THE ROBUST ASSET-LIABILITY MANAGEMENT PROBLEM UNDER RETURN AND INTEREST RATE UNCERTAINTY

by

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To my family and supervisors,

Thank you for your unwavering support and guidance.

This thesis is dedicated to you.

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Abstract

This dissertation centers around the robust asset-liability management (ALM) problem, comprising four interconnected articles. The first article conducts a comprehensive review of robust portfolio selection problems, categorizing and analyzing decades of work. The subsequent articles delve into ALM issues confronted by pension funds when uncertain parameters like asset returns and interest rates arise.

The second article introduces a novel mathematical model featuring a Worst-case Conditional Value-at-Risk (WCVaR) constraint, ensuring a high-probability funding ratio above regulatory thresholds. A tractable reformulation is devised, employing Worst-case Lower Partial Moment (WLPM) and a data-driven moment-based ambiguity set. Tested using real-world Canada Pension Plan data, the model excels in handling correlated uncertainties.

Conversely, the third article employs distributionally-robust optimization with various ambiguity sets: mixture-distribution, box, and Wasserstein-distance. These sets encapsulate asset return and liability uncertainty, offering tractable reformulations for the ALM problem. Numerical experiments, including the CPP dataset, showcase improved funding ratios and asset allocation.

The fourth article explores the K-adaptability problem, proposing a solution method through logic-based Benders decomposition, addressing min-max-min robust combinatorial optimization. An iterative algorithm handles solutions and adverse scenarios. Extensions encompass uncertain constraints and nonlinear functions, outperforming existing methods on benchmark instances.

In summary, this dissertation navigates the robust ALM problem with four interrelated articles. It surveys robust portfolio selection literature, introduces innovative models for ALM under uncertainty, employs distributionally-robust optimization with various ambiguity sets, and proposes solutions for the K-adaptability problem. Empirical validation through real-world and benchmark data consistently highlights the advantages of these methodologies.

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Chapter 1

Introduction

Financial institutions such as pension funds and insurance companies bear the crucial responsibility of prudently managing substantial amounts of assets. With the aim of ensuring long-term financial sustainability, decision-makers in these institutions face the complex task of striking a delicate balance between assets and liabilities. This challenge is addressed through the practice of ALM, which involves the optimal allocation of available funds to different assets to cover existing and future liabilities. The primary objective of the ALM problem is to minimize the contribution of employees while effectively managing the financial obligations of the institution (Zenios, 1995). This entails covering current and future liabilities, such as pension payments to retirees or insurance claims, while also adhering to regulatory requirements (Bodie et al., 1988). ALM strategies take into account various factors, including the investment horizon, risk tolerance, market conditions, and regulatory constraints. By strategically aligning assets and liabilities, financial institutions can mitigate risks associated with market fluctuations, interest rate changes, and unexpected events, thereby safeguarding the interests of their stakeholders (Holzmann, 2013).

For pension funds, in particular, ALM assumes heightened significance due to the nature of their obligations. Pension funds typically operate under a plan called *defined-benefit plan*, which guarantees specific paybacks to retirees based on predetermined formulas (Bodie et al., 1988). Meeting these obligations requires careful management of investment portfolios to generate returns that are sufficient to cover future pension liabilities. Failure to effectively manage the asset-liability relationship can lead to funding gaps, potential insolvency, and the inability to fulfill promised retirement benefits (Blake, 2003). Therefore, pension funds must employ sophisticated ALM techniques to optimize asset allocation, considering factors such as return expectations, risk tolerance, and the duration and magnitude of pension liabilities.

Pension funds play a significant role in the global financial landscape, managing a substantial portion of global assets. As of the end of 2021, pension funds worldwide controlled assets worth over $60.6 trillion, accounting for approximately 33% of global assets [1]. Remarkably, the assets held by pension funds in nine out of the 38 Organisation for Economic Co-operation and Development (OECD) countries exceeded their respective Gross Domestic Products (GDPs) [2]. This highlights the substantial financial influence and economic significance of pension funds in these nations.

Over the past decade, pension assets have witnessed considerable growth, increasing by an average of 5.7% from 2010 to 2020 [3]. This growth rate outpaced the global GDP growth rate of 2.6% during the same period [4]. This upward trajectory underscores the growing importance of retirement savings globally. With an aging population and individuals reaching retirement age, the outflows from pension funds to fulfill benefit obligations are accelerating. The ratio of total benefits paid from retirement savings plans to GDP varies across OECD countries, ranging from 0.5% to 8% [5]. These statistics illustrate the substantial financial commitment required to meet the retirement needs of individuals and the impact this has on the overall economy.

Given the magnitude of pension assets and their crucial role in securing retirees' financial well-being, ALM practices are of paramount importance. However, pension funds face the critical challenge of strategically investing the contributions they receive in order to generate sufficient returns and meet their future obligations (Gülpinar & Pachamanova, 2013). These funds typically construct diversified portfolios comprising various asset classes, including fixed-income securities, public and private equities, commodities, real estate, and infrastructure. However, such investments are not without risks, as they are susceptible to fluctuations in asset prices driven by market dynamics, sector-specific factors, and company-specific risks (Bogentoft et al., 2001).

The solvency of funded pension plans is greatly influenced by the assumptions made regarding expected returns and interest rates. These factors play a critical role in determining the funding status of pension plans and their ability to meet future

[1] https://www.thinkingaheadinstitute.org/research-papers/global-pension-assets-study-2022/

[2] https://www.statista.com/statistics/721151/average-growth-largest-pension-markets-worldwide/

[3] https://www.statista.com/statistics/721151/average-growth-largest-pension-markets-worldwide/

[4] https://www.macrotrends.net/countries/WLD/world/gdp-growth-rate

[5] https://www.oecd.org/finance/private-pensions/globalpensionstatistics.htm

obligations (Konstantin, 2018). This variability in factors of the ALM problem introduces challenges for pension managers when determining the optimal asset allocation strategy to adequately cover future liabilities (Gülpinar & Pachamanova, 2013). It highlights the importance of carefully assessing and incorporating the specific risk-return characteristics of different asset classes to mitigate the potential impact of investment performance on the fund's overall financial health. D'Addio et al. (2009) emphasized the substantial impact of uncertainty in asset returns on pension funds. This uncertainty underscores the need for a conservative approach to investment decisions, taking into account the potential variability and unpredictability of asset returns. Pension managers must carefully consider the level of uncertainty in asset returns and adopt risk management strategies that align with the long-term objectives and obligations of the pension fund.

Effectively managing uncertainty within the context of ALM is of utmost importance for institutions to make informed investment decisions and mitigate risk. Extensive research has been dedicated to developing models and methods that can quantify and address the various sources of uncertainty associated with the ALM problem. Notably, stochastic programming (SP) and robust optimization (RO) have emerged as the primary approaches in this domain. SP is a widely used approach in ALM, aiming to find an optimal solution that maximizes the expected value of the objective function while accounting for uncertainty. Several studies, such as those conducted in (Klaassen, 1997; Kouwenberg, 2001; Consigli, 2008; Duarte et al., 2017; Kopa et al., 2018; Barro et al., 2022), have successfully applied SP to ALM problems. However, it is important to note that SP requires knowledge of the distribution function of the random variables, which may not always be readily available. Furthermore, SP is a risk-neutral approach, meaning it does not provide protection against scenarios that turn out to be worse than expected. In some cases, SP solutions may also be infeasible for certain scenarios, posing practical challenges.

On the other hand, RO has emerged as an appealing method for addressing uncertainty in the context of ALM problems. Many references, including (Iyengar & Ma, 2016; Platanakis & Sutcliffe, 2017; Gülpinar & Pachamanova, 2013; Gülpınar et al., 2016), have proposed ALM models that incorporate RO to handle uncertainty. Compared to SP models, RO offers several advantages. It is a risk-averse approach that does not rely on explicit knowledge of the distribution function of uncertain parameters. By considering a range of possible scenarios and optimizing for the worst-case outcome, RO models aim to minimize the potential downside risk. However, a drawback of RO solutions is that they tend to be overly conservative. This conservatism can lead to decisions based on the worst-case scenario, resulting in higher opportunity costs for ALM problems. Researchers have recognized this trade-off and have been actively exploring methods to strike a balance between robustness and decision optimality. For those interested in delving further into the topic, references such as (Ben-Tal et al., 2009; Bertsimas et al., 2011; Gabrel et al., 2014; Ghahtarani et al., 2022) provide more comprehensive insights into the theory and applications of RO methods. These resources offer a deeper understanding of the techniques used to model uncertainty.

The existing literature on ALM optimization has explored RO and SP separately, but there are currently gaps in the literature that we address in this research. First: the existing literature overlooked the possibility of combining risk measures within the framework employed to handle uncertainty in ALM optimization, *e.g.,* SP or RO. This combination offers several advantages. Firstly, it allows for more comprehensive and accurate modeling of risks in pension fund management. The combination of risk measures and ambiguity of distribution functions of random variables enhances the decision-making process for asset allocation, considering both the uncertainty of returns and the associated risks. Secondly, it provides a more precise representation of the underlying probability distribution by utilizing an ambiguity set, which encompasses a range of possible distribution functions for random variables. This improved representation leads to better risk management and enhances the long-term financial stability of pension funds. Additionally, the integration of risk measures with uncertainty in ALM optimization yields more robust and reliable solutions, which are vital for ensuring the sustained financial health of pension funds.

Second: Despite the extensive availability of financial data, there is a notable research gap when it comes to applying distributionally robust optimization (DRO) approaches in ALM. While other ALM methods like SP and RO have been extensively explored, DRO has only recently gained attention in the literature. One reason for this is the inherent complexity of DRO models and their relatively recent development.

Nevertheless, DRO offers the potential to address certain limitations present in other ALM methods. For instance, it can mitigate the "optimizer's curse" associated with SP, where reliance on a single distribution estimate may lead to suboptimal outcomes (Smith & Winkler, 2006). Additionally, DRO provides means to explicitly incorporate uncertainty in the distribution of financial variables, which is particularly valuable in ALM problems where returns and interest rates are subject to significant uncertainty. By considering a range of possible distributions, DRO enables more robust decision-making and a more nuanced assessment of risk in ALM than deterministic and SP formulation of the ALM problem. This research direction holds promise for advancing the field and improving the management of assets and liabilities in various financial institutions.

Third: Additionally, in our research, we address the K-adaptability problem, which falls within the realm of adaptive robust optimization. This problem involves making preparations for K potential solutions under uncertain conditions and selecting the most suitable one once the uncertainty is resolved. An important application of this problem is the ALM model with binary decision variables. To tackle this problem, we propose a new approach specifically designed for cases with a linear objective and constraints, binary first-stage decision variables, second-stage objective uncertainty, and a polyhedral uncertainty set. To solve the K-adaptability problem, we employ a logic-based Benders decomposition technique. This approach allows us to handle the first-stage decisions in a master problem while transforming the Benders subproblem into a min-max-min robust combinatorial optimization problem. To solve the subproblem, we develop a double-oracle algorithm that iteratively generates adverse scenarios, determines recourse decisions, and assigns scenarios to K-subsets of the decisions by solving p-center problems.

1.1 Research themes

This dissertation explores four themes dealing with the ALM problem under uncertainty. Each theme is developed in a dedicated chapter. Theme 1 introduces a comprehensive critical review of the literature on the application of robust optimization in portfolio selection problems. Theme 2 develops the worst-case Conditional Value at Risk (WCVaR) for the ALM problem by using a moment-based ambiguity set. Theme 3 explores DRO formulations of the ALM problem. Theme 4 presents an algorithm for solving K-adaptability problem that can be used in the ALM problem with binary decision variables. In the remainder of this section, brief descriptions of the 4 themes are provided.

1.1.1 Robust Portfolio Selection Problems: A Comprehensive Review

In the past two decades, there has been a growing interest in robust portfolio selection problems (PSPs), leading to several attempts to review the literature in this field. One of the earliest reviews, conducted by Fabozzi et al. (2010), focused on the application of RO to basic mean-variance, mean-Value at Risk (mean-VaR), and mean-Conditional Value at Risk (mean-CVaR) problems. However, this review did not encompass newer variants of the problem such as robust index tracking, robust lower partial moment (LPM), robust mean absolute deviation (MAD), robust Omega ratio, and robust multi-objective PSPs. Scutellà & Recchia (2010) and Scutellà & Recchia (2013) also conducted reviews on robust mean-variance, robust VaR, and robust CVaR problems. However, similar to the previous review, they did not cover other types of robust PSPs such as ALM problems or risk-hedging PSPs. Likewise, Kim et al. (2014a) concentrated on worst-case formulations while neglecting other important classes, including relative robust models, robust regularization, net-zero alpha adjustment, and asymmetric uncertainty sets. Another review by Kim et al. (2018a) specifically focused on worst-case frameworks in bond portfolio construction, currency hedging, and option pricing. Although it briefly touched upon robust asset-liability management problems, log-robust models, and robust multi-period problems, it had a limited coverage of references in those areas. More recently, Xidonas et al. (2020) provided a comprehensive bibliographic review that categorized the literature and broadly covered the area of robust PSPs.

In this theme of research, our main focus is to provide a comprehensive classification of robust PSPs in multiple dimensions. The review methodology involves a systematic approach to gather and compile a comprehensive list of references for reviewing the literature on PSPs. The main steps of the review methodology are as follows. Compilation of keyword sets: Two sets of keywords were compiled. The first set included keywords related to financial problems. The second set comprised keywords related to RO. We conducted searches on prominent academic databases such as Scopus and Web of Science, as well as using the Google Scholar search engine. We explored all possible combinations of the first and second keyword sets to retrieve relevant references. Each retrieved reference was carefully examined to ensure its relevance to robust financial problems. We evaluated the alignment of each paper with the scope of the review. Whenever a relevant paper was identified, we extended our search by exploring the references cited within that paper and the papers that cited it, called backward search. The search process continued until no new references could be found, ensuring a thorough inclusion of relevant literature in the review.

Ultimately, we implemented a two-tier classification system for the papers. Initially, we classified them according to their relevance to specific financial problems. Within each class of PSPs, we further categorized the papers based on RO approaches, methods, uncertainty sets, uncertain parameters, and formulation choices. The main contribution of this review paper is the unique classification methodology, which sets it apart from other literature reviews. Notably, our review encompasses the most comprehensive range of financial problems compared to existing review papers. The main focus of this review paper is to identify the gaps present in the existing literature. The paper extensively discusses and analyzes multiple gaps, but it specifically highlights three significant gaps in the ALM problem. These gaps include the incorporation of risk measures and DRO formulation in ALM problems, the utilization of DRO in the context of ALM, and the creation of solution methods for ALM problems involving binary decision variables. This thesis thoroughly examines and tackles these gaps in a comprehensive manner. We address the following research questions in the first theme of the dissertation.

1. What are the main approaches and methodologies used in the literature to address robust PSPs?

2. What are the different types of risk measures and optimization objectives considered in robust PSPs?

3. What are the key factors and sources of uncertainty commonly incorporated in robust PSPs?

4. What are the recent advancements and developments in robust PSPs, including new variants, extensions, and applications?

5. What are the existing gaps and limitations in the literature on robust PSPs, and what are the potential areas for future research?

1.1.2 Worst-Case Conditional Value at Risk for Asset Liability Management: A Novel Framework for General Loss Functions

In this research theme, we address the ALM problem faced by financial institutions, specifically pension funds. Our goal is to prudently manage large amounts of assets and liabilities while striking a balance between minimizing contribution to the fund and controlling risk for long-term financial sustainability.

Our methodology involves integrating CVaR as a risk measure and leveraging DRO to handle distributional ambiguity in the ALM optimization process. We develop a theoretical framework that extends the concept of Worst-Case CVaR (WCVaR) to handle general loss functions encountered in the ALM problem. By incorporating WCVaR and considering the ambiguity of distribution functions, we aim to provide more realistic and robust solutions that enhance risk management and decision-making for pension funds.

To validate our methodology, we conduct empirical tests using real data from the Canada Pension Plan (CPP). This allows us to evaluate the effectiveness of our approach in managing risk and optimizing asset allocation in a practical setting. By leveraging the power of CVaR and DRO, we demonstrate the potential of our methodology to improve the long-term financial outcomes of pension funds.

Our contributions to the existing literature are twofold. Firstly, our study tackles the research gap by integrating risk measures with distributional ambiguity within the realm of ALM optimization. It is worth noting that WCVaR, being a risk measure, allows us to effectively address both risk and distributional ambiguity aspects. This integration enables a more comprehensive approach to risk management in pension fund management. Secondly, we extend the theoretical framework of WCVaR to handle general loss functions specific to the ALM problem. This extension opens up new possibilities for decision-making under uncertainty, not only in the ALM domain but also in other areas such as supply chain management and engineering design.

The findings of our study highlight the benefits of integrating risk measures and distributional ambiguity in ALM optimization. By considering both risk and distributional ambiguity, we can make more informed decisions on asset allocation, meet regulatory requirements, and improve long-term financial stability. Empirical tests using CPP data support the effectiveness of our methodology in managing risk and optimizing asset allocation for pension funds.

For practitioners in financial institutions, particularly pension funds, our methodology offers practical implications. By adopting our approach, they can enhance their risk management practices, optimize asset allocation strategies, and ensure compliance with regulatory requirements. The integration of risk measures and uncertainty modeling provides a more comprehensive framework for decision-making, leading to improved long-term financial outcomes. The following questions are considered:

1. How can risk measures and probability distribution ambiguity be incorporated into ALM to improve pension fund outcomes?

2. What are the computational challenges and techniques for implementing DRO with WCVaR in ALM?

3. How can the theoretical framework of WCVaR be extended for general loss functions in ALM, enabling realistic and robust decision-making under uncertainty?

4. How can DRO with WCVaR as a risk measure be applied to manage uncertainty in ALM with th Canada Pension Plan?

1.1.3 Distributionally Robust Asset Liability Management Problem

DRO is a relatively new approach that aims to minimize the impact of distribution ambiguity in ALM problems. Unlike other ALM methods like SP and RO, DRO considers a wider range of possible probability distributions, making the resulting investment strategy more robust to market changes. Despite the availability of financial data, the literature lacks sufficient exploration of DRO in ALM due to the complexity of DRO models and their recent development. However, DRO has the potential to overcome the limitations of SP and RO, offering explicit consideration of uncertainty in financial variables subject to significant fluctuations.

This theme proposes DRO formulations for ALM problems, which is the main mathematical modeling contribution in this chapter, exploring three approaches: mixture ambiguity sets with discrete scenarios, box ambiguity sets for discrete distribution functions, and Wasserstein metric ambiguity sets. The first approach, commonly used in PSPs, requires a large number of scenarios to represent complex distributions adequately. The second approach, employing box uncertain discrete distribution functions, offers a more flexible representation of uncertainty but may not capture a significant proportion of distribution functions. To address this limitation, the paper incorporates the Wasserstein metric into the ALM problem. We address the following research questions in this research theme.

1. How can mixture ambiguity sets effectively represent complex distributions in ALM?

2. What are the advantages and limitations of using box ambiguity sets for discrete distribution functions in ALM? How to address the limitation of capturing distribution functions?

3. How does incorporating the Wasserstein metric ambiguity sets enhance ALM's representation of uncertainty? What are the advantages of that?

4. How can applying DRO with real-world data contribute to a flexible and robust ALM framework?

1.1.4 A double-oracle, logic-based Benders decomposition approach to solve the K-adaptability problem

Recently, a new modeling approach called K-adaptability has been proposed as a conservative approximation for adaptive/adjustable robust optimization (ARO) with discrete recourse decisions. Rather than selecting any feasible recourse, K solutions are prepared in advance, and the best solution is chosen based on the realized parameter values. K-adaptability solutions are generally better than static RO solutions and are more acceptable to human users as they come from a small set of candidate solutions. This approach has been primarily explored for linear problems with polyhedral uncertainty sets, but it can be extended to other problem variants.

This theme of our research presents a new algorithm to solve the K-adaptability problem with binary or integer first-stage decisions. It employs a logic-based Benders algorithm to handle the first-stage decisions and an iterative double-oracle algorithm to generate worst-case scenarios and determine the optimal subset of solutions. The proposed approach demonstrates finite convergence when the uncertainty set is polyhedral. The algorithm is further extended to handle ARO problems with uncertainty in both stages and nonlinear functions. Extensive numerical experiments on benchmark problems highlight the computational superiority of the proposed approach compared to existing methods.

The contributions of this chapter include the development of an efficient solution method for the K-adaptability problem with binary or integer first-stage decisions. The proposed algorithm is capable of handling both affine and nonlinear functions and can be extended to handle uncertainty in both stages. The approach enjoys finite convergence and achieves computational superiority compared to existing solution methods. The findings of this study are supported by extensive numerical experiments on benchmark instances of various optimization problems.

The proposed algorithm for K-adaptability, which handles binary or integer first-stage decisions, has the potential to be applied to the ALM problem with binary decision variables, which involves making decisions regarding the allocation of assets and liabilities to ensure the long-term financial stability of an organization. By employing the logic-based Benders algorithm, the algorithm can effectively handle the complex decision-making process involved in ALM, specifically in determining the optimal allocation of assets under uncertainty.

For practitioners, the proposed methodology has several managerial implications. By employing the new approach, practitioners can effectively address optimization problems with discrete recourse decisions under parameter uncertainty. The ability to prepare multiple candidate solutions and select the best one based on realized values provides more flexibility and robustness in decision-making. The computational efficiency of the proposed algorithm allows practitioners to solve large-scale instances of the K-adaptability problem, enabling them to make more informed and optimized decisions in real-world applications. We address the following research questions in this research theme.

1. What is the computational efficiency of the logic-based Benders algorithm for handling first-stage decisions in K-adaptability problems?

2. Can the iterative double-oracle algorithm effectively handle uncertainty in both stages in K-adaptability problems?

3. How does the proposed algorithm compare to existing methods in terms of computational superiority and solution quality for benchmark problems, especially ALM problems?

Figure 1.1 depicts the interconnection among the proposed themes within the dissertation. As illustrated in the figure, the ALM problem under uncertainty can be bifurcated into two primary classes based on the nature of decision variables—continuous and discrete. Each class of the ALM problem can be effectively addressed through either a single-stage or two-stage RO/DRO methodology, the latter being referred to as the adaptive RO method. For the realm of continuous decision variables, the single-stage DRO approach can be employed, utilizing either moment-based or metric-based ambiguity sets. Notably, the single-stage RO ALM problems concerning continuous variables have already been explored in existing literature.

Conversely, the ARO formulation of the ALM problem incorporating discrete decision variables poses a challenging computational task. Consequently, we propose an innovative algorithm aimed at tackling the K-adaptability and MMMRCO versions of the ALM problem specifically in scenarios involving binary decision variables.

1.1.5 Dissertation outline

This dissertation follows a thesis-by-article format, consisting of four manuscripts. Two manuscripts have already been published, another two have been submitted for publication. Chapter 2 provides an extensive critical review of the literature on the subject of robust portfolio selection problems following the introductory Chapter

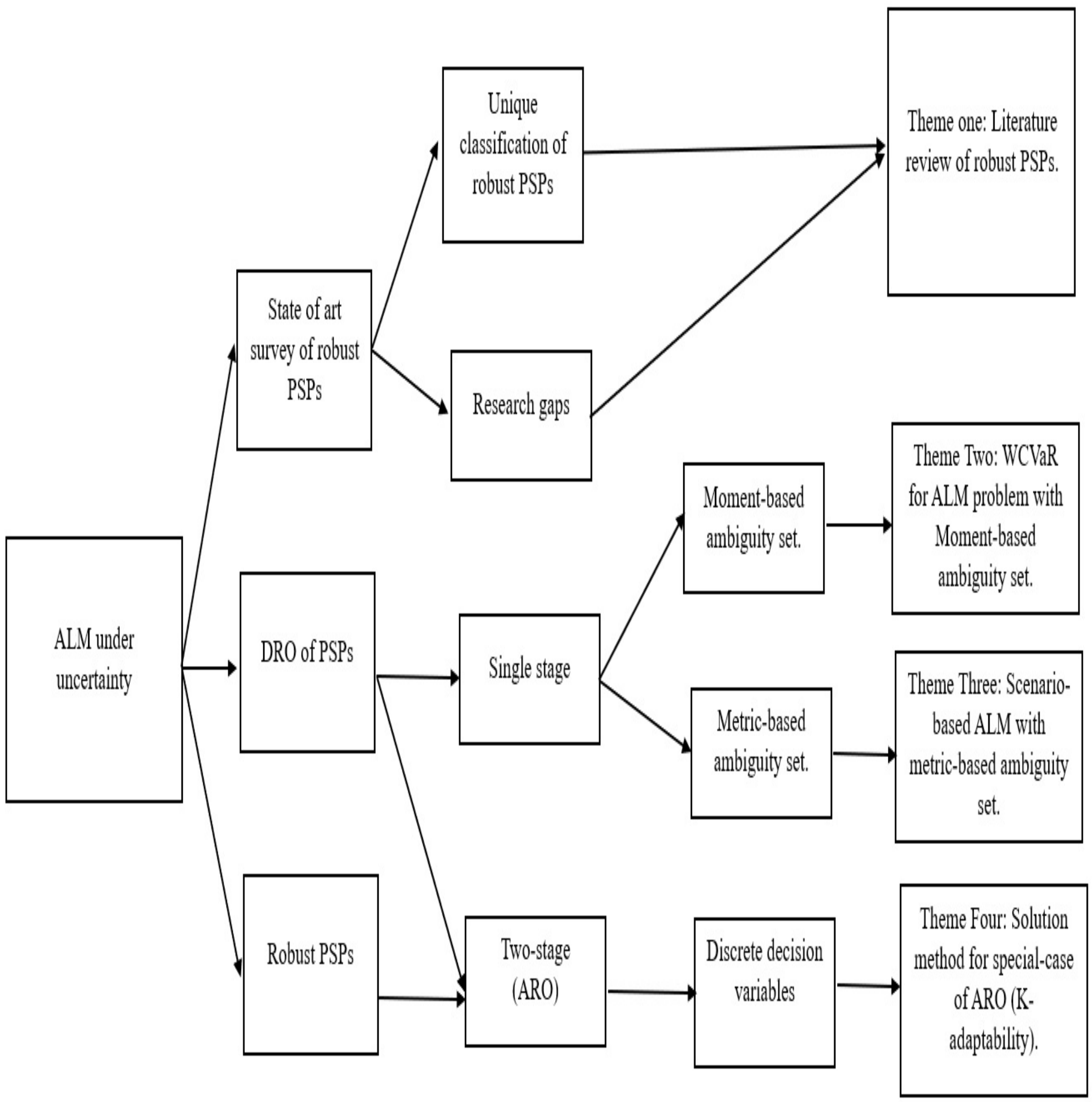

Figure 1.1: Connection between research themes

1. The content of Chapter 2 has been published in journal of *Operational Research* (Ghahtarani et al., 2022).

Chapter 3 introduces Worst-case Conditional Value at Risk for asset liability management: a novel framework for general loss functions. A preliminary version of this chapter was presented at the Canadian Operations Research Society (CORS) 2023. An extended version was submitted for publication in a peer-reviewed journal.

Chapter 4 introduces the distributionally robust asset-liability management problem. A manuscript resulting from this chapter has been submitted for publication in a peer-reviewed journal.

Chapter 5 presents a double-oracle, logic-based Benders decomposition approach to solve the K-adaptability problem. A manuscript resulting from this chapter has been published in the journal of *Computers & Operations Research* (Ghahtarani et al., 2023b).

In Chapter 6, the conclusions of this work are presented, highlighting significant observations and outlining future directions for further research based on the research outcomes.

Chapter 2

Robust Portfolio Selection Problems: A Comprehensive Review

2.1 Introduction

The PSP is a fundamental problem in finance that aims at optimally allocating funds among financial assets to maximize return and/or minimize risk. Different variants of the problem arise in reality due to the different risk attitudes of investors (risk-neutral *vs.* risk-averse), investment strategies, measures used to quantify risk (*e.g.,* variance, VaR), methods used to calculate return (*e.g.,* log-return) and planning horizon (single-period *vs.* multi-period), among other factors. Consequently, the PSP literature has grown considerably in terms of both size and diversity, allowing for several classification schemes to be employed.

An obvious classification is based on the risk measure to be optimized. Generally speaking, two broad classes of risk measures have been proposed: volatility-based and quantile-based. While variance, as the most popular volatility-based risk measure, has been the most widely-used risk measure in both theory and practice since the seminal work of Markowitz (1952), it has its deficiencies. First, it equally considers both positive and negative deviations around the expected return as undesirable risk, despite the desirability of the positive deviations for investors. Alternatively, downside risk measures that consider only the negative deviations of returns, like the lower partial moment (LPM), can be used. Furthermore, given that variance is a nonlinear risk measure, it leads to more complex formulations than those corresponding to linear risk measures like the mean absolute deviation (MAD) proposed by Konno & Yamazaki (1991). Related to volatility risk measures, Sharpe (1966) and Bernardo & Ledoit (2000), introduced *Sharpe ratio* and *Omega ratio*, respectively, to evaluate the performance of portfolios based on risk and return simultaneously. The most famous quantile-based risk measures are VaR and CVaR. The former quantifies the maximum loss at a specific confidence level, whereas the latter represents the expected value of losses greater than VaR at a confidence level. For details about quantile-based risk measures, interested readers are referred to Rockafellar et al. (2000).

Besides risk measures, PSPs can be classified based on investment strategies. For example, index tracking, first studied by Dembo & King (1992), is a passive investment strategy that tries to follow a market index. On the other hand, active investment strategies that involve ongoing buying and selling of assets are optimized by solving multi-period PSPs (see Dantzig & Infanger (1993) for an early example). Furthermore, hedging gives rise to a popular PSP in which an investment position is intended to offset potential losses or gains that may be incurred by a companion investment. Interested readers are referred to Lutgens et al. (2006) for a detailed account of financial hedging strategies. PSPs can be classified also according to return calculation methods. Goldfarb & Iyengar (2003) incorporated factors (macroeconomic, fundamental, and statistical) to determine market equilibrium and calculate the required rate of return, whereas Hull (2003) defined the *Log-return* as the equivalent, continuously-compounded rate of return of asset returns over a period of time.

Despite being a well-studied problem, a common feature of most PSPs addressed in the literature is that the problem parameters are assumed to be known with certainty. Ignoring the inherent uncertainty in parameters and instead using their point estimates often leads to suboptimal solutions. Two widely-used frameworks for dealing with uncertainty are *stochastic programming* (SP) and *robust optimization* (RO). SP focuses on the long-run performance of the portfolio by finding a solution that optimizes the expected value of the loss function. Despite its intuitive appeal and favorable convergence properties, SP requires the distribution function of the uncertain parameters to be known. Moreover, its risk-neutral nature does not provide protection from unfavorable scenarios, rendering it unsuitable for, typically, risk-averse investors. On the other hand, RO is a conservative approach that minimizes the loss function under the worst-case scenario (within an *uncertainty set*) and does not use information about the probability distribution of the uncertain parameters, making it an attractive alternative.

Given the rising interest in robust PSPs in the last two decades, several attempts have been made to review the growing robust PSP literature. Among the earliest reviews is that of Fabozzi et al. (2010), which concentrates on the application of RO on basic mean-variance, mean-VaR, and mean-CVaR problems, but does not cover more recent variants of the problem like robust index tracking, robust LPM, robust MAD, robust Omega ratio, and robust multi-objective PSPs. Scutellà & Recchia (2010) and Scutellà & Recchia (2013) also review robust mean-variance, robust VaR, and robust CVaR problems, but similarly, do not survey other robust PSPs. Likewise, Kim et al. (2014a) concentrate on worst-case formulations, while ignoring other important classes, including relative robust models, robust regularization, net-zero alpha adjustment and asymmetric uncertainty sets. Another review by Kim et al. (2018a) focuses on worst-case frameworks in bond portfolio construction, currency hedging, and option pricing, while covering a small number of references on robust asset-liability management problems, log-robust models, and robust multi-period problems. Recently, Xidonas et al. (2020) provided a categorized bibliographic review which broadly covers the area; their aim is to provide a rapid access to the topic for finance practitioners, and in general for those interested, but maybe not yet in the area.

The main contribution of this review paper is a multi-dimensional classification of robust PSPs. A dual-tier classification framework was established for this chapter. Initially, we organized papers based on their pertinence to distinct financial problems. Within each category of financial issues, a finer categorization was applied, taking into account various aspects such as robust optimization (RO) approaches, methods, uncertainty sets, uncertain parameters, and formulation. The principal innovation of this review paper lies in its distinct classification methodology, setting it apart from other literature reviews. Particularly noteworthy is our inclusion of an extensive spectrum of financial problems, making our review considerably more comprehensive than existing counterparts. The classification scheme of robust PSPs utilized in this review is illustrated in Figure 2.2. To search the literature, we first compiled two sets of keywords in both abbreviation and extension forms. The first set includes the following keywords related to financial problems: “portfolio selection”, “risk measures”, “VaR”, “CVaR”, “mean-variance”, “semi-variance”, “mean absolute deviation (MAD)”, “index tracking”, “factor-based portfolio”. The second set includes the robust optimization keywords: “robust optimization”, “distributionally robust optimization", "data-driven", and "uncertainty set". We then searched all pairs/combinations of the first and second keyword sets on both *Scopus* and *Web of Science databases*, and also using the *Google Scholar* search engine. The references retrieved from these searches were carefully screened to make sure that they are related to the robust financial problems. If a paper was deemed related to the scope of this review, we also conducted the backward references search. This process was repeated multiple times until no new references could be found. In addition to the primary sources, a selection of articles was also compiled from various supplementary sources, including pre-prints. Initially, a total of 369 papers were collected. Subsequently, after a thorough evaluation, 227 papers were excluded due to their lack of relevance to either robust optimization methods or financial issues. As a result of this screening process, a final count of 142 papers remained for the comprehensive review. Figure 2.1 shows the PRISMA flowchart of the review process.

Figure 2.3 portrays a breakdown of the reviewed articles by publication year, spanning between the years 2000 and 2021. We note that out of the 142 articles reviewed, 14 appeared in 2021, thus were not included in any of the previous reviews. Our review focuses on articles published in peer-reviewed journals. These articles appeared in a large number ($n = 54$) of finance and operations research (OR) journals. Figure 2.4 shows a breakdown of the reviewed papers by journal (sorted alphabetically). We note that most robust PSP articles were published in OR journals.

A major challenge when reviewing the robust PSP literature is the absence of a unified set of nomenclatures and notations for describing and formulating the problems. To be able to link and contrast different variants of the problem, we use, throughout our review, a unified set of most used notations, shown in Table 2.1. The notations that are used once are defined in the text. Our strategy for including mathematical formulas was to begin with the simplest and most general ones, then incrementally add new or alternative items (*e.g.,* terms in the objective function, constraints, risk measures, levels of optimization) at their first use in the robust PSPs literature. We also chose to include formulas that are commonly used and that constitute significant contributions, leaving behind some outliers and minor changes for brevity.

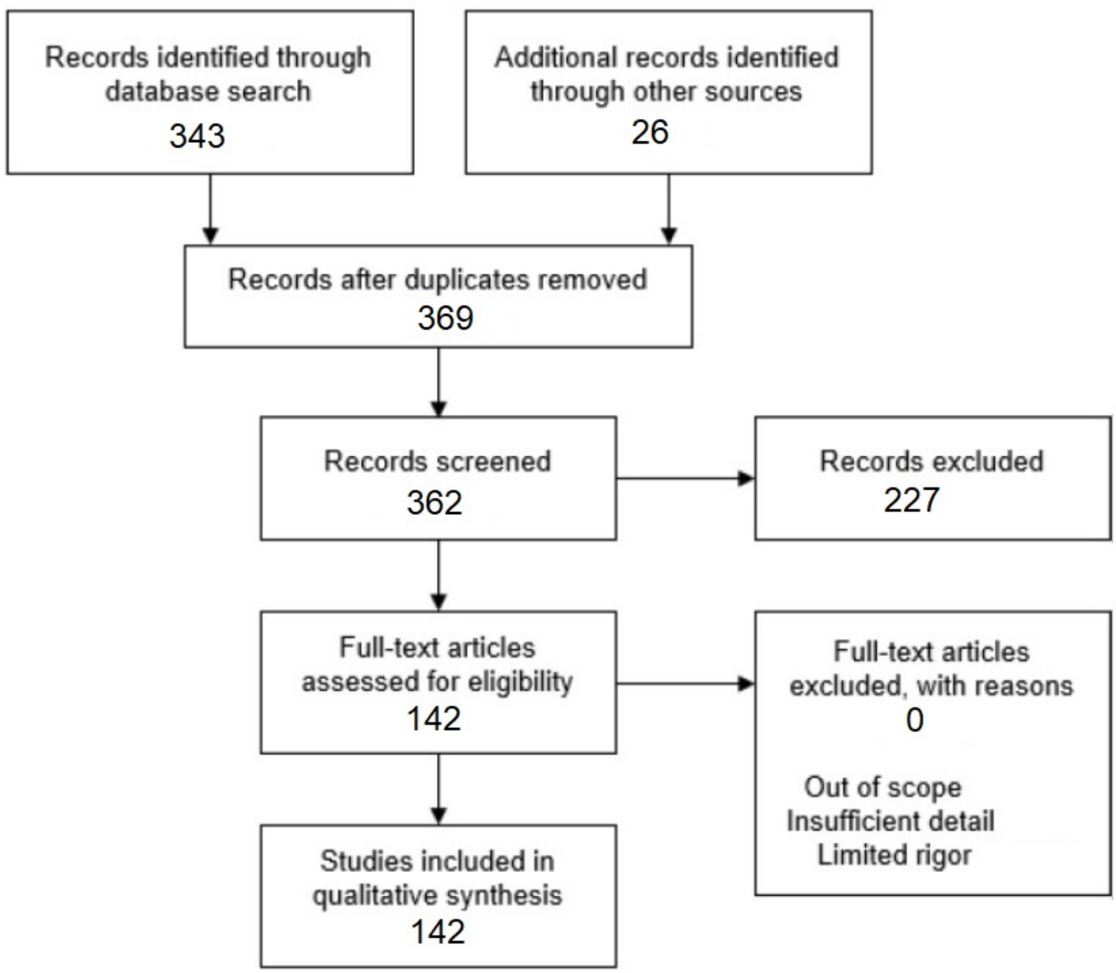

Figure 2.1: PRISMA flowchart of the review process

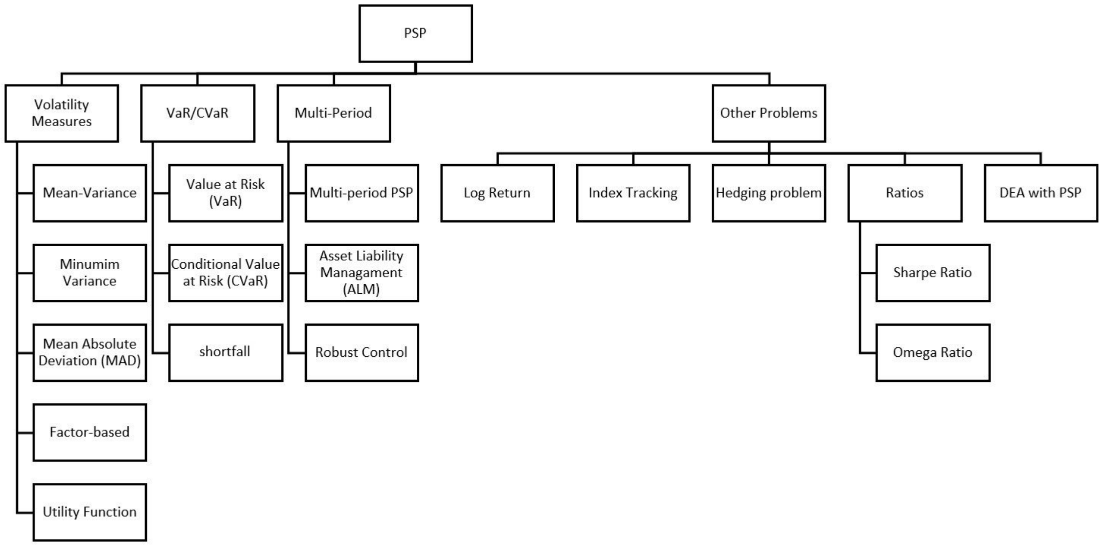

Figure 2.2: A schematic diagram of the classification scheme of robust PSPs review

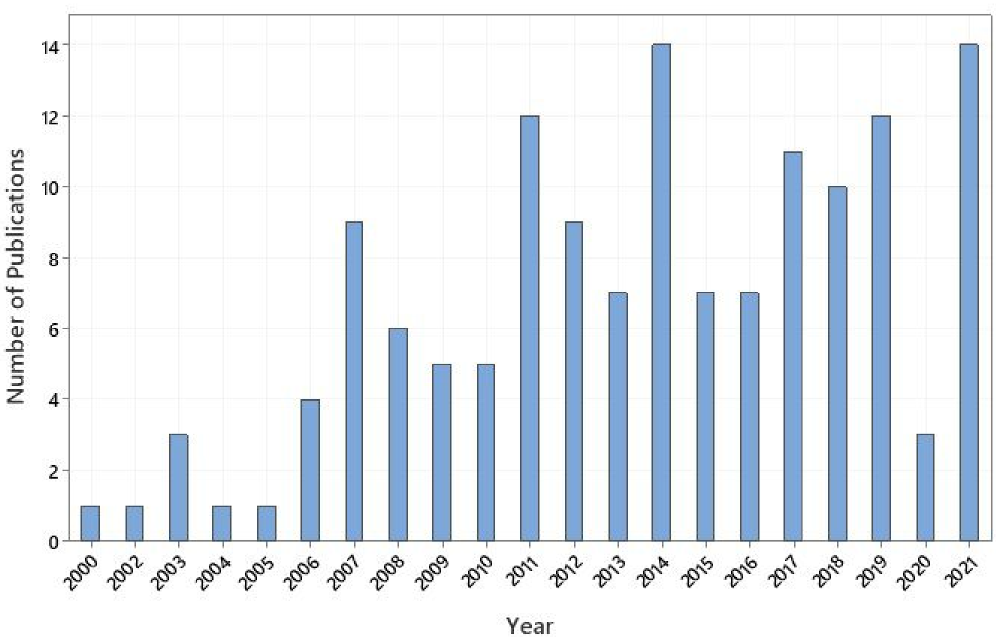

Figure 2.3: A breakdown of the reviewed article by publication year

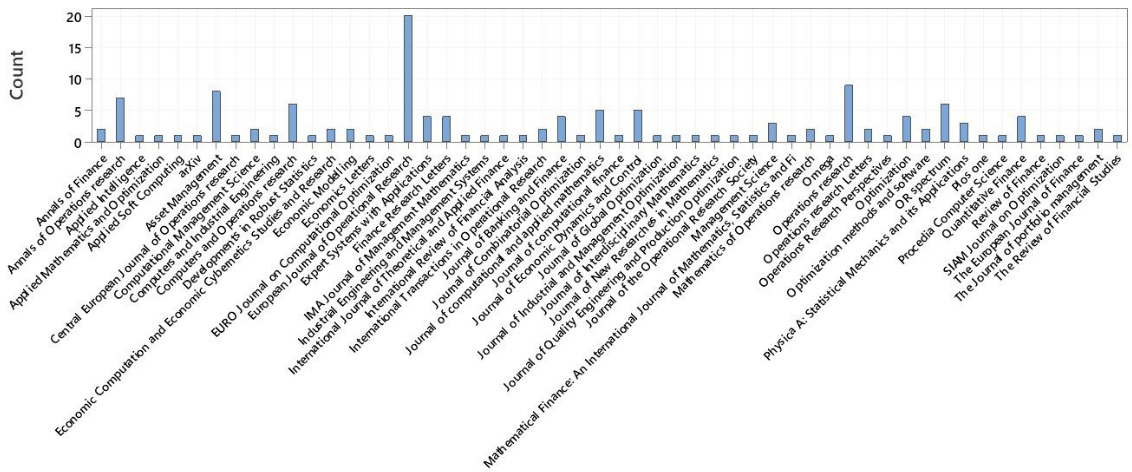

Figure 2.4: A breakdown of the reviewed articles by journal

Table 2.1: Notations and symbols

Symbol/Notation	Definition
$x \in \mathbb{R}^n$	Decision variable, x_j the proportion of the available budget invested in asset j
$r \in \mathbb{R}^n$	Asset return vector
$r^L \in \mathbb{R}^n$	Minimum asset return vector
$r^U \in \mathbb{R}^n$	Maximum asset return vector
$r_f \in \mathbb{R}$	Risk-free asset return
$Q \in \mathbb{R}^{n \times n}$	Variance-covariance matrix of the assets
$Q^L \in \mathbb{R}^{n \times n}$	Minimum variance-covariance matrix of the assets
$Q^U \in \mathbb{R}^{n \times n}$	Maximum variance-covariance matrix of the assets
$E \in \mathbb{R}$	Portfolio expected return
$v \in \mathbb{R}$	Portfolio variance
e	Vector of size n whose components are ones
$\lambda \in \mathbb{R}$	Risk aversion coefficient
U_Q	Uncertainty set of Q, which has the same dimensions of the uncertain parameter

Table 2.1: Notations and symbols

Symbol/Notation	Definition
U_r	Uncertainty set of r, which has the same dimensions of the uncertain parameter
$\Gamma \in \mathbb{R}^+$	Non-negative scalar that controls the size of uncertainty set
$Pr_j \in \mathbb{R}$	Price of asset j
$E_i \in \mathbb{R}$	Exchange rate of currency i
$\Sigma \in \mathbb{R}^{n \times n}$	Covariance matrix of the estimated expected returns
π	Nominal distribution function
p	True distribution function
ζ	A random variable
$A \in \mathbb{R}^{n \times n}$	A positive semi-definit matrix
η_i	A positive scalar
t	Indices of scenarios
$k_{min} \in \mathbb{R}^n$	Lower bound of decision variables
$k_{max} \in \mathbb{R}^n$	Upper bound of decision variables
c_j	Binary variable, if the asset j is selected it takes one, otherwise zero
$L \in \mathbb{Z}$, $H \in \mathbb{Z}$	Integer scalars that show the minimum and maximum number of assets in the portfolio
s	Indices of period
$W_0 \in \mathbb{R}$	Initial wealth of the investors
$\beta \in \mathbb{R}$	Confidence level
VaR	Value at Risk
CVaR	Conditional Value at Risk
$f(x, r)$	Loss function
$\Delta \in \mathbb{R}$, $\nu \in \mathbb{R}$	Transaction costs (buying and selling)

The remainder of this chapter is organized as follows. The next section provides a brief introduction to RO for non-specialists. Section 2.3 surveys robust PSP formulations based on volatility measures. Section 2.4 reviews quantile-based PSPs, which include Value at Risk (VaR), Conditional Value at Risk (CVaR), and their extensions with worst-case RO methods, relative RO and distributionally robust optimization (DRO). Furthermore, the relationship between uncertainty sets and risk measures, application of soft robust formulation with risk measures, worst-case CVaR and its relationship with uniform investment strategy, and robust arbitrage pricing theory with worst-case CVaR are also discussed in Section 2.4. Section 2.5 provides a review of RO in multi-period PSPs and asset-liability management (ALM) problems. Besides these two main problems, robust control formulations of investment problems are reviewed in this section. Section 2.6 reviews other financial problems that are not covered in the above-mentioned categories like robust log-return, index-tracking, hedging problem, risk-adjusted Sharpe ratio, robust scenario-based formulation, and robust data envelopment analysis. The last section provides conclusions and open issues in this context.

2.2 A Brief Introduction to Robust Optimization

This section provides a brief introduction to RO for readers who are not familiar with the topic. RO is a framework for dealing with the uncertainty of parameters in optimization problems by assuming that the parameters belong to an *uncertainty set* and optimizing over the worst realization in this set. The first RO formulation was developed by Soyster (1973) and used a *box* (hypercubic) uncertainty set that specifies an interval for each individual uncertain parameter. Even though this approach usually leads to tractable formulations, it is too conservative since it is based on the assumption that all parameters will take their worst possible values simultaneously, which rarely happens in reality. To overcome this issue, Ben-Tal & Nemirovski (1998) proposed an *ellipsoidal* uncertainty set that is centered at some nominal value and has a size (radius) that controls the conservatism of the solution based on the decision maker's aversion to uncertainty. Nevertheless, tractable reformulations of RO problems with ellipsoidal uncertainty sets give rise to nonlinear formulations that, generally, have a higher complexity than the nominal problem. Later, Bertsimas & Sim (2004) developed a special class of polyhedral uncertainty set, referred to as *budget*, that enables the level of conservatism to be controlled while preserving the tractability of the reformulated problems. All of the aforementioned uncertainty sets are *symmetric*, meaning that they are based on the assumption that forward and backward deviations around the nominal value are equal. Chen et al. (2007) argued that this assumption is not valid in many practical settings and proposed an *asymmetric* uncertainty set that is particularly suitable for financial applications.

Despite the protection it provides against adverse scenarios, classical RO is still considered overly conservative and pessimistic by many practitioners. To alleviate this concern, several RO variants have been developed. Scherer (2007) proposed adding a net-zero alpha adjustment constraint to any uncertainty set to guarantee that for any downward adjustment in the uncertain parameter, there is an offsetting upward adjustment, thus reducing the level of conservatism. Kouvelis & Yu (1997) proposed a *relative* RO approach that uses a regret function under the least desirable scenario. Although this approach provides solutions that perform better, on average, than classical RO, it suffers from intractability since it results in a three-level optimization problem.

Another way to achieve less conservative solutions is to use available partial information about the distribution function of the uncertain parameters rather than completely overlook them. A framework referred to as *distributionally robust optimization* (DRO), that dates back to the seminal work of Scarf (1958), has gained considerable attention recently. It assumes that the unknown probability distribution of the uncertain parameters belongs to a set of distributions called the *ambiguity set*, and optimizes the expected value of the objective function, where the expectation is taken with respect to the worst distribution in this set. Clearly, the tractability, convergence and out-of-sample performance guarantee offered by the DRO solution obtained depends on the ambiguity set used. Generally speaking, there are two main types of ambiguity sets: moment-based and discrepancy-based. The former includes distributions that enjoy some parametric properties, *e.g.,* mean or variance, whereas the latter include distributions that are within a certain "distance" (*e.g.,* the *Kullback–Leibler divergence* or the *Wasserstein metric*) from a reference distribution. The interested reader is referred to Rahimian & Mehrotra (2019); Esfahani & Kuhn (2018); Delage & Ye (2010) and the references therein for more information about DRO.

2.3 Robust PSPs with Volatility-based Risk Measures

In this section, we review the application of RO in PSPs with volatility-based risk measures, which include mean-variance, mean absolute deviation, lower partial moment, systematic risk, Omega ratio, and factor-based portfolio models.

2.3.1 Mean-Variance PSP

The mean-variance PSP was proposed by Markowitz (1952). In its general form, it assumes n risky assets, each has an expected rate of return denoted by the vector r, whereas v is the portfolio variance and Q is the variance-covariance matrix of the assets. In Markowitz's model, the variance of the portfolio is the risk measure to be optimized. The decision variable of this mathematical formulation is $[x_j]_{j=1,\dots,n}$, which represents the proportion of the available budget invested in asset j. When $x_j \geq 0$, it means that short selling is not allowed.

Moreover, $E = x^\intercal r$ is the portfolio expected return, $v = x^\intercal Qx$ is the portfolio variance, and E_0 is the minimum required expected rate of return. Then, the minimum variance PSP is $\min\limits_{x \in X}(v = x^\intercal Qx)$ s.t. $x^\intercal r \geq E_0$ where $X = \{x : \ e^\intercal x = 1, \ x_j \geq 0, \quad j = 1, ..., n\}$, and e is a vector of size n whose components are ones. Another version of the mean-variance PSP, called the risk-adjusted expected return, takes the form $\max\limits_{x \in X}(x^\intercal r - \lambda x^\intercal Qx)$. This formulation has the dual objectives of maximizing the portfolio return and minimizing its variance, where λ is a risk aversion coefficient set by the investor. In reality, however, the true values of the expected rate of return and the covariance matrix are not known with certainty. The general robust counterparts of the aforementioned PSPs are $\min\limits_{x \in X} \max\limits_{Q \in U_Q}(x^\intercal Qx)$ s.t. $\min\limits_{r \in U_r} x^\intercal r \geq E_0$, and $\max\limits_{x \in X} \min\limits_{r \in U_r,\, Q \in U_Q}(x^\intercal r - \lambda x^\intercal Qx)$, respectively, where U_Q and U_r are the uncertainty sets for the covariance matrix and the return vector, respectively. Tütüncü & Koenig (2004) used symmetric box uncertainty sets defined as $U_r := \{r : \ r^L \leq r \leq r^U\}$ and $U_Q =: \{Q : \ Q^L \leq Q \leq Q^U, Q \succeq 0\}$, where r^L and r^U are, respectively, the lower and upper bounds of the asset returns and Q^L and Q^U are the lower and the upper bounds of the covariance matrix elements while stipulating also that Q must remain positive semi-definite (PSD). It has been shown that, with these uncertainty sets, the robust counterparts could be tractably formulated as $\min\limits_{x \in X}(x^\intercal Q^U x)$ s.t. $x^\intercal r^L \geq E_0$, and $\max_{x \in X}(x^\intercal r^L - \lambda x^\intercal Q^U x)$, respectively. Khodamoradi et al. (2020) used similar uncertainty sets for a cardinality-constrained mean-variance PSP that allows short selling. Swain & Ojha (2021) also analyzed the robust mean-variance, and mean-semi-variance PSPs with box uncertainty sets, where both the expected return vector and the covariance matrix are uncertain parameters. However, Chen & Tan (2009) argued that deviations of the expected asset returns from their nominal values are not symmetric, meaning that the upside deviation is different from the downside deviation, thus are not accurately captured by classical symmetric uncertainty sets. Instead, they used non-symmetric interval uncertainty sets for the expected vector and covariance matrix of asset returns. The element-wise uncertainty interval was defined as $U_{r_i} = [\bar{r}_i - \theta_i^1, \bar{r}_i + \theta_i^2]$ and $U_{q_{ij}} = [\bar{q}_{ij} - \tau_{ij}^1, \bar{q}_{ij} + \tau_{ij}^2]$, where $\bar{r}_i$ and $\bar{q}_{ij}$ are elements of r and Q, respectively, that represent the nominal values of mean and covariance, whereas θ_i^1 and θ_i^2 are the downside and the upside deviations for the mean and τ_{ij}^1 and τ_{ij}^2 are the downside and the upside deviations for the covariance, respectively. To propose a robust counterpart, optimistic f_{opt} and pessimistic g_{pes} values are defined as $f_{opt} = \min_{r_i \in U_{r_i}}(x^\intercal r)$, and $g_{pes} = \max_{q_{ij} \in U_{q_{ij}}}(x^\intercal Q x)$, respectively. Alternatively, Fabozzi et al. (2007) defined an ellipsoidal uncertainty set for asset returns as $U_r := \{r : \; (r - \bar{r})Q^{-1}(r - \bar{r})^\intercal \leq \Gamma^2\}$, where $\hat{r}$ is the nominal return and Γ^2 is a non-negative scalar that controls the size of the uncertainty set. Hence, the robust counterpart can be tractably formulated as $\{\min_{x \in X} -\bar{r}^\intercal x + \Gamma\sqrt{x^\intercal Q x} + \lambda x^\intercal Q x\}$. However, the uncertainty of the covariance matrix was not considered, making the solution robust only against perturbations in the return vector. Pınar (2016) developed a robust mean-variance PSP with the same ellipsoidal uncertainty set while allowing short selling, which was also extended to the multi-period case.

Even though RO accounts for uncertainty in the problem parameters, Zymler et al. (2011) argued that if the uncertainty set is not set large enough, the solution might maintain its robustness only under normal market conditions, but not when the market crashes. Instead, they proposed using European-style options to hedge the mean-variance portfolios against abnormal market conditions. Two guarantee types were provided: weak, and strong. The weak guarantee applies under normal market conditions when the rate of the return is varying in an ellipsoid uncertainty set, whereas the strong guarantee applies to all possible asset returns by using the

European-style options in the form of constraints in the optimization problem. Hence, the strong guarantee is not based on RO formulation but on the mechanism of options to protect the portfolio in market crashes. Ashrafi & Thiele (2021) also used the idea of strong and weak guarantees. For the strong guarantee, an option is used in PSP, whereas for the weak guarantee, a budget uncertainty set for asset returns is used. Hence, the problem can be reformulation as a linear program (LP).

According to Lu et al. (2019), an important drawback of the mean-variance PSP is that its inputs are computed using only historical market returns, thus specific earnings announcements cannot be used to support the portfolio selection process. To overcome this issue, Black & Litterman (1990) proposed the BL method, which consists of both a market model and a view model. Lu et al. (2019) improved the view model of the BL method by using fuzzy logic to make it quantitative. Moreover, they incorporated multiple expert views instead of just one individual expert view in their formulation. To handle the heterogeneity of data collected from disparate sources, they applied RO with an ellipsoidal uncertainty set for the mean return vector and the return covariance matrix.

Fonseca & Rustem (2012) asserted that an important strategy in the PSP is diversification, which may eliminate some degree of risk since financial assets are less than perfectly correlated. One way to make a portfolio more diversified, investors can invest in foreign assets. However, foreign exchange rates' fluctuations may erode the investment's return. Moreover, both the asset returns and the currency rates are uncertain. Hence, Fonseca & Rustem (2012) and Fonseca et al. (2012) proposed a robust formulation for the international PSP with an ellipsoidal uncertainty set, which leads to a non-convex bilinear optimization problem. The problem considers n assets from m foreign currencies, where Pr_j^0 and Pr_j are, respectively, the current and future prices of asset j, and E_i^0 and E_i, respectively, are the current and future exchange rates of currency i. Therefore, the local return of asset j is $r_j^a = Pr_j/Pr_j^0$ and the exchange rate return of currency i is $r_i^e = E_i/E_i^0$. Using the auxiliary binary matrix $O = [o_{ji}]$, where o_{ji} equals 1 if asset j is traded in currency i and 0 otherwise, the international PSP is formulated as $\max_{x \in X} \min_{(r^a, r^e) \in U_{r^a, r^e}} [diag(r^a O r^e)]^\intercal x$, where the objective of this formulation is to maximize the worst-case return within all realizations in the uncertainty set U_{r^a,r^e}. A semi-definite programming (SDP) approximation is proposed to handle the non-linearity of the robust international PSP. Even though a robust international portfolio provides some level of guarantee against the uncertainty, investors might alternatively use forward contracts and quanto options (an exotic type of options translated at a fixed rate into another currency) to hedge risk. To make the formulation more practical, Fonseca & Rustem (2012) extended the robust international PSP with forward contracts and quanto options to reach a less conservative formulation.

Another classical uncertainty set used for robust PSPs is the budget uncertainty set proposed by Bertsimas & Sim (2004), which has the advantage of leading to tractable reformulations with the same complexity of the nominal problems. Sadjadi et al. (2012) considered a robust cardinality constrained mean-variance PSP with ellipsoidal, budget, and general-norm uncertainty sets and proposed a genetic algorithm to solve them. It was shown that using a budget uncertainty set has led to better rates of return compared to other uncertainty set types. Gregory et al. (2011) also tested a budget uncertainty set for the uncertain returns in a PSP to show the impact of the uncertainty set size on the portfolio return. The uncertainty set in this formulation is defined as $U_r = \{r : \; r = \bar{r} + \hat{r}\zeta, \; ||\zeta||_1 \leq \Gamma, \; |\zeta| \leq 1\}$, where $\bar{r}$ is nominal value, $\hat{r}$ is the deviation of return, ζ is the random variable, and Γ is the price of robustness that control the size of uncertainty set. The final formulation is $\max_{x \in X, \, z \geq 0, \, q \geq 0} \bar{r}^\intercal x - \Gamma z - e^\intercal q \; s.t. \; z + q_i \geq \hat{r}_i x_i, \; \forall i$. It has been shown that using the mean or the median of the asset returns as nominal values leads to the most robust portfolios.

Bienstock (2007) postulates that the solution methodology of RO is often chosen at the expense of the accuracy of the uncertainty model. Moreover, classical uncertainty sets might lead to overly conservative solutions. Alternatively, Bienstock (2007) proposed a data-driven approach to construct the uncertainty set by using uncertainty bands, each showing a different level of the return shortfall, which is an amount by which a financial obligation or liability exceeds the required amount of cash that is available. Hence, it is possible to specify rough frequencies of return shortfalls to approximate the return shortfall distribution. The robust models are formulated by allowing the uncertain parameter (asset returns) to deviate from the distribution by incorporating constraints related to the frequency of the return shortfall in different bands, an approach referred to as *robust histogram mean-variance* PSP. Although this formulation provides more flexibility than classical RO, the robust counterpart is an intractable mixed-integer program (MIP); thus, a cutting plane algorithm is proposed to solve it.

Fliege & Werner (2014) studied a robust multi-objective optimization (MOO) version of the mean-variance PSP while considering minimizing the variance and maximizing the mean return of the portfolio as the two objectives. Two MOO methods were applied: the ε-constraint scalarization (ECS) method, which pushes one of the objective functions, namely return maximization, to the constraints, and the weighted-sum scalarization (WSS) method, which combines the two objectives into a single one by assigning proper weights to them. Both methods lead to the same efficient frontier in the nominal case, but not for the robust problem. Fliege & Werner (2014) defined the location characteristics of the robust Pareto frontier with respect to the non-robust Pareto frontier, and demonstrated that standard techniques (ECS and WSS) from MOO can be used to construct the robust efficient frontier.

Alternatively, robustness in multi-objective PSPs can be achieved by a resampling method without classical uncertainty sets, which provides a wider range for uncertain parameters and solutions instead of the worst-case scenario RO with a specific uncertainty set. This approach requires replacing the parameters in the fitness functions at every generation. Hence, the evolution process would favor the solutions that show good performance in terms of risk and return over different scenarios (see *e.g.,* Shiraishi (2008), Ruppert (2014)). García et al. (2012) argued that one of the main problems portfolio managers face is uncertainty regarding the expected frontier derived from forecasts of future returns. Very often, expected frontiers lie far from the actual return, resulting in inaccurate forecasts of the portfolio risk/return profile. García et al. (2012) demonstrated that robustness of results can be achieved by avoiding optimization based on a single expected scenario that may produce solutions that are hyper-specialized and might be extremely sensitive to likely deviations. They tackle the problem of achieving robust or stable portfolios by using a multi-objective evolutionary algorithm that replaces the traditional fitness function with an extended one that uses a resampling mechanism and an implicit third objective to control the robustness of the solutions.

The formulations of Fliege & Werner (2014) and García et al. (2012) are based on differentiable functions. However, classical RO methods cannot be used on nonsmooth and non-differentiable functions. To address this issue, Fakhar et al. (2018) developed the necessary and sufficient optimality condition for a MOO problem with nonsmooth, *e.g.,* non-differentiable or discontinuous, functions, and proved that strong duality holds when these functions are convex. Moreover, they introduced the concept of saddle-point for MOO under uncertainty.

Ceria & Stubbs (2006) demonstrated that the mean-variance PSP is very sensitive to small variations in expected returns. They, instead, proposed a formulation for the robust PSP based on estimation errors. In this formulation, three distinctive Markowitz efficient frontiers were introduced: the true frontier calculated based on the true, yet unobservable, expected returns, the estimated frontier calculated based on the estimated return, and the actual frontier calculated based on the true expected returns of the portfolios on the estimated frontier. To have a portfolio as close as possible to its true frontier, the maximum difference between the estimated frontier and the actual frontier was minimized. Ceria & Stubbs (2006) modified the maximum difference between the estimated frontier and the actual frontier by adding a linear constraint. They assumed that the true returns lie inside the confidence region $(r-\bar{r})'\Sigma^{-1}(r-\bar{r}) \leq k^2$, where $k^2 \sim \chi_n^2$, and χ_n^2 is the inverse cumulative distribution function of the chi-squared distribution with n degrees of freedom. Points on the efficient frontier are calculated by solving $\max \bar{r}^\intercal x$ s.t. $x^\intercal Q x \leq v$, where v is the maximum acceptable variance. The optimal solution of this optimization problem is $x = \sqrt{\frac{\nu}{r^\intercal Q^{-1} r}} Q^{-1} r$. By considering r^* as the true, but unknown, expected return vector and $\bar{r}$ as an expected return, the true expected return of a portfolio on the estimated frontier is computed as $\sqrt{\frac{\nu}{\bar{r}^\intercal Q^{-1} \bar{r}}} r^{*\intercal} Q^{-1} \bar{r}$. Ceria & Stubbs (2006) assumed that $\tilde{x}$ is the optimal portfolio on the estimated frontier for a given target risk level. Then, the difference between the estimated expected return and the actual expected return of $\tilde{x}$ is $\bar{r}^\intercal \tilde{x} - r^{*\intercal} \tilde{x}$. Consequently, the maximum difference between the expected returns on the estimated efficient frontier and the actual efficient frontier is computed by solving $\max \ (\bar{r}^\intercal \tilde{x} - r^\intercal \tilde{x})$ s.t. $(r-\bar{r})^\intercal \Sigma^{-1} (r-\bar{r}) \leq k^2$. In this formulation $\tilde{x}$ is fixed and optimization is over r. Hence, the optimal solution is $r = \bar{r} - \sqrt{\frac{k^2}{\tilde{x}^\intercal \Sigma \tilde{x}}} \Sigma \tilde{x}$.

Moreover, the lowest value of the actual expected return is $\bar{r}^\intercal\tilde{x} = \bar{r}^\intercal\tilde{x} - k||\Sigma^{1/2}\tilde{x}||$. Finally, the maximum difference between the estimated frontier and the actual frontier is $\bar{r}^\intercal\tilde{x} - (\bar{r}^\intercal\tilde{x} - k||\Sigma^{1/2}\tilde{x}||) = k||\Sigma^{1/2}\tilde{x}||$. Effectively, minimizing the maximum difference between the actual and the true frontiers leads to a robust mean-variance PSP with an ellipsoidal uncertainty set, while the covariance matrix of estimation error is also captured using an ellipsoidal uncertainty set. Garlappi et al. (2007) also claimed that the estimation error is ignored in mean-variance PSPs. They propose a robust mean-variance PSP that is a special case of the PSP in Ceria & Stubbs (2006) since asset returns are assumed to be normally distributed. Multiple historical data sets are used to estimate the random variable of asset returns. The problem is formulated as $\max_{x\in X}\min_r x^\intercal r - \lambda x^\intercal Qx$ s.t. $f(r,\bar{r},Q) \leq \varepsilon$, where $\bar{r}$ is the estimated return, $f(.)$ is a vector-valued function, ε is a vector of constants that captures the investor's uncertainty- and ambiguity-aversion. The additional constraint, representing the confidence interval of the normal assets return, shows that the decision maker accepts the possibility of estimation error. Garlappi et al. (2007) compared their results for different $f(.)$ selections with the results of the traditional Bayesian models and showed that their models are risk-averse while the Bayesian models are risk-neutral towards the uncertainty in parameters.

Scherer (2007) analyzed the results and models of robust estimation error by Ceria & Stubbs (2006) and robust mean-variance with box uncertainty set by Tütüncü & Koenig (2004) and showed that the results of the robust mean-variance PSP are equivalent to the results of the mean-variance PSP with Bayesian shrinkage estimators for the uncertain parameters (for more details about Bayesian shrinkage estimators , see *e.g.,* Lemmer (1981)). The RO framework is criticized because it merely increases the complexity of the PSP while the solutions of the robust optimization, which depends on the choice of uncertainty set, are usually over-conservative. A method, referred to as *net-zero alpha adjustment*, is developed, by which adding a constraint to the uncertainty set ensures that for any downward adjustment in the uncertain vector, there is an offsetting upward adjustment. For example, with the uncertainty set $U = \{r = \bar{r}+\zeta : \zeta^\intercal\Sigma\eta \leq 1\}$, where Σ is the covariance matrix of estimation errors and ζ is a deviation vector, the constraint $e^\intercal\zeta = 0$ is added. Gülpınar et al. (2011) applied this method for a cardinality-constrained mean-variance PSP and found that adding a net-zero alpha adjustment to the ellipsoidal uncertainty set led to less conservative solutions than traditional robust mean-variance PSPs.

RO is a worst-case approach which assumes that the distribution function of uncertain parameters is unknown. However, partial information about the distribution function is often available, enabling less conservative distributionally robust optimization (DRO) formulations to be used. Several DRO models have been proposed for the mean-variance PSP.

Calafiore (2007) developed distributionally robust PSPs where two types of problems with different risk measures were addressed: the mean-variance PSP, which uses the mean and variance, and the mean absolute deviation PSP, which replaces the variance with the absolute deviation. Let us assume $r(1), ..., r(T)$ are T possible scenarios for the outcome of random return vector r, and p_t is the probability associated to the scenario $r(t)$, where $\{p_t \geq 0,\ t = 1, ..., T,\ \sum_{t=1}^{T} p_t = 1\}$. Then, the expected value is defined as $\mu(x,p) = \mathbb{E}[r^\intercal x] = \sum_{t=1}^{T} p_t r^\intercal(t)x = (\sum_{t=1}^{T} p_t r^\intercal(t))x = \bar{r}^\intercal(p)x$, where $\bar{r}(p) = \mathbb{E}[r] = \sum_{t=1}^{T} p_t r(t)$. A risk measure can be quantified as the variance: $v(x,p) = \mathbb{E}[(r^\intercal x - \mathbb{E}[r^\intercal x])^2] = x^\intercal Q(p)x$, where $Q(p)$ is the covariance matrix of r, and $Q(p) = \mathbb{E}[(r - \bar{r}(p))(r - \bar{r}(p))^\intercal] = \sum_{t=1}^{T} p_t(r(t) - \bar{r}(p))(r(t) - \bar{r}(p))^\intercal$. Another risk measure in this concept is the expected absolute deviation (EAD) $EDA(x,p) = \mathbb{E}[|r^\intercal x - E\{r^\intercal x\}|] = \sum_{t=1}^{T} p_t |r^\intercal(t)x - \mu(x,p)|$. By defining $\lambda \geq 0$ as a risk aversion ratio, then the mean-variance PSP is $\min_{x \in X} v(x,p) - \lambda\mu(x,p)$, and the PSP based on the absolute deviation measure is $\min_{x \in X} EDA(x,p) - \lambda\mu(x,p)$. A discrepancy-based ambiguity set based on the well-known Kullback-Leibler (KL) divergence, which measures the "distance" between a nominal distribution vector (π) and the unknown "true" distribution vector (p) is used, defined as $KL(p,\pi) = \sum_{t=1}^{T} p_t log \frac{p_t}{\pi_t}$. Then p is only known to lie within KL distance $d \geq 0$ from π, $K(\pi,d) = \{p :\ KL(p,\pi) \leq d\}$, where $K(\pi,d)$ is the ambiguity set for the return distribution. This ambiguity set leads to a SDP formulation for the mean-variance PSP that is solvable using interior-point methods. The distributionally robust absolute deviation PSP is convex in the decision variable for any given distribution function. Consequently, a sub-gradient method combined with a proposed cutting plane scheme was used to solve the worst-case mean absolute deviation PSP in polynomial-time. Baviera & Bianchi (2021) also applied KL divergence in the mean-variance PSP. However, unlike Calafiore (2007), they considered continuous distribution functions for the asset returns.

A limitation of Calafiore (2007) is that, while the probabilities of scenarios are uncertain, the scenarios themselves are assumed to be known with certainty, which is not always the case in reality. Pınar & Paç (2014) formulated a semi-deviation PSP while considering uncertainty in both asset returns (through an ellipsoidal uncertainty sets) and in the distribution function of returns (through a moment-based ambiguity set). Both single and multi-period cases were considered.

As Ding et al. (2018) argued, the Kullback–Leibler (KL) divergence used in Calafiore (2007) is a special case of Rényi divergence with order one, hence they used it in a more general DRO formulation of the mean-variance PSP. Besides the risky assets having a multivariate normal distribution, they considered a risk-free asset with a fixed rate of return r_f in their formulation. By allowing short selling and using E as a target average return of the portfolio, the problem is formulated as $\min_{x\in X} x^\intercal Qx$ s.t. $x^\intercal(r - r_f) \geq E - r_f$. In the nominal case, the empirical distribution, obtained from historical data, is used, assuming that $p \sim N(r, Q)$. Since there is ambiguity about the true distribution of returns, an ambiguity set is constructed that contains all distributions within a certain distance, measured using Rényi divergence, from the empirical distribution. Renyi divergence is defined as $D_r(p, \pi) = \frac{1}{r(r-1)} \ln \int p^r(\xi)\pi^{r-1}(\xi)d\xi,\ r \neq 0, 1$, where $\pi(\xi)$ and $p(\xi)$ are the probability density function under measures π and p, respectively. Hence, the final formulation of the distributionally robust PSP is $\max_{x\in X} \min_{r,Q}\ x^\intercal r - \lambda x^\intercal Qx + \lambda^\intercal D_r(p, \pi)$. Their model was solved in three cases: only the mean return vector is uncertain, only the covariance matrix is uncertain, and both are uncertain. It is worth mentioning that even though the ambiguity set used in Ding et al. (2018) is more general than the KL-divergence, their formulations are special cases from the distribution function perspective since both the empirical and the true distribution function of the asset returns are assumed to be multivariate normal.

Hauser et al. (2013) suggested that some professionals such as investment managers are frequently evaluated against their competitors and not on the absolute terms (worst-case solutions). The relative RO is the best possible approach to handle this situation where a regret function (the distance to the "winner" under the least desirable scenario) is used to propose an intractable three-level optimization problem. Hauser et al. (2013) incorporated a relative robust formulation into the mean-variance PSP where the regret function is $Rgrt_{U,B}(x) = \max_{Q\in U} l_B(x,Q) = \max_{Q\in U}(\sqrt{x^\intercal Qx} - \min_{b\in B}\sqrt{b^\intercal Qb})$, x is the decision variable, Q is the variance-covariance matrix, U is an uncertainty set, and $B = b_1, ..., b_m \subseteq R^n$ is the set of benchmarks. To solve the proposed model, a polynomial-time solvable approximation for the inner problem was developed. The formulation of Hauser et al. (2013) does not provide any control over regret value since the objective function is a regret function. Hence, Simões et al. (2018) extended a relative robust mean-variance PSP when a regret function is a constraint that provides more control over the regret value. Moreover, they defined proportional regret as an objective function, which is more perceivable by investors. Results show that the regret minimization seems to provide a greater degree of protection when it is compared to absolute robust optimization. Caçador et al. (2021) proposed a new methodology for computing relative-robust solutions for mean-variance and minimum variance PSPs. This solution methodology is based on a genetic algorithm (GA), allowing the transformation of the three-level optimization problem into a bi-level problem.

In the mean-variance PSP, it is assumed that there is a positive correlation between the expected return and the variance, which means more/less risk (variance) results in more/less profit (return). However, Baker et al. (2010) showed that in a long-term investment strategy, low-volatility portfolios outperform high-volatility portfolios. Consequently, a PSP that minimizes just the variance of the portfolio (*i.e.,* global minimum variance portfolio) might have a better performance than the mean-variance PSP. Maillet et al. (2015) showed that the optimal solution of the global minimum variance portfolio can be calculated by solving a least-square regression while the covariance matrix of assets is uncertain. Hence, a robust least-square regression is proposed where the uncertainty set is based on the Frobenius norm, leading to a second-order cone program (SOCP). Maillet et al. (2015) formulated the nominal global minimum variance PSP as $\min_{x\in X} x^\intercal \bar{Q}_S x$, where $\bar{Q}_S$ is an estimate of the covariance matrix, leading to the closed-form optimal solution $x^* = \frac{\bar{Q}_S^{-1}e}{e^\intercal \bar{Q}_S^{-1}e}$. They also showed that the optimal solution of this PSP can be computed as $x^* = \frac{e}{n} - M\bar{\zeta}^*$, where n is the number of stocks, M is an $n\times(n-1)$ matrix having the following properties: $M'e = 0$ and $M^\intercal M = I_{n-1}$, where I_{n-1} is the $(n-1)$ identity matrix. $\bar{\zeta}^*$ can be calculated based on the least square regression formulation $\bar{\zeta}^* = \arg\min_{\zeta} ||y - X\zeta||_2$, where $X = \bar{Q}_S^{1/2}M$, and $y = \bar{Q}_S^{1/2}\frac{e}{n}$. Moreover, $\bar{Q}_S^{1/2}$ is calculated from $\bar{Q}_S = \bar{Q}_S^{1/2}\bar{Q}_S^{1/2}$. For the robust PSP, Maillet et al. (2015) assumed that the pair (X, y) is uncertain and belongs to a family of matrices $(X + \Delta X, y + \Delta y)$, where $\Delta = [\Delta X, \Delta y]$ is a perturbation matrix while $||\Delta||_F = ||[\Delta X \;\; \Delta y]||_F \leq \rho$ is the uncertainty set, $||.||_F$ is the Frobenius norm and $\rho \geq 0$. Consequently, the robust counterpart of the least square regression is $\bar{\zeta}(\rho) = \arg\min_{\zeta} \max_{||\Delta X, \Delta y||_F \leq \rho} ||(y + \Delta y) - (X + \Delta X)\zeta||_2$. Monte Carlo simulation was used to test the robust formulation, showing that it dominates the non-robust one with respect to weight stability, portfolio variance, and risk-adjusted returns. To make the formulation of Maillet et al. (2015) more practical, Xidonas et al. (2017a) augmented it into the cardinality-constrained global minimum variance PSP using the approach proposed by Cornuejols & Tütüncü (2006), which uses scenarios instead of uncertainty sets to capture parameter uncertainty, making the formulation easier to handle. The problem is formulated as $\min_{x \in X} x^\intercal Q x$ s.t. $L \leq \sum_{j=1}^{n} c_j \leq H,\ c_j k_{min} \leq x_j \leq c_j k_{max},\ \ \forall j = 1, ..., n$. Xidonas et al. (2017a) defined a set of scenarios, indexed by $t \in T$, that describe the assets' performance, each has an expected return vector r_t and a covariance matrix Q_t. They also defined v_t^{2*} as the minimum variance of a portfolio under scenario t, which is calculated by solving the classical mean-variance PSP for $Q = Q_t$. The final formulation tries to find the optimal solution in the worst-case situation as $\min_{x \in X, s}$ s.t. $x^\intercal Q_t x \leq (1+s)v_t^{2*},\ \ \forall t \in T,\ L \leq \sum_{j=1}^{n} c_j \leq H,\ c_j k_{min} \leq x_j \leq c_j k_{max},\ \ \forall j = 1, ..., n$, where s is the relative worst variance aggravation based on the robust solution.

The risk parity or the equal risk contribution is a new asset allocation strategy in which all of the underlying assets in the portfolio contribute equally to the risk. It has been argued that risk parity results in a superior Sharpe ratio than the mean-variance PSP (see, *e.g.,* DeMiguel et al. (2009)). However, inputs of the risk parity formulations are often subject to uncertainty, which leads to sub-optimal solutions. DeMiguel et al. (2009) assumed that $\rho(.)$ is a continuously differentiable convex risk measure, and b_j are the risk budgets assigned by the investor. The risk budgeting problem becomes $x^* = x_j \frac{\partial \rho(x)}{\partial x_j} = b_j\ \forall j,\ x \in X$, where $\frac{\partial \rho(x)}{\partial x_j}$ is the marginal risk contribution and $x_j \frac{\partial \rho(x)}{\partial x_j}$ is the risk contribution of asset j, which has the optimal solution $x^* = \frac{w^*}{e^\intercal w^*}$, where $w^* = \arg\min_{w \geq 0} \{\rho(w) - \sum_{j=1}^{n} b_j \ln w_j\}$. Kapsos et al. (2018)

used the variance of the portfolio, which is uncertain and belongs to an uncertainty set U_Q, to quantify risk. With that, the robust counterpart of the risk budgeting problem becomes $\min_{x \in X} \max_{Q \in U_Q} \quad (x^{\intercal} Q x - \sum_{j=1}^{n} \ln b_j x_j)$, which is equivalent to $\min_{w \geq 0} \max_{Q \in U_Q} \quad (w^{\intercal} Q w - \sum_{j=1}^{n} \ln b_j w_j)$, where $x^* = \frac{w^*}{e^{\intercal} w^*}$ is a normalization of decision variables. Kapsos et al. (2018) proposed three robust risk budgeting formulations, for which the covariance matrix of assets belongs to; a discrete uncertainty set, a box uncertainty set while the upper bound is a PSD matrix, or a box uncertainty set without restrictions on its bounds. In the last case, the formulation is transformed to a semi-infinite problem that is solvable using an iterative procedure proposed by the authors.

Worst-case RO is an extreme case, which finds the optimal solution of an optimization problem for the worst possible situation. However, this approach is over-conservative. The goal of reducing the conservatism of RO solutions can be achieved by using other extreme cases than worst-case. Chen & Wei (2019) incorporated set order relations of solutions into a multi-objective mean-variance PSP with an ellipsoidal uncertainty set to show the relationship between optimization solutions and their efficiency by comparing multiple objective function values. These relations can be interpreted as extreme cases. The first relation, called "upper set less ordered relation", is the best solution for the worst-case situation, which is equivalent to the robust formulation. The second case is "lower set less ordered relation" which practically means the best-case solution. Third, "alternative set less ordered relation", is the intersection of the best-case and the worst-case solutions, *i.e.*, $X_{alternative} = X_{best-case} \cap X_{worst-case}$. This study assumed that the distributions of asset returns are normal. Chen & Zhou (2018), however, argued that practical and theoretical evidence shows that the distribution function of asset returns has a fat-tail. Hence, they applied the relation structure of Chen & Wei (2019) and the idea of other extreme cases in PSP without the normality assumption by using the higher moments (skewness and kurtosis) in their formulation. Both Chen & Wei (2019) and Chen & Zhou (2018) used a multi-objective particle swarm optimization algorithm to solve other extreme cases of the mean-variance PSP.

Extreme cases (worst-case, best-case, and the intersection of the best-case and the worst-case solutions, can be implemented in different market conditions. Bai et al. (2019) considered different realizations of the uncertain parameters in different market conditions by dividing the market situation into bull market, bear market, and steady market. In the bull market condition, it is assumed that the best-case scenario will happen, hence a best-case formulation (*i.e.,* min-min or max-max) is used. Conversely, in the bear market condition, it is assumed that the worst-case scenario will happen, leading to a typical worst-case RO. In the steady market, an alternative scenario, namely the intersection of solutions of the best-case and the worst-case scenarios is assumed to happen. In contrast to Chen & Zhou (2018) and Chen & Wei (2019), Bai et al. (2019) used a single objective mean-variance PSP.

An important criticism of the classical mean-variance PSP is its weak performance in out-of-sample data due to overfitting. It also has been shown that, for a large number of periods, the classical formulation of the mean-variance PSP amplifies the effects of noise, leading to an unstable and unreliable estimate of decision vectors. To reduce these effects, Dai & Wang (2019) proposed a *sparse* robust formulation for the mean-variance PSP, which places controls on the asset weights in the portfolio. The process of adding information to solve an ill-posed problem is called *regularization.* Dai & Wang (2019) defined $r_s = (r_{1s}, r_{2s}, ..., r_{ns}) \in \mathbb{R}^n$ as a vector of asset returns at time $s, (s = 1, ..., S)$. Moreover, $\mathbb{E}[r_s] = \bar{r}$ and $Q = \mathbb{E}[(r_s - \bar{r})(r_s - \bar{r})^\intercal]$ are mean vector and covariance matrix, respectively. The portfolio variance is $x^\intercal Q x = \mathbb{E}[|x^\intercal \bar{r} - x^\intercal r_s|^2] = \frac{1}{S}||x^\intercal \bar{r} e - Rx||_2^2$, where R is a matrix whose s^{th} row equal to r_s. If the expectations are replaced by the sample average, then the model can be expressed as a statistical regression, which takes the form $\min_{x \in X} \frac{1}{S}||x^\intercal \bar{r} e - Rx||_2^2$, where $||.||_2$ is the l_2 norm. If the size of R is large, then it amplifies the effects of noise, leading to an unstable and unreliable estimate of the vector x. To overcome this issue, a regularization is applied in the formulation as $\min_{x \in X}(\frac{1}{S}||x^\intercal \bar{r} e - Rx||_2^2 + \tau ||x||_1^1)$, where τ is the parameter for adjusting the relative importance of the l_1 norm penalty in the objective function. However, this sparse formulation does not consider return as an uncertain parameter. Consequently, two robust formulations of the mean-variance PSP with box and ellipsoidal uncertainty sets are proposed. The results showed that the sparse robust mean-variance PSP has better out-of-sample performance than other mean-variance formulations. Lee et al. (2020) extended the same concept to a robust sparse cardinality-constrained mean-variance PSP with ellipsoidal uncertainty set and l_2 norm regularization to achieve a better control over decision variables. This formulation results in a non-convex NP-hard problem. Hence, a relaxation to a SDP problem is proposed to make it more tractable.

Alternatively, it is possible to prevent the negative impact of noisy inputs by adding restrictions on the estimated parameters instead of restricting the decision variables. Plachel (2019) used the restricted estimators method with a box uncertainty set to derive a robust regularized minimum variance PSP based on the decomposition of covariance matrix proposed by Laloux et al. (1999). The proposed formulation was tested with the three major turmoils of the financial market (Black Monday, the Dotcom Bubble, and the Financial Crisis) and the results showed that the joint problem regularization and robustification outperforms the classical non-robust minimum variance and the non-regularized minimum variance PSP.

The classical mean-variance PSP is based on the Gaussian distribution assumption of asset returns. Based on historical evidence, Lauprete et al. (2003) showed that the returns of assets follow a heavy-tail distribution. Given that the uncertainty associated with the deviation of actual distribution functions from theoretical distribution functions might lead to sub-optimal solutions, they proposed a robust estimation that immunizes the estimators against uncertainty. DeMiguel & Nogales (2009) used two types of robust estimators (M-estimator and S-estimator) in the mean-variance PSP. M-estimator and S-estimator are based on convex symmetric and Tukey's bi-weight loss functions, respectively. The S-estimator has the advantage of not being sensitive to data scaling. DeMiguel & Nogales (2009) analyzed the sensitivity of M-portfolios and S-portfolios' (corresponding to M-estimator and S-estimator, respectively) weights with respect to the changes in the distribution of the asset returns. Results showed that these formulations are more robust than the traditional mean-variance PSP.

Kim et al. (2013b) identified a gap in the literature about the experimental evidence of the robust PSP. They analyzed the robust mean-variance PSP with box and ellipsoidal uncertainty sets. Results showed that weights in the robust mean-variance PSP align with assets having a higher correlation with the Fama-French three factors model which bets on fundamental factors of assets. Interested readers are refereed to Fama & French (2021) for more detail about Fama-French three factors model. Kim et al. (2014b) also concluded that robust solutions of the mean-variance PSP depend on fundamental factors movements. In another analysis, Kim et al. (2013a) showed that robust equity mean-variance portfolios have four advantageous characteristics compared to non-robust mean-variance PSPs: (1) fewer stocks, (2) less exposure to each stock (the amount of money that the investor could lose on an investment), (3) higher portfolio beta, and (4) large negative correlation between weight and stock beta. Kim et al. (2018b) concluded that the robust mean-variance PSP leads to the most efficient investment strategies that allocate risk efficiently. Kim et al. (2015) also illustrated that the robust approach is the best method for formulating the mean-variance PSP while the market switches between multiple states.

Similarly, Schöttle & Werner (2009) analyzed the Markowitz efficient frontier of robust mean-variance PSP with ellipsoidal and joint ellipsoidal uncertainty sets. They showed that the efficient frontiers of both robust formulations are exactly matched with the efficient frontier of the classical mean-variance PSP up to some level of risk. This means that the classical mean-variance PSP is already robust without applying RO methods. However, the robust mean-variance formulation identifies the unreliable upper part of the efficient frontier.

Recently, Yin et al. (2021) proposed a practical guide to robust portfolio optimization based on mean-variance formulations. They assumed that asset returns are uncertain and belong to either box or an ellipsoidal uncertainty sets. By using practical examples, they showed that the robust mean-variance PSP with an ellipsoidal uncertainty set provides a more robust formulation than its corresponding problem with a box uncertainty set.

2.3.2 Robust Mean Absolute Deviation

Konno & Yamazaki (1991) argued that calculating the covariance matrix in large mean-variance PSPs is a challenging task. Hence, they proposed the mean absolute deviation as an alternative volatility-based risk measure that reduces the computational complexity of the covariance matrix. The MAD PSP is formulated as $\min_{x\in X} \frac{1}{S}\sum_{s=1}^{S} |\sum_{j=1}^{n}(r_{js} - r_j)x_j|$ s.t. $\sum_{j=1}^{n} r_j x_j \geq EW_0$. This formulation can be transformed to an LP as $\min_{x\in X,y} \quad \frac{1}{S}\sum_{s=1}^{S} y_s$ s.t. $y_s + \sum_{j=1}^{n}(r_{js} - r_j)x_j \geq 0,\ \forall s,\ y_s - \sum_{j=1}^{n}(r_{js} - r_j)x_j \geq 0,\ \forall s,\ \sum_{j=1}^{n} r_j x_j \geq EW_0$. Besides being reformable as an LP, the MAD PSP has another advantage over the mean-variance PSP of not requiring the normality assumption for asset returns. However, MAD penalizes both positive and negative deviations equally, though positive deviations are desirable by investors. Moreover, in the classical MAD PSP, future asset returns are assumed to be known with certainty.

To handle the uncertainty of asset returns in the MAD PSP, Moon & Yao (2011) proposed a robust MAD PSP with a budget uncertainty set. However, Li et al. (2016) suggested that classical uncertainty sets do not capture the asymmetry in asset returns and, instead, proposed a robust MAD PSP with the asymmetric uncertainty set first introduced by Chen et al. (2007). Ghahtarani & Najafi (2018) developed a robust PSP based on m-MAD, a downside risk measure proposed by Michalowski & Ogryczak (2001) that penalizes only negative deviations. Chen et al. (2011a) proposed an alternative robust downside risk measure, referred to as *lower partial moment* (LPM), and used it, along with a moment-based ambiguity set, for single and multi-stage robust PSP that uses an S-shape value function. An important advantage of robust LPM over robust m-MAD is that the former can be used also to develop robust VaR/CVaR formulations with moment-based ambiguity sets. To avoid the over-conservatism of worst-case approaches, Xidonas et al. (2017b) employed a robust min-max regret approach in a multi-objective PSP. The objectives to be optimized are expected asset returns and MAD. The proposed approach results in solutions that do not have to be safe according to the worst realization of the parameters, but to the relevant optimum of each scenario.

2.3.3 Factor-Based Portfolio Models

Factor-based models are financial models that incorporate factors (macroeconomic, fundamental, and statistical) to determine the market equilibrium and calculate the required rate of return. Goldfarb & Iyengar (2003) developed a robust factor-based model for a PSP where uncertainty is considered by its sources, namely fundamental factors. The basic formulation of a factor-based model is $r = \mu + V^\intercal f + \epsilon$, where $\mu \in \mathbb{R}^n$ is the mean returns vector, $f \sim N(0, F)$ is the vector of the factors that drive the market, $V \in R^{m \times n}$ is the matrix of the factor loading of n assets, and $\epsilon \sim N(0, D)$ is the vector of the residual returns. Uncertain parameters are the mean return, the factor loading, and the covariance of residuals that belong to uncertainty sets with upper and lower bounds. Goldfarb & Iyengar (2003) defined uncertainty sets for these parameters as $U_d = \{D : D = diag(d), \ \ d_j \in [\underline{d}_j, \bar{d}_j], \ \forall j\}$, $U_\nu = \{V : V = V_0 + W, \ ||W_j||_g \leq \rho_j, \ \forall j\}$, and $U_m = \{\mu : \mu = \mu_0 + \xi, \quad |\xi_j| \leq \gamma_j, \ \forall j\}$, where W_j is the jth column of W and $||w||_g = \sqrt{w^\intercal G w}$ is an elliptic norm of w with respect to G. The return on a portfolio x is given by $r_x = r^\intercal x = \mu^\intercal x + f^\intercal V x + \epsilon^\intercal x \sim N(x^\intercal \mu, x^\intercal (V^\intercal F V + D)x)$. Both f and ϵ are assumed to follow normal distributions, thus r_x also follows a normal distribution. The robust factor-based model is developed based on two alternative assumptions. First, uncertainty in the mean is independent of the uncertainty in the covariance matrix of returns, which leads to a SOCP. Second, uncertainty in the mean depends on the uncertainty in the covariance matrix of the returns, which results in a SDP formulation for the worst-case VaR. It should be noted that the uncertainty sets in the robust factor-based models of Goldfarb & Iyengar (2003) are separable, leading to two important drawbacks: the results are conservative, and the robust portfolio constructed is not well diversified. Alternatively, Lu (2006) and Lu (2011) proposed robust factor-based models with a joint ellipsoidal uncertainty set that can be reformulated as a tractable cone programming problem. Additionally, Ling & Xu (2012) developed a robust factor-based model with joint marginal ellipsoidal uncertainty sets and options to hedge risks that generates robust portfolios with good wealth growth rates even if an extreme event occurs.

An important input to factor-based models is the “factor exposure”, which measures the reaction of factor-based models to risk factors. Kim et al. (2014c) argued that factor-based models are not robust against the uncertainty of risk factors such as macroeconomic factors. They proposed a robust factor-based model with an ellipsoidal uncertainty set that is robust against uncertainty and has the desired level of dependency on factor movements. This model manages the total portfolio risk by defining a robustness measure and a constraint that restricts the factor exposure of robust portfolios. Another evidence to support the use of robust factor-based models comes from Lutgens & Schotman (2010). They compared the Capital Asset Pricing Model (CAPM), the international CAPM, the international Fama, and the French factor-based models and showed that robust portfolios of factor models lead to better diversified portfolios.

2.3.4 Robust Utility Function PSP

Most PSPs are based on the return-risk trade-off concept. However, financial decisions might be made based on a utility function. Popescu (2007) developed a robust PSP for the expected utility function (the utility that an entity or aggregate economy is expected to reach under any number of circumstances) where the distribution function of asset returns is partially known and belongs to an ambiguity set with predetermined mean vector and covariance matrix. Natarajan et al. (2010) also proposed a less-complex robust formulation of the expected utility function PSP that uses a piecewise-linear concave function to model the investor's utility. Besides the ambiguity set of Popescu (2007), Natarajan et al. (2010) considered the case in which the mean vector and covariance matrix of uncertain parameters belong to box uncertainty sets. Ma et al. (2008) incorporated a robust factor model with a concave-convex utility function to seize the advantages of both approaches. They assumed that the mean returns vector, the factor loading covariance, and the residual covariance matrix are uncertain and belong to uncertainty intervals. The robust counterpart turned out to be a parametric quadratic programming problem that can be solved explicitly. Biagini & Pınar (2017) proposed a min-max robust utility function for Merton problem. Merton's portfolio problem is a well-known PSP problem where the investor must choose how much to consume and how allocate the remaining wealth between risky assets and a risk-free asset to maximize expected utility. An ellipsoidal uncertainty set is assumed to contain the drift from a compact values volatility realization.

2.4 Robust PSPs with Quantile-based Risk Measures

This section reviews robust quantile-based PSPs, which include PSPs based on the Value at Risk (VaR), Conditional Value at Risk (CVaR), and their extensions with worst-case RO, relative RO and distributionally robust optimization (DRO) methods. Furthermore, the relationship between uncertainty sets and risk measures, application of soft robust formulation with risk measures, worst-case CVaR and its relationship with the uniform investment strategy, and robust arbitrage pricing theory with worst-case CVaR are also discussed.

VaR is the maximum loss at a specific confidence level. In other words, VaR is the quantile of a loss distribution function, which is neither a convex nor coherent risk measure. A coherent risk measure is a function that satisfies the properties of monotonicity, sub-additivity, homogeneity, and translational invariance which provide computational advantages for a risk measure (see Artzner et al. (1999) for details about coherent risk measures). CVaR is a coherent risk measure that denotes expected loss greater than VaR for a specific confidence level. Let $f(x, r)$ be a loss function. For a given confidence level β, the Value at Risk is defined as $VaR_\beta(x) = \min\{\alpha \in \mathbb{R} : \Psi(x, \alpha) \geq \beta\}$, where $\Psi(x, \alpha) = \int_{f(x,r)\leq\alpha} p(r)dr$. Conditional Value at Risk is the expected loss that exceeds $VaR_\beta(x)$, mathematically defined as $CVaR_\beta(x) = \frac{1}{1-\beta}\int_{f(x,r)\geq VaR_\beta(x)} f(x, r)p(r)dr$. Rockafellar et al. (2000) proved that CVaR can be formulated as an optimization problem by defining an auxiliary function $F_\beta(x, \alpha) = \alpha + \frac{1}{1-\beta}\int_{y\in R^m}[f(x, r) - \alpha]^+ p(r)dr$, where $[.]^+ = \max\{., 0\}$ and $CVaR_\beta(x) = \min_{\alpha\in R} F_\beta(x, \alpha)$. They also proved that CVaR can be reformulated as an LP when using discrete scenarios for asset returns. An important input to these formulations is $p(.)$, which is often not known or only partially known. Considering the ambiguity of $p(.)$ leads to worst-case VaR and CVaR formulations.

2.4.1 Worst-Case VaR and CVaR

Ghaoui et al. (2003) were the first to propose a tractable reformulation for the worst-case VaR, defined as $VaR_p(x) = \min \alpha$ s.t. $\sup\{Prob\Psi(x, \alpha) \geq \beta\} \leq \epsilon$, where $VaR_p^{optimum} = \min VaR_p(x)\ s.t.\ x \in X$. The distribution function of asset returns is assumed to be partially known and belongs to one of the four moment-based ambiguity sets: 1) the first two moments (mean vector ($\hat{r}$) and covariance matrix (Σ)) of the loss distribution function are known and fixed. 2) the moments $(\Sigma, \hat{r})$ of the loss distribution function are known to belong to the convex set, , assuming that there is a point in U such that $\Sigma \succ 0$. By introducing $U_+ := \{(\Sigma, \hat{r}) \in U | \Gamma \succ 0\}$, the worst-case VaR for this case is formulated as $VaR_p(x) = \sup -r^\intercal x$ s.t. $(\Sigma, \hat{r}) \in U_+$. 3) polytopic uncertainty set defined as the convex hull of the vertices $(\hat{r}_1, \Sigma_1), ..., (\hat{r}_l, \Sigma_l)$. The polytope uncertainty set U is then constructed as $U = U_r \times U_\Sigma$, where $U_r = Co\{\hat{r}_1, ...\hat{r}_l\}$ and $U_\Sigma = Co\{\Sigma_1, ..., \Sigma_l\}$. By assuming that $\Sigma_i \succ 0, \quad i = 1, ..., l$, the worst-case VaR is formulated as $VaR_p(x) = k(\varepsilon)\sqrt{\max_{\Sigma\in U_\Sigma} x^\intercal \Sigma x} - \min_{\hat{r}\in U_r} \hat{r}^\intercal x =$

$\max_{1\leq i\leq l} k(\varepsilon)||\Sigma_i^{1/2}||_2 - \min_{1\leq i\leq l} \hat{r}_i^{\intercal} x$, where $k(\varepsilon) = \sqrt{\frac{1-\varepsilon}{\varepsilon}}$. Ghaoui et al. (2003) showed that this formulation can be transformed to a SOCP model. 4) component-wise bounds for moments. Ghaoui et al. (2003) also considered the worst-case VaR when the return of assets in the loss function is based on the factor model $r = Vf + \epsilon$, where f is an m-vector of random factors, ϵ is the residual (unexplained) return, and V is an $n \times m$ matrix of sensitivities of the returns. The covariance matrix of returns is stated as $\Sigma = D + VSV^{\intercal}$, where D is the diagonal covariance matrix of residuals and S is the covariance matrix of factors. Two cases of parameter certainty are considered: uncertainty in the factor's mean and covariance matrix, and uncertainty in the sensitivity matirx. In contrast to Ghaoui et al. (2003), the factor model of Goldfarb & Iyengar (2003) assumed that uncertainty in the mean is independent from that of the covariance matrix, leading the expected value of error term of the factor model to be equal to zero. This uncertainty structure leads to a SOCP reformulation, compared to the SDP reformulation of Ghaoui et al. (2003).

It is argued that the worst-case VaR is unrealistic and conservative. Therefore, a way to enforce the worst-case probability distribution to some level of smoothness was proposed by adding a relative entropy constraint (*i.e.,* KL divergence) with respect to a given "reference" probability distribution. Whereas Ghaoui et al. (2003) assumed that the return of assets follows a Gaussian distribution, Belhajjam et al. (2017) argued that the distribution function of return is asymmetric. Hence, extreme returns occur more frequently than would be under the normal distribution. Hence, they proposed a multivariate extreme Value at Risk (MEVaR) formula based on a multivariate minimum return that considers extremums of returns, *i.e.,* the lowest and highest daily returns. Since there is no guarantee that uncertain parameters belong to a symmetric uncertainty set, Natarajan et al. (2008) applied the asymmetric uncertainty set introduced by Chen et al. (2007) to develop a worst-case VaR measure. Results show that Asymmetry-Robust VaR (ARVaR) is an approximation of CVaR. Similar to Ghaoui et al. (2003), Natarajan et al. (2008) assumed that asset returns follow a factor model. Moreover, an asymmetric uncertainty set for the worst-case VaR leads to a tractable second-order cone program. Another less complex method to consider asymmetric uncertainty is to use interval random uncertainty sets. Chen et al. (2011b) developed a worst-case VaR assuming that the expected vector and covariance matrix of the returns are uncertain and belong to interval random uncertainty sets.

Huang et al. (2007) demonstrated that the exit time of investment (or the investment horizon) which is traditionally assumed to be deterministic, can, in reality, depend on market conditions. Consequently, they considered a conditional distribution function of the rate of return based on different exit times instead of the unconditional distribution function previously used in Ghaoui et al. (2003). Three robust portfolio formulations were proposed: 1) a portfolio formulation with componentwise uncertainty on moments of the conditional distribution function of exit time. 2) a portfolio formulation with semi-ellipsoidal uncertainty set on exit time. 3) moments of the conditional distribution function of exit time belonging to a polytope uncertainty set for each exit time. Huang et al. (2008) also assumed that the density function of exit time is only known to belong to an ambiguity set that covers all possible exit scenarios. They developed two formulations: a worst-case VaR with no information about exit time, and a formulation with partial information about exit time.

Kelly Jr (1956) proposed an investment strategy in the financial market (known as *Kelly Strategy*), which maximizes an expected portfolio growth rate. From a mathematical perspective, implementing the Kelly strategy is synonymous with solving a multi-period investment strategy, making it amenable to robust approaches for handling uncertainty. Rujeerapaiboon et al. (2016) considered Kelly's strategy under return uncertainty and proposed a formulation that includes the constraint $\mathbb{P}(total\ portfolio\ return\ \geq\ \gamma) \geq 1-\epsilon$, where γ is an expected total portfolio return, and $1-\epsilon$ is the confidence level. This chance constraint is, simply, the definition of VaR. In this formulation, the distribution function of asset returns is assumed to be uncertain and belongs to the class of moment-based ambiguity set introduced in Delage & Ye (2010). This ambiguity set leads to a SDP formulation for the worst-case VaR.

As mentioned earlier, VaR has a high computational complexity since it is not convex. Zhu & Fukushima (2009) proposed a PSP that maximizes the worst-case CVaR, defined as $\sup_{\pi \in P} CVaR_{\beta}(x)$, with three cases of uncertainty set for the probabilities of discrete return scenarios: a mixture distribution, a box uncertainty set, and an ellipsoidal uncertainty set. The last case led to a SOCP, whereas the first two cases resulted in LPs. A mixture distribution (P_M) is defined as $\pi \in P_M = \{\sum_{i=1}^{I} \eta_i p^i(.) : \sum_{i=1}^{I} \eta_i = 1, \;\; \eta_i \geq 0, \;\; i = 1, ..., I\}$, which leads to: $WCVaR_\beta(x) = \min_{\alpha\in\mathbb{R}} \max_{i\in l} F_\beta^i(x, \alpha)$, where $l = [1, ..., I]$. The box uncertainty set for probability distribution is defined as $\pi \in P_\pi^\beta = \{\pi : \;\; \pi = \pi^0 + \zeta, \;\; e^\intercal\zeta = 0, \;\; \underline{\zeta} \leq \zeta \leq \bar{\zeta}\}$, whereas the ellipsoidal uncertainty set for probability distribution function is defiend as $\pi \in P_\pi = \{\pi : \pi = \pi^0 + A\zeta, \; e^\intercal A\zeta = 0, \; \pi^0 + A\zeta \geq 0 \; ||\zeta|| \leq 1\}$, where $||\zeta|| = \sqrt{\zeta^\intercal\zeta}$ and π^0 is the nominal distribution. Doan et al. (2015) extended the worst-case CVaR formulation of Zhu & Fukushima (2009) by proposing a data-driven approach to construct a class of distributions for asset returns, known as Fréchet distributions, that leads to less conservative solutions than the worst-case CVaR. Moreover, Hasuike & Mehlawat (2018) incorporated the arbitrage pricing theory (APT) model, which is a multi-factor model, in a bi-objective PSP that aims at maximizing the expected return and minimizing the worst-case CVaR of a portfolio. Ghahtarani et al. (2018) proposed a robust CVaR formulation by considering the uncertainty of the return distribution's parameters. They proposed a robust mean-CVaR PSP with a chance constraint when asset returns follow a Gaussian distribution with uncertain moments. Hellmich & Kassberger (2011), in contrast, developed a worst-case CVaR model with asset returns that follows a heavy-tail multivariate generalized hyperbolic distribution. Their formulation can also capture the asymmetrical nature of asset returns.

One way to alleviate the over-conservatism of the worst-case VaR/CVaR solutions is to use a data-driven joint ellipsoidal uncertainty set in which the first two moments of the distribution function of asset returns are in an ellipsoid norm. Lotfi & Zenios (2018) proposed an algorithm for constructing data-driven ambiguity sets based on an optimization model to find the centers of joint ellipsoidal uncertainty sets. In another attempt, Liu et al. (2019) used the data-driven moment-based ambiguity set introduced in Delage & Ye (2010) to propose a worst-case CVaR in both single and multi-period PSPs. In this formulation, for each period there is a separate ambiguity set. They demonstrated that a robust counterpart of the multi-period mean-CVaR PSP can be solved as a sequence of optimization problems based on an adaptive robust formulation. Kang et al. (2019) argued that the ambiguity set of Delage & Ye (2010) leads to solutions that are too conservative. Therefore, they altered it by adding a zero-net adjustment constraint. Huang et al. (2021) proposed a distributionally robust mean-CVaR PSP with a moment-based ambiguity set. Besides DRO, they used an l_1 norm to limit the weights (decision variables) of the, so called, sparse PSP to limit the impact of noisy data. Results provide evidence that a sparce mean-CVaR PSP has better performance than a non-sparce formulation with respect to net portfolio return, Sharpe ratio, and cumulative return. Moreover, Zhao et al. (2021) formulated a cardinality-constrained rebalancing worst-case CVaR with a moment-based ambiguity set. The proposed formulation enhances the portfolio diversification.

Huang et al. (2010) claimed that investors usually do not want to pay the price of full robustness to protect their portfolios against the worst possible scenario. In an uncertain environment, investors may rather choose a strategy that avoids falling behind their competitors. According to this point of view, for each choice of decision variables and each scenario, the decision-maker compares the resulted objective value to the optimal value obtained under model uncertainty described by the scenario. The difference or the ratio of these two values is a regret measure. To minimize these regrets measures, Huang et al. (2010) developed a relative CVaR formulation, mathematically described as $RCVaR_\alpha(x) = \sup_{\pi \in P}\{CVaR_\alpha(x,\pi) - CVaR_\alpha(z^*(\pi),\pi)\}$, where $z^*(\pi) = \arg\min_{z \in X} CVaR_\alpha(z,\pi)$. However, since the true distribution (π) is not known, decision-makers try to make the relative CVaR as small as possible by considering all possible π values. Consequently, a finite number of forecasts for the distribution function of asset returns is considered. Results showed that the relative CVaR is less conservative than the worst-case CVaR for optimal portfolio return. Alternatively, Yu et al. (2017) proposed a relative CVaR and a worst-case CVaR by adjusting the required return from a fixed rate to a floating rate that changes according to market dynamics. Moreover, the formulation was extended by allowing short sale and adding a transaction cost constraint. Results showed that a relative CVaR yields slightly higher realized returns, lower trading costs, and better portfolio diversification than its corresponding worst-case CVaR model when the required return is fixed. Additionally, the out-of-sample performance of floating-return models compared to fixed-rate models is significantly better during periods when a market recovers from a financial crisis. Finally, robust floating-return models have a better asset allocation, save transaction costs, and attribute to superior profitability. Benati

& Conde (2021) proposed a model that minimizes the maximum regret on the expected returns while the conditional value-at-risk is bounded under different scenario settings. To solve this problem, a cutting plane approach was proposed.

An investment strategy that is widely used in financial markets is the *uniform investment strategy* or $1/N$ *rule*, which divides the budget among assets equally. Pflug et al. (2012) demonstrated that the uniform investment strategy is the best strategy for investment under uncertainty. They proposed robust mean-CVaR and mean-variance PSPs where the distribution function of asset returns is uncertain and belongs to a *Kantorovich* or *Wasserstein* metric-based ambiguity set. Results showed that when the size of the Wasserstein ambiguity set is infinity, solutions of the robust PSPs are equal to the uniform investment strategy. Hence, the optimal investment strategy in a high ambiguity situation is the uniform investment or $1/N$ rule. However, Pflug et al. (2012) assumed that all assets are subject to uncertainty though it is possible to use fixed-income assets with no ambiguity or uncertainty in the portfolio. Therefore, Paç & Pınar (2018) extended the robust uniform strategy of Pflug et al. (2012) by considering both ambiguous and unambiguous assets. They showed that by increasing the ambiguity level, measured by the radius of the ambiguity set, the optimal portfolio tends to use equal weights for all assets. Also, high levels of ambiguity result in portfolios that avoid ambiguous assets and favor unambiguous assets.

Finally, Natarajan et al. (2009) established the relationship between risk measures and uncertainty sets. They showed that using an ellipsoidal uncertainty set for asset returns corresponds to the classical mean-variance PSP, whereas the CVaR formulation results from using a special polyhedral uncertainty set. As discussed by Ben-Tal et al. (2010), in soft robust formulations, a penalty function is introduced such that if uncertain parameters fluctuate in the uncertainty set, the penalty function equals zero. Otherwise, the penalty function takes a positive value. Recchia & Scutellà (2014) proved that the definition of a convex risk measure is also based on a penalty function that is called norm-portfolio models, where using l_∞, l_1, and D-norm result in an LP for a norm-portfolio model. On the other hand, using a euclidean norm results in a SOCP, whereas applying a D-norm, proposed by Bertsimas et al. (2004b), for a penalty function with specific parameters leads to the CVaR formulation.

Worst-Case CVaR with Copula

Classical multivariate distribution functions make the worst-case CVaR computationally complex. One way to address this issue is to use copulas instead of multivariate distribution functions for asset returns. Copulas are multivariate distribution functions whose one-dimensional margins are uniformly distributed on a closed interval $[0, 1]$. One-dimensional margins of copulas can be replaced by univariate cumulative distributions of random variables. Hence, copulas consider the dependency between marginal distributions of random variables instead of focusing directly on dependency between random variables themselves. This characteristic makes them more flexible than standard distributions, and also an interesting candidate for the distribution function of the rate of return in the worst-case CVaR.

Kakouris & Rustem (2014) used Archimedean copulas to propose worst-case CVaR PSPs that avoids the shortcomings of worst-case CVaR PSPs based on a Gaussian distribution, which is a symmetric distribution for asset returns. There are three Archimedean copulas: the Clayton copula, the Gumbel copula, and the Frank copula. Kakouris & Rustem (2014) used a heuristic method to estimate copulas' parameters in the context of a multi-asset PSP. However, simulating data from three Archimedian copulas has computational challenges. On the other hand, Han et al. (2017) claimed that the formulation of Kakouris & Rustem (2014) is static, making it unable to deal with the dynamic nature of the financial market. They, instead, proposed a dynamic robust PSP with Archimedean copulas by using dynamic conditional correlation (DCC) copulas and copula-GARCH model to forecast the worst-case CVaR of bi-variate portfolios. Results show that dynamic worst-case CVaR models can put more weight on assets with lower volatility, which leads to a less aggressive trading strategy.

CVaR calculates the expected loss based on just one confidence level. However, decision-makers might prefer different confidence levels based on their risk attitude. One way to increase the flexibility of CVaR related to decision-makers' risk attitude is to use Mixed-CVaR and Mixed Deviation-CVaR. These mixed risk measures combine CVaRs with different confidence levels. Goel et al. (2019) proposed robust Mixed-CVaR and Mixed Deviation-CVaR Stable Tail-Adjusted Return Ratio (STARR), which is the portfolio return minus the risk-free rate of return divided by the expected tail loss (at a specific confidence level). Finally, a mixture copula set was used to consider distribution ambiguity, which resulted in an LP.

2.4.2 Robust Mean-CVaR/Shortfall PSP

Besides the worst-case VaR and CVaR, some researchers developed robust mean-CVaR PSPs where the distribution function of the loss function is assumed to be deterministic while returns of assets or weights of the mixture distribution function of the rate of return are uncertain. Thus, classical uncertainty sets are used to develop robust mean-CVaR PSPs. Quaranta & Zaffaroni (2008) proposed a robust mean-CVaR with a box uncertainty set that leads to an LP. Kara et al. (2019) also proposed a robust mean-CVaR PSP with a parallelepiped uncertainty set, developed by Özmen et al. (2011). The parallelepiped uncertainty set is practically a box uncertainty set while its elements are a convex hull of canonical vertices of an uncertain matrix. Elements of this uncertainty set are founded by the Cartesian product of uncertain intervals. An advantage of a parallelepiped uncertainty set over a box uncertainty set is that the lengths of intervals may vary among each other. Moreover, instead of a single price, it is possible to consider multiple and flexible varying prices of assets and also take into account likewise flexible returns. To reduce the conservatism of solutions of a robust mean-CVaR with a box uncertainty set, Guastaroba et al. (2011) developed a robust mean-CVaR with ellipsoidal and budget uncertainty sets, which lead to a SOCP and an LP, respectively. Besides the uncertainty of parameters, a mean-CVaR PSP has a multi-objective characteristic as it maximizes the expected return while minimizing the risk (CVaR). Then, a multi-objective formulation can capture the multiple-criteria nature of this problem. Rezaie et al. (2015) developed a robust bi-objective mean-CVaR PSP with a budget uncertainty set. An ideal and anti-ideal compromise programming approach was used to solve the proposed problem. This method seeks an answer as close as possible to the ideal value and as far as possible from the anti-ideal value of each objective. Ideal and anti-ideal values reflect investors' perspectives of the real world.

Another development of a robust mean-CVaR is based on mixture distribution functions. There are three reasons for using a mixture distribution function for asset returns. First, it is a combination of multiple distribution functions, thus enabling different market conditions with different distribution functions to be considered. Moreover, it replaces the estimation of the distribution function by a calculation of the distribution weights in a mixture distribution function. Finally, since any distribution function can be simulated by using a mixture of Gaussian distribution functions, a mixture distribution function has high flexibility. Zhu et al. (2014) used a mixture distribution function for asset returns to propose a robust mean-CVaR PSP. The uncertainty in their formulation is about the weights of distribution functions. For considering the uncertainty, ellipsoidal and box uncertainty sets were used. The former leads to a SOCP and the latter results in an LP.

Shortfall is also a quantile risk measure from the family of VaR and CVaR, introduced by Bertsimas et al. (2004a). Shortfall measures how great an expected loss will be if a portfolio return drops below the α-quantile of its distribution. Mathematically, it is defined as $S_\alpha = \mathbb{E}[r^\intercal x] - \mathbb{E}[r^\intercal x \mid r^\intercal x \leq q_\alpha(r^\intercal x)],\ \alpha \in (0,1)$, where q_α is the α-quantile of the distribution of random portfolio return. Like CVaR, shortfall can be reformulated as an LP while asset returns are subject to uncertainty. Later, Pachamanova (2006) developed a robust shortfall with an ellipsoidal uncertainty set, which can be reformulated as a SOCP. Their results showed that a robust shortfall PSP outperforms its nominal problem in the presence of uncertainty in terms of both return and risk. Another quantile-based measure is the conditional expectation type reward–risk performance measure developed by Ortobelli et al. (2019). This performance measure captures the portfolio's distributional behaviour on the tails. Kouaissah (2021) proposed a robust conditional expectation formulation where the asset returns are uncertain and belong to an ellipsoidal uncertainty set. Results of this robust formulation demonstrated better out-of-sample performance than its nominal counterpart.

2.5 Multi-Period PSP

Active strategies which involve ongoing buying and selling of assets are preferred by many investors. With an active strategy, investors continuously re-balance their portfolios by solving multi-period PSPs. In this section, we review applications of RO in this class of problems.

2.5.1 Robust Multi-Period PSP

Dantzig & Infanger (1993) proposed one of the most popular multi-period PSPs. Three types of decision variables are used in their formulation: x_j^s, y_j^s and z_j^s, denoting, respectively, the amounts of asset j at period s the investors hold, buy and sell. There are n risky assets and one risk-free asset. The problem is formulated as $\max \sum_{j=1}^{n+1} r_j^S x_j^S$ s.t. $x_j^s = r_j^{s-1} x_j^{s-1} - y_j^s + z_j^s,\ \forall j, s,\ x_{n+1}^s = r_{n+1}^{s-1} x_{n+1}^{s-1} + \sum_{j=1}^{n} (1 - \Delta_j^s) y_j^s - \sum_{j=1}^{n} (1 + \nu_j^s) z_j^s,\ y_j^s \geqslant 0,\ z_j^s \geqslant 0,\ \forall j, s,\ x_j^s \geqslant 0$, where the objective function maximizes the total wealth at the final period. The first constraint is for risky assets balancing, ensuring that the amount of risky assets held at period s equals the amount of assets carried forward from the previous period in addition to the net effect of transactions in the current period. The second constraint is for risk-free asset balancing, where $(1 - \Delta_j^s) y_j^s$ is the amount of cash investors receive from selling asset j at the beginning of the period s, whereas $(1 - \nu_j^s) z_j^s$ is the cash investors use to buy asset j at the beginning of period s. The uncertain parameters in this formulation are asset returns at each period. Ben-Tal et al. (2000) reformulated this multi-period PSP by defining cumulative asset returns $R_j^s = r_j^0 r_j^s ... r_j^{s-1}$, which become the new uncertain parameters. By considering these cumulative returns, Ben-Tal et al. (2000) defined new variables for their formulation as $\xi_j^s = \frac{x_j^s}{R_j^s}$, $\eta_j^s = \frac{y_j^s}{R_j^s}$, and $\zeta_j^s = \frac{z_j^s}{R_j^s}$. The final formulation becomes $\max \sum_{j=1}^{n+1} R_j^{S+1} \xi_j^S$ s.t. $\xi_j^s = \xi_j^{s-1} - \eta_j^s + \zeta_j^s,\ \forall j, s,\ \xi_{n+1}^s = \xi_{n+1}^{s-1} + \sum_{j=1}^{n} A_j^s \eta_j^s - \sum_{j=1}^{n} B_j^s \zeta_j^s,\ \forall s,\ \eta_j^s \geqslant 0,\quad \zeta_j^s \geqslant 0,\ \forall j, s,\ \xi_j^s \geqslant 0,\ \forall j, s$, where $A_j^s = (1 - \Delta_j^s) \frac{R_j^s}{R_{n+1}^s}$, and $B_j^s = (1 - \nu_j^s) \frac{R_j^s}{R_{n+1}^s}$. Both SP and RO were applied with the last nominal formulation. Interestingly, the RO problem was shown to be less complex than the SP one. An ellipsoidal uncertainty set was used in the robust problem, leading to a SOCP. Alternatively, Bertsimas & Pachamanova (2008) used a D-norm to define the uncertainty set for cumulative asset returns, thus leading to a tractable LP reformulation. To make the problem more appealing for practitioners and to increase its robustness against market volatility, Marzban et al. (2015) included American options in the robust formulation of Bertsimas & Pachamanova (2008). However, this change led to more conservative solutions in comparison to those of Ben-Tal et al. (2000) and Bertsimas & Pachamanova (2008).

Fernandes et al. (2016) added a loss function with a predetermined threshold as a constraint to the formulation of Dantzig & Infanger (1993), leading to a problem with a terminal wealth objective that requires a one-step-ahead asset return forecast as an input. A linear combination of chosen predictors is employed as a mixed-signals model that uses the last specific number of trading periods to forecast one-step ahead returns. A polyhedral set, constructed as the convex hull of the observed returns, is used as a data-driven uncertainty set. The proposed loss constraints adaptively generate different polyhedral feasible regions for investors' asset allocation decisions. Results showed that the data-driven problem led to less conservative solutions than classical RO.

To control the downside of losses, the lower partial moment (LPM) can also be used, which is more perceivable by investors than other risk measures. Ling et al. (2019) proposed a multi-period PSP similar to that of Dantzig & Infanger (1993) based on a downside risk measure with an asymmetrically distributed uncertainty set. The objective function combines the expected terminal wealth of the portfolio with its LPM. At each period $s = 0, \ldots, S$, returns are denoted as $r_0^s, r_1^s, ..., r_n^s$, where r_0^s is the deterministic risk-free return and r_j^s is the uncertain return of risky asset j. The decision variable x_j^s, $j = 0, \ldots, n$ denotes the dollar amount invested in asset j in period s. With that, the terminal value of the portfolio is given by $w^S = x_0^S(1+r_0^S)+(e+r^S)^\intercal x^S$, and the objective function is $\min -\mathbb{E}[W^T] + \lambda.\mathbb{E}[(\alpha - W^T)_+]$. Rebalancing constraints, similar to those used in Dantzig & Infanger (1993), are included.

Risk in a multi-period PSP can also be captured by the volatility of terminal wealth using mean-variance multi-period PSPs. Cong & Oosterlee (2017) considered discrete periods, indexed by $s \in \{0, \Delta s, ..., S - \Delta s\}$ for investment and denote by S the terminal period. Their formulation is based on maximizing the expected terminal wealth and minimizing the investment risk, quantified as $\hat{v}_0(W_0) = \max_{\{\hat{x}_s\}_{s=0}^{S-\Delta s}} \{E[W_S|W_0] - \lambda.Q[W_s|W_0]\}$, where $\hat{v}$ is the value function. In this formulation, W is the wealth, which is calculated as $W_{s+\Delta s} = W_s.(\hat{x}_s^\intercal r_t^e + r_f)$, $s = 0,\ \Delta s, ..., S-\Delta s$, whereas r_f is the return of the risk-free asset and $r_s^e = [r_s^e(1), ..., r_s^e(n)]$ is the vector of returns of the risky assets during $[s, s+\Delta s]$. Cong & Oosterlee (2017) argued that solving this problem using dynamic programming is difficult because of the non-linearity of conditional variance, so they replaced the dynamic mean-variance problem with a dynamic quadratic optimization problem. The new formulation is a target-based optimization since the risk aversion coefficient acts similar to an investment target in the problem. Moreover, solving the dynamic mean-variance PSP based on target-based optimization ensures time-inconsistency or "pre-commitment strategy", which means that the investor has committed to an initial investment strategy. However, in many cases, investors do not want to commit to an initial investment strategy. Therefore, Basak & Chabakauri (2010) suggested a time-consistency restriction in the formulation that can be solved in a backward recursive manner. Nevertheless, in both cases of pre-commitment and time-consistency strategies, the mean vector and the covariance matrix of returns of risky assets are subject to uncertainty. Hence, Cong & Oosterlee (2017) proposed robust pre-commitment and time-consistency strategies where stationary and non-stationary formulations generate portfolios with the same Sharpe ratio given the risk-free asset as a benchmark. Jiang & Wang (2021) proposed a multi-period, multi-objective PSP where the objectives are the expected value and variance of the portfolio returns. To consider parameter uncertainty, an ellipsoidal uncertainty set is used for asset returns, leading to a SOCP. Moreover, a weighted-sum approach is used to obtain the Pareto frontier of the solutions.

Volatility measures can be used to define an arbitrage opportunity, which is a portfolio that can be formed with a negative investment while its profit is positive. Pinar & Tütüncü (2005) considered n risky assets, where ν_j is the period-end value of \$1 invested in asset j at the beginning of the period. They used $\nu = (\nu_1, ..., \nu_n)$ as the vector of the end-of-period values, which $\bar{\nu}$ is its expected value and Q is its covariance matrix. The vector of return is defiend as $r = \nu - e$. If ν is known in advance, a portfolio x that satisfies $\bar{\nu}^\intercal x \geq 0,\ x^\intercal Q x = 0,\ e^\intercal x \leq 0$ corresponds to an arbitrage opportunity. Since $x^\intercal Q x = 0$ then there is not any deviation in the return of assets form their expected values. These conditions mean that there is a portfolio that can be formed with a negative investment while its profit is positive. In practice $x^\intercal Q x$ cannot be equal to zero. An investor can assume that a random number is "rarely" less than its mean minus θ times of its standard deviation as $\bar{\nu}^\intercal x - \theta\sqrt{x^\intercal Q x} \geq 0,\ e^\intercal x \leq 0$. Pinar & Tütüncü (2005) demonstrated that these conditions are related to an RO approach with an ellipsoidal uncertainty set. They also developed a multi-period PSP formulation by defining a self-financing constraint, in which the investment amount in the second period is based on the income of the first period. The end-of-period value of \$1 invested in an asset at each period is uncertain and belong to an ellipsoidal uncertainty set. An adjustable RO approach was used to handle uncertainty.

While most robust PSPs are modelled under the assumption that investors are perfectly rational beings, Liu et al. (2015) argued that the rationality assumption does not always hold. Studies of behavioral finance have found that the axioms of rationality are violated across a range of financial decision-making situations. The prospect theory delineates the behavior of investors and asserts that investors value gains and losses differently. Liu et al. (2015) proposed a robust multi-period PSP based on the premises of the prospect theory. Instead of classical utility or disutility functions, an S-shape value function, originally introduced by Kahneman & Tversky (2013), is used to model the investor perception towards return. To account for uncertainty in cumulative asset returns, a budget uncertainty set whose level of conservatism can be controlled is utilized. However, applying the prospect theory value function leads to a complex nonlinear programming model that is intractable. Therefore, an improved particle swarm optimization (PSO) algorithm was used to solve the problem.

Besides the uncertainty of individual asset returns at each period, macroeconomic conditions represent another source of uncertainty. Desmettre et al. (2015) proposed a formulation for a multi-period investment problem under uncertainty introduced by uncertain market crash sizes in an interval. The objective is to maximize the terminal wealth. This problem uses a min-max worst-case scenario formulation that can be solved analytically. However, an interval uncertainty set results in over-conservative solutions.

Another way to represent the uncertainty of parameters is by using discrete scenarios, which often lead to less complex formulations compared to those based on continuous uncertainty sets. The next section focuses on the use of discrete scenarios in robust multi-period PSPs.

2.5.2 Robust Discrete Scenarios and Decision Tree Models

Mulvey et al. (1995) developed a robust framework based on discrete scenarios in which infeasibility is allowed under some scenarios but is penalized in the objective function. Application of this approach to robust multi-period PSPs usually leads to less complex formulations than for robust problems that use continuous uncertainty sets. Pınar (2007) considered a two-period PSP in which the returns of risky assets are uncertain, and used a discrete scenario tree to model uncertainty. In another attempt, Oguzsoy & Güven (2007) proposed a robust multi-period PSP with rebalancing and transaction costs. The problem is formulated as an MIP since its decision variables are the number of shares. They also developed a scenario-based, multi-period, mean-variance PSP, in which a decision tree with different levels is used. Portfolio rebalancing can happen at any level of the decision tree, and each tree node shows different rival scenarios for returns and risk (variance). A min-max formulation is used to find the worst-case robust solution. The robust counterpart considers risk scenarios at each node, time period and return realization . Since it is very unlikely that the worst scenario across all dimensions is realized, this approach leads to overly conservative solutions. Conversely, Shen & Zhang (2008) used semi-variance as a disutility function (*i.e.,* risk measure), which penalizes only negative deviations. Both asset returns in each scenario and the conditional probabilities of scenarios are treated as uncertain parameters. Ellipsoidal uncertainty sets are for returns of assets at each scenario, which leads to a SOCP.

Two-stage stochastic programming is a practical framework for modeling uncertainty in optimization problems. In this approach, decision variables are divided into "here-and-now" and "wait-and-see" variables. The mathematical formulation of a two-stage stochastic programming is $\min_{x \in X} c^\intercal x + \mathbb{E}[F(x, \xi(w))]$, where $F(x, \xi(w)) = \min f(w)^\intercal y,\ \ s.t.\ A(w)x + Dy = b(w),\ \ y \geq 0$, where x is a "here and now" decision variable, y is a "wait and see" decision variable, $\mathbb{E}(.)$ is the expected value, $\xi(w) = (f(w), A(w), b(w))$ is the uncertain vectors, and D is the fixed recourse matrix. Ling et al. (2017) argued that because two-stage stochastic programming is a risk-neutral approach, it is not suitable for a certain setting, and developed a two-stage stochastic program with a mean-risk aversion concept as $\min_{x \in X} c^\intercal x + \mathbb{E}[F(x, \xi(w))] + \lambda \rho(F(x, \xi(w)))$, where ρ is a risk measure and $\lambda \geq 0$ is a trade-off coefficient that captures the risk-aversion attitude of the decision maker. To tackle the same problem, Ahmed (2006) used variance as a risk measure, while Ling et al. (2017) used CVaR as a risk measure leading to the less complex formulation

$\min_{x \in X} c^\intercal x + \mathbb{E}[F(x, \xi(w))] + \lambda CVaR_\alpha(F(x, \xi(w)))$. Ling et al. (2017) assumed that asset returns in the first stage belong to a set of scenarios with known probabilities, whereas the distribution function of asset returns in the second stage belongs to an ambiguity set with uncertainty about the first two moments. This approach results in a SDP formulation. Even though using discrete scenarios and a decision tree for a multi-period PSPs lead to tractable formulations, identifying all possible scenarios might be challenging.

2.5.3 Robust Regime Dependent Models

Liu & Chen (2014) argued that stock prices are affected by market conditions, which are assumed to follow a Markov regime-switching process. Specifically, in each market regime, financial parameters have different distribution functions. An approach to deal with parameter uncertainty in different market conditions is by using regime-dependent robust formulations. Liu & Chen (2014) described the time-varying properties of random returns by using a nonlinear dynamic model between periods. They assumed different uncertainty sets for each market situation. VaR is used as the basic risk measure in the formulation, where the distribution function of asset returns is uncertain and belongs to a moment-based ambiguity set. A restrictive assumption made in this study is that uncertainty sets of adjacent periods are independent and static, whereas in reality they usually are dynamic and dependent. Liu & Chen (2018) considered dependency of dynamic uncertainty sets between adjacent periods in their formulation. Moreover, instead of VaR, they used CVaR as the risk measure. Similar to the formulation of Liu & Chen (2014), it is assumed that moments of the loss function distribution are known and fixed, which leads to a SOCP formulation.

Yu (2016) also applied the regime-switching uncertainty set approach on a mean-CVaR PSP where the loss function is assumed to be the difference in wealth between times $s-1$ and s. This practically means that at each period, there is a different loss function which results in a different CVaR constraint. Because at each market state the risk-free rate of return can also change, risky asset returns, risk-free asset returns, and the distribution function of the loss function (probability of each scenario) are assumed to be uncertain and belong to ellipsoidal uncertainty sets. A three-step algorithm is used to find optimal solutions of the multi-period PSP based on different market states. An important advantage of this multi-period PSP is that it captures both regime-switching and parameter uncertainty simultaneously, leading to a more practical formulation than classical robust multi-period PSPs.

2.5.4 Asset-Liability Management Problem

Asset Liability Management (ALM) entails the allocation and management of assets, equity, interest rate, and credit risk (including risk overlays) to cover the commitments (*i.e.,* debts). In this section, we survey applications of RO in ALM problems.

Van Hest & De Waegenaere (2007) demonstrated that there are two types of investment strategies in an ALM problem: passive risk management, and active risk management. In the passive strategy, allocation of budget among different benchmarks such as equity, bonds, real estate *etc.* is the main decision. In active risk management, decisions are about tactical and operational investment activities that involve a number of investment managers, each is assigned a specific benchmark category. A formulation that calculates the total return of each manager by solving a mean-variance PSP based on the calculated expected value and variance of investment returns is proposed. These parameters are assumed to belong to ellipsoidal uncertainty sets. Practically, this robust ALM problem is a mean-variance PSP while the expected return and variance of asset returns belong to uncertainty sets.

Iyengar & Ma (2010) assumed that the source of uncertainty of asset returns are fundamental factors. Then, a factor model can capture the true uncertainty of asset returns instead of predefined nominal asset returns. Using robust factor models in ALM problems can enable the true sources of uncertainty to be captured, leading to more realistic formulations with better out-of-sample performance. Iyengar & Ma (2010) developed a RO formulation for pension fund management, which is an ALM problem with a constraint on funding ratio. This ratio indicates the value of assets to the present value of liabilities that are used in a chance constraint, where the probability that funding ratio is greater than a threshold should be greater than a confidence level. The present value of liabilities depends on the interest rate, whereas asset values depend on their rate of return. In the proposed formulation, the funding ratio is assumed to be an uncertain parameter that follows a factor model by a function that defines stochastic parameters. A Gaussian process for factors of uncertain parameters is considered. Parameters of factor models are assumed to belong to an ellipsoidal uncertainty set, which results in a SOCP. Platanakis & Sutcliffe (2017) proposed a factor model for asset returns and liabilities in which factor loading belongs to an ellipsoidal uncertainty set, asset returns and liabilities belong to box uncertainty sets, and the covariance matrix of disturbances has upper and lower bounds on its elements. It has been shown that this problem can be reformulated into a SOCP.

Gülpinar & Pachamanova (2013) used time-varying investment opportunities to propose a robust ALM. This method assumes that a future rate of return of an asset depends on its rate of return in a former period. They augmented the multi-period PSP formulation of Dantzig & Infanger (1993) by adding liabilities and a funding ratio constraints. The transformation of Ben-Tal et al. (2000) was also used to simplify the formulation, by which the cumulative rates of return of assets are the uncertain parameters that belong to an ellipsoidal uncertainty set. Asset returns and interest rates are assumed to follow a vector-autoregressive (VAR) process that captures the time-varying aspect of investment. Unlike the symmetric uncertainty sets assumption in other robust ALM problems, Gülpınar et al. (2016) developed a robust ALM problem using asymmetric uncertainty set, which captures the structure of uncertainty more accurately. Recently, Gajek & Krajewska (2021) proposed a robust ALM formulation where the interest rate is uncertain and the distribution function of the uncertain parameters belongs to a nonempty ambiguity set. This formulation bounds from above VaR of the change in the portfolio value due to interest rate model violation.

2.5.5 Robust Control Formulation

Robust control methods are designed to function properly provided that the uncertain parameters or disturbances are contained within some bounded/compact sets. Flor & Larsen (2014) developed a robust control formulation for an investment PSP. They assumed that an investor has access to stocks, bonds, and cash while interest rates are uncertain. In this formulation, a robust control, time-continuous formulation for the uncertainty of interest rate is developed. Results showed that the proposed model is more sensitive to the ambiguity about stocks than bonds. This problem is time-continuous, thus is formulated using differential equations.

Glasserman & Xu (2013) developed a robust control formulation for a multi-period PSP based on a factor model that is used to calculate the return of assets at the next period. They assumed that the factors are mean reverting and evolving and that their value at any time is a function of their previous time value and its residual. Two regression models were used. In the first model, a factor model calculates the return of assets at the next period. The second factor model calculates factor values at the next period. Sources of uncertainty in this PSP are the residuals of the two factor-models. Based on this formulation, the goal is to maximize the net present value of risk-adjusted excess gains by considering restricted transaction costs. Moreover, models are developed in two cases; finite-horizon investment, and infinite-horizon investment. A robust formulation based on the Bellman equation, leading to a dynamic programming model, is used. Results showed that the robust control formulation of Glasserman & Xu (2013) is more robust than deterministic formulations against perturbations of uncertain parameters.

Bo & Capponi (2017) applied a robust control approach for the credit portfolio, where the impact of credit risk model misspecification on the optimal investment strategies is measured. They proposed a formulation for a dynamic credit portfolio that accounts for robust decision rules against misspecifications of a model for the actual default intensity. Default intensity is defined as the probability of default for a certain time period conditional on no earlier default. In this formulation, an investor can invest in the money market and bonds by a pricing model of bonds that considers credit intensity. This portfolio formulation tries to maximize wealth while default intensity is uncertain.

2.6 Other Financial Problems

In this section, special PSP formulations are reviewed, including Log-robust portfolio selection, robust index tracking, hedging formulation, risk-adjusted Sharpe ratio, scenario-based formulation, and robust data envelopment analysis (DEA) for PSPs.

2.6.1 Log-Robust Portfolio Selection

Hull (2003) defined the Log-return as the equivalent, continuously-compounded rate of return of asset returns over a period of time. Log-return is calculated by taking the natural log of the ending stock price divided by the beginning value. It is based on a Levy process that represents the movements of a stock price whose successive displacements are random, independent, and statistically identical over different time intervals of the same length. Assume that Log-return of stock j at time S can be described as $Ln\frac{Pr_j(S)}{Pr_j(0)} = (\mu_j - \frac{\sigma_j^2}{2})S + \sigma_j\sqrt{S}z_j$, where S is the length of the time horizon, $Pr_j(0)$ is the initial price of stock j, $Pr_j(S)$ is the stock price at time S, μ_j is the drift of the Levy process for stock j, and σ_j is the standard deviation of the Levy process for stock j. Kawas & Thiele (2011a) proposed a Log-robust PSP where the scaled deviation belongs to a budget uncertainty set in two cases: correlated and uncorrelated assets. Let the uncertainty be represented as $\sum_{j=1}^{n}|\tilde{z}_j| \leq \Gamma,\ |\tilde{z}_j| \leq 1,\ \forall j$. Then, the robust problem can be formulated as $\max_{\tilde{x}}\min_{\tilde{z}}\sum_{j=1}^{n}\tilde{x}_j Pr_j(0) exp[(\mu_j - \frac{\sigma_j^2}{2})S + \sigma_j\sqrt{S}x\tilde{z}_j]$, s.t. $\sum_{j=1}^{n}|\tilde{z}_j| \leq \Gamma,\ |\tilde{z}_j| \leq 1\ \forall j,\ \sum_{j=1}^{n}\tilde{x}_j Pr_j(0) = B_0,\ \tilde{x}_j \geq 0\ \forall j$, where B_0 is available budget. Kawas & Thiele (2011a) transformed this formulation into an LP. They also considered a PSP with correlated assets, where $Ln\frac{Pr_j(S)}{Pr_j(0)} = (\mu_j - \frac{\sigma_j^2}{2})T + \sqrt{S}Z_j$, where Z has normal distribution with mean 0 and covariance matrix Q. They defined $Y = Q^{\frac{-1}{2}}Z$, where $Y \sim N(0, I)$. Kawas & Thiele (2011a) proposed a tracktable robust counterpart in the case correlated assets. Kawas & Thiele (2011b) extended the Log-robust PSP by allowing short selling, whereas Pae & Sabbaghi (2014) added a transaction cost constraint to make the formulation more realistic. Instead of using predefined uncertainty sets, Kawas & Thiele (2017) proposed a data-driven Log-robust PSPs for two cases, correlated and uncorrelated assets. In both cases, they optimized the worst-case PSP over the worst of finitely many polyhedral uncertainties sets using different estimation methods. Consequently, both the uncertainty of parameters and the ambiguity of uncertainty sets are considered. However, the robust formulations are based on the worst-case perspective and the solutions are still over-conservative. In contrast, Lim et al. (2012) proposed a relative robust log-return PSP which is less conservative than the worst-case Log-robust PSP, yet harder to solve.

Gülpinar et al. (2014) studied the robust PSP under supply disruption in the petroleum markets based on Log-return. They proposed a framework for portfolio management with a combination of commodities and stocks when the supply of commodities is uncertain. A geometric mean-reverting jump process is considered for prices to model the jumps (*i.e.,* large discrete movements). Both symmetric (ellipsoidal, and D-norm uncertainty sets) and asymmetric uncertainty sets for uncertain parameters are used. Results show that the D-norm uncertainty set leads to more extreme portfolio allocations with less diversification than the ellipsoidal and asymmetric uncertainty sets. Moreover, the asymmetric uncertainty set with a high price of robustness results in a high level of diversification.

2.6.2 Index-Tracking Portfolio Selection

Index tracking is a passive investment strategy where a portfolio is formed to follow an index benchmark. Hence, a logical index-tracking portfolio includes all stocks under an index based on their value weights. However, the need for frequent re-balancing transactions to closely track the index might lead to high transaction costs. Therefore, decision-makers might try to find the best possible combination of assets that follows a benchmark index with the lowest possible transaction cost, while also accounting for parameter uncertainty. Costa & Paiva (2002) developed two robust index-tracking PSPs where the return vector and the covariance matrix of risky assets are uncertain and belong to polytope uncertainty sets. Practically, the variance of tracking error (*i.e.,* the difference in actual performance between a portfolio and its corresponding benchmark) is used to capture the volatility of tracking error, leading to a quadratic programming (QP) formulation. In this formulation, the return of a given portfolio x is calculated as $x^\intercal r + (1 - x^\intercal)r_f$, whereas the return of the benchmark portfolio (index), denoted by x_B, is calculated as $x_B^\intercal r + (1 - x_B^\intercal)r_f$. With that, the tracking error is calculated as $tr(x) = (x - x_B)^\intercal r + (x_B - x)^\intercal r_f$, and the expected value and the variance of tracking error are $\rho_\varphi(x) = (x - x_B)^\intercal r + (x_B - x)^\intercal r_f = (x - x_B)^\intercal(\hat{r} - r_f)$, and $\sigma_Q^2(x) = (x - x_B)^\intercal Q(x - x_B)$, respectively. Hence, the problem is formulated as $\min_{x \in X} \ \sigma_Q^2(x)$ s.t. $\rho_\varphi(x) \geq E$, where E is the minimum acceptable target for the expected value of tracking error. Costa & Paiva (2002) assumed that r, r_f, and Q are not exactly known. Thus, they defined a set of all possible matrices

$\Phi \in Con[\Phi_1, ..., \Phi_n]$ where $\Phi = \begin{pmatrix} Q & r \\ 0 & r_f \end{pmatrix}$ and showed that the robust index tracking formulation can be transformed to a tractable formulation by using a linear matrix inequality. However, estimating the covariance matrix is computationally expensive in large problems. Hence, instead of the variance of tracking error, Chen & Kwon (2012) proposed a robust similarity measure that measures pairwise similarities between the assets and the targeted index, with a budget uncertainty set. Moreover, a cardinality constraint is used to limit the number of assets in the optimal portfolio, leading to a MIP.

The aforementioned robust index-tracking PSPs ignore the distribution function of asset returns. Alternatively, one can use partial information about the distribution of asset returns based on historical data. Ling et al. (2014) developed a distributionally robust downside risk measure formulation for index-tracking PSPs with a moment-based ambiguity set in two cases: 1) the first two moments (mean and covariance) are known and fixed, 2) the first two moments belong to ellipsoidal and polyhedral uncertainty sets, respectively. Results demonstrate that the distributionally robust index-tracking PSP provides less conservative solutions than classical robust index-tracking PSPs.

2.6.3 Robust Hedging

Hedging means an investment position intended to offset potential losses or gains that may be incurred by a companion investment. Options are important financial tools used for Hedging risk. An option is the opportunity, but not the obligation, for buying or selling underlying assets. Lutgens et al. (2006) used options to propose a RO formulation for hedging risk in two cases: a single stock and an option, and multiple assets and options. In the former case, they optimized the expected return while assuming that asset returns belongs to a discrete (scenario-based) uncertainty set. This formulation led to a max-min problem with a nonlinear inner optimization problem. In the second case, they assumed that the return vector belongs to an N-dimensional ellipsoidal uncertainty set, which results in a SOCP.

Gülpınar & Çanakoḡlu (2017) used weather derivatives in a PSP in which CVaR is the risk measure. Weather derivatives are traded as financial instruments between two parties. The seller agrees to bear the risk for a premium and makes a profit if nothing happens. However, if the weather turns out to be bad, then the buyer claims the agreed amount. The price of this specific derivative is a function of the weather. Gülpınar & Çanakoḡlu (2017) suggested a spatial temperature modeling where the correlation between the locations of weather derivatives under consideration are explicitly taken into account. Both symmetric (ellipsoidal) and asymmetric uncertainty sets are used to develop robust counterparts. Experimental results showed that a robust model with weather derivatives has better performance in the worst-case analysis.

2.6.4 Robust Sharpe and Omega Ratio

The Sharpe ratio (SR) is defined as a ratio of the expected excess return over the risk-free rate to the standard deviation of the excess return. However, parameters of the Sharpe ratio are subject to uncertainty. In practice, an estimate of the Sharpe ratio is used in optimization problems. To mitigate the estimations error, Deng et al. (2013) proposed a robust risk-adjusted Sharpe ratio and a robust VaR-adjusted Sharpe ratio (VaRSR), defined as the lowest Sharpe ratio consistent with the data in the observation period for a given confidence level. Based on the normality assumption of asset returns, Zymler et al. (2011) argued that an uncertainty set for a Sharpe ratio can take the form of an ellipse with exogenous parameters. They then showed that in one dimension, the uncertainty set is an interval where the inner-optimization solution in the robust formulation of a Sharpe ratio is exactly equal to a risk-adjusted Sharpe ratio. Results showed that VaRSR is more robust than SR when the return distribution is non-normal.

Maximizing Sharpe ratio is an important performance measures in PSPs. However, PSPs are prone to estimation errors and optimization amplifies estimation errors, resulting in portfolios with poor out-of-sample performance. One way to deal with this drawback is combination portfolios. Here, the portfolio is a linear combination of two or more prespecified portfolios. A proper combination can improve Sharpe ratio of the portfolio. Chakrabarti (2021) proposed a combination of robust minimum-variance and maximum Sharpe ratio based on a robust regret-minimizing portfolio.

They used box uncertainty sets for the asset returns and the covariance matrix. Finally, each portfolio is scored based on its worst-case regret and the optimal portfolio is the one with the smallest worst-case regret. Results showed that this portfolio is relatively close to the optimal combination portfolio for the actual parameter values.

Omega, an important ratio in finance proposed by Keating & Shadwick (2002), is the ratio of risk to return, assuming there is a predetermined threshold that partitions the returns into losses and gains. This ratio is an alternative to the Sharpe ratio and is based on information the Sharpe ratio discards. In practice, Sharpe ratio considers only the first two moments of the return distribution while Omega ratio considers all moments. Kapsos et al. (2014b) showed that the Omega ratio can be represented as an LP model. Kapsos et al. (2014a) introduced the worst-case Omega ratio (WCOR) when distribution functions of asset returns are partially known and belong to three different ambiguity sets. First, the underlying distribution is a mixture distribution with known continuous mixture components but unknown mixture weights. The second ambiguity set encompasses all possible distributions supported on a discrete set of scenarios. The third one uses box and ellipsoidal uncertainty sets for the probabilities of scenarios. Even though the Omega ratio considers both losses and gains, Sharma et al. (2017) argued that this approach is too sensitive to threshold used. Moreover, there is not any systematic way to specify this threshold. The formulation in Ghahtarani et al. (2019) uses the fundamental value of an asset as a threshold of the Omega ratio, which protects the portfolio against bubble conditions in the market. Sharma et al. (2017) redefined the Omega ratio by using a loss function instead of the return. Hence, it minimizes losses greater than a threshold and maximizes losses less than the same threshold when CVaR is used as the threshold. Furthermore, they developed a distributionally robust Omega-CVaR optimization formulation in which the probability of each scenario of the loss function is uncertain and belongs to three uncertainty sets: a mixed uncertainty set, a box uncertainty set, and an ellipsoidal uncertainty set. The first two uncertainty sets lead to LPs, whereas the last one results in a SOCP. Yu et al. (2019) compared results of the worst-case Omega ratio to those of the worst-case CVaR and relative CVaR formulations while adding transaction costs constraint and allowing short selling. Results show that the worst-case Omega portfolio yields lower loss values and higher market values compared to

CVaR-based models under various confidence levels. Georgantas et al. (2021) also compared the robust Omega ratio PSP proposed by Kapsos et al. (2014a) to the robust mean-variance PSP with box and ellipsoidal uncertainty sets and the robust CVaR PSP proposed by Zhu & Fukushima (2009). Results showed that robust PSPs are less diversified than their nominal counterparts. However, improvements were observed in the portfolio performance. Another comparison in this context has been done by Sehgal & Mehra (2021). They compared PSPs based on robust Omega ratio, semi-mean absolute deviation ratio, and weighted stable tail adjusted return ratio (STARR) with their non-robust counterparts. In these formulations, a budgeted uncertainty set is used for asset returns. Results showed that the robust formulations outperform the nominal problems with respect to standard deviation, value at risk (VaR), conditional value at risk (CVaR), Sharpe ratio, and stable tail adjusted return ratio (STARR).

Sharpe and Omega ratios are based on the absolute volatility of assets. However, some investors make decisions based on the volatility of an asset compared to the market and not on the absolute volatility itself. Beta is a measure of volatility that indicates whether an asset is more or less volatile compared to the market. Hence, Beta can be used as a decision criteria to capture the volatility of an asset compared to the market. A asset's Beta is calculated by dividing the product of the covariance of the asset returns and the market returns by the variance of the market returns over a specified period. However, this measure is subject to uncertainty since all components of the Beta formula are uncertain parameters. Ghahtarani & Najafi (2013) proposed a robust multi-objective PSP where the objectives are the portfolio rate of return and its systematic risk (Beta). A budget uncertainty set to model the uncertainty of Beta is used. The problem is reformulated as a tractable goal program. Results show that portfolios selected based on the robust Beta outperform non-robust Beta portfolios in terms of weight stability and return volatility.

2.6.5 Robust Scenario-Based Formulation

Unlike Section 2.5 which reviews scenario-based formulations of multi-period PSPs, this section focuses on the use of discrete scenarios to represent uncertainty in single-period PSPs. Kouvelis & Yu (1997) proposed a robust formulation for a discrete scenario-based uncertainty set. It optimizes an objective function based on the worst possible scenario, which leads to the worst-case conservative results. Roy (2010) proposed a new definition for robust scenario-based solutions in which a solution is robust if it exhibits good performance in most scenarios without ever exhibiting very poor performance in any scenario. Then, they developed *bw-robustness*, by taking into consideration minimum acceptable objective value and a target objective value to achieve, or exceed if possible. Gabrel et al. (2018) developed a robust scenario-based PSP by using both worst-case scenario and *bw*-robustness to maximize a portfolio's return while returns of assets belong to a discrete scenario-based uncertainty set. Moreover, they introduced a new robustness criterion called *pw-robustness*, in which instead of maximizing a proportion of scenarios that their values are greater than or equal to a threshold, the decision-maker specifies a fixed proportion of scenarios, and maximizes the value of the soft bound. The *pw*-robustness formulation is a MIP. To circumvent the computational time issue, Gabrel et al. (2018) proposed two heuristic methods that can be used to obtain quick solutions for problems of large sizes.

Some investors might invest based on their preferences of assets, where ranking information of assets are uncertain. Nguyen & Lo (2012) proposed a robust ranking mean-variance, which is similar to the classical mean-variance. However, the ranking of assets is used instead of the return of assets. Formulations were developed in two cases: the maximum ranking with and without risk (variance). The ranking of assets belongs to a discrete uncertainty set, which leads to a MIP solved by a constraint generation method.

2.6.6 Robust Data Envelopment Analysis and Portfolio Selection

One way for evaluating stocks or assets in the financial markets is data envelopment analysis (DEA), in which the efficiency of stocks or assets is evaluated based on a set of inputs and outputs (criteria). Based on this method, units (assets or stocks) are divided into two parts: efficient, and inefficient. Consequently, DEA calculates the efficacy rate of units. Peykani et al. (2016) demonstrated that the efficiency of stocks in DEA depends on inputs and outputs, which are uncertain. Consequently, they proposed robust DEA with a budget uncertainty set. Results of robust DEA are more robust than a non-robust DEA formulation with respect to the efficiency of stocks. However, their formulations can be used only for continuous uncertainty sets. Peykani et al. (2019) developed a robust DEA for a discrete scenario formulation with uncertainty, which expands the application of robust DEA to financial problems in the real-world. However, these robust DEA formulations provide the efficiency ratio without any detail about the amount of money invested in each asset while an investor needs to know the proportion of investment of funds invested in each asset. Peykani et al. (2020) proposed a two-phase portfolio selection process. At the first stage, the efficiency of candidate stocks is evaluated by robust DEA. In the second stage, the optimal portfolio is formed by using robust mean-semi variance-liquidity and robust mean-absolute deviation-liquidity models. In both phases, budget uncertainty sets are used for the uncertain parameters. This two-phase formulation provides two filters (robust DEA, and robust PSP) to find the optimal portfolio.

Table 2.2 lists all the reviewed articles (n=142) in chronological order and classifies them based on the problem type (PSP), uncertain parameters (UP), the structure of uncertainty or ambiguity sets used in the robust formulation (U/A set), the robust optimization method employed to deal with uncertainty (RO method) and the class of the tractably reformulated problem (Model).

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
1	Ben-Tal et al. (2000)								*						*						*						*							*				

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
2	Costa & Paiva (2002)											*			*	*					*						*						*					
3	Lauprete et al. (2003)	*													*																*		*					
4	Goldfarb & Iyengar (2003)	*				*		*					*				*				*						*							*				
5	Ghaoui et al. (2003)					*		*										*					*	*				*										
6	Tütüncü & Koenig (2004)	*													*	*					*						*						*					
7	Pinar & Tütüncü (2005)								*						*						*						*			*				*				*
8	Ceria & Stubbs (2006)	*													*	*					*										*			*				
9	Lu (2006)					*											*				*						*										*	
10	Pachamanova (2006)							*						*	*						*						*							*				
11	Lutgens et al. (2006)													*	*						*				*		*						*	*				
12	Fabozzi et al. (2007)	*													*						*						*							*				
13	Bienstock (2007)	*													*											*	*									*		
14	Garlappi et al. (2007)	*													*	*															*		*					
15	Calafiore (2007)	*		*														*						*				*									*	

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
16	Popescu (2007)					*		*										*					*					*					*					
17	Huang et al. (2007)							*										*		*			*					*							*			
18	Pınar (2007)								*						*										*		*					*						
19	Oguzsoy & Güven (2007)	*							*						*	*									*		*									*		
20	Van Hest & De Waegenaere (2007)	*								*					*						*						*							*				
21	Ma et al. (2008)					*											*				*						*						*					
22	Natarajan et al. (2008)							*									*					*					*							*				
23	Huang et al. (2008)							*										*		*						*		*				*						
24	Quaranta & Zaffaroni (2008)							*							*						*						*					*						
25	Bertsimas & Pachamanova (2008)								*						*						*						*					*						
26	Shen & Zhang (2008)								*					*	*						*				*		*							*				
27	Chen & Tan (2009)	*													*	*						*					*						*					
28	DeMiguel & Nogales (2009)	*													*																*		*					
29	Schöttle & Werner (2009)	*													*	*					*						*							*				

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
30	Zhu & Fukushima (2009)							*										*			*				*			*				*		*				
31	Natarajan et al. (2009)	*						*							*						*						*					*	*					
32	Lutgens & Schotman (2010)					*									*											*	*						*					
33	Natarajan et al. (2010)						*								*	*		*			*		*				*	*						*				*
34	Huang et al. (2010)							*										*							*				*			*		*				
35	Iyengar & Ma (2010)					*				*							*				*						*						*					
36	Iyengar & Ma (2010)									*					*						*						*							*				
37	Zymler et al. (2011)	*													*						*						*							*				
38	Gregory et al. (2011)	*													*						*						*						*					
39	Gülpınar et al. (2011)	*													*						*						*							*				
40	Moon & Yao (2011)			*											*						*						*					*						
41	Chen et al. (2011a)				*										*								*					*						*				
42	Lu (2011)					*											*				*						*							*				
43	Chen et al. (2011b)							*							*	*						*					*							*				
44	Hellmich & Kassberger (2011)							*										*			*												*					

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
45	Guastaroba et al. (2011)							*							*						*						*					*		*				
46	Kawas & Thiele (2011a)										*								*		*						*					*						
47	Kawas & Thiele (2011b)										*								*		*						*					*						
48	Zymler et al. (2011)												*	*	*						*						*							*				
49	Fonseca & Rustem (2012)	*													*					*	*						*							*				
50	Fonseca et al. (2012)	*													*					*	*						*							*				
51	Sadjadi et al. (2012)	*													*						*						*									*		
52	García et al. (2012)	*													*	*															*							
53	Ling & Xu (2012)					*											*				*						*							*				
54	Pflug et al. (2012)	*						*										*						*				*				*	*					
55	Lim et al. (2012)										*								*							*			*									
56	Chen & Kwon (2012)											*								*	*						*									*		
57	Nguyen & Lo (2012)	*																		*					*		*									*		
58	Hauser et al. (2013)	*													*	*					*								*					*				
59	Kim et al. (2013b)	*													*						*						*						*	*				
60	Kim et al. (2013a)	*													*						*						*						*	*				

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
61	Gülpinar & Pachamanova (2013)									*					*						*						*							*				
62	Glasserman & Xu (2013)					*			*								*														*							
63	Deng et al. (2013)							*					*		*											*	*						*					
64	Ghahtarani & Najafi (2013)												*							*	*						*					*						
65	Fliege & Werner (2014)	*													*	*					*						*							*				
66	Pınar & Paç (2014)	*													*			*			*		*				*	*						*				
67	Kim et al. (2014b)	*													*						*						*							*				
68	Kim et al. (2014c)					*											*				*						*							*				
69	Recchia & Scutellà (2014)														*						*						*					*		*				
70	Kakouris & Rustem (2014)							*										*								*		*									*	
71	Han et al. (2017)							*										*								*		*									*	
72	Zhu et al. (2014)							*										*		*	*						*	*				*		*				
73	Liu & Chen (2014)							*	*									*					*					*						*				
74	Flor & Larsen (2014)													*	*																*							
75	Pae & Sabbaghi (2014)										*								*		*						*					*						

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
76	Gülpinar et al. (2014)										*			*						*	*	*					*							*				
77	Ling et al. (2014)											*						*			*		*					*						*				
78	Kapsos et al. (2014a)												*					*			*				*			*				*		*				
79	Maillet et al. (2015)		*													*										*	*							*				
80	Kim et al. (2015)	*													*						*						*						*	*				
81	Doan et al. (2015)							*										*							*			*				*						
82	Rezaie et al. (2015)							*							*						*						*					*						
83	Marzban et al. (2015)								*						*						*						*					*						
84	Liu et al. (2015)						*		*						*						*						*						*					
85	Desmettre et al. (2015)								*											*						*	*					*						
86	Pınar (2016)	*													*						*						*							*				
87	Li et al. (2016)			*											*							*					*							*				
88	Rujeerapaiboon et al. (2016)							*										*					*					*							*			
89	Fernandes et al. (2016)								*						*											*	*					*						
90	Yu (2016)							*	*						*			*			*						*											
91	Gülpınar et al. (2016)									*					*						*	*					*							*				

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
92	Peykani et al. (2016)													*	*						*						*					*						
93	Xidonas et al. (2017a)		*													*									*		*									*		
94	Xidonas et al. (2017b)			*											*										*				*									*
95	Belhajjam et al. (2017)	*						*										*					*					*					*	*				
96	Yu et al. (2017)							*										*							*				*			*						
97	Cong & Oosterlee (2017)	*							*						*	*					*						*						*					
98	Ling et al. (2017)							*	*									*					*					*							*			
99	Platanakis & Sutcliffe (2017)					*				*					*					*	*						*							*				
100	Bo & Capponi (2017)													*																	*							
101	Kawas & Thiele (2017)										*								*		*						*	*				*						
102	Gülpınar & Çanakoğlu (2017)							*												*	*	*					*							*				
103	Sharma et al. (2017)							*					*					*			*				*			*				*		*				
104	Ding et al. (2018)	*													*	*								*				*									*	
105	Simões et al. (2018)	*													*	*					*								*					*				

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
106	Kapsos et al. (2018)		*													*					*				*		*						*					
107	Chen & Zhou (2018)	*													*	*					*						*				*			*				
108	Ghahtarani & Najafi (2018)			*											*						*						*					*						
109	Hasuike & Mehlawat (2018)					*		*										*					*					*					*					
110	Ghahtarani et al. (2018)							*										*					*					*				*						
111	Paç & Pınar (2018)	*						*										*						*				*				*	*					
112	Liu & Chen (2018)							*	*									*					*					*						*				
113	Gabrel et al. (2018)							*							*										*						*					*		
114	Lu et al. (2019)	*													*	*					*						*							*				
115	Chen & Wei (2019)	*													*	*					*						*				*			*				
116	Bai et al. (2019)	*													*	*					*						*				*			*				
117	Dai & Wang (2019)	*													*						*						*						*	*				
118	Plachel (2019)		*													*					*						*						*					
119	Liu et al. (2019)							*	*									*					*					*		*								*
120	Kang et al. (2019)							*										*					*					*						*				
121	Goel et al. (2019)							*										*								*		*				*						
122	Kara et al. (2019)							*							*											*	*					*						
123	Ling et al. (2019)				*	*			*						*							*					*							*				
124	Yu et al. (2019)							*					*					*			*				*			*				*						

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
125	Peykani et al. (2019)													*											*		*					*						
126	Khodamoradi et al. (2020)	*													*	*					*						*									*		
127	Lee et al. (2020)	*													*						*						*							*				
128	Peykani et al. (2020)	*		*											*						*						*					*						
129	Georgantas et al. (2021)	*						*					*		*			*			*						*	*				*						
130	Kouaissah (2021)													*	*						*						*							*				
131	Chakrabarti (2021)		*										*		*	*					*						*		*			*	*					
132	Benati & Conde (2021)							*							*										*				*									*
133	Ashrafi & Thiele (2021)													*	*						*						*					*						
134	Jiang & Wang (2021)	*							*						*						*						*							*				
135	Yin et al. (2021)	*													*						*						*						*	*				
136	Baviera & Bianchi (2021)	*																*						*				*					*					
137	Huang et al. (2021)							*							*								*					*				*						
138	Sehgal & Mehra (2021)			*				*					*		*						*						*					*						

Continued on Next Page

Table 2.2: Summary of the reviewed articles

#	Article	PSP type													UP [1]						U/A [2] set						RO Methods					Model						
		Mean-Variance	Minimum Variance	MAD	LPM	Factor-Based	Utility function	VaR/CVaR	Multi-Period	ALM	Log-return	Index tracking	Ratios	Others	Asset return	Variance-Covariance matrix	Factor-based parameters	Distribution function	Scale parameter (Log-Return)	Others	Classical	Asymmetrical	Moment-based	Metric-based	Discrete	Others	RO	DRO	Relative robust	Adaptive robust	Others	LP	NLP	SOCP	SDP	Mixed integer	Other Convex	Non-convex
139	Caçador et al. (2021)	*	*												*										*				*									*
140	Gajek & Krajewska (2021)							*		*								*							*			*					*					
141	Zhao et al. (2021)							*										*					*					*					*					
142	Swain & Ojha (2021)	*	*												*	*					*						*						*					

Moreover, Figures 2.5, 2.6, 2.7, 2.8, and 2.9 provide some statistics about the reviewed papers.

Figures 2.5, 2.6 show that %26 of published articles used mean-variance and %23 of published articles proposed robust VaR/CVaR formulations. Moreover, majority of robust PSPs leads to SOCP, and LP. The third robust counterpart type is NLP with %19. Figures 2.7, 2.8 show that %46 of published articles consider asset return as uncertain parameters. Another important classification of articles is based on type of uncertainty set. This figure also shows the distribution of uncertainty sets in published articles. It demonstrates that about %55 of articles use classical uncertainty sets include box, ellipsoidal, budgeted, and polyhedral. Figure 2.9 illustrates the distribution of RO methods in published articles. This figure shows that mostly classical RO and DRO are used in articles.

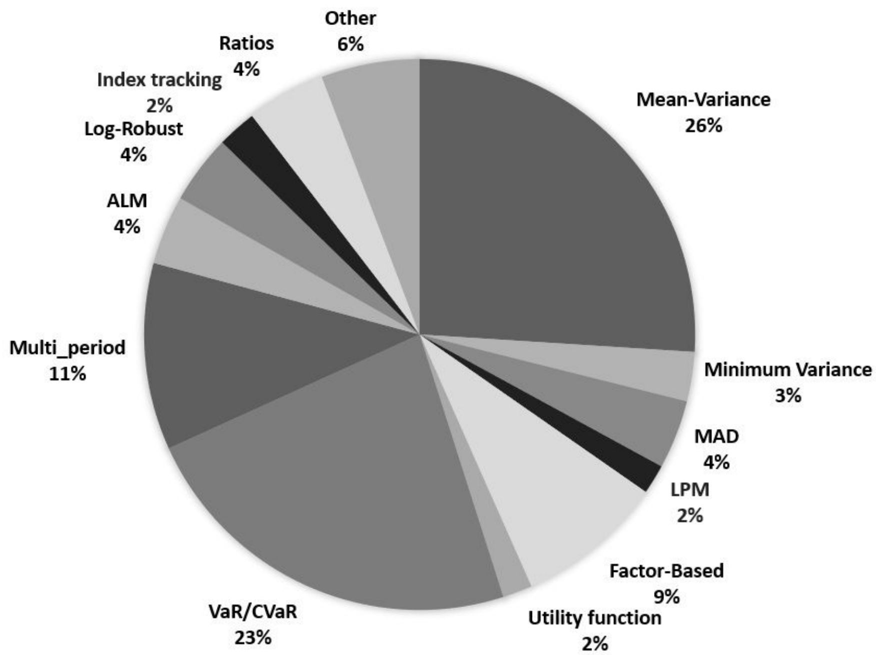

Figure 2.5: Distribution of articles based on PSP type

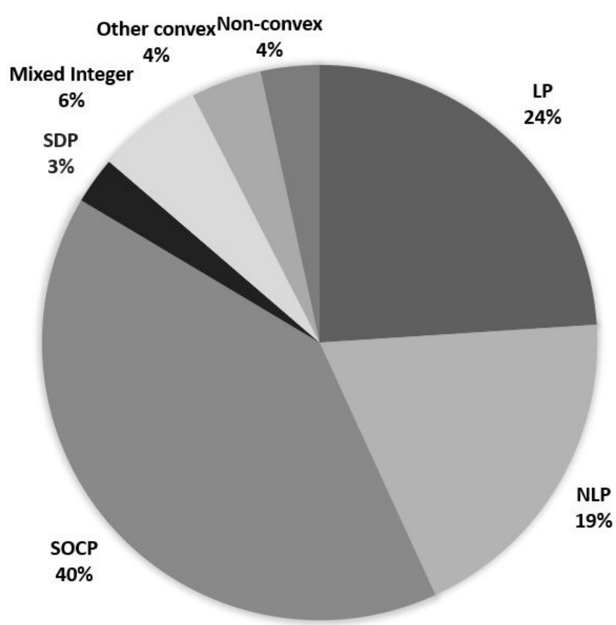

Figure 2.6: Distribution of articles based on optimization problem types

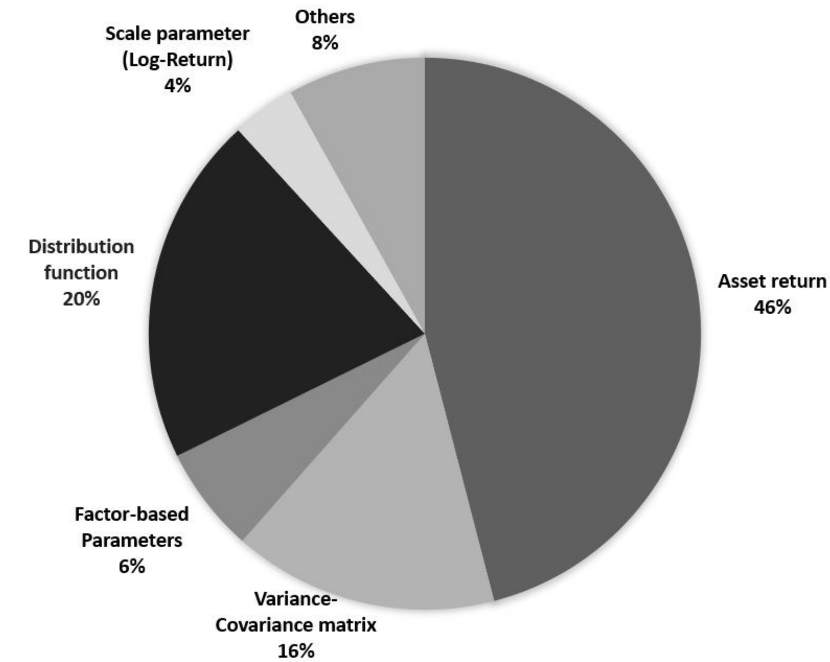

Figure 2.7: Distribution of articles based on uncertain parameters

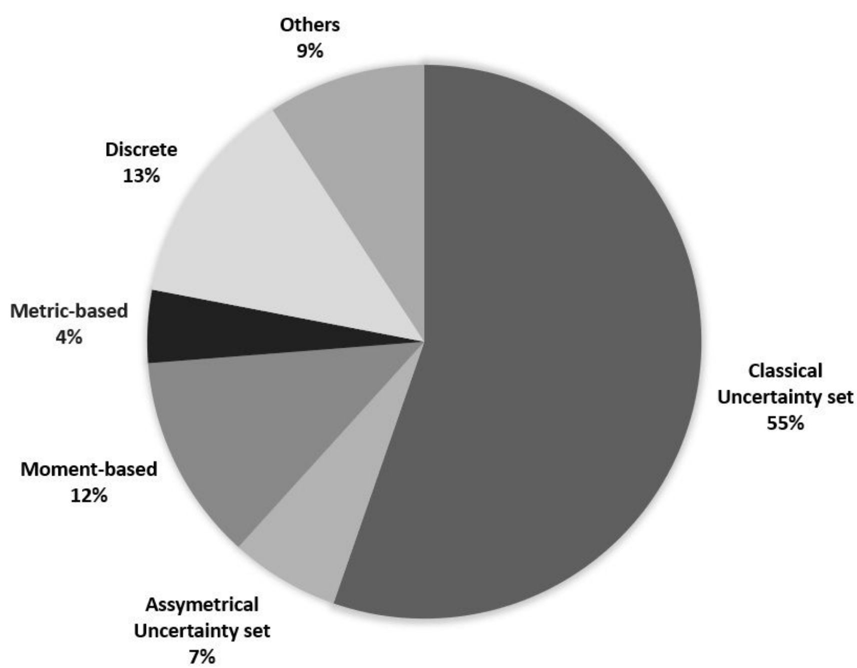

Figure 2.8: Distribution of articles based on uncertainty sets

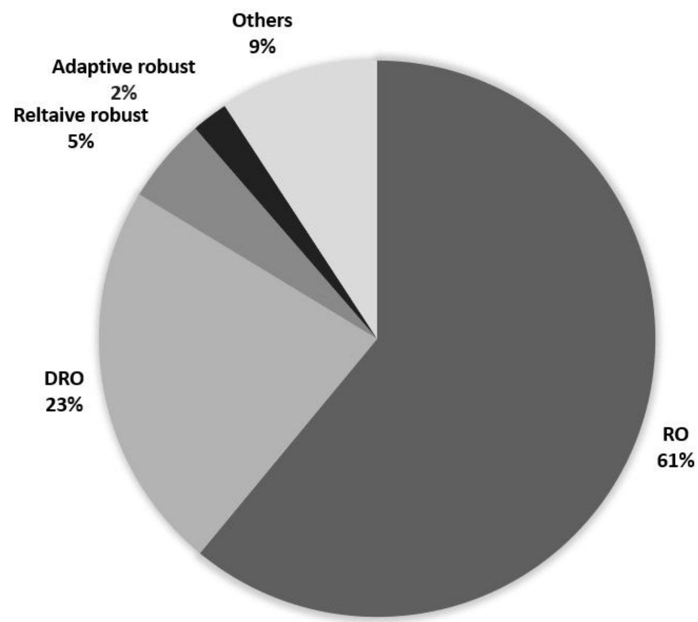

Figure 2.9: Distribution of articles based on RO methods

2.7 Conclusions and Future Research Directions

Portfolio selection has been a fertile area for applying modern RO techniques as evident by the large number of robust PSP articles published in the last two decades. The inherent uncertainty about future asset returns, the abundance of public data available and the risk-averse nature of most investors make RO an appealing approach in this area. As shown in this review paper, a wide range of robust PSP variants was studied, from a "plain vanilla" single-period, mean-variance PSP with a simple box uncertainty set (*e.g.,* Tütüncü & Koenig (2004)) to formulations that consider advanced risk measures (*e.g.,* Ghahtarani et al. (2018), Huang et al. (2010)), adaptive uncertainty sets (*e.g.,* Yu (2016)), real-life investment strategies (*e.g.,* Pflug et al. (2012), Paç & Pınar (2018)) and dynamic portfolio balancing (*e.g.,* Ling et al. (2019), Cong & Oosterlee (2017)). This variety of modeling assumptions and approaches and the overlaps among them make it difficult to develop a unifying framework for robust PSPs, yet we adopted a multi-dimensional classification scheme that depends on the risk measure to be optimized, the type of uncertain parameters, the approach used to capture uncertainty and the the planning horizon (*i.e.,* single- vs. multi-period).

Despite the surge of interest about robust PSPs in the research community, this area has received little attention from practitioners. A possible reason for such a rift between theory and practice is that research in this area was often driven by advancements in operations research methods rather than being in response to the real needs of the financial industry. Moreover, the value of using robust approaches might not be readily apparent to practitioners who are accustomed to classical PSP models. Therefore, experimental studies, like those presented in Kim et al. (2013b), Kim et al. (2014b), Kim et al. (2013a), Kim et al. (2018b), Kim et al. (2015), Schöttle & Werner (2009), are crucial for bridging this gap. The fact that tractable reformulations of most robust counterparts are more complex, both conceptually and computationally, than their corresponding deterministic formulations might make robust reformulations less attractive for practitioners (*e.g.,* Kouvelis & Yu (1997), Hauser et al. (2013), Simões et al. (2018), Lim et al. (2012), Huang et al. (2010)).

Nevertheless, the perception of robust optimization as an overly conservative portfolio selection approach is probably the major obstacle to its wide adoption by investment professionals. The reader can easily notice that this issue has received a lot of attention in the robust PSP research. Approaches proposed in the literature to attenuate the conservatism of robust formulations include: using controllable uncertainty sets (*e.g.,* ellipsoid (Fabozzi et al., 2007) or budget (Liu et al., 2015)), data-driven approaches (*e.g.,* Doan et al. (2015), Bienstock (2007)), alternative risk measures (*e.g.,* relative log-return (Lim et al., 2012) or (Huang et al., 2010)), distributionally robust optimization (*e.g.,* Ling et al. (2014)) and regime-dependent robust models (*e.g.,* Liu & Chen (2014), Yu (2016)). While these approaches can be effective in controlling conservatism and providing well-balanced solutions, they often lead to models that are challenging to handle since they increase the complexity of the problems.

Given that asset returns do not generally behave like independent random variables, but are instead dependent on common factors and have significant temporal correlations, trying to capture the variability of returns directly often leads to large uncertainty sets and hence conservative solutions. Instead, robust factor models deals with the uncertainty in the independent factors themselves, thus lead less conservative formulations. However, as Lu (2006) noted, selecting the suitable factors for the model and adjusting their weights are still worthy of further investigation. Another promising direction is to use *dynamic uncertainty sets*, that incorporate time-series models to capture auto-correlations in asset returns. Dynamic sets have been shown to result in less pessimistic solutions compared to static ones in other applications (Lorca & Sun, 2014, 2016).

An important advantage of financial markets is the abundance of historical data that can be used to build uncertainty and ambiguity sets for uncertain parameters. Although data-driven robust formulations for a few variants of the PSP have been proposed in the literature (*e.g.,* Bienstock (2007), Kawas & Thiele (2017), Rujeerapaiboon et al. (2016), Doan et al. (2015), Lotfi & Zenios (2018), Liu et al. (2019), Kang et al. (2019)), this is still a promising area for future research given the recently-proposed techniques for constructing and sizing uncertainty sets to achieve desirable properties (see *e.g.,* Bertsimas & Brown (2009), Bertsimas et al. (2018)). In a related matter, and as noted by Kang et al. (2019) in the context of robust CVaR optimization, it is still unclear which ambiguity sets should be used for DRO PSPs and how they should be sized to provide the best out-of-sample performance. With the plethora of ambiguity set structures proposed in recent year, investigating new variants of the distributionally robust PSPs is a plausible research direction.

Another promising research direction is the application of "soft" robust optimization methods to financial problems. A drawback of classical robust optimization is that it tries to capture most possible realizations of the parameters within the uncertainty set, which usually results in large sets and conservative solutions. Alternatively, one can construct smaller uncertainty sets that include only a subset of these possible realization and allow robust constraints to be violated, yet with penalties. Examples of these approaches include Globalized Robust Optimization (Ben-Tal et al., 2017), Robustness Optimization (Long et al., 2019) and Almost Robust Optimization (Baron et al., 2019). Soft robust optimization methods are still scarcely applied in the PSP literature (see Recchia & Scutellà (2014)), but have the potential for providing a trade-off between robustness and the quality of solutions.

Finally, in the context of the ALM problem, there are 3 main gaps in the literature. First and foremost, the integration of risk measures and DRO formulation in ALM problems is a crucial gap that requires thorough examination. Risk measures provide valuable insights into the uncertainty and potential losses associated with investment decisions. However, incorporating these risk measures effectively into ALM models and decision-making frameworks is a challenging task. The application of DRO in the context of ALM is another area that demands attention. DRO offers a powerful framework to address uncertainties and model ambiguity within optimization problems. However, its application and adaptation to ALM problems have not been extensively explored. Furthermore, the development of solution methods tailored specifically to ALM problems involving binary decision variables under uncertainty is an area that has received limited attention. Binary decision variables play a crucial role in ALM models, enabling the inclusion of strategic decisions such as asset allocation and liability hedging. However, finding efficient and effective solution techniques to optimize ALM problems with binary decision variables under uncertainty is a complex task.

Chapter 3

Worst-Case Conditional Value at Risk for Asset Liability Management: A Novel Framework for General Loss Functions

3.1 Introduction

Financial institutions like pension funds and insurance companies are mandated to prudently manage large amounts of assets and liabilities. Decision-makers in these institutions have to maintain a delicate balance between maximizing return and controlling risk to ensure their long-term financial sustainability. The Asset-liability Management (ALM) problem aims to achieve this goal by optimally allocating available funds to different assets such that profit is maximized while current and future liabilities are covered and any regulatory requirements are satisfied (Zenios, 1995). This problem is of particular concern for pension funds that must guarantee predefined payback to retirees (*i.e.,* defined benefit pension plans) (Bodie et al., 1988).

Pension funds control a sizable portion of global financial assets, in excess of \$60.6 trillion by the end of 2021, which represents 33% of the global assets [1]. At that time, the pension funds in 9 out of the 38 Organisation for Economic Co-operation and Development (OECD) countries had assets exceeding their respective GDPs. Furthermore, pension assets have grown by 5.7% in the last decade (2010-2020) [2] which exceeds the GDP growth rate of 2.6% over the same period [3], signifying the increasing importance of retirement savings globally. However, as large segments of the population have been reaching their retirement ages, outflows from pension funds to pay their benefits are also accelerating. The ratio of total benefits paid from retirement savings plans to GDP varies across OECD countries, ranging from 0.5%

[1] https://www.thinkingaheadinstitute.org/research-papers/global-pension-assets-study-2022/

[2] https://www.statista.com/statistics/721151/average-growth-largest-pension-markets-worldwide/

[3] https://www.macrotrends.net/countries/WLD/world/gdp-growth-rate

to 8% [4].

To meet their future obligations, pension funds need to invest collected contributions in diversified portfolios of assets (*i.e.,* fixed-income, public/private equities, real estate, and infrastructures) to generate sufficient returns. However, these investments come with inherent risks that can affect the portfolio's value and the fund's ability to meet its commitments. Like other investment portfolios, pension funds are exposed to asset price variations over time due to market, sector-specific, and company-specific risks. In contrast to classical investment portfolios, pension funds have defined future obligations and are subject to additional regulatory requirements that stipulate a minimum acceptable ratio between current assets and the present value of future liabilities (*i.e.,* the funding ratio). Hence, pension funds are also exposed to interest rate risks that severely affect the liabilities' present value, rendering classical portfolio selection problem (PSP) techniques unsuitable to manage them. Instead, ALM models that jointly consider asset returns and interest rate risks are used.

Several risk measures have been proposed in the literature to quantify this risk (Chen et al., 2008; Chiu & Li, 2006; Chiu & Wong, 2012; Leippold et al., 2004; Shen et al., 2020; Ferstl & Weissensteiner, 2011), among which is the Conditional Value at Risk (CVaR), which was first used in the context of ALM by Bogentoft et al. (2001). CVaR combines the risk level and the probability of an asset or portfolio's return falling below a specified threshold. In order to develop CVaR as a risk measure in the ALM problem, a loss function that considers the losses resulting from mismatches between asset returns and liabilities is required. The use of CVaR enables pension funds to control their risk exposure by managing the tail risk of their investments. Despite its usefulness, the ALM formulation proposed by Bogentoft et al. (2001) uses a sample-average-approximation (SAA) approach to model the uncertainty about asset returns, thus not capturing the full extent of variability of returns and interest rates and resulting in intractable formulations.

The solvency of funded pension plans is highly sensitive to the assumptions embedded in the expected returns and interest/discount rates (parameters of CVaR), as emphasized by Konstantin (2018). The discount rate is crucial in determining the funding status of pension plans. As bond yields have fallen over the past few decades,

[4] https://www.oecd.org/finance/private-pensions/globalpensionstatistics.htm

the discount rate should be adjusted downwards, while it remains highly aggressive in US public pension plans, the major holder of pension plans globally. Additionally, the expected fund performance varies significantly among different entities, even without necessarily different allocations. This variability in returns makes it challenging for pension managers to determine the optimal asset allocation to cover future liabilities. Furthermore, D'Addio et al. (2009) highlighted the significant impact of uncertainty in asset returns on pension funds, indicating the need for a conservative approach to investment based on asset returns uncertainty.

Managing the uncertainty associated with the ALM problem is critical for institutions to make better investment decisions and manage risk effectively. Therefore, research on ALM problems under uncertainty has focused on developing models and methods that can quantify and manage the various sources of uncertainty. As highlighted by Gülpinar & Pachamanova (2013), the most common approaches to this problem are stochastic programming (SP) and robust optimization (RO). SP is a risk-neutral approach that aims to find a solution that optimizes the expected value of the loss function. Various studies, such as (Klaassen, 1997; Kouwenberg, 2001; Consigli, 2008; Duarte et al., 2017; Kopa et al., 2018; Barro et al., 2022), have applied SP to ALM problems. However, SP requires that the distribution function of the random variables be known. Furthermore, the market environment is subject to continuous shifts, rendering historical data potentially inadequate for capturing the current market conditions necessary to construct the distribution function of asset returns. On top of that, the method is risk-neutral, meaning there is no immunity against scenarios that are worse than expected. Additionally, SP solutions may be infeasible for some scenarios. Despite its limitations, SP remains an intuitive approach with favorable convergence properties.

Another appealing method proposed for addressing uncertainty in ALM problems is RO. Researchers such as Iyengar & Ma (2016); Platanakis & Sutcliffe (2017); Gülpinar & Pachamanova (2013); Gülpınar et al. (2016) have used RO to develop ALM models under uncertainty. Despite the advantages of RO over SP models, such as being a risk-averse method and not requiring knowledge of the distribution function of uncertain parameters, the solutions produced by RO are usually overly conservative. This can increase the opportunity cost of ALM problems by basing the decisions on the worst-case scenario. Interested readers can refer to (Ben-Tal et al., 2009; Bertsimas et al., 2011; Gabrel et al., 2014; Ghahtarani et al., 2022) for more information about RO methods.

While RO and SP have been proposed for the ALM problem, there is currently no research in the literature that considers the combination of risk measures with the ambiguity of probability distribution in ALM optimization. This combination has several advantages. First, it allows for a more comprehensive risk modeling in pension fund management by considering a risk measure. Second, it enables pension fund managers to make more informed decisions on asset allocation, taking into account the uncertainty of returns and the associated risk. Third, it provides a more accurate representation of the underlying probability distribution by using a set of possible distribution functions for random variables (asset return) called the *ambiguity set*, which can lead to better risk management and improved long-term financial stability. Finally, the combination of risk measures with uncertainty in ALM optimization can lead to more robust and reliable solutions, which are essential for ensuring the long-term financial health of pension funds. This gap in the literature and the benefits of the combination of a risk measure and the ambiguity of distribution function motivates us to adopt distributionally robust optimization (DRO) approaches for the ALM problem. DRO considers the worst-case distribution within a set of candidate distributions that are compatible with available data. By using a risk measure (*e.g.,* CVaR) and accounting for ambiguity in the probability distribution through a DRO approach, more realistic solutions leading to better long-term financial outcomes for pension funds can be achieved. Combining CVaR with DRO leads to the worst-case CVaR (WCVaR) risk measure.

Although the literature suggests WCVaR as a valuable tool for PSPs, there is a gap in the theoretical framework that limits its applicability to more complex loss functions like that of the ALM problem. The loss function in the ALM problem is more intricate than that of the regular PSP due to the uncertainty of both asset returns and the present value of liabilities. On the other hand, the majority of research on CVaR in portfolio selection problems (PSP) assumes the availability of full knowledge of the distribution function of portfolio losses. However, the distribution functions of uncertain asset returns and the present value of liabilities in the ALM problem are not fully known due to the changing parameters based on market conditions. To address this issue, we have developed a novel theoretical framework that proposes the use of WCVaR for linear and nonlinear loss functions of random variables. Our theoretical development not only addresses the gap in the literature but also offers promising possibilities for extending WCVaR to other problem domains such as supply chain management and engineering design. With its enhanced versatility and applicability, WCVaR has the potential to become a go-to tool for a wider range of decision-making scenarios.

The remaining sections of this chapter are structured as follows. Section 3.2 provides a review related to the optimization formulation of the ALM problem using CVaR. In Section 3.3, we present an extension for the worst-case lower partial moment (WLPM) for functions of random variables. This extension is crucial in developing the WCVaR for more complex loss functions. Furthermore, in Section 3.3, we propose a formulation for WCVaR that is applicable to general loss functions. Section 3.4 delves into how to develop WCVaR for the ALM problem, along with an explanation of how to extend the data-driven moment-based ambiguity set. To test the proposed formulation on real data of the Canada Pension Plan (CPP), numerical experiments are conducted, and the results are presented in Section 3.5. Finally, Section 3.6 offers some conclusions and suggests potential areas for future research.

3.2 The ALM Problem with CVaR Constraints

In pension funds, premiums are collected from sponsors or currently active employees, and pensions are paid to retired employees. Moreover, available funds are invested in assets, which should be managed so that at each decision moment, the total value of all assets exceeds the fund's liabilities. The goal is to minimize the contribution rate by the sponsor and active employees of the fund (see Bogentoft et al. (2001)). Hence, the ALM problem for a pension fund tries to find the optimal contribution rate and allocation of funds in assets during an investment horizon of length T, which is divided into a set of decision moments $t = 0, \ldots, T$. At each decision moment t, decisions are made on the value of contributions to the fund and portfolio allocation. Let y_t be the contribution rate at decision moment t, which is a fraction of the sponsor and/or active employee's wage w_t at decision moment t. Besides, $x_{n,t}$ are decision variables of money invested in asset n in the t^{th} decision moment. The value of assets held by the fund at decision moment t is denoted by A_t. Payments made by the fund to retirees at decision moment t are liabilities and denoted by l_t. The present value of liabilities at decision moment t is calculated by $L_t = \sum_t^T \frac{l_t}{(1+\gamma)^t}$, $\forall t = 0, \ldots, T$, where γ is the discount rate. We consider a case in which benefit payments, *i.e.,* liabilities, are fixed and predefined. These kinds of pension funds are called *defined-benefit plans*. The present value of liabilities, L_t, is a random variable since the discount rate used to calculate it is, itself, a random variable. The funding ratio is defined as the ratio of the value of assets at decision moment t to the present value of liabilities at decision moment t. Finally, ψ is the minimum threshold of the funding ratio and is normally imposed by regulations. Model (3.1) shows the mathematical formulation of the ALM problem introduced by Bogentoft et al. (2001):

$$\min_{y_t, x_{n,t}} \quad h(y_1, \ldots, y_T), \tag{3.1a}$$

$$\text{s.t.} \quad \sum_{n=0}^{N} x_{n,t} = A_t + w_t y_t - l_t, \quad t = 0, \ldots, T-1, \tag{3.1b}$$

$$A_t \geq \psi L_t, \quad t = 0, \ldots, T, \tag{3.1c}$$

$$A_t = \sum_{n=0}^{N} x_{n,t-1}(1 + \xi_{n,t}), \quad t = 0, \ldots, T, \tag{3.1d}$$

$$x_{n,t} \in \mathcal{X}, y_t \in \mathcal{Y}, \quad t = 0, \ldots, T, n = 0, \ldots, N. \tag{3.1e}$$

In their paper, Bogentoft et al. (2001) introduced a function denoted by $h(y_0, \ldots, y_T)$, which serves as the objective function for the ALM problem expressed in (3.1). The function is defined in terms of the contribution rate and plays a crucial role in determining the optimal ALM strategy. The objective function (3.1a) can be the average contribution rate or the present value of all contributions. In this formulation, we consider the present value of contributions as the objective function, expressed as $h(y_0, \ldots, y_T) = \sum_{t=0}^T \frac{w_t y_t}{(1+\gamma)^t}$. Constraint (3.1b), called the balance constraint, ensures that the sum of all investments at decision moment t is equal to the assets held by the fund plus the contributions gathered at decision moment t minus liabilities in this decision moment. Constraint (3.1c), called the funding ratio, guarantees that the ratio of assets owned by the fund to the present value of liabilities at decision moment

t is greater than a minimum threshold ψ. Constraint (3.1d) calculates the value of assets owned by the fund at time t. In this formulation, the asset returns $\xi_{n,t}$ and the discount rate γ are uncertain parameters. Uncertainty of the discount rate γ leads to uncertainty in the present values of liabilities and future contributions. Finally, $\mathcal{X}$ and $\mathcal{Y}$ in (3.1e) are sets of regulatory constraints for the investment allocation and the contribution rate.

To make the formulation easier, we define $W_t = \frac{w_t}{(1+\gamma)^t}$, representing the present value of the sponsor and/or active employee's wages, which is also uncertain because it depends on the uncertain discount rate γ. The objective function of model (3.1) can be transformed into $\mathrm{W}^\intercal \mathrm{y}$, where $\mathrm{W} = \{W_0, \ldots, W_T\} \in \mathbb{R}^{T+1}$ and $\mathrm{y} = \{y_0, \ldots, y_T\} \in \mathbb{R}^{T+1}$ are the vectors of the present value of the active employee's wages and decision variables related to the contribution rates, respectively. We also define the vector $\mathrm{r}_t = \mathrm{e} + \boldsymbol{\xi}_t$, $t = 0, \ldots, T$, where e is an all-ones vector of size $N+1$. Additionally, the investment decision variable is defined as a vector in each decision moment, $\mathrm{x}_t = \{x_{0,t}, \ldots, x_{n,t}\}$. Using these notations, the ALM problem (3.1) can be transformed into a vector representation as follows:

$$\min_{\mathrm{y},\mathrm{x}_t} \quad \mathrm{W}^\intercal \mathrm{y}, \tag{3.2a}$$

$$\text{s.t.} \quad \mathrm{e}^\intercal \mathrm{x}_t = \mathrm{r}_t^\intercal \mathrm{x}_{t-1} + w_t y_t - l_t, \quad t = 0, \ldots, T-1, \tag{3.2b}$$

$$\mathrm{r}_t^\intercal \mathrm{x}_{t-1} \geq \psi L_t, \quad t = 0, \ldots, T, \tag{3.2c}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \quad t = 0, \ldots, T. \tag{3.2d}$$

In order to quantify the risk associated with an investment portfolio using the CVaR measure, it is essential to establish a loss function that captures the potential losses. Based on (Bogentoft et al., 2001), the loss function for problem (3.2) for each decision moment, t, is defined as $f_\psi(\mathrm{x}_t; \mathrm{r}_t, L_t) = \psi L_t - \mathrm{r}_t^\intercal \mathrm{x}_{t-1}$ as per constraint (3.2c). Note that the loss function and the CVaR are defined for each decision moment t. However, to simplify the formulations, we suppress the t subscript. The probability that $f_\psi(\mathrm{x}; \mathrm{r}, L)$ is not exceeding a threshold α is calculated as $\Psi(\mathrm{x}, \alpha) = \int_{f_\psi(\mathrm{x};\mathrm{r},L) \leq \alpha} p(\mathrm{r}, L) d(\mathrm{r}, L)$, where $p(\mathrm{r}, L)$ is the joint distribution function of the present value of liabilities and asset returns as random variables. It is worth noting that $p(\mathrm{r})$ is the marginal distribution function of asset returns and $p(L)$ is the marginal distribution of the present value of liabilities.

Value-at-Risk (VaR) is a measure of financial losses over a given time horizon under normal market conditions and a specified level of confidence. It provides an estimate of the maximum loss that an investor could expect to suffer over a given time horizon assuming that the portfolio is held to maturity and that market conditions remain stable. For a confidence level β and a fixed x, the VaR is formally represented as $VaR_\beta(\mathrm{x}) = \min\{\alpha \in \mathbb{R} : \Psi(\mathrm{x}, \alpha) \geq \beta\}$. CVaR is then defined as the expected loss that exceeds VaR, and is calculated as $CVaR_\beta(\mathrm{x}) = \frac{1}{1-\beta}\int_{f_\psi(\mathrm{x};\mathrm{r},L)\geq VaR_\beta(\mathrm{x})} f_\psi(\mathrm{x};\mathrm{r},L)p(\mathrm{r},L)d(\mathrm{r},L)$.

Borrowing the approach proposed by Rockafellar et al. (2000), we introduce an auxiliary function $G_\beta(\mathrm{x},\alpha) = \alpha + \frac{1}{1-\beta}\int_{\mathrm{r}\in\mathbb{R}^{n+1}, L\in\mathbb{R}}[f_\psi(\mathrm{x};\mathrm{r},L) - \alpha]^+ p(\mathrm{r},L)d(\mathrm{r},L)$, where $[.]^+ = max\{.,0\}$, and then $CVaR_\beta(\mathrm{x}) = \min_{\alpha\in\mathbb{R}} G_\beta(\mathrm{x},\alpha)$. To calculate $G_\beta(\mathrm{x},\alpha)$, a function to capture expected value of losses greater than α, it is necessary to have full knowledge about the joint distribution function of asset returns and the present value of liabilities, $p(\mathrm{r},L)$. However, in reality, full knowledge about this joint distribution function may not be available. Therefore, we apply a DRO approach that considers the ambiguity about the distribution function of these random variables. DRO offers a powerful framework for dealing with uncertainty in ALM by avoiding the assumption of a single distribution for randomly distributed variables. In this context, we have two key random variables: the present value of wages of active employees, W, and random variables in the loss function $f_\Psi(\mathrm{x};\mathrm{r},L)$ of the ALM problem. These variables have distinct distribution functions, namely q, the distribution function of the present value of the active employee's wages, and $p(\mathrm{r},L)$, the joint distribution function of asset returns and the present value of liabilities, respectively.

To account for the investor's ambiguity regarding the true distribution of the loss function and the present value of pension active employee wages, we introduce ambiguity sets of distributions. More specifically, we define Q as the ambiguity set of the distribution function of the present value of active employees' wages and $P(\mathrm{r},L)$ as the ambiguity set of the joint distribution function of asset returns and the present value of liabilities. Finally, $P(\mathrm{r})$ and $P(L)$ are the ambiguity sets of marginal distribution functions of asset returns and the present value of liabilities, respectively. Using these ambiguity sets, we can formulate the DRO version of the ALM problem

(3.2) as follows:

$$\min_{\mathrm{y},\mathrm{x}_t} \sup_{q \in Q} \quad \mathbb{E}_q[\mathrm{W}]^\intercal \mathrm{y}, \tag{3.3a}$$

$$\text{s.t.} \quad \mathrm{e}^\intercal \mathrm{x}_t = \inf_{p(\mathrm{r}) \in P(\mathrm{r})} \mathbb{E}_{p(\mathrm{r})}[\mathrm{r}_t]^\intercal \mathrm{x}_{t-1} + w_t y_t - l_t, \quad t = 0, \ldots, T-1, \tag{3.3b}$$

$$\sup_{p(\mathrm{r},L) \in P(\mathrm{r},L)} \min_{\alpha \in \mathbb{R}} G_\beta(\mathrm{x}_t, \alpha) \leq 0, \quad t = 0, \ldots, T, \tag{3.3c}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \quad t = 0, \ldots, T. \tag{3.3d}$$

The goal is to minimize the worst-case expected present value of future contributions to the fund, represented by the objective function (3.3a), subject to the balance constraint (3.3b), the WCVaR constraint (3.3c), and the regulatory constraint (3.3d). In the objective function (3.3a), the minimization is over the contribution rate y and the investment allocation in each decision moment x_t, while the maximization is over all probability distributions in the ambiguity set Q. The expected value is taken with respect to the probability distribution $q \in Q$. In the balance constraint (3.3b), the worst-case expected value is over the marginal distribution function of asset returns. The maximization of WCVaR is over the joint distribution function of asset returns and the present value of liabilities, and the minimization is over α, which is VaR here. By doing so, we obtain more robust results that are less sensitive to specific assumptions about the underlying probability distributions, making it particularly well-suited for managing financial risks in uncertain environments.

The subsequent task is to introduce WCVaR for ALM. However, the loss function for the ALM problem is more intricate than that of PSPs since the loss function of ALM has two random variables, asset returns and the present value of liabilities, while the loss function of PSP has just one random variable, asset returns. Therefore, an extension of the theoretical framework for the Worst-case Lower Partial Moment (WLPM) and WCVaR is necessary to apply them to more extensive loss functions.

3.3 WLPM and WCVaR for Linear Loss Functions

Chen et al. (2011a) proposed WLPM as a risk measure that has a close connection with WCVaR. Let ξ be a univariate random variable, with μ and σ being the first and second moments of ξ, and α is a fixed target. Chen et al. (2011a) proved that

$\sup_{\xi\sim(\mu,\sigma^2)} \mathbb{E}\left[(\alpha-\xi)^+\right] = \frac{\alpha-\mu+\sqrt{\sigma^2+(\alpha-\mu)^2}}{2}$ and showed that WCVaR can be defined based on WLPM. In particular, for a regular PSP with the loss function $-\mathrm{r}^\intercal\mathrm{x}$, where $\mathrm{r}\in\mathbb{R}^n$ is the asset returns vector, $\mathrm{x}\in\mathbb{R}^n$ is the vector of decision variables which is the proportion of investment in each asset, and $P(\mathrm{r})$ is an ambiguity set of the distribution function of asset returns, the WCVaR is defined as:

$$WCVaR_\beta(\mathrm{x}) = \sup_{p(\mathrm{r})\in P(\mathrm{r})} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(-\mathrm{r}^\intercal\mathrm{x}-\alpha)^+\right], \tag{3.4}$$

where $\sup_{p(\mathrm{r})\in P(\mathrm{r})} \mathbb{E}\left[(-\mathrm{r}^\intercal\mathrm{x}-\alpha)^+\right]$ is the WLPM. Here, the vector of asset returns, $\mathrm{r}\in\mathbb{R}^n$, is a random variable with mean $\hat{\mu}$ and covariance $\hat{\Sigma}\succ 0$ that belongs to a family of distributions

$$P(\mathrm{r}) = \left\{p\in M_+ | P(\mathrm{r}\in\Omega)=1, \mathbb{E}_p(\mathrm{r})=\hat{\mu},\ Cov_p(\mathrm{r})=\hat{\Sigma}\right\},$$

where M_+ is the set of all probability measures on the measurable space $(\mathbb{R}^n, B)$ with the Borel σ-algebra B on $\mathbb{R}^n$ and $\Omega\subseteq\mathbb{R}^n$ is a closed convex set known to contain the support of the random vector r. By using this ambiguity set, as proven by Chen et al. (2011a), the WCVaR evaluates to:

$$\max_{p(\mathrm{r})\in P(\mathrm{r})} CVaR_\beta(\mathrm{x},p) = -\hat{\mu}^\intercal\mathrm{x} + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\mathrm{x}^\intercal\hat{\Sigma}\mathrm{x}}. \tag{3.5}$$

The WCVaR formulation (3.5) is based on the assumption that the first two moments of the uncertain distribution function are known. However, there might be uncertainty about the moments when they are estimated using limited data samples. Kang et al. (2019) proposed the WCVaR with uncertain moments based on a data-driven moment-based ambiguity set defined as follows:

$$D_{P(\mathrm{r})}(\gamma_1,\gamma_2) = \{p\in M_+ | P(\mathrm{r}\in\Omega)=1, (\mathbb{E}_p(\mathrm{r})-\hat{\mu})^\intercal\hat{\Sigma}^{-1}(\mathbb{E}_p(\mathrm{r})-\hat{\mu})\leq\gamma_1,$$

$$||Cov_p(\mathrm{r})-\hat{\Sigma}||_F\leq\gamma_2, Cov_p(\mathrm{r})\succ 0\},$$

which is originally introduced by Delage & Ye (2010). In this ambiguity set, $\hat{\mu}$ and $\hat{\Sigma}$ are estimates of the mean vector and the covariance matrix of the random variable r, respectively. Kang et al. (2019) proved that WCVaR under moment uncertainty (as defined in $D_{P(\mathrm{r})}(\gamma_1, \gamma_2)$) is as follows:

$$\max_{p(\mathrm{r}) \in D_{P(\mathrm{r})}(\gamma_1,\gamma_2)} CVaR_\beta(\mathrm{x}, p) = -\hat{\mu}^\intercal \mathrm{x} + \sqrt{\gamma_1}\sqrt{\mathrm{x}^\intercal \hat{\Sigma} \mathrm{x}} + k\sqrt{\mathrm{x}^\intercal \left(\hat{\Sigma} + \gamma_2 I_n\right) \mathrm{x}}, \qquad (3.6)$$

where I_n is the identity matrix of size n, and $k = \sqrt{\frac{\beta}{1-\beta}}$.

The WCVaR reformulations (3.5) and (3.6) use the facts that the PSP loss function is a linear function of x and that r is the only random variable. However, the loss function can be more complex. As shown in Section 3.2, the loss function of the ALM problem includes a linear function of asset returns and the present value of future liabilities as random variables. To propose a tractable reformulation of the WCVaR constraint in the ALM problem, we are extending the WLPM and WCVaR formulations for the linear loss function of multiple random variables. For more clarity, we start with a linear loss function of a univariate random variable, then extend it to a linear function of multivariate random variables.

Lemma 1. *Let ξ be a univariate random variable, where $\mathbb{E}[\xi] = \mu$, $Var(\xi) = \sigma^2$, and $f(.)$ is a linear function of the random variable ξ that $f : \mathbb{R} \to \mathbb{R}$. Then, WLPM is as follows:*

$$\sup_{\xi \sim (\mu, \sigma^2)} \mathbb{E}\left[(\alpha - f(\xi))^+\right] = \frac{\alpha - f(\mu) + \sqrt{f'(\mu)^2 \sigma^2 + (\alpha - f(\mu))^2}}{2}.$$

Proof. The exact second-order Taylor expansion of $f(\xi)$ around $\mu = \mathbb{E}[\xi]$ for a linear function is as follows:

$$\mathbb{E}[f(\xi)] = \mathbb{E}\left[f(\mu) + f'(\mu)(\xi - \mu) + \frac{1}{2} f''(\mu)(\xi - \mu)^2\right].$$

It is known that $\mathbb{E}(a + b) = \mathbb{E}(a) + \mathbb{E}(b)$. Then:

$$\mathbb{E}[f(\xi)] = \mathbb{E}[f(\mu)] + f'(\mu)\mathbb{E}[\xi - \mu] + \frac{1}{2} f''(\mu)\mathbb{E}[\xi - \mu]^2,$$

where $\mathbb{E}\left[f\left(\mu\right)\right] = f\left(\mu\right)$, and $\mathbb{E}\left[\xi - \mu\right] = \mathbb{E}\left[\xi\right] - \mu = \mu - \mu = 0$. Then:

$$\mathbb{E}\left[f\left(\xi\right)\right] = f\left(\mu\right) + \frac{1}{2} f''\left(\mu\right) \mathbb{E}\left[\xi - \mu\right]^2 .$$

Since $\mathbb{E}\left[\xi - \mu\right]^2 = Var\left(\xi\right) = \sigma^2$, then:

$$\mathbb{E}\left[f\left(\xi\right)\right] = f\left(\mu\right) + \frac{1}{2} f''\left(\mu\right) \sigma^2$$

Because $f(.)$ is a linear function then $f''\left(.\right) = 0$, consequently $\mathbb{E}\left[f(\xi)\right] = f\left(\mu\right)$.

Next, we need to find $Var\left(f\left(\xi\right)\right)$. The first order Taylor expansion of $f\left(\xi\right)$ around $\mu = \mathbb{E}\left[\xi\right]$ is $f\left(\mu\right) + f'\left(\mu\right)\left(\xi - \mu\right)$. Then, $Var\left(f\left(\xi\right)\right)$ is as follows:

$$Var\left[f\left(\xi\right)\right] = Var\left[f\left(\mu\right) + f'\left(\mu\right)\left(\xi - \mu\right)\right] = Var\left[f\left(\mu\right) + f'\left(\mu\right)\xi - f'\left(\mu\right)\mu\right].$$

The first term, $f(\mu)$, is constant; then $Var\left(f\left(\mu\right)\right) = 0$. The third term, $Var\left(f'\left(\mu\right)\mu\right)$, is also a constant with a variance equal to zero. Consequently, the variance of $f\left(.\right)$ is as follows:

$$Var\left[f\left(\xi\right)\right] = Var\left[f'\left(\mu\right)\xi\right] = \left(f'\left(\mu\right)\right)^2 Var\left[\xi\right] = f'\left(\mu\right)^2 \sigma^2.$$

By substituting $\mathbb{E}\left[f\left(\xi\right)\right]$ and $Var\left(f\left(\xi\right)\right)$ into $\frac{\alpha - \mathbb{E}[f(\xi)] + \sqrt{Var(f(\xi)) + (\alpha - \mathbb{E}[f(\xi)])^2}}{2}$, the WLPM is as follows:

$$\mathbb{E}_{\xi \sim (\mu, \sigma^2)}\left[\left(\alpha - f\left(\xi\right)\right)^+\right] = \frac{\alpha - f\left(\mu\right) + \sqrt{f'\left(\mu\right)^2 \sigma^2 + \left(\alpha - f\left(\mu\right)\right)^2}}{2}.$$

□

Theorem 2. *Let ξ be a univariate random variable with mean μ and variance σ^2, and define the ambiguty set $P = \{p \in M_+ | P\left(\xi \in \Omega\right) = 1, \xi \sim (\mu, \sigma^2)\}$. Moreover, $f(\xi)$ is a linear loss function, where $f : \mathbb{R} \to \mathbb{R}$. Then WCVaR can be calculated as follows :*

$$WCVaR_\beta = \sup_{p(.) \in P} \min_{\alpha \in \mathbb{R}} \alpha + \frac{1}{1 - \beta} \mathbb{E}\left[\left(f\left(\xi\right) - \alpha\right)^+\right] = f\left(\mu\right) + \sqrt{\frac{\beta}{1 - \beta}} \sqrt{f'\left(\mu\right)^2 \sigma^2}.$$

Proof. Based on its definition, $WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi)-\alpha)^+\right]$. To reformulate the WCVaR, we need to calculate the WLPM term in the WCVaR definition. In Lemma 1, the LPM is in the form $\sup_{p(.)\in P}\mathbb{E}\left[(\alpha - f(\xi))^+\right]$. Hence, rearrange the WLPM term in CVaR as follows:

$$\sup_{p(.)\in P}\mathbb{E}\left[(f(\xi)-\alpha)^+\right] = \sup_{p(.)\in P}\mathbb{E}\left[(-\alpha-(-f(\xi)))^+\right].$$

By substituting the WLPM from Lemma 1 into WCVaR, we have:

$$WCVaR_\beta = \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\frac{-\alpha + f(\mu) + \sqrt{f'(\mu)^2\sigma^2 + (-\alpha + f(\mu))^2}}{2}.$$

The optimal value of α (α^*) can be calculated using the first-order optimality condition $\frac{\partial WCVaR_\beta}{\partial\alpha} = 0$. With that, we have:

$$\alpha^* = f(\mu) + \frac{2\beta - 1}{2\sqrt{\beta(\beta-1)}}\sqrt{f'(\mu)^2\sigma^2}.$$

By substituting α^* back, the WCVaR reduces to:

$$WCVaR_\beta = f(\mu) + \sqrt{\frac{\beta}{1-\beta}}\sqrt{f'(\mu)^2\sigma^2}.$$

□

Now let us consider the case when the loss function is a linear function of multivariate random variables, which is applicable in the context of the ALM problem.

Lemma 3. *Let* $\xi = \{\xi_1, \ldots, \xi_n\}$ *be a multivariate random variable, where* $\mathbb{E}[\xi_i] = \mu_i$, $Var(\xi_i) = \sigma_i^2$, $Cov(\xi_i, \xi_j) = \sigma_{ij}$, *and* $f(.)$ *is a linear function of the random variable* ξ *that* $f : \mathbb{R}^n \to \mathbb{R}$. *Then, WLPM is as follows:*

$$\sup_{\xi\sim(\mu,\Sigma_\xi)}\mathbb{E}\left[(\alpha - f(\xi))^+\right] = \frac{\alpha - f(\mu) + \sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j \sigma_{ij} + (\alpha - f(\mu))^2}}{2},$$

where $\mu = \{\mu_1, \ldots, \mu_n\}$ *is the mean vector, and* $d_i = \frac{\partial f(\xi)}{\partial \xi_i}\mid_{\xi=\mu}$, *in which* $\mid_{\xi=\mu}$ *means to evaluate the expression with* μ_i *replacing* ξ_i.

Proof. Based on the second-order Taylor series expansion of $f(.)$ around $\mu = \{\mu_1, \ldots, \mu_n\}$, the expected value of $f(\xi)$ is as follows:

$$\mathbb{E}[f(\xi)] = \mathbb{E}[f(\mu)] + \mathbb{E}[\nabla f(\mu)(\xi - \mu)] + \mathbb{E}\left[\frac{1}{2}(\xi - \mu)^{\intercal} H_f(\mu)(\xi - \mu)\right],$$

where $H_f = \frac{\partial^2 f(\xi)}{\partial \xi_i \partial \xi_j}$ is the Hessian matrix of f, and ∇f is the gradient of f. Since $f(.)$ is a linear function, then its second derivation is zero. Moreover, the second term of Taylor approximation is zero since $\mathbb{E}[\xi - \mu] = \mathbb{E}[\xi] - \mu = \mu - \mu = 0$. Hence, $\mathbb{E}[f(\xi)] = f(\mu)$. Moreover, the variance of $f(.)$ has to be calculated. Based on the first-order Taylor expression, the variance of $(f(\xi))$ is as follows:

$$Var(f(\xi)) = Var(f(\mu) + \nabla f(\mu)^{\intercal}(\xi - \mu)) = Var(f(\mu) + \nabla f(\mu)' \xi - \nabla f(\mu)' \mu).$$

Since $f(\mu)$, and $\nabla f(\mu)\mu$ are constants, their variances are zero. Hence, $Var(f(\xi)) = Var\left(\nabla f(\mu)' \xi\right)$ which is equivalent to $\nabla f'(\mu)^2 \Sigma_\xi$, where Σ_ξ is the covariance matrix. This formulation can be expanded as follows:

$$Var(f(\xi)) = \sum_i d_i^2 \sigma_i^2 + 2 \sum_i \sum_{j>i} d_i d_j \sigma_{ij},$$

where $d_i = \frac{\partial f(\xi)}{\partial \xi_i} |_{\xi=\mu}$.

By substituting $\mathbb{E}[f(\xi)]$ and $Var(f(\xi))$ into $\frac{\alpha - \mathbb{E}[f(\xi)] + \sqrt{Var(f(\xi)) + (\alpha - \mathbb{E}[f(\xi)])^2}}{2}$, then $\sup_{\xi \sim (\mu, \Sigma_\xi)} \mathbb{E}\left[(\alpha - f(\xi))^+\right]$ is calculated as follows:

$$\frac{\alpha - f(\mu) + \sqrt{\sum_i d_i^2 \sigma_i^2 + 2 \sum_i \sum_{j>i} d_i d_j \sigma_{ij} + (\alpha - f(\mu))^2}}{2}.$$

□

Theorem 4. *Let $\xi \in \mathbb{R}^n$ be a multivariate random variable with mean vector μ and covariance matrix Σ_ξ, where the ambiguity set is $P = \{p \in M_+ | P(\xi \in \Omega) = 1, \xi \sim (\mu, \Sigma_\xi)\}$. Moreover, $f(\xi)$ is a linear loss function, where $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Then WCVaR is defined as $WCVaR_\beta = \sup_{p(.) \in P} \min_{\alpha \in \mathbb{R}} \alpha + \frac{1}{1-\beta} \mathbb{E}[(f(\xi) - \alpha)^+]$ which is calculated* *by:*

$$WCVaR_\beta = \sup_{p(.)\in P} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi)-\alpha)^+\right] = f(\mu) + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j \sigma_{ij}}.$$

Proof. WCVaR is defined as $WCVaR_\beta = \sup_{p(.)\in P}\min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi)-\alpha)^+\right]$. In Lemma 3, we showed how to calculate the WLPM of a linear function of multivariate random variables as: $\sup_{p(.)\in P}\mathbb{E}\left[(f(\xi)-\alpha)^+\right] = \sup_{p(.)\in P}\mathbb{E}\left[(-\alpha-(-f(\xi)))^+\right]$, which is calculated by:

$$\frac{-\alpha + f(\mu) + \sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j\sigma_{ij} + (-\alpha+f(\mu))^2}}{2}, \tag{3.7}$$

to be substituted in WCVaR formulation. Then, the optimal value of α (α^*) is calculated using the first-order optimality condition $\frac{\partial WCVaR_\beta}{\partial\alpha} = 0$. With that, we have:

$$f(\mu) + \frac{2\beta-1}{2\sqrt{\beta(\beta-1)}}\sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j\sigma_{ij}}.$$

By substituting α^* back in (3.7), we get:

$$WCVaR_\beta = f(\mu) + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\sum_i d_i^2\sigma_i^2 + 2\sum_i\sum_{j>i} d_i d_j\sigma_{ij}}.$$

□

The theorems presented in this chapter, namely Theorems 2 and 4, offer a means of computing WCVaR for linear loss functions. However, WCVaR is not only applicable to financial problems but also to a variety of other fields where it is used as a risk measure for more general nonlinear loss functions. This chapter also extends the theorems to accommodate nonlinear loss functions and provides lemmas and proofs in the Appendix.

The extended WCVaR presented in this chapter has a wide range of potential applications, such as in supply chain and engineering problems, safety analysis, and healthcare. For readers interested in further exploring these applications, we recommend the following references: (Tao et al., 2021; Chaudhuri et al., 2022; Zhu et al., 2020; Chaudhuri et al., 2020; Chapman et al., 2021; von Schantz et al., 2020; Dehlendorff et al., 2010).

In the next section, we develop the WCVaR formulation for the ALM problem, in which a linear function of random variables is used as a loss function. We are using the theoretical results derived in this section to tractably reformulate the ALM problem with a WCVaR constraint.

3.4 WCVaR for ALM Problem

In this section, we use Theorem 4 to derive a tractable reformulation of the ALM problem with the WCVaR constraint (3.3c). This constraint ensures that the asset-liability mismatch is controlled in each decision moment, in the sense that the funding ratio remains above ψ with high probability, while accounting for the ambiguity surrounding the joint probability distribution of the asset returns and the present values of liabilities. Since the loss function $f_\psi(x; \mathrm{r}, L) = \psi L_t - \mathrm{r}_t^\intercal \mathrm{x}_{t-1}$ in this set of constraints is linear in the random variables r and L, Theorem 4 applies and the reformulation is exact. Recall that the random variables are defined as $L \sim \left(\bar{L}, \bar{\Sigma}_L\right)$, $\mathrm{r} \sim \left(\hat{\mathrm{r}}, \bar{\Sigma}_\mathrm{r}\right)$, and $Cov\left(L, \mathrm{r}\right) = \bar{\sigma}_{L,\mathrm{r}}$, where $\bar{L} \in \mathbb{R}$, $\bar{\Sigma}_L \in \mathbb{R}$, $\bar{\mathrm{r}} \in \mathbb{R}^{n+1}$, $\bar{\Sigma}_\mathrm{r} \in \mathbb{R}^{n+1 \times n+1}$, and $\bar{\sigma}_{L,\mathrm{r}} \in \mathbb{R}^{n+1}$. With that, we prove the following proposition.

Proposition 5. *For a given $t \in \{0, \ldots, T\}$, and using the ambiguity set $P\left(\mathrm{r}, L\right) = \{p\left(\mathrm{r}, L\right) \in M_+ | P\left(\mathrm{r}, L \in \Omega\right) = 1, \mathrm{r} \sim \left(\bar{\mathrm{r}}, \bar{\Sigma}_\mathrm{r}\right), L \sim \left(\bar{L}, \bar{\Sigma}_L\right), Cov\left(L, \mathrm{r}\right) = \bar{\sigma}_{L,\mathrm{r}}\}$, the left hand side (LHS) of the WCVaR constraint (3.3c) can be tractably reformulated as follows:*

$$\sup_{p(\mathrm{r},L) \in P(\mathrm{r},L)} \min_{\alpha \in \mathbb{R}} \alpha + \frac{1}{1-\beta} \mathbb{E}\left[\left(-\alpha - \left(\mathrm{x}^\intercal \mathrm{r} - \psi L\right)\right)^+\right] =$$

$$-\bar{\mathrm{r}}^\intercal \mathrm{x} + \psi \bar{L} + \sqrt{\frac{\beta}{1-\beta}} \sqrt{\psi^2 \bar{\Sigma}_L + \mathrm{x}^\intercal \bar{\Sigma}_\mathrm{r} x - 2\mathrm{x}^\intercal \bar{\sigma}_{(L,\mathrm{r})}}$$

Proof. Using the basic properties of mean and variance, it is easy to show that $\mathrm{r}^\intercal \mathrm{x} -$

$\psi L \sim (\bar{r}^\intercal \mathrm{x} - \psi \bar{L}, \psi^2 \bar{\Sigma}_L + \mathrm{x}^\intercal \bar{\Sigma}_\mathrm{r} \mathrm{x} - 2\mathrm{x}^\intercal \bar{\sigma}_{L,\mathrm{r}})$. Then WCVaR is defined as:

$$WCVaR_\beta = \sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(-\alpha - (\mathrm{x}^\intercal \mathrm{r} - \psi L))^+\right] \tag{3.8}$$

Based on Lemma 3, the WLPM is calculated as follows:

$$\sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} \mathbb{E}\left[(-\alpha - (\mathrm{r}^\intercal \mathrm{x} - \psi L))^+\right] =$$

$$\left[\frac{1}{2}\sqrt{\psi^2 \bar{\Sigma}_L + \mathrm{x}^\intercal \bar{\Sigma}_\mathrm{r} \mathrm{x} - 2\mathrm{x}^\intercal \bar{\sigma}_{L,\mathrm{r}} + \left(\bar{\mathrm{r}}^\intercal \mathrm{x} - \psi \bar{L} - \alpha\right)^2} + \frac{-\alpha - \left(\bar{\mathrm{r}}^\intercal \mathrm{x} - \psi \bar{L}\right)}{2}\right]. \tag{3.9}$$

By substituting (3.9) into the WCVaR formula (3.8), we obtain:

$$WCVaR_\beta(\mathrm{x}) = \alpha + \frac{1}{1-\beta}\left[\frac{1}{2}\sqrt{\psi^2 \bar{\Sigma}_L + \mathrm{x}^\intercal \bar{\Sigma}_\mathrm{r} \mathrm{x} - 2\mathrm{x}^\intercal \bar{\sigma}_{L,\mathrm{r}} + \left(\bar{\mathrm{r}}^\intercal \mathrm{x} - \psi \bar{L} - \alpha\right)^2} + \frac{-\alpha - \left(\bar{\mathrm{r}}^\intercal \mathrm{x} - \psi \bar{L}\right)}{2}\right].$$

In Theorem 4, we showed that $\alpha^*_\mathrm{x} = \frac{2\beta-1}{2\sqrt{\beta(1-\beta)}}\sqrt{\psi^2 \bar{\Sigma}_L + \mathrm{x}^\intercal \bar{\Sigma}_\mathrm{r} \mathrm{x} - 2\mathrm{x}^\intercal \bar{\sigma}_{L,\mathrm{r}}} - \bar{\mathrm{r}}^\intercal \mathrm{x} + \psi \bar{L}$. By substituting it back, the LHS of the WCVaR constraint (3.3c) can be written as follows:

$$WCVaR_\beta(\mathrm{x}) = -\bar{r}^\intercal \mathrm{x} + \psi \bar{L} + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\psi^2 \bar{\Sigma}_L + \mathrm{x}^\intercal \bar{\Sigma}_\mathrm{r} \mathrm{x} - 2\mathrm{x}^\intercal \bar{\sigma}_{(L,\mathrm{r})}}. \tag{3.10}$$

□

It should be noted that the WCVaR reformulation (3.10) is based on the assumption that the moments of random variables, asset returns, and the present value of future liabilities, are fixed and known. With $\bar{\mathrm{W}} = \mathbb{E}_p[\mathrm{W}]$ and $\bar{\mathrm{r}}_t = \mathbb{E}_{p(r)}[\mathrm{r}_t]$, model (3.3) is transformed into model (3.11) by substituting the WCVaR formula (3.10) in constraint (3.3c) to obtain:

$$\min_{\mathrm{y},\mathrm{x}_t} \quad \bar{\mathrm{W}}^\intercal \mathrm{y}, \tag{3.11a}$$

$$\text{s.t.} \quad \mathrm{e}^\intercal \mathrm{x}_t = \bar{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1} + w_t y_t - l_t, \qquad t = 0, \ldots, T-1, \tag{3.11b}$$

$$\bar{r}_t^\intercal \mathrm{x}_{t-1} + \psi \bar{L}_t + \sqrt{\frac{\beta}{1-\beta}} \sqrt{\psi^2 \bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal \bar{\Sigma}_{\mathrm{r}_t} \mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal \bar{\sigma}_{(L_t, \mathrm{r}_t)}} \leq 0, \quad t = 0, \ldots, T, \tag{3.11c}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \qquad t = 0, \ldots, T, \tag{3.11d}$$

which is a nonlinear program.

Even though we assumed that the moments of the uncertain distribution functions are known, the moments themselves might be uncertain. Moments of asset returns are uncertain because they depend on a variety of factors, such as market conditions, economic trends, and company performance. Moreover, moments of liabilities are also uncertain because they are affected by a variety of factors, such as interest rates, inflation, and changes in demographics. To address this case, we extend the moment-based ambiguity set, $D^t_{P(\mathrm{r},L)}(\gamma_1^t, \gamma_2^t, \gamma_3^t, \gamma_4^t, \boldsymbol{\gamma}_5^t)$ of Delage & Ye (2010) as follows:

$$\left\{ p \in M_+ | P(\mathrm{r}_t, \mathrm{L}_t \in \Omega) = 1, \quad \begin{array}{lllr} U_{\mathrm{r}_t} & = (\mathrm{r}_t - \hat{\mathrm{r}}_t)^\intercal \hat{\Sigma}_{\mathrm{r}_t}^{-1} (\mathrm{r}_t - \hat{\mathrm{r}}_t) & \leq \gamma_1^t, & \\ U_{\Sigma_{\mathrm{r}_t}} & = \| Cov_P(\mathrm{r}_t) - \hat{\Sigma}_{\mathrm{r}_t} \|_F & \leq \gamma_2^t, & Cov_P(\mathrm{r}_t) \succeq 0 \\ U_{L_t} & = (L_t - \hat{L}_t)^\intercal \hat{\Sigma}_{L_t}^{-1} (L_t - \hat{L}_t) & \leq \gamma_3^t, & \\ U_{\Sigma_{L_t}} & = \| Cov_P(L_t) - \hat{\Sigma}_{L_t} \|_F & \leq \gamma_4, & Cov_P(L_t) \succeq 0 \\ U_{\sigma_{(L_t,\mathrm{r}_t)}} & = \| \bar{\sigma}_{(L_t,\mathrm{x}_t)} \|_\infty & \leq \boldsymbol{\gamma}_5^t, & \end{array} \right\},$$

where $\gamma_1^t, \gamma_2^t, \gamma_3^t, \gamma_4^t \in \mathbb{R}$, and $\boldsymbol{\gamma}_5^t \in \mathbb{R}^{n+1}$. Moreover, $\hat{\mathrm{r}}_t$ and $\hat{L}_t$ are estimates of the mean of asset returns and the present value of future liabilities at decision moment t, respectively. Similarly, $\hat{\Sigma}_{\mathrm{r}_t}$ and $\hat{\Sigma}_{L_t}$ are estimates of the variance-covariance matrix of asset returns, and the present value of liabilities, respectively.

The proposed ambiguity set is designed to capture the uncertainty of moments in a data-driven manner. It consists of two ellipsoidal uncertainty sets for each decision moment t: U_{r_t} and U_{L_t}. The former represents the uncertainty set of the mean of asset returns, while the latter characterizes the uncertainty set of the mean of present values of future liabilities. To quantify the size of these sets, we use the parameters γ_1^t and γ_3^t. To capture the uncertainty of the second moments, the Frobenius norm is used to define two uncertainty sets: $U_{\Sigma_{\mathrm{r}_t}}$ and $U_{\Sigma_{L_t}}$. These sets represent possible variations in the real variance-covariance matrices of asset returns and the present value of future liabilities, respectively. Intuitively, the Frobenius norm measures the "size" of the matrices, and the uncertainty sets ensure that the real matrices are close to their estimates, up to a certain radius. The sizes of the uncertainty sets are determined by γ_2^t and γ_4^t, which represent the second and fourth moments of the estimation errors, respectively. Additionally, $U_{\sigma(L,\mathrm{r}_t)}$ denotes the box uncertainty set for the covariance of asset returns and the present value of future liabilities in each decision moment t. γ_5^t is the size of this uncertainty set. Finally, $Cov_P(\mathrm{r}_t)$ and $Cov_P(L_t)$ represent the actual variance-covariance matrices of asset returns and the present value of future liabilities that should be positive semi-definite.

The present value of active employee wages is also a random variable. Consequently, the data-driven moment-based ambiguity set for the present value of active employee wages is defined as follows:

$$Q(\gamma_6,\gamma_7) = \left\{ \begin{array}{lll} p \in M_+ | P\left(\mathrm{W} \in \Omega\right) = 1, & U_{\mathrm{W}} & = \left(\mathrm{W} - \hat{\mathrm{W}}\right)^{\intercal} \hat{\Sigma}_{\mathrm{W}}^{-1} (\mathrm{W} - \hat{\mathrm{W}}) \leq \gamma_6, \\ & U_{\Sigma_{\mathrm{W}}} & = \| Cov_P\left(\mathrm{W}\right) - \hat{\Sigma}_{\mathrm{W}} \|_F \leq \gamma_7 \end{array} \right\},$$

where an ellipsoidal uncertainty set U_{W} is used to represent the possible variations in the mean of the present value of active employee wages. Similarly, the uncertainty set $U_{\Sigma_{\mathrm{W}}}$ captures the variations in the variance-covariance matrix of the present value of active employee wages. To specify the sizes of these uncertainty sets, we use the parameters γ_6 and γ_7, where these parameters determine the radius of the uncertainty sets. Finally, $\bar{\mathrm{W}}$ and $\hat{\Sigma_{\mathrm{W}}}$ denote estimates of the mean and the variance-covariance matrices of the present value of active employee wages, respectively.

A tractable reformulation of the LHS of the WCVaR constraint (3.3c) with the proposed data-driven ambiguity set $D^t_{P(\mathrm{r},L)}\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\boldsymbol{\gamma}_5^t\right)$ is developed in proposition 6.

Proposition 6. *Considering that* $p\left(\mathrm{r},L\right) \in D^t_{P(\mathrm{r},L)}\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\boldsymbol{\gamma}_5^t\right)$*, the LHS of the*

WCVaR constraint (3.3c) with uncertain moments can be reformulated as follows:

$$\sup_{p(\mathrm{r},L)\in D^t_{P(\mathrm{r},L)}\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\gamma_5^t\right)} CVaR_\beta\left(\mathrm{x}_{t-1}\right) = -\left[\hat{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1} - \sqrt{\gamma_1^t}\sqrt{\mathrm{x}_{t-1}^\intercal \hat{\Sigma}_{\mathrm{r}_t} \mathrm{x}_{t-1}}\right] +$$

$$\psi\left[\hat{L}_t + \sqrt{\gamma_3^t}\sqrt{\psi^2\hat{\Sigma}_{L_t}}\right] + k\sqrt{\mathrm{x}_{t-1}^\intercal\left(\hat{\Sigma}_{\mathrm{r}_t} + \gamma_2^t I_{n+1}\right)\mathrm{x}_{t-1} + \psi^2\left(\hat{\Sigma}_{L_t} + \gamma_4^t\right) + 2\mathrm{x}_{t-1}^\intercal \gamma_5^t},$$

where $k = \sqrt{\frac{\beta}{1-\beta}}$.

Proof. In proposition (5), by fixing x_t, it was shown that:

$$\sup_{p(\mathrm{r},L)\in P(\mathrm{r},L)} CVaR_\beta\left(\mathrm{x}\right) = -\bar{\mathrm{r}}_t{}^\intercal \mathrm{x}_{t-1} + \psi\bar{L}_t + \sqrt{\frac{\beta}{1-\beta}}\sqrt{\psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(L_t,\mathrm{r}_t)}}.$$

Now, let us consider the case $p\left(\mathrm{r}, L\right) \in D^t_{P(\mathrm{r},L)}$, then the WCVaR $\sup_{p(\mathrm{r},L)\in D^t_{P(\mathrm{r},L)}} CVaR_\beta\left(\mathrm{x}_t\right)$ evaluates to:

$$\max_{\bar{\mathrm{r}}_t\in U_{\mathrm{r}_t}} -\bar{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1} + \max_{\bar{L}_t\in U_{L_t}} \psi\bar{L}_t + k \max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}} \sqrt{\psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}}.$$

The first term can be written as follows:

$$\max_{\bar{\mathrm{r}_t}\in U_{\mathrm{r}_t}} -\bar{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1} = -\min_{\bar{\mathrm{r}_t}\in U_{\mathrm{r}_t}} \bar{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1},$$

which is a classical robust optimization problem when an ellipsoidal uncertainty set is used for the uncertain parameter $\bar{\mathrm{r}}_t$. Consequently, its tractable reformulation is:

$$\min_{\bar{\mathrm{r}_t}\in U_{\mathrm{r}_t}} \bar{\mathrm{r}}_t^\intercal \mathrm{x}_{t-1} = \hat{\mathrm{r}_t}{}^\intercal \mathrm{x}_{t-1} - \sqrt{\gamma_1^t}\sqrt{\mathrm{x}_t^\intercal\hat{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1}}. \tag{3.12}$$

Likewise, the second term, related to the present value of future liabilities, can be tractably reformulated as follows:

$$\max_{\bar{L_{t]}}\in U_{L_t}} \psi\bar{L}_t = \psi\left[\hat{L}_t + \sqrt{\gamma_3^t}\sqrt{\psi^2\hat{\Sigma}_{L_t}}\right]. \tag{3.13}$$

Since the square root is a monotonically increasing function, then $\max_{z\in\mathcal{Z}}\sqrt{f(z)} =$

$\sqrt{\max_{z\in\mathcal{Z}} f(z)}$. Hence:

$$\max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}} \sqrt{\psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}} =$$

$$\sqrt{\max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}},\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}} \psi^2\bar{\Sigma}_{L_t} + \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}}. \quad (3.14)$$

Also, because the terms under the square root depend on different uncertainty sets, they are separable. Then, the expression (3.14) is equivalent to:

$$\sqrt{\max_{\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}}} \psi^2\bar{\Sigma}_{L_t} + \max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}}} \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - \min_{\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}} 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}}.$$

Kang et al. (2019) showed that $\max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}}} \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} = \mathrm{x}_{t-1}^\intercal\left(\hat{\Sigma}_{\mathrm{r}_t} + \gamma_2^t I_{n+1}\right)\mathrm{x}_{t-1}$, *i.e.,* the worst-case is obtained by perturbing the nominal variance-covariance matrix by the radius of the ambiguity set. By using the same proof developed in (Kang et al., 2019, Proposition 2.2), $\max_{\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}}} \psi^2\bar{\Sigma}_{L_t} = \psi^2\left(\hat{\Sigma}_{L_t} + \gamma_4^t\right)$. Finally, $\min_{\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}} 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}$ is a robust optimization problem with a box uncertainty set, which evaluates to $-2\mathrm{x}_{t-1}'\Upsilon_5^t$. With that, the third term can be tractably reformulated as follows:

$$\sqrt{\max_{\bar{\Sigma}_{L_t}\in U_{\Sigma_{L_t}}} \psi^2\bar{\Sigma}_{L_t} + \max_{\bar{\Sigma}_{\mathrm{r}_t}\in U_{\Sigma_{\mathrm{r}_t}}} \mathrm{x}_{t-1}^\intercal\bar{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1} - \min_{\bar{\sigma}_{(L_t,\mathrm{r}_t)}\in U_{\sigma_{(L_t,\mathrm{r}_t)}}} 2\mathrm{x}_{t-1}^\intercal\bar{\sigma}_{(\mathrm{r}_t,L_t)}} =$$

$$\sqrt{\mathrm{x}_{t-1}^\intercal\left(\hat{\Sigma}_{\mathrm{r}_t} + \gamma_2^t I_{n+1}\right)\mathrm{x}_{t-1} + \psi^2\left(\hat{\Sigma}_{L_t} + \gamma_4^t\right) + 2\mathrm{x}_{t-1}^\intercal\Upsilon_5^t} \quad (3.15)$$

Now, by combining (3.12), (3.13), and (3.15), the LHS of constraint (3.3c) is equivalent to:

$$\sup_{p(\mathrm{r},L)\in D^t_{P(\mathrm{r},L)}\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\Upsilon_5^t\right)} CVaR_\beta\left(\mathrm{x}_{t-1}\right) = -\left[\hat{\mathrm{r}}_t^\intercal\mathrm{x}_{t-1} - \sqrt{\gamma_1^t}\sqrt{\mathrm{x}_{t-1}^\intercal\hat{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1}}\right] +$$

$$\psi\left[\hat{L}_t+\sqrt{\gamma_3^t}\sqrt{\psi^2\hat{\Sigma}_{L_t}}\right]+k\sqrt{\mathrm{x}_{t-1}^{\intercal}\left(\hat{\Sigma}_{\mathrm{r}_t}+\gamma_2^t I_{n+1}\right)\mathrm{x}_{t-1}+\psi^2\left(\hat{\Sigma}_{L_t}+\gamma_4^t\right)+2\mathrm{x}_{t-1}^{\intercal}\boldsymbol{\gamma}_5^t},$$

where $k=\sqrt{\frac{\beta}{1-\beta}}$. □

Since $p(\mathrm{r},L)\in D_{P(\mathrm{r},L)}^t\left(\gamma_1^t,\gamma_2^t,\gamma_3^t,\gamma_4^t,\boldsymbol{\gamma}_5^t\right)$, $q\in Q\left(\gamma_6,\gamma_7\right)$, and based on proposition 6, the robust counterpart of model (3.3) is as follows:

$$min_{\mathrm{y}\in\mathcal{Y},\mathrm{x}_t\in\mathcal{X}}\quad \hat{\mathrm{W}}^{\intercal}\mathrm{y}+\sqrt{\gamma_6}\sqrt{\mathrm{y}^{\intercal}\hat{\Sigma}_{\mathrm{W}}\mathrm{y}}, \tag{3.16a}$$

$$\text{s.t.}\quad \mathrm{e}^{\intercal}\mathrm{x}_t=\hat{\mathrm{r}}_t^{\intercal}\mathrm{x}_{t-1}-\sqrt{\gamma_1^t}\sqrt{\mathrm{x}_{t-1}^{\intercal}\hat{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1}}+w_t y_t-l_t,\quad t=0,\dots,T-1, \tag{3.16b}$$

$$-\hat{\mathrm{r}}_t^{\intercal}\mathrm{x}_{t-1}+\sqrt{\gamma_1^t}\sqrt{\mathrm{x}_{t-1}^{\intercal}\hat{\Sigma}_{\mathrm{r}_t}\mathrm{x}_{t-1}}+\psi\hat{L}_t+\psi^2\sqrt{\gamma_3^t}\sqrt{\hat{\Sigma}_{L_t}}+$$
$$k\sqrt{\mathrm{x}_{t-1}^{\intercal}(\hat{\Sigma}_{\mathrm{r}_t}+\gamma_2^t I_{n+1})\mathrm{x}_{t-1}+\psi^2(\hat{\Sigma}_{L_t}+\gamma_4^t)+2\mathrm{x}_{t-1}^{\intercal}\boldsymbol{\gamma}_5^t}\leq 0,\qquad t=0,\dots,T. \tag{3.16c}$$

Model (3.16) represents a DRO version of the ALM model that accounts for moment uncertainty. This model is more complex than the original ALM problem, which was a linear programming model. The nonlinear nature of the model and the incorporation of moment-based ambiguity sets allow for a more accurate representation of the uncertainty inherent in the ALM problem. In the next section, we will evaluate the proposed model using real-world data, through which we can assess its effectiveness in providing robust solutions that improve the long-term financial outcomes of pension funds.

3.5 Numerical Results

In this research, we use data from the Canada pension plan (CPP) to conduct numerical experiments/tests. Contributions to CPP are compulsory for all working Canadians aged 18-70, based on CPP information [5]. Also, around 5.8 million individuals are receiving retirement benefits from CPP each month. On average \$811.21 are paid

[5]https://open.canada.ca/data/en/dataset/1fab2afd-4f3c-4922-a07e-58d7bed9dcfc

in month January 2023 to retired Canadians [6]. Moreover, $14,371,853$ individuals are contributing to CPP based on CPP investments report [7].

CPP is investing in 5 asset classes [8]: fixed income, private equity, public equity, infrastructure, and real estate. Moreover, CPP investments are geographically diversified in North America, Europe, and Asia. In our analysis, we use data from 10 major indexes from 2012 to 2022: $S\&P500$ index is used for public equities, Private Equity Index (PRIVEXD) is used for private equities, SP/TSX Capped Real Estate Index (GSPRTRE) is used for the real estate sector, Treasury Yield 10 Years (TNX) is used for fixed-income assets, and finally, $S\&P$ Global Infrastructure TR (SPGTINTR) is used for infrastructure investment. $S\&P$ TSX Composite is the index of the Canadian market. For public equities in Europe, FTSEurofirst 300 is used. STOXX Europe 20 is used for the private equity index in Europe. Shanghai Stock Exchange (SSE) and Nikkei-225 indexes are used as representatives of investment in Asia. The value of the total asset in CPP is \$539 B in 2022. Based on the most recent report of CPP, the projected earnings of contributors for 2022 have been $\$585,498$ M, where about %9.9 of that, $\$57,964$ M, is the contribution to CPP [9].

In order to apply model (3.16), individual WCVaR constraints are required for each decision moment. As a result, it is necessary to determine the moments of the uncertain distribution function of random variables for each decision moment. However, it is possible for asset returns to follow the same distribution in each period and for the mean/variance differences among periods to lack statistical significance. Consequently, the uncertain parameters in each decision moment may exhibit the same moments. Statistical analysis is conducted to find the distribution function and the first two moments of asset returns in each decision moment. The Individual Distribution Identification (IDI) feature in Minitab was used to conduct goodness-of-fit tests to identify the distribution function of returns with the maximum likelihood among a standard set of distribution functions. Table 3.1 shows the results of the goodness of fit for testing the distribution function of asset returns in each period. Based on the results illustrated in Table 3.1, we can conclude that the is not evidence

[6] https://www.canada.ca/en/services/benefits/publicpensions/cpp/cpp-benefit/amount.html
[7] https://www.cppinvestments.com/the-fund/our-performance/financial-results/f2022-annual-results
[8] https://ca.investing.com/
[9] https://www.osfi-bsif.gc.ca/Eng/oca-bac/ar-ra/cpp-rpc/Pages/cpp30.aspx

to reject the normality assumption of asset return since the p-values of the goodness-of-fit tests are greater than the significant level, $\alpha = 0.05$, in most periods.

We next test whether there are significant differences between the mean/variance values among different periods (months). Consequently, we test the equality of the mean/variance of asset returns in each period for all assets by using a one-way ANOVA test. Table 3.2 shows the results of this test. The null hypothesis for equality of variance is "All variances of an asset class in each period are equal", while its alternative is "At least one variance is different". Similarly, the null hypothesis for equality of the mean is "All means of an asset class in each period are equal" and the alternative one is "At least one mean is different". The Significance level for this test is 0.05. Based on the p-values illustrated in Table 3.2, we fail to reject the null hypotheses. Hence, we do not have any evidence to support the assumption of different means/variances across periods for the return of assets.

For solving the ALM problem, we consider a set of regulatory constraints. The contribution rate in each period is required to be between 5% to 10%. The investment in the US market cannot be greater than 60% of the whole fund. Investment in Canada must be at least 20% of the fund. At least 10% of the fund must be invested in fixed-income assets. Investment in Asia cannot be greater than 15% of the fund. Finally, the funding ratio should be at least 1.05. We provide in-sample and out-of-sample performance analyses to compare the results of the proposed DRO formulation in two cases, WCVaR of ALM problem where moments are uncertain (`UM`) (3.16) and WCVaR of ALM where moments are assumed to be known and fixed (`FM`) (3.11), in addition to the stochastic programming (SP) reformulation of the ALM problem with CvaR constraints (`SP`). In-sample performance analysis refers to evaluating the performance of a model on the same data that it was trained on. We are using historical data of CPP for in-sample analysis. On the other hand, out-of-sample performance analysis refers to evaluating the performance of a model on data that it has not seen during the training phase. We are using the simulation to generate data for out-of-sample analysis. Both in-sample and out-of-sample comparisons are based on the funding ratio and the fund return in each period.

Table 3.3 displays the in-sample performance of the funding ratio and fund return of the ALM problem under two different proposed approaches: `UM` and the `FM`, as

Table 3.1: p-value of the goodness of fit for testing the distribution function of asset returns in each period

Index	Distribution	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
PRIVEXD	Normal	0.688	0.343	<0.005	0.117	0.868	0.367	0.835	0.206	0.504	0.771	<0.005	0.127
	2-Parameter Exponential	0.053	0.035	<0.010	<0.010	0.035	<0.010	0.024	<0.010	0.011	0.05	0.146	<0.010
	3-Parameter Weibull	>0.500	0.298	0.072	>0.500	>0.500	>0.500	>0.500	0.182	>0.500	>0.500	0.093	0.466
	Smallest Extreme Value	>0.250	>0.250	0.124	>0.250	>0.250	>0.250	>0.250	0.089	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	0.208	<0.010	<0.010	>0.250	0.048	>0.250	0.094	0.231	>0.250	0.086	0.024
	Logistic	>0.250	>0.250	0.032	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.012	0.202
$S\&P500$	Normal	0.69	0.341	<0.005	0.118	0.869	0.367	0.835	0.206	0.504	0.77	<0.005	0.185
	2-Parameter Exponential	0.053	0.035	<0.010	<0.010	0.035	<0.010	0.024	<0.010	0.011	0.05	0.124	<0.010
	3-Parameter Weibull	>0.500	0.295	0.072	>0.500	>0.500	>0.500	>0.500	0.181	>0.500	>0.500	0.071	>0.500
	Smallest Extreme Value	>0.250	>0.250	0.124	>0.250	>0.250	>0.250	>0.250	0.088	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	0.208	<0.010	<0.010	>0.250	0.048	>0.250	0.094	0.231	>0.250	0.062	0.033
	Logistic	>0.250	>0.250	0.032	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.008	>0.250
GSPRTRE	Normal	0.11	0.137	<0.005	0.552	0.513	0.105	0.378	0.119	0.464	0.727	<0.005	0.124
	2-Parameter Exponential	0.041	<0.010	<0.010	0.03	0.023	<0.010	0.048	0.025	0.12	>0.250	0.159	>0.250
	3-Parameter Weibull	0.206	0.124	0.01	>0.500	>0.500	>0.500	>0.500	0.308	0.332	>0.500	0.115	0.389
	Smallest Extreme Value	0.05	0.2	0.024	>0.250	>0.250	>0.250	0.053	>0.250	>0.250	>0.250	<0.010	0.092
	Largest Extreme Value	0.208	0.078	<0.010	0.229	0.244	<0.010	>0.250	0.042	>0.250	>0.250	0.143	0.107
	Logistic	0.085	0.139	<0.005	>0.250	>0.250	>0.250	>0.250	0.12	>0.250	>0.250	0.022	0.098
$S\&P$/TSX Composite	Normal	0.008	0.197	<0.005	0.088	0.698	0.168	0.207	0.269	0.337	0.791	0.078	0.253
	2-Parameter Exponential	>0.250	<0.010	<0.010	<0.010	0.033	<0.010	>0.250	<0.010	>0.250	0.021	>0.250	<0.010
	3-Parameter Weibull	>0.500	0.244	0.072	0.093	>0.500	0.4	0.38	>0.500	0.367	>0.500	>0.500	>0.500
	Smallest Extreme Value	<0.010	>0.250	0.123	0.017	>0.250	>0.250	0.106	>0.250	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	0.054	<0.010	0.08	>0.250	0.04	>0.250	0.049	>0.250	>0.250	>0.250	0.081
	Logistic	0.083	0.238	0.021	0.149	>0.250	0.235	0.169	>0.250	>0.250	>0.250	0.211	>0.250
TNX	Normal	0.29	0.366	0.073	0.169	0.011	0.725	0.131	0.835	0.858	0.077	0.022	0.237
	2-Parameter Exponential	0.148	0.021	<0.010	>0.250	0.017	<0.010	0.023	0.017	0.13	0.097	>0.250	<0.010
	3-Parameter Weibull	0.226	0.479	0.054	>0.500	0.062	>0.500	0.181	>0.500	>0.500	0.408	>0.500	0.295
	Smallest Extreme Value	0.212	0.087	0.059	0.018	<0.010	>0.250	>0.250	>0.250	>0.250	<0.010	<0.010	>0.250
	Largest Extreme Value	0.244	>0.250	0.016	>0.250	0.108	0.173	0.063	>0.250	>0.250	>0.250	>0.250	0.035
	Logistic	0.23	>0.250	0.098	>0.250	0.056	>0.250	0.118	>0.250	>0.250	0.21	0.13	>0.250
SPGTINTR	Normal	0.005	0.026	<0.005	0.163	0.638	0.907	0.179	0.262	0.253	0.495	0.163	0.794
	2-Parameter Exponential	0.202	<0.010	<0.010	<0.010	0.024	0.017	0.018	0.012	<0.010	0.034	>0.250	>0.250
	3-Parameter Weibull	0.201	0.18	0.04	0.136	>0.500	>0.500	0.44	0.111	0.306	>0.500	>0.500	>0.500
	Smallest Extreme Value	<0.010	>0.250	0.075	0.04	>0.250	>0.250	>0.250	0.187	>0.250	>0.250	0.034	>0.250
	Largest Extreme Value	>0.250	<0.010	<0.010	0.077	>0.250	>0.250	0.043	0.201	0.037	0.228	>0.250	>0.250
	Logistic	0.052	0.042	0.009	>0.250	>0.250	>0.250	0.223	0.213	>0.250	>0.250	0.247	>0.250
FTSEurofirst 300	Normal	0.407	0.662	0.006	0.422	0.038	0.794	0.752	0.337	0.734	0.922	0.113	0.623
	2-Parameter Exponential	0.033	0.015	<0.010	0.101	<0.010	0.099	>0.250	<0.010	0.045	0.144	>0.250	0.063
	3-Parameter Weibull	>0.500	>0.500	0.099	0.485	0.299	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500
	Smallest Extreme Value	0.095	>0.250	0.174	0.159	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.013	>0.250
	Largest Extreme Value	>0.250	>0.250	<0.010	>0.250	<0.010	>0.250	>0.250	0.064	>0.250	>0.250	>0.250	>0.250
	Logistic	>0.250	>0.250	0.046	>0.250	0.101	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250
STOXX Europe 20	Normal	0.262	0.511	<0.005	0.731	0.892	0.393	0.79	0.467	0.05	0.901	<0.005	0.613
	2-Parameter Exponential	0.114	0.012	<0.010	0.021	0.057	<0.010	0.058	0.015	<0.010	0.031	0.246	0.046
	3-Parameter Weibull	>0.500	>0.500	0.038	>0.500	>0.500	>0.500	>0.500	0.439	0.203	>0.500	0.188	>0.500
	Smallest Extreme Value	0.046	0.138	0.071	>0.250	>0.250	>0.250	>0.250	0.227	>0.250	>0.250	<0.010	>0.250
	Largest Extreme Value	>0.250	>0.250	<0.010	>0.250	>0.250	0.061	>0.250	0.23	0.014	>0.250	0.178	>0.250
	Logistic	>0.250	>0.250	0.017	>0.250	>0.250	>0.250	>0.250	>0.250	0.07	>0.250	0.023	>0.250
SSE	Normal	0.045	0.146	0.048	<0.005	0.33	0.213	0.607	0.115	0.554	0.373	0.706	0.029
	2-Parameter Exponential	<0.010	0.043	>0.250	0.07	<0.010	<0.010	<0.010	<0.010	>0.250	0.016	>0.250	>0.250
	3-Parameter Weibull	>0.500	0.403	>0.500	0.049	0.174	>0.500	>0.500	0.476	0.482	0.413	>0.500	0.417
	Smallest Extreme Value	>0.250	0.011	0.016	<0.010	0.246	>0.250	>0.250	>0.250	>0.250	0.062	>0.250	<0.010
	Largest Extreme Value	<0.010	>0.250	0.205	0.082	0.167	0.069	0.169	0.019	>0.250	>0.250	>0.250	>0.250
	Logistic	0.226	>0.250	0.049	0.015	>0.250	0.221	>0.250	0.232	>0.250	>0.250	>0.250	0.136
Nikkei 225	Normal	0.487	0.57	0.435	0.676	0.544	0.152	0.892	0.78	0.055	0.451	0.35	0.102
	2-Parameter Exponential	0.09	0.057	<0.010	>0.250	<0.010	<0.010	0.018	<0.010	0.018	<0.010	0.023	<0.010
	3-Parameter Weibull	0.37	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500	>0.500	0.083	>0.500	0.495	0.278
	Smallest Extreme Value	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.145	>0.250	0.067	>0.250
	Largest Extreme Value	0.194	0.224	0.054	>0.250	0.097	0.014	>0.250	0.249	0.022	0.17	>0.250	0.015
	Logistic	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250	0.046	>0.250	>0.250	0.226

Table 3.2: p-values related to hypothesis test for equality of the mean/variance of asset returns in each period

Test	Equality of variances	Equality of means
PRIVEXD	0.971	0.407
$S\&P500$	0.974	0.422
GSPRTRE	0.831	0.965
$S\&P$/TSX Composite	0.813	0.496
TNX	0.275	0.868
SPGTINTR	0.797	0.733
FTSEurofirst 300	0.644	0.401
STOXX Europe 20	0.755	0.632
SSE	0.978	0.861
Nikkei 225	0.925	0.407

Table 3.3: In-sample performance of the ALM models

	UM		FM		SP	
Decision moments	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return
1	1.092	0.020	1.090	0.019	1.094	0.024
2	1.10	0.004	1.11	0.006	1.13	0.032
3	1.10	0.004	1.12	0.006	1.16	0.032
4	1.10	0.003	1.11	0.007	1.19	0.019
5	1.11	0.003	1.12	0.008	1.22	0.031
6	1.11	0.003	1.12	0.008	1.26	0.031
7	1.12	0.003	1.13	0.008	1.29	0.018
8	1.12	0.003	1.13	0.008	1.32	0.029
9	1.12	0.003	1.14	0.008	1.36	0.030
10	1.12	0.003	1.12	0.008	1.40	0.030
11	1.13	0.002	1.13	0.008	1.43	0.017

well as the risk-neutral approach of SP. It consists of 11 periods, each representing a specific time point. For the UM model, the highest funding ratio is 1.13 in the final period, while the lowest funding ratio is 1.09 in the first period. The corresponding fund return ranges from 0.002 to 0.02. The overall return in this investment horizon is 5.1%. For the FM model, the funding ratio ranges from 1.09 to 1.14, and the fund return ranges from 0.007 to 0.019 with an overall return of 9.9%. The funding ratios are slightly different in WCVaR models, which suggests that the uncertainty of moments affects the funding ratio and fund return.

For the SP model, the funding ratio ranges from 1.09 in the first period to 1.43 in the 11th period, and the fund return ranges from 0.017 to 0.032 overall return of 33%. The funding ratio and fund return of the SP model are higher than the UM and FM models, which indicates that the risk-neutral approach of SP is more optimistic than the WCVaR of ALM with fixed and uncertain moments.

Figure 3.1 shows the in-sample performance of the funding ratio of the SP, UM, and FM models. It illustrates that the SP has better performance than the FM and UM models based on funding ratio, which is predictable since the FM and UM models are more conservative than SP. Figure 3.2 demonstrates the fund return in each period. Although the SP has a higher return in each period than the two other models, it also has higher volatility. The UM and FM models show slightly different trends in the funding ratio and fund returns, which indicates that the uncertainty of moments has an impact on the performance of the ALM problem. Meanwhile, the SP model provides an optimistic scenario for the system's future performance with higher volatility of fund return in each period.

Asset allocation is a crucial decision in the ALM problem. It involves deciding how to distribute investments across different asset classes to achieve the desired level of return while minimizing risk. Figure 3.3 compares the optimal asset allocation of three models. As shown in Figure 3.3, the WCVaR models provide more diversified portfolios than the SP model, which leads to a less risky portfolio. The WCVaR models consider the probability distribution of returns and estimate the risk of the portfolio based on the worst-case scenario. As a result, the WCVaR models provide more robust and stable asset allocation over time. In contrast, the SP model does not account for the uncertainty of the distribution function and can lead to more volatile

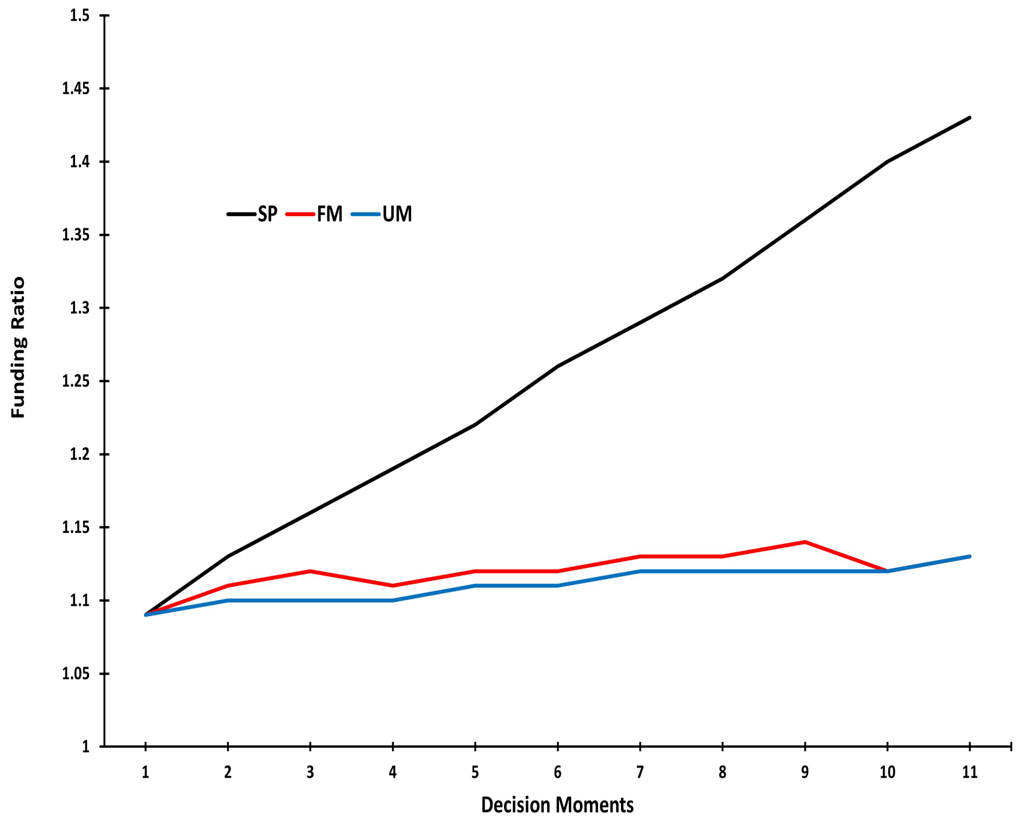

Figure 3.1: In-sample performance of funding ratio

asset allocation over the investment horizon. The comparison of the optimal asset allocation of the different models in Figure 3.3 highlights the advantages of using the WCVaR models, which provide more diversified and less risky portfolios compared to the SP model.

Another point of comparison is the contribution rate, which changes based on the funding ratio (FR) threshold. Table 3.4, shows the comparison of the optimal contribution rates of three models (UM, FM, and SP). As shown in Table 3.4, the optimal contribution rates of the three models differ depending on the FR parameter.

Table 3.4: Optimal contribution rates of three models based on funding ratio

Models	FR=1.02	FR=1.05	FR=1.07	FR=1.1	FR=1.15
UM	3.7%	5.7%	6.6%	7.7%	10.2%
FM	0.9%	2.4%	3.3%	4.8%	7.1%
SP	0.1%	2.3%	3.2%	4.6%	7.1%

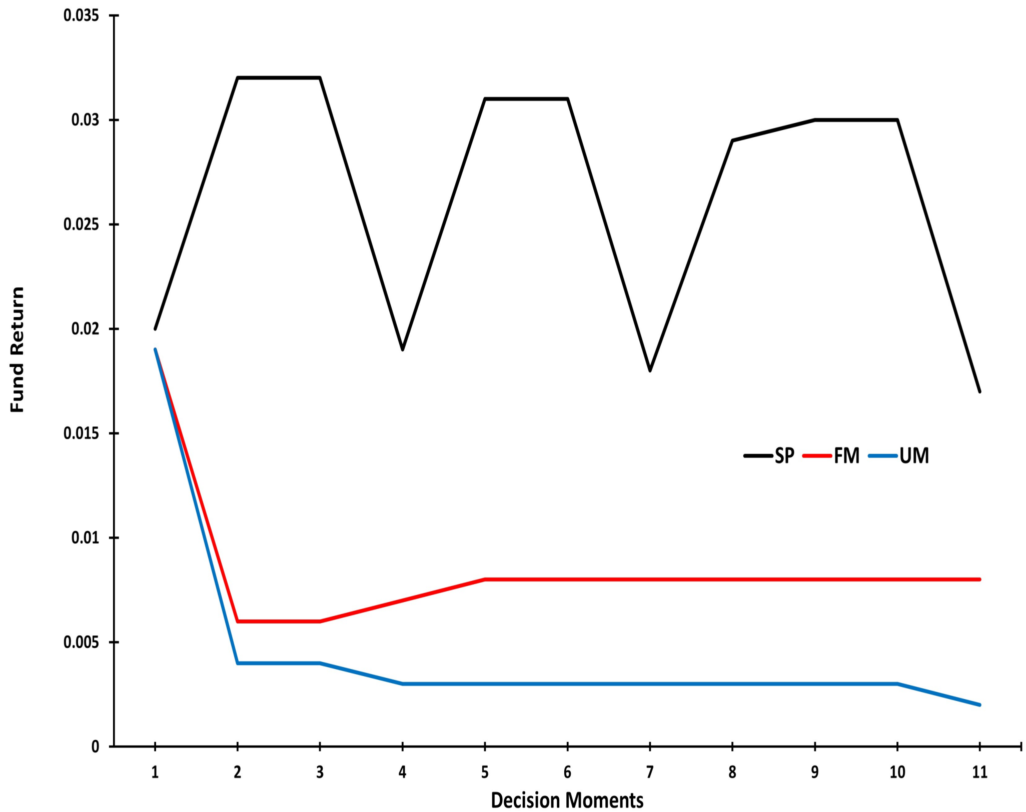

Figure 3.2: In-sample performance of fund return

For instance, when FR=1.02, the optimal contribution rates for the UM, FM, and SP models are 3.7%, 0.9%, and 0.1%, respectively. However, as FR is increased, the optimal contribution rates of all three models also increase. Furthermore, the UM model has the highest optimal contribution rates among the three models for all FR values. This suggests that this model may be the most conservative in managing risk under different FR scenarios. In contrast, the SP model has the lowest optimal contribution rates for FR values up to 1.1. However, when FR=1.15, the optimal contribution rates of the SP model become equal to that of the FM model.

Besides in-sample analysis, we are comparing the out-of-sample performance of the above-mentioned models using simulation. 1000 scenarios of asset returns are generated based on distribution functions of asset returns in Table 3.1. Then, the optimal investment strategies of the UM, FM, and SP models are used to compare the funding ratio and value of assets in each period. Table 3.5 presents the out-of-sample

Table 3.5: Out-of-sample performance of the ALM models

Decision moments	UM		FM		SP	
	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return
1	0.96	0.017	0.95	0.007	0.80	-0.157
2	0.97	0.006	0.94	-0.01	0.90	0.134
3	0.98	0.013	0.94	-0.004	0.93	0.028
4	1.02	0.038	0.94	0.004	0.85	-0.079
5	1.02	0.006	0.95	0.004	0.94	0.098
6	1.03	0.011	0.93	-0.013	0.94	0.00
7	1.04	0.011	0.96	0.026	0.92	-0.02
8	1.05	0.008	0.95	-0.01	0.91	-0.015
9	1.09	0.038	0.94	-0.003	0.85	-0.063
10	1.10	0.004	0.95	0.009	0.94	0.103
11	1.13	0.031	0.97	0.012	1.02	0.094

performance of three ALM models. In the first two columns, we have the results of the `UM` model, showing that the funding ratio ranges from 0.96 to 1.13 and the fund return ranges from 0.004 to 0.038 with an overall return of 9% in the investment horizon. The next two columns present the results of the `FM` model, where the funding ratio ranges from 0.93 to 0.97 and the fund return ranges from -0.013 to 0.026 with an overall return of -5%. Finally, the last two columns present the results of the `SP` model, where the funding ratio ranges from 0.8 to 1.02 and the fund return ranges from -0.157 to 0.134 with an overall return of -2% with very high volatility.

Figure 3.4 compares the out-of-sample performance of the `UM`, `FM`, and `SP` models based on the fund return in each period. When comparing the fund return, we can observe that the `FM` and `SP` models have 5 periods with a negative return rate. Moreover, the funding return of the `SP` model shows high volatility in comparison to other two models. The overall return of these two models, `SP` and `FM`, are negative: -2% and -5%, respectively. On the other hand, the `UM` model has a positive return in all periods with an overall average return of 9% which is very similar to the actual fund return of CPP last year which was 10% [10]. This indicates that the `UM` model is more effective in generating return compared to the `FM` and the `SP` models.

Figure 3.5 demonstrates the out-of-sample performance of models based on the

[10]https://www.cppinvestments.com/the-fund/our-performance/financial-results/f2022-annual-results

funding ratio. Comparing the three models based on the funding ratio, we can see that the `UM` model has higher funding ratios compared to the `FM` and `SP` models. Moreover, the `FM` model has better performance than the `SP` model except in the 6th period. This suggests that the `UM` model is more stable and has a better ability to meet its obligations than the two other models. On the other hand, the `SP` model has a lower funding ratio, indicating a higher risk of not being able to meet its obligations.

In conclusion, based on the results presented in Table 3.5, it appears that the `UM` model outperforms the `FM` and `SP` models in terms of funding ratio and fund return, implying better stability and asset management performance.

3.6 Conclusions

In this chapter, we proposed a theoretical foundation for developing the WCVaR formulation for the ALM problem. The proposed theoretical development can be used in any problem with general loss functions. Based on the proposed theoretical foundation of WCVaR, we introduced the DRO reformulation of the ALM problem where the loss function is a linear function of asset returns and the present value of liabilities. The DRO reformulation of the ALM problem is proposed in two cases. First, the moments of the uncertain distribution function are fully known and fixed. Second, the moments of the distribution function of random variables are uncertain and belong to the uncertainty set.

Real data of CPP are used to test and analyze the performance of optimal investment strategies obtained by solving the DRO reformulations. The analysis was based on the in-sample and out-of-sample performance of the models. The results showed that the SP reformulation of the ALM has better in-sample performance than the DRO reformulation of the ALM models with respect to the fund return and funding ratio in each period. However, out-of-sample performance analysis revealed that the investment strategy of the DRO formulation of the ALM problem with uncertain moments has a better funding ratio and higher overall average fund return than the DRO with fixed moments and SP models. Consequently, we can conclude that the investment strategy achieved from the DRO reformulation of the ALM problem with uncertain moments can handle the asset and liability balance of pension funds better than the investment strategies of the DRO with fixed moments and SP models.

Numerous avenues for future research exist to expand upon the contributions made by this work. To begin, while the effectiveness of the DRO reformulation of the ALM problem has been demonstrated using moment-based ambiguity sets, it is imperative to explore the performance of investment strategies derived from the DRO formulation of ALM utilizing ambiguity sets based on distances. Such an investigation can yield insights into how metric-based ambiguity sets influence the model's performance. In addition, while this study has evaluated the proposed models using authentic CPP data, subjecting them to further testing across a wider array of pension funds could enhance our understanding of the models' generalizability. Lastly, an avenue for extension involves incorporating other pivotal considerations into pension fund management within the proposed models. These considerations encompass taxes, transaction costs, and regulatory constraints. By integrating these elements, the models can establish a more comprehensive framework for pension fund management, capable of accommodating a broader spectrum of real-world constraints.

3.7 Appendix

Lemma 7. *Let ξ be a univariate random variable, where $\mathbb{E}[\xi] = \mu$, $Var(\xi) = \sigma^2$, and $f(.)$ is a nonlinear function of random variabnle ξ that $f : \mathbb{R} \to \mathbb{R}$. Then:*

$$\sup_{\xi \sim (\mu, \sigma^2)} \mathbb{E}\left[(\alpha - f(\xi))^+\right] \approxeq \frac{\alpha - \left(f(\mu) + \frac{1}{2} f''(\mu)\sigma^2\right) + \sqrt{f'(\mu)^2 \sigma^2 + \left(\alpha - \left(f(\mu) + \frac{1}{2} f''(\mu)\sigma^2\right)\right)^2}}{2},$$

where $f'(.)$ and $f''(.)$ are first and second derivation of $f(.)$, respectively.

Proof. First, we need to find the expected value of $f(\xi)$. The second-order Taylor approximation of $f(\xi)$ around μ is:

$$\mathbb{E}[f(\xi)] \approxeq \mathbb{E}\left[f(\mu) + f'(\mu)(\xi - \mu) + \frac{1}{2} f''(\mu)(\xi - \mu)^2\right].$$

It is known that $\mathbb{E}(a + b) = \mathbb{E}(a) + \mathbb{E}(b)$. Then we can expand the proposed second-order Taylor approximation as:

$$\mathbb{E}[f(\xi)] \approxeq \mathbb{E}[f(\mu)] + f'(\mu)\mathbb{E}[\xi - \mu] + \frac{1}{2} f''(\mu)\mathbb{E}[\xi - \mu]^2,$$

where $\mathbb{E}\left[f\left(\mu\right)\right] = f\left(\mu\right)$, and $\mathbb{E}\left[\xi - \mu\right] = \mu - \mu = 0$. Then:

$$\mathbb{E}[f(\xi)] \approxeq f(\mu) + \frac{1}{2} f''(\mu)\mathbb{E}[\xi - \mu]^2.$$

Since $\mathbb{E}\left[\xi - \mu\right]^2 = Var\left(\xi\right) = \sigma^2$, then:

$$\mathbb{E}\left[f\left(\xi\right)\right] \approxeq f(\mu) + \frac{1}{2} f''\left(\mu\right)\sigma^2.$$

Now, we need to approximate $Var\left(f\left(\xi\right)\right)$. The first order Taylor approximation of $f(\xi)$ around μ is:

$$f\left(\mu\right) + f'\left(\mu\right)\left(\xi - \mu\right).$$

Then $Var\left(f\left(\xi\right)\right)$ can be approximated as:

$$Var\left[f\left(\xi\right)\right] \approxeq Var\left[f\left(\mu\right) + f'\left(\mu\right)\left(\xi - \mu\right)\right] = Var\left[f\left(\mu\right) + f'\left(\mu\right)\xi - f'\left(\mu\right)\mu\right].$$

The first term, $f\left(\mu\right)$, is constant then $Var\left(f\left(\mu\right)\right) = 0$. The third term $Var\left(f'\left(\mu\right)\mu\right)$ is also constant with a variance of zero. Consequently:

$$Var\left[f\left(\xi\right)\right] \approxeq Var\left[f'\left(\mu\right)\xi\right] = \left(f'\left(\mu\right)\right)^2 Var\left[\xi\right] = f'\left(\mu\right)^2\sigma^2.$$

By substituting $\mathbb{E}\left[f\left(\xi\right)\right]$ and $Var\left(f\left(\xi\right)\right)$ into the WLPM reformulation by Chen et al. (2011a),

$$\sup_{\xi\sim(\mu,\sigma^2)} \mathbb{E}\left[\left(\alpha - f\left(\xi\right)\right)^+\right] = \frac{\alpha - \mathbb{E}\left[f\left(\xi\right)\right] + \sqrt{Var\left(f\left(\xi\right)\right) + \left(\alpha - \mathbb{E}\left[f\left(\xi\right)\right]\right)^2}}{2},$$

we obtained the desired result.

□

Theorem 8. *Let ξ be a univariate random variable with mean μ and variance σ^2, and define the ambiguty set $P = \{p \in M_+ | P\left(\xi \in \Omega\right) = 1, \xi \sim (\mu, \sigma^2)\}$. Moreover, $f(\xi)$ is a loss function, where $f : \mathbb{R} \to \mathbb{R}$. Then WCVaR can be approximated as* *follows:*

$$WCVaR_\beta \approxeq f(\mu) + \frac{1}{2} f''(\mu)\sigma^2 + \sqrt{\frac{\beta}{1-\beta}}\sqrt{f'(\mu)^2\sigma^2}$$

Proof. Based on the definition, $WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[(f(\xi)-\alpha)^+\right]$. To reformulate the WCVaR, we need to calculate the WLPM term in the WCVaR definition. In Lemma 7, the LPM is in the form $\sup_{p(.)\in P}\mathbb{E}\left[(\alpha - f(\xi))^+\right]$. Hence we need to rearrange the LPM in CVaR as:

$$\sup_{p(.)\in P}\mathbb{E}\left[(f(\xi)-\alpha)^+\right] = \sup_{p(.)\in P}\mathbb{E}\left[(-\alpha-(-f(\xi)))^+\right].$$

based on Lemma 7:

$$\sup_{p(.)\in P}\mathbb{E}\left[(-\alpha-(-f(\xi)))^+\right]$$

$$\approxeq \frac{-\alpha + \left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right) + \sqrt{f'(\mu)^2\sigma^2 + \left(-\alpha + \left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)\right)^2}}{2}.$$

Now, we can substitute this WLPM into the WCVaR formulation. Consequently:

$$WCVaR_\beta \approxeq \min_{\alpha\in\mathbb{R}} \alpha$$

$$+\frac{1}{1-\beta}\frac{-\alpha + \left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right) + \sqrt{f'(\mu)^2\sigma^2 + \left(-\alpha + \left(f(\mu)+\frac{1}{2}f''(\mu)\sigma^2\right)\right)^2}}{2}.$$

To evaluate the minimization over α in the WCVaR definition we use the first-order optimality condition $\frac{\partial WCVaR_\beta}{\partial\alpha} = 0$, resulting in:

$$\alpha^* = f(\mu) + \frac{1}{2}f''(\mu)\sigma^2 + \frac{2\beta-1}{2\sqrt{\beta(\beta-1)}}\sqrt{f'(\mu)^2\sigma^2}.$$

By substituting α^* back in the definition of WCAR, we obtain the desired result.

□

Lemma 7 and Theorem 8 are based on a function of a univariate random variable, while in many cases, loss functions are functions of multivariate random variables such as engineering design problems. Consequently, we extend this lemma/theorem to multivariate random variables.

Lemma 9. *Let* $\xi = \{\xi_1, \ldots, \xi_n\}$ *be a multivariate random variable, where* $\mathbb{E}[\xi_i] = \mu_i$, $Var(\xi_i) = \sigma_i^2$, $Cov(\xi_i, \xi_j) = \sigma_{ij}$, *and* $f(.)$ *is a nonlinear function of random variable* ξ *that* $f : \mathbb{R}^n \to \mathbb{R}$. *Then* $\sup_{\xi \sim (\mu, \Sigma_\xi)} \mathbb{E}\left[(\alpha - f(\xi))^+\right]$ *can be approximated by:*

$$\frac{1}{2}(\alpha - \left(f(\mu) + \sum_i e_i \frac{\sigma_i^2}{2} + \sum_i \sum_{j>i} e_{ij}\sigma_{ij}\right)$$

$$+ \sqrt{\sum_i d_i^2 \sigma_i^2 + 2\sum_i \sum_{j>i} d_i d_j \sigma_{ij} + \left(\alpha - \left(f(\mu) + \sum_i e_i \frac{\sigma_i^2}{2} + \sum_i \sum_{j>i} e_{ij}\sigma_{ij}\right)\right)^2}),$$

where $\mu = \{\mu_1, \ldots, \mu_n\}$ *is the mean vector,* $e_i = \frac{\partial^2 f(\xi)}{\partial^2 \xi_i} |_{\xi=\mu}$, $e_{ij} = \frac{\partial^2 f(\xi)}{\partial \xi_i \partial \xi_j} |_{\xi=\mu}$, $d_i = \frac{\partial f(\xi)}{\partial \xi_i} |_{\xi=\mu}$ *and* $|_{\xi=\mu}$ *means to evaluate the expression with* μ_i *replacing* ξ_i.

Proof. Based on the second-order Taylor series expansion of $f(.)$ around $\mu = \{\mu_1, \ldots, \mu_n\}$, the expected value of $f(.)$ is approximated by:

$$\mathbb{E}[f(\xi)] \approxeq \mathbb{E}[f(\mu)] + \mathbb{E}[\nabla f(\mu)(\xi - \mu)] + \mathbb{E}\left[\frac{1}{2}(\xi - \mu)^\intercal H_f(\mu)(\xi - \mu)\right],$$

where H_f is the Hessian matrix of f. The second term is zero since $\mathbb{E}[\xi - \mu] = \mathbb{E}[\xi] - \mu = \mu - \mu = 0$. In the last term, $\mathbb{E}\left[(\xi - \mu)^2\right] = \Sigma_\xi$ is the variance-covariance matrix of ξ, then $\mathbb{E}[f(\xi)] \approxeq f(\mu) + \frac{1}{2} H_f(\mu)\Sigma_\xi$. Expansion of this formulation is:

$$\mathbb{E}[f(\xi)] \approxeq f(\mu) + \sum_i e_i \frac{\sigma_i^2}{2} + \sum_i \sum_{j>i} e_{ij}\sigma_{ij}.$$

Moreover, based on the first-order Taylor approximation, the variance of $f(.)$ is:

$$Var(f(\xi)) \approxeq Var(f(\mu) + \nabla f(\mu)^\intercal (\xi - \mu)) = Var(f(\mu) + \nabla f(\mu)^\intercal \xi - \nabla f(\mu)^\intercal \mu).$$

Since $f(\mu)$, and $\nabla f(\mu)\mu$ are constants, their variances are zero. Hence, $Var(f(\xi)) \approxeq$

$Var\left(\nabla f\left(\mu\right)^{\intercal}\xi\right)$ which is equivalent to $\nabla f\left(\mu\right)^{2}\Sigma_{\xi}$. This formulation can be expanded as $Var\left(f\left(\xi\right)\right)\approxeq\sum_{i}d_{i}^{2}\sigma_{i}^{2}+2\sum_{i}\sum_{j>i}d_{i}d_{j}\sigma_{ij}$. By substituting $\mathbb{E}\left[f\left(\xi\right)\right]$ and $Var\left(f\left(\xi\right)\right)$ into $\frac{\alpha-\mathbb{E}[f(\xi)]+\sqrt{Var(f(\xi))+(\alpha-\mathbb{E}[f(\xi)])^{2}}}{2}$, we obtain the desired result.

□

Theorem 10. *Let $\xi\in\mathbb{R}^{n}$ be a multivariate random variable with mean vector μ and covariance matrix Σ_{ξ}, where the ambiguity set is $P=\{p\in M_{+}|P\left(\xi\in\Omega\right)=1,\xi\sim\left(\mu,\Sigma_{\xi}\right)\}$. Moreover, $f\left(\xi\right)$ is a loss function, where $f:\mathbb{R}^{n}\rightarrow\mathbb{R}$. Then WCVaR is defined as $WCVaR_{\beta}=\sup_{p(.)\in P}\min_{\alpha\in\mathbb{R}}\alpha+\frac{1}{1-\beta}\mathbb{E}\left[\left(f(\xi)-\alpha\right)^{+}\right]$ which is approximated by:*

$$f\left(\mu\right)+\sum_{i}e_{i}\frac{\sigma_{i}^{2}}{2}+\sum_{i}\sum_{j>i}e_{ij}\sigma_{ij}+\sqrt{\frac{\beta}{1-\beta}}\sqrt{\sum_{i}d_{i}^{2}\sigma_{i}^{2}+2\sum_{i}\sum_{j>i}d_{i}d_{j}\sigma_{ij}}.$$

Proof. WCVaR is defined as $WCVaR_{\beta}=\sup_{p(.)\in P}\min_{\alpha\in\mathbb{R}}\alpha+\frac{1}{1-\beta}\mathbb{E}\left[\left(f\left(\xi\right)-\alpha\right)^{+}\right]$. In Lemma 9, we showed how to approximate the WLPM of a function of multivariate random variables as $\sup_{p(.)\in P}\mathbb{E}\left[\left(f\left(\xi\right)-\alpha\right)^{+}\right]=\sup_{p(.)\in P}\mathbb{E}\left[\left(-\alpha-\left(-f\left(\xi\right)\right)\right)^{+}\right]$ which is approximated by:

$$\frac{1}{2}(-\alpha+\left(f\left(\mu\right)+\sum_{i}e_{i}\frac{\sigma_{i}^{2}}{2}+\sum_{i}\sum_{j>i}e_{ij}\sigma_{ij}\right)$$

$$+\sqrt{\sum_{i}d_{i}^{2}\sigma_{i}^{2}+2\sum_{i}\sum_{j>i}d_{i}d_{j}\sigma_{ij}+\left(-\alpha+\left(f\left(\mu\right)+\sum_{i}e_{i}\frac{\sigma_{i}^{2}}{2}+\sum_{i}\sum_{j>i}e_{ij}\sigma_{ij}\right)\right)^{2}}),$$

which can be substituted in WCVaR formulation instead of WLPM.

Minimization of WCVaR is over α. Then its optimal value α^{*} is needed. Optimal α^{*} can be calculated by using the first-order optimality condition $\frac{\partial WCVaR_{\beta}}{\partial\alpha}=0$. The α^{*} is as follows:

$$\alpha^{*}=f\left(\mu\right)+\sum_{i}e_{i}\frac{\sigma_{i}^{2}}{2}+\sum_{i}\sum_{j>i}e_{ij}\sigma_{ij}+\frac{2\beta-1}{2\sqrt{\beta\left(\beta-1\right)}}\sqrt{\sum_{i}d_{i}^{2}\sigma_{i}^{2}+2\sum_{i}\sum_{j>i}d_{i}d_{j}\sigma_{ij}}.$$

Finally, α^{*} can be used in the formulation of WCVaR instead of α which leads to the desired result.

□

A quadratic function is a special case of a nonlinear function. A loss function can be defined based on a quadratic function of a random variable. For example, tracking errors in index-tracking PSPs is an example of a quadratic function that can be used as a loss function. In Lemma 7 and theorem 8, both the mean and variance of the loss function are approximated by the Taylor approximation method. However, by using a quadratic function of a random variable as a loss function, the variance of the loss function should be approximated while the expected value of the loss function can be calculated based on exact formulation. Remark 1 shows how to calculate the WLPM for a quadratic function of a random variable.

Remark 1. *Let ξ be an univariate random variable, where $\mathbb{E}[\xi] = \mu$, and $Var(\xi) = \sigma^2$. Then, WLPM is approximated as follows:*

$$\sup_{\xi\sim(\mu,\sigma^2)} \mathbb{E}\left[\left(\alpha - \xi^2\right)^+\right] \approxeq \frac{\alpha - (\sigma^2 - \mu^2) + \sqrt{4\mu^2\sigma^2 + (\alpha - [\sigma^2 - \mu^2])^2}}{2}.$$

Proof. Based on definition of the first two moments of ξ, $Var\left(\xi\right) = \mathbb{E}\left[\xi^2\right] - \mathbb{E}\left[\xi\right]^2 = \sigma^2$. Then, $\mathbb{E}\left[\xi^2\right] = \sigma^2 - \mu^2$. The first-order Taylor approximation of ξ^2, around $\mathbb{E}\left[\xi\right]$, is $\mathbb{E}\left[\xi\right]^2 + 2\mathbb{E}\left[\xi\right]\left(\xi - \mathbb{E}\left[\xi\right]\right)$. Consequently, variance of ξ^2 is approximated as follows:

$$Var\left(\xi^2\right) \approxeq Var\left(\mathbb{E}\left[\xi\right]^2 + 2\mathbb{E}\left[\xi\right]\left(\xi - \mathbb{E}\left[\xi\right]\right)\right)$$

$$= Var\left(\mathbb{E}\left[\xi\right]^2 + 2\mathbb{E}\left[\xi\right]\xi - 2\mathbb{E}\left[\xi\right]\mathbb{E}\left[\xi\right]\right) = Var\left(\mu^2 + 2\mu\xi - 2\mu^2\right).$$

The first and third terms are constants, then their variances are zero. Hence,

$$Var\left(\xi^2\right) \approxeq Var\left(2\mu\xi\right) = 4\mu^2 Var\left(\xi\right).$$

By using $\mathbb{E}\left[\xi^2\right]$ and $Var\left(\xi^2\right)$, WLPM is as follows:

$$\sup_{\xi\sim(\mu,\sigma^2)} \mathbb{E}\left[\left(\alpha - \xi^2\right)^+\right] \approxeq \frac{\alpha - (\sigma^2 - \mu^2) + \sqrt{4\mu^2\sigma^2 + (\alpha - [\sigma^2 - \mu^2])^2}}{2}.$$

□

The WCVaR for quadratic loss function is defined based on remark 2.

Remark 2. *Let ξ be a univariate random variable with mean μ and variance σ^2, where the ambiguity set is $P = \{p \in M_+ | P(\xi \in \Omega) = 1, \xi \sim (\mu, \sigma^2)\}$. Moreover, ξ^2 is a loss function. Then WCVaR is defined as:*

$$WCVaR_\beta \approxeq \sigma^2 - \mu^2 + 2\mu\sigma\sqrt{\frac{\beta}{1-\beta}}$$

Proof. Based on definition, $WCVaR_\beta = \sup_{p(.)\in P} \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\mathbb{E}\left[\left(\xi^2 - \alpha\right)^+\right]$. The WLPM of ξ^2 is defined based on remark 1. Hence:

$$\sup_{p(.)\in P} \mathbb{E}\left[\left(\xi^2 - \alpha\right)^+\right] = \sup_{p(.)\in P} \mathbb{E}\left[(-\alpha - (-\xi^2))^+\right] \approxeq \frac{-\alpha + (\sigma^2 - \mu^2) + \sqrt{4\mu^2\sigma^2 + \left(-\alpha + [\sigma^2 - \mu^2]\right)^2}}{2}.$$

By using the approximation of WLPM in WCVaR formulation, WCVaR of the quadratic loss function is as follows:

$$WCVaR_\beta \approxeq \min_{\alpha\in\mathbb{R}} \alpha + \frac{1}{1-\beta}\frac{-\alpha + (\sigma^2 - \mu^2) + \sqrt{4\mu^2\sigma^2 + \left(-\alpha + [\sigma^2 - \mu^2]\right)^2}}{2}.$$

Minimization of WCVaR is over α, hence its optimal value is needed which can be calculated by solving the first-order optimality condition, $\frac{\partial WCVaR_\beta}{\partial \alpha} = 0$. The optimal α^* is as follows:

$$\alpha^* = \sigma^2 - \mu^2 + \frac{2\beta - 1}{2\sqrt{\beta(\beta-1)}}\sqrt{4\mu^2 Var(\xi)}.$$

By using α^* in $WCVaR$ instead of α, WCVaR is approximated as desired result.

□

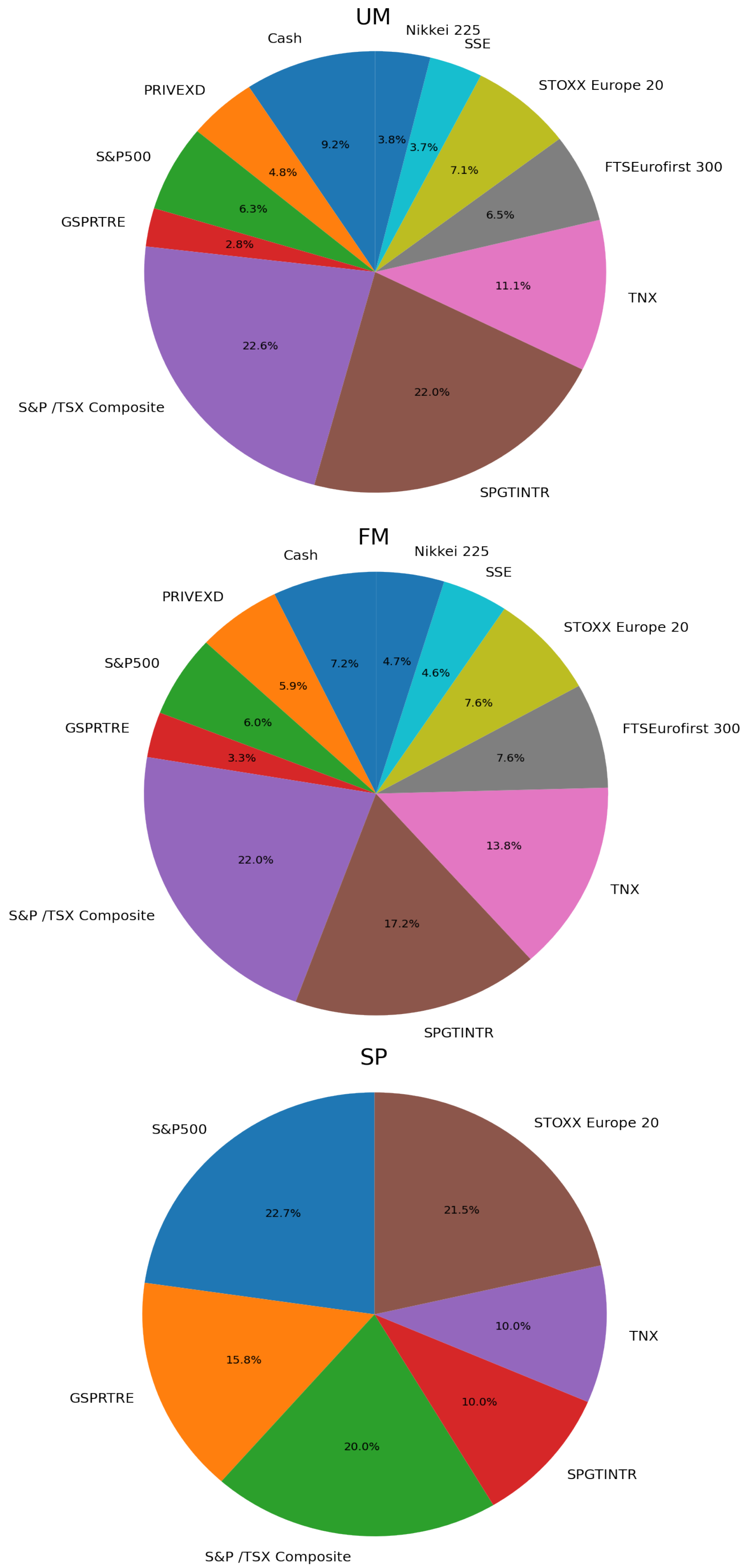

Figure 3.3: Comparision of optimal asset allocation

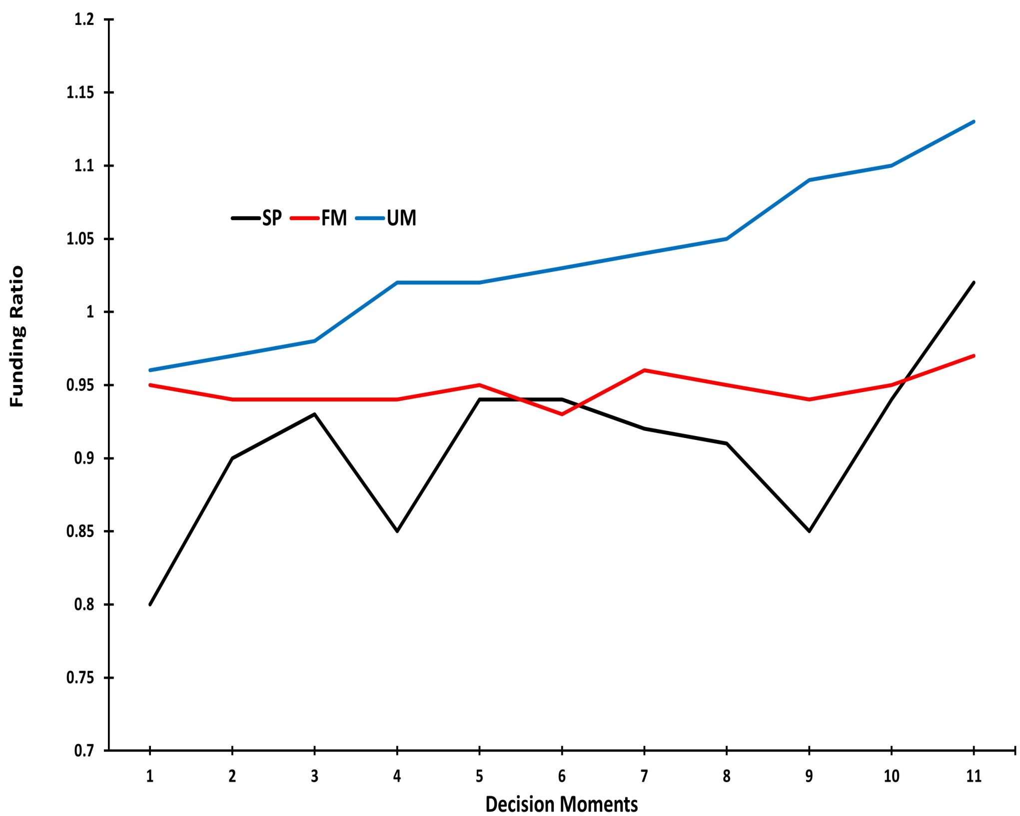

Figure 3.4: Out-of-sample performance of fund return

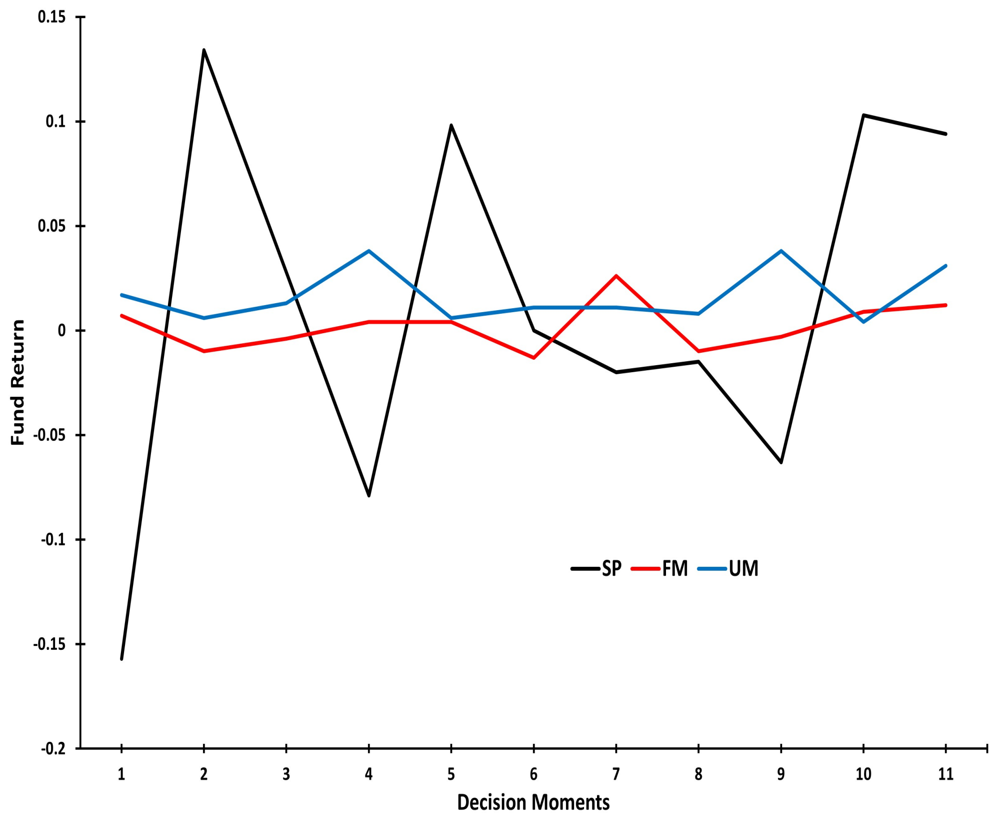

Figure 3.5: Out-of-sample performance of funding ratio

Chapter 4

Distributionally Robust Asset Liability Management Problem

4.1 Introduction

Asset-liability management (ALM) refers to the challenge of managing the assets and liabilities of an entity in a way that ensures the entity can meet its financial obligations in the future (Zenios, 1995). The ALM problem typically arises in financial institutions such as banks, insurance companies, and pension funds, which have significant liabilities that must be met over a long period of time. In particular, ALM is a critical concern for pension funds that must ensure they meet specific obligations to individuals who have contributed to their funds while also generating investment returns (Bodie et al., 1988).

Pension funds play a crucial role in the global financial landscape as evidenced by their substantial assets which exceeded \$60.6 trillion by the end of 2021, accounting for 33% of global assets [1]. This magnitude is exemplified by the fact that in nine out of the 38 Organisation for Economic Co-operation and Development (OECD) countries, pension fund assets surpassed their respective GDPs. In the last decade (2010-2020), pension assets have grown by 5.7% [2], outpacing the 2.6% GDP growth rate over the same period [3] and underscoring the increasing importance of retirement savings worldwide. However, as the aging population grows, outflows from pension funds to cover benefits are also accelerating. The ratio of benefits paid from retirement savings plans to GDP varies across OECD countries, ranging from 0.5% to 8% [4]. This growth in both assets and payouts highlights the need for prudent and sustainable management of pension funds to ensure that retirees receive their benefits without

[1] https://www.thinkingaheadinstitute.org/research-papers/global-pension-assets-study-2022/

[2] https://www.statista.com/statistics/721151/average-growth-largest-pension-markets-worldwide/

[3] https://www.macrotrends.net/countries/WLD/world/gdp-growth-rate

[4] https://www.oecd.org/finance/private-pensions/globalpensionstatistics.htm

putting undue stress on the funds' assets.

Pension funds are essential for ensuring retirement income security; however, they encounter challenges arising from demographic changes, low-interest rates, and increasing life expectancy. To address these challenges, reforms have been implemented, including raising the retirement age and promoting private pension plans (Holzmann, 2013). However, one of the major challenges for pension fund managers is the uncertainty surrounding future asset returns and liabilities. The asset returns and the value of liabilities can fluctuate due to factors like inflation, interest rates, and market conditions. To mitigate this risk, effective ALM strategies are necessary. These strategies involve monitoring and managing investment portfolios to ensure they are well-aligned with future obligations. By optimizing the ALM problem under uncertainty, pension funds can enhance long-term sustainability and avoid financial distress (Gülpınar et al., 2016).

Among the powerful techniques for managing uncertainty in ALM problems is stochastic programming (SP), which explicitly models uncertainty through probability distributions of asset return and liability values and finds optimal asset allocation strategies of the portfolio under different scenarios. This approach allows investors to better hedge against unexpected changes in asset returns and liability values, while still maintaining a desirable level of return (Kouwenberg, 2001). SP can also be used to optimize dynamic asset allocation strategies, where the asset allocation is periodically adjusted in response to changing market conditions and liability values (Consigli & Dempster, 1998; Dempster & Consigli, 1996; Hibiki, 2006). By modeling the stochastic behavior of asset returns and liability values, investors can make more informed decisions about the optimal timing and size of asset allocation adjustments, and better manage the risk of underfunding future liabilities. For more details on the application of SP in ALM problems, we refer to (Klaassen, 1997; Kouwenberg, 2001; Consigli, 2008; Duarte et al., 2017; Kopa et al., 2018; Barro et al., 2022). Despite its intuitive appeal and favorable convergence properties, SP requires large amounts of data on asset returns and liability values to construct their probability distributions, which may be difficult to obtain or may not be available. Furthermore, SP is a risk-neutral approach and thus does not provide sufficient protection against adverse scenarios.

Another popular framework for dealing with uncertainty is robust optimization (RO), which seeks to find solutions that perform optimally under worst-case scenarios, in contrast to SP that aims to optimize the expected performance (Ben-Tal et al., 2009; Gabrel et al., 2014; Ghahtarani et al., 2022). In the context of ALM, RO can be used to find an asset allocation strategy that is most robust to uncertainty in asset returns and liability values.

A few recent attempts have been made to apply RO to mitigate uncertainty in the ALM problem. Iyengar & Ma (2016) introduced a robust factor model to capture the true uncertainty of asset returns in pension fund management. By incorporating a factor model with stochastic parameters, they developed an ALM formulation with a constraint on the funding ratio. The funding ratio, representing the assets' value relative to the present value of liabilities, is subject to uncertainty. The proposed formulation assumes the funding ratio as an uncertain parameter and utilizes a Gaussian process for factors. Platanakis & Sutcliffe (2017) extended this approach by considering ellipsoidal uncertainty sets for factor loading, box ambiguity sets for asset returns and liabilities, and upper and lower bounds for the covariance matrix of disturbances, which enabled the problem to be reformulated as a second-order cone programming (SOCP) model. Based on the results, these robust factor models and formulations enhance the out-of-sample performance of ALM problems. Alongside robust factor models for ALM problems, Gülpinar & Pachamanova (2013) proposed a robust ALM using time-varying investment opportunities. They extended the multiperiod PSP formulation of Dantzig & Infanger (1993) by including liabilities and funding ratio constraints. In this formulation, cumulative rates of return of assets are treated as uncertain parameters within an ellipsoidal uncertainty set. Moreover, asset returns and interest rates are modeled by using the vector-autoregressive process to capture the dynamic nature of investments. In contrast to other robust ALM approaches, Gülpınar et al. (2016) developed an asymmetric uncertainty set to better reflect the actual uncertainty structure. Gajek & Krajewska (2022) proposed a robust ALM formulation with uncertain interest rates, where the distribution function of the uncertain parameters belongs to a nonempty ambiguity set. This formulation provides an upper bound on VaR (Value at Risk) for portfolio value changes caused by violations of the interest rate model. Finally, the ALM problem with discrete recourse decision and parameter uncertainty has been addressed by Ghahtarani et al. (2023b) using the K-adaptability approach in Chapter 5. However, despite being a risk-averse and distribution-free approach, RO usually results in overly conservative investment strategies, which can lead to missed opportunities for higher returns, thus negatively impacting the long-term performance of pension funds.

Disadvantages of the application of either SP or RO in ALM problems provide motivation for the application of a relatively new approach to ALM problems called distributionally robust optimization (DRO) (Rahimian & Mehrotra, 2019). Like RO, DRO aims to minimize the impact of uncertain scenarios on investment decisions. However, DRO goes one step further by enabling the available information about the probability distribution of random variables, albeit limited and imperfect, to be incorporated into the decision-making process, thus leading to less conservative and more stable investment strategies (Lin et al., 2022). Unlike SP and RO, which have been applied to the ALM problem, DRO is yet to be leveraged in this context. One reason for this is the inherent complexity of DRO models and their comparatively recent development. Notably, a study by Ghahtarani et al. (2023a) focused on moment-based ambiguity sets for the ALM problem and introduced worst-case Conditional Value at Risk (CVaR) as a risk measure to deal with parameter uncertainty in the problem. DRO has the potential to address some of the limitations of other methods, including the optimizer's curse in SP and the over-conservatism in RO. Moreover, DRO provides a way to explicitly consider the ambiguity in the distribution of financial variables.

In this chapter, we aim to fill this gap in the literature by proposing DRO formulations for the ALM problem. We explore scenarios-based approaches to address the uncertainty of parameters in the ALM problem. Numerous studies have suggested that scenario-based analysis is superior to prediction-based analysis in financial problems. Boender (1997) argues that scenarios explicitly record assumptions about the future and provide a common framework for discussion. By utilizing scenarios, we can create a better understanding between managers and stakeholders, which can ultimately contribute to more effective decision-making. The main goal of this chapter is to develop DRO scenario-based formulations and compare them against each other. The first formulation uses mixture ambiguity sets, each representing a convex combination of multiple distributions, each having multiple scenarios, which is commonly used in portfolio selection problems (Zhu & Fukushima, 2009). In the second formulation, we investigate the case that the probabilities of scenarios are interval-bounded, but also there is a requirement that they add up to 1, which basically means that we are using a polyhedral set to represent the uncertain probabilities. Lastly, we incorporate the Wasserstein ambiguity set into the ALM problem, which is a metric-based ambiguity set. These ambiguity sets have been specifically chosen due to their suitability for utilization in scenario-based DRO formulation.

We demonstrate, through the application of DRO in an ALM context using real-world data, the potential advantages of this approach. Specifically, we develop DRO models for an ALM problem that accounts for the ambiguity about the distribution of asset returns and interest rates. We also compare the performance of our DRO models to the traditional SP formulation of the ALM problem to demonstrate the advantages and limitations of each approach. By doing so, we hope to contribute to the development of a more robust and flexible framework for ALM that can better account for the uncertainty and variability in financial markets.

The set of most used notations is shown in Table 4.1, whereas the notations that are used once are defined in the text.

Table 4.1: Notations and symbols

Symbol/Notation	Definition
$t \in \{0, \ldots, T\}$	Indices of decision moments
T	Investment horizon
$s \in \{1, \ldots, S\}$	Indices of discount rate scenarios
$k \in \{1, ..., K\}$	Indices of asset return scenarios
y_t	Contribution rate at the moment t, the fraction of sponsor and/or active employees' wages
$y_{t,s}$	Contribution rate at the moment t based on scenario s
$n \in \{0, \ldots, N\}$	Indices of assets, where $n = 0$ represents risk-free asset or cash
$x_{n,t}$	Money invested in asset n at the moment t

Table 4.1: Notations and symbols

Symbol/Notation	Definition
$x_{n,t,k}$	Money invested in asset n at the moment t based on scenario k
A_t	Value of assets owned by the fund at the moment t
W_t	Wages earned by active members at the moment t
l_t	Payments made by the fund to retirees at the moment t
L_t	Net present value of liabilities of the fund at the moment t
$L_{t,s}$	Net present value of liabilities of the fund at the moment t based on scenario s
$\xi_{n,t}$	Return on investment in asset n at the moment t
$\xi_{n,t,k}$	Return on investment in asset n at the moment t based on scenario k
ψ	Minimum threshold of funding ratio
γ	Discount rate for calculating the present value
W_t	Net present value of wages at the moment t
p	Discrete distribution function of discount rate
q	Discrete distribution function of asset returns
P	Ambiguity set of the distribution function of discount rate
Q	Ambiguity set of the distribution function of asset returns

The remaining sections of this chapter are structured as follows. Section 4.2 introduces the mathematical formulation of the ALM problem. In Section 4.3, we present DRO formulations of the ALM problem based on the mixture distribution, the box, and the Wasserstein ambiguity sets. To test the proposed formulation, numerical experiments using real data from the Canada Pension Plan (CPP) are conducted, and the results are presented in Section 4.4. Finally, Section 4.5 offers some conclusions and suggests potential areas for future research.

4.2 ALM Model for Pension Funds

The ALM problem under consideration aims to find an optimal investment strategy that achieves a trade-off between augmenting investment returns and reducing the risk of insolvency. The objective of the ALM for a pension fund is to minimize the contribution rate by both the sponsor and active employees of the fund (*i.e.,* the contributors), as defined in previous studies (Bogentoft et al., 2001). The financial burden is reduced by reducing the contribution rate, while the efficient investment strategy balances risk and returns over the investment horizon. The optimization process involves selecting the optimal mix of asset classes, such as stocks, bonds, and alternative investments, and the corresponding contribution rate for each period of the investment horizon. By finding the optimal solution to the ALM problem, the pension fund can ensure that it meets its future obligations while minimizing the financial burden on its stakeholders.

The investment horizon for the ALM problem under consideration, denoted as T, encompasses a series of decision moments represented by $t = 0, \ldots, T$. Several variables and decision-making components are at play in the ALM problem for pension funds. At moment t, the contribution rate is denoted by y_t, which is the fraction of the contributor's wage w_t collected. Additionally, the decision variables $x_{n,t}$ represent the amount of money invested in asset n at moment t, while $\xi_{n,t}$ is the return of asset n in t^{th} moment. The value of assets held by the fund at moment t is represented by A_t, while the liabilities at that moment, which are payments made by the fund to retirees, are denoted by l_t. The present value of liabilities at moment t is given by L_t, which is calculated by $\sum_{t=0}^{T} \frac{l_t}{(1+\gamma)^t}$, $\forall t = 0, \ldots, T$. It is worth noting that in this case, benefit payments and liabilities are fixed and predefined, which classifies this type of pension fund as a *defined-benefit plan.* The discount rate, γ, for the calculation of the present value of liabilities is a random variable. The funding ratio, a crucial parameter in the ALM problem, ensures that the ratio of assets owned by the fund to the present value of liabilities at moment t is maintained above a minimum threshold ψ. This means that the fund has sufficient resources to meet its future obligations.

Model (4.1) shows the mathematical formulation of the ALM problem:

$$\min_{y_t, x_{n,t}} \quad h(y_1, \ldots, y_T), \tag{4.1a}$$

$$\text{s.t.} \quad \sum_{n=0}^{N} x_{n,t} = A_t + w_t y_t - l_t, \quad t = 0, \ldots, T-1, \tag{4.1b}$$

$$A_t \geq \psi L_t, \quad t = 0, \ldots, T, \tag{4.1c}$$

$$A_t = \sum_{n=0}^{N} x_{n,t-1}(1 + \xi_{n,t}), \quad t = 0, \ldots, T, \tag{4.1d}$$

$$x_{n,t} \in \mathcal{X}, y_t \in \mathcal{Y}, \quad t = 0, \ldots, T, n = 0, \ldots, N. \tag{4.1e}$$

The objective function of the ALM problem (4.1), introduced by Bogentoft et al. (2001), is denoted by $h(y_0, \ldots, y_T)$, which is a function defined in terms of the contribution rate and plays a crucial role in determining the optimal ALM strategy. In particular, the objective function (4.1a) can be defined as the present value of all contributions, *i.e.,* $h(y_0, \ldots, y_T) = \sum_{t=0}^{T} W_t y_t$, where $W_t = \frac{w_t}{(1+\gamma)^t}$. The balance constraint (4.1b) ensures that the sum of all investments in period t is equal to the assets held by the fund plus the contributions gathered in period t minus liabilities in this period. The funding ratio constraint (4.1c) guarantees that the ratio of assets owned by the fund to the present value of liabilities at period t is greater than a minimum threshold ψ. Constraint (4.1d) describes how to calculate the value of assets owned by the fund at moment t as a function of their values at the moment $(t-1)$ and the return rates. Any regulatory or practical (*e.g.,* nonnegativity) restrictions on the investment strategy and the contribution rates are encapsulated in the sets $\mathcal{X}$ and $\mathcal{Y}$, respectively, and are enforced by constraint (4.1e). In formulation (4.1), the uncertain parameters are the discount rate, γ, and the asset returns, $\xi_{n,t}$. The uncertainty of the discount rate makes the present value of future liabilities, L_t, and the present value of wages, W_t, uncertain, since they both depend on γ.

To simplify the formulation, we express the objective function of model (4.1) using the vectors $\mathrm{W} = [W_0, \ldots, W_T]^\intercal \in \mathbb{R}^{T+1}$ and $\mathrm{y} = [y_0, \ldots, y_T]^\intercal \in \mathbb{R}^{T+1}$, which represent the present value of the contributors' wages and the contribution rate decision variables, respectively. The objective function can then be written as $\mathrm{W}^\intercal \mathrm{y}$. We also introduce the vector $\mathrm{r}_t = \mathrm{e} + \xi_t$ for $t = 0, \ldots, T$, where e is an all-ones vector of size $N+1$ and ξ_t is an uncertain vector that captures the variation in the plan's funding status. Additionally, we define the investment decision variable as a vector $\mathrm{x}_t = [x_{0,t}, \dots, x_{n,t}]^\intercal$ for each decision moment t. Using these notations, we can transform the ALM problem (4.2) into a vector representation as follows:

$$\min_{\mathrm{y},\mathrm{x}_t} \quad \mathrm{W}^\intercal \mathrm{y}, \tag{4.2a}$$

$$\text{s.t.} \quad \mathrm{e}^\intercal \mathrm{x}_t = \mathrm{r}_t^\intercal \mathrm{x}_{t-1} + w_t y_t - l_t, \quad t = 0, \dots, T-1, \tag{4.2b}$$

$$\mathrm{r}_t^\intercal \mathrm{x}_{t-1} \geq \psi L_t, \quad t = 0, \dots, T, \tag{4.2c}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \quad t = 0, \dots, T. \tag{4.2d}$$

The linear programming model (4.2) incorporates three uncertain parameters: W, r_t, and L_t. To ensure the robustness of the model, it is important to account for the uncertainty associated with these parameters. In the following section, we present a DRO reformulation of (4.2) to address this issue.

4.3 Distributionally Robust ALM

Before providing the DRO formulation, we begin by presenting a scenario-based SP formulation of the ALM problem. As previously discussed, the present value of wages, W, and the present value of future liabilities, L_t, are both influenced by the uncertain discount rate, while all other parameters affecting them are assumed deterministic. As a result, they are perfectly correlated and thus can both be represented using a single set of discrete scenarios $\{s\}_{s=1,\dots,S}$ having a given distribution function $p(.)$. Likewise, we use a finite set of scenarios $\{k\}_{k=1,\dots,K}$, having the discrete distribution function $q(.)$, to capture the uncertainty of asset returns. With that, model (4.3) presents an SP formulation of the ALM based on these scenario sets and distribution functions. Note that the additional subscript (s or k) for the uncertain parameters denotes the scenario.

$$
\begin{aligned}
\min_{\mathrm{y},\mathrm{x}_t} \quad & \mathbb{E}_p(\mathrm{W}_s^{\intercal}\mathrm{y}), && && \text{(4.3a)}\\
\text{s.t.} \quad & \mathrm{e}^{\intercal}\mathrm{x}_t = \mathbb{E}_q(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}) + w_t y_t - l_t, && t = 0,\dots,T-1, && \text{(4.3b)}\\
& \mathbb{E}_q(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}) \geq \psi \mathbb{E}_p(L_{t,s}), && t = 0,\dots,T, && \text{(4.3c)}\\
& \mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} && t = 0,\dots,T. && \text{(4.3d)}
\end{aligned}
$$

In model (4.3), it is essential to have knowledge of the distribution functions for accurate analysis and decision-making. This requirement arises due to the significant impact of uncertainty in the problem domain. Additionally, the tractability of this problem is closely linked to the number of scenarios considered. As the number of scenarios increases, the complexity of the problem grows exponentially. Consequently, the computational burden and resource requirements for solving the problem also increase. Therefore, carefully considering the number of scenarios is crucial to balance accuracy and computational feasibility in tackling the model, which is a major limitation of this SP formulation. However, in reality, these distribution functions may not be fully known, and therefore, we propose DRO as an alternative to SP for the ALM problem since the former does not require exact knowledge of the probability distributions. Moreover, DRO provides a robust solution by accounting for a range of possible different distributions, which enables decision-makers to hedge against various plausible distributional scenarios, leading to more reliable and stable solutions that are less sensitive to uncertain input parameters. Consequently, we assume that the distribution functions belong to the sets that represent a range of possible probability distributions, called ambiguity sets. Let P and Q be ambiguity sets for the distribution functions of asset return and discount rate, respectively. Then, the DRO formulation of the ALM is presented in model (4.4):

$$\min_{\mathrm{y},\mathrm{x}_t} \sup_{p\in P} \quad \mathbb{E}_p(\mathrm{W}_s^{\intercal}\mathrm{y}), \tag{4.4a}$$

$$\text{s.t.} \quad \mathrm{e}^{\intercal}\mathrm{x}_t = \inf_{q\in Q} \mathbb{E}_q(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}) + w_t y_t - l_t, \quad t = 0,\dots,T-1, \tag{4.4b}$$

$$\inf_{q\in Q} \mathbb{E}_q(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}) \geq \psi \sup_{p\in P} \mathbb{E}_p(L_{t,s}), \quad t = 0,\dots,T, \tag{4.4c}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \quad t = 0,\dots,T. \tag{4.4d}$$

Model (4.4) is based on the worst-case expected value of random variables. For the remainder of the paper, we will explore various ambiguity sets that can be applied to the formulation presented in the model (4.4).

4.3.1 Mixture Distribution

We are dealing with ambiguous discrete distribution functions, Whereas the scenarios themselves are deterministically defined. One approach to address this ambiguity of the distribution function is to use a set to represent the possible discrete distribution function. It is common to consider uncertain discrete distributions in portfolio selection problems (for more details, see Costa & Paiva (2002), Ghaoui et al. (2003), Ghahtarani et al. (2022)). In this case, the ambiguity set is considered a mixture of predetermined likelihood distributions. Based on Zhu & Fukushima (2009), these ambiguity sets are defined as $P_M := \left\{p : p = \sum_{i=1}^{I} \lambda_i p^i; \sum_{i=1}^{I} \lambda_i = 1; \lambda_i \geq 0; i = 1,\dots,I\right\}$, where p^i is the i^{th} likelihood distribution and I is the number of likelihood distributions. Likewise, $Q_M := \left\{q : q = \sum_{j=1}^{J} \lambda_j q^j; \sum_{j=1}^{J} \lambda_j = 1; \lambda_j \geq 0, ; j = 1,\dots,J\right\}$, where q^j is the j^{th} likelihood distribution and J is the number of likelihood distributions.

Although the ambiguity set P_M includes all convex combinations of the I likelihood distributions $p^i,\ i = 1,\dots,I$, it is easy to show that the worst-case distribution for any given values of the decision variables is one of I likelihood distributions themselves. This *maximal solution* property is due to the fact that finding the worst-case distribution is analogous to solving a binary knapsack problem with unit-sized items and a knapsack capacity of 1, where only one item (having the highest value) is selected. A similar argument can be made for the ambiguity set Q_M, though with the lowest value item selected. Thus, to reformulate model (4.4) with the mentioned ambiguity sets, we introduce the auxiliary variables θ, μ_t, and ω_t, where $\theta \geq \sum_{s=1}^{S}(\mathrm{W}_s^{\intercal}\mathrm{y})p_s^i$, $\mu_t \leq \sum_{k=1}^{K}(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1})q_k^j$, and $\omega_t \geq \sum_{s=1}^{S} L_{t,s}p_s^i$. Using these ambiguity sets and auxiliary variables, the DRO problem can be written in the epigraph form as follows:

$$\min_{\mathrm{y},\mathrm{x}_t,\theta,\mu_t,\omega_t} \quad \theta, \tag{4.5a}$$

$$\text{s.t.} \quad \theta \geq \sum_{s=1}^{S}(\mathrm{W}_s^{\intercal}\mathrm{y})p_s^i, \qquad i = 1,\ldots,I, \tag{4.5b}$$

$$\mathrm{e}^{\intercal}\mathrm{x}_t = \mu_t + w_t y_t - l_t, \quad t = 0,\ldots,T-1, \tag{4.5c}$$

$$\mu_t \geq \psi\omega_t, \qquad t = 0,\ldots,T, \tag{4.5d}$$

$$\mu_t \leq \sum_{k=1}^{K}(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1})q_k^j \qquad t = 0,\ldots,T, j = 1,\ldots,J, \tag{4.5e}$$

$$\omega_t \geq \sum_{s=1}^{S} L_{t,s}p_s^i, \qquad t = 0,\ldots,T, i = 1,\ldots,I, \tag{4.5f}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \qquad t = 0,\ldots,T. \tag{4.5g}$$

Model (4.5) is a linear programming model, which is a tractable model and captures the ambiguity of discrete distribution functions.

4.3.2 Discrete Distribution with Box Ambiguity

The mixture-distribution ambiguity set proposed in subsection 4.3.1 has two main drawbacks. First, it confines the ambiguity about the discrete probability distributions to finite sets of elements (distribution functions) and their convex combinations while ignoring the possibility that the true distribution functions can take other forms. Although this issue can be partially alleviated by increasing the value of I, *i.e.,* using a large number of distribution functions that cover a wider range of possibilities, the problem size inevitably grows, thus reducing its tractability, which is the second drawback. Alternatively, a box ambiguity set can be used for the discrete distribution function, which provides does not need a large number of possible distribution functions in a convex set. Note that for ambiguity sets of discrete distributions, model

(4.4) can be expanded to:

$$\min_{\mathrm{y},\mathrm{x}_t} \sup_{p \in P} \quad \sum_{s=1}^{S} (\mathrm{W}_s^{\intercal}\mathrm{y}) p_s, \tag{4.6a}$$

$$\text{s.t.} \qquad \mathrm{e}^{\intercal}\mathrm{x}_t = \inf_{q \in Q} \sum_{k=1}^{K} (\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}) q_k + w_t y_t - l_t, \qquad t = 0, \ldots, T-1, \tag{4.6b}$$

$$\inf_{q \in Q} \sum_{k=1}^{K} (\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}) q_k \geq \psi \sup_{p \in P} \sum_{s=1}^{S} (L_{t,s}) p_s, \quad t = 0, \ldots, T, \tag{4.6c}$$

$$\mathrm{x}_t \in \mathcal{X}, \mathrm{y} \in \mathcal{Y} \qquad t = 0, \ldots, T. \tag{4.6d}$$

In model (4.6), we are dealing with two ambiguity sets P and Q that contain probability distributions of the random variables. Specifically, $p(\cdot) \in P$, which is defined as $P := \{p : p_s = p_s^0 + \eta_s : \sum_{s=1}^{S} \eta_s = 0, \underline{\eta}_s \leq \eta_s \leq \overline{\eta}_s\}$, where p_s^0 is the nominal probability of scenario s, and $\eta_s \in [\underline{\eta}_s, \overline{\eta}_s]$ is a bounded perturbation from it. Likewise, $q(\cdot) \in Q$, which is defined as $Q := \{q(.) : q_k = q_k^0 + \xi_k : \sum_{k=1}^{K} \xi_k = 0, \underline{\xi}_k \leq \xi_k \leq \overline{\xi}_k\}$, where q_k^0 is the nominal probability of scenario k, while $\xi_k \in [\underline{\xi}_k, \overline{\xi}_k]$.

To reformulate model (4.6), we apply LP duality to the inner problems. Specifically, the inner problem of the uncertain objective function (4.6a) can be expressed as follows:

$$\max_{\eta_s} \quad \sum_{s=1}^{S} (\mathrm{W}_s^{\intercal}\mathrm{y})(p_s^0 + \eta_s), \tag{4.7a}$$

$$\text{s.t.} \quad \sum_{s=1}^{S} \eta_s = 0, \qquad (z), \tag{4.7b}$$

$$\eta_s \leq \overline{\eta}_s, \qquad s = 1, \ldots, S, \quad (d_s^+), \tag{4.7c}$$

$$-\eta_s \leq -\underline{\eta}_s, \qquad s = 1, \ldots, S, \quad (d_s^-), \tag{4.7d}$$

where z, d_s^+, and d_s^- are the dual variables of their respective constraints. Problem (4.7) aims to optimize over the perturbation parameters η_s. Thus, the dual form of model (4.7) can be written as follows:

$$\sum_{s=1}^{S}(\mathrm{W}_s^{\intercal}\mathrm{y})p_s^0 + \min_{d_s^+\geq 0, d_s^-\geq 0, z} \quad \sum_{s=1}^{S}(d_s^+\overline{\eta}_s - d_s^-\underline{\eta}_s), \tag{4.8a}$$

$$\text{s.t.} \qquad z + d_s^+ - d_s^- \geq \mathrm{W}_s^{\intercal}\mathrm{y}, \quad s = 1, \ldots, S. \tag{4.8b}$$

Likewise, constraints (4.6b) and (4.6c) involve the inner optimization of uncertain parameters related to asset returns. Specifically, this inner optimization can be expressed as follows:

$$\min_{\xi_k} \quad \sum_{k=1}^{K}(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1})(q_k^0 + \xi_k), \tag{4.9a}$$

$$\text{s.t.} \quad \sum_{k=1}^{K}\xi_k = 0, \qquad (\Gamma), \tag{4.9b}$$

$$-\overline{\xi}_k \leq -\xi_k, \qquad k = 1, \ldots, K, \quad (\omega_k^+), \tag{4.9c}$$

$$\underline{\xi}_k \leq \xi_k, \qquad k = 1, \ldots, K, \quad (\omega_k^-), \tag{4.9d}$$

where Γ, ω_k^+, and ω_k^- are dual variables. The problem described in (4.9) optimizes over the perturbation variables ξ_k. To achieve this goal, the objective function (4.9a) is reformulated as $\sum_{k=1}^{K}(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1})q^0 + \min_{\xi_k}\sum_{k=1}^{K}(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1})\xi_k$, which expresses the minimum value of the linear combination $\sum_{k=1}^{K}(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1})\xi_k$ over all possible values of ξ_k. The dual form of the model presented in (4.9) can be expressed as follows:

$$\sum_{k=1}^{K}(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1})q^0 + \max_{\omega_k^+\geq 0, \omega_k^-\geq 0, \Gamma} \quad \sum_{k=1}^{K}(\omega_k^-\underline{\xi}_k - \omega_k^+\overline{\xi}_k), \tag{4.10a}$$

$$\text{s.t.} \qquad \Gamma + \omega_k^- - \omega_k^+ \leq \mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}, \quad k = 1, \ldots, K. \tag{4.10b}$$

Finally, the inner optimization model related to uncertainty of $L_{t,s}$ for each t in constraint (4.6c) is as follows:

$$\max_{\eta_s} \quad \sum_{s=1}^{S} L_{ts}(p_s^0 + \eta_s), \tag{4.11a}$$

$$\text{s.t.} \quad \sum_{s=1}^{S} \eta_s = 0, \qquad (z), \tag{4.11b}$$

$$\eta_s \leq \overline{\eta}_s, \qquad s = 1, \ldots, S, \quad (d_s^+), \tag{4.11c}$$

$$-\eta_s \leq -\underline{\eta}_s, \qquad s = 1, \ldots, S, \quad (d_s^-). \tag{4.11d}$$

The optimization problem (4.11) is over η_s. Then the objective function (4.11a) is transformed to $\sum_{s=1}^{S} L_{ts} p_s^0 + \max_{\eta_s} \sum_{s=1}^{S} L_{ts}\eta_s$. The dual form of model (4.11) for each decision moment t is as follows:

$$\sum_{s=1}^{S} L_{ts} p_s^0 + \min_{d_s^+ \geq 0, d_s^- \geq 0, z} \quad \sum_{s=1}^{S} (d_s^+ \overline{\eta}_s - d_s^- \underline{\eta}_s), \tag{4.12a}$$

$$\text{s.t.} \qquad z + d_s^+ - d_s^- \geq L_{t,s}, \quad s = 1, \ldots, S. \tag{4.12b}$$

The DRO of the ALM model with box ambiguity set can be formulated by substituting the dual forms of the optimization problems (4.8), (4.10), and (4.12) into the original optimization problem expressed in (4.6). The resulting final formulation is as follows:

$$\min_{\mathrm{y}, \mathrm{x}_t, d_s^+ \geq 0, d_s^-, \omega_k^+, \omega_k^- \geq 0, \Gamma, z} \quad \sum_{s=1}^{S} (\mathrm{W}_s^\intercal \mathrm{y})\, p_s^0 + \sum_{s=1}^{S} \left(d_s^+ \overline{\eta}_s - d_s^- \underline{\eta}_s \right), \tag{4.13a}$$

$$\text{s.t.} \quad \mathrm{e}^\intercal \mathrm{x}_t = \sum_{k=1}^{K} \left(\mathrm{r}_{t,k}^\intercal \mathrm{x}_{t-1} \right) q_k^0 + \sum_{k=1}^{K} \left(\omega_k^- \underline{\xi}_k - \omega_k^+ \overline{\xi}_k \right) + w_t y_t - l_t, \qquad t = 0, \ldots, T-1, \tag{4.13b}$$

$$\sum_{k=1}^{K}\left(\mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}\right)q_k^0+\sum_{k=1}^{K}\left(\omega_k^-\underline{\xi}_k-\omega_k^+\overline{\xi}_k\right)\geq$$
$$\psi\left(\sum_{s=1}^{S}L_{t,s}p_s^0+\sum_{s=1}^{S}\left(d_s^+\overline{\eta}_s-d_s^-\underline{\eta}_s\right)\right),\qquad t=0,\ldots,T,\tag{4.13c}$$

$$z+d_s^+-d_s^-\geq \mathrm{W}_s^{\intercal}\mathrm{y},\qquad s=1,\ldots,S,\tag{4.13d}$$
$$\Gamma+\omega_k^--\omega_k^+\leq \mathrm{r}_{t,k}^{\intercal}\mathrm{x}_{t-1}\qquad t=0,\ldots,T,k=1,\ldots,K,\tag{4.13e}$$
$$z+d_s^+-d_s^-\geq L_{t,s}\qquad t=0,\ldots,T,s=1,\ldots,S.\tag{4.13f}$$

Problem (4.13) is the tractable reformulation of (4.6) with box ambiguity sets.

4.3.3 Wasserstein Ambiguity Set

An ambiguity set that has drawn a lot of attention recently due to its favorable properties (*i.e.,* finite sample guarantee, asymptotic consistency, and tractability) is that based on the Wasserstein metric (Mohajerin Esfahani & Kuhn, 2018). Unlike the mixture-distribution and box ambiguity sets utilized earlier, which consider only distributions that are supported on the same support set of the empirical distribution (*i.e.,* use the same set of scenarios), the Wasserstein ambiguity set includes all distributions, discrete or continuous, that are sufficiently close to the empirical distribution. Thus, it offers higher flexibility and a more realistic representation of the uncertainty of the random problem parameters. In other words, we do not only consider the "original" scenarios on which the empirical distribution is supported, but also other scenarios not seen before.

The Wasserstein ambiguity set can be constructed using the discrete empirical probability distribution $\hat{p}=\frac{1}{S}\sum_{s\in S}\delta_{\hat{\mathrm{W}}_s}$, where δ is an indicator function that takes the value 1 for elements of the discrete set of scenarios $\hat{\Xi}_{\mathrm{W}}:=\hat{\mathrm{W}}_1,\ldots,\hat{\mathrm{W}}_s\subset\Xi_{\mathrm{W}}$ and 0 elsewhere. Specifically, the ambiguity set is defined as $D(\hat{p},\epsilon^1)=\{p\in M\mid P(\hat{\mathrm{W}}\in\Xi_{\mathrm{W}})=1,\ dw(\hat{p},p)\leq\epsilon^1\}$, where $dw(\hat{p},p)$ is the Wasserstein distance between the discrete empirical distribution $\hat{p}$ and a probability distribution p, and ϵ^1 is the radius of the ambiguity set. This ambiguity set is designed to capture a range of probability distributions within a certain distance of the empirical distribution.

Similarly, the ambiguity set for the vector of asset returns random variables $\mathbf{r}_t$, denoted as $D(\hat{q}_t, \epsilon_t^2)$, is constructed using the discrete empirical probability distribution $\hat{q}_t = \frac{1}{K}\sum_{k\in K}\delta_{\hat{\mathrm{r}}_t}$. Here, $\hat{\Xi}_{\mathrm{r}_t} := \hat{\mathrm{r}}_{t,1}, ..., \hat{\mathrm{r}}_{t,k} \subset \Xi_{\mathrm{r}_t}; \forall t$ is the set of empirical realizations of the vector of random variables $\mathbf{r}_t$. The ambiguity set $D(\hat{q}_t, \epsilon_t^2)$ is defined as $D(\hat{q}_t, \epsilon_t^2) = \{q_t \in M \mid P(\hat{\mathrm{r}_t} \in \Xi\mathrm{r}_t) = 1,\ \ dw(\hat{q}_t, q_t) \leq \epsilon_t^2\}$, where $dw(\hat{q}_t, q_t)$ is the Wasserstein distance between the discrete empirical distribution $\hat{q}_t$ and the probability distribution q_t, and ϵ_t^2 is the radius of the ambiguity set. To calculate the Wasserstein distance between two probability metrics q_1 and q_2, we use the integral representation $dw(q_1, q_2) = \int_{\Xi^2} ||\xi_1, \xi_2||Q(d\xi_1, d\xi_2)$, where Q is the joint distribution of ξ_1 and ξ_2 with marginal probabilities q_1 and q_2, respectively. This distance measure is used to capture the similarity between two probability distributions, where a smaller Wasserstein distance corresponds to a higher similarity between the distributions. Based on Mohajerin Esfahani & Kuhn (2018), under a convexity condition of the support set $\Xi_{\mathrm{W}} := \{\mathbb{C}\mathrm{W} \leq \mathrm{d}\}$, where $\mathbb{C} \in \mathbb{R}^{m\times(t+1)}$, $\mathrm{d} \in \mathbb{R}^m$, constraint (4.4a) is transformed into:

$$\inf_{\lambda,\nu_s,\gamma_s\geq 0} \quad \lambda\epsilon^1 + \frac{1}{S}\sum_{s\in S}\nu_s, \tag{4.14a}$$

$$\text{s.t.} \quad \hat{\mathrm{W}}_s^\intercal \mathrm{y} + (\mathrm{d} - \mathbb{C}\hat{\mathrm{W}}_s)^\intercal \boldsymbol{\gamma}_s \leq \nu_s, \quad s = 1, \ldots, S, \tag{4.14b}$$

$$\|\mathbb{C}^\intercal \boldsymbol{\gamma}_s - \mathrm{y}\|_* \leq \lambda, \quad s = 1, \ldots, S, \tag{4.14c}$$

where $\boldsymbol{\gamma}_s \in \mathbb{R}^m$, and $\|.\|_*$ is the dual norm of $\|.\|$, the norm used in the Wasserstein metric definition. With that, the right-hand side of constraint (4.4c), which has the p-distributed random parameter L_t with the support set $\Xi_{L_t} := \{\mathrm{f}_t L_t \leq \mathrm{b}_t,\ \forall t \in T\}$, where $\mathrm{f}_t \in \mathbb{R}^m$, $\mathrm{b}_t \in \mathbb{R}^m$ and $L_t \in \mathbb{R}$, reduces to:

$$\inf_{\theta_t,\upsilon_{s,t},\delta_{s,t}\geq 0} \quad \theta_t\epsilon_t^1 + \frac{1}{S}\sum_{s\in S}\upsilon_{s,t}, \tag{4.15a}$$

$$\text{s.t.} \quad \hat{L}_{s,t} + (\mathrm{b}_t - \mathrm{f}_t\hat{L}_{s,t})^\intercal \updelta_{s,t} \leq \upsilon_{s,t}, \quad s = 1, \ldots, S, t = 0, \ldots, T, \tag{4.15b}$$

$$\|\mathrm{f}_t^\intercal \updelta_{s,t} - 1\|_* \leq \theta_t, \quad s = 1, \ldots, S, t = 0, \ldots, T, \tag{4.15c}$$

where $\updelta_{s,t} \in \mathbb{R}^m$.

In constraints (4.4b) and (4.4b), $\inf_{q\in Q}\mathbb{E}_q(\mathrm{r}_{t,k}^\intercal \mathrm{x}_{t-1}) = -\sup_{q\in Q}\mathbb{E}_q(-\mathrm{r}_{t,k}^\intercal \mathrm{x}_{t-1})$. Then, by assuming that $\Xi_{\mathrm{r}_t} := \{\mathbb{M}_t \mathrm{r}_t \leq \mathrm{u}_t,\ t \in T\}$, where $\mathbb{M}_t \in \mathbb{R}^{m\times(n+1)}$ and $\mathrm{u}_t \in \mathbb{R}^m$, constraints (4.4b) and (4.4b) are reformulated as:

$$\inf_{\phi_t,\varphi_{k,t},\zeta_{k,t}\geq 0} \quad \phi_t\epsilon_t^2 + \frac{1}{K}\sum_{k\in K}\varphi_{k,t}, \tag{4.16a}$$

$$\text{s.t.} \quad -\hat{\mathrm{r}}_{k,t}^\intercal \mathrm{x}_{t-1} + (\mathrm{u}_t - \mathbb{M}_t\hat{\mathrm{r}}_{k,t})^\intercal \zeta_{k,t} \leq \varphi_{k,t}, \quad k = 1,\ldots,K, t = 0,\ldots,T, \tag{4.16b}$$

$$\|\mathbb{M}_t^\intercal \zeta_{k,t} - \mathrm{x}_{t-1}\|_* \leq \phi_t, \quad k = 1,\ldots,K, t = 0,\ldots,T, \tag{4.16c}$$

where $\zeta_{k,t} \in \mathbb{R}^m$. By substituting (4.14), (4.15), and (4.16) in model (4.4), we have the DRO counterpart of the ALM as follows:

$$\inf_{\lambda,\nu_s,\gamma_s,\phi_t,\varphi_{k,t},\zeta_{k,t},\theta_t,\upsilon_{s,t},\delta_{s,t}\geq 0} \lambda\epsilon^1 + \frac{1}{S}\sum_{s\in S}\nu_s, \tag{4.17a}$$

$$\text{s.t.} \quad \mathrm{e}^\intercal \mathrm{x}_t = -\phi_t\epsilon_t^2 - \frac{1}{K}\sum_{k\in K}\varphi_{k,t} + w_t y_{t,s} - l_t, \quad t = 0,...,T-1, \tag{4.17b}$$

$$-\phi_t\epsilon_t^2 - \frac{1}{K}\sum_{k\in K}\varphi_{k,t} \geq \theta_t\epsilon_t^1 + \frac{1}{S}\sum_{s\in S}\upsilon_{s,t},\ t = 0,\ldots,T, \tag{4.17c}$$

$$\hat{\mathrm{W}}_s^\intercal \mathrm{y} + (\mathrm{d} - \mathbb{C}\hat{\mathrm{W}}_s)^\intercal \gamma_s \leq \nu_s,\ s = 1,\ldots,S, \tag{4.17d}$$

$$\|\mathbb{C}^\intercal \gamma_s - \mathrm{y}\|_* \leq \lambda,\ s = 1,\ldots,S, \tag{4.17e}$$

$$-\hat{\mathrm{r}}_{k,t}^\intercal \mathrm{x}_{t-1} + (\mathrm{u}_t - \mathbb{M}_t\hat{\mathrm{r}}_{k,t})^\intercal \zeta_{k,t} \leq \varphi_{k,t},\ k = 1,\ldots,K, t = 0,\ldots,T, \tag{4.17f}$$

$$\|\mathbb{M}_t^\intercal \zeta_{k,t} - \mathrm{x}_{t-1}\|_* \leq \phi_t,\ k = 1,\ldots,K, t = 0,\ldots,T, \tag{4.17g}$$

$$\hat{L}_{s,t} + (\mathrm{b}_t - \mathrm{f}_t\hat{L}_{s,t})^\intercal \delta_{s,t} \leq \upsilon_{s,t},\ s = 1,\ldots,S, t = 0,\ldots,T, \tag{4.17h}$$

$$\|\mathrm{f}_t^\intercal \delta_{s,t} - 1\|_* \leq \theta_t,\ s = 1,\ldots,S, t = 0,\ldots,T, \tag{4.17i}$$

$$\mathrm{x}_t \in \mathcal{X},\ \mathrm{y} \in \mathcal{Y}. \tag{4.17j}$$

The tractability of problem (4.17) depends on the dual norm $\|.\|_*$. In this chapter, we are using norm 2, $\|.\|_2$. Moreover, we use a box support set, where $\mathbb{M} = -\mathbb{I}$ and $\mathbb{C} = \mathbb{I}$, and $\mathcal{I}$ is the identity matrix that leads the box support sets for the uncertain parameters.

4.4 Numerical Results

For our study, we utilize data from the Canada Pension Plan (CPP) to conduct a series of numerical experiments. As a mandatory requirement for all employed Canadians aged 18-70, the CPP receives contributions from a vast majority of the working population. According to CPP's official website [5], approximately 5.8 million individuals currently receive retirement benefits from CPP, with an average payout of $811.21 in January 2023 [6]. Additionally, CPP's investments report [7] indicates that around 14,371,853 individuals are contributing to CPP.

CPP invests in a diverse portfolio of five asset classes, as per information from investing.com [8]. These asset classes include fixed income, private equity, public equity, infrastructure, and real estate, which are geographically diversified in North America, Europe, and Asia. For our analysis, we have used data from ten major indexes from 2012 to 2022. The S&P 500 index represents public equities, while the Private Equity Index (PRIVEXD) represents private equities. In addition, we use the SP/TSX Capped Real Estate Index (GSPRTRE) for the real estate sector, Treasury Yield 10 Years (TNX) for fixed-income assets, and S&P Global Infrastructure TR (SPGTINTR) for infrastructure investment. The S&P/TSX Composite is used as the index for the Canadian market, while the FTSEurofirst 300 represents public equities in Europe. For the private equity index in Europe, we use the STOXX Europe 20. The Shanghai Stock Exchange (SSE) and Nikkei-225 indexes have been utilized as representatives of investment in Asia. As of 2022, the total value of assets under CPP management is estimated to be $539 billion. Based on the most recent report from CPP [9], the projected earnings of contributors for 2022 have been calculated to be $585,498 million, of which $57,964 million (approximately 9.9%) represents the contribution to CPP.

The inputs of our proposed models are the value of liabilities in each decision moment, contributions of employees at each decision moment, scenarios of asset returns,

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[5] https://open.canada.ca/data/en/dataset/1fab2afd-4f3c-4922-a07e-58d7bed9dcfc

[6] https://www.canada.ca/en/services/benefits/publicpensions/cpp/cpp-benefit/amount.html

[7] https://www.cppinvestments.com/the-fund/our-performance/financial-results/f2022-annual-results

[8] https://ca.investing.com/

[9] https://www.osfi-bsif.gc.ca/Eng/oca-bac/ar-ra/cpp-rpc/Pages/cpp30.aspx

benefit paid by pension, and scenarios of interest rates. To generate scenarios of asset returns, we use Monte Carlo simulation based on the geometric Brownian motion (GBM), which is a common approach to generate random data in financial problems (McLeish, 2011). The formula for GBM is as follows:

$$\frac{\Delta Pr}{Pr} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}, \tag{4.18}$$

where Pr is the asset price, ΔPr is the change in asset price, μ represents the expected return, σ denotes the standard deviation of returns, ϵ is a normally-distributed random variable, and Δt is the elapsed time period. Our analysis is based on the monthly returns of the ten mentioned indexes spanning from November 2012 to November 2022. During this period, we identified four distinct market regimes based on the long-term mean and standard deviation of the last 30 years. The first period, from November 2012 to February 2018, was characterized by steady growth with low volatility. The second period, from March 2018 to January 2020, experienced higher volatility than the previous period but still maintained a positive trend. The third period, from January 2020 to December 2021, was marked by high volatility and significant fluctuations. Finally, the post-pandemic period from January 2022 to November 2022 saw a return to high volatility, albeit with a different market trajectory. Using these historical data, we constructed different scenarios for our simulation. Our analysis, based on the k-mean clustering method of Horvath et al. (2021), reveals that during the observed period, 51% of the time the market exhibited steady growth with low volatility (LV), 22% of the time it had medium volatility (MV) but still showed growth, 17% of the time it was characterized by increasing high volatility (IHV) with positive returns and 10% of the time it was a decreasing high volatility (DHV) decreasing market with negative returns. We generated 1000 scenarios for each asset in each period, corresponding to the 4 market regimes. The average return of these 1000 scenarios for each asset in each period, based on each market condition, is considered as the asset's return in that period for the 4 regimes.

The sets $\mathcal{X}$ and $\mathcal{Y}$ in model (4.4) are defined using regulatory constraints based on the last decade's real investment structure in CPP. These constraints ensure that the contribution rate in each period falls within a range of 5% to 10%, and the investment in the US market cannot exceed 60% of the total fund, while the allocation to Canada must be at least 20%. Additionally, we mandate that a minimum of 10% of the fund be invested in fixed-income assets. The allocation to Asia must not exceed 15% of the fund, and the funding ratio must be at least 1.05.

To evaluate the efficacy of our proposed DRO formulation, we conduct in-sample and out-of-sample performance analyses for four models. The first model is the mixture distribution ALM (`MD`) (4.5) that incorporates four market conditions: LV, MV, IHV, and DHV. To determine the discrete probabilities associated with each market condition, we leverage the trends observed in historical data spanning the last 30 years. For the period from 2012 to 2022, the distribution functions of the market conditions are as follows: LV with a probability of 0.51, MV with a probability of 0.22, IHV with a probability of 0.17, and DHV with a probability of 0.1. Similarly, for the period from 2002 to 2012, the distribution functions are: LV with a probability of 0.35, MV with a probability of 0.39, IHV with a probability of 0.15, and DHV with a probability of 0.11. Lastly, for the period from 1992 to 2002, the distribution functions are: LV with a probability of 0.41, MV with a probability of 0.19, IHV with a probability of 0.20, and DHV with a probability of 0.06. Additionally, we consider a case where equal probabilities are assigned to each market condition, resulting in a probability of 0.25 for each LV, MV, IHV, and DHV. The second model is the box discrete distribution ALM (`BD`) (4.13), where half of the range of possible probabilities in each market condition is considered as volatility range in the box ambiguity set. The third mode is the Wasserstein metric ALM (`WM`) (4.17), where the radius of the Wasserstein ball is half of the range of possible probabilities in each market condition times to mean of asset return. The return on the market in the last 3 decades has been almost 10%. Then we consider 10% volatility for parameters [10]. Finally, the stochastic programming of ALM (`SP`) is the last model for comparison to the proposed models.

4.4.1 In-Sample Performance Analysis

In-sample performance analysis involves assessing the model's performance using the same data on which it was trained. For this analysis, we use historical data from CPP. Table 4.2 presents the in-sample performance of four ALM models: `MD`, `BD`, `WM`,

[10] https://www.officialdata.org/us/stocks/s-p-500/2002

and SP. The performance is measured by the funding ratio and fund return for each model over 12 months, and zero is the current decision moment. The funding ratio is a measure of solvency, which indicates the extent to which the value of the assets exceeds the present value of the liabilities. A value greater than one is desirable, indicating more assets than liabilities in the fund. Conversely, a value less than one is undesirable, indicating more liabilities than assets. The fund return, on the other hand, measures the rate of return earned on the fund's assets. Table 4.2 is organized as follows. The first column lists the periods for which the models' performance is reported. The second column reports the funding ratio and fund return for the MD model. Similarly, the third and fourth columns report the funding ratio and fund return for the BD model, respectively. The fifth and sixth columns report the funding ratio and fund return for the WM model, respectively. The last two columns report the funding ratio and fund return for the SP model, respectively. Overall, the table indicates that all four models have generally good performance in terms of both funding ratio and fund return. However, there are some differences in performance across models and periods.

In terms of the funding ratio, the SP model has the highest funding ratio in most of the periods, followed by the MD model, WM model, and BD model. The SP model has a funding ratio of between 1.094 to 1.236, indicating that it has more assets than liabilities. The MD model shows the second-best performance in terms of funding ratio, with a range between 1.093 to 1.190. The WM model also has a funding ratio of between 1.1 and 1.135, indicating that it is also solvent. Table 4.2 indicates that the BD model has the most conservative approach, with a funding ratio range between 1.093 to 1.107. In terms of the fund return, the SP model has the highest fund return in most periods, followed by the MD model, WM model, and BD model. The SP model has a fund return range between 0.003 to 0.028, indicating that it is earning a relatively high rate of return on its assets. The MD model has a fund return between 0.001 to 0.019, indicating that it is also earning a positive rate of return on its assets. The WM and BD models exhibit fund returns within a range of -0.001 to 0.017 and -0.001 to 0.013, respectively, but they are slightly lower than those of the MD and SP models.

The average funding ratio and fund return provide insights into the average performance of each ALM model which are mentioned in Table 4.2. Among the models,

Table 4.2: In-sample performance of the ALM models

	MD		BD		WM		SP	
Decision moments	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return
0	1.093	0.002	1.093	-0.001	1.100	-0.001	1.094	0.003
1	1.101	0.001	1.093	0.000	1.102	0.001	1.102	0.005
2	1.110	0.012	1.093	0.001	1.102	0.002	1.104	0.015
3	1.126	0.007	1.095	0.002	1.102	0.007	1.155	0.005
4	1.138	0.016	1.097	0.003	1.112	0.008	1.157	0.025
5	1.141	0.013	1.099	0.004	1.116	0.008	1.164	0.016
6	1.141	0.015	1.099	0.004	1.122	0.009	1.170	0.018
7	1.142	0.013	1.100	0.005	1.122	0.010	1.183	0.018
8	1.158	0.016	1.103	0.006	1.126	0.011	1.194	0.021
9	1.167	0.013	1.103	0.007	1.131	0.012	1.195	0.028
10	1.187	0.014	1.106	0.013	1.132	0.012	1.219	0.015
11	1.190	0.019	1.107	0.013	1.135	0.017	1.236	0.022
Average	1.141	0.012	1.099	0.005	1.117	0.008	1.164	0.016
Std. Dev.	0.033	0.006	0.005	0.005	0.014	0.005	0.046	0.008

the mean funding ratio is highest for SP with a value of 1.164, indicating a higher ratio of assets to liabilities on average. This is followed by WM with a mean funding ratio of 1.117. MD and BD have lower mean funding ratios of 1.141 and 1.099, respectively. In terms of the mean fund return, SP also has the highest value of 0.016, indicating a higher average return compared to the other models. MD has the second highest mean fund return at 0.012, followed by WM at 0.008, and BD with the lowest mean fund return of 0.005.

The standard deviation values provide insights into the variability or dispersion of the performance metrics. SP has the highest standard deviation for both the funding ratio, 0.046, and fund return, 0.008, indicating a wider range of performance outcomes compared to the other models. MD and WM have moderate standard deviations, with MD having a standard deviation of 0.033 for the funding ratio and 0.006 for the fund return, and WM having a standard deviation of 0.014 for the funding ratio and 0.005 for the fund return. BD has the lowest standard deviation values among the models, with a standard deviation of 0.005 for both the funding ratio and fund return, suggesting relatively stable performance across different periods.

We can conclude that the SP model is the most optimistic and the BD model is the most pessimistic. The SP model has the highest funding ratio and fund return in most of the periods, indicating that it is performing better than the other models. On the other hand, the BD model has the lowest funding ratio range among the models, indicating that it is the least solvent. Additionally, the BD model has the lowest fund return range among the models, suggesting that it is earning the lowest rate of return on its assets. The MD and WM models have intermediate levels of performance between the SP and BD models, indicating that they are neither as optimistic as the SP model nor as pessimistic as the BD model.

Asset allocation is a crucial component in addressing the ALM problem. It requires determining how to distribute investments among different asset classes to achieve the desired return while minimizing risk. To compare the optimal asset allocation of four models, Figure 4.1 has been prepared.

As shown in Figure 4.1, the WM and BD models offer more diversified portfolios than the SP and BD models, resulting in less risky portfolios. The WM and BD models take into account the wider range of possible probability distributions in their ambiguity set. Therefore, these models provide a more robust and stable asset allocation over time. On the other hand, the MD model just considers a set of scenarios for their ambiguous distribution, and SP assumes the discrete distribution function of uncertain parameters, asset return, and the present value of future liabilities, are known.

The funding ratio (FR) threshold is another important factor that affects the optimal contribution rate in asset liability management (ALM). Table 4.3 and Figure 4.2 present a comparison of the optimal contribution rates of four models (MD, BD, WM, and SP) under different FR values. Figure 4.2 indicates that the optimal contribution rates for all models increase as the FR threshold increases. This is expected because a higher FR threshold implies a higher level of required funding, which in turn requires higher contribution rates to meet the threshold. Moreover, the table indicates that the optimal contribution rates for each model are different for different FR values. For instance, the SP model has the lowest optimal contribution rates among the four models for all FR values, while the BD model has the highest optimal contribution rates for FR=1.15. Figure 4.2 shows that the WM and BD models have relatively higher optimal contribution rates than the MD and SP models for all FR values. This suggests that the former models may be less conservative in managing risk and require higher contributions to ensure funding adequacy.

Table 4.3: Optimal contribution rates of four models based on funding ratio

Models	FR=1.02	FR=1.05	FR=1.07	FR=1.1	FR=1.15
MD	1.7%	5.4%	6.6%	8.7%	10.2%
BD	3.1%	6.4%	9.3%	11.8%	14.8%
WM	2.1%	6.1%	8.9%	9.5%	10.6%
SP	1.1%	3.4%	5.2%	8.1%	10.1%

4.4.2 Out-of-Sample Performance Analysis

In addition to the in-sample analysis, we evaluated the out-of-sample performance of the aforementioned models using a simulation to generate the testing data. Out-of-sample analysis refers to a method of evaluating the performance and robustness of a statistical or predictive model using data that is separate from the data used to develop or train the model. Specifically, we generated 1000 scenarios of asset returns based on their distribution functions reported in (Ghahtarani et al., 2023a). We then employed the optimal investment strategies of the MD, BD, WM, and SP models to compare the funding ratio and asset value in each period. The results of this analysis are presented in Table 4.4.

The out-of-sample analysis reveals that the DRO formulations employed in the ALM, WM, and BD models exhibit superior performance compared to the MD and SP models. Looking at the funding ratio for MD, the ratio starts at 1.0769 at the moment zero, the current decision moment, and decreases to 0.820 in period nine. However, the funding ratio then increases in decision moments 10 and 11, ending at 1.030. The fund return for MD starts at 0.009 at the current decision moment, decreases to -0.002 at moment four, and then increases to 0.014 in period five. The fund return fluctuates between positive and negative values for the remaining periods. For BD, the funding ratio starts at 0.9570 and gradually increases to 1.067 in period 11. The fund return for BD starts at 0.006 and steadily increases to 0.024 in period 11. WM starts with a funding ratio of 0.963, which increases to 1.088 in period 10 and then decreases slightly to 1.114 in period 11. The fund return for WM increases from 0.013 to 0.025 in period 10 and then remains constant at 0.025 in period 11. Finally, SP starts with a funding ratio of 0.822, which increases to 0.992 in period 11. The fund return for SP starts at 0.002, decreases to -0.007 in period three, and then increases to 0.004 in period 10.

The average funding ratio for MD is 0.976, indicating an average funding ratio below 1. This suggests that, on average, the assets are lower than the liabilities for MD. BD has a mean funding ratio of 1.006, indicating a slightly higher average ratio where assets are closer to liabilities. WM has the highest mean funding ratio among the models at 1.040, suggesting a relatively higher average ratio of assets to liabilities. SP has the lowest mean funding ratio of 0.860, indicating a lower average ratio where liabilities are higher than assets. In terms of the mean fund return, MD has a mean value of 0.004, suggesting a slightly positive average return. BD has a higher mean fund return of 0.013, indicating a relatively higher average return compared to MD. WM has a mean fund return of 0.018, suggesting a slightly higher average return among the models. SP has the lowest mean fund return value of 0.001, indicating a very low average return.

The standard deviation values provide insights into the variability or dispersion of the performance metrics. For the funding ratio, MD has a standard deviation of 0.083, indicating a relatively higher variability compared to the other models. BD has the lowest standard deviation of 0.039, suggesting a lower variability in funding ratio performance. WM and SP have standard deviations of 0.048 and 0.104, respectively, indicating moderate to high variability. Regarding the fund return, MD has a standard deviation of 0.009, suggesting a relatively higher variability in returns. BD has the lowest standard deviation of 0.005, indicating a lower variability in fund return performance. WM has a standard deviation of 0.005, similar to BD, suggesting relatively stable fund return outcomes. SP has a standard deviation of 0.003, indicating a lower variability compared to the other models.

Based on the results presented in Table 4.4, it is evident that the WM model outperforms the other models in terms of both fund return and funding ratio. With the highest mean values across the investment horizon, WM demonstrates superior performance from both perspectives. In particular, the WM model exhibits a higher mean funding ratio compared to MD and SP, indicating a more favorable financial position. This implies that WM manages to maintain a healthier balance between assets and liabilities, resulting in a greater ability to meet financial obligations. The higher funding ratio suggests a more robust and secure asset-liability management strategy. Additionally, WM achieves the highest mean fund return among all the models, surpassing

Table 4.4: Out-sample performance of the ALM models

	MD		BD		WM		SP	
Decision moments	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return	Funding ratio	Fund return
0	1.077	0.009	0.957	0.006	0.963	0.013	0.822	0.002
1	1.043	0.001	0.959	0.010	0.972	0.014	0.981	0.001
2	1.016	0.001	0.960	0.010	0.983	0.014	0.983	0.004
3	1.022	0.007	0.965	0.011	1.036	0.015	0.774	-0.007
4	0.816	-0.002	1.022	0.011	1.044	0.021	0.730	-0.004
5	0.931	0.014	1.014	0.010	1.058	0.023	0.923	0.001
6	0.970	0.001	1.028	0.018	1.048	0.009	0.752	-0.003
7	0.976	0.004	1.016	0.013	1.049	0.018	0.770	0.001
8	0.978	0.013	1.012	0.015	1.059	0.014	0.781	0.004
9	0.820	-0.015	1.045	0.019	1.077	0.025	0.859	0.002
10	1.035	0.017	1.038	0.013	1.088	0.025	0.982	0.005
11	1.030	-0.002	1.067	0.024	1.114	0.025	0.992	0.003
Average	0.976	0.004	1.006	0.013	1.040	0.018	0.860	0.0007
Std. Dev.	0.083	0.009	0.039	0.005	0.048	0.005	0.104	0.003

BD, MD, and SP. This indicates that WM generates more favorable investment returns on average throughout the investment horizon. A higher mean fund return suggests better investment performance, potentially leading to higher profits and returns for the ALM strategy. Therefore, based on the mean funding ratio and fund return values, it can be concluded that WM exhibits better performance than the other models in terms of stability and asset management. Its higher funding ratio indicates a stronger financial position, while the superior mean fund return reflects better investment outcomes. These findings highlight the effectiveness of the WM model in achieving both financial stability and favorable investment returns, making it a preferable choice among the options considered.

4.5 Conclusions

Pension funds play a vital role in ensuring retirement income security for workers globally. However, they face challenges such as uncertainty of asset return and liability values. To address these issues, an effective asset-liability management (ALM) strategy must be implemented, balancing the competing objectives of generating returns and meeting future obligations. In this chapter, we addressed the uncertainty of parameters in the ALM problem by exploring three different approaches: using mixture ambiguity sets with discrete scenarios and box uncertain discrete distribution functions. However, both of these approaches have limitations, and to overcome them, we incorporated the Wasserstein metric into the ALM problem. By incorporating the Wasserstein metric, we provide a more comprehensive and reliable approach to dealing with the limitations of ambiguity sets while maintaining the desirable properties of finite sample guarantee, asymptotic consistency, and tractability.

This study has used data from the CPP to conduct a series of numerical experiments and tests to simulate different market scenarios and their impact on the plan. Monte Carlo simulation based on geometric Brownian motion was used to generate scenarios of asset returns. The analysis revealed four distinct market regimes during the observed period from November 2012 to November 2022.

The in-sample performance analysis of four ALM models (`MD`, `BD`, `WM`, and `SP`) reveals valuable insights into their performance in terms of funding ratio and fund return. The results indicate that all four models generally demonstrate good performance, but there are variations across models and periods. The `SP` model consistently exhibits the highest funding ratio and fund return, indicating its optimistic performance. It has a higher ratio of assets to liabilities on average and earns a relatively high rate of return on its assets. On the other hand, the `BD` model displays the most conservative approach, with the lowest funding ratio range and lowest fund return range among the models.

The out-of-sample analysis of the ALM models reveals that the `WM` model demonstrates superior performance compared to the `MD`, `BD`, and `SP` models in terms of both funding ratio and fund return. The superior performance of the `WM` model in terms of funding ratio and fund return can be attributed to its robust optimization approach and risk management capabilities. By incorporating worst-case distribution functions based on the scenarios of asset returns, the `WM` model is designed to handle extreme market conditions and mitigate potential risks. The higher mean funding ratio of the `WM` model indicates a more conservative approach to asset-liability management, ensuring that the liabilities are well-covered by the available assets. This conservative stance provides a buffer against unexpected market fluctuations and reduces the likelihood of financial instability.

Furthermore, the higher mean fund return of the `WM` model suggests that it is able to capture profitable investment opportunities more effectively than the other models. This can be attributed to the optimization framework of the `WM` model, which aims to maximize investment returns while considering the constraints imposed by liabilities and risk tolerance. The stability of the `WM` model's funding ratio and fund return is also evident from its lower standard deviation values compared to the other models. A lower standard deviation implies less variability and a more consistent performance over time. This stability is crucial for long-term financial planning and managing the risks associated with asset-liability mismatches.

Prospective research avenues could further enrich the comprehension and utilization of ALM strategies for pension funds, encompassing several key directions.

Initially, an intriguing path involves expanding the analysis to encompass risk measures within the ALM problem. By assimilating such preferences into the ALM framework, a more intricate assessment of the interplay between risk and return can emerge. This approach could unveil how diverse investors, driven by varying risk appetites, might adopt distinct ALM strategies.

Subsequently, while the present study harnessed the Wasserstein metric to bolster the resilience of ALM models, alternative distance metrics like the Kullback-Leibler divergence or the Total Variation distance remain unexplored. Pioneering investigations could delve into the utilization of diverse distance metrics, evaluating their efficacy within the ALM context to discern any performance disparities.

Lastly, though the current study centered on the CPP, an intriguing trajectory involves extending the inquiry to encompass other pension funds across diverse regions. This endeavor would illuminate the adaptability and efficacy of ALM models within distinct contexts. Scrutinizing the performance of ALM strategies across assorted pension funds could elucidate how varying market dynamics and regulatory landscapes influence the effectiveness of these strategies.

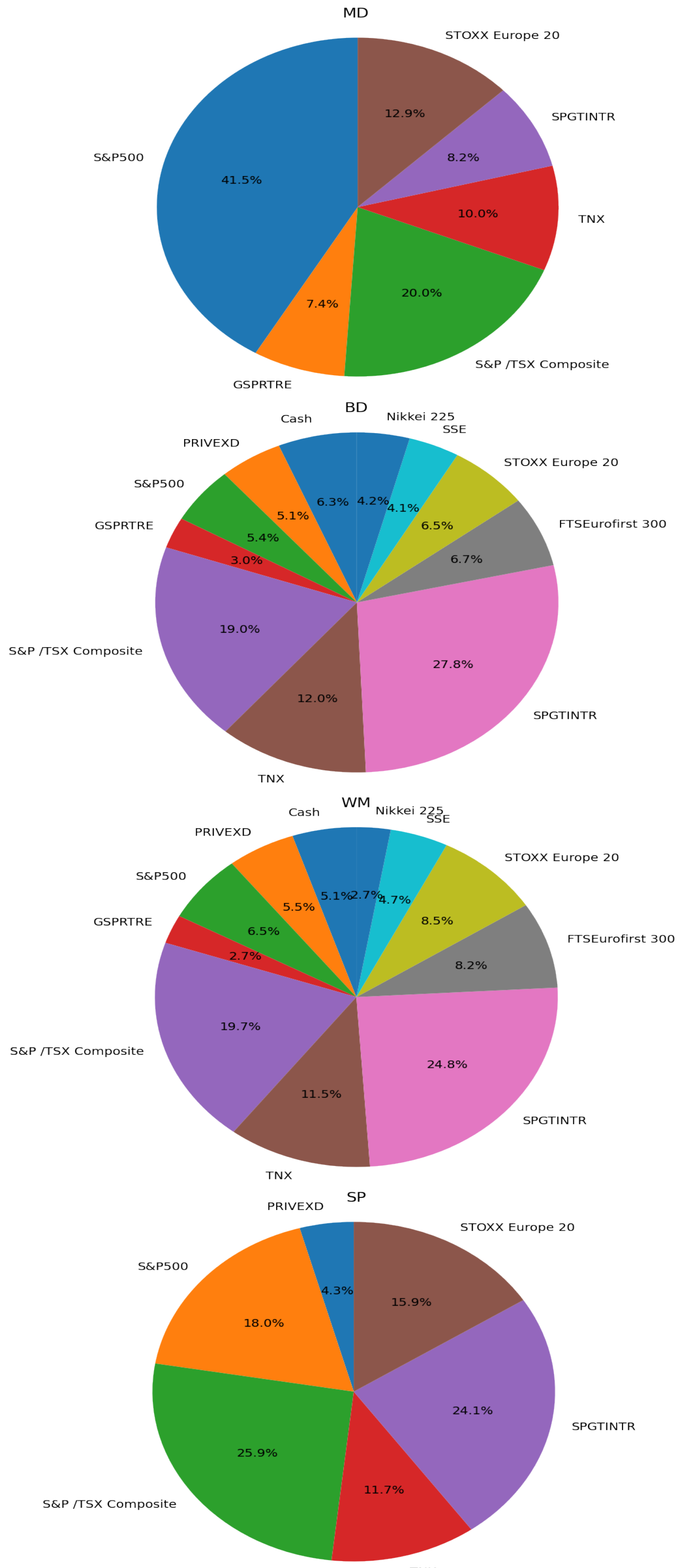

Figure 4.1: Comparision of optimal asset allocation

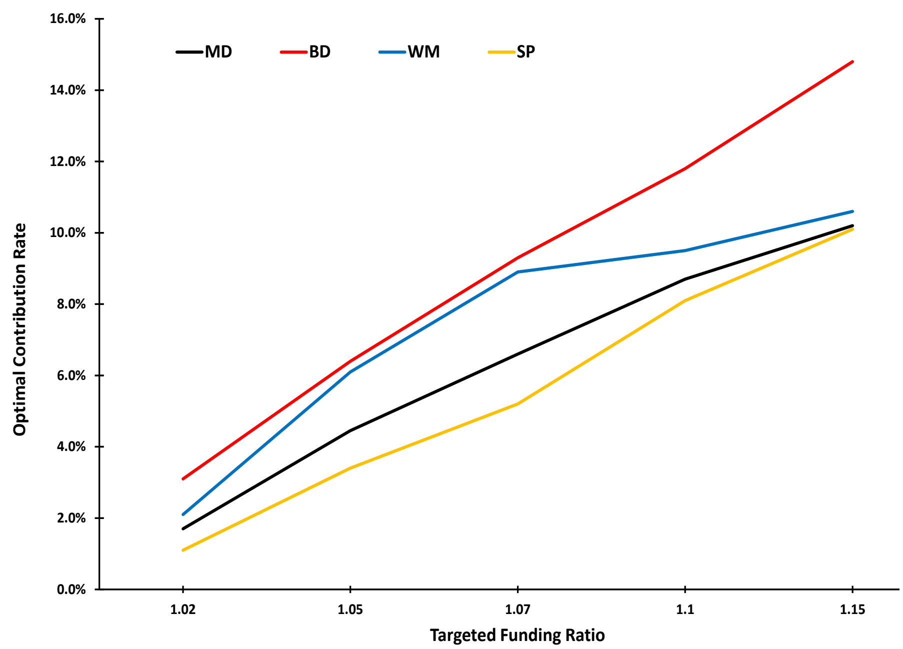

Figure 4.2: Optimal contribution rate for targeted funding ratios

Chapter 5

A Double-Oracle, Logic-Based Benders Decomposition Approach to Solve the K-adaptability Problem

5.1 Introduction

Robust Optimization (RO) has become a classical framework for dealing with parameter uncertainty in optimization problems (Bertsimas et al., 2011). In RO, parameter uncertainty is captured through an *uncertainty set* of proper structure and size, and the optimization is conducted with respect to the *worst-case* realization in it. An important class of RO problems that has gained considerable attention recently is adaptive/adjustable robust optimization (ARO), in which some decisions are assumed to be delayable until the realized values of uncertain parameters become partially or fully known. Whereas ARO formulations often lead to better (less pessimistic) solutions than their corresponding static RO models, they are computationally intractable (Ben-Tal et al., 2004). However, several exact and approximate algorithms have been proposed to solve important ARO classes such as linear two-stage RO problems (Thiele et al., 2009; Chen & Zhang, 2009; Kuhn et al., 2011; Zhao & Zeng, 2012; Bertsimas et al., 2012; Jiang et al., 2012; Iancu et al., 2013). With a few exceptions, these algorithms use duality to handle the second-stage (recourse) problem. Hence, they can be used only with continuous recourse decisions. Although some attempts have been made to develop efficient solution algorithms for ARO problems with discrete recourse (see, *e.g.,* (Dhamdhere et al., 2005; Georghiou et al., 2015; Bertsimas & Georghiou, 2015, 2018)), the literature for this class of problems is still sparse.

Recently, an alternative modeling approach referred to as K*-adaptability*, has been proposed as a conservative approximation for ARO problems with discrete recourse (Hanasusanto et al., 2015; Subramanyam et al., 2020). Rather than allowing any feasible integer recourse to be selected, the decision-maker *prepares* K solutions in advance (under uncertainty). Then, upon full knowledge of the realized value of the uncertain parameters, the best among these K solutions is selected. Apart from being better in general compared to the solutions of static RO, Buchheim & Kurtz (2017) argued that K-adaptability solutions are more easily accepted by a human user as they do not change each time but are taken from a relatively small set of candidate solutions.

Similar to Hanasusanto et al. (2015) and Subramanyam et al. (2020), we initially focus on a linear version of the problem, in which both the objective function and constraints in the first- and second-stages are affine functions of the decision variables, the uncertain parameters affect the second-stage objective function, and the uncertainty set is polyhedral. However, we later show how the algorithm that we propose can be adapted to solve other variants of the problem. Formally, the linear K-adaptability problem under consideration is formulated as follows:

$$\min_{\mathrm{x}\in\mathcal{X},\{\mathrm{y}^k\}_{k\in[K]}} \quad \mathrm{c}^\intercal\mathrm{x} + \max_{\xi\in\Xi}\min_{k\in[K]}\left\{\xi^\intercal\mathrm{Q}\mathrm{y}^k:\ \mathrm{T}\mathrm{x}+\mathrm{W}\mathrm{y}^k\leq\mathrm{b}\right\} \tag{5.1a}$$

$$\text{s.t.} \quad \mathrm{y}^k\in\mathcal{Y},\ k\in[K], \tag{5.1b}$$

where $\mathrm{c}\in\mathbb{R}^n$, $\mathrm{Q}\in\mathbb{R}^{q\times m}$, $\mathrm{T}\in\mathbb{R}^{s\times n}$, $\mathrm{W}\in\mathbb{R}^{s\times m}$, $\mathrm{b}\in\mathbb{R}^s$, and $[K]:=\{1,\ldots,K\}$. In this formulation, ξ denotes the uncertain parameters contained in the polyhedral set $\Xi\subset\mathbb{R}^q$, whereas $\mathrm{x}\in\mathcal{X}\subset\{0,1\}^n$ and $\mathrm{y}^k\in\mathcal{Y}\subset\mathbb{N}^m$ represent the *here-and-now* (first-stage) and the *wait-and-see* (second-stage) decision variables, respectively.

A special case of the K-adaptability problem, known in the literature as the *min-max-min robust combinatorial optimization* (MMMRCO) problem, arises when the only decision that is made under uncertainty is the pre-selection of the K possible recourse actions. In other words, the problem does not have "real" first-stage decisions. This problem has many practical applications such as parcel delivery and finding routes in hazardous situations (Arslan et al., 2022). The basic MMMRCO problem (without constraint uncertainty) is formulated as follows:

$$\min_{\mathrm{y}^k\in\mathcal{Y}}\max_{\xi\in\Xi}\min_{k\in[K]}\left\{\xi^\intercal\mathrm{Q}\mathrm{y}^k:\ \mathrm{W}\mathrm{y}^k\leq\mathrm{b}\right\}. \tag{5.2}$$

So far, two solution approaches have been developed for the K-adaptability problem. Hanasusanto et al. (2015) proposed a mixed-integer linear programming approximation derived using linear programming (LP) duality, which leads to a monolithic formulation involving bilinear terms. A McCormick envelope is used to linearize the bilinear terms, which requires a large number of new variables to be introduced. Moreover, the number of binary variables increases significantly as K becomes larger. In another attempt to solve the K-adaptability problem, Subramanyam et al. (2020) proposed a branch-and-bound algorithm that enjoys asymptotic convergence in general, but has finite convergence under specific conditions. This algorithm works by generating a relevant subset of uncertainty realizations and enumerating their assignment to the pre-selected K solutions. Nevertheless, both approaches were found to be ineffective for solving large instances of the shortest path problem, *e.g.,* with more than 25 nodes.

Besides the aforementioned methods that can handle the general case (*i.e.,* with first-stage decisions), a few approaches to solve variants of Problem (5.2) have been proposed. Chassein et al. (2019) developed a branch-and-bound algorithm that can solve large instances of the MMMRCO problem, yet only with budget uncertainty sets. They also proposed a heuristic solution algorithm based on the formulation of Hanasusanto et al. (2015). However, instead of using the McCormick linearization approach to handle the bilinear terms, they used the block-coordinate descent algorithm, which has no optimality guarantee. Moreover, their algorithms can hardly solve any instance of the shortest path problem with $K > 3$. For the same special case, Goerigk et al. (2020) developed an integer programming formulation and an exact row-and-column-generation algorithm that is suitable only for small instances, in addition to two heuristics that can handle larger ones. Recently, Arslan et al. (2022) proposed a solution approach that iteratively generates scenarios of the uncertain parameters and assigns them to solutions by solving a p-center problem. However, it works well only if there is an effective way to restrict and enumerate the search space.

In this chapter, we present a new approach to solve the K-adaptability problem with binary or integer first-stage decisions. The scenario generation step in the proposed approach enjoys finite convergence when the uncertainty set is polyhedral, but the approach can be used with any convex uncertainty set. Although we focus initially on problems with affine functions, the proposed algorithm can be extended to the case of nonlinear recourse problems. The proposed approach uses a *logic-based* (also referred to as *combinatorial*) Benders algorithm to handle the first-stage decisions such that the remaining subproblem is an MMMRCO that is solved iteratively to generate optimality cuts. To solve this subproblem, we propose a double-oracle algorithm that iterates between solving an adversary problem to iteratively generate worst-case scenarios for a K-subset of feasible solutions and determine the scenario-solution assignment and solving a decision-maker problem to find the optimal K-subset of solutions for all the scenarios generated so far. Although the way scenarios are generated and assigned to solutions is similar to that proposed by Arslan et al. (2022), our approach uses a more efficient way (*i.e.,* by solving an optimization problem) to identify the optimal K-subset of recourse solutions in every iteration. We note that the K-adaptability problem formulation provided in (5.1) is based on that introduced by Bertsimas & Caramanis (2010), *i.e.,* with a first-stage problem that is not subject to uncertainty, whereas Hanasusanto et al. (2015) addressed an extended version in which both stages are affected by the same uncertain parameters. Hence, we show how the proposed algorithm can be extended to handle K-adaptability problems with uncertainty in both stages (whether the two stages depend on the same or different uncertain parameters). Finally, extensive numerical experiments are conducted on benchmark instances of several classical optimization problems, and the computational superiority of the proposed approach *vis-á-vis* state-of-the-art solution methods is demonstrated.

The remainder of this chapter is organized as follows. Section 5.2 presents the approach proposed to solve problem (5.1) (*i.e.,* with linear objective function and constraints and with uncertainty affecting only the recourse objective function). Section 5.3 studies the convergence properties of the proposed algorithm and proves its finite convergence. In Section 5.4, we show how the proposed algorithm can be modified to solve different variants of the K-adaptability problem, namely, problems with integer first-stage decision variables, problems with nonlinear functions, and problems affected by uncertainty in both stages. The numerical experiments conducted to test the proposed algorithm on benchmark problems, and a detailed discussion of their results, are presented in Section 5.5. Finally, Section 5.6 provides some conclusions and suggests future research directions.

Notation. We use upright lower and upper case letters, respectively, for vectors and matrices. Individual elements of these vectors and matrices are denoted using *italic* versions of the same letters. For example, elements of the I-dimension vector x are denoted as x_i. Depending on the context, upper case letters might be used also to denote scalars (*e.g.,* I), whereas lower case letters might denote also functions (*e.g.,* $g(\cdot)$). The calligraphic font is used for sets (*e.g.,* $\mathcal{X}$). $[J]$ is used as a shorthand for the set of integers $\{1, 2, \ldots, J\}$. A partial set of a given set $[J]$ is denoted as $[J']$. The symbol e is used to denote an all-ones vector of appropriate size and e_j is the j-th column of the identity matrix $\mathbb{I}_j$.

5.2 The Proposed Solution Approach

In this section, we present the proposed approach to solve the K-adaptability problem (5.1) with binary first-stage decision variables and recourse objective uncertainty. First, we show how a logic-based Benders decomposition is applied to deal with the discrete first-stage variables. Then, we describe a double-oracle algorithm to solve the subproblem.

5.2.1 A Logic-based Benders Decomposition

We apply Benders decomposition by projecting the model onto the subspace defined by the first-stage variables x to get the master problem (**MP**):

$$\min_{\mathrm{x} \in \mathcal{X} \cap \mathcal{V}} \mathrm{c}^{\intercal}\mathrm{x} + \nu(\mathrm{x}), \tag{5.3}$$

where $\mathcal{V} := \{\mathrm{x} : \mathrm{W}\mathrm{y} \leq \mathrm{b} - \mathrm{T}\mathrm{x} \text{ for some } \mathrm{y} \in \mathcal{Y}\}$. For a given $\bar{\mathrm{x}} \in \mathcal{X} \cap \mathcal{V}$, $\nu(\bar{\mathrm{x}})$ is the optimal value of the sub-problem (**SP**):

$$\min_{\{\mathrm{y}^k\}_{k \in [K]}} \max_{\xi \in \Xi} \min_{k \in [K]} \quad \xi^{\intercal}\mathrm{Q}\mathrm{y}^k \tag{5.4a}$$

$$\text{s.t.} \qquad \mathrm{y}^k \in \mathcal{Y},\ \mathrm{W}\mathrm{y}^k \leq \mathrm{b} - \mathrm{T}\bar{\mathrm{x}} \qquad \forall k \in [K]. \tag{5.4b}$$

We note that since $\mathrm{x} \in \mathcal{X} \cap \mathcal{V}$, the problem enjoys *relatively complete recourse*, *i.e.,* **SP** has feasible solutions for all $\bar{\mathrm{x}} \in \mathcal{X} \cap \mathcal{V}$. Without this property, the **SP** might be infeasible for some $\bar{\mathrm{x}}$, thus requiring feasibility cuts to be generated.

The basic idea of the classical Benders algorithm is to approximate the function $\nu(\mathrm{x})$ using hyper-planes (referred to as *optimality cuts*) generated by solving the dual **SP** for fixed values of x. However, since **SP** has integer decision variables, it is not possible to use the duality theory to generate cuts. Assuming that we have an oracle to solve **SP**, in any iteration r, the r-th feasible solution x^r is used to define the sets $\mathcal{S}_r := \{i \in [n] : x_i^r = 1\}$ and to evaluate its corresponding worst-case second-stage objective function value θ_r. We use this solution and value to generate the valid combinatorial cut, first proposed by Laporte & Louveaux (1993),

$$\theta \geq (\theta_r - L_r)\left(\sum_{i \in \mathcal{S}_r} x_i - \sum_{i \notin \mathcal{S}_r} x_i\right) - (\theta_r - L_r)\left(|\mathcal{S}_r| - 1\right) + L_r, \tag{5.5}$$

where $|\mathcal{S}_r|$ is the cardinality of $\mathcal{S}_r$, L_r is a lower bound of $\nu(\mathrm{x})$ for all $\mathrm{x} \in \mathcal{X}$, and θ is a **MP** decision variable that defines the epigraph of $\nu(\mathrm{x})$. Hence, **MP** can be written as follows:

$$\min_{\mathrm{x} \in \mathcal{X}, \theta} \ \mathrm{c}^\intercal \mathrm{x} + \theta \tag{5.6a}$$

$$\text{s.t. } \theta \geq (\theta_r - L_r)\left(\sum_{i \in \mathcal{S}_r} x_i - \sum_{i \notin \mathcal{S}_r} x_i\right) - (\theta_r - L_r)\left(|\mathcal{S}_r| - 1\right) + L_r \quad \forall r \in \mathcal{R}, \tag{5.6b}$$

where $\mathcal{R} := \{1, \ldots, |\mathcal{X}|\}$ is a set of indexes that enumerate all the members of $\mathcal{X}$, i.e. $\{x^r\}_{r=1}^{|\mathcal{X}|} = \mathcal{X}$. Note that $|\mathcal{X}|$ is necessarily finite since the feasible set $\mathcal{X}$ is discrete and bounded. For completeness, we refer the reader to Appendix 5.7 for more details on the equivalence between (5.3) and (5.6).

The Logic-based Benders decomposition algorithm is summarized as follows:

Initiate with an arbitrary feasible $\bar{\mathrm{x}}$, set $UB = \infty$, $LB = -\infty$, $r = 1$

while $UB - LB \geq \epsilon$ **do**

Solve **SP** (5.4) with $\bar{\mathrm{x}}$ and find θ_r

Find a lower bound L_r for **SP** (5.4) as will be explained later

Generate the optimality cut:

$\theta \geq (\theta_r - L_r) \left(\sum_{i \in \mathcal{S}_r} x_i - \sum_{i \notin \mathcal{S}_r} x_i \right) - (\theta_r - L_r) (|\mathcal{S}_r| - 1|) + L_r$

Update the upper bound: $UB = \min \{UB, \mathrm{c}^\intercal \bar{\mathrm{x}} + \theta_r\}$

Solve **MP** (5.6) with the new optimality cut added and find the optimal partial solution x^*

Set LB equal to the optimal value of **MP** (5.6)

Use x^* as $\bar{\mathrm{x}}$ in the next iteration and set $r = r + 1$

end

Return: Declare the pair $\left(\mathrm{x}^*, \mathrm{y}^{k^*}\right)$ as the optimal solution

Algorithm 1: Logic-based Benders decomposition algorithm for solving K-adaptability problem

We note that **SP** (5.4) is an MMMCRO problem. In the next section, we propose a novel approach to solve it and obtain the lower bound L_r.

5.2.2 A Double-oracle Algorithm for Solving SP

Arslan et al. (2022) showed that **SP** (5.4) can be reformulated as a p-center problem over the entire sets $\mathcal{Y}$ and Ξ, denoted herein by $\mathbf{P}(\mathcal{Y}, \Xi)$. It should be noted that both $\mathcal{Y}$ and Ξ have exponential numbers of elements and vertices, respectively. Hence, rather than considering all their elements/vertices at the outset, the recourse solutions and scenarios are generated and added to the formulation iteratively. Let us define the partial sets $\mathcal{Y}' \subseteq \mathcal{Y}$, with $|\mathcal{Y}| \geq K$, and $\Xi' \subset \Xi$ and use $j \in [J]$ ($[J']$) and $h \in [H]$ ($[H']$) to index the elements of $\mathcal{Y}$ ($\mathcal{Y}'$) and the vertices of Ξ (Ξ'), also referred to as "scenarios", respectively. Hence, a *reduced* subproblem over the partial sets $\mathcal{Y}'$ and

Ξ' can be stated as the p-center problem $\mathbf{P}(\mathcal{Y}', \Xi')$:

$$\min_{\{z_j\}_{j\in[J']}, \{v_{jh}\}_{j\in[J'],h\in[H']}, w} \quad w \tag{5.7a}$$

$$\text{s.t.} \quad w \geq \sum_{j\in[J']} (\xi_h^\intercal \mathrm{Q} \mathrm{y}_j)\, v_{jh} \quad \forall h \in [H'] \tag{5.7b}$$

$$\sum_{j\in[J']} v_{jh} = 1 \quad \forall h \in [H'] \tag{5.7c}$$

$$\sum_{j\in[J']} z_j = K \tag{5.7d}$$

$$v_{jh} \leq z_j \quad \forall j \in [J'], \forall h \in [H'] \tag{5.7e}$$

$$v_{jh}, z_j \in \{0, 1\} \quad \forall j \in [J'], \forall h \in [H']. \tag{5.7f}$$

The binary variable z_j takes value 1 if the feasible solution y_j is selected to be among the K "prepared" solutions, and v_{jh} takes value 1 if scenario ξ_h is assigned to solution y_j, and 0 otherwise. Constraint (5.7b) finds the scenario-solution pair with the worst cost among all assignments. Constraint (5.7c) ensures that each scenario is assigned to exactly one solution, whereas (5.7d) and (5.7e), respectively, stipulate that K solutions are selected and that scenarios can be assigned to selected solutions only.

To solve **SP**, the following algorithm is proposed:

1. Solve the problem $\mathbf{P}(\mathcal{Y}', \Xi)$, *i.e.*, the problem with the subset $\mathcal{Y}'$ of all recourse solutions generated so far (carried forward from Step 2 in the previous iteration) and all scenarios in Ξ to obtain an upper bound UB. To solve this problem, we begin with a subset Ξ' of scenarios and perform the following steps.

   (a) Solve the problem $\mathbf{P}(\mathcal{Y}', \Xi')$ (*i.e.*, Problem (5.7)) to find z^*, v^* and w^*. Identify the optimal K-subset of recourses as $\left\{\mathrm{y}^k \in \mathcal{Y}' : z_k^* = 1\right\}$.

   (b) Given the current optimal K-subset $\left\{\mathrm{y}^k\right\}_{k\in[K]}$ of recourses, try to find a scenario $\xi_{|H'|+1} \in \Xi$ that violates (5.7b) by solving the following problem:

$$\max_{\xi\in\Xi,\eta} \quad \eta \tag{5.8a}$$

$$\text{s.t.} \quad \eta \leq \xi^\intercal \mathrm{Q} \mathrm{y}^k \quad k \in [K]. \tag{5.8b}$$

(c) If $\eta^* > w^*$, add the new scenario to Ξ' and repeat Steps (a) and (b). Otherwise, stop and move to Step 2.

2. In this step, we find the optimal K-subset $\left\{\mathrm{y}^{k^*}\right\}_{k\in[K]}$ of feasible recourses that minimize the worst-case loss for the discrete scenario set Ξ' by solving the following problem:

$$\begin{aligned}
\min_{\{\mathrm{y}^k\}_{k\in[K]},\gamma,\{u_{kh}\}_{k\in[K],h\in[H']}} \quad & \gamma && && (5.9a)\\
\text{s.t.} \quad & \xi_h^\intercal \mathrm{Q}\mathrm{y}^k \le \gamma + M(1-u_{kh}) && \forall k\in[K],\, \forall h\in[H'] && (5.9b)\\
& \sum_{k\in[K]} u_{kh} = 1 && \forall h\in[H'] && (5.9c)\\
& \mathrm{y}^k \in \mathcal{Y} && \forall k\in[K] && (5.9d)\\
& \mathrm{W}\mathrm{y}^k \le \mathrm{b} - \mathrm{T}\bar{\mathrm{x}} && \forall k\in[K] && (5.9e)\\
& u_{kh}\in\{0,1\} && \forall k\in[K],\, \forall h\in[H'], && (5.9f)
\end{aligned}$$

where u_{kh} takes the value 1 if scenario h is assigned to recourse k, in which case constraint (5.9b) reduces to $\xi_h^\intercal \mathrm{Q}\mathrm{y}^k \le \gamma$; otherwise it becomes redundant. We set $M > \left\{\max_{\mathrm{y}^k\in\mathcal{Y}} \xi_h^\intercal \mathrm{Q}\mathrm{y}^k,\ \text{s.t. } \mathrm{W}\mathrm{y}^k \le \mathrm{b} - \mathrm{T}\bar{\mathrm{x}}\right\}$. The solution pool is then updated as $\mathcal{Y}' \leftarrow \mathcal{Y}' \cup \left\{\mathrm{y}^{k^*}\right\}_{k\in[K]}$, where $\left\{\mathrm{y}^{k^*}\right\}_{k\in[K]}$ is the partial optimal solution of problem (5.9). Moreover, we set $LB = \gamma$.

3. Iterate between Steps (1) and (2) until $UB - LB < \varepsilon$. Declare the incumbent $\left\{\mathrm{y}^{k^*}\right\}_{k\in[K]}$ as the optimal solution.

Initialization: $\mathcal{Y}', \Xi', LB = -\infty, UB = +\infty$

Iteration:

while $UB - LB \geq \epsilon$ **do**

  while *Scenario-added*=***true*** **do**

    Compute w^*, z^*, v^*, and y^{k^*} by solving (5.7)

    Compute $\xi_{|H|+1} \in \Xi$, and η^* by solving (5.8)

    if $\eta^* > w^*$ **then**

      $\Xi' \leftarrow \Xi' \cup \left\{\xi_{|H|+1}\right\}_{k\in[K]}$

      Scenario-added=**true**

    else

      $\Xi' \leftarrow \Xi'$

      Scenario-added=**false**

    end

  end

  Return: Ξ', $UB = w^*$

  Compute $\{\mathrm{y}^{k^*}\}_{k\in[K]}$, γ^* by solving (5.9)

  $\mathcal{Y}' \leftarrow \mathcal{Y}' \cup \left\{\mathrm{y}^{k^*}\right\}_{k\in[K]}$

  $LB = \gamma^*$

end

Return: $\{y^{k^*}\}_{k\in[K]}$

Algorithm 2: The Double-Oracle (DO) algorithm for solving **SP**

To generate an optimality cut, the logic-based Benders decomposition algorithm explained in the previous section requires a valid lower bound on the optimal value of the second-stage problem that satisfies $L_r \leq \min_{\mathrm{x}}\{\nu(\mathrm{x})|\ \mathrm{x} \in \mathcal{X}\}$. In every iteration of the proposed algorithm, we calculate a lower bound on the optimal value of **SP** for a fixed first-stage decision $\bar{\mathrm{x}}$ by solving (5.9). However, a lower bound on the optimal value of **SP** for all $\mathrm{x} \in \mathcal{X}$ is required, which can be obtained by solving the following problem:

$$
\begin{aligned}
\min_{\{\mathrm{y}^k\}_{k\in[K]},\gamma,\{u_{kh}\}_{k\in[K],h\in[H']},\mathrm{x}} \quad & \gamma && && \text{(5.10a)}\\
\text{s.t.} \quad & \xi_h^\intercal \mathrm{Q}\mathrm{y}^k \le \gamma + M(1-u_{kh}) && \forall k\in[K],\ \forall h\in[H'] && \text{(5.10b)}\\
& \sum_{k\in[K]} u_{kh} = 1 && \forall h\in[H'] && \text{(5.10c)}\\
& \mathrm{y}^k \in \mathcal{Y} && \forall k\in[K] && \text{(5.10d)}\\
& \mathrm{W}\mathrm{y}^k \le \mathrm{b} - \mathrm{T}\mathrm{x} && \forall k\in[K] && \text{(5.10e)}\\
& \mathrm{x}\in\mathcal{X} && && \text{(5.10f)}\\
& u_{kh}\in\{0,1\} && \forall k\in[K],\ \forall h\in[H']. && \text{(5.10g)}
\end{aligned}
$$

Note that the lower bound changes at each Benders iteration since we solve problem (5.10) in each iteration by using an updated subset of scenarios Ξ'. We also suggest warm-starting **SP** in every Benders iteration by re-using some of the y variables generated in previous iterations. Given a subset $\{\mathrm{y}_j\}_{j\in[J']}$ of recourse solutions, one can "filter" them using constraint (5.9e) and reuse the ones that satisfy this constraint for the new $\bar{\mathrm{x}}$ in problem (5.7) right away. Likewise, the scenarios (vertices of Ξ) generated in an iteration can be re-used in subsequent iterations of the Benders algorithm since they do not depend on $\bar{\mathrm{x}}$. Such warm-staring techniques can substantially improve the performance of the DO algorithm. Figure 5.1 illustrates the proposed approach.

There are similarities and differences between our approach and both the scenario generation (SG) algorithm developed by Arslan et al. (2022) and the row-and-column-generation (RCG) algorithm proposed by Goerigk et al. (2020). Neither SG nor RCG can handle the K-adaptability problem as they are developed for the MMMRCO problem only. Both algorithms use the p-center problem (5.7) to assign scenarios to solutions. However, SG uses a *decision version* of problem (5.7) that is solved through a binary search algorithm, in contrast to the more-efficient double-oracle algorithm proposed in this chapter. RCG uses a formulation similar to (5.7) as the master problem. However, it assumes that y^k are also decision variables, thus has to deal with the bilinear terms $\mathrm{v}^\intercal \mathrm{y}^k$, which are linearized by defining additional variables and constraints. Moreover, RCG is developed specifically for a discrete budgeted

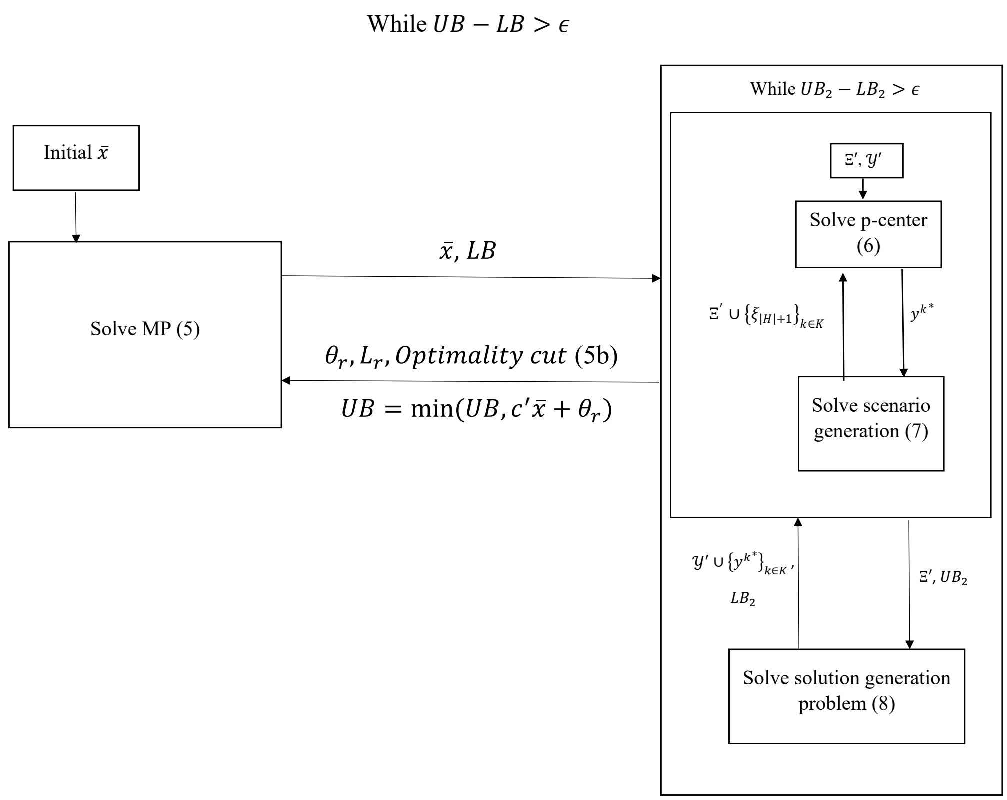

Figure 5.1: Double-oracle, logic-based Benders decomposition approach

uncertainty set, whereas our algorithm does not have this limitation. Later, we show how our algorithm can handle different extensions of the problem that SG and RCG cannot.

5.3 Finite Convergence

The proposed approach consists of two loops. The outer loop handles the first-stage decisions through a logic-based Benders decomposition algorithm. The inner loop, which has two steps: scenario generation and solution generation, is used to solve **SP**, which is a MMMRCO problem. We show that both the outer loop and the scenario generation step in the inner loop terminate after a finite number of iterations, and that the solution generation step leads to the convergence of LB and UB. Note that the following lemmas do not exploit the linearity of the cost and recourse constraint functions in Problem (5.1).

Lemma 11. *By assuming that $\mathcal{X}$ is a bounded discrete set, the number of iterations of the outer loop is finite.*

Proof. Since $\mathcal{X}$ is a bounded discrete set, there is a finite number of feasible solutions $\bar{\mathrm{x}}$ for the first-stage problem (5.6). Each first-stage solution corresponds to a single optimality cut (5.6b), generated by solving **SP** to obtain x^r, θ_r and L_r. Hence, the number of optimality cuts is finite, and so is the number of outer loop iterations. $\square$

Lemma 12. *By assuming that $\mathcal{Y}$ is a bounded discrete set, the maximum number of scenarios $\{\xi_h\}_{h\in H}$ that can be generated through the double oracle algorithm is finite and equal to $\frac{|J|!}{K!(|J|-K)!}$.*

Proof. Since $\mathcal{Y}$ is a bounded discrete set, its elements (feasible solutions) are finite and thus can be enumerated. The p-center problem (5.7) selects K solutions and assigns those selected solutions to scenarios. Moreover, those K selected solutions are used to generate worst-case scenarios through problem (5.8); and if the generated scenario violates constraint (5.7b), it is added to the set of scenarios. However, if all possible worst-case scenarios of all feasible $(\bar{\mathrm{x}}, \mathrm{y}^k)$ assignments in our subset of scenarios are available, then no generated scenario from problem (5.8) can violate (5.7b) and the iteration between (5.7) and (5.8) will terminate. If all combinations of K out of $|\mathcal{Y}|$ are selected and used in problem (5.8), all possible worst-case scenarios for all feasible $(\bar{\mathrm{x}}, \mathrm{y}^k)$ are generated. Since $\mathcal{Y}$ is a bounded discrete set, there are exactly $\binom{|J|}{k}$ possible ways to select K elements from the set $\mathcal{Y}$. Consequently, at most there are $\binom{|J|}{k} = \frac{|J|!}{k!(|J|-k)!}$ possible solutions for (5.7). Hence, in the worst-case situation, with the finite number of iterations between the p-center problem, (5.7), and the scenario generation problem, (5.8), all possible worst-case scenarios are generated. therefore, no new scenario can violate (5.7b) and then, this step terminates. $\square$

Lemma 13. *The upper and lower bounds, obtained respectively by solving problems $P(\mathcal{Y}', \Xi)$ and $P(\mathcal{Y}, \Xi')$, converge after a finite number of iterations.*

Proof. The upper bound is found by solving $\mathbf{P}(\mathcal{Y}', \Xi)$. To get the optimal assignment, problem (5.7) with $\mathcal{Y}'$ must be solved. However, its optimal value, which is the upper bound, can be obtained alternatively by solving $UB = \max_{h\in[H]} \min_{j\in[J']} \xi_h^\intercal \mathrm{Q}\mathrm{y}_j$ (see

Proposition 15 in Appendix 5.7). On the other hand, by using a fixed set of scenarios, a K-subset of solutions is generated by solving (5.9), which can be reformulated as $LB = \max_{h\in[H']} \min_{\mathrm{y}^k\in\mathcal{Y}} \xi_h^\intercal \mathrm{Q}\mathrm{y}^k$ (see Proposition 16 in Appendix 5.7). There are two cases:

Case 1: The optimal solutions y^{k^*} already exist in the subset of solutions $\mathcal{Y}'$, in which case the optimal pair of solutions and scenarios will be found by solving (5.7) and its objective value equals $UB = \max_{h\in[H]} \min_{j\in[J']} \xi_h^\intercal \mathrm{Q}\mathrm{y}_j$. On the other hand, the optimization problem related to the lower bound (5.9) will generate y^{k^*} as the optimal solution since the scenarios are fixed for both the lower and upper bound problems. In this case, $LB = UB$ and the algorithm terminates.

Case 2: The optimal solutions y^{k^*} are not in the subset of solutions $\mathcal{Y}'$. In this case, by using fixed scenarios and solving (5.9), a K-subset of solutions is generated that includes y^{k^*}. These solutions are added to the subset of solutions in the p-center problem (5.7). Consequently, the optimal pair of solutions and scenarios will be in the $\mathcal{Y}'$, and Ξ', respectively. Hence, by solving the p-center problem (5.7), the optimal pair of solutions and scenarios will be selected. Consequently, $LB = UB$ and the algorithm will terminate. □

Based on these lemmas, we can conclude that all three loops in the proposed algorithm will terminate in a finite number of iterations. Hence, the following theorem is proven.

Theorem 14. *The double-oracle, logic-based Benders decomposition algorithm enjoys finite convergence.*

5.4 Extensions

So far, we focused on the K-adaptability problem with linear objectives and constraints, binary first-stage and integer second-stage decision variables, and with objective uncertainty only in the second-stage. In this section, we show how the proposed algorithm can be extended to more general cases beyond the basic setting outlined earlier.

5.4.1 Second-stage Constraint Uncertainty

Similar to the algorithm proposed by Arslan et al. (2022) to solve the MMMRCO problem, our approach can be extended to the case when uncertainty affects both the objective function and constraints of the recourse problem. The extended problem can be formulated as follows:

$$\min_{\mathrm{x}\in\mathcal{X},\{\mathrm{y}^k\}_{k\in[K]}} \quad \mathrm{c}^\intercal\mathrm{x} + \sup_{\xi\in\Xi}\min_{k\in[K]}\left\{\xi^\intercal\mathrm{Q}\mathrm{y}^k :\ \mathrm{T}\mathrm{x}+\mathrm{W}(\xi)\mathrm{y}^k \leq \mathrm{b}\right\} \tag{5.11a}$$

$$\text{s.t.} \quad \mathrm{y}^k \in \mathcal{Y},\ k\in[K], \tag{5.11b}$$

where $\mathrm{W}(\xi)$ is an affine mapping of uncertain parameters. Note the dependency of the **SP** constraints on ξ. To solve this problem, we use the same iterative algorithm explained earlier but with a modified p-center problem $\mathbf{P}'(\mathcal{Y}',\Xi')$ by adding (5.12f), as follows:

$$\min_{\{z_j\}_{j\in[J']},\{v_{jh}\}_{j\in[J'],h\in[H']},w} \quad w \tag{5.12a}$$

$$\text{s.t.} \quad w \geq \sum_{j\in[J']}\xi_h^\intercal\mathrm{Q}\mathrm{y}_j v_{jh} \quad \forall h\in[H'] \tag{5.12b}$$

$$\sum_{j\in[J']} v_{jh} = 1 \quad \forall h\in[H'] \tag{5.12c}$$

$$\sum_{j\in[J']} z_j = K \tag{5.12d}$$

$$v_{jh} \leq z_j \quad \forall j\in[J'],\forall h\in[H'] \tag{5.12e}$$

$$v_{jh} = 0 \quad \forall j\in[J'],\ h\in[H'] : \exists s\in[S] \ \text{s.t. } \mathrm{e}_s^\intercal(\mathrm{W}(\xi_h)\mathrm{y}_j - \mathrm{b} + \mathrm{T}\bar{\mathrm{x}}) > 0 \tag{5.12f}$$

$$v_{jh}, z_j \in \{0,1\} \quad \forall j\in[J'],\forall h\in[H']. \tag{5.12g}$$

In this formulation, there are s constraints with uncertain parameters, *i.e.,* $\mathrm{b}\in\mathbb{R}^s$. The quantifier constraint (5.12f) is not a problem in terms of modeling, but it prevents infeasible assignment of solution-scenario pairs. Moreover, problem (5.8) is modified by adding constraint (5.13c) as follows:

$$\max_{\xi\in\Xi,\eta,\lambda\in\{0,1\}^K} \quad \eta \tag{5.13a}$$

$$\text{s.t.} \quad \eta \leq \xi^{\intercal}\mathrm{Q}\mathrm{y}^k + M\lambda^k \qquad k\in[K] \tag{5.13b}$$

$$\mathrm{W}(\xi)\mathrm{y}^k > \mathrm{b} - \mathrm{T}\bar{\mathrm{x}} - M(1-\lambda^k) \quad k\in[K], \tag{5.13c}$$

where $\lambda^k = 1$ if the scenario ξ is such that y^k is infeasible for any of the s uncertain constraints, thus enforces that we do not consider $\xi^{\intercal}\mathrm{Q}\mathrm{y}^k$ for calculating the upper bound on η. Moreover, the solution generation problem (5.9) is modified by considering constraint (5.14e) with uncertain parameters.

$$\min_{\{\mathrm{y}^k\}_{k\in[K]},\gamma,\{u_{kh}\}_{k\in[K],h\in[H']}} \quad \gamma \tag{5.14a}$$

$$\text{s.t.} \quad \xi_h'\mathrm{Q}\mathrm{y}^k \leq \gamma + M(1-u_{kh}) \qquad \forall k\in[K],\ \forall h\in[H'] \tag{5.14b}$$

$$\sum_{k\in[K]} u_{kh} = 1 \qquad \forall h\in[H'] \tag{5.14c}$$

$$\mathrm{y}^k \in \mathcal{Y} \qquad \forall k\in[K] \tag{5.14d}$$

$$\mathrm{W}(\xi_h^{\intercal})\mathrm{y}^k \leq \mathrm{b} - \mathrm{T}\bar{\mathrm{x}} + M(1-u_{kh}) \quad \forall k\in[K], \forall h\in[H'] \tag{5.14e}$$

$$u_{kh}\in\{0,1\} \qquad \forall k\in[K],\ \forall h\in[H']. \tag{5.14f}$$

One should note that even though $\mathcal{V}$ can be modified to ensure that **MP** satisfies relatively complete recourse, it might happen that for some $\bar{\mathrm{x}}\in\mathcal{X}\cap\mathcal{V}$, there exist no set of K solutions $\{\mathrm{y}^k\}_{k=1}^K$ that ensure that some y^k is always feasible under all $\xi\in\Xi$. For this reason, at the r-th iteration of the logic-based Benders decomposition algorithm, **SP** might become infeasible for x_i^r, which is identified when problem (5.14) becomes infeasible for some Ξ'. At this point, Algorithm 1 should be modified to generate and add to **MP** a feasibility cut of the form:

$$\sum_{i\in\mathcal{S}_r}\mathrm{x}_i - \sum_{i\notin\mathcal{S}_r}\mathrm{x}_i \leq |\mathcal{S}_r| - 1,$$

in order to discard x_i^r from the set of feasible candidates, instead of producing an optimality cut of the form (5.5).

5.4.2 Bounded First-stage Integer Decision Variables

The outer loop in the proposed algorithm depends on the first-stage variables being binary (*i.e.,* $\mathrm{x} \in \{0,1\}^n$) to generate logic-based Benders cuts of the type (5.5). However, if the first-stage integer variables are not binary, but rather bounded general integer, *i.e.,* $\mathcal{X} \in \mathbb{Z}_+^n$, one can simply apply the transformation $x_i = \sum_{p_i=0}^{P_i} 2^p u_{ip_i},\ i = 1,\dots,n$, where $u_{ip_i} \in \{0,1\}$, and P_i depends on the upper bound of x_i. Clearly, this generalization comes at the expense of increasing the number of variables in the first-stage problem, thus it might be efficient only for small values of P_i.

5.4.3 First-stage Objective Uncertainty

Even though Bertsimas & Caramanis (2010) defined K-adaptability such that the first-stage objective is deterministic, we extend our algorithm to the case when there is first-stage objective uncertainty, similar to Hanasusanto et al. (2015). A practical example of the K-adaptability problem with uncertainty in the first-stage is the multi-period portfolio selection problem when decisions about the allocation of budget among assets have to be made at the beginning of the investment horizon, thus are first-stage decisions. Indeed, asset returns are uncertain even at the outset, and the initial capital allocation decision cannot be postponed until this uncertainty is revealed. Hence, the here-and-now decisions are directly affected by uncertain parameters. In this section, we differentiate between two cases of first-stage objective uncertainty. First: when uncertain parameter is independent of the second-stage uncertainty, which means uncertain parameters of first and second-stage objective functions are two different and independent parameters. Second: when some (or all) uncertain parameters affect both stages, *i.e.,* dependent uncertainty, which means they depend on the same uncertain parameters. We show how the proposed approach is tailored for each case.

Independent Uncertainty

Let us consider the following K-adaptability problem:

$$\min_{\mathrm{x}\in\mathcal{X}} \max_{\xi\in\Xi,\omega\in\Omega} \min_{k\in K}\{\omega^\intercal \mathrm{Cx} + \xi^\intercal \mathrm{Qy}^k :\ \mathrm{Tx}+\mathrm{Wy}^k \leq \mathrm{b}\} \tag{5.15a}$$

$$\text{s.t.} \qquad \mathrm{y}^k \in \mathcal{Y},\ k\in[K], \tag{5.15b}$$

where $\omega \in \mathbb{R}^c$, $C \in \mathbb{R}^{c\times n}$, and Ω is a compact and convex uncertainty set. Other variables and parameters are the same as those used in formulation (5.1). We assume that ω and ξ are disjoint sets of uncertain parameters. In this case, **MP** (5.6) is modified as follows:

$$\min_{\mathrm{x}\in\mathcal{X},\,\theta} \max_{\omega\in\Omega} \omega^\intercal \mathrm{Cx} + \theta \tag{5.16a}$$

$$\text{s.t.} \qquad \theta \geq (\theta_r - L)\left(\sum_{i\in\mathcal{S}_r} x_i - \sum_{i\notin\mathcal{S}_r} x_i\right) - (\theta_r - L_r)\left(|\mathcal{S}_r| - 1\right) + L, \tag{5.16b}$$

which is a static RO problem that can be tractably reformulated by applying convex duality on the inner maximization. The SP does not change, and we can apply the double-oracle algorithm described in Section 5.2.2 to solve it.

Dependent Uncertainty

Next, we consider the K-adaptability problem variant addressed by Hanasusanto et al. (2015), where the same uncertain parameters affect both stages, formulated as follows:

$$\min_{\mathrm{x}\in\mathcal{X},\{y^k\}_{k\in[K]}} \max_{\xi\in\Xi} \min_{k\in K} \xi^\intercal \mathrm{Cx} + \xi^\intercal \mathrm{Qy}^k \tag{5.17a}$$

$$\text{s.t.} \qquad \mathrm{y}^k \in \mathcal{Y},\ \mathrm{T}x + \mathrm{Wy}^k \leq \mathrm{b},\ k\in[K], \tag{5.17b}$$

where $\mathrm{C} \in \mathbb{R}^{q\times n}$ and the rest of the parameters and variables are the same as in (5.1). Since both first and second-stage decision variables depend on the same uncertain parameter ξ, we can reformulate model (5.17) into model (5.1). To propose this reformulation, we defined $\tilde{\mathrm{y}} \in \{0,1\}^n \times \mathcal{Y}$. Moreover, let $\tilde{\mathrm{Q}} = [\mathrm{C},\mathrm{Q}] \in \mathbb{R}^{q\times(n+m)}$,

$\tilde{\mathrm{T}} = \begin{bmatrix} \mathbb{I} \\ \mathbb{I} \\ \mathrm{T} \end{bmatrix}$, $\tilde{\mathrm{W}} = \begin{bmatrix} -\mathbb{I} & 0 \\ \mathbb{I} & 0 \\ 0 & \mathrm{W} \end{bmatrix}$, and $\tilde{b} = \begin{bmatrix} 0 \\ 0 \\ \mathrm{b} \end{bmatrix}$. We have that problem (5.17) can be equivalently formulated as:

$$\min_{\mathrm{x}\in\mathcal{X},\{\tilde{\mathrm{y}}^k\}_{k\in[K]}} \max_{\xi\in\Xi} \min_{k\in K} \xi^\intercal \tilde{\mathrm{Q}} \tilde{\mathrm{y}}^k \tag{5.18a}$$

$$\text{s.t.} \quad \tilde{\mathrm{y}} \in \{0,1\}^n \times \mathcal{Y},\ \tilde{\mathrm{T}}x + \tilde{\mathrm{W}}\tilde{\mathrm{y}}^k \leq \tilde{\mathrm{b}},\ k \in [K]. \tag{5.18b}$$

In problem (5.18), $\tilde{\mathrm{y}}$ is the second-stage decision variable while x is the first-stage decision variable.

5.4.4 Nonlinear Objective and Constraint Functions

In principle, our algorithm can be extended to address nonlinear K-adaptability problems of the form:

$$\min_{\mathrm{x}\in\mathcal{X}} f(\mathrm{x}) + \max_{\xi\in\Xi} \min_{k\in K} \{g(\xi, \mathrm{y}^k) :\ h(\xi, \mathrm{x}, \mathrm{y}^k) \leq \mathrm{b}\} \tag{5.19a}$$

$$\text{s.t.} \quad \mathrm{y}^k \in \mathcal{Y},\ k \in [K], \tag{5.19b}$$

where $f : \mathcal{X} \mapsto \mathbb{R}$ is convex in x, $g : \Xi \times \mathcal{Y} \mapsto \mathbb{R}$ is affine in ξ and convex in y, and $h : \Xi \times \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ is affine in ξ and jointly convex in x and y. In this case, **MP** (5.6) is modified by using $f(\mathrm{x})$ instead of $\mathrm{c}^\intercal\mathrm{x}$. Moreover, the p-center problem (5.12) is modified by replacing $\xi_h^\intercal \mathrm{Q}\mathrm{y}_j$ and $\mathrm{W}(\xi_h)\mathrm{y}_j + \mathrm{T}\bar{\mathrm{x}}$ with $g(\xi_h, \mathrm{y}_j)$ and $h(\xi_h, \mathrm{y}_j, \bar{\mathrm{x}})$, respectively. In the p-center problem, v_{jh} and z_j are decision variables while y_j, ξ_h, and $\bar{\mathrm{x}}$ are constants. Consequently, regardless of the type of functions $g(\cdot,\cdot)$ and $h(\cdot,\cdot,\cdot)$, the p-center problem finds the optimal K solutions and assigns them to scenarios. Moreover, the scenario generation problem (5.8) is rewritten as follows:

$$\max_{\xi\in\Xi,\eta,\lambda\in\{0,1\}^K} \quad \eta \tag{5.20a}$$

$$\text{s.t.} \quad \eta \leq g(\xi, \mathrm{y}^k) + M\lambda^k \qquad k \in [K] \tag{5.20b}$$

$$h(\xi, \mathrm{y}^k, \bar{\mathrm{x}}) > \mathrm{b} - M(1-\lambda^k) \quad k \in [K], \tag{5.20c}$$

which again takes the form of a mixed integer LP given our assumption that $g(\cdot,\cdot)$ and $h(\cdot,\cdot,\cdot)$ be affine in ξ. Finally, the solution generation problem (5.9) is changed by replacing $\xi_h^\intercal \mathrm{Q}\mathrm{y}^k$ and $\mathrm{W}\mathrm{y}^k + \mathrm{T}\bar{\mathrm{x}}$ with $g(\xi_h, \mathrm{y}^k)$ and $h(\xi_h, \mathrm{y}^k, \bar{\mathrm{x}})$, respectively.

5.5 Numerical Results

This section presents and analyzes the numerical results obtained by implementing the proposed algorithm to solve four problems: the shortest path problem, the knapsack problem, a generic K-adaptability problem, and the asset-liability management problem. These results are compared (when possible) to those obtained from the following state-of-the-art algorithms.

- `MILP`: the mixed-integer linear programming reformulation of Hanasusanto et al. (2015);
- `IA`: the iterative algorithm of Chassein et al. (2019);
- `RCG`: the row-and-column generation algorithm of Goerigk et al. (2020);
- `SG`: the scenario generation approach of Arslan et al. (2022); and
- `BB`: the branch-and-bound method of Subramanyam et al. (2020).

The proposed `DO` algorithm, `MILP`, `IA`, and `RCG` were coded on Python 3.10.4 using Jupyter. `SG` is available here[1], and `BB` is also available here[2]. The subproblems were solved using CPLEX called through CPLEX-CMD on a Linux laptop with an 8th generation Intel Core i7 7700 processor and 16 GB RAM. The time limit was set to two hours (7200 seconds).

5.5.1 Shortest Path Problem

The first problem used to evaluate the proposed algorithm is the adaptive shortest path problem, previously studied in Hanasusanto et al. (2015) and Chassein et al. (2019). This problem aims to select a subset of network arcs with the least total cost to form a path from a source s to a destination t when arc costs are uncertain. In

[1] https://github.com/mjposs/min-max-min
[2] https://github.com/AnirudhSubramanyam/KAdaptabilitySolver

the K-adaptability variant of the problem, K paths are pre-formed and the shortest (least costly) among them is selected once the actual costs are realized. We used test instances from Arslan et al. (2022), available here[3].

Formally, the problem can be described as follows: A network $(\mathcal{V}, \mathcal{A})$ has the cost of each arc $(i, j) \in \mathcal{A}$ characterized as $c_{ij} = \bar{c}_{ij} + \xi_{ij}\hat{c}_{ij}$, where $\bar{c}_{ij}$ is the nominal cost and $\hat{c}_{ij}$ is the maximal deviation. The primary uncertain parameter ξ belongs to the budgeted uncertainty set $\Xi = \left\{\xi \in [0, 1] | \sum_{(i,j)\in A} \xi_{ij} \leq \Gamma\right\}$, where Γ is an *uncertainty budget* that controls the size of uncertainty set. With that, the problem is formulated as follows:

$$\min_{\mathrm{x}\in\{0,1\}^n} \quad \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} \tag{5.21a}$$

$$\text{s.t.} \quad \sum_{(i,j)\in\delta^+(i)} x_{ij} - \sum_{(i,j)\in\delta^-(i)} x_{ij} = b_i, \quad \forall i \in \mathcal{V}, \tag{5.21b}$$

where $b_s = -1$, $b_t = 1$, and $b_i = 0$ for $i \in \mathcal{V}/\ \{s, t\}$ and the sets δ_i^+ and δ_i^- represent the forward and backward stars of node $i \in \mathcal{V}$, respectively.

We solve the problem in different sizes of $|\mathcal{V}| \in \{20, 25, 40, 50\}$. For each problem size, we considered $K \in \{2, 3, 4, 5, 6\}$ and $\Gamma \in \{3, 6\}$. Ten randomly-generated instances were solved for each combination of $|\mathcal{V}|$, K, and Γ. We compare the results of our algorithm with those obtained using `MILP`, `RCG`, `IA` and `SG`. Table 5.1 shows the number of solved instances out of 10 instances by these algorithms for $\Gamma = 3$, and $\Gamma = 6$.

Intuitively, as the problem size increases (in terms of both $|\mathcal{V}|$ and K), fewer instances were solved to optimality by all algorithms. Nevertheless, `DO` showed better performance than all other algorithms. For example, it solved all instances with $\Gamma = 3$ and $K \in \{2, 3, 4, 5\}$, while none of the other algorithms could solve all of these instances within the cut-off time. Moreover, `DO` solved 30-60% of the instances with $|\mathcal{V}|$ =50, Γ=3, and $K \in \{2, 3, 4, 5, 6\}$, while the next best algorithm was `SG` that could not solve instances with $|\mathcal{V}|$ =50 and K >3. The comparison results for Γ=6 exhibit the same pattern. `DO` had a significant performance over the benchmark

[3] https://github.com/mjposs/min-max-min

Table 5.1: Number of solved instances and average CPU time for the shortest path problem

			Number of solved instances					Average CPU time (s)				
Γ	$\lvert\mathcal{V}\rvert$	Alg	K=2	K=3	K=4	K=5	K=6	K=2	K=3	K=4	K=5	K=6
3	20	`DO`	10	10	10	10	10	1.55	1.56	1.69	1.82	1.99
		`MILP`	10	10	8	8	8	0.46	0.96	3.09	6.54	43.08
		`IA`	9	9	7	7	6	0.35	0.41	0.38	0.40	0.43
		`RCG`	10	10	8	8	8	1.37	2.21	3.81	6.21	4.87
		`SG`	10	10	8	8	8	1.62	1.59	1.77	1.85	1.99
3	25	`DO`	10	10	10	10	10	2.87	2.80	3.26	3.70	7.23
		`MILP`	10	10	4	1	1	12.69	53.75	524.45	2.50	16.55
		`IA`	10	10	8	4	4	3.44	3.42	3.23	4.08	4.36
		`RCG`	8	8	8	7	7	2.59	2.61	6.08	7.57	8.70
		`SG`	10	10	10	10	10	2.93	2.95	3.07	3.94	6.52
3	40	`DO`	10	10	10	10	10	8.63	7.50	9.04	11.97	86.95
		`MILP`	9	8	4	0	0	90.61	299.26	873.76	TL	TL
		`IA`	10	10	10	8	7	68.45	81.99	79.13	65.68	77.08
		`RCG`	9	9	6	5	4	4.54	8.81	13.19	15.14	59.29
		`SG`	10	10	8	8	8	6.91	6.51	8.93	10.85	82.07
3	50	`DO`	7	5	5	4	4	9.00	17.04	17.15	15.36	17.78
		`MILP`	1	0	0	0	0	355.99	TL	TL	TL	TL
		`IA`	2	1	0	0	0	102.68	118.69	TL	TL	TL
		`RCG`	1	1	0	0	0	90.56	232.81	TL	TL	TL
		`SG`	3	1	0	0	0	28.93	TL	TL	TL	TL
6	20	`DO`	10	10	10	10	9	1.54	1.54	1.54	1.61	1.85
		`MILP`	10	10	8	7	7	0.72	1.26	3.31	7.21	119.40
		`IA`	9	8	7	6	6	0.28	0.35	0.36	0.40	0.42
		`RCG`	8	8	8	7	7	1.27	2.21	3.74	6.23	4.77
		`SG`	8	8	6	6	5	1.55	1.57	1.61	1.64	1.93
6	25	`DO`	10	10	10	10	10	4.43	4.47	5.08	8.19	9.23
		`MILP`	10	10	4	1	0	14.31	75.40	151.24	389.98	TL
		`IA`	10	10	10	6	4	3.39	4.34	4.84	5.28	14.11
		`RCG`	6	5	4	4	4	2.69	3.13	3.21	14.28	17.19
		`SG`	9	9	7	7	7	4.40	4.66	4.91	8.16	10.10
6	40	`DO`	9	9	8	8	8	10.59	11.27	14.58	22.20	26.96
		`MILP`	8	7	10	0	0	70.59	664.92	1044.60	TL	TL
		`IA`	10	10	10	9	5	54.03	73.24	74.91	81.40	85.20
		`RCG`	5	4	2	2	1	7.32	14.41	39.41	45.17	125.01
		`SG`	8	8	6	5	5	11.41	11.81	12.20	20.76	24.96
6	50	`DO`	6	5	5	4	4	10.46	15.45	15.70	18.01	22.08
		`MILP`	0	0	0	0	0	TL	TL	TL	TL	TL
		`IA`	1	0	0	0	0	81.04	TL	TL	TL	TL
		`RCG`	2	0	0	0	0	TL	TL	TL	TL	TL
		`SG`	3	1	0	0	0	377.54	TL	TL	TL	TL

algorithms, except IA, when the uncertainty budget Γ was doubled. IA solved 100% of the instances with $|\mathcal{V}|$=40, $K \in \{2,3,4\}$, and Γ=6, while DO solved 80% to 90% of the instances of the same size. However, for $|\mathcal{V}|$=50, the performance of IA deteriorated, as it could only solve 0-10% of the instances, whereas DO solved 40-70% of these instances. These results clearly show the performance advantage of the proposed approach over other algorithms proposed in the literature, especially for large-size problems. It is worth noting that IA, unlike DO, cannot guarantee global optimality since it uses a fixing heuristic to handle a bilinear term in each iteration.

Another important performance metric for comparing algorithms is the processing (CPU) time. The average CPU time (in seconds) for Γ=3, Γ=6 are shown in Table 5.1. When TL is shown, it indicates that the problem could not be solved within the cut-off time. It can be seen that IA had better performance than DO in small-size instances (with an average difference of about 2 seconds). However, for the largest problem size ($|\mathcal{V}|$=50), IA could not solve any instance with K >3 within the cut-off time, whereas DO solved instances of the same size with K=6 in less than 20 seconds. We observe that the DO had much smaller CPU times for large instances in comparison to the iterative algorithm, which also does not guarantee optimality. The average CPU times for DO and SG were almost identical. For example, the average CPU times of DO for instances with $|\mathcal{V}| \in \{20,25,40\}$, Γ=3, and $K = \{2,3,4,5,6\}$ were between 1.55 and 86.95 seconds, while those for SG were 1.62-82.07 seconds. However, SG could not solve any instances with $|\mathcal{V}|$=50, and K >3, while DO solved some of these instances within the cut-off time.

Furthermore, all tested algorithms (except MILP) provide lower bounds on the optimal value for instances that could not be solved to optimality. Hence, we compare the best lower bound achieved by each algorithm to the best lower bound of DO based on 10 instances for different $|\mathcal{V}|$ and Γ values. The relative gaps shown in Table 5.2 are calculated as $\frac{LB_{\texttt{DO}}-LB_{\texttt{Alg}}}{LB_{\texttt{DO}}}$, $\texttt{Alg} \in \{\texttt{IA}, \texttt{RCG}, \texttt{SG}\}$. A positive relative gap means that DO reached a better (higher) LB than the benchmark algorithm. We notice that the average gaps of RCG and SG relative to DO were up to 3% in small-size instances, but reached 13% in large instances. As shown earlier, RCG, SG and DO could solve the problem to optimality for $|\mathcal{V}|$ =20, $K \in \{2,3\}$ and $\Gamma \in \{3,6\}$, resulting in zero gaps between them. Moreover, SG and DO generated the same results for instances with

Table 5.2: Gap analysis for the shortest path problem

Γ	$\lvert\mathcal{V}\rvert$	Alg	Relative gap (%)					Ratio of better LB by `DO`				
			K=2	K=3	K=4	K=5	K=6	K=2	K=3	K=4	K=5	K=6
3	20	`IA`	2.69	3.43	11.98	13.45	16.71	1/1	1/1	3/3	2/3	4/4
		`RCG`	-	-	-1.36	1.82	1.41	-	-	2/2	2/2	1/2
		`SG`	-	-	-1.10	1.90	2.97	-	-	1/2	1/2	1/2
3	25	`IA`	-	-	5.86	13.74	16.71	-	-	2/2	4/6	5/6
		`RCG`	1.12	3.43	-1.36	4.97	2.41	2/2	2/2	1/2	3/3	2/3
		`SG`	-	-	-	-	-	-	-	-	-	-
3	40	`IA`	7.92	11.59	15.94	14.77	17.15	-	-	1/1	2/2	2/3
		`RCG`	2.11	2.07	0.48	3.42	2.31	1/1	1/1	3/4	5/5	5/6
		`SG`	-	-	1.08	0.87	1.45	-	-	2/2	1/2	1/2
3	50	`IA`	12.27	13.15	15.68	18.09	19.56	7/8	9/9	10/10	10/10	9/10
		`RCG`	4.89	5.33	6.85	7.60	9.82	8/9	8/9	10/10	10/10	8/10
		`SG`	3.23	5.58	7.54	5.39	8.54	4/7	5/9	6/10	4/10	5/10
6	20	`IA`	4.66	6.65	5.06	5.09	5.14	1/1	2/2	2/3	4/4	3/4
		`RCG`	1.29	2.19	-1.47	3.11	2.7	2/2	2/2	1/2	3/3	2/3
		`SG`	1.43	2.32	1.74	-0.29	1.29	2/2	2/2	3/4	2/4	3/5
6	25	`IA`	4.19	5.13	9.19	8.87	11.41	-	-	-	2/2	2/6
		`RCG`	2.28	2.99	2.57	3.63	1.73	4/4	5/5	4/6	4/6	3/6
		`SG`	2.43	3.43	3.93	2.98	-3.78	1/1	1/1	3/3	2/3	1/3
6	40	`IA`	1.51	3.41	3.21	4.23	6.79	-	-	-	1/1	2/5
		`RCG`	2.38	4.56	5.28	6.43	9.67	3/5	5/6	7/8	7/8	9/9
		`SG`	1.94	5.12	3.56	6.17	8.79	1/2	2/2	2/4	4/5	4/5
6	50	`IA`	11.95	17.37	21.02	25.03	27.55	9/9	10/10	10/10	9/10	9/10
		`RCG`	6.14	7.78	10.89	11.47	13.1	6/8	9/10	9/10	10/10	7/10
		`SG`	4.95	-1.67	2.78	7.46	8.74	5/7	5/9	6/10	7/10	7/10

$|\mathcal{V}|$ =25, $K \in \{2,3,4,5\}$, and Γ =3. On the other hand, IA, a heuristic with no optimality guarantee, had relative gaps of up to 26%.

Table 5.2 also shows the number of instances for which DO reached a better LB than the benchmark algorithm among the instances that could not be solved within the cut-off time. We note that RCG and SG provided the same LB as DO for instances with $|\mathcal{V}|$ =20 and Γ=3. Moreover, IA, SG, and DO provided the same LB for instances with $|\mathcal{V}| \in \{25, 40\}$, Γ ∈{3,4}, and K ∈{2,3,4}. However, DO frequently provided better lower bounds than the other algorithms in large instances. For example, it outperformed IA in terms of LB in 7/8 to 10/10 instances with $|\mathcal{V}| = 50$ that could not be solved to optimality. The ratios for the same set of instances were 8/9 to 10/10 compared to RCG and 4/7 to 7/10 compared to SG. Clearly, DO reached the best LB in the vast majority of cases.

5.5.2 Adaptive Knapsack Problem

We then tested on the adaptive knapsack problem, which aims to prepare K different combinations of items that respect the capacity constraint without exact knowledge of their profit. Let $[n]$ be the set of items, w_i and p_i, respectively, be the weight and profit of item $i \in [n]$, and b is the available budget. The feasible set is defined as $\mathcal{X} = \{\mathrm{x} \in \{0,1\}^n | \sum_{i\in[n]} w_i x_i \leq b\}$. The goal is to find the best combination of items that maximize the profit $\mathrm{p}^\intercal \mathrm{x}$. The uncertain parameter p_i is assumed to follow $p_i = \left(1 + \sum_{j\in[m]} \frac{\Phi_{ij}\xi_j}{2}\right) \bar{p}_i$, where $\bar{p}_i$ is the nominal profit, m is the number of uncertain factors and $\Phi \in \mathbb{R}^{n\times m}$ is the factor loading matrix. The i-th row of Φ is characterized by the set $\left\{\Phi_i \in [-1,1]^m \mid \sum_{j\in[m]} |\Phi_{ij}| = 1\right\}$. As a result, the realized profit of each object $i \in [n]$ remains within the interval $[\frac{\bar{p}_i}{2}, \frac{3\bar{p}_i}{2}]$. We solve the problem in different sizes $n \in \{100, 150, 200, 300\}$ and different values of $K \in \{2,3,4,5,6\}$. Ten instances of each combination of n, and K are solved, and the results obtained from our algorithm are compared to those of IA, MILP and SG.

The percentage of solved knapsack problem instances are shown in Tables 5.3. It can be seen that for different problem sizes, DO and SG were able to solve almost the same percentage of test instances. However, MILP and IA could hardly solve any instances with $K \geq 3$ or $n \geq 200$. By increasing n and K, the percentage of solved instances by SG and DO dropped to 50-60%, showing the significant impact of n and

Table 5.3: Number of solved instances and average CPU time for the adaptive knapsack problem

n	Alg	Number of solved instances					Average CPU time				
		K=2	K=3	K=4	K=5	K=6	K=2	K=3	K=4	K=5	K=6
100	`DO`	10	10	10	10	10	16.29	25.92	49.01	49.60	128.35
	`MILP`	2	0	0	0	0	3,829.95	TL	TL	TL	TL
	`IA`	2	1	0	0	0	999.12	1,460.23	TL	TL	TL
	`SG`	10	10	10	10	10	22.60	28.85	70.23	118.96	212.19
150	`DO`	10	10	10	10	10	83.93	92.21	93.19	93.45	110.52
	`MILP`	1	0	0	0	0	4,072.81	TL	TL	TL	TL
	`IA`	1	0	0	0	0	70.94	TL	TL	TL	TL
	`SG`	10	10	10	10	10	72.73	88.41	82.61	102.92	124.05
200	`DO`	7	7	7	6	6	457.84	515.72	545.24	567.05	574.33
	`MILP`	0	0	0	0	0	TL	TL	TL	TL	TL
	`IA`	0	0	0	0	0	TL	TL	TL	TL	TL
	`SG`	7	7	7	6	6	537.63	548.42	573.13	622.36	630.55
300	`DO`	6	6	5	5	5	1,734.74	2,129.96	2,673.10	3,382.99	4,875.86
	`MILP`	0	0	0	0	0	TL	TL	TL	TL	TL
	`IA`	0	0	0	0	0	TL	TL	TL	TL	TL
	`SG`	5	3	3	4	3	1,733.66	2,438.93	3,138.38	4,568.43	5,233.18

K on their performance. Table 5.3 also shows the average CPU time in seconds for the solved knapsack problem instances. It can be seen that the average CPU times for `DO` and `SG` are very close, with no clear advantage for either algorithm. However, our algorithm is much faster than `IA` and `MILP`.

Table 5.4 shows the relative gaps of `IA` and `SG` with regard to the best upper bound (UB) found by `DO`. This relative gap is calculated as $\frac{UB_{\texttt{Alg}} - UB_{\texttt{DO}}}{UB_{\texttt{DO}}}$, $\texttt{Alg} \in \{\texttt{IA}, \texttt{SG}\}$. Thus, positive relative gaps indicate that `DO` provides tighter upper bounds than the benchmark algorithm. Moreover, the ratio of unsolved instances for which `DO` provides a better UB than the other algorithm is provided. We note that the upper bounds provided by `IA` are consistently looser than those of `DO`, with 8-10 out of the 10 unsolved instances having better `DO` upper bounds. On the other hand, both `DO` and `SG` solved all instances with $n \in \{100, 150\}$ to optimality, thus leading to zero relative gaps between the two algorithms. However, ratios for the instances with $n \in \{200, 300\}$ ranged between 1/3 to 5/6. Results in Table 5.4 clearly demonstrate that `DO` provides better upper bounds for the knapsack problem than the other algorithms, especially for large instances.

Table 5.4: Gap analysis for the adaptive knapsack problem

n	Alg	Relative gap (%)					Ratio of better UB by DO				
		K=2	K=3	K=4	K=5	K=6	K=2	K=3	K=4	K=5	K=6
100	IA	4.32	6.11	12.79	14.36	19.89	8/10	9/10	10/10	10/10	10/10
	SG	-	-	-	-	-	-	-	-	-	-
150	IA	2.12	4.33	5.83	7.46	9.37	9/10	10/10	10/10	10/10	10/10
	SG	-	-	-	-	-	-	-	-	-	-
200	IA	8.66	9.22	7.03	8.82	8.58	10/10	10/10	10/10	10/10	10/10
	SG	2.44	-1.28	-1.39	3.58	1.09	2/3	1/3	3/3	4/4	2/4
300	IA	10.57	16.31	17.36	18.22	21.77	10/10	10/10	10/10	10/10	10/10
	SG	3.21	2.56	3.06	3.24	6.19	3/4	4/5	4/4	4/5	5/6

5.5.3 Generic K-adaptability Problem

So far, we presented the results obtained for instances in the form of MMMRCO problems, which do not have actual first-stage decision variables. In this section, we test on binary K-adaptability problems with actual first-stage decisions in the form of model (5.22) to implement the logic-based Benders decomposition (LBD) presented in Algorithm 1.

$$\min_{\mathrm{x},\{\mathrm{y}^k\}_{k\in[K]}} \max_{\xi\in\Xi} \min_{k\in[K]} \sum_i a_i x_i + \sum_j c_j y_j^k \tag{5.22a}$$

$$\text{s.t.} \quad \sum_i x_i = b, \tag{5.22b}$$

$$-\sum_i d_i x_i + \sum_j f_j y_j \geq l, \tag{5.22c}$$

where $\mathrm{x} \in \{0,1\}^n, \mathrm{y} \in \{0,1\}^m$ are the first- and second-stage decision variables, respectively. Moreover, we set $c_j = \bar{c}_j + \xi_j \hat{c}_j$, where $\bar{c}_j$ is the nominal value that is drawn at random from the uniform distribution $\mathcal{U}(8,12)$ and $\hat{c}_j$ is the maximal deviation, set equal to 25% of the nominal value. Moreover, a_i, d_i f_j are generated randomly based on the uniform distributions $\mathcal{U}(8,12)$, $\mathcal{U}(50,100)$ and $\mathcal{U}(80,90)$, respectively. Finally, we set b=10, and l=0. The uncertain parameter ξ_j is contained in the set $\Xi = \left\{\xi \in [0,1] \mid \sum_j \xi_j \leq \Gamma\right\}$. Ten random instances of each size and uncertainty budget combination were solved. We compared our approach against the other algorithms in the literature that can handle first-stage decision variables, namely MILP

Table 5.5: Number of solved instances and average CPU time for generic K-adaptability instances

(n,m)	Alg	Number of solved instances					Average CPU time				
		K=2	K=3	K=4	K=5	K=6	K=2	K=3	K=4	K=5	K=6
(20, 20)	LBD	10	10	10	10	10	12.17	16.51	22.11	28.48	37.52
	MILP	10	10	10	10	10	0.82	1.47	0.50	21.93	70.54
	BB	10	10	10	10	10	12.47	17.75	24.29	26.62	35.78
(30, 30)	LBD	10	10	10	10	10	31.84	41.69	42.13	57.68	58.45
	MILP	10	10	10	10	10	1.10	2.78	128.62	325.99	402.52
	BB	10	10	10	10	10	32.62	44.82	96.30	153.90	355.75
(40, 40)	LBD	10	10	10	10	10	50.67	62.05	73.02	189.71	294.91
	MILP	10	8	8	0	0	1.50	112.55	196.46	TL	TL
	BB	10	10	8	8	7	81.91	660.71	800.25	1,077.28	TL
(50, 50)	LBD	10	8	8	7	7	74.63	86.93	186.54	308.29	530.35
	MILP	6	0	0	0	0	3.87	TL	TL	TL	TL
	BB	10	6	5	2	1	176.47	930.46	1,205.01	TL	TL

and BB.

The percentages of solved instances by each algorithm are presented in Table 5.5. For small instances (*i.e.,* $n = m$ =20, 30), the three algorithms were able to solve all instances. However, as the problem sizes were increased, the performance advantage of our algorithm became clear, especially for large values of K. For example, MILP could solve 60% of the instances of size $n = m$ =50 only with K =2, but none when K >2. In contrast, LBD solved the vast majority of instances of the same size to proven optimality within the cut-off time even with $K = 6$. Table 5.5 also shows the CPU times of all algorithms for the instances that were solved within the cutoff time. Again, it is clear that while MILP could solve small instances with small K values efficiently, LBD significantly outperformed it in large instances. Furthermore, the proposed algorithm scales well in K, thus can result in high adaptability in the face of parameter uncertainty. Similar insights could be drawn when comparing LBD to BB.

5.5.4 Asset Liability Management Problem

Some modifications to the proposed algorithm were presented in Section 5.4 to handle extensions of the basic problem. The most complicated among these extensions is the K-adaptability problem with objective uncertainty in both stages alongside uncertainty in the second-stage constraints, as outlined in Sections 5.4.1 and 5.4.3. To demonstrate the validity of the modifications proposed for this extension, we solve an adaptive asset-liability management (ALM) problem.

Bogentoft et al. (2001) defined the ALM problem as a multi-period asset allocation problem to balance assets and liabilities. The goal is to find the optimal asset allocation that maximizes profit in each period and covers liabilities. The investment horizon has length T, which is divided into a set of decision moments $t = 0, ..., T$. Let us consider n asset classes to select for investment. Then, $\mathrm{x}_t \in \{0, 1\}^n$ is a binary decision vector in the t^{th} period, where $x_{n,t} = 1$ if the n^{th} asset class in period t is selected for investment, otherwise $x_{n,t} = 0$. The value of assets owned by the fund at time t is denoted by a_t. Payments made by the fund to retirees at time t are liabilities and are denoted by d_t. The present value of these liabilities at time t is d_t. The minimum threshold of the funding ratio, which is the ratio of asset values to the present value of liabilities in each period, is given by ψ. Finally, $\mathrm{r}_t \in \mathbb{R}^n$ and $\mathrm{c}_t \in \mathbb{R}^n$ are vector of asset values and investment costs in the t^{th} period, respectively. the ALM problem with binary decision variables is formulated as follows:

$$\max_{\{\mathrm{x}_t \in \mathcal{X}_t, a_t\}_{t=0,\dots,T}} \quad A_T \tag{5.23a}$$

$$\text{s.t.} \quad a_t = a_{t-1} + \mathrm{r}_t^\intercal \mathrm{x}_{t-1} - \mathrm{c}_t^\intercal \mathrm{x}_t - d_t \quad \forall t = 1, \dots, T \tag{5.23b}$$

$$a_0 = b - \mathrm{c}_0^\intercal \mathrm{x}_0 - d_0 \tag{5.23c}$$

$$a_t \geq \psi d_t \quad \forall t = 0, \dots, T \tag{5.23d}$$

$$\mathrm{x}_t \in \mathcal{X}_t \quad \forall t = 0, \dots, T. \tag{5.23e}$$

The objective function (5.23a) tries to maximize the asset value in the final period. Constraint (5.23b), referred to as the balance constraint, ensures that the value of assets owned by the fund at period t is equal to the assets carried from period $t - 1$ minus liabilities and costs at this period. Constraint (5.23c) is the balance constraint of the current period, which is the current budget for investment b minus costs of investment in the current period and liabilities. Constraint (5.23d), referred to as the funding ratio, guarantees that the ratio of assets owned by the fund to the present value of liabilities at period t is greater than a minimum threshold ψ. Finally, constraint (5.23e) encompasses the regulatory constraints governing the fund.

Note that the asset values can be stated as follows:

$$a_t = b + \sum_{\tau=1}^{t} \mathrm{r}_\tau^\intercal \mathrm{x}_{\tau-1} - \sum_{\tau=0}^{t} (\mathrm{c}_\tau^\intercal \mathrm{x}_\tau + d_\tau) \qquad \forall t = 1, \ldots, T.$$

Hence, by substituting in (5.23), the adaptive ALM problem becomes

$$\max_{\mathrm{x}_t \in \mathcal{X}_t} \quad b + \sum_{t=1}^{T} \mathrm{r}_t^\intercal \mathrm{x}_{t-1} - \sum_{t=0}^{T} (\mathrm{c}_t^\intercal \mathrm{x}_t + d_t) \tag{5.24a}$$

$$\text{s.t.} \quad b + \sum_{\tau=1}^{t} \mathrm{r}_\tau^\intercal \mathrm{x}_{\tau-1} - \sum_{\tau=0}^{t} (\mathrm{c}_\tau^\intercal \mathrm{x}_\tau + d_\tau) \geq \psi D_t \quad \forall t = 0, \ldots, T \tag{5.24b}$$

$$\mathrm{x}_t \in \mathcal{X}_t \qquad \forall t = 0, \ldots, T.. \tag{5.24c}$$

We consider an adaptive ALM problem with only two periods in this round of experiments, *i.e.,* $t \in \{0, 1\}$, which leads to the following formulation:

$$\max_{\mathrm{x}_0 \in \mathcal{X}_0, \mathrm{x}_1 \in \mathcal{X}_1} \quad b + \mathrm{r}_1^\intercal \mathrm{x}_0 - \mathrm{c}_0^\intercal \mathrm{x}_0 - \mathrm{c}_1 \mathrm{x}_1 - d_0 - d_1 \tag{5.25a}$$

$$\text{s.t.} \quad b + \mathrm{r}_1^\intercal \mathrm{x}_0 - \mathrm{c}_0^\intercal \mathrm{x}_0 - \mathrm{c}_1 \mathrm{x}_1 - d_0 - d_2 \geq \psi D_1, \tag{5.25b}$$

where x_0 and x_1 are first and second-stage decision variables, respectively. The objective function (5.25a) includes both first and second-stage decision variables with uncertain coefficients, c_1, and r_1. We assume that c_0 is deterministic while r_1 is contained in the box uncertainty set $\{\mathrm{r}_1 = \bar{\mathrm{r}}_1 + \xi \hat{\mathrm{r}}_1,\ \xi \in [-1, 1]^n\}$, where $\bar{\mathrm{r}}_1$ is the nominal asset value and $\hat{\mathrm{r}}_1$ is the maximum deviation of asset value. Moreover $\mathrm{c}_1 = \bar{\mathrm{c}}_1 + \xi \hat{\mathrm{c}}_1$, where $\bar{\mathrm{c}}_1$ is the nominal value of costs, and $\hat{\mathrm{c}}_1$ is the deviation of costs.

For the regulatory constraints defining $\mathcal{X}_t$, we use the cardinality constraint $\mathrm{e}_t^\intercal \mathrm{x}_t = \lceil n/2 \rceil\ \forall t$, where e_t is a vector of ones of size n. We considered $n \in \{100, 150, 200, 250\}$ in our experiment. Nominal asset values and nominal cost are generated randomly based on the uniform distributions $\mathcal{U}(1000, 2000)$ and $\mathcal{U}(20, 50)$, respectively. Moreover, we set $d_t = \frac{\sum_{i=1}^{n} r_{it}}{8}\ \forall t$, $b = \frac{\sum_{i=1}^{n} r_{i1}}{2}$ and $\psi = 1.05$, and use an interest rate of 5% per anuum to calculate the present value of liabilities.

Table 5.6: Number of solved instances and average CPU time for adaptive ALM problem

	Number of solved instances					Average CPU time				
n	K=2	K=3	K=4	K=5	K=6	K=2	K=3	K=4	K=5	K=6
100	10	10	10	10	10	34.03	42.89	55.89	69.12	83.35
150	10	10	10	10	10	62.14	88.25	114.43	140.12	172.66
200	10	10	10	10	9	114.91	165.42	224.57	279.93	327.66
250	10	10	10	9	9	156.31	206.19	566.16	699.83	917.70

Table 5.6 shows the number of ALM instances solved to proven optimality by our algorithm within the cut-off time. The results demonstrate that `LBD` was able to solve the vast majority of instances (between 90-100%) for different combinations of n and K. We also provide the average CPU time for the solved instances, which clearly depends on n and K as well. For instances with n =100, the average CPU time was 34.03-83.35 seconds, whereas it was between 4.5 and 11 times larger for instances with n =250, amounting to 156.31-917.70 seconds. To our best knowledge, no other algorithm can handle the K-adaptability problem with uncertainty in the recourse constraints and the first and second-stage objective functions. Consequently, it was not possible to compare `LBD` to other algorithms for this problem.

5.6 Conclusion

A new approach for solving K-adaptability problems is presented in this chapter. A logic-based Benders decomposition is used first to isolate the first-stage variables in a master problem such that the remaining Benders subproblem, taking the form an MMMRCO problem, is solved to iteratively generate Benders optimality cuts. A novel double-oracle algorithm that iterates between generating new adverse scenarios and assigning them to K-subsets of recourses, and generating new recourse solutions is devised. Unlike other algorithms proposed in the literature, the recourse solutions are generated by solving MILP problems, and thus can be performed effectively. We proved the finite convergence of the proposed approach. Furthermore, we showed how it can be extended to other important variants of the MMMRCO and K-adaptability problems, some of which are unsolvable by existing algorithms. These extensions include problems with second-stage constraint uncertainty, bounded first-stage integer decision variables, first-stage objective uncertainty (independent or dependent from the second-stage uncertainty), and general convex objective and constraint functions.

Numerical experiments on several standard benchmark test problems from the literature clearly demonstrated the performance advantage of the proposed approach *vis-à-vis* state-of-the-art solution algorithms. We were able to solve large instances of the shortest path problem with 50 nodes and K =6, not solvable by other algorithms, in an average of 22 seconds. Likewise, the proposed approach was capable of solving large instances of the adaptive knapsack problem with n =300 and K of up to 6, outperforming all other algorithms. Even for the instances that could not be solved within the cut-off time, our algorithm consistently provided stronger bounds than the benchmark algorithms. We also solved large K-adaptability problems with real first-stage variables in shorter times compared to the existing `MILP` and `BB` approaches. Finally, to demonstrate the validity of the proposed extensions, we solved a new adaptive ALM problem with constraint uncertainty in the recourse problem and with first and second-stage objective uncertainty and showed that it can handle large instances of up to 250 assets in less than 16 minutes on average.

Our forthcoming research endeavors encompass a comprehensive exploration of the potential extensions and enhancements to the proposed algorithm. Primarily, we intend to delve into the extension of the algorithm to encompass general RO problems involving combinatorial recourse, as well as K-adaptability problems with continuous first-stage variables. Recognizing that the crux of our approach's efficacy lies in resolving the p-center assignment problem, especially for vast sets of scenarios and recourse solutions, we are dedicated to investigating strategies that can effectively address this challenge. This exploration holds the promise of substantially bolstering the overall efficiency of our proposed algorithm.

Furthermore, a notable avenue for improvement pertains to the combinatorial Benders decomposition technique. Acknowledging its susceptibility to sluggish convergence due to the generation of weak cuts, we anticipate that the incorporation of enhancement techniques specific to Benders decomposition could notably enhance the performance of our proposed approach.

Concurrently, we advocate for the utilization of streamlined methodologies in tackling the p-center problem, with the aim of optimizing the efficiency of our algorithm. Additionally, our future endeavors will be directed toward refining the lower bound employed in the optimality cut, as a more robust lower bound has the potential to expedite convergence towards the optimal solution within our proposed algorithm.

5.7 Appendix

Proposition 15. *The objective function value, w, of the p-center problem (5.7) can be achieved by solving:*

$$w^* = \max_{h\in[H]} \min_{j\in[J']} \xi_h^\intercal \mathrm{Q}\mathrm{y}_j.$$

Proof. There are $|H|$ constraints in the form of (5.7b) while constraints (5.7c) force the problem to select one pair of scenario and solution in each constraint (5.7b). The objective value should be minimum of w that is greater than selected pairs of scenarios and solutions in each h constraint (5.7b). To find the optimal value of w, minimum of $\xi_h^\intercal \mathrm{Q}\mathrm{y}_j$ for each $h \in [H']$ is selected, then optimal w^* will be maximum of selected pairs of scenarios and solutions in each constraint (5.7b). If $w^* > \max_{h\in[H]} \min_{j\in[J']} \xi_h^\intercal \mathrm{Q}\mathrm{y}_j$, then it cannot be optimal because there is a feasible solution with lower objective value which is $\max_{h\in[H]} \min_{j\in[J']} \xi_h^\intercal \mathrm{Q}\mathrm{y}_j$. On the other hand, if $w^* < \max_{h\in[H]} \min_{j\in[J']} \xi_h^\intercal \mathrm{Q}\mathrm{y}_j$, then some of constraints (5.7b) whose pair of solutions and scenarios are greater than w will be violated. Consequently, $UB = w^* = \max_{h\in[H]} \min_{j\in[J']} \xi_h^\intercal \mathrm{Q}\mathrm{y}_j$. □

Proposition 16. *The objective function value, γ, of the problem (5.9) can be achieved by solving:*

$$\gamma^* = \max_{h\in[H']} \min_{\mathrm{y}^k\in\mathcal{Y}} \xi_h^\intercal \mathrm{Q}\mathrm{y}^k.$$

Proof. Let y^{k^*} be the optimal solution of problem (5.9). There are $|K| \times |H'|$ constraints of (5.9b). For each $h \in [H']$ there are $|K|$ constraints in form of (5.9b). Constraints (5.9c) force the problem to select one pair of scenarios and solutions for each $h \in [H']$. The objective function is minimization, consequently, minimum of $(\mathrm{y}^{k^*}, \xi_h^\intercal)$ for each $h \in [H']$ are selected. Since γ should be greater than $\xi_h^\intercal \mathrm{Q}\mathrm{y}^{k^*}$, then maximum of selected pairs will be optimal objective value γ^*. Consequently, the optimal value of objective function, γ^* can be achieved by solving $\gamma^* = \max_{h\in[H']} \min_{\mathrm{y}^k\in\mathcal{Y}} \xi_h^\intercal \mathrm{Q}\mathrm{y}^k$. □

Proposition 17. *Problem (5.3) and (5.6) are equivalent.*

Proof. First, given that θ serves as an epigraph variable, it is clear that problem (5.6) is equivalent to:

$$\min_{\mathrm{x}\in\mathcal{X}\cap\mathcal{V}} \mathrm{c}^{\intercal}\mathrm{x} + \max_{r\in\mathcal{R}} \rho(\mathrm{x}, r),$$

where $\rho(x,r) := (\theta_r - L_r)\left(\sum_{i\in\mathcal{S}_r} x_i - \sum_{i\notin\mathcal{S}_r} x_i\right) - (\theta_r - L_r)(|\mathcal{S}_r| - 1) + L$. We are therefore left with showing that for all $\bar{x}\in\mathcal{X}\cap\mathcal{V}$ we have that $\nu(x) = \max_{r\in\mathcal{R}} \rho(\mathrm{x}, r)$. To do so, we let $\bar{r}\in\mathcal{R}$ be the index such that $x^{\bar{r}} = \bar{x}$ and consider two cases for the values returned by $\rho(x,r)$. First, in the case that $r = \bar{r}$:

$$\begin{aligned}
\rho(\bar{x},\bar{r}) &= (\theta_{\bar{r}} - L_r)\left(\sum_{i\in\mathcal{S}_{\bar{r}}} \bar{x}_i - \sum_{i\notin\mathcal{S}_{\bar{r}}} \bar{x}_i\right) - (\theta_{\bar{r}} - L_r)(|\mathcal{S}_{\bar{r}}| - 1) + L_r \\
&= (\theta_{\bar{r}} - L_r)(|\mathcal{S}_{\bar{r}}| - |\mathcal{S}_{\bar{r}}| + 1) + L_r = \theta_{\bar{r}} = \nu(x^{\bar{r}}) = \nu(\bar{x}).
\end{aligned}$$

Next, if $r \neq \bar{r}$, then

$$\begin{aligned}
\rho(\bar{x},r) &= (\theta_r - L_r)\left(\sum_{i\in\mathcal{S}_r} \bar{x}_i - \sum_{i\notin\mathcal{S}_r} \bar{x}_i\right) - (\theta_r - L_r)(|\mathcal{S}_r| - 1) + L_r \\
&= \theta_r - (\theta_r - L_r)\left(|\mathcal{S}_r| - \sum_{i\in\mathcal{S}_r} \bar{x}_i + \sum_{i\notin\mathcal{S}_r} \bar{x}_i\right) \\
&= \theta_r - (\theta_r - L_r)\left(\sum_{i\in\mathcal{S}_r}(1 - \bar{x}_i) - \sum_{i\notin\mathcal{S}_r} \bar{x}_i\right) \leq \theta_r - (\theta_r - L_r)\cdot 1 = L_r \leq \nu(\bar{x}),
\end{aligned}$$

where in the first inequality we exploited the fact that when $\bar{x}\neq x^r$, it must be that $\sum_{i\in\mathcal{S}_r}(1-\bar{x}_i) - \sum_{i\notin\mathcal{S}_r} \bar{x}_i \geq 1$, whereas we also have that $0 \leq \nu(x^r) - L_r = \theta_r - L_r$. We can thus conclude that :

$$\max_{r\in\mathcal{R}} \rho(\bar{x}, r) \leq \nu(\bar{x}) = \rho(\bar{x},\bar{r}) \leq \max_{r\in\mathcal{R}} \rho(\bar{x}, r),$$

so $\nu(\bar{x}) = \max_{r\in\mathcal{R}} \rho(\bar{x}, r)$.

□

Chapter 6

Conclusions

6.1 Summary of Research Themes

This dissertation explored four themes dealing with the optimization of ALM problems under uncertainty. The first theme provided a critical review of robust PSPs, identifying challenges and potential areas for future research. The second theme presented theoretical frameworks for developing worst-case CVaR for general loss functions that can be used in ALM problems where the loss function includes the present value of future liabilities and asset returns. Moreover, a moment-based ambiguity set is developed for the ALM problem to propose the DRO formulation of ALM with WCVaR. The third theme addressed DRO formulations of the ALM problem based on the mixture distribution function, box ambiguity sets of the discrete probability distribution, and the Wasserstein ambiguity set. The fourth theme introduced a double-oracle, logic-based Benders decomposition approach to solve the K-adaptability problem that can be used in ALM problems with discrete/binary decision variables. The conclusions and future research extensions for the four themes are discussed below.

While there has been a significant increase in research in the field of PSP under uncertainty in the last 20 years, driven by advances in RO, few recent reviews of the robust PSP literature exist to cover the mathematical and theoretical aspects of this problem. In Chapter 2 dealing with Theme 1, a state-of-the-art literature review of robust PSPs was conducted to provide a comprehensive understanding of the field and its advancements. The study examined recent research articles on robust PSP optimization using a systematic classification and analysis framework to classify and analyze the models. The review also identified the limitations and potential areas for future research in the field.

In Chapter 2, we identified some research gaps related to robust PSPs. Moreover, we highlighted that existing methods for addressing ALM problems are insufficient when it comes to incorporating uncertain parameters and discrete decision variables. Furthermore, the current theoretical frameworks do not adequately accommodate the loss function in the ALM problem, hindering the proposal of worst-case CVaR. Additionally, the literature lacks studies on DRO formulations of ALM problems due to the absence of applicable theoretical frameworks. These ALM research gaps are only some of the gaps identified through the review.

Theme 2 presents a theoretical framework for the development of a WCVaR formulation in the context of ALM problems. The theoretical foundation put forth in this study is applicable to a wide range of problems featuring general loss functions. By leveraging this theoretical framework, we introduced a data-driven DRO formulation of the ALM problem, where the loss function is a linear combination of asset returns and the present value of future liabilities. The DRO formulation of the ALM problem was proposed in two distinct cases. Firstly, we considered situations where the moments of the distribution function are fully known and fixed. Secondly, we explored the case where the moments of the distribution function of random variables are uncertain and belong to an uncertainty set. This formulation accounted for the inherent uncertainty associated with the underlying distributions and offers a more robust approach than SP of the ALM problem.

To evaluate the performance of the optimal investment strategies derived from solving the DRO reformulations in Chapter 3, we used real data from the CPP as a case study. The analysis encompasses both in-sample and out-of-sample performance assessments of the models. The in-sample performance evaluation compared the DRO formulation with the fixed moments, the DRO formulation with uncertain moments, and the traditional SP ALM model in terms of fund returns and funding ratios across different periods. The results demonstrated that the SP formulation of the ALM exhibits superior in-sample performance compared to the DRO formulation with fixed and uncertain moments with respect to fund returns and funding ratios within each period. However, the out-of-sample performance analysis revealed a different picture. The investment strategy derived from the DRO formulation with uncertain moments consistently outperformed both the DRO formulation with fixed moments and the SP model. It exhibited a higher overall average fund return and achieves a better funding ratio over time. Based on these findings, we can confidently conclude that the investment strategy derived from the DRO formulation of the ALM problem with uncertain moments offers a superior approach to managing the asset and liability balance of pension funds. The robustness of the DRO approach, combined with its ability to account for uncertain moments, enables more effective decision-making in ALM problems and contributes to the stability and success of pension fund management.

In theme 3, we tackled the inherent uncertainty in ALM problems by exploring three different DRO approaches: mixture ambiguity sets with discrete scenarios, box ambiguity sets for a discrete distribution function and Wasserstein ambiguity set. To validate our approach, we utilized data from the CPP to conduct a series of numerical experiments and tests, simulating various market scenarios and their impacts on the plan. Applying Monte Carlo simulation based on geometric Brownian motion, we generated scenarios of asset returns. The analysis revealed four distinct market regimes observed from November 2012 to November 2022.

We presented an in-sample performance analysis of four ALM models: `MD`, `BD`, `WM`, and `SP`, covering a 12-period timeframe. The funding ratio and fund return serve as crucial metrics for evaluating the solvency and investment performance of the fund. Overall, the results indicated that all four models perform well in terms of both funding ratio and fund return, with the `SP` model exhibiting the highest performance in most periods. However, variations in performance across models and periods exist, with the `BD` model adopting a more conservative approach. Furthermore, the `WM` and `BD` models demonstrated more diversified portfolios, posing less risk compared to the `SP` and `MD` models. This is attributed to the `WM` and `BD` models considering a broader range of possible probability distributions in their ambiguity sets, thus providing a more robust and stable asset allocation strategy over time. Conversely, the `MD` model considers a limited set of scenarios, while the `SP` model assumes complete knowledge of the discrete distribution function of uncertain parameters, asset returns, and the present value of future liabilities. Consequently, the results suggest that the `WM` and `BD` models are better suited for investors seeking less risky and diversified portfolios, whereas the `SP` and `MD` models may appeal to investors targeting higher returns at the expense of increased risk. In addition to the in-sample analysis, we conducted an out-of-sample evaluation of the four models: `MD`, `BD`, `WM`, and `SP`. The results indicated that the `BD` model achieves the highest average funding ratio, followed by `WM`, `MD`, and

SP. Moreover, the WM model exhibited the highest average fund return, followed by BD, MD, and SP. Consequently, these models offered valuable insights for investors and fund managers seeking to make optimal asset allocation decisions and manage portfolios efficiently.

Under Theme 4 in Chapter 5, we presented a new method for tackling K-adaptability problems. Departing from conventional approaches, our technique utilized a logic-based Benders decomposition to isolate the first-stage variables in a master problem. By doing so, we effectively transformed the remaining Benders subproblem, which takes the form of an MMMRCO problem. Iteratively solving this subproblem enabled the generation of Benders optimality cuts. To further enhance the efficiency and effectiveness of our approach, we devised a novel double-oracle algorithm. This algorithm engaged in a cyclical process that involves generating new adverse scenarios and assigning them to K-subsets of resources. It also focused on generating new recourse solutions. Notably, unlike other algorithms proposed in previous studies, our approach utilized MILP problems to generate recourse solutions. This choice facilitates a more streamlined and effective resolution process. To establish the validity and potential of our method, we have rigorously proven the finite convergence of the proposed approach. Additionally, we have demonstrated its versatility by extending it to address other significant variants of the MMMRCO and K-adaptability problems. Notably, our approach successfully tackled problems with second-stage constraint uncertainty, bounded first-stage integer decision variables, first-stage objective uncertainty (independent or dependent on the second-stage uncertainty), as well as problems with general convex objective and constraint functions. These extensions address limitations that existing algorithms have been unable to overcome.

To substantiate the practical advantages of our approach, we conducted comprehensive numerical experiments using various standard benchmark test problems drawn from the literature. The results unequivocally showcased the superior performance of our proposed technique when compared to state-of-the-art solution algorithms. We sought to validate the effectiveness of our proposed extensions by solving a novel adaptive ALM problem. This particular problem entailed constraint uncertainty in the recourse problem, as well as first and second-stage objective uncertainty. Our approach demonstrated its capabilities by handling large instances, involving up to 250 assets, in under 16 minutes on average. Moreover, when solving large instances of the shortest path problem with 50 nodes and $K = 6$, our approach outperformed alternative algorithms by achieving an average solution time of just 22 seconds. Similarly, when tackling large instances of the adaptive knapsack problem with $n = 300$ and K ranging up to 6, our technique consistently outperformed other algorithms. Even in cases where our algorithm did not meet the cutoff time, it consistently provided stronger bounds than the benchmark algorithms. Moreover, when confronted with large-scale K-adaptability problems involving real first-stage variables, our approach outperformed the existing `MILP` and `BB` approaches in terms of solution times. In summary, the last theme introduces a groundbreaking approach to solving K-adaptability problems. Through novel methodologies and rigorous experimentation, we have demonstrated its superiority over existing algorithms and highlighted its potential for addressing various problem variants. By offering more efficient and effective solutions, our approach paves the way for advancements in decision-making under uncertainty and resource allocation optimization.

6.2 Managerial Insights

The realm of ALM within pension funds and investment portfolios is complex, especially in the presence of uncertainty. This study delves into a comprehensive analysis of ALM problems, offering valuable insights that can significantly impact decision-making processes for fund managers and investors.

6.2.1 Key Takeaways

The research underscores the importance of a data-driven approach to ALM. By incorporating real-world data from the CPP, the study demonstrates the practical application of theoretical frameworks. This approach aligns investment strategies with actual market dynamics, improving the accuracy of fund performance evaluation.

The introduction of the DRO formulation represents a significant advancement in ALM. This approach acknowledges the uncertainty inherent in distributions, providing a robust and effective alternative to traditional SP. The ability to adapt to uncertain moments contributes to the resilience of investment strategies, particularly during turbulent market periods.

The comparison between different DRO formulations (MD, BD, WM) and SP highlights the value of diversification in managing risk. The models with broader ambiguity sets exhibit more stable and less risky asset allocation strategies over time. This insight can guide investors seeking a balance between risk and return.

The study's out-of-sample analysis reveals the long-term efficacy of the DRO formulation with uncertain moments. While traditional SP may excel in in-sample performance, the DRO approach consistently outperforms in terms of average fund return and funding ratio over time. This finding highlights the importance of considering the broader market context for sustainable performance.

The study's approach extends beyond ALM problems, showcasing the efficacy of logic-based Benders decomposition and the double-oracle algorithm in addressing K-adaptability problems. The technique's adaptability to diverse scenarios and its ability to provide efficient solutions make it a promising tool for complex decision-making under uncertainty.

6.2.2 Managerial Implications

- Enhanced Decision-Making: Fund managers can leverage the DRO formulation with uncertain moments to develop more robust investment strategies. This approach facilitates effective ALM by accounting for market volatility and uncertainty, ultimately leading to more stable and sustainable fund performance.

- Risk-Adjusted Strategies: Investors seeking to manage risk while aiming for attractive returns should consider models with broader ambiguity sets, such as BD and WM. These models provide diversified portfolios that can weather market fluctuations more effectively.

- Long-Term Perspective: While SP may demonstrate superior in-sample performance, the study emphasizes the need for a long-term perspective. Investments informed by the DRO formulation with uncertain moments exhibit resilience over time, making them valuable choices for investors focused on sustained growth.

- Adaptability to Complex Scenarios: The introduced methodology offers a versatile solution for various decision-making problems beyond ALM. Decision-makers facing intricate resource allocation challenges and uncertainty can benefit from this approach's efficiency and effectiveness.

6.3 Future Research Direction

For future research endeavors, we suggest exploring potential extensions of the double-oracle, logic-based Benders decomposition algorithm to solve the K-adaptability problem to encompass general RO problems featuring combinatorial recourse, as well as K-adaptability problems incorporating continuous first-stage variables. It is worth noting that the proposed approach encounters a bottleneck in solving the p-center assignment problem, particularly when dealing with large sets of scenarios and recourse solutions. Thus, it becomes crucial to investigate methods that effectively address this problem, ultimately enhancing the overall efficiency of the proposed algorithm. Another avenue for improvement lies in the combinatorial Benders decomposition, which is known to exhibit slow convergence due to the generation of weak cuts. Introducing enhancement techniques to the Benders decomposition methodology holds significant potential in substantially improving the performance of the proposed approach. These research directions offer promising avenues for future investigations and hold the potential to advance the capabilities and efficiency of our proposed algorithm.

Another important avenue for future research in this context is to explore the application of sequential robust optimization techniques to address the multi-stage nature of the ALM problem. The ALM problem typically involves decision-making over multiple time periods, where decisions made in earlier stages can have a significant impact on future stages. Sequential robust optimization provides a framework for making decisions sequentially while considering uncertainty in future stages (Havinga et al., 2017; Houska & Diehl, 2013). By incorporating this approach into the ALM problem, researchers can develop more dynamic and adaptive strategies that account for evolving market conditions and changing investment opportunities over time.

Another potential direction for research is to investigate the integration of robust optimization within a rolling horizon framework. In this approach, the ALM problem is solved iteratively over a series of time periods, with decisions being made for each period based on the available information at that time. As new information becomes available, the optimization problem is re-evaluated and updated decisions are made, taking into account the evolving uncertainty based on robust approximate dynamic programming (Jiang & Jiang, 2013; Mei et al., 2022; Wei et al., 2021).

Moreover, while the performance of the proposed models has been analyzed using real-world data from the CPP, conducting further testing on a diverse set of pension funds can offer a better understanding of the generalizability and robustness of the proposed models. Lastly, the proposed models can be extended to incorporate other important considerations in pension fund management, including taxes, transaction costs, and regulatory constraints. These extensions can contribute to a more comprehensive framework for pension fund management that can effectively handle a wider range of real-world constraints and provide more realistic and actionable strategies.

By addressing these avenues for future research, scholars can deepen their understanding of ALM under uncertainty and contribute to the development of more sophisticated and practical models for managing pension funds.

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