A MINIMAL MORSE RESOLUTION OF PATH IDEALS OF LINES OF PROJECTIVE DIMENSION 2

by

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Abstract

We study the connection between a special class of monomial ideals and CW complexes. Let I denote the ideal $I_t(L_{2t+1})$, i.e., a path ideal of line graph of projective dimension 2. We study cellular resolutions and discrete Morse theory as a tool to find a CW complex that supports the minimal free resolution of I. As a result, we have constructed an explicit Morse matching that induces a CW complex supporting the minimal free resolution of I. We also used the results from Bayer and Sturmfels [6] to prove that the minimal free resolution of I is supported on a solid $(t+2)$-gon.

List of Symbols Used

The following list describes the symbols that are used in this thesis.

β_i	Betti number
$\beta_{i,j}$	Graded Betti number
$\beta_{i,m}$	Multigraded Betti number
$\mathrm{gp}(a, b)$	Gradient path from a to b
$\mathbb{N}$	The set of non-negative integers
$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers
$\mathbf{F}$	Free resolution of a R-module
$\mathbf{m}$	The maximal ideal $(x_1, \ldots, x_n)$ in S
$\mathcal{C}$	Chain complex
$\mathscr{T}$	Topology on a set
k	Arbitrary field
R	The quotien ring S/I, where I is a graded ideal in S
S	The polynomial ring $k[x_1, \ldots, x_n]$
X^n	The collection of n-dimensional faces of the CW complex X
X_n	The n-skeleton of the CW complex X

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Chapter 1

Introduction

Monomial ideals and CW (or cell) complexes are common objects in algebra and topology, respectively. In this research, we study the connection between free resolutions of monomial ideals and CW complexes. A free resolution of a monomial ideal I is a sequence of free modules connected by module homomorphisms, i.e., differentials, which describes relations among elements in I. A chain complex of a CW complex has exactly the same structure with different terminologies. Thus, one can observe some correspondence between monomial ideals and CW complexes. For example, the projective dimension and generators of a monomial ideal correspond to the dimension and vertices of a CW complex, respectively. Moreover, we can construct a free resolution of a monomial ideal I from the chain complex of some CW complex X, through a process called homogenization. In this case, we say that I has a free resolution supported on X, or that I has a cellular resolution. We will discuss this in detail in Chapter 4.

The concept of describing monomial ideals using topological objects was first introduced by Taylor [25] in the case of simplicial complexes, which are a special class of CW complexes. Taylor concluded that every monomial ideal has a simplicial resolution, i.e., a free resolution supported on a simplicial complex. In particular, any monomial ideal with r generators has a free resolution supported on an $(r-1)$-simplex, which we call the Taylor complex, and this free resolution is called the Taylor resolution [25]. Another famous simplicial resolution is the Lyubeznik resolution [21], which is "smaller" than the Taylor resolution as the chosen simplicial complex is a subcomplex of the Taylor complex. However, both Taylor and Lyubeznik resolutions are not minimal in general. The minimal free resolution is important, because it is unique (up to isomorphism) and therefore reveals the most amount of information of a monomial ideal, compared to other free resolutions. Thus, people started looking for simplicial complexes that support the minimal free resolution of a monomial ideal after Taylor's study.

In 1998, Bayer, Peeva, and Sturmfels [5] provided criteria for a simplicial resolution to be minimal. Based on this result, it is not hard to conclude that not every monomial ideal has a minimal simplicial resolution, e.g., path ideals of line graphs and cycles of projective dimension 2 [27]. Later in the same year, Bayer and Sturmfels [6] extended the study from simplicial complexes to (regular) CW complexes. In particular, they introduced the concept of cellular resolutions and generalized the criteria in [5] to CW complexes. This allows us to explore minimal cellular resolutions of monomial ideals, especially those that do not have minimal simplicial resolution. Although CW complexes are much more general than simplicial complexes, they are not enough to describe all monomial ideals, because we know from Velasco [26] that some monomial ideals do not have a minimal free resolution supported even on a CW complex. Based on the study of Bayer and Sturmfels [6], people have been trying to find out what classes of monomial ideals have a minimal cellular resolution. Below are some results from relevant studies.

In 2002, Batzies and Welker [4] constructed minimal cellular resolutions for two special classes of monomial ideals (i.e., generic and shellable ideals) based on Chari's reformulation [9] of Forman's discrete Morse theory [14]. In 2017, Faridi and Hersey [13] found that every monomial ideal with projective dimension 1 has a minimal simplicial resolution. In 2020, Fernández-Ramos and Gimenez [1] also used discrete Morse theory to develop a pruning algorithm that produces a cellular resolution, which is not always minimal, from the Taylor resolution. In the same year, Barile and Macchia [3] presented a construction of minimal free resolutions for edge ideals of forests using discrete Morse theory. Later on, we [27] attempted to extend the result from [13] to projective dimension 2, but instead, we concluded that path ideals of line graphs and cycles with projective dimension 2 do not have minimal simplicial resolution.

This research is a continuation of [27]. We want to know whether these latter two classes of monomial ideals have minimal cellular resolutions. In this thesis, we focus on path ideals of line graphs of projective dimension 2 (i.e., $I_t(L_{2t+1})$). We will also implement discrete Morse theory, which provides us with a process to obtain a cellular resolution by reducing the Taylor complex using homogeneous acyclic matchings. We will then determine whether the obtained cellular resolution is minimal.

We start with preliminaries in Chapter 2. In this chapter, we define graded modules and (minimal) free resolutions. We will also introduce some invariants of monomial ideals (e.g., projective dimension and graded Betti numbers). Chapter 3 focuses on basics of Algebraic topology, which includes the construction of CW complexes and chain complexes of regular CW complexes. We will also provide examples for better illustration. In Chapter 4, we connect CW chain complexes with free resolutions of monomial ideals by constructing cellular resolutions. Then we build a homogeneous acyclic matching and conclude that this matching produces the minimal cellular resolution of $I_t(L_{2t+1})$. Based on the critical cells, which we obtained from our matching, we believe that the minimal free resolution of $I_t(L_{2t+1})$ is supported on a solid $(t+2)$-gon, which we verified in the final section of Chapter 4 using the criterion from Bayer and Sturmfels [6]. In summary, we have found a homogeneous acyclic matching that induces the minimal cellular resolution, which is supported on a solid polygon, of path ideals of line graphs with projective dimension 2.

Chapter 2

Graded Objects

2.1 Graded polynomial rings and modules

Definition 2.1.1 ([2], p.102)**.** A ($\mathbb{N}$-) *graded ring* is a ring A that has a direct sum decomposition $A = \oplus_{i=0}^{\infty} A_i$ where $(A_i)_{i=0}^{\infty}$ is a family of additive subgroups of A such that $A_i A_j \subset A_{i+j}$ for all $i, j \geq 0$. Let A be a graded ring and M an A-module. Then M is *graded* if it has a direct sum decomposition $M = \oplus_{i=0}^{\infty} M_i$ where $(M_i)_{i=0}^{\infty}$ is a family of subgroups of M such that $A_i M_j \subset M_{i+j}$ for all $i, j \geq 0$. An element $x \in M$ is *homogeneous* of *degree* i if $x \in M_i$ for some $i \geq 0$. Every element $y \in M$ can be uniquely written as $\sum_i y_i$ where all but finitely many y_i's are 0. The non-zero components in the sum are the *homogeneous components* of y.

For any polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k, we introduce a *standard grading* from Peeva ([24], p. 1-2): For any monomial $m = x_1^{c_1} \cdots x_n^{c_n}$, we define $\deg(m) = c_1 + \cdots + c_n$. We have $S = \oplus_{i=0}^{\infty} S_i$ where each S_i is the k-vector space spanned by all monomials of degree i. For any polynomial $f \in S$, we say that f is homogeneous of degree i if $f \in S_i$ for some $i \geq 0$. Also, f can be uniquely written as a finite sum $\sum_i f_i$, and the non-zero components in the sum are the *homogeneous components* of f (of degree i).

The polynomial ring $S = k[x_1, \ldots, x_n]$ is also $\mathbb{N}^n$-graded by

$$\text{mdeg}(x_i) = e_i$$

where the symbol mdeg represents multidegree and e_i is the i-th standard vector in $\mathbb{N}^n$ ([24], p. 101). We have $S = \oplus_{m \text{ is a monomial}} S_m$ and $S_m S_{m'} = S_{mm'}$ for all monomials m, m', where each S_m is a k-vector space spanned by the monomial m. In this case, we say that S is *multigraded*, instead of $\mathbb{N}^n$-graded.

Similarly, an S-module M is *multigraded* if it has a direct sum decomposition $M = \oplus_{m \text{ is a monomial}} M_m$ where $(M_m)_{m \text{ is a monomial}}$ is a family of subgroups of M such that $S_m T_{m'} \subset M_{mm'}$ for all monomials m, m'.

Example 2.1.2. Let $A = k[x, y, z]$, $f = xyz + 3x^2y$, and $g = yz + x^3 + 2x^2y - 5z^4$. Then f is homogeneous of degree 3, but g is not homogeneous. The homogeneous components of g are yz, $x^3 + 2x^2y$, and $-5z^4$. Moreover, $\mathrm{mdeg}(xyz) = (1, 1, 1)$ and $\mathrm{mdeg}(x^2y) = (2, 1, 0)$.

Definition 2.1.3 ([24], p.6)**.** Let A be a ring and M be a finitely generated graded A-module. For any $p \in \mathbb{N}$, we denote by $M(-p)$ the graded A-module such that $M(-p)_i = M_{i-p}$ for all i. Then $M(-p)$ is called the module M *shifted* p *degrees*, and p is the *shift*.

In the case of multigrading, we denote by $M(m)$ the graded S-module such that $M(m)_{m'} = M_{m'/m}$ where $M_{m'/m} = 0$ if $m \nmid m'$.

Proposition 2.1.4 ([24], p.2)**.** *The following are equivalent.*

(1) For each $f \in J$, all homogeneous components of f also belong to J.

(2) $J = \oplus_{i=0}^{\infty} J_i$, where $J_i = J \cap S_i$.

(3) If $\tilde{J}$ is the ideal generated by all homogeneous elements in J, then $J = \tilde{J}$.

(4) J has a system of homogeneous generators.

Proof. (1) $\implies$ (2): (1) implies $J \subset \oplus_{i=0}^{\infty} J_i$, and the other inclusion is clear.

(2) $\implies$ (3): (2) implies $J \subset \tilde{J}$. Since J contains the generators of $\tilde{J}$, $\tilde{J} \subset J$.

(3) $\implies$ (4): The homogeneous generators of $\tilde{J}$ also generate J as $J = \tilde{J}$.

(4) $\implies$ (1): Every homogeneous component of f is generated by the system of homogeneous generators in (4) and therefore in J. □

Definition 2.1.5. A proper ideal J in S is *graded* if it satisfies any of the four conditions in Proposition 2.1.4.

Let $S = k[x_1, \ldots, x_n]$, where k is a field. From now on, we denote the quotient ring S/I by R, where I is a graded ideal in S. Also, when we say "module", we mean R-module, unless otherwise specified.

2.2 Graded homomorphisms and free resolutions

Definition 2.2.1 ([24], p.6-7)**.** Let R be a ring and M, N graded R-modules. A homomorphism $\phi : M \to N$ has *degree* i if $\deg(\phi(x)) = i + \deg(x)$ for every homogeneous element $x \in M - \ker(\phi)$. Let $\mathrm{Hom}_i(M, N)$ denote the set of homomorphisms from M to N that have degree i. Then a homomorphism $\phi \in \mathrm{Hom}(M, N)$ is *graded* or *homogeneous* if $\phi \in \mathrm{Hom}_i(M, N)$ for some $i \in \mathbb{N}$.

Example 2.2.2. Let $A = k[x, y, z]$. The homomorphism

$$A \oplus A(-1) \oplus A(-2) \xrightarrow{\begin{pmatrix} x^3 & y^4 & z^5 \end{pmatrix}} A$$

has degree 3.

Definition 2.2.3. Let M be a finitely generated R-module. A *free resolution* of M is an exact sequence

$$\mathbf{F}: \quad \ldots \to F_i \xrightarrow{d_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

such that each F_i is a finitely generated free R-module. We say that $\mathbf{F}$ is *graded* if M is graded and each d_i have degree 0.

A graded free resolution can be constructed by induction as follows.

Construction 2.2.4 ([24], p.17-18)**.** Let M be a finitely generated graded R-module.

Step 0: Pick a set of homogeneous generators $m_1, \ldots, m_r$ of M with degree $\alpha_1, \ldots, \alpha_r$, respectively. Set $F_0 = R(-\alpha_1) \oplus \cdots \oplus R(-\alpha_r)$ and define $d_0 : F_0 \to M$ by $1_j \mapsto m_j$ for each $1 \leq j \leq r$, where 1_j denotes the 1-generator of $R(-\alpha_j)$.

Step 1: Pick a set of homogeneous generators $f_1, \ldots, f_\ell$ of $\ker(d_0)$ with degree $\xi_1, \ldots, \xi_\ell$, respectively. Set $F_1 = R(-\xi_1) \oplus \cdots \oplus R(-\xi_\ell)$ and define $d_1 : F_1 \to F_0$ by $1_j \to f_j$ for each $1 \leq j \leq \ell$, where 1_j denotes the 1-generator of $R(-\xi_j)$.

Now assume by induction that F_i and d_i are defined in *Step* i.

Step $i+1$: Pick a set of homogeneous generators $g_1, \ldots, g_s$ of $\ker(d_i)$ with degree $\gamma_1, \ldots, \gamma_s$, respectively. Set $F_{i+1} = R(-\gamma_1) \oplus \cdots \oplus R(-\gamma_s)$ and define $d_{i+1} : F_{i+1} \to F_i$ by $1_j \mapsto g_j$ for each $1 \leq j \leq s$, where 1_j denotes the 1-generator of $R(-\gamma_j)$.

One can check with definition that the above construction indeed gives us a graded free resolution of M.

Example 2.2.5. Let $A = k[x,y,z]$ and $B = (y, xz, x^3)$. Then

$$A(-5) \xrightarrow{\begin{pmatrix} x^2 \\ -z \\ y \end{pmatrix}} \begin{matrix} A(-3) \\ \oplus \\ A(-4) \\ \oplus \\ A(-4) \end{matrix} \xrightarrow{\begin{pmatrix} xz & x^3 & 0 \\ -y & 0 & x^2 \\ 0 & -y & -z \end{pmatrix}} \begin{matrix} A(-1) \\ \oplus \\ A(-2) \\ \oplus \\ A(-3) \end{matrix} \xrightarrow{\begin{pmatrix} y & xz & x^3 \end{pmatrix}} A \to A/B \to 0$$

is a free resolution of A/B (as an A-module).

Given a monomial ideal J, similar to Construction 2.2.4, we can construct a free resolution of J in the multigraded setting. The only difference is the notation of the graded R-modules.

Construction 2.2.6 (Multigraded free resolution)**.** Let J be a finitely generated monomial ideal in S.

Step 0: Pick a (minimal) set of generators $m_1, \ldots, m_r$ of J. Set $F_0 = R(m_1) \oplus \cdots \oplus R(m_r)$ and define $d_0 : F_0 \to M$ by $1_j \mapsto m_j$ for each $1 \leq j \leq r$, where 1_j denotes the 1-generator of $R(m_j)$.

Step 1: Pick a set of generators $f_1, \ldots, f_\ell$ of $\ker(d_0)$. Set $F_1 = R(\alpha_1) \oplus \cdots \oplus R(\alpha_\ell)$, where $\alpha_i = \operatorname{lcm}\{m_j : a_j \neq 0\}$ and a_j is a coefficient in the expression $f_i = \sum_{j=1}^r a_j 1_j$. Define $d_1 : F_1 \to F_0$ by $1'_j \to f_j$ for each $1 \leq j \leq \ell$, where $1'_j$ denotes the 1-generator of $R(\alpha_j)$.

Now assume by induction that F_i and d_i are defined in *Step* i.

Step $i+1$: Pick a set of generators $g_1, \ldots, g_s$ of $\ker(d_i)$. Set $F_{i+1} = R(\xi_1) \oplus \cdots \oplus R(\xi_s)$, where each ξ_i is defined similarly as in *Step* 1. Define $d_{i+1} : F_{i+1} \to F_i$ by $1''_j \mapsto g_j$ for each $1 \leq j \leq s$, where 1_j denotes the 1-generator of $R(\xi_j)$.

Example 2.2.7. The free resolution of Example 2.2.5 in the multigraded setting is given by

$$A(x^3yz) \xrightarrow{\begin{pmatrix} x^2 \\ -z \\ y \end{pmatrix}} \begin{matrix} A(xyz) \\ \oplus \\ A(x^3y) \\ \oplus \\ A(x^3z) \end{matrix} \xrightarrow{\begin{pmatrix} xz & x^3 & 0 \\ -y & 0 & x^2 \\ 0 & -y & -z \end{pmatrix}} \begin{matrix} A(y) \\ \oplus \\ A(xz) \\ \oplus \\ A(x^3) \end{matrix} \xrightarrow{\begin{pmatrix} y & xz & x^3 \end{pmatrix}} A \to A/B \to 0.$$

Definition 2.2.8 ([24], p.29)**.** A graded free resolution $\mathbf{F}$ of a finitely generated graded R-module M is *minimal* if for all $i \geq 0$, $d_{i+1}(F_{i+1}) \subset \mathbf{m}F_i$, where $\mathbf{m}$ denotes the maximal ideal $(x_1, \ldots, x_n)$ in S. Equivalently, the chosen set of homogeneous generators of $\ker(d_i)$ is minimal at each step in Construction 2.2.4.

In fact, the graded free resolution in Example 2.2.5 is minimal.

Definition 2.2.9. Let M and N be R-modules with free resolutions $\mathbf{F}$ and $\mathbf{G}$, respectively. We say that $\mathbf{F}$ and $\mathbf{G}$ are isomorphic if there is a collection of isomorphisms $\phi_i : F_i \to G_i$ such that $\phi_{i-1}d_i = \partial_i\phi_i$ for each i. In other words, the following diagram commutes.

$$\begin{tikzcd}
\mathbf{F}: & \cdots \arrow[r, "d_{i+1}"] & F_i \arrow[r, "d_i"] \arrow[d, "\phi_i"'] & F_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "\phi_{i-1}"] & \cdots \arrow[r, "d_1"] & F_0 \arrow[r, "d_0"] \arrow[d, "\phi_0"'] & M \arrow[r] \arrow[d, "\phi_{-1}"] & 0 \\
 & & & & \cdots & & & \\
\mathbf{G}: & \cdots \arrow[r, "\partial_{i+1}"] & G_i \arrow[r, "\partial_i"] & G_{i-1} \arrow[r, "\partial_{i-1}"] & \cdots \arrow[r, "\partial_1"] & G_0 \arrow[r, "\partial_0"] & N \arrow[r] & 0
\end{tikzcd}$$

Theorem 2.2.10 ([16])**.** *Let M be a finitely generated graded R-module. Then there is a unique minimal graded free resolution of M, up to isomorphism.*

By "uniqueness" from the above theorem, for a finitely generated graded R-module M, we may say "the" minimal graded free resolution of M. Furthermore, we can now introduce some numerical invariants of the minimal graded free resolution, such as Betti numbers.

2.3 Graded Betti numbers

Definition 2.3.1. Let M be a finitely generated graded R-module and $\mathbf{F}$ the minimal graded free resolution of M. The *i-th Betti number of M over R* is defined as

$$\beta_i^R(M) = \operatorname{rank}(F_i).$$

The *graded Betti numbers* of M are defined as

$$\beta_{i,j}^R(M) = \text{the number of summands in } F_i \text{ of the form } R(-j).$$

The *multigraded Betti numbers* of M are defined as

$$\beta^R_{i,m}(M) = \text{the number of summands in } F_i \text{ of the form } R(m),$$

where m is a monomial.

When there is no ambiguity about M and R, we simply write β_i, instead of $\beta^R_i(M)$, similar for $\beta_{i,j}$ and $\beta_{i,m}$.

Remark 2.3.2. If our graded free resolution also admits a multigrading, by definition, we have

$$\beta_i = \sum_j \beta_{i,j} = \sum_j \sum_{\deg(m)=j} \beta_{i,m}.$$

We usually summarize the Betti numbers as a table called the Betti diagram.

	β_0	β_1	β_2	$\dots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\dots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\dots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\dots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.1: The Betti diagram

The number at the j-th row and i-th column represents the graded Betti number $\beta_{i,i+j}$. A zero is denoted by "-". Notice that we do not have $\beta_{i,j}$ for $j < i$ in the diagram, because they are all 0 (see [24], p.43).

In general, it is not easy to compute these Betti numbers by hand. One reason is that the minimal graded free resolution is hard to construct, since we need to determine at each step whether the generating set we pick is minimal. Thus, we usually need to use softwares to compute the free resolutions, as well as the Betti numbers. In particular, we use Macaulay2 [15] for most computation in this research.

Another invariant of a minimal graded free resolution is its "length", which is called "projective dimension". In other words, if we are building a minimal graded free resolution by Construction 2.2.4, the "length" represents the number of steps needed until we get $\ker(d_i) = 0$ (therefore $F_{i+1} = 0$) for some i. By Hilbert's syzygy theorem [16], we know that this number is finite, and in fact, at most n (recall $S = k[x_1, \dots, x_n]$). This gives rise to the following definition.

Definition 2.3.3. Let M be a finitely generated graded R-module and $\mathbf{F}$ the minimal graded free resolution of M. The *projective dimension* of M is defined as

$$\mathrm{pd}(M) = \max\{i : \beta_i^R(M) \neq 0\}.$$

Sometimes, $\mathrm{pd}(M)$ is also called the *length* of $\mathbf{F}$.

Remark 2.3.4. Let I be a monomial ideal (as an S-module). Let $\mathbf{F}$ and $\mathbf{G}$ be the minimal graded free resolution of I and S/I, respectively. The only difference between $\mathbf{F}$ and $\mathbf{G}$ is that $\mathbf{G}$ has one extra component $S \to S/I$, while the rest are exactly the same as $\mathbf{F}$, i.e., $F_i = G_{i+1}$ for all $i \geq 0$. It follows that

$$\beta_{i,j}^S(I) = \beta_{i+1,j}^S(S/I) \qquad \text{and} \qquad \mathrm{pd}(I) = \mathrm{pd}(S/I) - 1,$$

for all i, j with $i \geq 0$. Furthermore, we always have $\beta_{0,0}^S(S/I) = 1$.

Since either of $\mathbf{F}$ or $\mathbf{G}$ will provide all information of the other, we may say that computing the minimal graded free resolutions of I and S/I are equivalent.

Example 2.3.5. Consider the graded free resolution from Example 2.2.5, which, we know, is minimal. The graded Betti numbers are

$$\beta_{0,0} = \beta_{1,1} = \beta_{1,2} = \beta_{1,3} = \beta_{2,3} = \beta_{3,5} = 1, \qquad \beta_{2,4} = 2.$$

We have $\mathrm{pd}(A/B) = 3$. The Betti diagram is given by:

	1	3	3	1
0	1	1	-	-
1	-	1	1	-
2	-	1	2	1

Table 2.2: Betti diagram of A/B

For B, it follows that $\mathrm{pd}(B) = 2$ and the Betti diagram for B is:

	3	3	1
1	1	-	-
2	1	1	-
3	1	2	1

Table 2.3: Betti diagram of B

Resolutions are useful because they give algebraic invariants of monomial ideals. An effective method to calculate them is to homogenize the chain complex of topological objects, and then use discrete homotopy theory to "shrink" them. We will see more details in Chapter 4.

Next, we introduce the basic concepts of topological objects that are needed in this research.

Chapter 3

Basics from Algebraic Topology

3.1 Topological space

Definition 3.1.1. A *topology* on a set X is a collection $\mathscr{T}$ of subsets of X having the following properties:

(1) $\varnothing, X \in \mathscr{T}$.

(2) $\mathscr{T}$ is closed under arbitrary union and finite intersection.

A set X together with a specified topology $\mathscr{T}$ is called a *topological space*. A subset U of X is an *open set* if $U \in \mathscr{T}$. In this case, we say that U is open in X.

Example 3.1.2. Let $X = \{a, \infty, \varnothing\}$. The following are topologies on X.

(a) $\mathscr{T}_1 = \{\varnothing, X\}$

(b) $\mathscr{T}_2 = \{\varnothing, \{a\}, X\}$

(c) $\mathscr{T}_3 = \{\varnothing, \{\infty, \varnothing\}, X\}$

(d) $\mathscr{T}_4 = \{\varnothing, \{a\}, \{\infty\}, \{\varnothing\}, \{a, \infty\}, \{a, \varnothing\}, \{\infty, \varnothing\}, X\}$

Definition 3.1.3. Let X be a set and $\mathscr{B}$ a collection of subsets of X such that:

(1) For each $x \in X$, there is a $B \in \mathscr{B}$ containing x.

(2) If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathscr{B}$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

We define the *topology* $\mathscr{T}$ *generated by* $\mathscr{B}$ as follows: A subset U of X is open in X if for each $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subset U$. The collection $\mathscr{B}$ is called the *basis* for $\mathscr{T}$, and the elements in $\mathscr{B}$ are called the *basis elements*.

It is not hard to check that the set $\mathscr{T}$ constructed in the above definition is indeed a topology on X.

Example 3.1.4. Given any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define $||x|| = \sqrt{\sum_{i=1}^{n} x_i^2}$. The Euclidean space $\mathbb{R}^n$ is the topological space generated by the open balls $B(a, \delta) = \{x \in \mathbb{R}^n : ||x - a|| < \delta\}$, where $\delta > 0$.

Definition 3.1.5. Let X be a topological space and Y a subset of X, then the collection $\{U \cap Y : U \text{ is open in } X\}$ is a topology on Y, which is called the *subspace topology*. With this topology, Y is called a *subspce* of X.

Again, one can verify that the collection in the above definition is indeed a topology on Y using the following equations.

$$\bigcup_{\alpha\in A}(Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha\in A} U_\alpha\right)$$
$$\bigcap_{j=1}^{n}(Y \cap U_j) = Y \cap \left(\bigcap_{j=1}^{n} U_j\right)$$

Example 3.1.6. The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ is a subspace of the Euclidean space $\mathbb{R}^{n+1}$.

Definition 3.1.7. Let X, Y be topological spaces. A function $f : X \to Y$ is *continuous* if for any open set V in Y, the set $f^{-1}(V)$ is open in X. If f is invertible and both f and f^{-1} are continuous, f is called a *homeomorphism*. Furthermore, if there is a homeomorphism between X and Y, we say that X and Y are *homeomorphic* and denote this by $X \cong Y$.

Definition 3.1.8. A topological space that is homeomorphic to a (closed unit) n-dimensional ball, i.e., $B^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$, is called an *$n$-cell*.

Another version of the above definition uses open balls. In this research, we use the "closed" version as it makes the construction of a CW complex easier to explain.

Example 3.1.9. The space $[-10, 10] \subset \mathbb{R}$ is a 1-cell, as it is homeomorphic to $B^1 = [-1, 1]$ (with homeomorphism $x \mapsto x/10$).

Proposition 3.1.10. *Let X be a topological space and $\sim$ an equivalence relation on X. Let $[x]$ denote the equivalence class of $x \in X$. Define $X/\!\sim\, = \{[x] : x \in X\}$ and $\pi : X \to X/\!\sim$ by $x \mapsto [x]$. Then $\mathscr{T} = \{U \subset X/\!\sim\, : \pi^{-1}(U) \text{ is open in } X\}$ is a topology on $X/\!\sim$.*

Proof. Notice that $\varnothing \in \mathscr{T}$ is vacuously true and $X/\!\sim\, \in \mathscr{T}$ by the fact that $\pi^{-1}(X/\!\sim) = X$. Also, $\mathscr{T}$ is closed under arbitrary union and finite intersection because the topology on X is. $\square$

The topological space $X/\!\sim$ is called *the quotient space of X under $\sim$*, and the topology $\mathscr{T}$ on $X/\!\sim$ is called the *quotient topology.*

Example 3.1.11. Consider $X = [0, 1]$ and $\sim$ defined by $x \sim y \iff x - y \in \mathbb{Z}$. Then $X/\!\sim\, \cong S^1$ with homeomorphism $f : X/\!\sim\, \to S^1$ defined as $[x] \mapsto (\cos 2\pi x, \sin 2\pi x)$. This can be illustrated as follows.

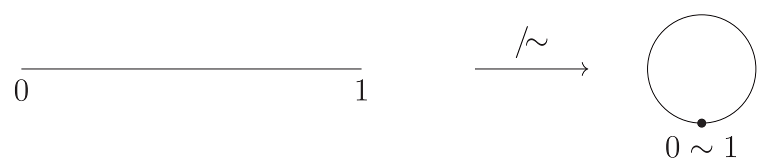

Figure 3.1: The quotient space of $[0, 1]$ under $\sim$

Definition 3.1.12 ([19], p.63)**.** Let X, Y be disjoint topological spaces, $A \subset X$, and $f : A \to Y$ a continuous function. Let $\sim$ be the equivalence relation generated by $x \sim f(x)$ for all $x \in A$. Then the topological space $Z = (X \cup Y)/\!\sim$ is said to be obtained by *attaching X to Y over f*, denoted by $Y \cup_f X$. The space Z is called the *adjunction space.*

Example 3.1.13. Let $X = Y = B^2$. Below picture illustrates the space obtained by attaching X to Y over the (identity) embedding $\mathrm{id} : \partial B^2 = S^1 \hookrightarrow Y$.

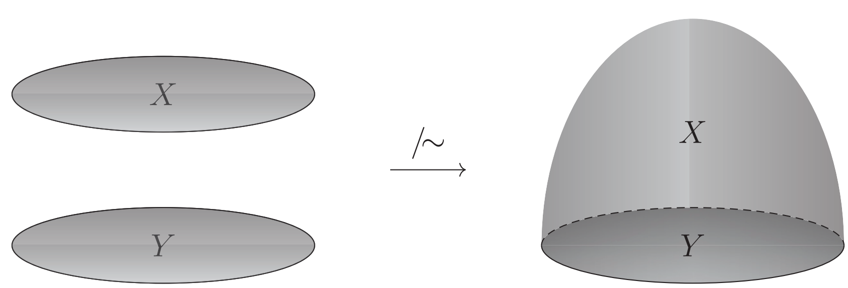

Figure 3.2: Topological space obtained by attaching two disks over the identity map

3.2 CW complexes

Now we are ready to construct a CW complex.

Construction 3.2.1 ([19], p.65)**.** The construction of a CW complex X is by induction on the dimension n.

Step 0: Let X_0 be a non-empty discrete (i.e., finite or countably infinite) set of points (i.e., 0-cells).

Assume by induction that X_n has been constructed from X_{n-1} in *Step* n.

Step $n+1$: Let X_{n+1} be the topological space obtained by simultaneously attaching $(n+1)$-cells (i.e., each homeomorphic to B^{n+1}) to X_n along the boundaries (i.e., over continuous map $S^n \to X_n$).

Finally, we define $X = \bigcup_n X_n$. This procedure can either stop after a finite number of steps or continue infinitely. The *dimension* of X is the same as the dimension of the cell in X that has the highest dimension. Each X_n constructed in *Step* n is called the *n-skeleton* (of X).

The j-cells in X are called *faces* of X, also, we say that $\varnothing$ is a face of X of dimension -1. If σ, τ are two faces of X such that $\sigma \subset \tau$, we say that σ is a *subface* of τ, which we denote by $\sigma \leq \tau$. If $\sigma \subsetneq \tau$, we say that σ is a *proper subface* of τ, denoted by $\sigma < \tau$. A CW complex is *regular* if at each step j of Construction 3.2.1, the j-cells are attached to X_{j-1} via homeomorphisms.

By construction, we have

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

and in general, the dimension of X can be infinite. To distinguish from the n-skeleton, we denote by X^n the collection of n-dimensional faces of X.

Notice that by our definition, "$\sigma \subset \tau$" is equivalent to "$\sigma \leq \tau$". If the j-cells are defined as the open balls, then we say that $\sigma \leq \tau$ if $\bar{\sigma} \subset \bar{\tau}$, where $\bar{\sigma}$ denotes the closure of σ, i.e., $\bar{\sigma} = \sigma \cup \partial\sigma$.

Example 3.2.2 (A non-regular CW complex)**.** The 2-sphere S^2 is a 2-dimensional CW complex obtained by attaching B^2 along the boundary to a point via the constant map.

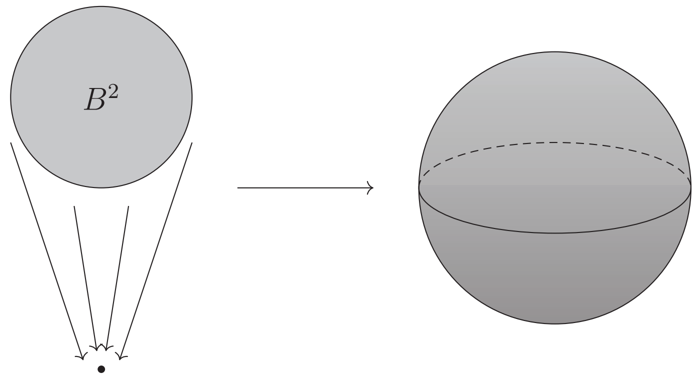

Figure 3.3: S^2 obtained by attaching the boundary of B^2 to a point

The S^2 is this example is not regular, since the (constant) attaching map is not one-to-one (therefore not a homeomorphism).

Observe that S^2 can also be regular, depending on its CW structure (i.e., construction), if we obtain S^2 by attaching 2 vertices, 2 edges, and 2 disks together via (identity) embeddings (see Kozlov [18], p.35, for a detailed discussion about S^n).

Example 3.2.3. A solid square is a 2-dimensional CW complex attached by identity maps (the 2-cell is chosen to be a closed square, as it is homeomorphic to D^2).

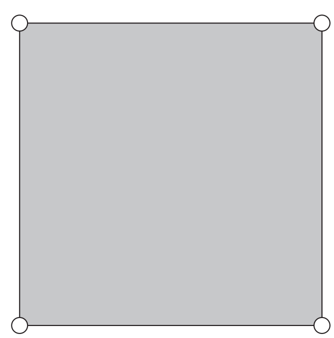

Figure 3.4: A solid square as a CW complex homeomorphic to a 2-cell

Furthermore, this is a regular CW complex.

The most commonly known example of a regular CW complex is a (non-empty) simplicial complex.

Definition 3.2.4 (Abstract simplicial complex)**.** Let V be a finite set and Δ a subset of the power set of V. Then Δ is called an *(abstract) simplicial complex* if for every

$F \in \Delta$, $G \subset F$ implies $G \in \Delta$. Every $F \in \Delta$ is called a *face* of Δ. The dimension of F is defined by

$$\dim(F) = |F| - 1$$

and

$$\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}.$$

Further, let $n = |V|$, then Δ is called an $(n-1)$-*simplex* if it contains exactly one maximal face.

Example 3.2.5. Every non-empty simplicial complex X is a regular CW complex. The n-skeleton of X consists of all i-simplices in X for $i \leq n$.

Definition 3.2.6 ([23], p.114)**.** Let X be a CW-complex and σ a face of X. A face $\sigma' < \sigma$ in X is called a *facet of* σ if there is no face τ of X such that $\sigma' < \tau < \sigma$. If there is no face τ of X such that $\sigma < \tau$, then σ is called a *facet of* X.

Below is a supplementary lemma for the definition of an incidence function.

Lemma 3.2.7 ([8], p.264)**.** *Let X be a regular CW complex. Let σ, τ be two faces of X such that $\tau \leq \sigma$ and $\dim(\sigma) - \dim(\tau) = 2$, then there are exactly two facets of σ (with dimension $\dim(\sigma) - 1$) that contain τ.*

3.3 Cellular chain complex of regular CW complexes

We first define as follows the incidence function ε on a CW complex, which is essential for defining boundary maps.

Definition 3.3.1 ([8], p.264-265)**.** Let X be a regular CW complex with vertex set V and Γ the set of faces of X. We say that $\varepsilon : \Gamma \times \Gamma \to \{-1, 0, 1\}$ is an *incidence function* on X if:

(1) For all $v \in V$, $\varepsilon(\{v\}, \varnothing) = 1$.

(2) For any two faces F, G of X, $\varepsilon(F, G) \neq 0$ if and only if $G \leq F$ and $\dim(F) - \dim(G) = 1$.

(3) For any two faces F, G of X with $G \leq F$ and $\dim(F) - \dim(G) = 2$,

$$\varepsilon(F, H_1)\varepsilon(H_1, G) + \varepsilon(F, H_2)\varepsilon(H_2, G) = 0,$$

where H_1, H_2 are the two facets of F that contain G.

Lemma 3.3.2 ([8], p.265)**.** *Every regular CW complex can be equipped with an incidence function.*

Definition 3.3.3 ([6])**.** Let X be a regular CW complex equipped with an incidence function ε. Define the free k-module

$$\mathcal{C}_i(X; k) = \bigoplus_{F \in X^i} kF,$$

where kF denotes the k-module generated by the face F and k is an arbitrary field. Let $i \geq 1$ and define the boundary map $\partial_j : \mathcal{C}_i(X; k) \to \mathcal{C}_{i-1}(X; k)$ by

$$\partial_i(F) = \sum_{G \in X^{i-1}} \varepsilon(F, G)G,$$

where $F \in X^i$. Then the chain connected by the boundary maps

$$\mathcal{C}(X; k): \quad \cdots \xrightarrow{\partial_{i+1}} \mathcal{C}_i(X; k) \xrightarrow{\partial_i} \mathcal{C}_{i-1}(X; k) \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} \mathcal{C}_1(X; k) \xrightarrow{\partial_1} \mathcal{C}_0(X; k) \to 0$$

is called the *(oriented) chain complex of* X. The chain

$$\tilde{\mathcal{C}}(X; k): \quad \cdots \xrightarrow{\partial_{i+1}} \mathcal{C}_i(X; k) \xrightarrow{\partial_i} \mathcal{C}_{i-1}(X; k) \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} \mathcal{C}_0(X; k) \xrightarrow{\partial_0} \mathcal{C}_{-1}(X; k) \to 0,$$

where ∂_0 is defined by $\{v\} \mapsto \varnothing$ for each $\{v\} \in X^0$, is called the *augmented chain complex of* X, where $\mathcal{C}_{-1}(X; k) = k\varnothing$, i.e., the k-module generated by the element $\varnothing$. For each integer $i \geq -1$, the i-th homology $\tilde{H}_i(X; k) = \ker \partial_i / \mathrm{im}\partial_{i+1}$ of $\tilde{\mathcal{C}}(X; k)$ is called *the reduced cellular homology group of degree* i. If $\tilde{H}_i(X; k) = 0$ for all $i \geq -1$, we say that X is *acyclic*.

The only difference between the chain complex and augmented chain complex of X is that the latter contains an additional k-module $\mathcal{C}_{-1}(X; k)$. Moreover, it is not hard to verify that $\partial^2 = 0$ by the definition of an incidence function.

Remark 3.3.4. Let X be a CW complex. Then $\tilde{H}_{-1}(X; k) = 0$ because the ∂_0 is onto as $X_0 \neq \varnothing$.

In fact, when X is a simplicial complex, then ([22], p.9)

$$\tilde{H}_{-1}(X;k) = \begin{cases} k & X = \{\varnothing\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\{\varnothing\}$ is called the *empty (abstract) simplicial complex*, which is not a CW complex by Construction 3.2.1.

Example 3.3.5. Consider the solid square from Example 3.2.3, which we denote by X.

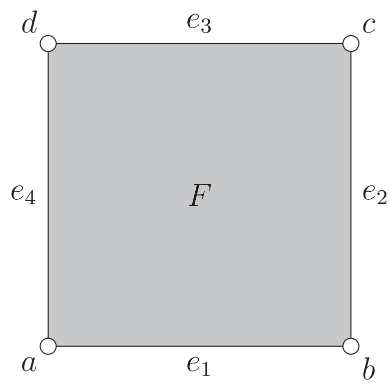

Figure 3.5: A solid square as a 2-dimensional CW complex

The following defines an incidence function ε on X:

$$\varepsilon(\{a\},\varnothing) = \varepsilon(\{b\},\varnothing) = \varepsilon(\{c\},\varnothing) = \varepsilon(\{d\},\varnothing) = 1,$$
$$\varepsilon(e_1,\{a\}) = \varepsilon(e_2,\{b\}) = \varepsilon(e_3,\{c\}) = \varepsilon(e_4,\{d\}) = -1,$$
$$\varepsilon(e_1,\{b\}) = \varepsilon(e_2,\{c\}) = \varepsilon(e_3,\{d\}) = \varepsilon(e_4,\{a\}) = 1,$$
$$\varepsilon(F,e_1) = \varepsilon(F,e_2) = \varepsilon(F,e_3) = \varepsilon(F,e_4) = 1.$$

The oriented chain complex $\mathcal{C}(X;k)$ is given by

$$0 \to kF \xrightarrow{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}} \begin{matrix} ke_1 \\ \oplus \\ ke_2 \\ \oplus \\ ke_3 \\ \oplus \\ ke_4 \end{matrix} \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}} \begin{matrix} k\{a\} \\ \oplus \\ k\{b\} \\ \oplus \\ k\{c\} \\ \oplus \\ k\{d\} \end{matrix} \to 0.$$

The augmented chain complex $\tilde{\mathcal{C}}(X;k)$ is given by

$$0 \to kF \xrightarrow{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}} \begin{matrix} ke_1 \\ \oplus \\ ke_2 \\ \oplus \\ ke_3 \\ \oplus \\ ke_4 \end{matrix} \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}} \begin{matrix} k\{a\} \\ \oplus \\ k\{b\} \\ \oplus \\ k\{c\} \\ \oplus \\ k\{d\} \end{matrix} \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}} k\varnothing \to 0.$$

Furthermore, one can show that X is acyclic by verifying $\ker \partial_i = \mathrm{im}\partial_{i+1}$.

Lemma 3.3.6. *The line graph L_n and solid n-gon are acyclic for any integer $n \geq 2$.*

Proof. We will prove this lemma directly using the definition. Consider any solid n-gon (see below), which we denote by X.

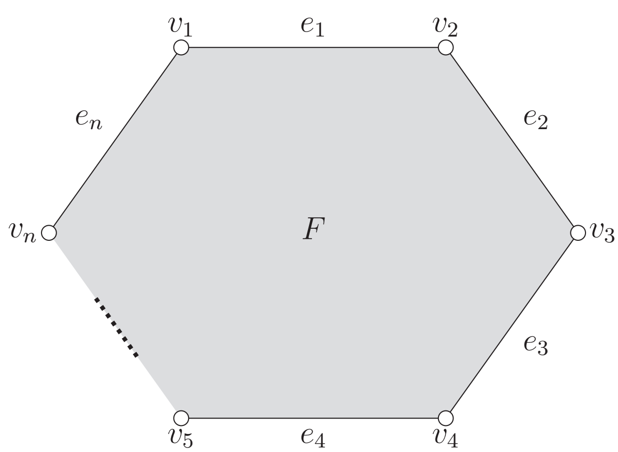

Figure 3.6: A solid n-gon as a 2-dimensional CW complex

Define the incidence function ε on X by

$$\begin{aligned} 1 &= \varepsilon(\{v_i\}, \varnothing) \text{ for each } i = 1, \ldots n, \\ -1 &= \varepsilon(e_i, \{v_i\}) \text{ for each } i = 1, \ldots, n, \\ 1 &= \varepsilon(e_n, \{v_1\}) = \varepsilon(e_i, \{v_{i+1}\}) \text{ for each } i = 1, \ldots, n-1, \\ 1 &= \varepsilon(F, e_i) \text{ for each } i = 1, \ldots, n. \end{aligned}$$

Then the augmented chain complex of X is given by

$$0 \to kF \xrightarrow{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}} \begin{matrix} ke_1 \\ \oplus \\ ke_2 \\ \oplus \\ \vdots \\ \oplus \\ ke_n \end{matrix} \xrightarrow{\begin{pmatrix}-1 & 0 & \cdots & 1\\ 1 & -1 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & -1\end{pmatrix}} \begin{matrix} k\{v_1\} \\ \oplus \\ k\{v_2\} \\ \oplus \\ \vdots \\ \oplus \\ k\{v_n\} \end{matrix} \xrightarrow{\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}} k\varnothing \to 0.$$

By Remark 3.3.4, $\tilde{H}_{-1}(X;k) = 0$. It remains to check that $\ker \partial_i \subset \mathrm{im}\partial_{i+1}$ for $i = 0, 1, 2$. Using matrix algebra, we can calculate the explicit forms of the kernels:

$$\begin{aligned}
\ker \partial_0 &= \{(-a_1 - \cdots - a_{n-1})\{v_1\} + a_1\{v_2\} + \cdots + a_{n-1}\{v_n\} : a_1, \ldots, a_{n-1} \in k\},\\
\ker \partial_1 &= \{b(e_1 + e_2 + \cdots + e_n) : b \in k\},\\
\ker \partial_2 &= 0 = \mathrm{im}\partial_3.
\end{aligned}$$

Let $a_1, \ldots, a_{n-1}, b \in k$. Observe that

$$\begin{aligned}
\partial_1\left(\sum_{i=1}^{n-1}\sum_{j=i}^{n-1} a_j e_i\right) &= (-a_1 - \cdots - a_{n-1})\{v_1\} + a_1\{v_2\} + \cdots + a_{n-1}\{v_n\},\\
\partial_2(bF) &= b(e_1 + e_2 + \cdots + e_n),
\end{aligned}$$

which shows that $\ker \partial_0 \subset \mathrm{im}\partial_1$ and $\ker \partial_1 \subset \mathrm{im}\partial_2$. This completes the proof for solid n-gon. We know that L_n, as a 1-dimensional simplicial complex, is acyclic, since it is a tree graph. □

3.4 Polyhedral cell complex

A polyhedral cell complex X, which consists of convex polytopes, is a special case of (regular) CW complex, since we can build it using Construction 3.2.1 by choosing the n-cells in X to be the n-polytopes and attaching them using certain homeomorphisms (see Kozlov [18], p.25, for details).

Instead of constructing a polyhedral cell complex as a CW complex, we can define it in a way that is easier to understand, i.e., less abstract.

Definition 3.4.1. (1) ([8], p.223) The convex hull of a finite set $A \subset \mathbb{R}^n$, i.e., the smallest convex set that contains A, is called a *polytope*.

(2) ([22], p.62) A (finite) collection X of polytopes is called a *polyhedral cell complex* if it satisfies the following properties.

(a) If $P \in X$, then every face of P belongs to X.

(b) For all $P, Q \in X$, $P \cap Q$ is a face of both P and Q.

Each polytope in X is called a *face* of X. The *dimension* of X is defined as

$$\dim(X) = \max\{\dim(P) : P \in X\}.$$

By convention, $\dim(\varnothing) = -1$.

Example 3.4.2. A solid square is a 2-dimensional polyhedral cell complex. In fact, it can be represented as

$$X = \{\{a, b, c, d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \varnothing\}.$$

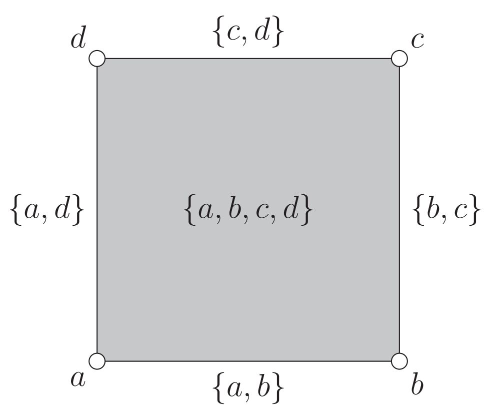

Figure 3.7: A solid square as a 2-dimensional polyhedral cell complex

Example 3.4.3. Every (finite) simplicial complex is a polyhedral cell complex.

Chapter 4

Cellular Resolutions of Monomial Ideals

Now we improve this study by exploring more general topological objects, namely, CW complexes. In this research, we want to determine whether I has minimal cellular resolutions, based on the idea from Bayer and Sturmfels [6].

Throughout this thesis, $I = (m_1, \ldots, m_r)$ means that $\{m_1, \ldots, m_r\}$ is the minimal monomial generating set of I.

4.1 Cellular Resolutions

In this section, we introduce the connection between cellular chain complex of regular CW complexes and free resolutions of monomial ideals by defining a free resolution supported on a regular CW complex.

We start by constructing a chain complex of a monomial ideal from a regular CW complex.

Construction 4.1.1 (Homogenization [6, 7])**.** Let $I = (m_1, \ldots, m_r)$ be a monomial ideal in S and X a (finite) regular CW complex with r vertices $v_1, \ldots, v_r$ labeled by the monomials $m_1, \ldots, m_r$, respectively. For each face F of X, we label F by the monomial

$$m_F = \operatorname{lcm}\{m_i : \{v_i\} \leq F\}.$$

We also say that F has multidegree m_F.

Consider any chain complex $\mathcal{C}(X; k)$ of X.

$$\mathcal{C}(X;k): \quad \cdots \xrightarrow{\partial_{i+1}} \mathcal{C}_i(X;k) \xrightarrow{\partial_i} \mathcal{C}_{i-1}(X;k) \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} \mathcal{C}_0(X;k) \to 0$$

For each $i \geq 0$, define

$$C_i = \bigoplus_{F \in X^i} S(m_F),$$

i.e., the free S-module with basis $\{F : F \in X^i\}$. Also, define the map $d_i : C_i \to C_{i-1}$ (where $i \geq 1$) by

$$d_i(F) = \sum_{G \in X^{i-1}} \varepsilon(F, G) \frac{m_F}{m_G} 1_G,$$

where F is a basis element of C_i, ε is the incident function for $\mathcal{C}(X; k)$ on X, and 1_G denotes the 1-generator of $S(-m_G)$, which is a summand of $C_{i-1} = \bigoplus_{F \in X^{i-1}} S(m_F)$.

Finally, the sequence of S-modules

$$C_X : \quad \cdots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} I \to 0,$$

where $d_0 = \begin{pmatrix} m_1 & m_2 & \cdots & m_r \end{pmatrix}$, is a (multigraded) chain complex of I. We also call it a *cellular complex* of I. The above process is called *homogenization* of cellular chain complex.

If the cellular complex C_X in Construction 4.1.1 is a free resolution of I, we say that X *supports a free resolution of* I or I *has a free resolution supported on* X. In this case, C_X is called a *cellular resolution* of I.

Example 4.1.2. Consider the chain complex in Example 3.3.5 and recall Figure 3.5.

$$0 \to kF \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \begin{matrix} ke_1 \\ \oplus \\ ke_2 \\ \oplus \\ ke_3 \\ \oplus \\ ke_4 \end{matrix} \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}} \begin{matrix} k\{a\} \\ \oplus \\ k\{b\} \\ \oplus \\ k\{c\} \\ \oplus \\ k\{d\} \end{matrix} \to 0.$$

To construct a cellular complex of $I = (xy, yz, zw, wu)$, we start by labeling the faces of X.

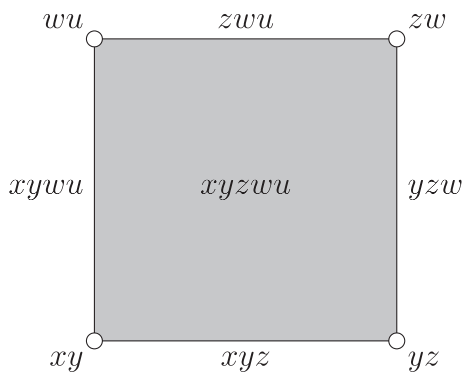

Figure 4.1: A solid square with faces labeled by monomials

The cellular complex C_X obtained from Construction 4.1.1 is

$$0 \to S(xyzwu) \xrightarrow{\begin{pmatrix} wu \\ xu \\ xy \\ z \end{pmatrix}} \begin{matrix} S(xyz) \\ \oplus \\ S(yzw) \\ \oplus \\ S(zwu) \\ \oplus \\ S(xywu) \end{matrix} \xrightarrow{\begin{pmatrix} -z & 0 & 0 & wu \\ x & -w & 0 & 0 \\ 0 & y & -u & 0 \\ 0 & 0 & z & -xy \end{pmatrix}} \begin{matrix} S(xy) \\ \oplus \\ S(yz) \\ \oplus \\ S(zw) \\ \oplus \\ S(wu) \end{matrix}$$

$$\xrightarrow{\begin{pmatrix} xy & yz & zw & wu \end{pmatrix}} I \to 0.$$

In fact, this is the (multigraded) minimal free resolution of I. Therefore, we conclude that X supports the minimal free resolution of I.

In general, it is not easy to determine whether a regular CW complex supports a free resolution of a monomial ideal, especially when we need to further verify the minimality of the free resolution, as we have to go through complicated calculation. Therefore, we will introduce a theorem that helps us simplify the problem, as well as the criterion for the minimality of the cellular resolution.

First, we need the concept of "induced subcomplex". Let $I = (m_1, \ldots, m_r)$ and X a regular CW complex on r vertices, whose faces are labeled by monomials (the same way in Construction 4.1.1).

Definition 4.1.3. Let m be a monomial in S. The *induced subcomplex* $X_{\leq m} \subset X$ is defined as

$$X_{\leq m} = \{F : F \text{ is a face of } X \text{ such that } m_F \mid m\},$$

which is a CW complex that is contained in X and consists of the faces of X whose label divides m.

Example 4.1.4. Recall the labeled solid square X from Example 4.1.2 and consider $m = xyzw$. We have

$$m_{\{a\}} = xy, m_{\{b\}} = yz, m_{\{c\}} = zw, m_{e_1} = xyz, m_{e_2} = yzw,$$

which are all faces of X with multidegrees dividing $xyzw$, so $X_{xyzw} = \{\{a\}, \{b\}, \{c\}, e_1, e_2\}$, which is illustrated below.

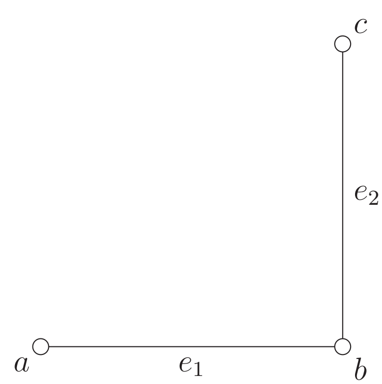

Figure 4.2: The induced subcomplex $X_{\leq xyzw}$

Theorem 4.1.5 ([6]). *The regular CW complex X supports a free resolution of I if and only if the subcomplex $X_{\leq m}$ is either acyclic or empty for every monomial $m \in S$. Moreover, the free resolution of I supported on X is minimal if and only if for all faces F, G of X such that $G < F$, we have $m_F \neq m_G$.*

Note that this theorem generalizes Lemma 2.1 in [5] from simplical complexes to regular CW complexes.

Remark 4.1.6. To determine whether a regular CW complex supports a free resolution of a monomial ideal I using Theorem 4.1.5, it suffices to check that $X_{\leq m}$ is acyclic for all monomials m in the lcm lattice (see Definition 4.1.7) of I.

Definition 4.1.7. Let P be a poset. Then P is called a *lattice* if for any $a, b \in P$, the set $\{a, b\}$ has a join (i.e., least upper bound) and a meet (i.e., greatest lower bound).

Let I be a monomial ideal in S. The lcm lattice of I is the poset of lcm's among the (minimal) monomial generators of I with the partial order "$|$". It is usually represented as a Hasse diagram.

Example 4.1.8. Let $I = (x_1x_2, x_2x_3, x_3x_4)$.

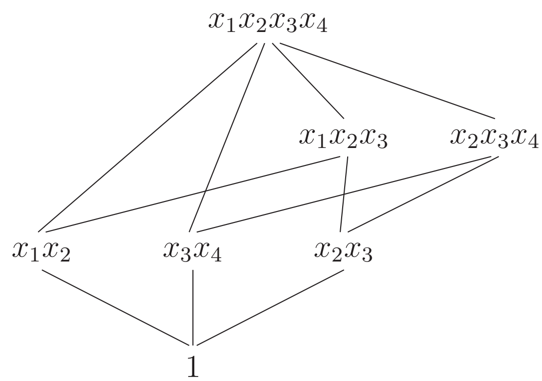

Figure 4.3: The lcm lattice of I

Example 4.1.9. The CW complex from Example 4.1.2 satisfies the conditions in Theorem 4.1.5.

If we know that the minimal free resolution of a monomial ideal I is supported on a CW complex X, then we can directly read all the Betti numbers of I from the labeling of X.

Example 4.1.10. Recall the solid square X (Figure 4.1) labeled by the generators of $I_2(L_5)$ from Example 4.1.2. We know that X has one 2-dimensional face labeled by $xyzwu$, which corresponds to $S(xyzwu)$ in the minimal free resolution of $I_2(L_5)$. This tells us that

$$\beta_{2,xyzwu} = \beta_{2,5} = 1.$$

Similarly, the four 1-dimensional faces, i.e., edges, of X labeled by $xyz, yzw, zwu, xyuw$ corresponds to the Betti numbers

$$\beta_{1,xyz}, \beta_{1,yzw}, \beta_{1,zwu}, \beta_{1,xywu} = 1,$$

respectively, which tells us $\beta_{1,3} = 3$ and $\beta_{1,4} = 1$, and so on. From the labeling of X, we can directly obtain the Betti diagram of $I_2(L_5)$:

	4	4	1
1	-	-	-
2	4	3	-
3	-	1	1

Table 4.1: Betti diagram of $I_2(L_5)$

Simplicial complexes are a special class of CW complexes, and they are useful for calculating algebraic invariants of monomial ideals. As we previously mentioned, a simplicial complex Δ supporting the minimal free resolution of I implies that Δ is acyclic and the (graded) Betti numbers of I agree with the labels of Δ.

Example 4.1.11. Recall the ideal B from Example 2.2.5. We knew that the graded Betti numbers of B are

$$\beta_{0,1} = \beta_{0,2} = \beta_{0,3} = \beta_{1,3} = \beta_{2,5} = 1, \qquad \beta_{1,4} = 2.$$

The minimal free resolution of B is supported on the following simplicial complex Δ, which we know to be acyclic.

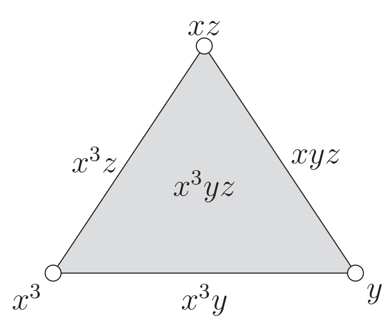

Figure 4.4: (Labeled) Δ that supports the minimal free resolution of B

The Betti numbers agree with the labels of Δ because $\beta_{0,1} = 1$ corresponds to Δ having 1 vertex of degree 1, namely, $\deg(y) = 1$; $\beta_{1,4} = 2$ corresponds to Δ having 2 edges of degree 4, namely, $\deg(x^3y) = \deg(x^3z) = 4$, and so on. In other words, given a graded Betti number $\beta_{i,j}$, the integers i and j correspond to the dimension and degree, respectively, of a face of Δ.

In 1966, Taylor [25] found that for any monomial ideal $I = (m_1, \ldots, m_r)$, I has a free resolution, which we call the *Taylor resolution*, supported on an $(r-1)$-simplex. This result can also be verified by Theorem 4.1.5, since $X_{\leq m}$ is always a simplex, therefore acyclic. The free resolution from Example 4.1.11 is a Taylor resolution.

However, in general, the Taylor resolution is not minimal. In other words, a simplex "resolves everything", but it does not always give the minimal free resolution. Then we want to know what simplicial complexes do and what classes of monomial ideals have minimal simplicial resolutions. Faridi and Hersey [13] have partially answered this question. They concluded that all monomial ideals of projective dimension 1 have a minimal free resolution supported on a 1-dimensional simplicial complex (not necessarily unique).

Example 4.1.12. Let $I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4)$. The minimal free resolution of I is supported on both of the below simplicial complexes, in particular, trees.

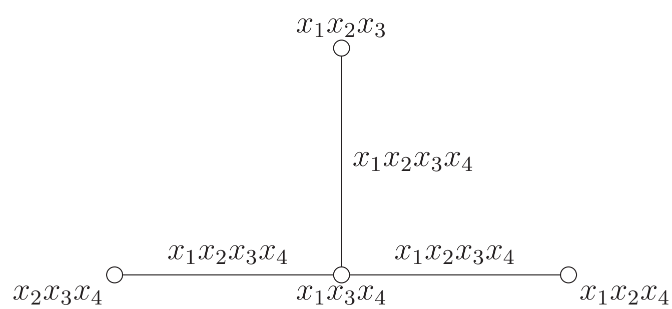

Figure 4.5: A tree supporting the minimal free resolution of I

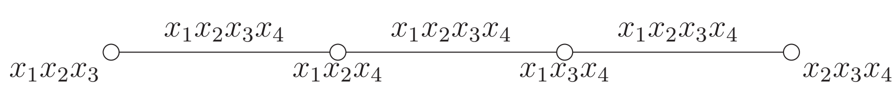

Figure 4.6: Another tree supporting the minimal free resolution of I

Then a natural follow-up question would be: How about higher projective dimensions, e.g., projective dimension 2?

We [27] gave a negative answer to this question in projective dimension 2. Let I denote the path ideal of line graphs $I_t(L_n)$ or cycles $I_t(C_n)$. In [27], for $\mathrm{pd}(I) = 2$, we calculated the Betti numbers of I, which are given in the form $\beta_0 = \beta_1 = m$ and

$\beta_2 = 1$ for some $m \in \mathbb{Z}$. Using the well-known Kruskal-Katona theorem [20, 17] in combinatorics, which characterizes f-vectors (see Definition 4.1.13) of acyclic simplicial complexes, we found that there exist acyclic simplicial complexes that have the f-vector of the form $(m, m, 1)$, but none of them support the minimal free resolution of I. This means that I does not have a minimal simplicial resolution.

Definition 4.1.13. Let Δ be a simplicial complex and $r = \dim(\Delta)$. The vector $f = (f_0, f_1, \dots, f_r)$, where f_i denotes the number of i-dimensional faces of Δ, is called the *f-vector* of Δ.

Example 4.1.14. Let $I = (xy, yz, zw, wu)$. We know from [27] that the Betti diagram of I is given as follows.

	4	4	1
1	-	-	-
2	4	3	-
3	-	1	1

Table 4.2: Betti diagram of I

If the minimal free resolution of I is supported on a simplicial complex Δ, then the f-vector of Δ would be $(4, 4, 1)$. It is not hard to check that below is only possible form of Δ.

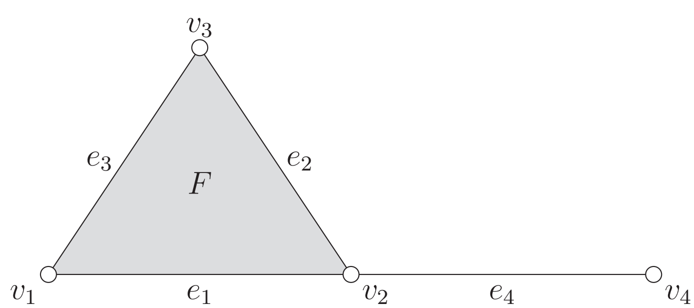

Figure 4.7: Acyclic Δ with f-vector $(4, 4, 1)$

Since $\beta_{2,5} = 1$, the 2-dimensional face F of Δ must be labeled by $xyzwu$, which is the only monomial of degree 5 in the lcm lattice of generators of I. It follows that two of the vertices v_1, v_2, v_3 are labeled by xy and wu, respectively. Then one can check that there is no way to label the third vertex without contradicting the graded Betti numbers. Hence, I does not have a minimal simplicial resolution.

4.2 Resolutions from matchings

Now we introduce a method to reduce the Taylor resolution of a monomial ideal. First consider the example below.

Example 4.2.1. Let $I = (xy, yz, zw)$. By Taylor [25], the 2-simplex Δ below supports a free resolution of I.

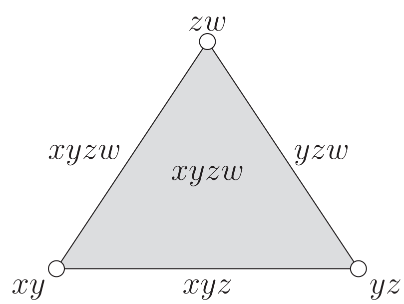

Figure 4.8: The 2-simplex Δ that supports a free resolution of I

In fact, I has a minimal simplicial resolution, but it is not supported on Δ. Instead, the minimal free resolution of I is supported on the below 1-dimensional simplicial complex Γ.

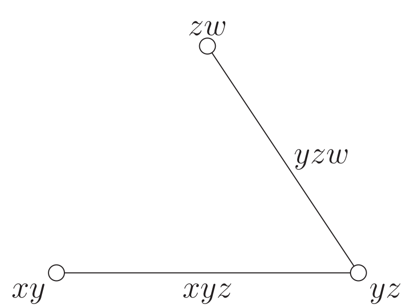

Figure 4.9: The 1-dimensional simplicial complex Γ

Notice that the facet and an edge of Δ have the same monomial label $xyzw$, so we may think of Γ as a smaller simplicial complex that is obtained from Δ by removing the facet and edge that are both labeled by $xyzw$.

In other words, we want to obtain a simplicial or CW complex that is smaller than the Taylor complex of I by "matching out" the faces that have the same monomial labels. This idea of reducing a simplex can be interpreted via a method in discrete homotopy theory, which is called the discrete Morse theory.

Now we start introducing the details with the definition of a matching in a graph.

Definition 4.2.2. Let $G = (V, E)$ be a graph and $M \subset E$. If no edges in M have common endpoints and M contains no loops, then M is called a *matching* in G.

Example 4.2.3. Consider the directed graph $G = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{(a, b), (b, c), (a, c), (c, a), (c, c), (b, d), (d, a)\}$.

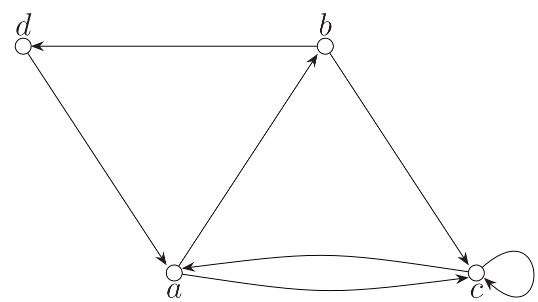

Figure 4.10: The picture of the digraph G

Then $\{(a, c), (b, d)\}$, $\{(d, a), (b, c)\}$, and $\{(a, b)\}$ are matchings in G.

Terminology 4.2.4 ([12])**.** Let $I = (m_1, \ldots, m_n)$ be a monomial ideal and X the Taylor complex of I, i.e., the simplex on the vertex set $\{m_1, \ldots, m_n\}$. We define G_X to be the directed graph with vertex set

$$V_X = \{\sigma : \sigma \in X\}$$

and edge set

$$E_X = \{(\sigma, \sigma') : \sigma' \subset \sigma \text{ and } \dim(\sigma) = \dim(\sigma') + 1\}.$$

Let M be a matching on G_X and define G_X^M to be the directed graph with the same vertex set V and edge set

$$E_X^M = (E_X - M) \cup \{(\sigma', \sigma) : (\sigma, \sigma') \in M\}.$$

We say that M is *homogeneous* if for all $(\sigma, \sigma') \in M$, we have $\mathrm{lcm}(\sigma) = \mathrm{lcm}(\sigma')$, where $\mathrm{lcm}(\sigma) = \mathrm{lcm}\{m : m \in \sigma\}$, and that M is *acyclic* if G_X^M contains no directed cycles. If M is acyclic, a cell $\sigma \in X$ is called M-*critical* if it does not appear in M.

A (directed) path from σ to σ' in the graph G_X^M is called a *gradient path*, denoted by $\mathrm{gp}_M(\sigma, \sigma')$.

We can obtain a cellular resolution of a monomial ideal I from homogeneous acyclic matchings by the following theorem.

Theorem 4.2.5 ([4]). *Let X be the Taylor complex of I and M a homogeneous acyclic matching in G_X. Then there exists a CW complex X_M that supports a free resolution of I. Furthermore, the n-cells (faces) of X_M are in one-to-one correspondence with the M-critical n-cells of X.*

Below lemma provides us with the inclusion relation among the cells of X_M, which is not covered by Theorem 4.2.5.

Lemma 4.2.6 ([4]). *Let σ'' and σ be two M-critical cells of X such that* $\dim(\sigma) = \dim(\sigma'') + 1$, *and σ''_M, σ_M be their corresponding faces of X_M. Then $\sigma''_M \leq \sigma_M$ if and only if (i) $\sigma'' \subset \sigma$, or (ii) there is a gradient path* $\mathrm{gp}_M(\sigma', \sigma'')$ *for some $\sigma' \subset \sigma$ with* $\dim(\sigma') = \dim(\sigma'')$.

Example 4.2.7. Let $I = (xy, yz, zw, wu)$ and X the Taylor complex of I. Then G_X is given by the graph below.

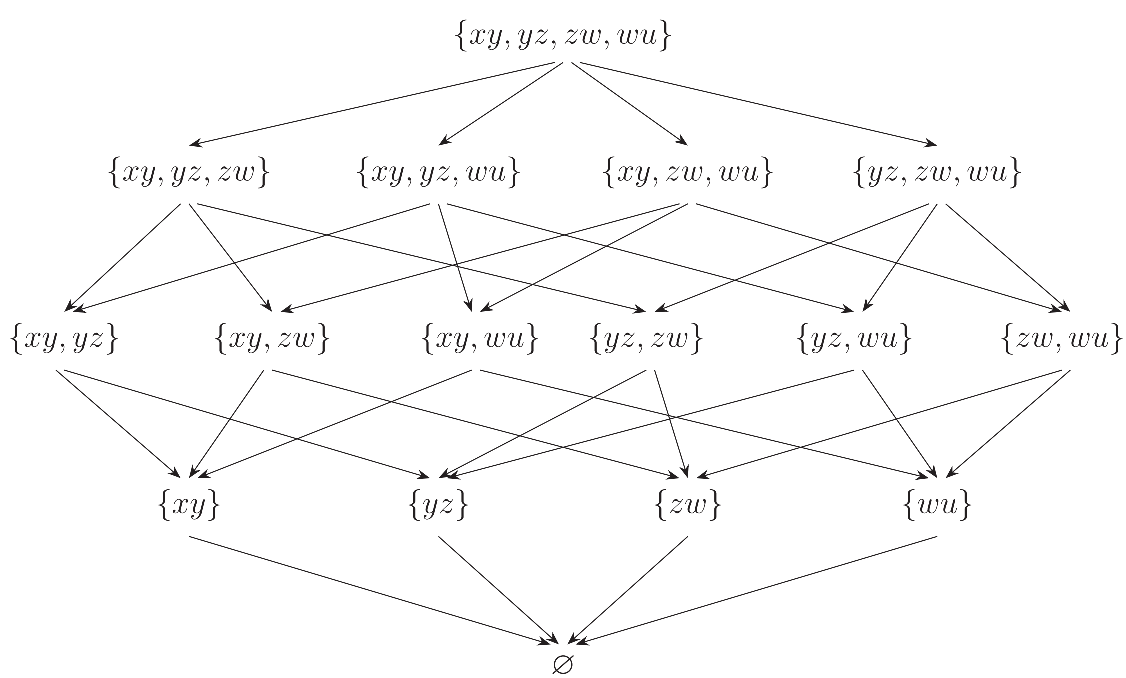

Figure 4.11: The graph of G_X

Consider the matching $M = \{(\{xy, yz, zw, wu\}, \{xy, zw, wu\}), (\{xy, yz, zw\}, \{xy, zw\}), (\{yz, zw, wu\}, \{yz, wu\})\}$ in G_X. Then G_X^M is given by the graph below.

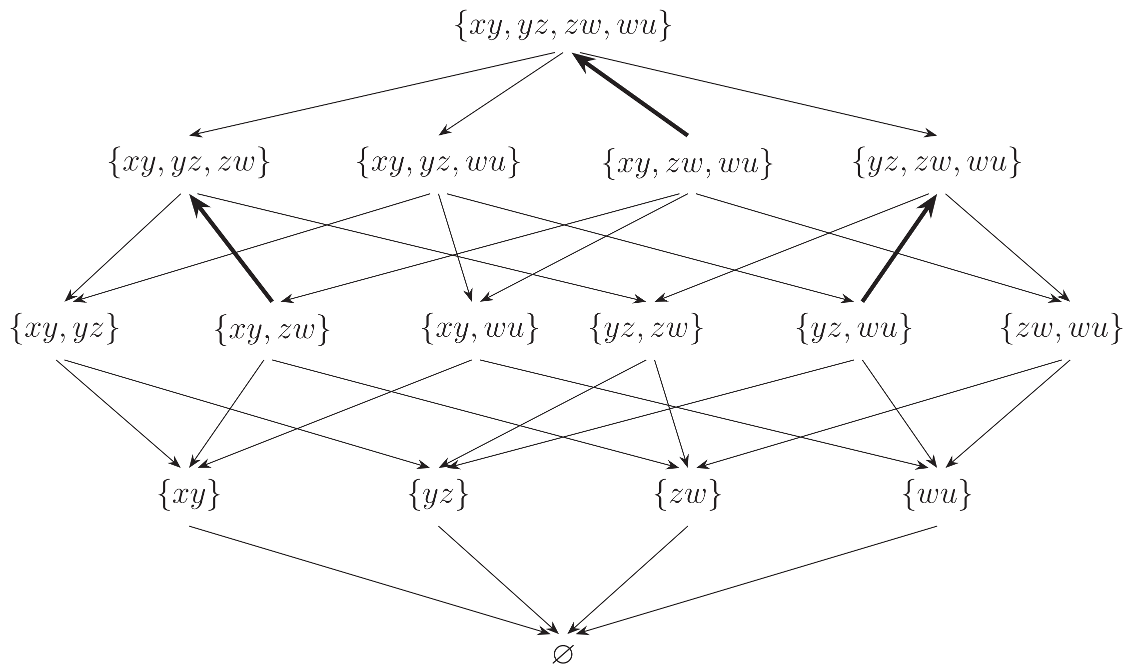

Figure 4.12: The graph of G_X^M

The M-critical cells are $\{xy, yz, wu\}, \{xy, yz\}, \{xy, wu\}, \{yz, zw\}, \{zw, wu\}, \{xy\}, \{yz\}, \{zw\}, \{wu\}, \varnothing$. For each M-critical cell σ, we denote its corresponding cell in X_M by σ_M. By Lemma 4.2.6, we have

$$\begin{aligned}
&\varnothing_M \leq \{xy\}_M, \{yz\}_M, \{zw\}_M, \{wu\}_M,\\
&\{xy\}_M \leq \{xy, yz\}_M, \{xy, wu\}_M,\\
&\{yz\}_M \leq \{xy, yz\}_M, \{yz, zw\}_M,\\
&\{zw\}_M \leq \{yz, zw\}_M, \{zw, wu\}_M,\\
&\{wu\}_M \leq \{xy, wu\}_M, \{zw, wu\}_M,\\
&\{xy, yz\}_M, \{xy, wu\}_M, \{yz, zw\}_M, \{zw, wu\}_M \leq \{xy, yz, wu\}_M.
\end{aligned}$$

We have $\{yz, zw\}_M \leq \{xy, yz, wu\}_M$ because of the gradient path $(\{yz, wu\}, \{yz, zw, wu\}, \{yz, zw\})$. By Theorem 4.2.5, I has a free resolution supported on a CW complex consisting of 4 vertices, 4 edges, and one 2-dimensional face with the above relation.

4.3 Homogeneous acyclic matching for $I_t(L_n)$

Definition 4.3.1. Let G be a simple graph with vertices $v_1, \dots, v_n$. The ideal

$$I_t(G) = (x_{i_1} \cdots x_{i_t} : (v_{i_1}, \dots, v_{i_t}) \text{ is a simple path in } G)$$

is called the *path ideal of G with length $t-1$*.

In this research, we focus on $I_t(L_n)$ with projective dimension 2. By [27], we have the following results.

Theorem 4.3.2 ([27])**.** *Let $I = I_t(L_n)$ with $t \geq 2$. If $\operatorname{pd}(I) = 2$, we have $n = 2t+1$. Moreover, the Betti diagram of I is given by:*

	$t+2$	$t+2$	1
$\vdots$	-	-	-
t	$t+2$	$t+1$	-
$\vdots$	-	-	-
$2t-1$	-	1	1

Table 4.3: Betti diagram of $I_t(L_{2t+1})$

Now we construct a homogeneous acyclic matching using the result that $n = 2t+1$ to induce the minimal cellular resolution of I.

Construction 4.3.3. Let $I = I_t(L_{2t+1})$ and X the Taylor complex of I. Let the generators of I be

$$m_1 = x_1 \cdots x_t, \quad m_2 = x_2 \cdots x_{t+1}, \dots, \quad m_{t+2} = x_{t+2} \cdots x_{2t+1}.$$

We impose the order $m_1 < m_2 < \cdots < m_{t+2}$ on the generators (i.e., $m_i < m_j \iff i < j$). For each $\sigma \in X$, we denote by $\sigma[j]$ the j-th smallest element in σ. For each $j \in \mathbb{Z}$, define

$$U_j = \{\sigma \in X : \dim(\sigma) = j\}.$$

Now we construct the matching M in G_X "from top to bottom".

Define

$$M_{t+1} = \{(\sigma, \sigma - \{\sigma[2]\}) : \sigma \in U_{t+1}\}.$$

For each $2 < j < t+1$, define

$$T_j = \{\sigma \in U_j : (\tau, \sigma) \in M_{j+1} \text{ for some } \tau \in U_{j+1}\}$$

and

$$M_j = \{(\sigma, \sigma - \{\sigma[2]\}) : \sigma \in U_j - T_j\}.$$

Define

$$M_2 = \{(\sigma, \sigma - \{\sigma[2]\}) : \sigma \in U_2 - (T_2 \cup \{\{m_1, m_2, m_{t+2}\}\})\},$$

where T_2 is defined similarly as above.

Finally, we take

$$M = \bigcup_{j=2}^{t+1} M_j.$$

By construction, we have the following properties about M.

Remark 4.3.4. Let $(\sigma, \tau) \in M$ and write $\sigma = \{m_{i_1}, m_{i_2}, \ldots, m_{i_p}\}$, $\tau = \{m_{j_1}, m_{j_2}, \ldots, m_{j_q}\}$, where $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$.

(1) The set M is indeed a matching.

(2) We have $i_2 - i_1 = 1$.

(3) We have $j_2 - j_1 > 1$.

(4) We have $m_{i_1} = m_{j_1}$, i.e., $\sigma[1] = \tau[1]$.

In the above remark, (2) is true since otherwise we would have $(\sigma \cup \{m_{k+1}\}, \sigma) \in M$, i.e., σ becomes an endpoint of an element in M, instead of an initial point. Also by definition of M, (3) and (4) are true.

Lemma 4.3.5. *Let $i < j < k$. We have $m_j \mid \operatorname{lcm}\{m_i, m_k\} \iff k - i \leq t$.*

Proof. Assume $m_j \mid \operatorname{lcm}\{m_i, m_k\}$. If $j > i + t - 1$. Then for each $\ell = j, \ldots, j+t-1$, we have $x_\ell \nmid m_i$. It follows by our assumption that $x_\ell \mid m_k$, i.e., $m_j \mid m_k$, which contradicts that $m_1, \ldots, m_{t+2}$ minimally generates I. Hence, $j \leq i + t - 1$. We have

$$j < i + t \leq j + t - 1 \implies x_{i+t} \mid m_j \xrightarrow{x_{i+t} \nmid m_i} x_{i+t} \mid m_k \implies k \leq i + t, \text{ i.e., } k - i \leq t.$$

Assume conversely that $k-i \leq t$, i.e., $i \geq k-t$. Then for each $\ell = j, \ldots, j+t-1$, we have

$$i < j \leq \ell \leq j+t-1 < k+t-1 \implies i < \ell < k+t-1.$$

If $\ell \leq i+t-1$, then $x_\ell \mid m_i$. If $\ell > i+t-1$, i.e., $\ell \geq i+t$, by our assumption,

$$\ell \geq k-t+t = k \implies x_\ell \mid m_k.$$

Therefore, we have either $x_\ell \mid m_i$ or $x_\ell \mid m_k$, which means that $m_j \mid \mathrm{lcm}\{m_i, m_k\}$. $\square$

Proposition 4.3.6. *The matching M in Construction 4.3.3 is homogeneous.*

Proof. Let $(\sigma, \sigma - \{\sigma[2]\}) \in M$ and write $\sigma = \{m_{i_1}, m_{i_2}, \ldots, m_{i_p}\}$, where $i_1 < i_2 < \cdots < i_p$. By Remark 4.3.4(2), $i_2 - i_1 = 1$. Further, it follows by Lemma 4.3.5 that

$$\mathrm{lcm}(\sigma) = \mathrm{lcm}(\sigma - \{\sigma[2]\}) \iff i_3 - i_1 \leq t.$$

By the structure of $I_t(L_{2t+1})$, we observe that $\sigma = \{m_1, m_2, m_{t+2}\}$ is the only face of X such that $i_2 - i_1 = 1$ and $i_3 - i_1 > t$, but it does not appear in our matching.

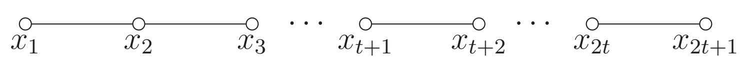

Figure 4.13: The graph of L_{2t+1}

$\square$

Proposition 4.3.7. *The matching M in Construction 4.3.3 is acyclic.*

Proof. By construction of M, the only possible form of directed cycles in G_X^M is shown in Figure 4.14 because we cannot have consecutive edges in G_X^M pointing upward by definition of a matching.

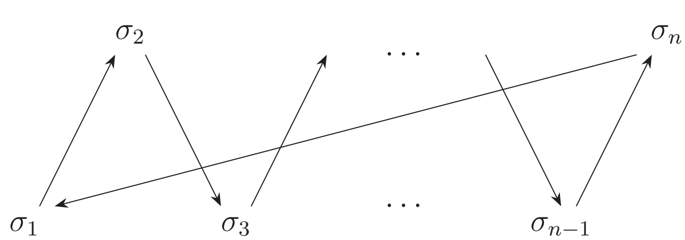

Figure 4.14: The only possible form of a directed cycle in G_X^M

Towards a contradiction, assume that there is a cycle in G_X^M of the above form. Write $\sigma_2 = \{m_{i_1}, m_{i_2}, m_{i_3}, \ldots, m_{i_p}\}$, where $i_1 < i_2 < \cdots < i_p$ and $i_2 - i_1 = 1$. Then $\sigma_1 = \{m_{i_1}, m_{i_3}, \ldots, m_{i_p}\}$ and $\sigma_3 = \sigma_2 - \{m_{i_j}\}$, where $1 \leq j \leq p$ and $j \neq 2$. Furthermore, $\sigma_1 \subset \sigma_n$.

If $m_{i_1} \in \sigma_3$, since $m_{i_2} \in \sigma_3$, we get a contradiction (see Remark 4.3.4(3)). Thus, $m_{i_1} \notin \sigma_3$ and $\sigma_3[1] = m_{i_2} = \sigma_4[1]$. We will prove that $\sigma_{2k}[1] \geq m_{i_2}$ for all $2 \leq kn/2$ by induction on k.

The base case is already shown. Now assume that the $\sigma_{2k}[1] \geq m_{i_2}$ for some $2 \leq k < n/2$. Since $\sigma_{2k+1} \subset \sigma_{2k}$, we have $\sigma_{2k+1}[1] \geq \sigma_{2k}[1] \geq m_{i_2}$. By Remark 4.3.4(4), $\sigma_{2k+2}[1] = \sigma_{2k+1}[1] \geq m_{i_2}$.

$$\begin{tikzcd}
\cdots & \sigma_{2k} \arrow[dr] & & \sigma_{2k+2} \\
\cdots \arrow[ur] & & \sigma_{2k+1} \arrow[ur] &
\end{tikzcd}$$

Therefore, $\sigma_n[1] \geq m_{i_2} > m_{i_1}$. However, $m_{i_1} \notin \sigma_n$ contradicts that $\sigma_1 \subset \sigma_n$. Hence, the matching M is acyclic. $\square$

4.4 CW complex for $I_t(L_n)$

Before we prove the results about the M-critical cells of X, we need a supplementary lemma.

Lemma 4.4.1. *For any $2 \leq k \leq t$, G_X^M contains a gradient path from $\{m_2, m_{t+2}\}$ to $\{m_k, m_{k+1}, m_{t+2}\}$.*

Proof. We prove this lemma by an induction on k. First, we have the gradient path

$$(\{m_2, m_{t+2}\}, \{m_2, m_3, m_{t+2}\}),$$

which proves the base case. Now assume that the lemma is true for some $2 \leq k \leq t-1$, i.e., we have a gradient path $\mathrm{gp}_M(\{m_2, m_{t+2}\}, \{m_k, m_{k+1}, m_{t+2}\})$. Connecting this path with $(\{m_{k+1}, m_{t+2}\}, \{m_{k+1}, m_{k+2}, m_{t+2}\})$ gives us a gradient path from $\{m_2, m_{t+2}\}$ to $\{m_{k+1}, m_{k+2}, m_{t+2}\}$, which completes the induction step.

$$
\begin{tikzcd}
 & \{m_2, m_3, m_{t+2}\} \arrow[dr] & \cdots & \{m_k, m_{k+1}, m_{t+2}\} \arrow[dr, very thick] & & \{m_{k+1}, m_{k+2}, m_{t+2}\} \\
\{m_2, m_{t+2}\} \arrow[ur] & & \cdots \arrow[ur] & & \{m_{k+1}, m_{t+2}\} \arrow[ur, very thick] &
\end{tikzcd}
$$

$\square$

Proposition 4.4.2. *(a) Let* $I = I_t(L_{2t+1}) = (x_1 \cdots x_t, x_2 \cdots x_{t+1}, \dots, x_{t+2} \cdots x_{2t+1})$ *and* X *the Taylor complex of* I*. Let* M *be the homogeneous acyclic matching built from Construction 4.3.3. Denote* $x_1 \cdots x_t, x_2 \cdots x_{t+1}, \dots, x_{n-t+1} \cdots x_{2t+1}$ *by* $m_1, m_2, \dots,$ m_{t+2}*, respectively. Then the* M*-critical cells of* X *are*

$$
\begin{aligned}
&\varnothing, \\
&\{m_1\}, \{m_2\}, \dots, \{m_{t+2}\}, \\
&\{m_1, m_2\}, \{m_2, m_3\}, \dots, \{m_{t+1}, m_{t+2}\}, \{m_1, m_{t+2}\} \\
&\{m_1, m_2, m_{t+2}\}.
\end{aligned}
$$

(b) For each M*-critical cell* σ*, we denote its corresponding cell in* X_M *by* σ_M*. Then we have*

$$
\begin{aligned}
&\varnothing_M \le \{m_1\}_M, \{m_2\}_M, \dots, \{m_{t+2}\}_M, \\
&\{m_1\}_M \le \{m_1, m_2\}_M, \{m_1, m_{t+2}\}_M, \\
&\{m_2\}_M \le \{m_1, m_2\}_M, \{m_2, m_3\}_M, \\
&\vdots \\
&\{m_{t+2}\}_M \le \{m_1, m_{t+2}\}_M, \{m_{t+1}, m_{t+2}\}_M, \\
&\{m_1, m_2\}_M, \{m_2, m_3\}_M, \dots, \{m_{t+1}, m_{t+2}\}_M, \{m_1, m_{t+2}\}_M \le \{m_1, m_2, m_{t+2}\}_M.
\end{aligned}
$$

Proof. (a) Let S denote the set of the claimed M-critical cells. We want to show that every cell σ of X is unmatched if and only if $\sigma \in S$. By construction of M, we already know that all elements in S are unmatched. Let σ be an n-cell of X that is not in S and write $\sigma = \{m_{i_1}, m_{i_2}, \dots, m_{i_n}\}$, where $i_1 < i_2 < \cdots < i_n$.

If $i_2 - i_1 = 1$, since $\sigma \neq \{m_1, m_2, m_{t+2}\}$, we have $(\sigma, \sigma - \{m_{i_2}\}) \in M$.

If $i_2 - i_1 > 1$, we have $(\sigma \cup \{m_{i_1+1}\}, \sigma) \in M$.

By construction, σ is matched by M in either case.

(b) By Lemma 4.2.6(i), we have

$$
\begin{aligned}
&\varnothing_M \leq \{m_1\}_M, \{m_2\}_M, \ldots, \{m_{t+2}\}_M, \\
&\{m_1\}_M \leq \{m_1, m_2\}_M, \{m_1, m_{t+1}\}_M, \\
&\{m_2\}_M \leq \{m_1, m_2\}_M, \{m_2, m_3\}_M, \\
&\vdots \\
&\{m_{t+2}\}_M \leq \{m_1, m_{t+2}\}_M, \{m_{t+1}, m_{t+2}\}_M, \\
&\{m_1, m_2\}_M, \{m_1, m_{t+2}\}_M \leq \{m_1, m_2, m_{t+2}\}_M.
\end{aligned}
$$

It remains to show that $\{m_2, m_3\}_M, \ldots, \{m_{t+1}, m_{t+2}\}_M \leq \{m_1, m_2, m_{t+2}\}_M$ by Lemma 4.2.6(ii).

Now consider any $2 \leq k \leq t$. By Lemma 4.4.1, G_X^M contains a gradient path $\mathrm{gp}_M(\{m_2, m_{t+2}\}, \{m_k, m_{k+1}, m_{t+2}\})$. Since $\{m_k, m_{k+1}\} \leq \{m_k, m_{k+1}\}$, this gradient path can be extended to $\{m_k, m_{k+1}\}$.

$$
\begin{tikzcd}
& \{m_2, m_3, m_{t+2}\} \arrow[dr] & \cdots & \{m_k, m_{k+1}, m_{t+2}\} \arrow[dr, thick] & \\
\{m_2, m_{t+2}\} \arrow[ur] & & \cdots \arrow[ur] & & \{m_k, m_{k+1}\}
\end{tikzcd}
$$

By Lemma 4.2.6(ii), we have $\{m_k, m_{k+1}\}_M \leq \{m_1, m_2, m_{t+2}\}$. Moreover, Lemma 4.4.1 is also true for $k = t$ in particular, i.e., there is a gradient path

$$\mathrm{gp}_M(\{m_2, m_{t+2}\}, \{m_t, m_{t+1}, m_{t+2}\}),$$

which can be extended to $\{m_{t+1}, m_{t+2}\}$.

$$
\begin{tikzcd}
& \{m_2, m_3, m_{t+2}\} \arrow[dr] & \cdots & \{m_t, m_{t+1}, m_{t+2}\} \arrow[dr, thick] & \\
\{m_2, m_{t+2}\} \arrow[ur] & & \cdots \arrow[ur] & & \{m_{t+1}, m_{k+2}\}
\end{tikzcd}
$$

Hence, we also have $\{m_{t+1}, m_{t+2}\}_M \leq \{m_1, m_2, m_{t+2}\}_M$ by Lemma 4.2.6(ii). $\square$

Based on the information of the critical cells, we naturally conjecture that the CW complex X_M in Proposition 4.4.2 is a solid $(t+2)$-gon (see Figure 4.15). Although we cannot prove that X_M is precisely the polygon we claimed it is, this conjecture provides us with the clue to a CW complex that supports the minimal free resolution of $I_t(L_{2t+1})$.

The reason that we mention "minimal" is that the monomial labels of Figure 4.15 totally agree with the Betti numbers of $I_t(L_{2t+1})$. Therefore, with the inspiration of our conjecture, we come up with Theorem 4.4.3, which we will prove using Theorem 4.1.5.

Theorem 4.4.3. *The minimal free resolution of the path ideal $I = I_t(L_{2t+1})$ is supported on a solid $(t+2)$-gon.*

Proof. Let $I = I_t(L_{2t+1}) = (m_1, m_2, \ldots, m_{t+2})$, where

$$\begin{aligned} m_1 &= x_1 \cdots x_t, \\ m_2 &= x_2 \cdots x_{t+1}, \\ &\vdots \\ m_{t+2} &= x_{t+2} \cdots x_{2t+1}. \end{aligned}$$

Let X be a $(t+2)$-gon with vertices labeled by $m_1, m_2, \ldots, m_{t+2}$ (see below).

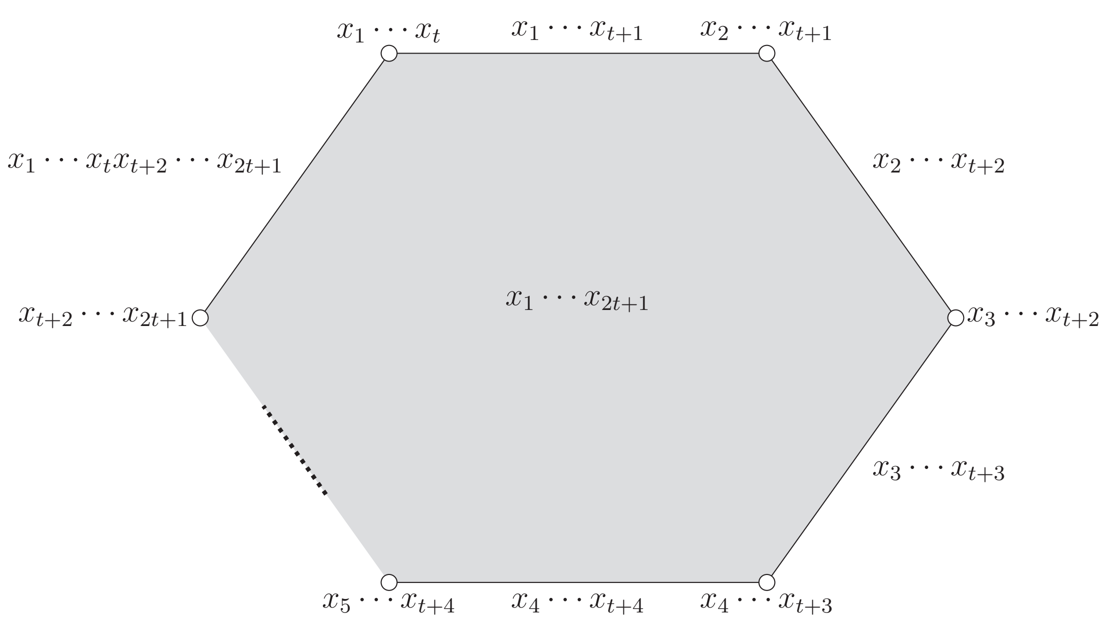

Figure 4.15: The solid $(t+2)$-gon labeled by the generators of $I_t(L_{2t+1})$

Let m be any monomial in the lcm lattice of $m_1, \ldots, m_{t+2}$. In our case, $X_{\leq m}$ has three possible forms: (i) a single vertex, (ii) a line L_n $(2 \leq n \leq t+1)$, and (iii) the entire complex X, because by the structure of $I_t(L_{2t+1})$, we have

$$X_{\leq \operatorname{lcm}\{m_i, m_j\}} = \begin{cases} L_{|i-j|+1} & |i-j| < t+1, \\ L_2 & |i-j| = t+1. \end{cases}$$

In other words, $X_{\leq m}$ is always connected.

A single vertex is cyclic, which is clear. Also, cases (ii) and (iii) are covered in Lemma 3.3.6. Therefore, we conclude that $X_{\leq m}$ is acyclic for all monomials $m \in S$. From the labeling of X, we also have $m_F \neq m_G$ for all faces F, G of X with $G < F$. By Theorem 4.1.5, X supports the minimal free resolution of I. $\square$

Chapter 5

Conclusion

Let $I = I_t(L_{2t+1})$, i.e., a path ideal of line graph with projective dimension 2. In summary, we have found a homogeneous acyclic matching in the graph G_X, which by discrete Morse Theory, gives us a CW complex X that supports a free resolution of I. Since by Proposition 4.4.2, the number of faces of X matches the Betti number of I in each dimension, we conclude that our matching gives us the minimal cellular resolution of I, i.e., the Morse resolution cannot be smaller.

Furthermore, based on Proposition 4.4.2, which tells us some properties of the structure of the induced Morse complex X, we conjectured that X is a solid $(t+2)$-gon. Using Bayer and Sturmfels' criteria (see Theorem 4.1.5), we finally proved that the $(t+2)$-gon from our conjecture indeed supports the minimal free resolution of I.

As a result of this research, we conclude that path ideals of line graph with projective dimension 2 have a minimal cellular resolution supported on a solid polygon.

Recall from [27] that path ideals of line graphs and cycles with projective dimension 2 do not have a minimal simplicial resolution. We wanted to know whether these two classes of monomial ideals have a minimal cellular resolution. Now we have solved one part of this problem, while the other part remains unsolved, i.e., the minimal cellular resolution of path ideals of cycles. Since we know from [27] that $I_t(C_n)$ of projective dimension 2 has Betti numbers $n, n, 1$, if the minimal free resolution of $I_t(C_n)$ is supported on a CW complex X, then X has n vertices, n edges, and 1 2-dimensional face. Thus, we can make a reasonable conjecture that the minimal cellular resolution of $I_t(C_n)$ is also supported on a solid polygon (i.e., n-gon). For example, one can prove the below lemma using Theorem 4.1.5.

Lemma 5.0.1. *The minimal free resolutions of $I_2(C_4)$ and $I_2(C_5)$ are supported on a solid square and pentagon, respectively.*

However, the case for cycles is not simple, because when having projective dimension 2, the parameters n and t of $I_t(C_n)$ are much more flexible than those of $I_t(L_n)$.

In particular, we need to consider n such that $t+2 \leq n \leq 2t+1$ [27]. This makes it much harder to check acyclicity of $X_{\leq m}$ in Theorem 4.1.5. If we approach this problem using discrete Morse theory, the homogeneous acyclic matching for $I_t(C_n)$ is also hard to build, because the patterns of the matching are difficult to observe due to the flexibility of n and t. Then it would be worthwhile to try some known algorithms (e.g., [1] and [10]) to construct the desired matching for $I_t(C_n)$. We are currently working on an algorithm based on Barile-Macchia resolution [3] to find a Morse matching that induces the minimal cellular resolution for general $I_t(L_n)$ and $I_t(C_n)$ [11] and hoping to resolve the difficulties that are previously mentioned.

Moreover, for $I = I_t(L_{2t+1})$, there is no reason to believe that the solid polygon we proposed in Theorem 4.4.3 is the unique CW complex (up to homotopy equivalence) that supports the minimal free resolution of I, because two different CW complexes can support the same free resolution (recall Example 4.1.12). For further research, it would also be interesting to explore other CW complexes that support the minimal free resolution of $I_t(L_{2t+1})$.

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