VIII.—On the Graphical Treatment of the Inertia of the Connecting Rod—By Prof. J. G. MacGregor, D. Sc., Dalhousie College, Halifax, N. S.

(Received June 15th, 1892).

In slow-speed steam engines, no great error is introduced in calculating the effort of the connecting rod on the crank-pin, on the assumption that the connecting rod is without mass. In high-speed engines, however, a ccnsiderable error is thus introduced; and it is therefore desirable to have a method of determining the actual effort. In this paper a graphical method of making the determination is described.

The effort transmitted by the connecting rod is affected by the weight of the rod as well as by its inertia, and also by the friction of the surfaces in contact. The effect of the weight of the rod and of friction, however, may be found by well known methods. I shall therefore assume the rod to be weightless (though not massless), and the surfaces in contact to be smooth.

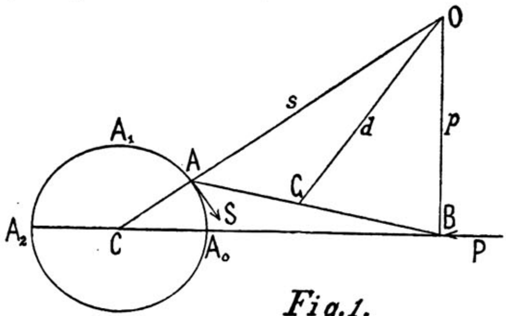

Fig. 1.

Let CA be the centre line of the crank of the ordinary steam engine, and AB that of the connecting rod, BC being thus the line of the piston's motion. The end B of AB therefore moves to and fro in the line BC, while the crank-pin A moves in the circle $A_0 A_1 A_2$. The motion of these points is regulated by the flywheel. If the engine have a flywheel of sufficiently great moment of inertia, A will move in its circle with practically uniform speed. If the moment of inertia be not sufficiently great for this purpose, fluctuations of speed will occur, which, apart from the solution of the present problem, may be determined approximately. We may therefore regard the velocity of the crank-pin A as known for all positions of the crank.

The motion of the connecting rod is thus one of the data of the problem. As is well known, it may be regarded as rotating instantaneously about a fixed axis whose position is the intersection O, of the line CA produced, with a line through B perpendicular to BC. The distances of O from A and B for any position of the crank may be found by drawing to scale a diagram similar to Fig. 1 and measuring the lengths of the lines AO and BO. We shall use the symbols s and p to indicate these distances respectively.

The forces acting on the connecting rod are (its weight being neglected) the force exerted on the end B by the crosshead of the piston rod, and the resistance of the crank-pin acting on the end A, which is of course equal and opposite to the force exerted by the rod on the crank-pin. As we are neglecting friction, these forces may be considered as acting through the points B and A, the centres of the pins. They may be resolved into components in and perpendicular to the lines of motion of B and A respectively. Let P and S be the components in the lines of motion. The indicator diagram, the area of the piston, and the mass of the reciprocating parts, being given, P may readily be determined for all positions of the crank. S is the force which it is desired to determine.

The simplest relation between these forces and the kinetic changes which the connecting rod is given as undergoing, is that expressed in the equation of energy. Let dc be the length of arc described by the crank-pin A, during any small displacement of the rod. Then, as the rod is instantaneously rotating about O, $(p/s)dc$ will be the distance traversed in the same time by B. Hence the work done by the forces acting on the rod is

$$P(p/s)dc - Sdc\,;$$

for the component forces perpendicular to the lines of motion of A and B do no work. The work done must be equal to the increment of the kinetic energy of the rod. As the rod is rotating about the point O, instantaneously fixed, its kinetic energy is $\frac{1}{2}\omega^2 mk^2$, where ω is its angular velocity about O, m its mass, and k its radius of gyration about an axis through O perpendicular to the plane of motion. Hence the equation of energy is:

$$\left(P\frac{p}{s} - S\right) dc = d\left(\tfrac{1}{2}\,\omega^2 mk^2\right),$$

or

$$P\frac{p}{s} - S = \frac{d}{dc}\left(\tfrac{1}{2}\,\omega^2 mk^2\right).$$

In this equation P, p and s are known as pointed out above, for all crank positions, *i. e.*, for all values of c. The mass m is known. The angular velocity ω may easily be found; for it is equal to the linear velocity of the crank-pin divided by s, the distance of the pin from O; and the angular velocity of the crank being given, together with its length, the linear velocity of the crank-pin may be obtained at once. The radius of gyration, k, about O is equal to the square root of the sum of the squares of the radius of gyration, h, about a parallel axis through G, the centre of mass of the rod, which is constant and may be calculated, the form and dimensions of the rod being given, and of the distance, d, of G from O, which may be found for all crank positions by measuring the length of the line GO in diagrams similar to Fig. 1. All the variable quantities of the above equation except S may thus be expressed as functions of c. It is therefore sufficient for the determination of S.

Usually, however, the problem under consideration is presented in this way:—By what amount is the component, normal to the crank, of the effort on the crank-pin too great, when calculated on the assumption that the connecting rod has no inertia? Or in other words, what pull normal to the crank must the crank-pin exert on the rod, in order that the rod may move in the given way?

The equation of energy modified so as to be a direct answer to this question takes a somewhat simpler form. For if S' be the component normal to the crank of the effort on the crank-pin, calculated on the assumption referred to, we have, putting $m=0$,

$$P\,(p/s) - S' = 0.$$

Hence the amount by which the required force is too great when calculated in this way, is given by the equation :

$$S' - S = \frac{d}{dc}(\tfrac{1}{2}\,\omega^2 m k^2).$$

This expression lends itself readily to graphical treatment. For this purpose we find ω for various crank positions, by drawing diagrams similar to Fig. 1 for as many positions of the crank as may be desired, measuring the lengths s in these positions and dividing the values of the velocity, V, of the pin for these positions by the corresponding values of s. We then plot a curve with distances traversed by the crank-pin from some initial position such as A_0, as abscissæ, and the corresponding values of ω as ordinates, thus obtaining a curve which gives us the values of ω for all crank positions. Then selecting points on this curve, whose ordinates have simple values, such as can be raised to the square "in the head," or by reference to a table of squares, we obtain a series of values of ω^2 for the selected crank positions; and a second curve giving the variation of ω^2 with the crank position may be plotted. Or as $\omega^2 = \omega\,V/s$, we may obtain the ω^2 curve from the ω curve by the construction by which we obtain the curve vv from the curve VV below, V and s taking the place of the k and s used in that construction.

A similar curve for k^2 must next be obtained. As $k^2 = h^2 + d^2$, its values for different positions of the crank may be obtained by finding the values of d from the diagrams similar to Fig. 1, already drawn, and adding their squares to the square of the constant h. For this purpose draw a right-angled triangle abc (a diagram is not necessary), whose sides ab and bc, containing the right angle, represent on any convenient scale the quantities h and d. Then the hypothenuse ac will represent to the same scale $\sqrt{h^2 + d^2}$ or k. From a point e in ac at a distance from a or c, say a, of one scale division, draw a line ef, in any direction, equal to ac; join af, and through c draw a line parallel to ef and meeting af produced in g. The line cg will represent $h^2 + d^2$ or k^2 to the same scale as ab and bc represent h and d.

The ω^2 curve and the k^2 curve must now be combined so as to give an $\omega^2 k^2$ curve. This may readily be done, either by the ordinary process of graphical multiplication or by selecting points in either curve which have ordinates of simple value and multiplying by them the corresponding ordinates of the other. The corresponding ordinates of the $\omega^2 k^2$ curve are thus obtained, and the curve may then be plotted. The quantity $\frac{1}{2} m$ being a constant, this curve, read to the proper scale, will also be a curve giving the values of $\frac{1}{2}m\omega^2k^2$ for all crank positions; and the tangent of the inclination to the axis of crank positions, of the tangent to this curve at any point, is the value of $S' - S$ for the corresponding crank position.

This process, however, is laborious, and the above equation may be thrown into a form which gives a much simpler graphical treatment. For this purpose the two variable quantities, ω and k, are combined in one, the product, ωk, being obviously the velocity of any point rigidly connected with the rod and at a distance from O equal to k. If we call this velocity v we have

$$S' - S = \frac{d}{dc}\left(\tfrac{1}{2}mv^2\right),$$
$$= mv\frac{dv}{dc}.$$

In this expression there is but one quantity, v, varying with c. It leads, therefore, to a very simple graphical treatment.

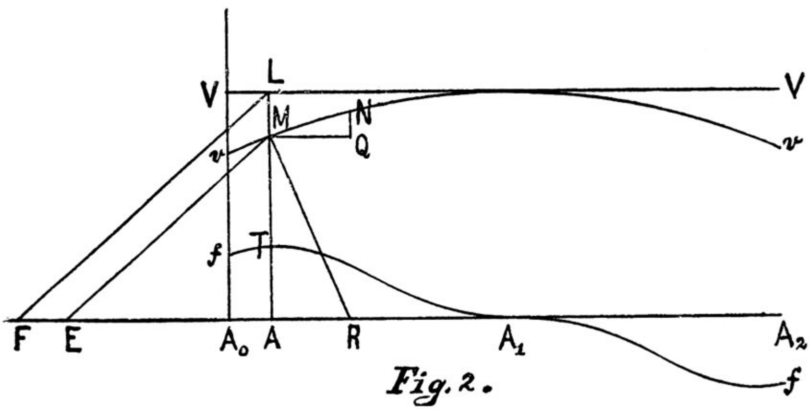

Fig. 2.

Let A_0A_2 (Fig. 2) be the straight line or axis on which distances (c) traversed by the crank-pin are represented, A_0A, for example, representing, on some convenient scale of distances, the length of the arc A_0A in Fig. 1. Let the ordinate AL represent, on some convenient scale of velocities, the velocity of the crank-pin in the crank position represented by A (Figs. 1 and 2.) A smooth curve, VV, drawn through L and a sufficient number of points similarly determined, gives the velocity of the crank-pin in all crank positions. As seen above, if the moment of inertia of the fly-wheel be sufficiently great, it will be practically a straight line; if not, it will be a known curve.

The value of the velocity, V, of the crank-pin being known for all positions of the crank, the velocity, v, of the above formula, may be obtained at once; for, as the rod is instantaneously rotating about O, we have

$$v = \omega k = \frac{V}{s}k.$$

To find the value of v corresponding to the crank position A, cut off from AA_0 produced (Fig. 2), AE and AF representing on any scale k and s respectively (obtained as on p. 196 and from Fig. 1); join FL, and through E draw a line parallel to FL and intersecting AL in M. Then we have

$$AM = AL\frac{AE}{AF} = V\frac{k}{s}.$$

Hence AM represents v to the same scale as AL represents V; and M is therefore a point on the curve which gives the values of v for all crank positions. Other points may be similarly determined, and a smooth curve, $v\,v$, may then be drawn through them. Its form is roughly indicated in Fig. 2, which is not, however, drawn to scale. It will obviously touch the VV curve at the crank position A_1, (Figs 1 and 2), the rod in that position having a motion of translation only (O being at an infinite distance). In all other crank positions between A_0 and A_2 its ordinates will be less than the ordinates of the VV curve. Obviously the lines FL and EM need not be actually drawn in the above construction; it is sufficient to mark their end points.

From this curve we may find the values of vdv/dc for all crank positions. Thus for the position A:—Let N be a point on the curve $v\,v$, near M. From M and N draw MQ and NQ parallel to the axes of crank positions and of velocities respectively. Then MQ and NQ are the values of dc and dv for a small displacement of the rod from the crank position A; and when the displacement is made indefinitely small, NQ/MQ becomes ultimately the dv/dc of the above formula, and MN becomes a straight line. From M draw MR a normal to the curve at M. Then, since MN, NQ and QM are perpendicular to MR, RA and AM respectively, the triangles MNQ and MRA are similar. Hence

$$AR = AM\frac{NQ}{MQ} = v\,\frac{dv}{dc}.$$

From AL cut off a part AT equal to AR. Then T is a point on a curve whose ordinates represent on some scale to be determined, the values of vdv/dc for all crank positions. Other points may be similarly determined, and a smooth curve drawn through them. Its form is roughly indicated in Fig. 2 by the curve ff. The dv and dc at the crank position A, being both increments, and vdv/dc being therefore positive, AT is drawn upwards. Between A_1 and A_2, dv is a decrement; vdv/dc is thus negative; and hence the ordinates of ff are there drawn downwards. Obviously the lines MQ and NQ do not require to be drawn in making the construction. They are introduced above for purposes of proof. Nor does MR require to be drawn. It is necessary only to mark its end point R. If the drawing be made on co-ordinate paper, the line MA will be a line on the paper.

The mass m being a constant, $S' - S$ is proportional to vdv/dc, and its values in different crank positions will therefore be represented by the same straight lines which represent the values of vdv/dc. Hence the curve ff gives not only the values of vdv/dc, but also, if read to the proper scale, the values of $S' - S$ for all crank positions. If this scale be determined therefore the problem is solved.

There are four steps in the above method of obtaining values of $S' - S$. First the curve VV is drawn from data of the problem. In order to draw it, scales of velocity and of distance or length must first be selected; which means that we must select convenient units of velocity and length for the purposes of our geometrical constructions. Let us suppose we select 9 inches (equal to $\frac{3}{4}$ ft.) to be represented by one division of the distance scale, and 10 feet per second to be represented by one division of the velocity scale.

Secondly, the curve, *vv*, is obtained from the curve, *VV;* and as seen above the scales of the two curves are the same.

Thirdly, the curve, *ff*, is obtained from *vv* by applying a geometrical construction, which is the equivalent of the algebraic operation indicated by the expression vdv/dc. Hence the scale of the ordinates of the curve *ff* must be related to the velocity and distance scales in the same way as the unit of vdv/dc is related to the units of v and c. Now vdv/dc has the dimensions of an acceleration; and the magnitude of a unit of acceleration is always equal to the quotient of the square of the magnitude of the unit of velocity by the magnitude of the unit of length, provided these units are units of some one derived system, and their magnitudes are expressed in terms of the same units of length and time. Hence our unit of acceleration must be $10^2 \div \frac{3}{4} = \frac{400}{3}$ feet-per-second per second. The scale of *ff*, therefore, considered as giving values of vdv/dc for the various crank positions is $\frac{400}{3}$ ft.-sec. units to a division.

Finally, in employing *ff* as a curve of force, we apply the equation:

$$S' - S = mv\frac{dv}{dc},$$

without any further geometrical construction. It is obvious from this equation that if vdv/dc have the value $\frac{400}{3}$, $S' - S$ will have the value $\frac{400\ m}{3}$. Hence the scale of the ordinates of *ff*, considered as a force curve, will be $\frac{400\ m}{3}$, the value of m varying with the unit of mass in terms of which the mass of the rod is expressed. Also the above equation holds only provided all quantities in it be expressed in terms of derived units. Hence the unit of force in terms of which the scale of the force curve will be expressed will be the unit of force of the system derived from the unit of mass selected and the units of length and time employed above. Thus, the scale of the ordinates of ff, considered as an acceleration curve, having been found, with the velocity and distance scales originally selected, to be $\frac{400}{3}$ ft-per-sec per sec. to a division, if the mass of the connecting rod be 1 cwt. (British) and if we select the pound as unit of mass, the force scale will be $112 \times \frac{400}{3}$ poundals to a division. If we wish the force scale to be expressed in pounds-weight, we must express m in terms of the unit of mass of the gravitational system, viz., 32 lbs., in which case the force scale is $\frac{112}{32} \times \frac{400}{3}$ pounds-weight to a division.

The $S' - S$ curve having thus been obtained and its scale determined, it is easy to obtain the value of S when P is given, by a graphical method. For, as seen above,

$$P\,(p/s) = S',$$

and P being given for all crank positions, and p and s being found from the diagrams similar to Fig. 1, the values of $P(p/s)$ may be found by the ordinary graphical methods of multiplication and division, and a curve of $P(p/s)$ or S' plotted in the same way as the above curves. Then the excess of the length of any ordinate of the S' curve over that of the corresponding ordinate of the $S' - S$ curve, proper regard being had to sign, will be the length of the corresponding ordinate of the S curve.

Sometimes instead of requiring to find the amount by which the effort on the crank-pin, S, is diminished by the inertia of the connecting rod, the " nett forward piston pressure," P, being given, we have to find the amount by which this latter force must be increased in order that the effort on the crank-pin may not be diminished by the inertia of the rod. To make this determination additional calculation is necessary. For we have as before

$$P\frac{p}{s} - S = mv\frac{dv}{dc};$$

and therefore if P' be the value of P, calculated on the assumption of a massless connecting rod, requisite to produce a given S,

$$P'(p/s) - S = o.$$

Hence

$$P - P' = \frac{s}{p} mv\frac{dv}{dc} = \frac{s}{p}(S' - S.)$$

If, therefore, the ordinates of ff be increased in the ratio of s/p, we obtain a curve giving the values of $P - P'$ to the same scale as that on which ff gives the values of $S' - S$. We may obviously obtain the $P - P'$ curve from the $S' - S$ curve, by a process similar to that by which we obtained the curve $v\,v$ from the curve VV above.

The portion of the literature of graphical methods which is accessible to me is very small; and I am not at all sure that the above method is new. I have been led to submit it to the Institute by finding, in so recent and so authoritative a work as that of Prof. A. B. W. Kennedy, on the Mechanics of Machinery (1886), a graphical method of solving the above problem which appears to me to be erroneous. Prof. Kennedy obtains the curve $v\,v$ in the way shewn above, and then, after pointing out that it is not what is known as a "velocity curve to a distance base," he proceeds to treat it in the way in which a curve to that kind would be treated in order to obtain from it a force curve, though without justifying this course. He thus obtains a graphical construction for solving the above problem, which is inconsistent with that given above, and which seems to me to give inaccurate results.