_1 and have degree d or larger. Thus, $d' < d$.

Let $T \subseteq S$ be the ideal generated by $\text{gens}(L) \cup \{x_1^{d-d'} m'\} \setminus \{m\}$. Note that $x_1^{d-d'} m'$ is not in L as its squarefree part – the product of its support – is m' which is not in L . Therefore $\dim_{\mathbb{k}}(T_d) = \dim_{\mathbb{k}}(L_d)$ since $x_1^{d-d'} m'$ has degree d .

Let $A = \text{gens}(\nabla L_d)$ and $B = \text{gens}(\nabla T_d)$ be the sets of degree $d + 1$ monomials lying above L_d and T_d respectively. If L were Gotzmann, it would follow that $|A| \leq |B|$. We will show that instead $|B| < |A|$.

We claim that $B \setminus A = \{x_1^{d-d'} m' x_i : x_i \text{ divides } m'\}$.

First, take a monomial $\mu = x_1^{d-d'} m' x_i$ with x_i dividing m' . The squarefree part of μ is m' which is not in L . Since L is squarefree, we can conclude that μ is not in L and therefore $\mu \notin A$. That is $\{x_1^{d-d'} m' x_i : x_i \text{ divides } m'\} \subseteq B \setminus A$.

In the other direction, take $\mu \in B \setminus A$. If μ is not divisible by $x_1^{d-d'} m'$ then it is clearly in the shadow of some generator of L_d and hence μ is in A . Thus, μ must be divisible by $x_1^{d-d'} m'$. If μ has the form $x_1^{d-d'} m' x_i$ for some i and x_i does not divide m' , then $m' x_i$ is a squarefree monomial of degree $d' + 1$ which is divisible by x_1 . By the choice of m' , we have $m' x_i \in L$ and hence $x_1^{d-d'} m' x_i \in A$, proving the claim. In particular, $|B \setminus A| = d'$.

Similarly, monomials in $A \setminus B$ must have the form $x_i m$ for some i . If x_i divides m then $x_i m$ has support m and hence is not in B . Thus

$$A \setminus B \supseteq \{x_i m : x_i \text{ divides } m\}$$

which has cardinality at least d .

As $|B \setminus A| = d' < d \leq |A \setminus B|$, it follows that $\dim_{\mathbb{k}}(\nabla T_d) = |B| < |A| = \dim_{\mathbb{k}}(\nabla L_d)$, and so L is not Gotzmann. \square

Lemma 5.7. *Let $I \subseteq S$ be a Gotzmann squarefree monomial ideal with $I \subseteq (x_i)$. Then $\frac{1}{x_i} I$ is Gotzmann in S .*

Proof. Let L be the (non-squarefree) lexification of I . Since (x_1) is the lexification of (x_i) and (x_i) contains I , it is clear that $L \subseteq (x_1)$.

So, the lexification of $\frac{1}{x_i} I$ is $\frac{1}{x_1} L$. Since I is Gotzmann, by Theorem 3.37, I and L have the same number of minimal generators in each degree. The number of minimal generators of $\frac{1}{x_i} I$ in degree d is the same as the minimal number of generators of I in degree $d + 1$, and likewise for $\frac{1}{x_1} L$ and L . Therefore, $\frac{1}{x_1} L$ and $\frac{1}{x_i} I$ have the same number of generators in each degree. Using Theorem 3.37 again, we see that $\frac{1}{x_i} I$ is Gotzmann. \square

Lemma 5.8. *Let $I \subseteq S$ be a Gotzmann squarefree monomial ideal with $(x_i) \subseteq I$. The image of I in the quotient ring $S/(x_i)$ is a Gotzmann squarefree monomial ideal.*

Proof. By renaming the variables if necessary, we may assume that $(x_1) \subseteq I$. Let \bar{I} be the image of I in $S/(x_1)$ (or, equivalently, the squarefree monomial ideal of $\mathbb{k}[x_2, \dots, x_n]$ generated by every generator of I other than x_1).

Let L be the (non-squarefree) lexification of I in S . The lexification L contains (x_1) as I does. Let \bar{L} be the image of L in $S/(x_1)$. Then \bar{L} is the lexification of \bar{I} . Observe that $\text{gens}(\bar{I}) = \text{gens}(I) \setminus \{x_1\}$ and similarly for L . Thus, applying Theorem 3.37 twice, \bar{I} is Gotzmann. \square

Lemma 5.6 allows us to characterize the squarefree monomial ideals which are Gotzmann.

Theorem 5.9. *Suppose $I \subset S$ is a squarefree monomial ideal. Then I is Gotzmann if and only if*

$$I = m_1(x_{1,1}, \dots, x_{1,r_1}) + m_1 m_2(x_{2,1}, \dots, x_{2,r_2}) \\ + \dots + m_1 \cdots m_s(x_{s,1}, \dots, x_{s,r_s})$$

for some squarefree monomials m_1, \dots, m_s and variables $x_{i,j} \in \{x_1, \dots, x_n\}$ all having pairwise disjoint support and satisfying

- $\deg m_i \geq 1$ for $1 < i \leq s$,
- $r_i \geq 1$ for $1 \leq i < s$,
- $r_s \neq 1$ and
- $\deg m_s \geq 2$ when $r_s = 0$.

Furthermore, this representation for I is unique. Note that if $r_s = 0$, we use the convention that $m_1 \cdots m_s(\emptyset) = (m_1 \cdots m_s)$.

Proof. Assume that I is Gotzmann. This proof will be by induction on the number v of variables that appear in the generators of I . If $v = 0$ then the generators of I cannot be divisible by any variables. Since I is proper, I must be the zero ideal. The zero ideal can be written in the desired form by letting $s = 0$.

If $v > 0$ then, by Lemma 5.6, either $(x_j) \subseteq I$ or $I \subseteq (x_j)$ for some j .

If $I \subseteq (x_j)$ then $\frac{1}{x_j}I$ is Gotzmann in S and its generators are supported on $\{x_1, \dots, \hat{x}_j, \dots, x_n\}$. If $\frac{1}{x_j}I$ is the unit ideal, then $I = (x_j)$ and is of the desired form with $s = 1$, $r_1 = 1$, $m_1 = 1$ and $x_{1,1} = x_j$. Otherwise, $\frac{1}{x_j}I$ is proper and, by induction on v , $\frac{1}{x_j}I$ may be written as

$$m_1(x_{1,1}, \dots, x_{1,r_1}) + m_1 m_2(x_{2,1}, \dots, x_{2,r_2}) \\ + \dots + m_1 \cdots m_s(x_{s,1}, \dots, x_{s,r_s})$$

where x_j does not appear in this expression. Thus, I can be expressed in the desired form by replacing m_1 with $x_j m_1$.

Alternatively, if we suppose that $(x_j) \subseteq I$ then we can write $I = (x_j) + J$ where J is Gotzmann in the ring $\mathbb{k}[x_1, \dots, \hat{x}_j, \dots, x_n]$. By induction on v , J may be written in the desired form and so $I = (x_j) + J$ has the desired form as well (with $m_1 = 1$).

The other direction, that any ideal of the given form is Gotzmann, follows from Theorem 4.14.

We now show that uniqueness of this representation follows from the uniqueness of the minimal monomial generators of our Gotzmann squarefree monomial ideal I . The monomial m_1 is determined by the g.c.d. of the generators of I . The set of variables $\{x_{1,1}, \dots, x_{r_1}\}$ is the set of degree one generators of $\frac{1}{m_1}I$. After removing these degree one generators of $\frac{1}{m_1}I$, m_2 is determined by the g.c.d. of the generators that remain. We can continue in this way to uniquely determine each m_i and each set $\{x_{i,1}, \dots, x_{i,r_i}\}$ with $i < s$. At the last stage, we have either a single monomial m_s or a set of two or more monomials. If we have a single monomial it must have degree at least two as otherwise, it would be a degree one generator and have been removed at the previous step. That is to say, we either have $r_s = 0$ and $\deg m_s \geq 2$ or we have $r_s \geq 2$. \square

Corollary 5.10. *A squarefree monomial ideal in the polynomial ring is Gotzmann if and only if it is the facet ideal of a star-shaped complex.*

See Figure 5.1 for an example of a star-shaped complex and a factorization of its facet ideal so that it appears as in Theorem 5.9.

There is a small amount of freedom in correspondence between the monomials m_1, \dots, m_s in Theorem 5.9 and the chain of faces $\emptyset \subset F_0 \subset \dots \subset F_{d-1}$ in the

definition of a star-shaped complex. If we index the variables in each of the m_i together so that

$$\begin{aligned} m_1 &= x_{i_1} \cdots x_{i_{d_1}}, \\ m_2 &= x_{i_{d_1+1}} \cdots x_{i_{d_1+d_2}}, \\ &\vdots \\ m_s &= x_{i_{\sum_{k=1}^{s-1} d_k+1}} \cdots x_{i_{\sum_{k=1}^s d_k}}, \end{aligned}$$

where m_i has degree d_i , then we can let $F_j = \{x_{i_1}, \dots, x_{i_{j+1}}\}$, though permuting the order of the variables in any m_i leads to other acceptable chains of faces.

Corollary 5.11. *For every Gotzmann squarefree monomial I ideal in $S = \mathbb{k}[x_1, \dots, x_n]$ there exists an ordering of the variables of S so that I is a squarefree lex ideal. That is to say, for some ordering of the variables, I^{sf} is a lex ideal in the Kruskal-Katona ring Q .*

Proof. Using the notation of Theorem 5.9,

$$I = m_1(x_{1,1}, \dots, x_{1,r_1}) + \cdots + m_1 \cdots m_s(x_{s,1}, \dots, x_{s,r_s}).$$

Reorder the variables of S so that the variables dividing m_1 precede the variables $x_{1,j}$ which are then followed by the variables dividing m_2 and so on. That is, write I as

$$\begin{aligned} I &= x_1 \cdots x_{d_1}(x_{d_1+1}, \dots, x_{d_1+r_1}) \\ &\quad + x_{d_1+r_1+1} \cdots x_{d_1+r_1+d_2}(x_{d_1+r_1+d_2+1}, \dots, x_{d_1+r_1+d_2+r_2}) \\ &\quad + \cdots \\ &\quad + x_{1+\sum_{j=1}^{s-1}(d_j+r_j)} \cdots x_{d_{s-1}+\sum_{j=1}^{s-1}(d_j+r_j)}(x_{1+d_s+\sum_{j=1}^{s-1}(d_j+r_j)}, \dots, x_{d_s+r_s+\sum_{j=1}^{s-1}(d_j+r_j)}) \end{aligned}$$

where d_i is the degree of m_i .

The ideal I is generated in degrees $d = 1 + \sum_{j=1}^i d_j$ for integers $i = 1, \dots, s$. We claim that in these degrees, $I_d^{\text{sf}} = \text{Lex}_{\geq m_1 \cdots m_i x_{i,r_i}}^Q$.

If we take a squarefree monomial m' which appears lexicographically before $m = m_1 \cdots m_i x_{i,r_i}$ then there must be a variable appearing before x_{i,r_i} which divides m'

but not m . As each m_j with $j \leq i$ divides m , the variable dividing m' must be $x_{j,k}$ for some $j \leq i$ and $k < r_i$. If this is the first variable where m' and m differ, then by our order on the variables, $m_1 \cdots m_j$ must also divide m' . Thus, $m' \in I$ as $x_1 \cdots m_j x_{j,k}$ divides m' . This proves our claim that homogeneous components of I^{sf} are lex segments in the degrees of its generators. As the other homogeneous components are shadows of the lower degrees, and since the shadows of lex segments are lex segments, we see that I^{sf} is a lex ideal. \square

In the next section we use our classification of Gotzmann squarefree monomial ideals of S to enumerate them.

5.2 Enumerating Gotzmann Squarefree Monomial Ideals

Our classification of Gotzmann squarefree monomial ideals allows us to begin the process of their enumeration. In doing so, we will use a variety of interesting tools from enumerative combinatorics.

Let \mathbf{G}_n be the **set of all Gotzmann squarefree monomial ideals** in a polynomial ring over n variables. From Theorem 5.9, we know that every proper ideal $I \in \mathbf{G}_n$ is of the form

$$I = m_1(x_{1,1}, \dots, x_{1,r_1}) + \cdots + m_1 \cdots m_s(x_{s,1}, \dots, x_{s,r_s}).$$

where $\deg m_i \geq 1$ for $1 < i \leq s$, $r_i \geq 1$ for $1 \leq i < s$, $r_s \neq 1$ and $\deg m_s \geq 2$ when $r_s = 0$.

Example 5.12. Figure 5.3 lists all 19 Gotzmann squarefree monomial ideals in $\mathbb{K}[x_1, x_2, x_3]$.

In order to enumerate Gotzmann squarefree monomial ideals, we will break them into the two cases $r_s = 0$ and $r_s \geq 2$ and then relate them to certain set compositions.

Definition 5.13 (Set Composition). Let $[n] = \{1, \dots, n\}$. A **weak set composition** (or ordered set partition) of $[n]$ is a sequence

$$\sigma = (\sigma_1, \dots, \sigma_k)$$

Form	Gotzmann Ideals			
$s = 0$	(0)			
$s = 1, r_1 = 0$	(1)	(x_1)	(x_2)	(x_3)
	(x_1x_2)	(x_1x_3)	(x_2x_3)	$(x_1x_2x_3)$
$s = 1, r_1 \geq 2$	(x_1, x_2)	(x_1, x_3)	(x_2, x_3)	
	$x_3(x_1, x_2)$	$x_2(x_1, x_3)$	$x_1(x_2, x_3)$	(x_1, x_2, x_3)
$s = 2$	$(x_1) + (x_2x_3)$	$(x_2) + (x_1x_3)$	$(x_3) + (x_1x_2)$	

Figure 5.3: Gotzmann squarefree monomial ideals in $\mathbb{k}[x_1, x_2, x_3]$.

of pairwise disjoint sets whose union is $[n] = \sigma_1 \cup \dots \cup \sigma_k$. The set σ_i is called the ***i*-th part** of σ and a weak set composition of $[n]$ is called a ***k*-composition** if it has k parts.

Given a sequence of k non-negative integers $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$, we say that a weak k -composition σ of $[n]$ is ***a*-restricted** if $|\sigma_i| \geq a_i$ for each i . We let $\mathbf{H}_{n,\mathbf{a}}$ be the **set of all *a*-restricted *k*-compositions of $[n]$** .

The difference between set compositions and weak set compositions is that set compositions are defined to have non-empty parts (i.e., they are $(1, \dots, 1)$ -restricted). Since we will primarily work with restricted set compositions, where the minimum sizes of the parts are made explicit, this distinction should not cause confusion.

The set of all weak k -compositions of $[n]$ is equal to $\mathbf{H}_{n,\mathbf{0}}$ where $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^k$, since the size of each part of a weak set composition is always non-negative.

A weak k -composition of $[n]$ can be determined by taking each $i \in [n]$ and choosing a set σ_j , $1 \leq j \leq k$ for i to belong to. That is to say, for every weak k -composition σ of $[n]$, we can construct a function $\phi_\sigma : [n] \rightarrow [k]$ where $\phi_\sigma(i) = j$ for $i \in \sigma_j$. Thus, there are k^n different weak k -compositions of $[n]$.

Example 5.14. In the following example of restricted set compositions, we write compositions without set braces for brevity; e.g., $(\{1, 4\}, \emptyset, \{2, 3\}) = (14, \emptyset, 23)$. If we

let $n = 3$ then

$$\begin{aligned} \mathbf{H}_{3,(0)} &= \{(123)\}, \\ \mathbf{H}_{3,(0,0)} &= \left\{ (123, \emptyset), (12, 3), (13, 2), (23, 1), \right. \\ &\quad \left. (1, 23), (2, 13), (3, 23), (\emptyset, 123) \right\}, \\ \mathbf{H}_{3,(0,0,2)} &= \left\{ (1, \emptyset, 23), (2, \emptyset, 13), (3, \emptyset, 12), \right. \\ &\quad \left. (\emptyset, 1, 23), (\emptyset, 2, 13), (\emptyset, 3, 12), (\emptyset, \emptyset, 123) \right\}, \text{ and} \\ \mathbf{H}_{3,(0,0,1,2)} &= \{(\emptyset, \emptyset, 1, 23), (\emptyset, \emptyset, 2, 13), (\emptyset, \emptyset, 3, 12)\}. \end{aligned}$$

It is easy to see connections between set compositions and Gotzmann squarefree monomial ideals. We now give the definitions needed to make this correspondence precise.

Given a weak set composition $\sigma = (\sigma_1, \dots, \sigma_k)$, we can construct a Gotzmann squarefree monomial ideal I_σ given by

$$I_\sigma = \begin{cases} m_1(x_{1,1}, \dots, x_{1,r_1}) + \dots + m_1 \cdots m_s(x_{s,1}, \dots, x_{s,r_s}) & k \text{ odd,} \\ m_1(x_{1,1}, \dots, x_{1,r_1}) + \dots + m_1 \cdots m_{s-1}(x_{s-1,1}, \dots, x_{s-1,r_{s-1}}) + (m_1 \cdots m_s) & k \text{ even.} \end{cases}$$

where $s = \lfloor k/2 \rfloor$ and for $1 \leq i \leq s$,

$$m_i = \prod_{j \in \sigma_{2i}} x_j, \quad (5.2.1)$$

$$(x_{i,1}, \dots, x_{i,r_i}) = (x_j \mid j \in \sigma_{2i+1}) \quad (5.2.2)$$

and where σ_1 is the set of indices of variables not appearing in the generators of I_σ . In other words, parts of σ with even indices determine the monomials, while parts with odd indices (other than σ_1) determine the sets of variables. Two particular cases to note are $I_{(\emptyset)} = (0)$ and $I_{(\emptyset, \emptyset)} = (1)$.

In the next theorem we show that the map $\sigma \rightarrow I_\sigma$ is a bijection between certain restricted set compositions and Gotzmann squarefree monomial ideals in a ring with n variables. The restrictions needed are given by the vectors $\mathbf{a}(k)$, for integers $k \geq 1$,

which are defined as

$$\mathbf{a}(k) = \begin{cases} (0) & \text{if } k = 1, \\ (0, 0) & \text{if } k = 2, \text{ and} \\ (0, 0, \underbrace{1, \dots, 1}_{k-3 \text{ times}}, 2) & \text{if } k \geq 3. \end{cases}$$

Note that $\mathbf{H}_{n, \mathbf{a}} = \emptyset$ when the sum of the entries of \mathbf{a} is larger than n . So, for $n \neq 0$ and $k \geq n + 2$, $\mathbf{H}_{n, \mathbf{a}(k)} = \emptyset$.

Theorem 5.15. *Let $\mathbf{a}(k) = (0, 0, 1, \dots, 1, 2) \in \mathbb{N}^k$ with $\mathbf{a}(1) = (0)$ and $\mathbf{a}(2) = (0, 0)$. The set \mathbf{G}_n of all Gotzmann squarefree monomial ideals in a polynomial ring with n variables is in bijection with the disjoint union,*

$$\bigcup_{k=1}^{\infty} \mathbf{H}_{n, \mathbf{a}(k)}.$$

Proof. By Theorem 5.9, each of Gotzmann squarefree monomial ideal $I \in \mathbf{G}_n$ can be written as

$$I = m_1(x_{1,1}, \dots, x_{1,r_1}) + m_1 m_2(x_{2,1}, \dots, x_{2,r_2}) \\ + \dots + m_1 \cdots m_s(x_{s,1}, \dots, x_{s,r_s})$$

in a unique way where m_i and r_i are subject to the following restrictions: $\deg m_i \geq 1$ for $1 < i \leq s$, $r_i \geq 1$ for $1 \leq i < s$, $r_s \neq 1$ and $\deg m_s \geq 2$ when $r_s = 0$.

The unique presentation of each ideal $I \in \mathbf{G}_n$ determines a unique weak k -composition σ of $[n]$ with $I_\sigma = I$, where

$$k = \begin{cases} 2s & \text{if } r_s = 0, \text{ and} \\ 2s + 1 & \text{if } r_s \geq 2. \end{cases}$$

The parts of σ are determined by equations (5.2.1) and (5.2.2).

The compositions σ that are produced in this way must have $|\sigma_{2i}| \geq 1$ for $1 < i \leq s$ as $\deg m_i \geq 1$ for $1 < i \leq s$. These compositions must also satisfy the restriction $|\sigma_{2i+1}| \geq 1$ for $1 \leq i < s$ as $r_i \geq 1$ for $1 \leq i < s$. The last part σ_k of σ corresponds to the indices of m_s when $r_s = 0$ and to the indices of $\{x_{s,1}, \dots, x_{s,r_s}\}$ when $r_s \geq 2$. In both of these cases, we must have $|\sigma_k| \geq 2$ due to the restrictions from Theorem 5.9.

Thus, the unique weak set composition determined by each ideal $I \in \mathbf{G}_n$ is $\mathbf{a}(k)$ -restricted. Since, by Theorem 5.9, every $\mathbf{a}(k)$ -restricted composition σ also gives a Gotzmann squarefree monomial ideal I_σ , we have the desired bijection. \square

For an example of this relationship between Gotzmann squarefree monomial ideals and restricted set compositions, compare the ideals in Example 5.12 with the compositions in Example 5.14 and note that they are in one-to-one correspondence.

We have already mentioned the correspondence between k -compositions σ of $[n]$ and functions $\phi_\sigma : [n] \rightarrow [k]$ with $\phi_\sigma(i) = j$ for $i \in \sigma_j$. One last way to represent k -compositions of $[n]$ is to write them as words in a way similar to the correspondence between subsets of a finite set and words in 0's and 1's. An **alphabet** A is simply a finite set whose elements are called **letters**. A **word** (or string) over an alphabet A is a finite sequence of elements of A . This is standard terminology in the study of formal languages and combinatorics on words [Lot97].

Given a word $c_1 \cdots c_n$ of length n over the alphabet $[k]$, we can define a composition $\sigma = (\sigma_1, \dots, \sigma_k)$ by $i \in \sigma_j$ if $c_i = j$. This forms a bijective correspondence between k -compositions of $[n]$ and words of length n over an alphabet of size k .

For example, with $n = 7$ and $k = 3$, the word 1223132 over the alphabet $\{1, 2, 3\}$ corresponds to the k -composition $(15, 237, 46)$ of $\{1, \dots, 7\}$.

If we have an \mathbf{a} -restricted k -composition σ then, in the word representation of σ , the letter i must occur at least a_i times. From these word representations, it is easy to find a recurrence for the size of $\mathbf{H}_{n,\mathbf{a}}$:

Proposition 5.16. *For all $n \in \mathbb{N}$ and any sequence $\mathbf{a} \in \mathbb{N}^k$, we have*

$$|\mathbf{H}_{n,\mathbf{a}}| = \sum_{i=1}^k |\mathbf{H}_{n-1,(a_1,\dots,\max(0,a_i-1),\dots,a_k)}|$$

where

$$|\mathbf{H}_{0,\mathbf{a}}| = \begin{cases} 1 & \text{if } \mathbf{a} = (0, \dots, 0), \text{ and} \\ 0 & \text{if there is some } a_i \neq 0. \end{cases}$$

Proof. Think of elements of $\mathbf{H}_{n,\mathbf{a}}$ as words $c_1 \cdots c_n$ over the alphabet $[k]$ with the letter i occurring at least a_i times. Thus, $\mathbf{H}_{n,\mathbf{a}}$ can be written as the disjoint union

$$\mathbf{H}_{n,\mathbf{a}} = \bigsqcup_{i=1}^k \mathbf{K}_i$$

where K_i is the set of words in $H_{n,\mathbf{a}}$ with i as their first letter. If we remove the first letter from a word in K_i then the remaining subword must contain letter i at least $\max(0, a_i - 1)$ times. Thus there is a bijection between K_i and $H_{n-1, (a_1, \dots, \max(0, a_i - 1), \dots, a_k)}$ given by removing the first letter of a word in K_i .

In the cases where $n = 0$, the empty word corresponds to the composition $(\emptyset, \dots, \emptyset)$, and this composition is an \mathbf{a} -restricted composition if and only if each $a_i = 0$. \square

The remainder of this section is devoted to producing a generating function which encodes $|G_n|$, the number of Gotzmann squarefree monomial ideals in a polynomial ring of n variables, for each n . We begin with some notation and basic facts about ordinary and exponential generating functions, and then we build a bivariate generating function which encodes $|H_{n,\mathbf{a}(k)}|$ for each n and k . Finally, we apply Theorem 5.15 to build a generating function for $|G_n|$.

Let A be a set of (combinatorial) objects and let $\omega : A \rightarrow \mathbb{N}$ be a function called a **weight** function. The ω -**weight** of an element $a \in A$ is simply $\omega(a)$.

For a given weight function ω , we are interested in the number of elements of A with weight n for each $n \in \mathbb{N}$ or, in other words, we want to count the size of $\omega^{-1}(n)$. It will be useful to encode these numbers in the coefficients of formal power series, however there are two ways to do this: ordinary and exponential generating functions. Algebraic operations on generating functions correspond to combinatorial operations on the objects they count. We will need Cartesian products and shuffle products in order to count restricted set compositions. These two products correspond to multiplication of ordinary and exponential generating functions respectively. Therefore we will need to use both types of generating functions.

Definition 5.17 (Generating Functions). Let A be a set with weight function ω . The **ordinary generating function** of (A, ω) is defined to be,

$$\Phi_{A,\omega}(t) = \sum_{a \in A} t^{\omega(a)} = \sum_{i=0}^{\infty} |\omega^{-1}(i)| t^i.$$

The **exponential generating function** of (A, ω) is defined as,

$$\Psi_{A,\omega}(t) = \sum_{a \in A} \frac{1}{\omega(a)!} t^{\omega(a)} = \sum_{i=0}^{\infty} \frac{|\omega^{-1}(i)|}{i!} t^i.$$

If we can express a generating function for (\mathbf{A}, ω) as a rational or analytic function, we can use partial differentiation to extract formulas for the number of elements of \mathbf{A} with weight d . For the task of coefficient extraction, we define two operators

$$[t^d]\Phi_{\mathbf{A},\omega}(t) = d! \frac{\partial^d}{\partial t^d} \Phi_{\mathbf{A},\omega}(0) = |\omega^{-1}(d)|$$

and

$$\llbracket t^d \rrbracket \Psi_{\mathbf{A},\omega}(t) = \frac{\partial^d}{\partial t^d} \Psi_{\mathbf{A},\omega}(0) = |\omega^{-1}(d)|.$$

The main difference between ordinary and exponential generating functions, besides how their coefficients are extracted, is that they multiply differently. Recall that the product of two power series $f(t) = \sum_{i=0}^{\infty} a_i t^i$ and $g(t) = \sum_{i=0}^{\infty} b_i t^i$ is defined as

$$f(t)g(t) = \sum_{i=0}^{\infty} \sum_{j=0}^i a_j b_{i-j} t^i.$$

If instead we have $f(t) = \sum_{i=0}^{\infty} \frac{a_i}{i!} t^i$ and $g(t) = \sum_{i=0}^{\infty} \frac{b_i}{i!} t^i$ then their product can be written as

$$\begin{aligned} f(t)g(t) &= \sum_{i=0}^{\infty} \sum_{j=0}^i \frac{a_j b_{i-j}}{j!(i-j)!} t^i \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^i \binom{i}{j} \frac{a_j b_{i-j}}{i!} t^i. \end{aligned}$$

That is, $\llbracket t^n \rrbracket f(t)g(t) = \sum_{j=0}^i \binom{i}{j} a_j b_{i-j}$ where $a_j = \llbracket t^j \rrbracket f(t)$ and $b_j = \llbracket t^j \rrbracket g(t)$.

Let \mathbf{A} and \mathbf{B} be sets of combinatorial objects with weight functions $\omega : \mathbf{A} \rightarrow \mathbb{N}$ and $\nu : \mathbf{B} \rightarrow \mathbb{N}$, respectively. The **Cartesian product** $\mathbf{A} \times \mathbf{B}$ is naturally weighted by $\omega + \nu : \mathbf{A} \times \mathbf{B} \rightarrow \mathbb{N}$ where $(\omega + \nu)(a, b) = \omega(a) + \nu(b)$. The product of ordinary generating functions gives the generating function of the Cartesian product:

$$\Phi_{\mathbf{A},\omega}(t)\Phi_{\mathbf{B},\nu}(t) = \Phi_{\mathbf{A} \times \mathbf{B},\omega+\nu}(t).$$

The combinatorial interpretation of the product of two exponential generating functions is slightly more complicated, so we will only describe the product for sets of words.

Take a word w over the alphabet A and let B be a subset of A . The **subword** of w in the alphabet B is defined to be the word w_B formed from all letters in w that

are in B while preserving their order. For example, the subword of $bxabybxa$ in the alphabet $\{a, b\}$ is $babba$.

Let A and B be two sets of words over disjoint alphabets A and B , respectively. We define the **shuffle product** $A * B$ to be the set of words w over the alphabet $A \cup B$ with $w_A \in A$ and $w_B \in B$.

Example 5.18. If $A = \{aa, aba\}$ and $B = \{xx, xy\}$ then

$$A * B = \left\{ \begin{array}{l} aaxx, axax, axxa, xaax, xaxa, xxaa, \\ aaxy, axay, axya, xaay, xaya, xyaa, \\ abaxx, abxax, abxxa, axbax, axbxa, axxba, xabax, xabxa, xaxba, xxaba, \\ abaxy, abxay, abxya, axbay, axbya, axyba, xabay, xabya, xayba, xyaba \end{array} \right\}.$$

Example 5.19. Recall that the sets $H_{n,\mathbf{a}}$, with $\mathbf{a} \in \mathbb{N}^k$, correspond to words of length n over the alphabet $[k]$ where the letter i must occur at least a_i times. These restricted sets of words can built by shuffling together restricted words over single letter alphabets:

$$\begin{aligned} \bigcup_{n=0}^{\infty} H_{n,(1,2)} &\cong \left\{ \begin{array}{l} abb, bab, bba, \\ aabb, abab, abba, baab, baba, bbaa, \\ abbb, babb, bbab, bbba, \dots \end{array} \right\} \\ &= \{a, aa, aaa, \dots\} * \{bb, bbb, \dots\} \\ &\cong \left(\bigcup_{n=0}^{\infty} H_{n,(1)} \right) * \left(\bigcup_{n=0}^{\infty} H_{n,(2)} \right) \end{aligned}$$

The following lemma tells us that the product of two exponential generating functions for sets of words gives another exponential generating function that counts words in the shuffle product:

Lemma 5.20. *Let A and B be sets of words over disjoint alphabets. Then,*

$$\Psi_{A*B,\omega}(t) = \Psi_{A,\nu}(t)\Psi_{B,\mu}(t)$$

where $\omega : A * B \rightarrow \mathbb{N}$, $\nu : A \rightarrow \mathbb{N}$ and $\mu : B \rightarrow \mathbb{N}$ all map a word to its length.

Proof. Take two words $w \in \mathbf{A}$ and $w' \in \mathbf{B}$ of lengths i and j respectively. There are $\binom{i+j}{i}$ ways to shuffle together w and w' since the resulting words have length $i+j$ and we need to i positions for letters in w . Thus, the number of words of length d in $\mathbf{A} * \mathbf{B}$ is $\sum_{i=0}^d \binom{d}{i} a_i b_{d-i}$ where a_i and b_i are the number of words of length i from \mathbf{A} and \mathbf{B} , respectively. Since $[[t^d]]\Psi_{\mathbf{A},\nu}(t)\Psi_{\mathbf{B},\mu}(t)$ is also equal to $\sum_{i=0}^d \binom{d}{i} a_i b_{d-i}$, we get the desired result. \square

Let \mathbf{A} be a set and let $\omega, \nu : \mathbf{A} \rightarrow \mathbb{N}$ be two weight functions. We define the **bivariate generating function** of $(\mathbf{A}, \omega, \nu)$ to be

$$\Phi_{\mathbf{A},\omega,\nu}(s, t) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{|\omega^{-1}(k) \cap \nu^{-1}(n)|}{n!} s^k t^n$$

which is ordinary in s , but exponential in t . The coefficient of $\frac{s^k t^n}{n!}$ is given by $[s^k][[t^n]]\Phi_{\mathbf{A},\omega,\nu}(s, t)$.

Theorem 5.21. *Let $\mathbf{a}(k) = (0, 0, 1, \dots, 1, 2) \in \mathbb{N}^k$ with $\mathbf{a}(1) = (0)$ and $\mathbf{a}(2) = (0, 0)$.*

Let \mathbf{H} be the disjoint union

$$\mathbf{H} = \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \mathbf{H}_{n,\mathbf{a}(k)}.$$

Let σ be an $\mathbf{a}(k)$ -restricted k -composition of $[n]$ and let $\omega, \nu : \mathbf{H} \rightarrow \mathbb{N}$ be the weight functions $\omega(\sigma) = k$ and $\nu(\sigma) = n$. Then

$$\Phi_{\mathbf{H},\omega,\nu}(s, t) = se^t + s^2 e^{2t} + \frac{s^3 e^{2t}(e^t - t - 1)}{1 + s - se^t}$$

is the bivariate generating function of \mathbf{H} which is ordinary in s and exponential in t .

Proof. Recall that for $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$, $\mathbf{H}_{n,\mathbf{a}}$ is in correspondence with words of length n in k letters where the i -th letter of the alphabet occurs at least a_i times.

Therefore, the set $\mathbf{H}_{\mathbf{a}} = \cup_{n=0}^{\infty} \mathbf{H}_{n,\mathbf{a}}$ corresponds to words of any length, again with the restriction that the i -th letter occurs at least a_i times.

For instance $\mathbf{H}_{(0)} \cong \{\epsilon, z, zz, zzz, \dots\}$ is the set of all words in a single letter. Here, ϵ denotes the empty word. Since, in $\mathbf{H}_{(0)}$, there is one word of each length n , its exponential generating function using length as the weight is

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

Similarly, the exponential generating function of $H_{(1)} \cong \{z, zz, zzz, \dots\}$ is

$$e^t - 1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

and the exponential generating function of $H_{(2)} \cong \{zz, zzz, \dots\}$ is

$$e^t - t - 1 = \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

Using the word representation of $H_{\mathbf{a}(k)}$ gives

$$H_{\mathbf{a}(k)} \cong H_{(0)} * H_{(0)} * \underbrace{H_{(1)} * \dots * H_{(1)}}_{k-3 \text{ times}} * H_{(2)}$$

as every word can be decomposed into its subwords over single letter alphabets.

Restricting the number of times a letter occurs in these single letter subwords is the same as restricting the number of times a letter occurs in the whole word.

Consequently, the exponential generating function of $H_{\mathbf{a}(k)}$ is

$$\begin{aligned} \Psi_{H_{\mathbf{a}(k)}, \nu}(t) &= e^t e^t (e^t - 1)^{k-3} (e^t - t - 1) \\ &= e^{2t} (e^t - 1)^{k-3} (e^t - t - 1) \end{aligned}$$

when $k \geq 3$. Since $\mathbf{a}(1) = (0)$ and $\mathbf{a}(2) = (0, 0)$, we have $\Psi_{H_{\mathbf{a}(1)}, \nu}(t) = e^t$ and $\Psi_{H_{\mathbf{a}(2)}, \nu}(t) = e^{2t}$.

The bivariate generating function of $H = \bigcup_{k=1}^{\infty} H_{\mathbf{a}(k)}$ is

$$\begin{aligned} \Phi_{H_{\mathbf{a}(k)}, \omega, \nu}(s, t) &= \sum_{k=1}^{\infty} s^k \Psi_{H_{\mathbf{a}(k)}, \nu}(t) \\ &= se^t + s^2 e^{2t} + \sum_{k=3}^{\infty} s^k e^{2t} (e^t - 1)^{k-3} (e^t - t - 1) \\ &= se^t + s^2 e^{2t} + s^3 e^{2t} (e^t - t - 1) \sum_{i=0}^{\infty} s^i (e^t - 1)^i \\ &= se^t + s^2 e^{2t} + \frac{s^3 e^{2t} (e^t - t - 1)}{1 - s(e^t - 1)}. \end{aligned}$$

□

Corollary 5.22. *Let G_n be the set of Gotzmann squarefree monomial ideals in a polynomial ring in n variables. Let the disjoint union of these sets be $G = \bigcup_{n=0}^{\infty} G_n$ and let $\nu : G \rightarrow \mathbb{N}$ be the weight function $\nu(I) = n$ for $I \in G_n$. Then*

$$\Psi_{G, \nu}(t) = \frac{e^t(2 - te^t)}{2 - e^t}.$$

Proof. By Theorem 5.15, we have $|\mathbf{G}_n| = \sum_{k=1}^{\infty} |\mathbf{H}_{n,\mathbf{a}(k)}|$. Therefore,

$$\begin{aligned} \Psi_{\mathbf{G},\nu}(t) &= \sum_{n=0}^{\infty} \frac{|\mathbf{G}_n|}{n!} t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{|\mathbf{H}_{n,\mathbf{a}(k)}|}{n!} t^n \\ &= \Phi_{\mathbf{H},\omega,\nu}(1, t) \\ &= e^t + e^{2t} + \frac{e^{2t}(e^t - t - 1)}{2 - e^t} \\ &= \frac{e^t(2 - te^t)}{2 - e^t}. \end{aligned} \quad \square$$

From this generating function, one can extract the number of Gotzmann squarefree monomial ideals in $\mathbb{k}[x_1, \dots, x_n]$. For $0 \leq n \leq 5$, these numbers are 2, 3, 6, 19, 96, and 669. This sequence does not appear in the ‘‘On-Line Encyclopedia of Integer Sequences’’. If we subtract two from each integer in the above sequence, we get the number of proper and non-zero Gotzmann squarefree monomial ideals in a polynomial ring in n variables (e.g., 0, 1, 4, 17, 94, 667, ...). The exponential generating function of this modified sequence can be computed as

$$\begin{aligned} \frac{e^t(2 - te^t)}{2 - e^t} - 2e^t &= \frac{e^t(-2 + 2e^t - te^t)}{2 - e^t} \\ &= t + 4t^2 + 17t^3 + 94t^4 + 667t^5 + \dots \end{aligned}$$

In the ‘‘On-Line Encyclopedia of Integer Sequences’’, this modified sequence and its exponential generating function appear as the number of distinct resistances possible with at most n arbitrary resistors connected in series or in parallel [Slo03, A123750].

5.3 Gotzmann Ideals of the Kruskal-Katona Ring

The problem of classifying all Gotzmann monomial ideals of the Kruskal-Katona ring Q turns out to be much more difficult. Since the Gotzmann squarefree monomial ideals of S are Gotzmann in Q (Proposition 3.35), we might hope to prove some squarefree analog of Lemma 5.6 in Q ; then, arguing as in the previous section, we would be able to prove that Gotzmann ideals of Q are lex segments or, perhaps, initial segments in some other monomial order (these alternative orders are discussed

in [Mer06]). Unfortunately such an approach is doomed to fail, as the following examples show.

Example 5.23. The ideal $I = (ab, ac, bd, cd)$ is Gotzmann in the Kruskal-Katona ring $Q = \mathbb{k}[a, b, c, d]/(a^2, b^2, c^2, d^2)$ but is not a lex ideal with respect to any order on the variables. To see that I is Gotzmann, note that $\nabla I_2 = Q_3$ and the shadow of the lex segment $L = \text{span}_{\mathbb{k}}\{ab, ac, ad, bc\}$ is also Q_3 .

Thinking of I and (L) as edge ideals, I corresponds to a four cycle while L corresponds to a three cycle with an extra edge. Since these graphs are not isomorphic, there is no permutation of the variables that will make I a lex ideal of Q .

Also, note that every ideal I' generated by three generators of I is the same up to a permutation of the variables. So, setting $I' = (ab, ac, bd)$ we see that $\nabla I'_2 = Q_3$ again, while the lex segment $L' = \text{span}_{\mathbb{k}}\{ab, ac, ad\}$ has a strictly smaller shadow – the shadow does not contain bcd . Thus, I' is not Gotzmann.

Consider all possible orders m_1, \dots, m_6 of the monomial basis of Q_2 for which every initial segment $\text{span}_{\mathbb{k}}\{m_1, \dots, m_k\}$ is Gotzmann. Though I is Gotzmann, it is not generated by an initial segment under any of these orders as I' is not Gotzmann.

The ideal I above is (up to symmetry) the only monomial Gotzmann ideal of $\mathbb{k}[a, b, c, d]/(a^2, b^2, c^2, d^2)$ which is not lex in some order. Thus we might hope that it is the only such ideal, or at least is the first instance of a one-parameter family of exceptions. This hope is dashed as well as soon as we add a fifth variable.

Example 5.24. The ideal $I = (abc, abd, abe, acd, ace, bcd, bce)$ is Gotzmann in Q but is not lex with respect to any order on the variables. If there were some order of the variables for which I was lex then, like the dimension six lex segment of degree three in a ring with five variables, there should be a variable dividing all but one generator of I .

Throughout this section, all ideals will be monomial ideals of Q . Since we no longer work with the polynomial ring, we can dispense with the notation I^{sf} to indicate that an ideal lives in Q , and will simply write I, J , etc. Many of our arguments are technical, so for ease of notation we work mostly with monomial vector spaces rather than ideals.

We will show that every Gotzmann monomial vector space $V \subseteq Q_d$ can be decomposed as the direct sum of two monomial vector spaces which are Gotzmann in a Kruskal-Katona ring with one fewer generator. This decomposition relates to the operation of compression (see [MP06] or [Mer08]). We begin by setting the necessary notation.

Given a (fixed) variable x_i , let $\mathfrak{n} = (x_1, \dots, \hat{x}_i, \dots, x_n)$ be the maximal ideal in $R = Q/(x_i)$ which is a squarefree ring on $n - 1$ variables. If V is a homogeneous vector space in R_d , we use $\nabla_{\mathfrak{R}} V$ to denote the **upper shadow** $(\mathfrak{n}(V))_{d+1} \subseteq R_{d+1}$ within R .

Definition 5.25 (x_i -decomposition). Let $V \subseteq Q_d$ be a monomial vector space and fix a variable x_i . The monomial basis of V can be partitioned as $A \cup B$ where A contains the monomials divisible by x_i and B contains those not divisible by x_i .

Let V_0 be the monomial vector space spanned by B and let V_1 be the monomial vector space spanned by $\{m \mid x_i m \in A\}$. We write V as the direct sum

$$V = V_0 \oplus x_i V_1$$

which we call the **x_i -decomposition** of V .

We view the monomial vector spaces V_0 and V_1 as subspaces of R_d and R_{d-1} respectively.

Definition 5.26 (x_i -compression). Let $V = V_0 \oplus x_i V_1$ be the x_i -decomposition of the monomial vector space $V \subseteq Q_d$. Let L_0 and L_1 be the squarefree lex-segments in $R = Q/(x_i)$ with the same degrees and dimensions as V_0 and V_1 . The **x_i -compression** of V is the monomial vector space

$$L = L_0 \oplus x_i L_1.$$

We recall the following important fact about compressions from [MP06]:

Proposition 5.27 [MP06]. *If L is the x_i -compression of the monomial vector space $V \subseteq Q_d$, then*

$$\dim_{\mathbb{k}}(\nabla L) \leq \dim_{\mathbb{k}}(\nabla V).$$

Lemma 5.28. *If $V = V_0 \oplus x_i V_1 \subseteq Q_d$ then the x_i -decomposition of ∇V is*

$$\nabla V = \nabla_{\mathfrak{R}} V_0 \oplus x_i(V_0 + \nabla_{\mathfrak{R}} V_1),$$

where $R = Q/(x_i)$.

Proof. Since $x_i^2 = 0$, we have $\nabla(x_i V_1) = \nabla_R(x_i V_1)$. Thus,

$$\begin{aligned} \nabla V &= \nabla(V_0 + x_i V_1) \\ &= \nabla_R V_0 + x_i V_0 + x_i \nabla_R V_1 \\ &= \nabla_R V_0 \oplus x_i(V_0 + \nabla_R V_1). \end{aligned}$$

This sum is direct since the second summand is contained in (x_i) while the monomials in the basis of the second summand are not divisible by x_i . \square

Proposition 5.29. *Let $V \subseteq Q_d$ be a Gotzmann monomial vector space and let $V = V_0 \oplus x_i V_1$ be its x_i -decomposition. Then V_0 is Gotzmann in $R = Q/(x_i)$.*

Proof. Let L be the x_i -compression of V . As V is Gotzmann $\dim_{\mathbb{k}}(\nabla V) \leq \dim_{\mathbb{k}}(\nabla L)$ and so $\dim_{\mathbb{k}}(\nabla V) = \dim_{\mathbb{k}}(\nabla L)$ by Proposition 5.27.

Thus we have

$$\dim_{\mathbb{k}}(\nabla_R V_0) + \dim_{\mathbb{k}}(V_0 + \nabla_R V_1) = \dim_{\mathbb{k}}(\nabla_R L_0) + \dim_{\mathbb{k}}(L_0 + \nabla_R L_1) \quad (5.3.1)$$

from the previous lemma.

As we mentioned in Section 3.1, the shadow of a lex segment is a lex segment. So L_1 and $\nabla_R L_0$ are both lex segments of the same degree, meaning that one contains the other. If $\nabla_R L_1 \subseteq L_0$ then

$$\dim_{\mathbb{k}}(L_0 + \nabla_R L_1) = \dim_{\mathbb{k}}(L_0) = \dim_{\mathbb{k}}(V_0) \leq \dim_{\mathbb{k}}(V_0 + \nabla_R V_1).$$

Similarly, if $L_0 \subseteq \nabla_R L_1$ then

$$\dim_{\mathbb{k}}(L_0 + \nabla_R L_1) = \dim_{\mathbb{k}}(\nabla_R L_1) \leq \dim_{\mathbb{k}}(\nabla_R V_1) \leq \dim_{\mathbb{k}}(V_0 + \nabla_R V_1).$$

In both cases $\dim_{\mathbb{k}}(L_0 + \nabla_R L_1) \leq \dim_{\mathbb{k}}(V_0 + \nabla_R V_1)$. From the equality above we see that $\dim_{\mathbb{k}}(\nabla_R V_0) \leq \dim_{\mathbb{k}}(\nabla_R L_0)$ and hence V_0 is Gotzmann by Proposition 3.34. \square

Lemma 5.30. *Let V be Gotzmann in Q with x_i -decomposition $V = V_0 \oplus x_i V_1$ and let $L = L_0 \oplus x_i L_1$ be its x_i -compression. Then either V_1 is Gotzmann in $R = Q/(x_i)$ or $\nabla_R L_1 \subset L_0$.*

Proof. We know from the previous proposition that V_0 is Gotzmann in R and hence $\dim_{\mathbb{k}}(\nabla_{\mathbb{R}} V_0) = \dim_{\mathbb{k}}(\nabla_{\mathbb{R}} L_0)$. Thus, the equality (5.3.1) gives

$$\dim_{\mathbb{k}}(V_0 + \nabla_{\mathbb{R}} V_1) = \dim_{\mathbb{k}}(L_0 + \nabla_{\mathbb{R}} L_1).$$

If $\nabla_{\mathbb{R}} L_1 \not\subseteq L_0$ then $L_0 \subseteq \nabla_{\mathbb{R}} L_1$ as they are both lex segments in the same degree. Thus

$$\dim_{\mathbb{k}}(\nabla_{\mathbb{R}} V_1) \leq \dim_{\mathbb{k}}(V_0 + \nabla_{\mathbb{R}} V_1) = \dim_{\mathbb{k}}(L_0 + \nabla_{\mathbb{R}} L_1) = \dim_{\mathbb{k}}(\nabla_{\mathbb{R}} L_1)$$

which proves that V_1 is Gotzmann. \square

If $\nabla_{\mathbb{R}} L_1 \subset L_0$, then V_1 need not be Gotzmann. For example,

$$V = \text{span}_{\mathbb{k}}\{abc, abd, acd, bcd, bce, bde, cde\}$$

is Gotzmann in $Q = \mathbb{k}[a, b, c, d, e]/(a^2, \dots, e^2)$, but $V_1 = \text{span}_{\mathbb{k}}\{bc, bd, cd\}$ from the a -decomposition of V is not Gotzmann in $R = Q/(a)$.

Though V_1 is not always Gotzmann, we will see that it is always possible to choose some x_i such that V_1 is Gotzmann.

Lemma 5.31. *Let V be Gotzmann with x_i -decomposition $V = V_0 \oplus x_i V_1$ and compression $L = L_0 \oplus x_i L_1$. Also, let $R = Q/(x_i)$. If $\nabla_{\mathbb{R}} L_1 \subseteq L_0$ then for every monomial $m \in V$ with $x_i | m$ and variable x_j not dividing m , we have $\frac{x_j}{x_i} m \in V$.*

Proof. Applying equality (5.3.1), we have $\dim_{\mathbb{k}}(\nabla_{\mathbb{R}} V_1 + V_0) = \dim_{\mathbb{k}}(\nabla_{\mathbb{R}} L_1 + L_0) = \dim_{\mathbb{k}}(L_0) = \dim_{\mathbb{k}}(V_0)$, i.e., $\nabla_{\mathbb{R}} V_1 \subseteq V_0$. The desired property follows. \square

Theorem 5.32. *Suppose $V \subseteq Q_d$ is a Gotzmann monomial vector space. Then x_i may be chosen so that both summands*

$$V_0 = \text{span}_{\mathbb{k}}\{m \in \text{gens}(V) \mid x_i \text{ does not divide } m\}$$

and

$$V_1 = \text{span}_{\mathbb{k}}\{m/x_i \mid m \in \text{gens}(V), x_i \text{ divides } m\}$$

of the x_i -decomposition of V are Gotzmann in $R = Q/(x_i)$ and $V_0 \subseteq \nabla_{\mathbb{R}} V_1$.

Proof. Suppose that x_i cannot be chosen so that the summands L_1 and L_0 of the x_i -compression satisfy $L_0 \subseteq \nabla_R L_1$. Then Lemma 5.31 applies for all variables x_i . That is, for all monomials $m \in V$, all variables x_i dividing m and all variables x_j not dividing m , we have $\frac{x_j}{x_i}m \in V$.

The only subspaces of Q_d satisfying this property are (0) and Q_d . In either case, we have $L_0 \subseteq \nabla_R L_1$ for any x_i .

Thus, x_i may be chosen such that $L_0 \subseteq \nabla_R L_1$. Then by Lemma 5.30, V_1 and V_0 are Gotzmann in R . Applying equality (5.3.1), we have $\dim_{\mathbb{k}}(V_0 + \nabla_R V_1) = \dim_{\mathbb{k}}(L_0 + \nabla_R L_1) = \dim_{\mathbb{k}}(\nabla_R L_1) = \dim_{\mathbb{k}}(\nabla_R V_1)$, i.e., $V_0 \subseteq \nabla_R V_1$. \square

In fact, the obvious choice of variable works:

Lemma 5.33. *Suppose $V \subseteq Q_d$ is a Gotzmann monomial vector space, and let x_i be such that $\dim_{\mathbb{k}}(V \cap (x_i))$ is maximal. Let $V = V_0 \oplus x_i V_1$ be the x_i -decomposition of V . Then V_0 and V_1 are both Gotzmann in $R = Q/(x_i)$ and $V_0 \subseteq \nabla_R V_1$.*

Proof. Let L_0 and L_1 be the lexifications in R of V_0 and V_1 , respectively.

By Theorem 5.32, there exists a variable x_j such that we may decompose $V = W_0 \oplus x_j W_1$ with both W_0 and W_1 Gotzmann in R and $W_0 \subseteq \nabla_R W_1$.

We have

$$\dim_{\mathbb{k}}(L_0) \leq \dim_{\mathbb{k}}(W_0) \leq \dim_{\mathbb{k}}(\nabla_R W_1) \leq \dim_{\mathbb{k}}(\nabla_R L_1),$$

the first inequality by construction, the second by Theorem 5.32, and the third because $\dim_{\mathbb{k}}(W_1) \leq \dim_{\mathbb{k}}(L_1)$ and both are Gotzmann. By Lemma 5.30, V_1 is Gotzmann. Applying equality (5.3.1) again, we obtain $V_0 \subseteq \nabla_R V_1$. \square

Unfortunately the converse to Theorem 5.32 does not hold in general. For example, let $V = \text{span}_{\mathbb{k}}\{ab, ac, bc\}$ in $Q = \mathbb{k}[a, b, c, d]/(a^2, \dots, d^2)$. Then V is not Gotzmann in Q but, decomposing with respect to a , $V_0 = \text{span}_{\mathbb{k}}\{bc\}$ and $V_1 = \text{span}_{\mathbb{k}}\{b, c\}$ are both Gotzmann in $R = \mathbb{k}[b, c, d]/(b^2, c^2, d^2)$.

We can, however, prove the following partial converse.

Theorem 5.34. *Let V_0 and V_1 be Gotzmann monomial vector spaces in $R = Q/(x_i)$ with $V_0 = \nabla_R V_1$. Then $V = V_0 \oplus (x_i \nabla_R V_1)$ is Gotzmann in Q .*

Proof. Choose any lex order in which x_i comes last, and let $L = L_0 + x_i L_1$ be the x_i -compression of V . We have

$$\begin{aligned}
 \dim_{\mathbb{k}}(\nabla V) &= \dim_{\mathbb{k}}(\nabla_{\mathbb{R}} V_0) + \dim_{\mathbb{k}}(V_0 + \nabla_{\mathbb{R}} V_1) \\
 &= \dim_{\mathbb{k}}(\nabla_{\mathbb{R}} V_0) + \dim_{\mathbb{k}}(V_0) \\
 &= \dim_{\mathbb{k}}(\nabla_{\mathbb{R}} L_0) + \dim_{\mathbb{k}}(L_0) \\
 &= \dim_{\mathbb{k}}(\nabla_{\mathbb{R}} L_0) + \dim_{\mathbb{k}}(L_0 + \nabla_{\mathbb{R}} L_1) \\
 &= \dim_{\mathbb{k}}(\nabla L).
 \end{aligned}$$

Thus, it suffices to show that L is lex.

Indeed, suppose that $u \in L$ and v is a monomial of the same degree which precedes u in the lex order. If both or neither of u, v are divisible by x_i , then clearly $v \in L$. Now suppose that u is divisible by x_i but v is not. Then we may write $u = u'x_i$. By construction, v precedes u' in the (ungraded) lex order. Let $v' = \frac{v}{x_j}$, where x_j is the lex-last variable dividing v . Then v' precedes u' in the lex order as well, so $u' \in L_1$ implies $v' \in L_1$ and in particular $v \in \nabla_{\mathbb{R}} L_1 = L_0$. A similar argument shows that $v \in L$ if v is divisible by x_i but u is not. \square

Example 5.35. Consider the Gotzmann vector space

$$V_1 = \text{span}_{\mathbb{k}}\{ab, bc, cd, ad\}$$

in $R = \mathbb{k}[a, b, c, d]/(a^2, \dots, d^2)$. Let $V_0 = \nabla_{\mathbb{R}} V_1$:

$$V_0 = \text{span}_{\mathbb{k}}\{abc, abd, acd, bcd\}.$$

In $Q = \mathbb{k}[a, b, c, d, e]/(a^2, \dots, e^2)$, the monomial vector space $V = V_0 + eV_1$ is Gotzmann but is not lex with respect to any order of the variables.

5.4 Alexander Duality of Gotzmann Ideals

Recall that the Alexander dual of a simplicial complex Δ on vertices $V = \{x_1, \dots, x_n\}$ is

$$\Delta^{\vee} = \{F \subseteq V \mid V \setminus F \notin \Delta\}.$$

Through the Stanley-Reisner correspondence, we define the Alexander dual of a squarefree monomial ideal $I \subseteq Q$ to be $I^\vee = I_{\Delta^\vee} \subseteq Q$.

The d -th homogeneous component of I^\vee is determined by the $(n-d)$ -th component of I ;

$$(I^\vee)_d = \text{span}_{\mathbb{k}}\{m \in \text{gens } Q_d \mid \frac{x_1 \cdots x_n}{m} \notin I_{n-d}\}.$$

So, it makes sense to define the Alexander dual of a monomial vector space $V \subseteq Q_d$ similarly.

Definition 5.36. Let Q be the Kruskal-Katona ring in n variables. The **Alexander dual** of a monomial vector space $V \subseteq Q_d$ is the subspace $V^\vee \subseteq Q_{n-d}$ spanned by the monomials $\{m \in \text{gens } Q_{n-d} \mid \frac{x_1 \cdots x_n}{m} \notin V\}$.

The Alexander dual of a monomial ideal $I \subseteq Q$ can now be written as

$$I^\vee = \bigoplus_{d=0}^n (I_d)^\vee.$$

Alexander duality has many nice properties. For example, duality turns generators into associated primes and the duals of lex or Borel-fixed ideals are always lex or Borel-fixed, respectively. As lex ideals are Gotzmann, it is natural to ask if the Alexander duals of Gotzmann ideals are also Gotzmann. We will show in Theorem 5.38 that this is not the case; lex ideals are the only Gotzmann ideals with Gotzmann duals.

Just like we defined upper shadows $\nabla_R V$ of homogeneous vector spaces $V \subseteq R_d$ with $R = Q/(x_i)$, we can define the **lower shadow** $\Delta_R V$ of a homogeneous vector space $V \subseteq R_d$ to be

$$\begin{aligned} \Delta_R V &= \text{span}_{\mathbb{k}}\{f \in R_{d-1} \mid \forall j \in \{1, \dots, \hat{i}, \dots, n\}, x_j f \in V\} \\ &= (V : \mathfrak{n}_1), \end{aligned}$$

where $\mathfrak{n} = (x_1, \dots, \hat{x}_i, \dots, x_n)$ is the homogeneous maximal ideal of R .

Theorem 5.37. *Let $V \subseteq Q_d$ be a monomial vector space whose Alexander dual is Gotzmann. Then there exists a variable x_i such that both summands V_0 and V_1 of the x_i -decomposition have Gotzmann Alexander duals in $R = Q/(x_i)$, and also $\Delta_R V_0 = (V_0 : \mathfrak{n}_1)$ is contained in V_1 .*

Proof. Let $W = V^\vee$. Then Theorem 5.32 applies to W , so we may choose x_i such that W_0 and W_1 are Gotzmann in R and $W_0 \subseteq \nabla_R W_1$.

The Alexander duals of V_0 and V_1 from the x_i -decomposition of V are

$$V_0 = (W_1)^\vee \quad \text{and} \quad V_1 = (W_0)^\vee,$$

proving the first part of the theorem.

Finally, suppose that $m \in \Delta_R V_0$. We will show that $m \in V_1$. By construction, $mx_j \in V_0$ for all $x_j \neq x_i$ and not dividing m , so $\frac{x_1 \cdots x_n}{mx_j} \notin W_1$ for any such x_j . Therefore $\frac{x_1 \cdots x_n}{m} \notin \nabla_R W_1$. Since $W_0 \subseteq \nabla_R W_1$, we have $\frac{x_1 \cdots x_n}{m} \notin W_0$. Thus $m \in V_1$, as desired. \square

Thus, any recursive enumeration of ideals with Gotzmann duals should look similar to any recursive enumeration of Gotzmann ideals. However, they will not be identical. In fact, monomial vector spaces which are simultaneously Gotzmann and have Gotzmann duals are quite rare, as the next theorem shows.

Theorem 5.38. *If $V \subset Q_d$ is Gotzmann and its Alexander dual V^\vee is Gotzmann as well, then V is a lex segment for some order of the variables of Q .*

Proof. Suppose that V not a lex segment in any order of the variables. Then there exists a counterexample $V \subset Q_d$ where $Q = \mathbb{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ with n minimal.

Let x_i be chosen to maximize $\dim_{\mathbb{k}}(V \cap (x_i))$. This choice of x_i minimizes the dimension of $V + (x_i)_d$. One can check that $V^\vee \cap (x_i) = (V + (x_i)_d)^\vee$, and thus $\dim_{\mathbb{k}}(V^\vee \cap (x_i))$ is maximal as well, and Lemma 5.33 applies to both V and V^\vee .

Thus V_0 and V_1 are both Gotzmann and their duals are Gotzmann, so, by the minimality of n , both are lex in $R = Q/(x_i)$. Since V is not lex, we have $V_0 \neq 0$ and $V_1 \neq R_{d-1}$. Since $V_0 \neq 0$, we have $\mathbf{m}_{n-d-1}V = Q_{n-1}$. Thus the lexification of V (in any order where x_i comes first) must contain at least one monomial not divisible by x_i . Similarly, the lexification of V^\vee must contain at least one monomial not divisible

by x_i . Thus, if L and L^\vee are the lexifications of V and V^\vee , respectively, we have

$$\begin{aligned} \dim_{\mathbb{k}}(L) + \dim_{\mathbb{k}}(L^\vee) &\geq \dim_{\mathbb{k}}(R_{d-1}) + 1 + \dim_{\mathbb{k}}(R_{n-d-1}) + 1 \\ &\geq \dim_{\mathbb{k}}(R_{d-1}) + \dim_{\mathbb{k}}(R_d) \\ &= \dim_{\mathbb{k}}(Q_d). \end{aligned}$$

On the other hand, $\dim_{\mathbb{k}}(L) + \dim_{\mathbb{k}}(L^\vee) = \dim_{\mathbb{k}}(V) + \dim_{\mathbb{k}}(V^\vee) = \dim_{\mathbb{k}}(Q_d)$. Thus, such a minimal counterexample cannot exist. \square

Note that Theorem 5.38 is not a theorem about ideals. If I is a Gotzmann monomial ideal with a Gotzmann dual, then Theorem 5.38 guarantees that every homogeneous component I_d is lex in some order, but does not guarantee a consistent order.

Example 5.39. The ideal $I \subset \mathbb{k}[a, \dots, e]/(a^2, \dots, e^2)$ given by

$$I = (bc, abd, abe, acd, ace, ade)$$

is Gotzmann and its Alexander dual I^\vee is also Gotzmann, but I is not a lex ideal in any order. The component I_k is lex with respect to the order $a > b > c > d > e$ for $k \neq 2$, and with respect to the order $b > c > a > d > e$ for $k < 3$, but no lex order works in both degrees two and three.

Conclusion

There are a number of interesting lines of research that are suggested by the results on Gotzmann ideals contained in this thesis.

Perhaps the most obvious question one might ask is how to characterize all Gotzmann monomial ideals of the polynomial ring S . Our characterization of Gotzmann squarefree monomial ideals relied on the interplay between the polynomial ring and the Kruskal-Katona ring. Much like the Kruskal-Katona ring, the Clements-Lindström ring,

$$C = \mathbb{k}[x_1, \dots, x_n]/(x_1^{a_1}, \dots, x_n^{a_n}),$$

satisfies a version of Macaulay's theorem in which lex ideals model Hilbert functions. If we want to use techniques similar to those in Section 5.1 to classify all Gotzmann monomial ideals, then we would have to leverage the correspondence between monomial ideals of S and monomial ideals in C .

For any fixed monomial ideal $I \subseteq S$ we can choose a Clements-Lindström ring with each a_i sufficiently large that each minimal generator of I appears as a minimal generator of its image IC . On the other hand, if we choose a_i to be too large then there is no information to be gained by comparing I to IC . This is because, for each $d < \min\{a_1, \dots, a_n\}$, we have $S_d \cong C_d$.

Another interesting problem to pose is to generalize the connection between Macaulay's theorem for the polynomial ring and the Kruskal-Katona theorem for the exterior algebra. The polynomial ring and the exterior algebra are both quadratic quotients of the tensor algebra: if V is a vector space with basis $\{e_1, \dots, e_n\}$ then

$$\begin{aligned} S &\cong T(V)/(e_i \otimes e_j - e_j \otimes e_i \mid 1 \leq i < j \leq n), \text{ and} \\ E &= T(V)/(e_i \otimes e_j + e_j \otimes e_i \mid 1 \leq i < j \leq n). \end{aligned}$$

These two algebras are dual in the sense that their generating relations are orthogonal. If we have two dual quadratic quotients of the tensor algebra, R_1 and R_2 , and if the Hilbert functions of two sided ideals in R_1 are achieved by lexicographic ideals, then

are Hilbert functions of ideals in R_2 achieved by lexicographic ideals of R_2 ? One may need to assume that the initial ideals of the ideals defining R_1 and R_2 both contain $e_j \otimes e_i$ with $i < j$ so that R_1 and R_2 are “nearly commutative”.

One final and broader project, first posed by Mermin and Peeva in a variety of forms [MP07], is to understand which algebras satisfy some version of Macaulay’s theorem. We have already seen that the polynomial ring, Kruskal-Katona ring, exterior algebra and Clements-Lindström ring satisfy Macaulay’s theorem in the sense that all Hilbert functions of ideals in these rings are modelled by lexicographic ideal. Gasharov, Horwitz and Peeva [GHP08] showed that this also holds true for the rational normal curve, $T = \mathbb{k}[y, xy, x^2y, \dots, x^{n-1}y]$.

One could also look for rings satisfying weaker versions of Macaulay’s theorem. For instance, in the paper “Lexlike Sequences”, Mermin models Hilbert functions using ideals whose components come from initial intervals in orders other than the lexicographic order [Mer06].

It would be interesting to have examples of rings where Hilbert functions cannot be modelled in a componentwise fashion. To be more precise, take a standard graded algebra R and two Hilbert functions $\text{HF}_{R/I}$ and $\text{HF}_{R/I'}$ with $\text{HF}_{R/I}(d) = \text{HF}_{R/I'}(d)$ for some d . Is it possible that for every ideal J with $\text{HF}_{R/J} = \text{HF}_{R/I}$ and every ideal J' with $\text{HF}_{R/J'} = \text{HF}_{R/I'}$ that we have $J_d \neq J'_d$? When Macaulay’s theorem holds, the lexifications of I and J always have the same homogeneous component in degree d . So, it would be interesting to find algebras where this component cannot be shared.

Inspiration for building algebras which satisfy Macaulay’s theorem can also be found in the combinatorial literature (see [Eng97]). There are many known posets, called Macaulay posets, which abstract the poset of monomials ordered by divisibility and which satisfy a combinatorial variant of Macaulay’s theorem. From these known Macaulay posets, it would be interesting to know which can actually be realized as the monomial bases of standard graded algebras satisfying Macaulay’s theorem.

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Appendix

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