GROUND STATE DEGENERACY IN THE TWISTED SECTORS OF CONWAY MOONSHINE

by

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To Gus and George.

Contents

Abstract

We discover a new and unexpected pattern in the ground state degeneracy of the N = 1 Conway Moonshine SVOA twisted sectors. The main goal of this thesis is to calculate the NS- and R- characters of the Co_0-twisted R-sector of $V^{f\natural}$. We find empirically that in Conway Moonshine, for about 95% of possible Co_0-twistings, all ground states have the same fermionic parity, whereas 5% of them have ground states of bosonic and fermionic parities. We prove that the characters of the Ramond-sector twisted by an action of Co_0 must be constant. Additionally, we prove that the Ramond-character of the g-twisted R-sector is equal to the trace of g.

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Chapter 1

Introduction

"I can explain what ... Moonshine is in one sentence, it is the voice of God."

— Simon P. Norton

The concept of "Moonshine" began with a celebrated observation involving empirical features of exceptional groups: a remarkable relationship between the Monster finite simple group $\mathbb{M}$ and the theory of modular functions [37].

This led to a whole mathematicalization of conformal field theory, known under the name vertex operator algebra (VOA). Vertex operator algebras are an axiomatization of a two-dimensional conformal field theory [40]. These days, VOAs are important throughout mathematics and particularly appear in geometric representation theory. In the early days, vertex operators describing the propagation of string states appeared in string theory. Analogs of vertex operators were found in the representation theory of affine Kac-Moody algebras (essentially the Lie algebras of loop groups). The notion of vertex operator, first generalized in 1986 by Richard Borcherds in [4], was motivated by the construction of an infinite-dimensional Lie algebra in [32]. A modified version, that of vertex operator algebra, was introduced in [33, 34] by Frankel, Lepowksi, and Meurman in 1988, where they constructed what is notably the most famous of all vertex operator algebras: the monstrous Moonshine module $V^\natural$ (see Appendix C).

A number of closely related Moonshines have been constructed for the children of the Monster (Appendix B), and they are all largely described by the physics of $V^\natural$. For the children of Conway's largest sporadic group, modern Moonshines have also been found, albeit most of them still lack a detailed physical description ([29], [11], [16], [27] are nice examples). These are all hoped to be explained by an intriguing supersymmetric conformal field theory $V^{f\natural}$ uncovered by Duncan [28]. Encouraged by the existence and conjectured uniqueness of $V^\natural$, Duncan established an analogous super VOA $V^{f\natural}$, hereafter called Conway Moonshine, whose automorphism group is

Conway's group Co_1.

Mathematicians continue to make progress in these questions by doing empirical mathematics. By compiling interesting data about these structures, we search and hope for surprising patterns that uncover parts of the narrative. Having a pretty good visualization of some parts of the story, physics can help guide us to possibly more interesting places to look.

In this thesis, we discover a brand-new and unexpected pattern in Conway Moonshine. The most likely behavior of a supersymmetric field theory is that it will produce a single isolated ground state for each conceivable twisting. If there is more than one ground state, there will typically be both bosonic and fermionic ground states. The presence of a second symmetry, which prevents the ground state degeneracy from being broken in the manner of a fine-structure constant, typically accounts for the occurrence of these numerous ground states. In the majority of similar physical situations, this additional symmetry is a supersymmetry, which forces the same number of bosonic and fermionic ground states. No physical mechanism that is currently understood would compel a ground state degeneracy where all ground states have the same fermion parity. Nevertheless, we find empirically that in Conway Moonshine, for about 95% of possible twistings, all ground states have the same fermion parity, whereas 5% of them have ground states of both parities (Appendix A).

We begin this thesis with an exploration of supersymmetry (§2.1). We review quantum mechanics models and define their partition functions (§2.2). We follow with a review of the principal definitions of vertex operator algebras (§3.1) and their corresponding structures and concepts (§3.2-3). Some important groups and properties that lead to the uncovering of Moonshine are presented (§4.1). We review the concept of lattice SVOAs and derive their characters (§4.2). Following a brief history of its research and discovery, we define Conway Moonshine (§4.4). We then calculate the Neveu-Schwarz and Ramond characters of the Co_0-twisted Ramond sector of $V^{f\natural}$ (§5).

Chapter 2

Background

2.1 Supersymmetry

The leitmotif of this paper is the concept of supersymmetric objects, mainly super vertex operator algebras. All experimentally known symmetries in nature relate bosons to bosons and fermions to fermions. Theoretical physicists have hypothesized the existence of a type of symmetry, called *supersymmetry*, that relates bosons and fermions [49]. In the physical sense, supersymmetry is a spacetime symmetry between two fundamental types of particles, bosons and fermions, which have integer-valued spin and half-integer-valued spin respectively. According to supersymmetry, every particle from one class would have a *superpartner* particle from the other whose spin differs by a half-integer.

For any group G, a *G-graded algebra* is an algebra A whose additive group can be represented in the form of a direct sum of groups $\{A_g\}_{g\in G}$ where $A_g A_h \subseteq A_{g+h}$ for any $g, h \in G$:

$$A = \bigoplus_{g\in G} A_g \tag{2.1.0.1}$$

The basic idea of super mathematics is that all objects are $\mathbb{Z}_2$-graded, meaning they can be decomposed into "even" and "odd" pieces. An element of the even piece is called a *boson* and an element of the odd piece is called a *fermion*.

One may regard $\mathbb{Z}_2$-grading as splitting an object into its even and odd parts. For example, the complex numbers $\mathbb{C}$ are a $\mathbb{Z}_2$-graded algebra over the real numbers $\mathbb{R}$. The degree 0 (even) elements of $\mathbb{C}$ are the purely real numbers, and the degree 1 (odd) elements of $\mathbb{C}$ are the purely imaginary numbers.

Let $p(a)$ represent the *parity*, 0 or 1, of a homogeneous element in a super object. The general principle to extending axioms of algebraic objects onto super algebraic objects is straightforward. In a formula, if there are monomials of elements with interchanged terms, then in the equivalent formula in the super setting, every interchange of neighboring terms, say a and b, is multiplied by a factor of $(-1)^{p(a)p(b)}$. For example, in any algebra, the commutator is defined by $[a, b] := ab - ba$. Whereas, in a superalgebras the supercommutator is defined as $[a, b] := ab - (-1)^{p(a)p(b)}ba$. An algebra A is *commutative* if the commutator vanishes for all $a, b \in A$, i.e. $[a, b] = 0$. A superalgebra A is *supercommutative* if the supercommutator vanishes for all $a, b \in A$. It is easy to see that, although $\mathbb{C}$ is commutative, it is not supercommutative with its nontrivial $\mathbb{Z}_2$-grading.

In this thesis, any definition where the term "super" is omitted can be turned into a super object by including a $\mathbb{Z}_2$-grading and the above sign convention.

2.2 Partition Functions

The *spectrum* of an operator A is a generalization of the set of eigenvalues. It is the set $\sigma(A)$ of values α for which the operator $A - \alpha I$ is not invertible, where I is the identity operator. A quantum mechanics model consists of a complex *Hilbert space* with a *Hamiltonian* operator, representing the total energy of our model. The spectrum of a Hamiltonian $\hat{H}$ on a Hilbert space $\mathcal{H}$ is required to be real and bounded below by definition. The *ground state* is the eigenspace of minimal eigenvalue. The dimension of this eigenspace is called the *ground state degeneracy.* In the Schrödinger picture, $\hat{H}$ generates the time evolution of quantum states. Working in Planck units and letting $\hbar = 1$, the *Minkowski time-evolution operator*

$$U_t = e^{it\hat{H}}, \tag{2.2.0.1}$$

is unitary and does not have a well-defined trace. There is also a *Euclidean time-evolution operator*, for purely imaginary t such that $t = is$,

$$U_s = e^{-s\hat{H}}. \tag{2.2.0.2}$$

Assuming U_s is trace-class for all $s > 0$, the function

$$Z(\mathcal{H}) : s \longmapsto tr_{\mathcal{H}}(U_s) \tag{2.2.0.3}$$

is called the *partition function*. Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a super Hilbert space where $\mathcal{H}^+$ and $\mathcal{H}^-$ are the bosonic and fermionic states respectively. We define the operator $(-1)^f : \mathcal{H} \to \mathcal{H}$, which acts as +id on $\mathcal{H}^+$ and -id on $\mathcal{H}^-$. The Hamiltonian $\hat{H}$ of $\mathcal{H}$ is required to be a bosonic operator corresponding to the total energy of $\mathcal{H}$. It maps $\mathcal{H}^+$ to $\mathcal{H}^+$ and $\mathcal{H}^-$ to $\mathcal{H}^-$. Since we are in a super Hilbert space, there are two different traces we can take: the one that adds the even and odd parts, and the one that subtracts them. We call the first the *Neveu-Schwarz* partition function

$$Z^{NS}(\mathcal{H}) = tr_{\mathcal{H}}(e^{-s\hat{H}}), \tag{2.2.0.4}$$

and the second the *Ramond* partition function

$$Z^R(\mathcal{H}) = tr_{\mathcal{H}}((-1)^f e^{-s\hat{H}}). \tag{2.2.0.5}$$

A *supersymmetry* operator $\hat{G}$ is a fermionic operator, i.e. it maps $\mathcal{H}^+$ to $\mathcal{H}^-$ and $\mathcal{H}^-$ to $\mathcal{H}^+$:

$$(-1)^f \psi = \psi \tag{2.2.0.6}$$

for $\psi \in \mathcal{H}^+$ and

$$(-1)^f \varphi = -\varphi \tag{2.2.0.7}$$

for $\varphi \in \mathcal{H}^-$. Equivalently, $\hat{G}$ must anticommute with $(-1)^f$:

$$(-1)^f \hat{G} + \hat{G}(-1)^f = 0, \tag{2.2.0.8}$$

$$(-1)^f \hat{G} = -\hat{G}(-1)^f. \tag{2.2.0.9}$$

Additionally, $\hat{G}$ must commute with the Hamiltonian operator $\hat{H}$,

$$\hat{G}\hat{H} - \hat{H}\hat{G} = 0.$$

Finally, from $N =$ *1 supersymmetric quantum mechanics*, we require

$$\hat{G}^2 = \hat{H}. \tag{2.2.0.10}$$

Theorem 2.2.1. For some super Hilbert space $\mathcal{H}$ with Hamiltonian $\hat{H}$, if a supersymmetry operator $\hat{G}$ exists, $\mathrm{Z}^R(\mathcal{H})$ is a constant.

We will give two proofs:

Proof. (version 1) The spectrum of $\hat{H}$ is in $\mathbb{R}$. We diagonalize $\hat{H}$ and for each eigenvalue λ of $\hat{H}$, denote by $\mathcal{H}_\lambda$ the eigenspace of λ. Since $\mathcal{H}$ is a super Hilbert space and $\hat{H}$ preserves the super structure, $\mathcal{H}_\lambda$ decomposes as a sum of bosonic and fermionic pieces, $\mathcal{H}_\lambda = \mathcal{H}_\lambda^+ \oplus \mathcal{H}_\lambda^-$. Letting $q = e^{-s\hat{H}}$ the trace function is then $\sum_\lambda q^\lambda(\dim(\mathcal{H}_\lambda^+) \pm \dim(\mathcal{H}_\lambda^-))$, where addition is Neveu-Schwarz and subtraction is Ramond as usual. Now, $\hat{G}$ commutes with $\hat{H}$ so $\hat{G}$ preserves the eigenspaces. Also, $\hat{G}^2 = \hat{H}$, so, on the λ eigenspace, $\hat{G}^2$ is multiplication by λ. Then, if $\lambda \neq 0$, $\hat{G}^2$ is an isomorphism so $\hat{G}$ is also an isomorphism. Recall that $\hat{G}$ is odd in the sense that it sends $\mathcal{H}_\lambda^+$ to $\mathcal{H}_\lambda^-$ and $\mathcal{H}_\lambda^-$ and $\mathcal{H}_\lambda^+$. So, if $\hat{G}$ is an isomorphism, we have $\dim(\mathcal{H}_\lambda^+) = \dim(\mathcal{H}_\lambda^-)$. It follows that the Ramond trace is $Z^R(\mathcal{H}) = q^0(\dim(\mathcal{H}_0^+) - \dim(\mathcal{H}_0^-))$, which is a constant. □

Proof. (version 2) Recall that $Z^R(\mathcal{H}) = \mathrm{tr}_\mathcal{H}((-1)^f e^{-s\hat{H}})$ and $Z^{NS}(\mathcal{H}) = \mathrm{tr}_\mathcal{H}(e^{-s\hat{H}})$. and take the derivatives. Now, we take the partial derivatives in s and we obtain:

$$\begin{aligned}
\frac{\partial Z^{\overset{\mathrm{NS}}{\mathrm{R}}}(\mathcal{H})}{\partial s} &= \frac{\partial}{\partial s}\mathrm{tr}_\mathcal{H}((\pm 1)^f e^{-s\hat{H}}) \\
&= -\mathrm{tr}_\mathcal{H}((\pm 1)^f e^{-s\hat{H}}\hat{H}) \\
&= -\mathrm{tr}_\mathcal{H}((\pm 1)^f e^{-s\hat{H}}\hat{G}^2) && \hat{G}^2 = \hat{H} \\
&= \mathrm{tr}_\mathcal{H}(\hat{G}(\pm 1)^f e^{-s\hat{H}}\hat{G}) \\
&= \pm\mathrm{tr}_\mathcal{H}((\pm 1)^f \hat{G} e^{-s\hat{H}}\hat{G}) && \hat{G}(\pm 1)^f = \pm(\pm 1)^f\hat{G} \\
&= \pm\mathrm{tr}_\mathcal{H}((\pm 1)^f e^{-s\hat{H}}\hat{G}^2) \\
&= \pm\mathrm{tr}_\mathcal{H}((\pm 1)^f e^{-s\hat{H}}\hat{H}) \\
&= \pm\frac{\partial Z^{\overset{\mathrm{NS}}{\mathrm{R}}}(\mathcal{H})}{\partial s}
\end{aligned}$$

Then,

$$Z^{NS}(\mathcal{H}) = \mathrm{tr}_\mathcal{H}((\pm 1)^f e^{-s\hat{H}}\hat{H})$$

and

$$Z^R(\mathcal{H}) = -\mathrm{tr}_{\mathcal{H}}((\pm 1)^f e^{-s\hat{H}}\hat{H}).$$

Thus, $\frac{\partial}{\partial s}Z^R(\mathcal{H}) = -\frac{\partial}{\partial s}Z^R(\mathcal{H}) = 0$ and $Z^R(\mathcal{H})$ is a constant. □

Proof 1 moreover shows that $Z^R(\mathcal{H}) \in \mathbb{Z}$. This integer is called the *Witten index* of $\mathcal{H}$ [56].

Since $\hat{H}$ is a square, its spectrum is bounded below by 0. Thus, if $\hat{G}$ exists, then $Z^{NS}(\mathcal{H})$ is smooth at $s = 0$ and $Z^{NS}(\mathcal{H})(s = 0)$ counts the number of states of eigenvalue 0. Since $Z^{NS}(\mathcal{H})$ adds the bosons and fermions to each other if $Z^{NS}(\mathcal{H})(s = 0) = 0$, then the ground state is strictly above zero and we have an equal number of boson and fermion ground states. Now, if $Z^{NS}(\mathcal{H})(s = 0) \neq 0$, then we can retrieve the number of boson and fermion ground states via

$$\#\text{bosons} = \frac{Z^{NS}(\mathcal{H})(s = 0) + Z^R(\mathcal{H})}{2}, \tag{2.2.0.11}$$

and,

$$\#\text{fermions} = \frac{Z^{NS}(\mathcal{H})(s = 0) - Z^R(\mathcal{H})}{2}. \tag{2.2.0.12}$$

Chapter 3

Super Vertex Operator Algebras

3.1 Super Vertex Operator Algebras

Although there is no mathematically concrete definition of a quantum field theory (QFT) in general, we can define 2-dimensional conformal field theories and topological QFTs [3]. In conformal field theory, we deal with vertex operators which are analogous to operators in QFT. This means we can write Taylor-like expansions of two vertex operators, called the *operator product expansions* (OPE), which gives the QFT analog of two fields interacting.

We will only consider super vertex operator algebras (SVOA) of "CFT-type". Such a SVOA comes with a grading by $\frac{1}{2}\mathbb{Z} = \{\frac{n}{2} \mid n \in \mathbb{Z}\}$:

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$$

that we call *spin*. The spin-0 space V_0 is spanned by the vacuum vector.

For mathematicians, some notable characteristics of vertex algebras make them stand out from other algebras. Being motivated by theoretical physics, the definition of vertex algebra can be tedious and complicated. As such, many papers have slightly differing definitions. In [19], De Sole and Kac laid 5 definitions of vertex algebras. We use the one more commonly used by Duncan in his papers [25, 27],.

Suppose V is a vector space over $\mathbb{C}$ which admits a $\frac{1}{2}\mathbb{Z}$-grading $V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$ [16], which we call spin. By letting

$$V_{\bar{0}} = \bigoplus_{n \in \mathbb{Z}} V_n \qquad V_{\bar{1}} = \bigoplus_{n \in \frac{1}{2}+\mathbb{Z}} V_n$$

we define a natural $\mathbb{Z}_2$-grading [41]. We say $V_{\bar{0}}$ is the bosonic (even) part of V and its elements are bosons. Respectively, $V_{\bar{1}}$ is the fermionic (odd) part of V and its elements are fermions. An element $a \in V$ has *parity* $p(a) \in \mathbb{Z}_2$ if $a \in V_{\overline{p(a)}}$. If $\dim V < \infty$, then we let the *superdimension* of V be sdim $V = \dim V_{\bar{0}}$ - $\dim V_{\bar{1}}$. Superdimension then calculates the difference between the bosons and the fermions.

Let U be a vector space. We write $U[[z^{\pm 1}]]$ for the vector space series

$$U[[z^{\pm 1}]] = \left\{ \sum_{n\in\mathbb{Z}} u_n z^n \mid u_n \in U \right\}$$

whose elements are bi-infinite formal power series in $z^{\pm 1}$ with coefficients in U. Additionally, we denote $U((z))$ for the vector space

$$U((z)) = U[[z]][z^{-1}] = \bigcup_{N\to-\infty} \left\{ \sum_{n=N}^{\infty} u_n z^n \mid u_n \in U \right\}$$

whose elements are Laurent series in z with coefficients in U. We take U to be an associative algebra with identity such that multiplication is defined. From now on, we will usually fix $U = \mathrm{End}(V)$.

A formal power series

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-(n+1)} \qquad \in \mathrm{End}(V)[[z^{\pm 1}]] \tag{3.1.0.1}$$

where $a_{(n)} \in$ End(V) and z is a formal variable, is called a *field* or *vertex operator* if, for all $v \in V$, there is a K such that

$$a_{(n)}v = 0 \qquad \text{for all } n \geq K. \tag{3.1.0.2}$$

A field $Y(a,z)$ has *parity* $p(a) \in \mathbb{Z}_2$ if

$$a_{(n)}V_\alpha \subset V_{\alpha+p(a)}$$

for all $\alpha \in \mathbb{Z}_2$ and $n \in \mathbb{Z}$. We denote the *space of fields* $\mathcal{E}(V)$. Note that $\mathcal{E}(V)$ is closed under the formal derivative $\frac{\partial}{\partial z}$.

A *super vertex algebra*, SVA, is the following collection of data:

a *space of states* — a superspace V;

a *vacuum vector* — a vector $|0\rangle \in V_{\bar{0}}$;

an *infinitesimal translation operator* — an even parity preserving endomorphism $T : V \mapsto V$ that raises the $\frac{1}{2}$-grading by 1

a *state-operator* correspondence — a parity preserving linear map of V to the space of fields

$$\begin{aligned} Y(\cdot, z)\colon V &\to \mathcal{E}(V) \\ a &\mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-(n+1)} \end{aligned}$$

The state-operator correspondence provides a product operator defined by

$$\begin{aligned} V \otimes V &\to V((z)) \\ a \otimes b &\mapsto Y(a, z)b = \sum_{n\in\mathbb{Z}} a_{(n)} b z^{-(n+1)} \end{aligned}$$

Super vertex algebras must satisfy the following axioms.

Axiom 3.1.0.1. (SVA1.) For all $a \in V$,

(1.1) $Y(|0\rangle, z) = \mathrm{Id}_V$,

(1.2) $Y(a, z)|0\rangle|_{z=0} = a$,

(1.3) $Y(a, z) = 0$ if and only if $a = 0$.

The above implies that if $Y(a, z)b = 0$ for all $b \in V$, then $a = 0$.

Axiom 3.1.0.2. (SVA2.) For all $a \in V$, $[T, Y(a, z)] = \partial_z Y(a, z)$, $T|0\rangle = 0$ where $T \in \mathrm{End}(V)$ is defined by

$$T(a) = a_{(-2)}|0\rangle.$$

Axiom 3.1.0.3. (SVA3.) For all a and $b \in V$, $Y(a, z)$ and $Y(b, z)$ are mutually local, meaning

$$(z - w)^N Y(a, z)Y(b, w) = (-1)^{p(a)p(b)}(z - w)^N Y(b, w)Y(a, z)$$

for $N \gg 0$.

Axiom 3.1.0.4. (Jacobi Identity, SVA4.) For any $a, b \in V$,

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y(a, z_1)Y(b, z_2) - (-1)^{p(a)p(b)}z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y(b, z_2)Y(a, z_1)$$
$$= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y(Y(a, z_0)b, z_2), \tag{3.1.0.3}$$

where δ is the formal series

$$\delta(\frac{y - x}{z}) := \sum_{s\geq 0, r\in\mathbb{Z}} \binom{r}{s}(-1)^s y^{r-s}x^s z^{-r}. \tag{3.1.0.4}$$

A *super vertex operator algebra* of central charge $c \in \mathbb{C}$ is an SVA equipped with a distinguished vector ω called the *conformal* or *Virasoro vector*. The conformal vector is even such that

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-(n+2)} = \sum_{n\in\mathbb{Z}} \omega_{(n)}z^{-(n+1)}.$$

Super vertex operator algebras must satisfy the four main axioms of super vertex algebras along with the following.

Axiom 3.1.0.5. (SVOA1.) Writing $L_n := \omega_{(n+1)}$, the operators $\{L_n\}_{n\in\mathbb{Z}} \in \mathrm{End}(V)$ satisfy

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \tag{3.1.0.5}$$
$$[L_m, c] = 0.$$

for some $c \in \mathbb{C}$.

This axiom is can be translated as the $\{L_n\} \in \mathrm{End}(V)$ furnishing a representation of the Virasoro algebra. The *Virasoro algebra* is defined by the basis elements $\{L_n, c\}_{n\in\mathbb{Z}}$, where c is a *central charge* and L_n are *Virasoro modes*. It satisfies axiom 3.1.0.5.

Axiom 3.1.0.6. (SVOA2.) For all $a \in V$, $[L_{-1}, Y(a, z)] = Y(L_{-1}a, z) = \partial_z Y(a, z)$, for $a \in V$, meaning

$$T = L_{-1}.$$

Axiom 3.1.0.7. (SVOA3.) The operator L_0 is diagonalizable on V with eigenvalues contained in $\frac{1}{2}\mathbb{Z}$ and bounded below and the eigenspaces are finite-dimensional.

The L_0-eigenvalue of an eigenvector $a \in V$ is called the *conformal weight* of a, denoted h_a. Note that $h_{|0\rangle} = 0$ and $h_\omega = 2$.

An automorphism of V is a vector space automorphism $g : V \longrightarrow V$, such that

$$gY(a, z)g^{-1} = Y(ga, z), \tag{3.1.0.6}$$

$$g\omega = \omega. \tag{3.1.0.7}$$

Let V be an SVOA of central charge c. The character of V is the series

$$Z(V) = tr_V q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_n dim(V_n) q^n, \tag{3.1.0.8}$$

for $q = e^{2i\pi\tau}$ and $\tau \in \mathbb{H}$. One can also consider a character that is twisted by an automorphism g of V,

$$Z_g(V) = tr_V(gq^{L_0 - \frac{c}{24}}), \tag{3.1.0.9}$$

for $q = e^{2i\pi\tau}$ and $\tau \in \mathbb{H}$. For all SVOAs considered in this thesis, these characters converge absolutely with radius 1 [57]. Following [9], we nevertheless treat them as formal power series. Notice here that these characters use the mathematical notions of "trace" and "dimension". Accordingly, some authors will prefer those terms over "character" depending on the context.

An *$N = 1$ super vertex operator algebra* is an SVOA of central charge $c \in \mathbb{C}$ and conformal vector ω equipped with a distinguished spin-$\frac{3}{2}$ vector τ called the *superconformal vector*. The superconformal vector is odd, so

$$Y(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \tau_n z^{-(n+1)} = \sum_{n \in \mathbb{Z}} G_n z^{-(n+\frac{3}{2})}.$$

The N = 1 SVOA satisfy the axioms of SVOAs along with:

Axiom 3.1.0.8. (SVOA4.) Writing $L_m := \omega_{(m+1)}$, and $G_n := \tau_{(n+\frac{1}{2})}$, $\{L_m\}_{m \in \mathbb{Z}}$ and $\{G_n\}_{n \in \mathbb{Z}} \in \mathrm{End}(V)$ satisfy

$$[L_m, G_n] = \frac{m - 2n}{2} G_{m+n}, \tag{3.1.0.10}$$

$$[G_m, G_n] = 2L_{m+n} + \frac{4m^2-1}{12}\delta_{m+n,0}c, \qquad (3.1.0.11)$$

for the central charge $c \in \mathbb{C}$.

We remind the reader that G is odd, so

$$[G_m, G_n] = -(-1)^{p(G_m)p(G_n)}[G_n, G_m] = [G_n, G_m].$$

Axiom 3.1.0.8 is translated as the $\{L_m\}_{m\in\mathbb{Z}}$ and $\{G_n\}_{n\in\mathbb{Z}+\frac{1}{2}} \in \mathrm{End}(V)$ furnish a representation of the *Neveu-Schwarz algebra*.

An SVOA V may not have a superconformal vector. If it exists, a superconformal vector for an SVOA V is generally not unique. In particular, if $\tau \in V_{\bar{1}}$ is a superconformal element, then $\frac{1}{2}\tau_0\tau = \frac{1}{2}G_{-\frac{1}{2}}\tau \in V_{\bar{0}}$ is a conformal element. Accordingly, we always assume that a superconformal element for V is chosen such that $\frac{1}{2}G_{-\frac{1}{2}}\tau = \omega$.

The N = 1 SVOAs are fully defined by the following theorem.

Theorem 3.1.1. [40] Let V be an SVOA with a conformal element ω and central charge. Then $\tau \in V_{3/2}$ is a superconformal element if and only if, for $Y(\tau, z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}} z^{-(n+2)}$, the following properties hold:

(i) $G_0\tau = 2\omega$,

(ii) $G_2\tau = \frac{2}{3}c|0\rangle$,

(iii) $G_k\tau = 0$ for $k > 2$.

Further readings on SVOAs and their properties can be done in [40] and [41].

3.2 Modules

Let V be an SVOA. An *SVOA-module* is a superspace $M = M_{\bar{0}} \oplus M_{\bar{1}}$ along with a parity preserving linear map

$$\begin{aligned} &V \to \mathcal{E}(M) \\ &a \mapsto Y^{(M)}(a, z) = \sum_{n\in\mathbb{Z}} a^{(M)}_{(n)} z^{-(n+1)} \end{aligned}$$

The state-operator correspondence provides a product operator defined by

$$\begin{aligned} V \otimes M &\to M((z)) \\ a \otimes u &\mapsto Y^{(M)}(a,z)u = \sum_{n\in\mathbb{Z}} a^{(M)}_{(n)} u z^{-(n+1)} \end{aligned}$$

Modules must satisfy the following axioms.

Axiom 3.2.0.1. (SVAOM1.) The vertex operator $Y^{(M)}(|0\rangle, z)$ is the identity on M:

(1) $Y^{(M)}(|0\rangle, z) = Id_M$,

(2) $Y^{(M)}(a, z)|0\rangle|_{z=0} = a^{(M)}$,

(3) $Y^{(M)}(a, z) = 0$ if and only if $a = 0$.

Axiom 3.2.0.2. (Jacobi identity, SVAOM2.) For any $a, b \in V$,

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y^{(M)}(a, z_1)Y^{(M)}(b, z_2) - (-1)^{p(a)p(b)} z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y^{(M)}(b, z_2)Y^{(M)}(a, z_1)$$
$$= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y^{(M)}(Y(a, z_0)b, z_2), \tag{3.2.0.1}$$

where δ is the series defined in axiom 3.1.0.4.

If $a \in V_{\bar{0}}$, then $a^{(M)}_{(n)}$ is a bosonic operator for all $n \in \mathbb{Z}$. Respectively, if $a \in V_{\bar{1}}$, then $a^{(M)}_{(n)}$ is a fermionic operator for all $n \in \mathbb{Z}$.

When we write $Y^{(M)}(a, z)u = \sum_{n\in\mathbb{Z}} a^{(M)}_{(n)} u z^{-(n+1)}$, we have $a^{(M)}_{(n)} u \in M_{p(a)+p(u)}$ when $a \in V_{p(a)}$ and $u \in M_{p(u)}$. Also, $Y^{(M)}(a, z)u = 0$ for all $u \in M$ implies that $a = 0$ [25].

The V-module pair $\{M, Y^{(M)}\}$, or just M when the context is clear, is called an *admissible V-module* if the additional axiom is satisfied:

Axiom 3.2.0.3. (SVOAM3.) Setting $Y^{(M)}(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-(n+2)}$, we have

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \tag{3.2.0.2}$$

$$Y^{(M)}((L_{-1}a, z) = \partial_z Y^{(M)}(a, z), \tag{3.2.0.3}$$

for all $a \in V$.

An admissible V-module is *irreducible* if it admits no nontrivial proper graded submodules. We say an SVOA V is *rational* if any admissible V-module M can be decomposed as the direct sum of irreducible admissible V-modules. We say V is *simple* if, up to isomorphism, V is the only irreducible admissible module over itself. An SVOA V is *holomorphic* if V is rational, simple, and irreducible. We emphasize here that all SVOAs treated in this thesis are holomorphic.

Further notions of modules, submodules, quotient modules, etc., work in the usual way. For further readings on modules and their properties, see [24].

3.2.1 Twisted Modules

One can define modules that are twisted by a symmetry of the vertex algebra. We define here the general case and focus on a special case later. Let V be an SVOA and g be an automorphism of finite order k. A *g-twisted module* or simply *twisted module* of V is a superspace $M = M_{\bar{0}} + M_{\bar{1}}$ along with a parity preserving map

$$\begin{aligned} &V \to \mathrm{End}(M)[[z^{\pm\frac{1}{k}}]] \\ &a \mapsto Y_g^{(M)}(a, z^{\frac{1}{k}}) = \sum_{n\in\frac{1}{k}\mathbb{Z}} a_{(n),g}^{(M)} z^{-(n+1)} \end{aligned}$$

such that $Y_g^{(M)}(a,z)b \in M((z^{\frac{1}{k}}))$ for all $a \in A$ and $b \in M$, such that the following axioms hold.

Axiom 3.2.1.1. (TSVOAM1.) Setting

$$V^m = \{v \in V | gv = e^{-2\pi i \frac{m}{k}} v\},$$

for $0 \leq m \leq k-1$, we have

$$Y_g^{(M)}(a, z^{\frac{1}{k}}) = \sum_{n\in\frac{m}{k}+\mathbb{Z}} a_{(n),g}^{(M)} z^{-(n+1)} \tag{3.2.1.1}$$

for $v \in V^m$.

Axiom 3.2.1.2. (TSVOAM2.) The twisted vertex operator $Y_g^{(M)}(|0\rangle, z^{\frac{1}{k}})$ is the identity on M:

(1) $Y_g^{(M)}(|0\rangle, z^{\frac{1}{k}}) = Id_M$,

(2) $Y_g^{(M)}(a, z^{\frac{1}{k}})|0\rangle|_{z=0} = a^{(M)}$,

(3) $Y_g^{(M)}(a, z^{\frac{1}{k}}) = 0$ if and only if $a = 0$.

Axiom 3.2.1.3. (Jacobi identity, TSVOAM3.) For any $a, b \in V$,

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y_g^{(M)}\Big(a, z_1^{\frac{1}{k}}\Big)Y_g^{(M)}\Big(b, z_2^{\frac{1}{k}}\Big)$$

$$- (-1)^{p(a)p(b)} z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y_g^{(M)}\Big(b, z_2^{\frac{1}{k}}\Big)Y_g^{(M)}\Big(a, z_1^{\frac{1}{k}}\Big) \tag{3.2.1.2}$$

$$= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y_g^{(M)}\Big(Y\Big(a, z_0^{\frac{1}{k}}\Big)b, z_2^{\frac{1}{k}}\Big), \tag{3.2.1.3}$$

where δ is the series defined in axiom 3.1.0.4.

Axiom 3.2.1.4. (TSVOAM4.) Setting $Y_g^{(M)}(\omega, z^{\frac{1}{k}}) = \sum_{n\in\mathbb{Z}} L_n, g z^{-(n+2)}$, we have

$$Y_g^{(M)}((L_{-1}a, z^{\frac{1}{k}}) = \partial_z Y^{(M)}(a, z^{\frac{1}{k}}) \tag{3.2.1.4}$$

for all $a \in V$.

Axiom 3.2.1.5. (TSVOAM5.) For all $a \in V$ and $u \in M$,

$$a_{(n),g}u = 0 \qquad \text{for } n \text{ sufficiently large.} \tag{3.2.1.5}$$

If g is a finite order automorphism of V, then V is *g-rational* if every g-twisted V-module is the direct sum of simple g-twisted V-modules. The main theorem by Carnahan and Miyamoto in [9] implies that a rational SVOA is g-rational for any finite-order automorphism g.

All SVOAs V admit an order two involution, called the *canonical involution*, $(-1)^f$, which is the identity on $V_{\bar{0}}$ and acts as -1 on $V_{\bar{1}}$.

Let $(-1)^f = \mathrm{Id}_{V_{\bar{0}}} \oplus (-\mathrm{Id}_{V_{\bar{1}}})$ be the canonical involution on an SVOA $V = V_{\bar{0}} + V_{\bar{1}}$. A *canonically twisted module* of V is a superspace $M = M_{\bar{0}} + M_{\bar{1}}$ along with a parity preserving map

$$\begin{aligned} V &\to \mathrm{End}(M)[[z^{\pm\frac{1}{2}}]] \\ a &\mapsto Y^{(M)}_{(-1)^f}(a, z^{\frac{1}{2}}) = \sum_{n\in\frac{1}{2}\mathbb{Z}} a^{(M)}_{(n),(-1)^f} z^{-(n+1)} \end{aligned}$$

where multiplication is defined by such that $Y^{(M)}_{(-1)^f}(a, z)u \in M((z^{\frac{1}{2}}))$ for all $a \in V$ and $u \in M$. The canonically twisted modules are simply $(-1)^f$-twisted modules.

3.2.2 Existence and Uniqueness

SVOA-modules, whether g-twisted or not, have been studied in depth (for example [41]). Despite the fact that the theory of twisted modules contains the theory of ordinary modules, it is common and useful to call to the untwisted theory if $g = 1$, and to the twisted theory in all other cases.

Theorem 3.2.1. [22, 23] Let V be a rational super vertex operator algebra and $g \in \mathrm{Aut}(V)$ be of finite order. Then V has at least one simple g-twisted V-module and only finitely many isomorphism classes of simple g-twisted V-module.

In this thesis, the SVOAs that are studied are all holomorphic. These particular modules have the following useful property.

Theorem 3.2.2. [23] Let V be a holomorphic super vertex operator algebra and $g \in \mathrm{Aut}(V)$ be of finite order. Then, there is a unique isomorphism class of simple g-twisted V-module.

This uniqueness allows us to use the term "the" when speaking about some specifically important g-twisted V-modules. Following the physical language, we will call the unique irreducible g-twisted module the *g-twisted sector*. We will also use the term *Neveu-Schwarz sector* and *Ramond sector* for the 1-twisted and $(-1)^f$-twisted sectors. The *g-twisted Ramond sector* is the $((-1)^f g)$-twisted sector.

3.3 Characters

Any possibly-twisted SVOA module M has two *characters* or *graded dimensions*. The *NS-character* is the graded dimension

$$Z^{NS}(M) = tr_M(q^{L_0-\frac{c}{24}}),$$

that adds the bosonic and fermionic subspaces of M. The *R-character* is the graded dimension

$$Z^R(M) = str_M(q^{L_0-\frac{c}{24}}) = tr_M((-1)^f q^{L_0-\frac{c}{24}}),$$

that subtracts the fermionic subspaces from the bosonic subspaces of M. We treat q as a formal variable. In fact, these characters typically converge to meromorphic functions. We denote the R-character of the NS- and R-sectors $Z^{R,R}(M)$ and $Z^{R,NS}(M)$ respectively and the NS-character of the NS- and R-sectors $Z^{NS,NS}(M)$ and $Z^{NS,R}(M)$ respectively.

We can interpret these characters as partition functions of quantum mechanics models. The Hilbert space is a Hilbert completion of M and the Hamiltonian of M is $L_0 - \frac{c}{24}$. Then M has a partition function described by 2.2.0.3. The Neveu-Schwarz partition function of M is

$$Z^{NS}(M) = \mathrm{tr}_M(e^{-s(L_0-\frac{c}{24})}), \tag{3.3.0.1}$$

and the Ramond partition function of M is

$$Z^{NS}(M) = \mathrm{tr}_M((-1)^f e^{-s(L_0-\frac{c}{24})}). \tag{3.3.0.2}$$

Letting $q = e^{-s}$, we can see that the definitions of partition functions and characters are the same.

Let V be an N = 1 SVOA with conformal vector ω and central charge c. Recall that, this means that among the operators of V is a superconformal vector τ. In the Neveu-Schwarz sector, the expansion of τ is

$$Y(\tau, z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} G_n z^{-(n+\frac{3}{2})}, \tag{3.3.0.3}$$

where the shift by $\frac{1}{2}$ comes from the fact that τ is fermionic. In the Ramond sector, the expansion of τ is

$$Y(\tau, z) = \sum_{n \in \mathbb{Z}} G_n z^{-(n+\frac{3}{2})}. \tag{3.3.0.4}$$

Here, the relative shift by $\frac{1}{2}$ comes from the $(-1)^f$-twisting, since τ is fermionic. Suppose g is some automorphism of V that fixes τ. Then the mode expansion of τ in the g-twisted NS- and R- sectors does not pick up any further shift from the twisting. Thus 3.3.0.3 and 3.3.0.4 still hold in the g-twisted NS- and R- sectors respectively.

The equations 3.1.0.10 and 3.1.0.11 hold in any module, with the degrees m, n ranging over different sets. For example, in the g-twisted R-sector, G_n is integer-graded. In particular, there exists the operator G_0. Expanding 3.1.0.11 for $m, n = 0$,

$$[G_0, G_0] = 2L_0 - \frac{c}{12} \tag{3.3.0.5}$$

$$= L_0 - \frac{c}{24}, \tag{3.3.0.6}$$

so $G_0^2 = L_0 - \frac{c}{24}$ and G_0 is a supersymmetry operator by 2.2.0.10.

Thus, these quantum mechanics models are supersymmetric. By theorem 2.2.1, $Z_{R,R}$ is constant.

Chapter 4

Group Theory

4.1 Interesting Groups: GL_n, O_n, SO_{2n}, Maximal Tori

It goes without saying that group theory is central to the study of abstract algebra. The action of some groups on algebraic structures will be crucial to our study of Moonshine. Moonshine consists of an astonishing set of relationships between the Monster finite simple group and the modular functions in number theory, and is reviewed in detail in appendix C. In this section, we define some interesting groups that will underline some of our most important definitions. It is notable to say that all the following groups are compact Lie groups and thus have particularly well-behaved properties. By definition, a *Lie group* is a group with a compatible smooth manifold structure.

Let $\mathrm{GL}_n(\mathbb{R})$ be the *general linear group*, the set of $n \times n$ invertible matrices equipped with ordinary matrix multiplication. The *orthogonal group*, O_n is a subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of all $n \times n$ orthogonal matrices.

$$\mathrm{O}_n = \{\text{all } n \times n \text{ matrices } M \text{ such that } MM^T = I_n\}.$$

For any element $M \in \mathrm{O}_n$, the determinant of M is ± 1.

The *special orthogonal group*, SO_n, is the identity component of O_n and contains all $n \times n$ orthogonal matrices with determinant 1. We observe the short exact sequence

$$1 \longrightarrow \mathrm{SO}_n \longrightarrow \mathrm{O}_n \xrightarrow{det} \{\pm 1\} \longrightarrow 1. \qquad (4.1.0.1)$$

The orthogonal group and the special orthogonal group of dimension n share the same Lie algebra.

Another notable class of groups is the *spin groups*, $\mathrm{Spin}(n)$, for $n \in \mathbb{N}$. The group $\mathrm{Spin}(n)$ is the unique connected double cover of SO_n: there exists a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}_n \longrightarrow 1. \tag{4.1.0.2}$$

We can construct $\mathrm{Spin}(n)$ from the set of even invertible elements in the Clifford algebra $\mathrm{Cliff}(n)$, as is done in [27].

A *torus* T in a compact Lie group G is a compact, connected abelian Lie subgroup of G. A *maximal torus* is one maximal among such subgroups. It is typically not unique but is essentially unique: all maximal tori are conjugate. In SO_{2n}, the maximal torus T consists of all block-diagonal matrices with 2×2 blocks, such that each block is a rotation matrix [42].

4.2 Lattices

A *full-rank lattice*, Λ, is a discrete subgroup of the Cartesian space $\mathbb{R}^n$ that spans $\mathbb{R}^n$ as a vector space over $\mathbb{R}$. As a group, a lattice is a finitely-generated free abelian group and thus isomorphic to $\mathbb{Z}^n$.

Consider $\mathbb{R}^n$ with inner-product $\langle \cdot, \cdot \rangle$. Let $B = [x_1, ..., x_n] \in \mathbb{R}^n$ be linearly independent vectors in $\mathbb{R}^n$. The lattice generated by B is the set

$$\Lambda = \sum_{i=1}^{n} a_i x_i, \tag{4.2.0.1}$$

for $a_i \in \mathbb{Z}$, of all integer linear combinations of the columns of B. We call B a *basis* of Λ. The dimension n is called the *rank* of Λ. When clarity is needed, we denote by Λ_n a lattice of rank n.

A lattice Λ is *integral* when the inner products of lattice vectors are integral meaning

$$\langle \ell, \ell' \rangle \in \mathbb{Z} \text{ for all } \ell, \ell' \in \Lambda. \tag{4.2.0.2}$$

A lattice is *even* if $\|\ell\|^2 := \langle \ell, \ell \rangle \in 2\mathbb{Z}$ for all lattice vectors ℓ. In that case, the lattice is automatically integral. A lattice is *odd* if it is integral but not even. The dual $\Lambda^\vee$ of Λ is the lattice

$$\Lambda^\vee = \{x \in \mathbb{R}^n | \langle x, \ell \rangle \in \mathbb{Z} \text{ for all } \ell \in \Lambda\}. \tag{4.2.0.3}$$

It is easy to see that, Λ is integral if and only $\Lambda \subseteq \Lambda^\vee$. A lattice is called *unimodular* if $\Lambda = \Lambda^\vee$ [13]. We denote by $\mathcal{V}_n(\Lambda)$ the number of vectors in Λ with squared length $2n$,

$$\|\ell\|^2 = \langle \ell, \ell \rangle = n. \tag{4.2.0.4}$$

A *root* in an even lattice is a vector of squared length 2

$$\|r\|^2 = \langle r, r \rangle = 2. \tag{4.2.0.5}$$

We highlight two famous lattices in the study of Moonshine. The first, the *Leech lattice* Λ_{24}, was discovered by John Leech in 1965 [44]. It is the unique lattice in 24-dimensional Euclidean space $\mathbb{R}^{24}$ such that Λ_{24} is unimodular, even and has no roots. The Leech lattice is constructed in detail in [17].

In fact, Conway showed with the following theorem that Λ_{24} is the only lattice in fewer than 32 dimensions with $\mathcal{V}_2 = 0$ that is unimodular and even.

Theorem 4.2.1. [13] If Λ is even and unimodular and has a dimension less than 32 with $\mathcal{V}_2(\Lambda) = 0$, then

(i) The dimension of Λ is 24

(ii) $\mathcal{V}_4(\Lambda) = 196\ 560$

(iii) $\mathcal{V}_6(\Lambda) = 16\ 773\ 120$

(iv) $\mathcal{V}_8(\Lambda) = 398\ 034\ 000$

Another special family of lattices are the *root lattices*. They are the even lattices generated by roots. The E_8 lattice, also known as D_8^+ in $\mathbb{R}^8$ is an example of such a lattice. It is the unique even, unimodular lattice of rank 8.

Any integral lattice Λ can be assigned a *Theta series*

$$\Theta_\Lambda(\tau) = \sum_{\ell \in \Lambda} q^{\frac{\langle \ell, \ell \rangle}{2}}, \tag{4.2.0.6}$$

where $q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$, the upper-half plane $\mathbb{H} := \{\tau \in \mathbb{C}, \Im(\tau) > 0\}$. We can also define a *signed Theta series*, which will remember the parity of the ℓ's

$$\Theta_\Lambda^S(\tau) = \sum_{\ell \in \Lambda} (-1)^{p(\ell)} q^{\frac{\langle \ell, \ell \rangle}{2}}. \tag{4.2.0.7}$$

Note that we will usually treat these series as formal q-series. That said, since the number of lattice vectors of length R^2 grows like a power of R, these lattice Theta series in fact have radius of convergence 1. It is standard to think of $q = e^{2\pi i\tau}$ and think of Theta series, and indeed all q-series, as functions of τ; the radius of convergence translates into the statement that $\Theta(\tau)$ is holomorphic in the upper half plane $\mathbb{H}$.

The coefficients of q^n give us the number of vectors in Λ with norm n, $\mathcal{V}_n$. Some common examples are

$$\Theta_{\mathbb{Z}}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} = 1 + 2\sqrt{q} + 2q^2 + 2q^{\frac{9}{2}} + 2q^8 + 2q^{\frac{25}{2}} + ... \tag{4.2.0.8}$$

$$\Theta^S_{\mathbb{Z}}(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} = 1 - 2\sqrt{q} + 2q^2 - 2q^{\frac{9}{2}} + 2q^8 - 2q^{\frac{25}{2}} + ... \tag{4.2.0.9}$$

for the integer lattice $\mathbb{Z}$. These series are known as the Jacobi theta series θ_3 and θ_4 respectively.

We can talk about signed and unsigned Theta series for shifted lattices. Say we shift the integer lattice by $\frac{1}{2}$, we obtain the lattice $\mathbb{Z} + \frac{1}{2}$ and the Theta series are

$$\Theta_{\mathbb{Z}+\frac{1}{2}}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(n+\frac{1}{2})^2}{2}} = 2q^{\frac{1}{8}} + 2q^{\frac{9}{8}} + 2q^{\frac{25}{8}} + 2q^{\frac{49}{8}} + 2q^{\frac{81}{8}} + q^{\frac{121}{8}} + ... \tag{4.2.0.10}$$

$$\Theta^S_{\mathbb{Z}+\frac{1}{2}}(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(n+\frac{1}{2})^2}{2}} \equiv 0 \tag{4.2.0.11}$$

These series are known as the Jacobi theta series θ_2 and θ_1 respectively.

The Theta series for E_8 is

$$\Theta_{E_8} = \frac{1}{2}(\theta_1(\tau)^8 + \theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8) \tag{4.2.0.12}$$

whereas for Λ_{24}, we have

$$\Theta_{\Lambda_{24}} = \frac{1}{2}(\theta_1^{24} + \theta_2(\tau)^{24} + \theta_3(\tau)^{24} + \theta_4(\tau)^{24}) - \frac{69}{16}(\theta_2(\tau)\theta_3(\tau)\theta_4(\tau))^8 \tag{4.2.0.13}$$

Given a positive definite lattice Λ one can construct a *lattice SVOA* V_Λ and its corresponding modules, which is done in detail in [33, 34], [48] and [7]. We will not review the detailed construction but mention that V_Λ is Λ-graded. As a vector space, V_Λ is an infinitely generated polynomial algebra

$$V_\Lambda \cong \mathbb{C}\begin{bmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{(n)} & x_2^{(n)} & x_3^{(n)} & \dots \end{bmatrix} \otimes \bigoplus_{\ell\in\Lambda} \mathbb{C}e^\ell$$

where n is the rank of Λ and e^ℓ is the basis vector of the ℓ^{th} graded piece. The $\frac{1}{2}\mathbb{Z}$-grading is compatible with this isomorphism through:

- $x_k^{(i)}$ is graded with degree k
- e^ℓ is graded with degree $\frac{||\ell||^2}{2}$

For example, $x_2^{(1)}x_4^{(3)}x_6^{(5)}e^\ell$ is in degree $2+4+6+\frac{||\ell||^2}{2}$.

Theorem 4.2.2. [20] Let Λ be an integral lattice and let V_Λ be the lattice SVOA associated with Λ. Then, every simple V_Λ-module is isomorphic to some module $V_{L+\Lambda}$ for $L+\Lambda \in \Lambda^\vee/\Lambda$ and $V_{0+\Lambda} \cong V_\Lambda$ as V_Λ-modules.

Corollary 4.2.1. [48] Let Λ be a unimodular lattice. Then, V_Λ is holomorphic.

From the above description of V_Λ, we can extract the NS- and R-characters. Indeed, dimension is multiplicative for tensor product, so

$$Z\Big(\mathbb{C}[x_1,x_2,\dots]^{\otimes n} \otimes \bigoplus_{\ell\in\Lambda}\mathbb{C}e^\ell\Big) = Z\Big(\mathbb{C}[x_1,x_2,\dots]\Big)^n Z\Big(\bigoplus_{\ell\in\Lambda}\mathbb{C}e^\ell\Big) \tag{4.2.0.14}$$

$$= \Big(\prod_{k=1}^\infty Z(\mathbb{C}[x_k])\Big)^n \Theta_\Lambda \tag{4.2.0.15}$$

$$= \Big(\prod_{k=1}^\infty \sum_{m=0}^\infty q^{mk}\Big)^n \Theta_\Lambda \tag{4.2.0.16}$$

$$= \Big(\prod_{k=1}^\infty (1-q^k)^{-1}\Big)^n \Theta_\Lambda \tag{4.2.0.17}$$

We use $\eta(\tau)$ or just η to denote the Dedekind eta function,

$$\eta(\tau) = e^{\frac{i\pi\tau}{12}}\prod_{k=1}^\infty(1-e^{2k\pi i\tau}) = q^{\frac{1}{24}}\prod_{k=1}^\infty(1-q^k) \tag{4.2.0.18}$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$. The Ramanujan Delta function is defined as

$$\Delta(\tau) = \eta^{24}(\tau). \tag{4.2.0.19}$$

It follows from the above that the NS-character of V_Λ is

$$Z^{NS}(V_\Lambda) = \frac{\Theta_\Lambda(\tau)}{\eta(\tau)^n}, \tag{4.2.0.20}$$

and the R-character is

$$Z^{R}(V_\Lambda) = \frac{\Theta^S_\Lambda(\tau)}{\eta(\tau)^n}, \tag{4.2.0.21}$$

where n is the rank of Λ. It turns out that $Z(V_\Lambda)$ is a modular function for some congruence group. The characters of the simple V_Λ-modules are

$$Z^{NS}(L) = \frac{\Theta_L(\tau)}{\eta(\tau)^n} \tag{4.2.0.22}$$

and

$$Z^{R}(L) = \frac{\Theta^S_L(\tau)}{\eta(\tau)^n} \tag{4.2.0.23}$$

for $L \in \Lambda^\vee/\Lambda$ where

$$\Theta_L = \sum_{\ell \in \Lambda + L} q^{\frac{\ell^2}{2}} \tag{4.2.0.24}$$

and

$$\Theta^S_L = \sum_{\ell \in \Lambda + L} (-1)^{p(\ell)} q^{\frac{\ell^2}{2}}. \tag{4.2.0.25}$$

Let g be an automorphism of a lattice Λ. We can lift g to an automorphism of the lattice SVOA V_Λ [48]. If g is an automorphism of Λ without fixed-points, then all lifts of g to automorphisms of V_Λ are conjugate to each other [8]. The automorphism g acts naturally on the group $\Lambda^\vee/\Lambda$ and we write

$$(\Lambda^\vee/\Lambda)^g = \{L + \Lambda \in \Lambda^\vee/\Lambda | (\mathrm{id} - g)L \in \Lambda\}$$

for the fixed points under the action of g.

4.3 Conway's Groups

While studying Λ_{24}, John Conway analyzed the automorphism group $\mathrm{Aut}(\Lambda_{24})$ which he denoted $\cdot 0$. It is now stantard to call this group Co_0. He also discovered three new sporadic simple groups of interest, Co_1, Co_2 and Co_3 [14], [12]. The first and largest, Co_0 consists of those operations of the orthogonal group $\mathrm{O}_{24}(\mathbb{Q})$, which preserve the Leech lattice.

$$\mathrm{Co}_0 := \mathrm{Aut}(\Lambda_{24}).$$

In the literature, Co_0 is most often referred to as *Conway's group*. With an impressive dimension of $2^{22} \times 3^9 \times 5^4 \times 7^2 \times 11 \times 13 \times 23$, the group Co_0 has a center of order 2, $Z(\mathrm{Co}_0) = \{\pm 1\}$. If we reflect the Leech lattice, we get an object which is not rotationally related to the original Leech lattice. There is only one conjugacy class of non-trivial maps

$$\mathrm{Co}_0 \longrightarrow \mathrm{O}_{24}$$

but two conjugacy classes of non-trivial maps

$$\mathrm{Co}_0 \longrightarrow \mathrm{SO}_{24}$$

which are exchanged by reflection.

The simple group denoted Co_1, Conway's largest sporadic simple group, is obtained by taking the quotient

$$\mathrm{Co}_1 := \mathrm{Co}_0/Z(\mathrm{Co}_0)$$

of Co_0 by its center and has dimension $2^{21} \times 3^9 \times 5^4 \times 7^2 \times 11 \times 13 \times 23$. The Conway group Co_0 is a double cover of Co_1. The sporadic simple groups Co_2 and Co_3 are certain maximal subgroups of Co_1 and will not play a role in this thesis.

For any finite group G, $\mathrm{H}_1(G) = G/[G,G]$ is the *abelianization* of G, and $\mathrm{H}_2(G)$ is the *Schur multiplier* of G. It was calculated by Conway that $\mathrm{H}_1(\mathrm{Co}_0) = \mathrm{H}_2(\mathrm{Co}_0) = 0$ and the following lemma follows immediately.

Lemma 4.3.1. Let $G \subset \mathrm{SO}_{24}$ be isomorphic to Co_0. Then, there is a unique lift of G to $\mathrm{Spin}(24)$.

Proof. From [15], notice that the Schur multiplier of Co_0 is trivial. Then, the inverse image of G under the natural homomorphism $\mathrm{Spin}(24) \longrightarrow \mathrm{SO}_{24}$ contains a copy of Co_0, which defines a lift

$$\hat{G} = \mathrm{Co}_0 \xrightarrow{\cong} G.$$

Assume there exists another lift $\hat{G}'$ of G. Given $g \in G$, we write $\hat{g}$ and $\hat{g}'$ for the corresponding element of $\hat{G}$ and $\hat{G}'$ respectively such that $\hat{g}(\cdot) = \hat{g}'(\cdot) = g$. Now, $\hat{g}' = \pm\hat{g}$ as elements of $\mathrm{Spin}(24)$, so $\hat{G} \cap \hat{G}'$ is a normal subgroup of $\hat{G}$ and of $\hat{G}'$ containing all their elements of odd order respectively. The only proper nontrivial normal subgroup of Co_0 is its center of order 2. Thus, $\hat{G} \cap \hat{G}' = \hat{G}$. It follows that $\hat{G} = \hat{G}'$. Thus $\hat{G}$ is the unique lift of G to $\mathrm{Spin}(24)$. □

Given $G \subset \mathrm{SO}_{24}$, isomorphic to Co_0, let $\hat{G}$ be the unique lift of G to $\mathrm{Spin}(24)$. We write

$$G \xrightarrow{\cong} \hat{G}$$
$$g \longmapsto \hat{g}$$

for the *inverse lift*, the inverse of the isomorphism $\hat{G} \xrightarrow{\cong} G$ obtained by restricting $\mathrm{Spin}(24) \longrightarrow \mathrm{SO}_{24}$.

4.4 Conway Moonshine

We review in detail the story of monstrous Moonshine in appendix C. Briefly, there exists a particular holomorphic VOA, called $V^\natural$, constructed from the Leech lattice by [33, 34]. The automorphism group of $V^\natural$ is the Monster sporadic simple group $\mathbb{M}$ explored in appendix B. The characters of the g-twisted sectors $Z_{1,g}(V^\natural)$ have remarkable number theoretic properties.

In 1979, Conway and Norton suggested that a similar phenomena to Moonshine may happen for other groups [16]: see in particular Queen and Kondo's work [43, 50]. Many of these early Moonshines are now known to have monstrous origin.

However, there are two well-known examples of modern Moonshine, Mathieu Moonshine [29], [11] and Conway Moonshine [16], which are not known to have monstrous origin. In this thesis, we focus on Conway Moonshine first introduced by

Duncan [28].

4.4.1 Construction

The construction of $V^\natural$ in [33, 34] makes manifest a certain maximal subgroup of $\mathbb{M}$ of shape $2^{1+24}.\mathrm{Co}_1$. Inside Conway's group Co_1 is an analogous maximal subgroup of shape $2^{1+8}.\mathrm{O}_8^+(2)$, where, up to some $\{\pm 1\}$, $\mathrm{O}_8^+(2)$ is the automorphism group of the E_8 lattice. This would suggest that we can repeat the method used to construct $\mathrm{V}^\natural$ in [33, 34] starting with the E_8 lattice. This was done by Duncan in his Ph.D. thesis [28]. Specifically, Duncan built the *Conway Moonshine module*, $\mathrm{V}^{f\natural}$, starting with the N = 1 supersymmetric E_8 lattice. The result is an N = 1 SVOA $\mathrm{V}^{f\natural}$ with automorphism group Co_1.

There are more direct constructions of $V^{f\natural}$, such as was done by Duncan and Mack-Rane in [27]. They showed that $V^{f\natural}$ admits an SVOA structure by explicitly describing the vertex operator correspondence, defining a state-field correspondence, as required by the axioms,

$$V^{f\natural} \otimes V^{f\natural} \longrightarrow \mathcal{E}(V^{f\natural}) \tag{4.4.1.1}$$

$$a \otimes b \longmapsto Y(a, z)b. \tag{4.4.1.2}$$

Now that the existence of $V^{f\natural}$ is known, one can ask about gauging fermion parity and discover that $V^{f\natural}$ can be constructed from the SVOA built functorially from $\mathbb{R}^{24}$, called *Fer(24)* [38]. The automorphism group of Fer(24) is O_{24}. We are not going to spell out the construction of Fer($2n$). It is explained in chapter 4 of [40]. A reader who knows about Clifford algebras should think of Fer($2n$) as their SVOA analog. As explained in [40], Fer($2n$) is isomorphic to the lattice SVOA for the $\mathbb{Z}^n$ lattice. In section §4.2, we laid out everything there is to know about the characters of Fer($2n$) as a lattice SVOA. It follows that, for Fer(24), the NS-sector is the $\mathbb{Z}^{12}$ lattice and the R-sector is the $(\mathbb{Z}+\frac{1}{2})^{12} \subseteq \mathbb{R}^n$ lattice.

This construction leads to another notation for $V^{f\natural}$, $\mathrm{Fer}(24)/\!\!/(-1)^f$. In his paper [25], Duncan studied this construction and built the self-dual N = 1 SVOA, denoted $A^{f\natural}$ whose full automorphism group is Conway's largest sporadic group, Co_1.

There are two ways to orbifold Fer(24) by $(-1)^f$, which are exchanged by reflection. When orbifolding Fer(24) by $(-1)^f$, we take the fermions in the NS-sector to the R-sector and move half of the R-sector to the NS-sector. An important note here is that the R-sector is not well-defined as a super vector space. It splits as a sum of two vector spaces, but there is no right way to tell which one is the bosons and which one is the fermions. One way to visualize this is with the $\mathrm{Fer}(4) = \mathbb{Z}^2$ lattice:

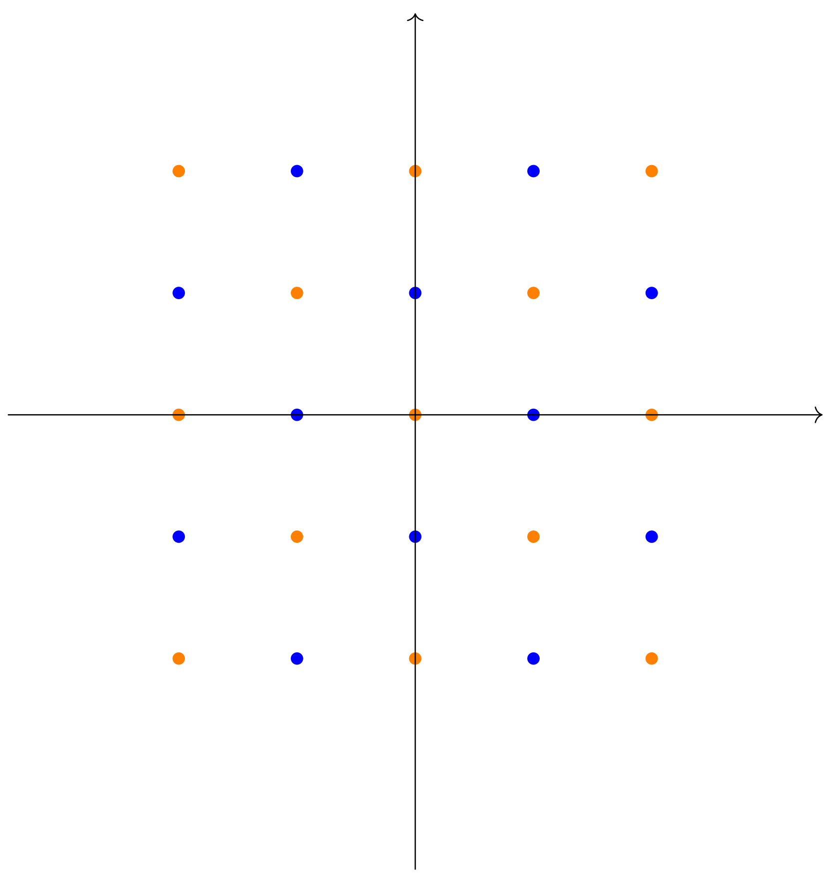

Figure 4.1: ●: bosons, ●: fermions

The eigenvector of $(-1)^f$ in SO_4 is $(\frac{1}{2}, \frac{1}{2})$. Thus, the R-sector is the $\mathbb{Z}^2$ lattice, shifted by the vector $(\frac{1}{2}, \frac{1}{2})$, which is the $(\mathbb{Z} + \frac{1}{2})^2$ lattice colored in purple below.

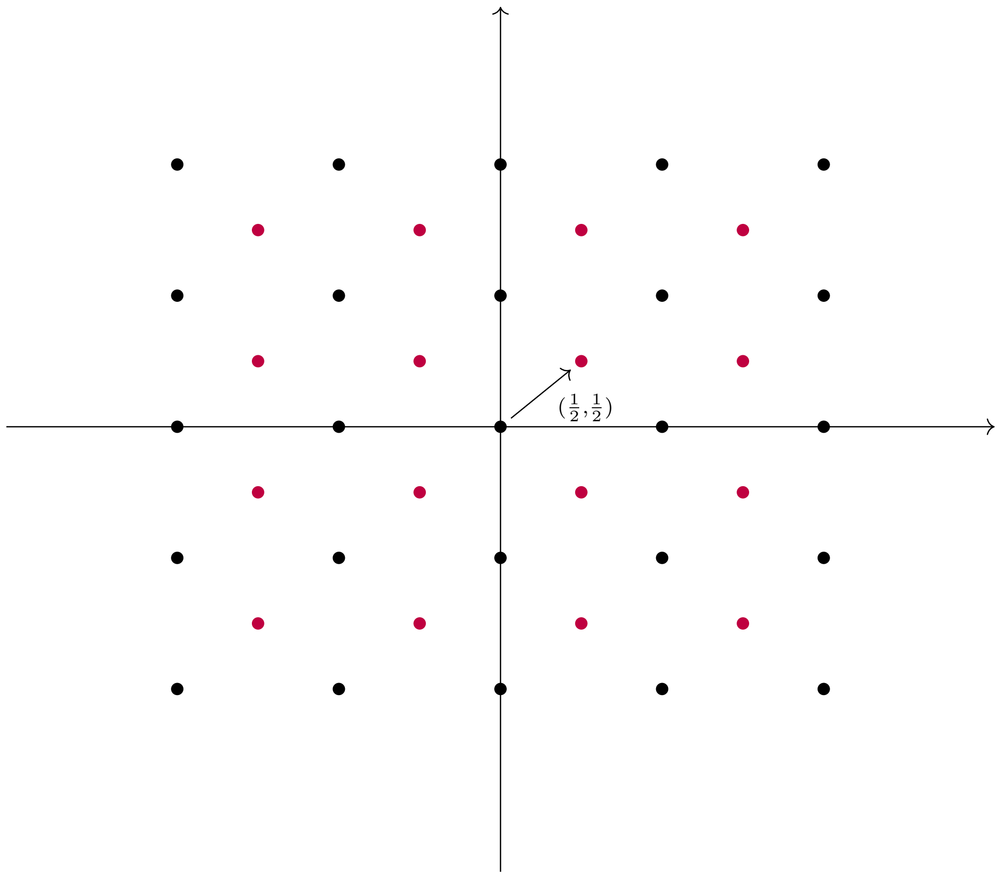

Figure 4.2: •: R-sector, •: NS-sector

We must arbitrarily decide whether *even* + $\frac{1}{2}$ or *odd* + $\frac{1}{2}$ is the fermions or respectively the bosons. Indeed, reflection acts by $x \longrightarrow -x$ on the lattice but -(*even* + $\frac{1}{2}$) = *odd* + $\frac{1}{2}$ and -(*odd* + $\frac{1}{2}$) = *even* + $\frac{1}{2}$. In figure 4.3, we let *even* + $\frac{1}{2}$ be the bosons and in figure 4.4, the bosons are *odd* + $\frac{1}{2}$.

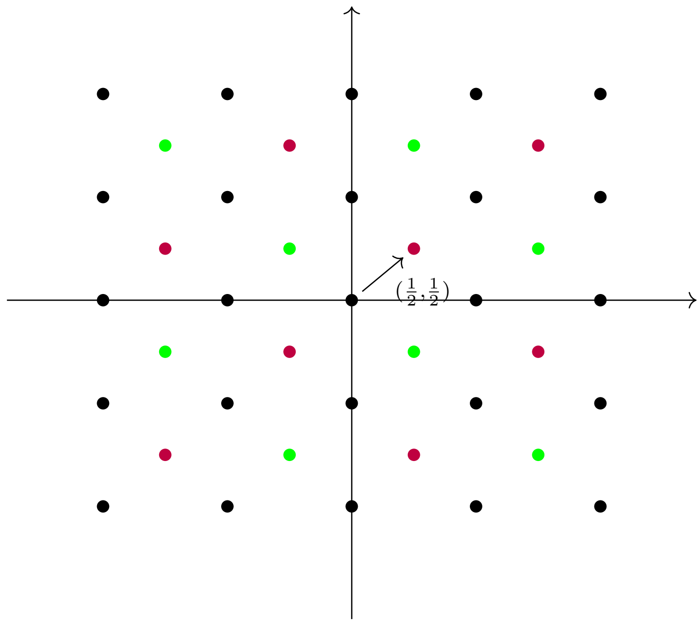

Figure 4.3: •: R-sector boson, •: R-sector fermion, •: NS-sector

$\updownarrow$ reflection

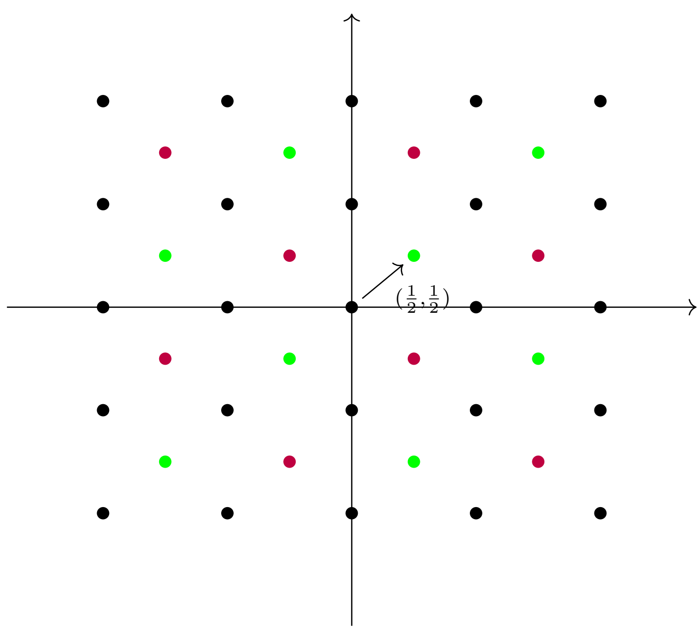

Figure 4.4: •: R-sector boson, •: R-sector fermion, •: NS-sector

Then, arbitrarily choosing the parity assignment from figure 4.4 to the R-sector and assigning fermion and boson parity to the NS-sector sectors we obtain figure 4.5:

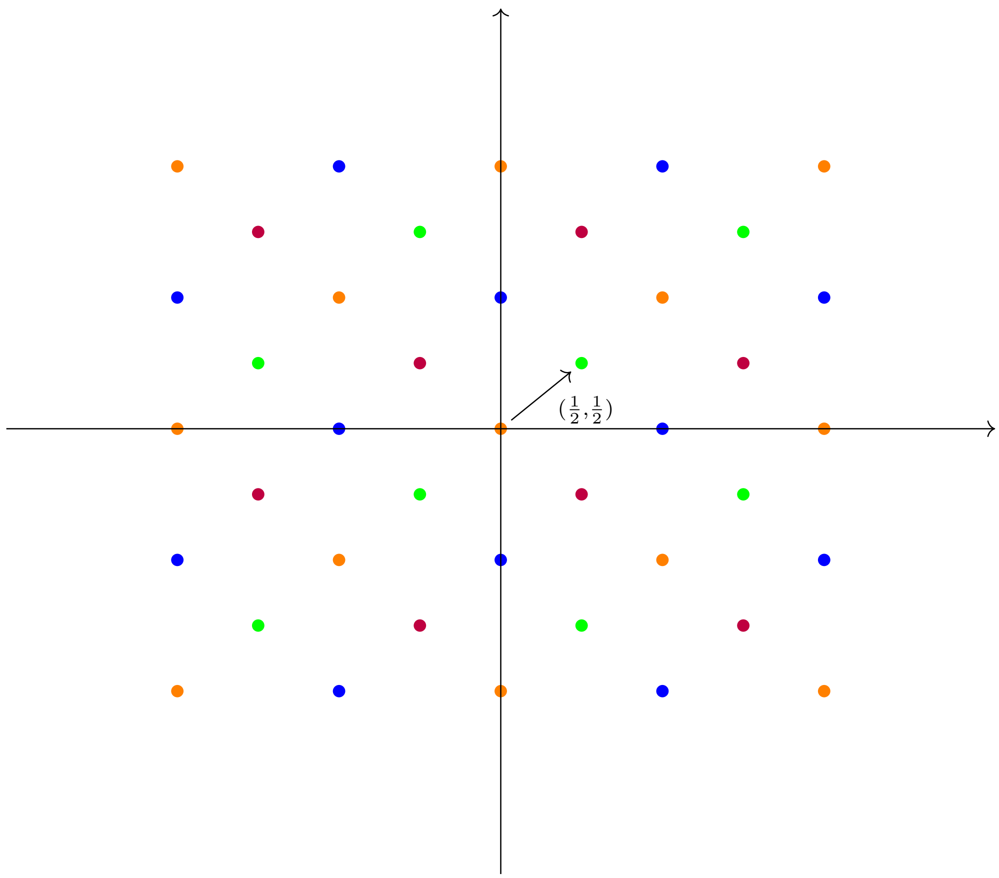

Figure 4.5: ●: R-sector boson, ●: R-sector fermion, ●: NS-sector boson, ●: NS-sector fermion

Recall that the NS-character adds bosons and fermions and the R-character subtracts them. The above picture allows us to visualize what the NS- and R-characters of the NS- and R sectors are calculating. By 4.2.0.20, and 4.2.0.21:

$$Z^{NS,NS=\mathbb{Z}^2}\Big(\mathrm{Fer}(4)\Big) = \dim(\bullet) + \dim(\bullet) \tag{4.4.1.3}$$

$$= \frac{\Theta_{\mathbb{Z}^2}(\tau)}{\eta(\tau)^2} \tag{4.4.1.4}$$

$$= \frac{\Theta_{\mathbb{Z}}(\tau)^2}{\eta(\tau)^2} \tag{4.4.1.5}$$

$$Z^{R,NS=\mathbb{Z}^2}\Big(\mathrm{Fer}(4)\Big) = \dim(\bullet) - \dim(\bullet) \tag{4.4.1.6}$$
$$= \frac{\Theta^S_{\mathbb{Z}^2}(\tau)}{\eta(\tau)^2} \tag{4.4.1.7}$$
$$= \frac{\Theta^S_{\mathbb{Z}}(\tau)^2}{\eta(\tau)^2} \tag{4.4.1.8}$$

$$Z^{NS,R=(\mathbb{Z}+\frac{1}{2})^2}\Big(\mathrm{Fer}(4)\Big) = \dim(\bullet) + \dim(\bullet) \tag{4.4.1.9}$$
$$= \frac{\Theta_{(\mathbb{Z}+\frac{1}{2})^2}(\tau)}{\eta(\tau)^2} \tag{4.4.1.10}$$
$$= \frac{\Theta_{\mathbb{Z}+\frac{1}{2}}(\tau)^2}{\eta(\tau)^2} \tag{4.4.1.11}$$

$$Z^{R,R=(\mathbb{Z}+\frac{1}{2})^2}\Big(\mathrm{Fer}(4)\Big) = \dim(\bullet) - \dim(\bullet) \tag{4.4.1.12}$$
$$= \frac{\Theta^S_{(\mathbb{Z}+\frac{1}{2})^2}(\tau)}{\eta(\tau)^2} \tag{4.4.1.13}$$
$$= \frac{\Theta^S_{\mathbb{Z}+\frac{1}{2}}(\tau)^2}{\eta(\tau)^2} \tag{4.4.1.14}$$

The gauging procedure cuts the $\mathbb{Z}^n$ lattice to down to the D_n lattice and adds half of $(\mathbb{Z}+\frac{1}{2})^n$. The resulting lattice is D_n^+. For $\mathrm{Fer}(24)^{NS} \cong \mathbb{Z}^{12}$ and $\mathrm{Fer}(24)^R \cong (\mathbb{Z}+\frac{1}{2})^{12}$, we move the fermions in $\mathrm{Fer}(24)^{NS}$ to $\mathrm{Fer}(24)^R$ resulting in the D_{12} lattice and bring half of $\mathrm{Fer}(24)^R$ into $\mathrm{Fer}(24)^{NS}$, resulting in the D_{12}^+ lattice.

We can visualize the gauging procedure with $\mathrm{Fer}(4) = \mathbb{Z}^2$, represented by figure 4.2, and build $\mathrm{Fer}(4)/\!\!/(-1)^f$. We move the NS-sector fermions into the R-sector and obtain figure 4.6:

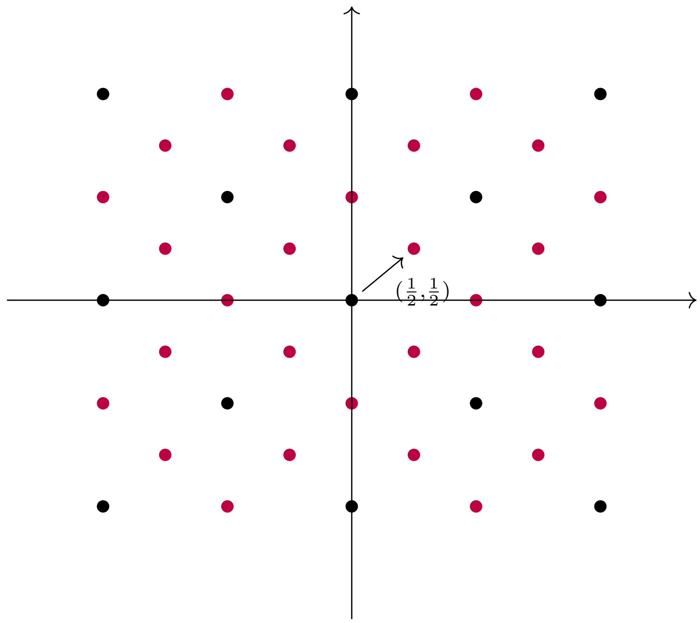

Figure 4.6: •: R-sector, •: NS-sector

Now, moving half of the R-sector into the NS-sector:

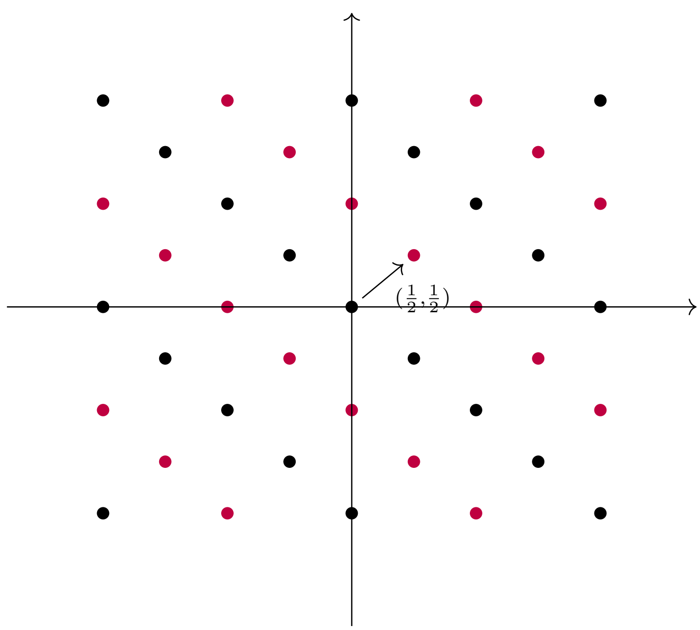

Figure 4.7: •: R-sector, •: NS-sector

Given we have no way of coherently assigning fermion parity to the R-sector, we make a random choice.

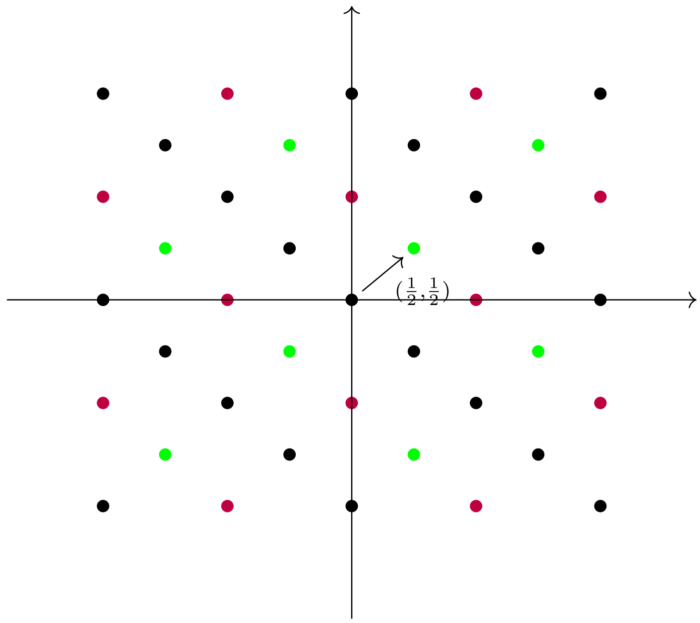

Figure 4.8: •: R-sector boson, •: R-sector fermion, •: NS-sector

Both options are related by reflection so our choice does not truly matter but it breaks the O_{24} symmetry of Fer(24) to an SO_{24} symmetry. This shows we have two fundamentally different ways to have a Co_0 action on the orbifold, which builds what we call $V^{f\natural}$ and $V^{s\natural}$ (as is done in [27]). We can tell them apart by looking at the action of $Z(Co_0)$. For one choice, $V^{f\natural}$, the action of Co_0 on the NS-sector factors through Co_1 and $Z(Co_0)$ acts by -1 on the R-sector. For the other, $V^{s\natural}$, the action on the bosons in both sectors factors through Co_1 and the action on the fermions is trivial. As SVOAs, we have

$$A^{f\natural} \cong V^{f\natural} \cong V^{s\natural}$$

but as Co_0-modules,

$$A^{f\natural} \cong V^{f\natural} \not\cong V^{s\natural}$$

since $Z(Co_0)$ acts trivially on $V^{f\natural}$ [27].

Any supersymmetry is generated by a fermion in the NS-sector. In $V^{s\natural}$, Co_0 acts non-trivially on all fermions, so we do not have a Co_0-invariant supersymmetry. In $A^{f\natural}$ and $V^{f\natural}$, Duncan and Mack-Rane showed that there is precisely one fermion of the appropriate spin fixed by Co_0, satisfying 3.1.0.10 and generating supersymmetry [25], [27]. Thus $V^{f\natural}$ has a distinguished supersymmetry fixed by Co_0 but $V^{s\natural}$ does not have a distinguished one. Accordingly, we focus our attention on $V^{f\natural}$.

From figure 4.8, we can see that the NS-character of the R-sector is

$$Z^{NS,R}(Fer(4) /\!\!/ (-1)^f) = \dim({\color{purple}\bullet}) + \dim({\color{green}\bullet}) \tag{4.4.1.15}$$

and the R-character is

$$Z^{R,R}(Fer(4) /\!\!/ (-1)^f) = \dim({\color{purple}\bullet}) - \dim({\color{green}\bullet}) \tag{4.4.1.16}$$

Now, using 4.4.1.3,4.4.1.6, 4.4.1.9 and 4.4.1.12, we have

$$\dim({\color{purple}\bullet}) = \frac{1}{2}\left(Z^{NS,NS}\Big(\mathrm{Fer}(4)\Big) - Z^{R,NS}\Big(\mathrm{Fer}(4)\Big)\right) \tag{4.4.1.17}$$

$$\dim({\color{green}\bullet}) = \frac{1}{2}\left(Z^{NS,R}\Big(\mathrm{Fer}(4)\Big) - Z^{R,R}\Big(\mathrm{Fer}(4)\Big)\right) \tag{4.4.1.18}$$

Thus,

$$\begin{aligned} Z^{NS,R}(Fer(4) /\!\!/ (-1)^f) = \frac{1}{2}\Bigg[&\left(Z^{NS,NS}\Big(\mathrm{Fer}(4)\Big) - Z^{R,NS}\Big(\mathrm{Fer}(4)\Big)\right) \\ &+ \left(Z^{NS,R}\Big(\mathrm{Fer}(4)\Big) - Z^{R,R}\Big(\mathrm{Fer}(4)\Big)\right)\Bigg] \end{aligned} \tag{4.4.1.19}$$

$$\begin{aligned} Z^{R,R}(Fer(4) /\!\!/ (-1)^f) = \frac{1}{2}\Bigg[&\left(Z^{NS,NS}\Big(\mathrm{Fer}(4)\Big) - Z^{R,NS}\Big(\mathrm{Fer}(4)\Big)\right) \\ &- \left(Z^{NS,R}\Big(\mathrm{Fer}(4)\Big) - Z^{R,R}\Big(\mathrm{Fer}(4)\Big)\right)\Bigg] \end{aligned} \tag{4.4.1.20}$$

Chapter 5

Calculations

The purpose of this section is to calculate the Ramond and Neveu-Schwarz characters of the R-sector of $V^{f\natural}$, $Z^{R,R}(V^{f\natural})$ and $Z^{NS,R}(V^{f\natural})$. Then, we add an action of $g \in \mathrm{Co}_0$ to the R-sector and calculate the corresponding g-twisted characters. As explained in section §3.3, these calculations will tell us the number of ground states i.e. the ground state degeneracy of the g-twisted R-sector of $V^{f\natural}$.

It is easy to see that, if the number of ground states match up, all ground states have the same fermion parity. To differentiate between the absence or presence of a g-twisting in the R-sector, we denote the characters $Z_{1,1}^{R,R}(V^{f\natural})$, $Z_{1,1}^{NS,R}(V^{f\natural})$ and $Z_{1,g}^{R,R}(V^{f\natural})$, $Z_{1,g}^{NS,R}(V^{f\natural})$ respectively. There are more twistings and spin structures that we could take into account, hence the overloaded notation.

5.1 Frame shape

Although the order of Co_0 is 8,315,553,613,086,720,000, there are only 167 conjugacy classes. Accordingly, we describe elements by the conjugacy class to which they belong. An important tool in our explicit computations of $Z^{R,R}(V^{f\natural})$ and $Z^{NS,R}(V^{f\natural})$, twisted and non-twisted, is the Frame shape [30]. Since Co_0 is the automorphism group of the integral Leech lattice Λ_{24}, in some basis, its elements are given by an integer matrix. Thus, the characteristic polynomial for the action of g is an integer polynomial and factors as

$$p_g = \det(g - xI_{24}) = \prod_{k>0}(1-x^k)^{p_k} \tag{5.1.0.1}$$

with all but finitely many $p_k = 0$ such that the p_k are possibly-negative integers and $\sum_{p_k} = 24$. The *Frame shape* of g is the product

$$\pi_g = \prod_{k>0} k^{p_k} \tag{5.1.0.2}$$

It encodes the characteristic polynomial of g and hence its conjugacy class in $\mathrm{GL}_{24}(\mathbb{C})$. For example, the trace of g can be read as

$$tr(g) = p_1 = \text{number of cycles of length 1} \tag{5.1.0.3}$$

The Frame shapes for elements of Co_0 were calculated in [43]. Notice that, out of the 167 conjugacy classes of Co_0, some fuse to produce 160 Frame shapes.

There is another important feature about an element $g \in \mathrm{Co}_0$ that can be read off of its Frame shape. Let $l(g)$ be the smallest k such that the exponent p_k of k is not equal to zero in the Frame shape of g. Respectively, let $L(g)$ be the largest. Let $\epsilon(g) = \pm 1$ be the sign of $p_{l(g)}$ and $N = l(g)L(g)$. According to [39], we say g is *balanced* if $p_k = \epsilon(g) p_{\frac{N}{k}}$ for all k.

Example 5.1.0.1. Let $g \in \mathrm{Co}_0$ be an element of conjugacy class 8B with Frame shape $8^4 2^{-4}$. Then, $l(g) = 2, L(g) = 8$ and $\epsilon(g) = -1$. For $k = 2$, $p_2 = -4 = (-1)4 = (-1)p_8 = (-1)p_{\frac{16}{2}}$ and for $k = 8$, $p_8 = 4 = (-1)(-4) = (-1)p_2 = (-1)p_{\frac{16}{8}}$. So elements of 8B are balanced.

Example 5.1.0.2. Let $g \in \mathrm{Co}_0$ be an element of conjugacy class 4D with Frame shape $4^8 2^{-4}$. Again, $l(g) = 2, L(g) = 8$ and $\epsilon(g) = -1$. Not surprisingly, $k = 2$, $p_2 = -4 = (-1)4 \neq (-1)p_{\frac{16}{2}}$. So elements of 4D are unbalanced.

In $\mathrm{GL}_{12}(\mathbb{C})$ and O_{24}, since the Frame shape of an element g encodes its characteristic polynomial, it encodes its conjugacy class. Another way to find the conjugacy class of g is to diagonalize it. Let $g \in \mathrm{SO}_{24}$. Since we can work up to conjugacy, we can diagonalize g over O_{24} and obtain 2×2 blocks

$$\begin{bmatrix} \begin{bmatrix} \cos(2\pi\lambda_1) & \sin(2\pi\lambda_1) \\ -\sin(2\pi\lambda_1) & \cos(2\pi\lambda_1) \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} \cos(2\pi\lambda_{12}) & \sin(2\pi\lambda_{12}) \\ -\sin(2\pi\lambda_{12}) & \cos(2\pi\lambda_{12}) \end{bmatrix} \end{bmatrix}$$

where each block is a rotation matrix. The $\{\lambda_i\}_{i=1}^{12}$ are (the logarithms of, up to a factor of 2π) the eigenvalues of g i.e. the angles of the roots of the characteristic polynomial of g. All objects of this shape are the maximal torus, T of SO_{24} as laid out in section §4.1. Note that

$$\begin{bmatrix} \cos(2\pi\lambda_n) & \sin(2\pi\lambda_n) \\ -\sin(2\pi\lambda_n) & \cos(2\pi\lambda_n) \end{bmatrix}$$

is conjugate to

$$\begin{bmatrix} \cos(2\pi(-\lambda_n)) & \sin(2\pi(-\lambda_n)) \\ -\sin(2\pi(-\lambda_n)) & \cos(2\pi(-\lambda_n)) \end{bmatrix}$$

so the λs are only well-defined up to sign and modulo 1.

5.2 Eigenvalues

The condensed and simplified recipe to extract the 12 λs from the Frame shape of $g \in \mathrm{Co}_0$ is:

(i) Extract all 24 eigenvalues of g

(ii) Since the λs pair up by sign, we only keep half

We really will want the conjugacy classes in Spin(24), which will give us more information since Spin(24) is the double cover of SO_{24}. By lemma 4.3.1, there exist a unique lift from Co_0 to Spin(24). Thus, each conjugacy class in Co_0 gives a conjugacy class in Spin(24). The Frame shape only records the conjugacy class in O_{24}.

These calculations are a piece of experimental mathematics. We have no way of coherently choosing the 12 values that represent the conjugacy classe of g in Spin(24). In the next section, we will find a formula for $Z_{1,g}^{R,R}(V^{f\natural})$ in terms of the list of λs. From theorem 2.2.1, we know $Z_{1,g}^{R,R}(V^{f\natural})$ is independent of τ. Then, in step (ii), this allows us to keep the 12 values for λ that arrange this.

Let $\pi_g = \prod_{k|m} k^{p_k}$ be the Frame shape of an element $g \in \mathrm{Co}_0$. Since the eigenvalues of g are the roots of p_g, we can use π_g to extract them in an "easier" way. Indeed, the roots of $1 - x^k$ are the k^{th} roots of unity, their angles are $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}$. In other words, for each k, we have $|p_k|$ copies of all fractions

$$x = \begin{cases} \frac{n}{k} & \text{if } p_k > 0 \\ -\frac{n}{k} & \text{if } p_k < 0 \end{cases}$$

for all $0 \leq n < k$. Then, simplifying the numbers x and $-x$ mod 1, we always end up with a list of 24 eigenvalues contenders, $CE_g := \{\mu_i\}_{i=1}^{24}$. We have two main separate cases to deal with.

First and most straightforward, the Frame shapes with all even powers. For all $\mu_i = \frac{n}{k} \in CE_g$, we have $p_k = 2m$ copies of μ_i. Keeping only $\frac{p_k}{2} = m$ of each μ_i, we get our 12 eigenvalues for g, $E_g = \{\lambda_i\}_{i=1}^{12}$.

Example 5.2.0.1. Let $g \in \mathrm{Co}_0$ be an element in the conjugacy class 12E, equivalent to 12B in Co_1. The Frame shape of g is $\pi_g = 2^2 12^4 4^{-4} 6^{-2}$.

For 2^2, we have 2×0 and $2 \times \frac{1}{2}$.

For 12^4, we have 4×0,$4 \times \frac{1}{12}$,$4 \times \frac{1}{6}$, $4 \times \frac{1}{4}$, $4 \times \frac{1}{3}$, $4 \times \frac{5}{12}$, $4 \times \frac{1}{2}$, $4 \times \frac{7}{12}$, $4 \times \frac{2}{3}$, $4 \times \frac{3}{4}$, $4 \times \frac{5}{6}$, $4 \times \frac{11}{12}$.

For 4^{-4}, we have 4×-0,$4 \times -\frac{1}{4}$, $4 \times -\frac{1}{2}$, $4 \times -\frac{3}{4}$.

For 6^{-2}, we have 2×-0, $2 \times -\frac{1}{6}$, $2 \times -\frac{1}{3}$, $2 \times -\frac{1}{2}$, $2 \times -\frac{2}{3}$, $2 \times -\frac{5}{6}$.

Simplifying and subtracting opposites when possible, we end up with

$$CE_g = \left\{ \boxed{\frac{1}{12}}, \boxed{\frac{1}{12}}, \frac{1}{12}, \frac{1}{12}, \boxed{\frac{1}{6}}, \frac{1}{6}, \boxed{\frac{1}{3}}, \frac{1}{3}, \boxed{\frac{5}{12}}, \boxed{\frac{5}{12}}, \frac{5}{12}, \frac{5}{12}, \boxed{\frac{7}{12}}, \boxed{\frac{7}{12}}, \frac{7}{12}, \frac{7}{12}, \boxed{\frac{2}{3}}, \frac{2}{3}, \boxed{\frac{5}{6}}, \frac{5}{6}, \boxed{\frac{11}{12}}, \boxed{\frac{11}{12}}, \frac{11}{12}, \frac{11}{12} \right\}$$

Keeping only half of each value of μ_i, we get

$$E_g = \left\{ \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{5}{12}, \frac{5}{12}, \frac{7}{12}, \frac{7}{12}, \frac{2}{3}, \frac{5}{6}, \frac{11}{12}, \frac{11}{12} \right\}$$

.

Now, let us tackle the Frame shapes with odd powers. The steps for obtaining CE_g are the same but the choice of the λ_i,s to keep for E_g varies depending on the contents of CE_g.

For each $\mu_i \in CE_g$ such that there are $p_k = 2m+1$ copies of $\mu_i = \frac{n}{k}$ and $p_k = 2m+1$ copies of $\overline{\mu_i} = \frac{k-n}{k}$. Without loss of generality, we let $\mu_i < \overline{\mu_i}$. Then, we keep $\lceil \frac{p_k}{2} \rceil$

copies of $\overline{\mu_i}$ and $\lfloor \frac{p_k}{2} \rfloor$ copies of μ_i. If there are even powers in the Frame shape, we deal with those the same way as outlined beforehand.

Example 5.2.0.2. Let $g \in \mathrm{Co}_0$ be an element in the conjugacy class 14B, equivalent to 7B in Co_1. The Frame shape of g is $\pi_g = 2^3 14^3 1^{-3} 7^{-3}$. We have

$$CE_g = \left\{ \boxed{\frac{1}{14}}, \frac{1}{14}, \frac{1}{14}, \boxed{\frac{3}{14}}, \frac{3}{14}, \frac{3}{14}, \boxed{\frac{5}{14}}, \frac{5}{14}, \frac{5}{14}, \boxed{\frac{1}{2}}, \boxed{\frac{1}{2}}, \boxed{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2}, \right.$$
$$\left. \frac{1}{2}, \boxed{\frac{9}{14}}, \boxed{\frac{9}{14}}, \frac{9}{14}, \boxed{\frac{11}{14}}, \boxed{\frac{11}{14}}, \frac{11}{14}, \boxed{\frac{13}{14}}, \boxed{\frac{13}{14}}, \frac{13}{14} \right\}$$

We get

$$E_g = \left\{ \frac{1}{14}, \frac{3}{14}, \frac{5}{14}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{9}{14}, \frac{9}{14}, \frac{11}{14}, \frac{11}{14}, \frac{13}{14}, \frac{13}{14} \right\}$$

Example 5.2.0.3. Let $g \in \mathrm{Co}_0$ be an element in the conjugacy class 24E, equivalent to 24B in Co_1. The Frame shape of g is $\pi_g = 1^2 4^1 6^1 24^2 2^{-1} 3^{-2} 8^{-2} 12^{-1}$. We have

$$CE_g = \left\{ \boxed{\frac{1}{24}}, \frac{1}{24}, \frac{1}{12}, \boxed{\frac{1}{6}}, \frac{1}{6}, \boxed{\frac{5}{24}}, \frac{5}{24}, \boxed{\frac{7}{24}}, \frac{7}{24}, \frac{5}{12}, \boxed{\frac{11}{24}}, \frac{11}{24}, \boxed{\frac{13}{24}}, \frac{13}{24}, \right.$$
$$\left. \boxed{\frac{7}{12}}, \boxed{\frac{17}{24}}, \frac{17}{24}, \boxed{\frac{19}{24}}, \frac{19}{24}, \frac{5}{6}, \boxed{\frac{5}{6}}, \boxed{\frac{11}{12}}, \boxed{\frac{23}{24}}, \frac{23}{24} \right\}$$

We get,

$$E_g = \left\{ \frac{1}{24}, \frac{1}{6}, \frac{5}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}, \frac{13}{24}, \frac{7}{12}, \frac{17}{24}, \frac{19}{24}, \frac{5}{6}, \frac{11}{12}, \frac{23}{24} \right\}$$

As simple as this calculation looks, some elements require an extra step to find final values in E_g. Assuming E_g is ordered in ascending order, we need to change the twelfth element, λ_{12} to $1 - \lambda_{12} \pmod 1 = \lambda_1$.

Example 5.2.0.4. One such occurrence is for $g \in \mathrm{Co}_0$, an element in the conjugacy class 10C, equivalent to 5C in Co_1. While our usual methods would produce

$$E_g = \left\{ \frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{2}, \frac{1}{2}, \frac{7}{10}, \frac{7}{10}, \frac{7}{10}, \frac{9}{10}, \frac{9}{10}, \frac{9}{10} \right\},$$

running the code on this vector does not produce an integer value for $Z^{R,R}_{1,g}$. This lets us know that there is something happening. By trial and error, we figured out that we need to switch out $\lambda_{12} = \frac{9}{10}$ for $1 - \frac{9}{10} = \frac{1}{10}$. The eigenvector is thus

$$E_g = \left\{ \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{2}, \frac{1}{2}, \frac{7}{10}, \frac{7}{10}, \frac{7}{10}, \frac{9}{10}, \frac{9}{10} \right\}$$

Looking at the table of eigenvectors in appendix A, one might notice that this only happens for a handful of classes. While we used trial and error to find which elements needed this change, there is a mathematical explanation for this. The Frame shape tells us the conjugacy class of $\pm g$ in O_{24}. Now, by 4.1.0.1, $\pm g$ also lives in SO_{24}, but elements can be conjugate in O_{24} and not in SO_{24}. The eigenvector E_g, considered mod 1, tells us the conjugacy class in SO_{24}. By conjugating in SO_{24}, we can fully rearrange the entries in E_g and switch the signs of any even number of λ_is. In O_{24}, we can switch any number of signs. Thus, looking at the Frame shape, we get two conjugacy classes in SO_{24}, which differ by switching one λ_i to $1 - \lambda_i$ (mod 1).

Now, for g in SO_{24}, using 4.3.1, we can lift g to Spin(24), and perhaps map back down to SO^+_{24}. To identify a conjugacy class in Spin(24), we write down E_g (up to permutations and switching evenly-many signs) but consider it not mod 1, but rather as an integer vector whose total sum of entries is even. So, whereas for SO_{24}, every λ_i is congruent to one in $[0,1)^{12}$, in Spin(24) every λ_i is congruent to one in $[0,2) \times [0,1)^{11}$. In other words, the two lifts to SO_{24} differ by changing one λ_i to $1 + \lambda_i$.

All conjugacy classes of elements in Co_0 and Co_1, as well as their Frame shapes and eigenvalues are compiled in section 1 of appendix A.

5.3 Result Expectations

Recall from section 2.2 that $Z^R(\mathcal{H})$ is an integer. In $V^{f\natural}$, we know specifically which integer $Z^R(V^{f\natural})$ is:

Theorem 5.3.1. Let $V^{f\natural}$ be the N = 1 SVOA as described above and g be an element of Co_0 with Frame shape $\pi_g = \prod_{k|m} k^{p_k}$. Then $\mathrm{Z}^{R,R}_{1,g}(V^{f\natural}) = p_1$.

Proof. Recall that

$$Z^R(\mathcal{H}) = \mathrm{tr}_{\mathcal{H}}((-1)^f e^{-s\hat{H}}) \tag{5.3.0.1}$$

and

$$Z^{R,R}_{g,1}(V^{f\natural}) = \mathrm{tr}_{V^{f\natural}_R}((-1)^f g q^{L_0 - \frac{c}{24}}) \tag{5.3.0.2}$$

The Ramond Hilbert space is

$$V^{f\natural}_R = \mathbb{R}^{24} \oplus \text{higher spaces}, \tag{5.3.0.3}$$

where $\mathbb{R}^{24}$ is the ground state.

By 2.2.1, $Z^{R,R}_{g,1}(V^{f\natural})$ must be constant as a function of τ. So,

$$Z^{R,R}_{g,1}(\tau) = tr_{V^{f\natural}_R}((-1)^f g q^{L_0 - \frac{c}{24}}) = tr_{\mathbb{R}^{24}}(g), \tag{5.3.0.4}$$

which we can read off the Frame shape of g as the exponent of 1.

Zhu's theorem states that if V is any rational SVOA and M is any V-module, then Z(M) is a scalar-valued modular function for some subgroup of the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ [57]. A modular function is invariant respect to Γ. Zhu's theorem describes how characters of modules transform under the action of Γ. Specifically, the characters transform into linear transformations of other characters. By the main theorem by Carnahan and Miyamoto in [9], rationality implies g-rationality for g of finite order, so Zhu's theorem extends to twisted and twined characters.

The group Γ is generated by

$$\left\{ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} | S^2 = c = (ST)^3 \right\}$$

A full definition can be found in [47], while a thorough exploration can be found in [31]. It acts on functions via $S(f(\tau)) = f(-\frac{1}{\tau})$ and $T(f(\tau)) = f(\tau + 1)$. By an application of Zhu's theorem,

$$\begin{aligned} S[Z^{R,R}_{1,g}(\tau)] &= Z^{R,R}_{1,g}(-\frac{1}{\tau}) \\ &= Z^{R,R}_{g,1}(\tau) \\ &= \mathrm{tr}_{\mathbb{R}^{24}}(g) \end{aligned} \tag{5.3.0.5}$$

□

In layman's terms, we expect $Z^{R,R}_{1,g}(V^{f\natural})$ to be equal to the exponent of 1 in the Frame shape of $g \in \mathrm{Co}_0$, which is the trace of g in $\mathbb{R}^{24}$.

The R-character $Z^{NS,R}_{1,g}$ calculates the ground state of the $\lambda = 0$ eigenspace. We are looking to find out if the bosonic and fermionic ground states cancel out.

5.4 No twist: $Z^{R,R}_{1,1}$ and $Z^{NS,R}_{1,1}$

Let $\{\theta_i\}_{i=1}^4$ be the original Jacobi θ-functions, in terms of the nome $q = e^{2i\pi\tau}$ for $\tau \in \mathbb{H}$.

$$\theta_1 = \sum_{n\in\mathbb{Z}} (-1)^n q^{2\frac{(n+\frac{1}{2})^2}{2}} \tag{5.4.0.1}$$

$$\theta_2 = \sum_{n\in\mathbb{Z}} q^{\frac{(n+\frac{1}{2})^2}{2}} \tag{5.4.0.2}$$

$$\theta_3 = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} \tag{5.4.0.3}$$

$$\theta_4 = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} \tag{5.4.0.4}$$

Up to isomorphism, there are only three holomorphic SVOAs with central charge 12 [18]. Two of these are well-known to us:

1. $V^{f\natural}$ – the unique holomorphic SCFT with c = 12 and no spin-$\frac{1}{2}$ fields.

2. Fer(24) – the unique SVOA generated by a 24-dimensional vector space of free fermions.

We've seen previously in section §4.4.1 that we can obtain $V^{f\natural}$ from Fer(24) and all symmetries, up to some $\{\pm 1\}$ are already present. The NS-sector of Fer(24) is the $\mathbb{Z}^{12}$ lattice. Then, from section §4.2, we know

$$Z_{1,1}^{NS,NS}(Fer(24)) = \frac{\Theta_{\mathbb{Z}}^{12}}{\eta(\tau)^{12}} = \frac{\theta_3^{12}}{\eta(\tau)^{12}} \tag{5.4.0.5}$$

$$Z_{1,1}^{R,NS}(Fer(24)) = \frac{(\Theta_{\mathbb{Z}}^{S})^{12}}{\eta(\tau)^{12}} = \frac{\theta_4^{12}}{\eta(\tau)^{12}} \tag{5.4.0.6}$$

The R-sector of Fer(24) is the $(\mathbb{Z}+\frac{1}{2})^{12}$ lattice. Then,

$$Z_{1,1}^{NS,R}(Fer(24)) = \frac{\Theta_{\mathbb{Z}+\frac{1}{2}}^{12}}{\eta(\tau)^{12}} = \frac{\theta_2^{12}}{\eta(\tau)^{12}} \tag{5.4.0.7}$$

$$Z_{1,1}^{R,R}(Fer(24)) = \frac{(\Theta_{\mathbb{Z}+\frac{1}{2}}^{S})^{12}}{\eta(\tau)^{12}} = \frac{\theta_1^{12}}{\eta(\tau)^{12}} \tag{5.4.0.8}$$

Since $V^{f\natural} = Fer(24) /\!\!/ (-1)^f$, we can write all formulas in terms of Fer(24) and just play around with some $\{\pm 1\}$. Then, using the formulas from section §4.4.1, we get

$$\begin{aligned} Z_{1,1}^{R,R}(V^{f\natural}) &= \frac{1}{2}\Bigg[\bigg(Z_{1,1}^{NS,NS}\Big(\text{Fer}(24)\Big) - Z_{1,1}^{R,NS}\Big(\text{Fer}(24)\Big)\bigg) \\ &\quad - \bigg(Z_{1,1}^{NS,R}\Big(\text{Fer}(24)\Big) - Z_{1,1}^{R,R}\Big(\text{Fer}(24)\Big)\bigg)\Bigg] \qquad (5.4.0.9) \\ &= \frac{1}{2}\frac{1}{\eta(\tau)^{12}}\Big[\Big(\theta_3^{12} - \theta_4^{12}\Big) - \Big(\theta_2^{12} - \theta_1^{12}\Big)\Big] \qquad (5.4.0.10) \end{aligned}$$

and,

$$\begin{aligned} Z_{1,1}^{NS,R}(V^{f\natural}) &= \frac{1}{2}\Bigg[\bigg(Z_{1,1}^{NS,NS}\Big(\text{Fer}(24)\Big) - Z_{1,1}^{R,NS}\Big(\text{Fer}(24)\Big)\bigg) \\ &\quad + \bigg(Z_{1,1}^{NS,R}\Big(\text{Fer}(24)\Big) - Z_{1,1}^{R,R}\Big(\text{Fer}(24)\Big)\bigg)\Bigg] \qquad (5.4.0.11) \\ &= \frac{1}{2}\frac{1}{\eta(\tau)^{12}}\Big[\Big(\theta_3^{12} - \theta_4^{12}\Big) + \Big(\theta_2^{12} - \theta_1^{12}\Big)\Big] \qquad (5.4.0.12) \end{aligned}$$

Notice that the only difference between $Z^{R,R}_{1,1}(V^{f\natural})$ and $Z^{NS,R}_{1,1}(V^{f\natural})$ is a sign, highlighting the subtraction versus addition of bosonic and fermionic ground states respectively as explained in section §2.2. Then,

$$\begin{aligned} Z^{R,R}_{1,1} &= \frac{1}{2}\Big(\frac{1}{\eta(\tau)}\Big)^{12}\Big((\theta_3^{12} - \theta_4^{12}) - (\theta_2^{12} - \theta_1^{12})\Big) \\ &= \frac{1}{2\sqrt{\Delta(\tau)}}\Big((\theta_3^{12} - \theta_4^{12}) - (\theta_2^{12} - \theta_1^{12})\Big) \qquad (5.4.0.13) \\ &= 24 \end{aligned}$$

Recall that we can view "no twisting" as an action by the identity automorphism of Co_0, which is in the conjugacy class 1A. An element in 1A has Frame shape 1^{24}. Our result for $Z^{R,R}_{1,1}$ follows what we would expect from 5.3.1.

$$\begin{aligned} Z^{NS,R}_{1,1} &= \frac{1}{2}\Big(\frac{1}{\eta(\tau)}\Big)^{12}\Big((\theta_3^{12} - \theta_4^{12}) + (\theta_2^{12} - \theta_1^{12})\Big) \\ &= \frac{1}{2\sqrt{\Delta(\tau)})}\Big((\theta_3^{12} - \theta_4^{12}) + (\theta_2^{12} - \theta_1^{12})\Big) \qquad (5.4.0.14) \\ &= 24 + 4096q + 98304q^2 + 1228800q^3 + 10747904q^4 \qquad (5.4.0.15) \\ &+ 74244096q^5 + 432144384q^6 + 2204860416q^7 + ... \end{aligned}$$

We can see that

$$Z^{R,R}_{1,1} = Z^{NS,R}_{1,1}(q = 0) = 24$$

5.5 Twist: $Z^{R,R}_{1,g}$ and $Z^{NS,R}_{1,g}$

Now, we twist the R-sector by $g \in Co_0$. This results in the $(\mathbb{Z} + \lambda_i)^{12}$ shifted lattice, where the $\{\lambda_i\}_{i=1}^{12}$ are the eigenvalues of $g \in Co_0$ obtained using the methods in section §5.2.

We define the "g-twisted" Jacobi θ-functions.

$$\begin{aligned} \theta_\lambda &= \sum_{n\in\mathbb{Z}} q^{\frac{(n+\lambda)^2}{2}} \qquad (5.5.0.1) \\ &= ... + q^{\frac{(-2+\lambda)^2}{2}} + q^{\frac{(-1+\lambda)^2}{2}} + q^{\frac{\lambda^2}{2}} + q^{\frac{(1+\lambda)^2}{2}} + q^{\frac{(2+\lambda)^2}{2}} + q^{\frac{(3+\lambda)^2}{2}} + ... \end{aligned}$$

$$\theta^S_\lambda = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(n+\lambda)^2}{2}} \tag{5.5.0.2}$$
$$= ... + q^{\frac{(-2+\lambda)^2}{2}} - q^{\frac{(-1+\lambda)^2}{2}} + q^{\frac{\lambda^2}{2}} - q^{\frac{(1+\lambda)^2}{2}} + q^{\frac{(2+\lambda)^2}{2}} - q^{\frac{(3+\lambda)^2}{2}} + ...$$

$$\theta_{\lambda+\frac{1}{2}} = \sum_{n\in\mathbb{Z}} q^{\frac{(n+\lambda+\frac{1}{2})^2}{2}} \tag{5.5.0.3}$$
$$= ... + q^{\frac{(-\frac{3}{2}+\lambda)^2}{2}} + q^{\frac{(-\frac{1}{2}+\lambda)^2}{2}} + q^{\frac{(\frac{1}{2}+\lambda)^2}{2}} + q^{\frac{(\frac{3}{2}+\lambda)^2}{2}} + q^{\frac{(\frac{5}{2}+\lambda)^2}{2}} + ...$$

$$\theta^S_{\lambda+\frac{1}{2}} = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(n+\lambda+\frac{1}{2})^2}{2}} \tag{5.5.0.4}$$
$$= ... + q^{\frac{(-\frac{3}{2}+\lambda)^2}{2}} - q^{\frac{(-\frac{1}{2}+\lambda)^2}{2}} + q^{\frac{(\frac{1}{2}+\lambda)^2}{2}} - q^{\frac{(\frac{3}{2}+\lambda)^2}{2}} + q^{\frac{(\frac{5}{2}+\lambda)^2}{2}} - ...$$

We can find $Z^{R,R}_{1,g}(V^{f\natural})$ and $Z^{NS,R}_{1,g}(V^{f\natural})$ in a way similar as before. The g-twisted NS-sector of Fer(24) is the $\mathbb{Z}^{12}$ lattice shifted by $\{\lambda_i\}_{i=1}^{12}$. Then,

$$Z^{NS,NS}_{1,g}(Fer(24)) = \frac{(\theta_{\lambda_i})^{12}}{\eta(\tau)^{12}} \tag{5.5.0.5}$$
$$Z^{R,NS}_{1,g}(Fer(24)) = \frac{(\theta^S_{\lambda_i})^{12}}{\eta(\tau)^{12}} \tag{5.5.0.6}$$

The g-twisted R-sector of Fer(24) is the $(\mathbb{Z}+\frac{1}{2})^{12}$ lattice shifted by $\{\lambda_i\}_{i=1}^{12}$. Then,

$$Z^{NS,R}_{1,g}(Fer(24)) = \frac{(\theta_{\lambda_i+\frac{1}{2}})^{12}}{\eta(\tau)^{12}} \tag{5.5.0.7}$$
$$Z^{R,R}_{1,g}(Fer(24)) = \frac{(\theta^S_{\lambda_i+\frac{1}{2}})^{12}}{\eta(\tau)^{12}} \tag{5.5.0.8}$$

We have,

$$\begin{aligned}
Z_{1,g}^{R,R}(V^{f\natural}) &= \frac{1}{2}\Bigg[\bigg(Z_{1,g}^{NS,NS}\Big(\mathrm{Fer}(24)\Big) - Z_{1,g}^{R,NS}\Big(\mathrm{Fer}(24)\Big)\bigg) \\
&\quad - \bigg(Z_{1,g}^{NS,R}\Big(\mathrm{Fer}(24)\Big) - Z_{1,g}^{R,R}\Big(\mathrm{Fer}(24)\Big)\bigg)\Bigg] \qquad (5.5.0.9) \\
&= \frac{1}{2}\frac{1}{\eta(\tau)^{12}}\left[\left(\prod_{i=1}^{12}\theta_{\lambda_i} - \prod_{i=1}^{12}\theta_{\lambda_i}^{S}\right) - \left(\prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}} - \prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}}^{S}\right)\right] \\
&= \frac{1}{2\sqrt{\Delta(\tau))}}\left[\left(\prod_{i=1}^{12}\theta_{\lambda_i} - \prod_{i=1}^{12}\theta_{\lambda_i}^{S}\right) - \left(\prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}} - \prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}}^{S}\right)\right]
\end{aligned}$$

and,

$$\begin{aligned}
Z_{1,g}^{NS,R}(V^{f\natural}) &= \frac{1}{2}\Bigg[\bigg(Z_{1,g}^{NS,NS}\Big(\mathrm{Fer}(24)\Big) - Z_{1,g}^{R,NS}\Big(\mathrm{Fer}(24)\Big)\bigg) \\
&\quad + \bigg(Z_{1,g}^{NS,R}\Big(\mathrm{Fer}(24)\Big) - Z_{1,g}^{R,R}\Big(\mathrm{Fer}(24)\Big)\bigg)\Bigg] \qquad (5.5.0.10) \\
&= \frac{1}{2}\frac{1}{\eta(\tau)^{12}}\left[\left(\prod_{i=1}^{12}\theta_{\lambda_i} - \prod_{i=1}^{12}\theta_{\lambda_i}^{S}\right) + \left(\prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}} - \prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}}^{S}\right)\right] \\
&= \frac{1}{2\sqrt{\Delta(\tau))}}\left[\left(\prod_{i=1}^{12}\theta_{\lambda_i} - \prod_{i=1}^{12}\theta_{\lambda_i}^{S}\right) + \left(\prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}} - \prod_{i=1}^{12}\theta_{\lambda_i+\frac{1}{2}}^{S}\right)\right]
\end{aligned}$$

Again, the only difference between $Z_{1,1}^{R,R}(V^{f\natural})$ and $Z_{1,1}^{NS,R}(V^{f\natural})$ is a sign.

5.6 Coding Process

There were two main parts of coding to obtain the tables of values. The first part consisted of finding the vectors E_g for all 160 conjugacy classes of elements in Co_0. We used Java as a programming language as it was efficient and easy to use for large arrays and could keep track of rational numbers, called doubles in programming terms. While we tried to do this part solely through code, which worked for about half of the conjugacy classes, we had to do many of them by hand. There was no simple way to get a computer to make some of the arbitrary choices. Identifying the elements that fall into the last case was easy, if $\mathrm{Z}_{1,g}^{R,R}$ was not equal to the power of 1 in the Frame shape, then the change had to be made.

Surprisingly, most of the troubles came while mapping the calculation using Maple. Maple is a user-friendly numeric computing environment equipped with its own multi-paradigm programming language. While choosing a programming language, the primary concern was the use of large polynomials with fractional exponents in our calculations which posed an issue when dividing. Although computers can handle positive integer exponents easily, we had to produce many methods to deal with our tricky cases. The first simple major method changed all exponents in a given polynomial to integer values only, keeping track of the shift and applying it to $\sqrt{\Delta}$. While we initially thought this would be simple enough, Maple cannot apply its division function to negative exponents in polynomials. This resulted in us having to produce a method that only made the exponents positive integers. Again, keeping track of the shift so we can apply it to $\sqrt{\Delta}$.

Our main issue was the sheer size of the memory needed to run these calculations. To avoid issues we kept the infinite sums and products to $\sum_{-5}^{5}$ and $\prod_{1}^{10}$ respectively and truncated polynomials to the 10^{th} power. Unfortunately, the program routinely disconnected from the kernel, meaning it thought it was running on infinite loops even though it was not. We also tried to connect directly to the Maple servers through the university's access but continued to disconnect. This just meant that we could only run up to 20 calculations at a time.

Chapter 6

Results

The results are compiled in section 2 of appendix A. It was already predicted that for all $g \in Co_0$, $Z^{R,R}_{1,g}$ = "exponent of 1 in the Frame shape". The observed data validates this forecast.

Our main line of questioning was around the Ramond sector of Conway Moonshine and whether or not all g-twistings had a ground state of the same fermionic parity. As a consequence of our calculations, we can say that our question was answered positively for most, but not all $g \in Co_1, \pm g \in Co_0$. Indeed, we found a total of six outliers. The conjugacy classes 3C, 4B, 5C, 6F, 8D, and 9B in Co_1 have

$$Z^{NS,R}_{1,g}(q = 0) \neq Z^{R,R}_{1,g},$$

suggesting these twistings have ground states of both bosonic and fermionic parities. One important fact to note about these 6 outliers in the pattern is that they are all unbalanced. Other than this observation, we have no mathematical or physical reasoning for this almost-pattern.

Finding out that an experiment leads to some expected result is always gratifying. Even more rewarding to mathematicians is when results hold almost always, but not exactly always. This is a recurring storyline in Moonshine and this thesis is one in a vast array of examples. The empirical results of our calculation exceeded our expectations. Moonshine is riddled with mysterious conspiracies and almost-patterns. As such, discovering these for Conway Moonshine has opened some exciting doors.

Appendix A

Tables

A.1 Frame shapes and Eigenvalues

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$1A$	$1A$	1^{24}	0	0	0	0	0	0	0	0	0	0	0	0
$2A$	$1A$	$2^{24}1^{-24}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$2B$	$2A$	$1^{8}2^{8}$	0	0	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$2C$	$2A$	$2^{16}1^{-8}$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$4A$	$2B$	$4^{12}2^{-12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$2D$	$2C$	2^{12}	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$3A$	$3A$	$3^{12}1^{-12}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$6A$	$3A$	$1^{12}6^{12}2^{-12}3^{-12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$3B$	$3B$	$1^{6}3^{6}$	0	0	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$6B$	$3B$	$2^{6}6^{6}1^{-6}3^{-6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$3C$	$3C$	$3^{9}1^{-3}$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$6C$	$3C$	$1^{3}6^{9}2^{-3}3^{-9}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$3D$	$3D$	3^{8}	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$6D$	$3D$	$6^{8}3^{-8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$4B$	$4A$	$1^{8}4^{8}2^{-8}$	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$4C$	$4A$	$4^{8}1^{-8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$4D$	$4B$	$4^{8}2^{-4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$4E$	$4C$	$1^4 2^2 4^4$	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
$4F$	$4C$	$2^6 4^4 1^{-4}$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$4G$	$4D$	$2^4 4^4$	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
$8A$	$4E$	$8^6 4^{-6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{7}{8}$
$4H$	$4F$	4^6	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
$5A$	$5A$	$5^6 1^{-6}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$10A$	$5A$	$1^6 10^6 2^{-6} 5^{-6}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{7}{10}$	$\frac{7}{10}$	$\frac{7}{10}$	$\frac{9}{10}$	$\frac{9}{10}$	$\frac{9}{10}$
$5B$	$5B$	$1^4 5^4$	0	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$10B$	$5B$	$2^4 10^4 1^{-4} 5^{-4}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{10}$	$\frac{7}{10}$	$\frac{9}{10}$	$\frac{9}{10}$
$5C$	$5C$	$5^5 1^{-1}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$10C$	$5C$	$1^1 10^5 2^{-1} 5^{-5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{10}$	$\frac{7}{10}$	$\frac{7}{10}$	$\frac{9}{10}$	$\frac{9}{10}$	$\frac{1}{10}$
$6E$	$6A$	$3^4 6^4 1^{-4} 2^{-4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$
$6F$	$6A$	$1^4 6^8 2^{-8} 3^{-4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$12A$	$6B$	$2^6 12^6 4^{-6} 6^{-6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{11}{12}$	$\frac{11}{12}$	$\frac{11}{12}$
$6G$	$6C$	$2^5 3^4 6^1 1^{-4}$	0	0	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$
$6H$	$6C$	$1^4 2^1 6^5 3^{-4}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$6I$	$6D$	$1^5 3^1 6^4 2^{-4}$	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$
$6J$	$6D$	$2^1 6^5 1^{-5} 3^{-1}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$
$6K$	$6E$	$1^2 2^2 3^2 6^2$	0	0	0	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$
$6L$	$6E$	$2^4 6^4 1^{-2} 3^{-2}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$
$6M$	$6F$	$3^3 6^3 1^{-1} 2^{-1}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$6N$	$6F$	$1^1 6^6 2^{-2} 3^{-3}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$6O$	$6G$	$2^3 6^3$	0	0	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$
$12B$	$6H$	$12^4 6^{-4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{11}{12}$	$\frac{11}{12}$
$6P$	$6I$	6^4	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$
$7A$	$7A$	$7^4 1^{-4}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$
$14A$	$7A$	$1^4 14^4 2^{-4} 7^{-4}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{3}{14}$	$\frac{3}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$\frac{5}{14}$
$7B$	$7B$	$1^3 7^3$	0	0	0	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{6}{7}$
$14B$	$7B$	$2^3 14^3 1^{-3} 7^{-3}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{9}{14}$	$\frac{9}{14}$	$\frac{11}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{13}{14}$
$8B$	$8A$	$8^4 2^{-4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{7}{8}$
$8C$	$8B$	$2^4 8^4 4^{-4}$	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	$\frac{7}{8}$
$8D$	$8C$	$1^4 8^4 2^{-2} 4^{-2}$	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{7}{8}$
$8E$	$8C$	$2^2 8^4 1^{-4} 4^{-2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{7}{8}$
$8F$	$8D$	$8^4 4^{-2}$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{7}{8}$
$8G$	$8E$	$1^2 2^1 4^1 8^2$	0	0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{7}{8}$
$8H$	$8E$	$2^3 4^1 8^2 1^{-2}$	0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{7}{8}$
$8I$	$8F$	$4^2 8^2$	0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{7}{8}$
$9A$	$9A$	$9^3 1^{-3}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{8}{9}$
$18A$	$9A$	$1^3 18^3 2^{-3} 9^{-3}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{11}{18}$	$\frac{11}{18}$	$\frac{13}{18}$	$\frac{13}{18}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{17}{18}$	$\frac{17}{18}$
$9B$	$9B$	$9^3 3^{-1}$	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{8}{9}$
$18B$	$9B$	$3^1 18^3 6^{-1} 9^{-3}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{1}{2}$	$\frac{11}{18}$	$\frac{11}{18}$	$\frac{13}{18}$	$\frac{13}{18}$	$\frac{5}{6}$	$\frac{17}{18}$	$\frac{17}{18}$
$9C$	$9C$	$1^3 9^3 3^{-2}$	0	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{8}{9}$

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$18C$	$9C$	$2^3 3^2 18^3 1^{-3} 6^{-2} 9^{-3}$	$\frac{1}{18}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{11}{18}$	$\frac{11}{18}$	$\frac{13}{18}$	$\frac{13}{18}$	$\frac{5}{6}$	$\frac{17}{18}$	$\frac{17}{18}$
$10D$	$10A$	$5^2 10^2 1^{-2} 2^{-2}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
$10E$	$10A$	$1^2 10^4 2^{-4} 5^{-2}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{2}{5}$
$20A$	$10B$	$2^3 20^3 4^{-3} 10^{-3}$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{7}{20}$	$\frac{9}{20}$	$\frac{11}{20}$	$\frac{11}{20}$	$\frac{13}{20}$	$\frac{13}{20}$	$\frac{17}{20}$	$\frac{17}{20}$	$\frac{19}{20}$	$\frac{19}{20}$
$20B$	$10C$	$4^2 20^2 2^{-2} 10^{-2}$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{7}{20}$	$\frac{9}{20}$	$\frac{11}{20}$	$\frac{13}{20}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{17}{20}$	$\frac{19}{20}$
$10F$	$10D$	$2^3 5^2 10^1 1^{-2}$	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{9}{10}$
$10G$	$10D$	$1^2 2^1 10^3 5^{-2}$	0	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	$\frac{9}{10}$
$10H$	$10E$	$1^3 5^1 10^2 2^{-2}$	0	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{9}{10}$
$10I$	$10E$	$2^1 10^3 1^{-3} 5^{-1}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	$\frac{9}{10}$
$10J$	$10F$	$2^2 10^2$	0	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$
$11A$	$11A$	$1^2 11^2$	0	0	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{3}{11}$	$\frac{3}{11}$	$\frac{4}{11}$	$\frac{4}{11}$	$\frac{5}{11}$	$\frac{5}{11}$
$22A$	$11A$	$2^2 22^2 1^{-2} 11^{-2}$	0	$\frac{1}{22}$	$\frac{1}{11}$	$\frac{3}{22}$	$\frac{2}{11}$	$\frac{5}{22}$	$\frac{3}{11}$	$\frac{7}{22}$	$\frac{4}{11}$	$\frac{9}{22}$	$\frac{5}{11}$	$\frac{1}{2}$
$12C$	$12A$	$2^4 3^4 12^4 1^{-4} 4^{-4} 6^{-4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{5}{12}$
$12D$	$12A$	$1^4 12^4 3^{-4} 4^{-4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{5}{12}$
$12E$	$12B$	$2^2 12^4 4^{-4} 6^{-2}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{5}{12}$
$12F$	$12C$	$6^2 12^2 2^{-2} 4^2$	0	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{2}$
$12G$	$12D$	$2^1 3^3 12^3 1^{-1} 4^{-1} 6^{-3}$	0	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{11}{12}$	$\frac{11}{12}$
$12H$	$12D$	$1^1 12^3 3^{-3} 4^{-1}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{11}{12}$
$12I$	$12E$	$1^2 3^2 4^2 12^2 2^{-2} 6^{-2}$	0	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{5}{12}$
$12J$	$12E$	$4^2 12^2 1^{-2} 3^{-2}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{2}$
$24A$	$12F$	$4^3 24^3 8^{-3} 12^{-3}$	$\frac{1}{24}$	$\frac{5}{24}$	$\frac{7}{24}$	$\frac{11}{24}$	$\frac{13}{24}$	$\frac{13}{24}$	$\frac{17}{24}$	$\frac{17}{24}$	$\frac{19}{24}$	$\frac{19}{24}$	$\frac{23}{24}$	$\frac{23}{24}$

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$12K$	$12G$	$4^2 12^2 2^{-1} 6^{-1}$	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{11}{12}$
$12I$	$12H$	$1^1 2^2 3^1 12^2 4^{-2}$	0	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{11}{12}$
$12M$	$12H$	$2^3 6^1 12^2 1^{-1} 3^{-1} 4^{-2}$	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{11}{12}$
$12N$	$12I$	$2^2 3^2 4^1 12^1 1^{-2}$	0	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{11}{12}$
$12O$	$12I$	$1^2 4^1 6^2 12^1 3^{-2}$	0	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{11}{12}$
$12P$	$12J$	$2^1 4^1 6^1 12^1$	0	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{11}{12}$
$12Q$	$12K$	$1^3 12^3 2^{-1} 3^{-1} 4^{-1} 6^{-1}$	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{11}{12}$
$12R$	$12K$	$2^2 3^1 12^3 1^{-3} 4^{-1} 6^{-2}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{11}{12}$
$24B$	$12L$	$24^2 12^{-2}$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{24}$	$\frac{7}{24}$	$\frac{3}{8}$	$\frac{11}{24}$	$\frac{13}{24}$	$\frac{5}{8}$	$\frac{17}{24}$	$\frac{19}{24}$	$\frac{7}{8}$	$\frac{23}{24}$
$12S$	$12M$	12^2	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{2}$
$13A$	$13A$	$13^2 1^{-2}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{2}{13}$	$\frac{3}{13}$	$\frac{3}{13}$	$\frac{4}{13}$	$\frac{4}{13}$	$\frac{5}{13}$	$\frac{5}{13}$	$\frac{6}{13}$	$\frac{6}{13}$
$26A$	$13A$	$1^2 26^2 2^{-2} 13^{-2}$	$\frac{1}{26}$	$\frac{3}{26}$	$\frac{5}{26}$	$\frac{7}{26}$	$\frac{9}{26}$	$\frac{11}{26}$	$\frac{15}{26}$	$\frac{17}{26}$	$\frac{19}{26}$	$\frac{21}{26}$	$\frac{23}{26}$	$\frac{25}{26}$
$28A$	$14A$	$2^2 28^2 4^{-2} 14^{-2}$	$\frac{1}{28}$	$\frac{3}{28}$	$\frac{5}{28}$	$\frac{9}{28}$	$\frac{11}{28}$	$\frac{13}{28}$	$\frac{15}{28}$	$\frac{17}{28}$	$\frac{19}{28}$	$\frac{23}{28}$	$\frac{25}{28}$	$\frac{27}{28}$
$14C$	$14B$	$1^1 2^1 7^1 14^1$	0	0	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{9}{14}$	$\frac{5}{7}$	$\frac{11}{14}$	$\frac{6}{7}$	$\frac{13}{14}$
$14D$	$14B$	$2^2 14^2 1^{-1} 7^{-1}$	0	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{9}{14}$	$\frac{5}{7}$	$\frac{11}{14}$	$\frac{6}{7}$	$\frac{13}{14}$
$15A$	$15A$	$1^3 15^3 3^{-3} 5^{-3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{8}{15}$	$\frac{11}{15}$	$\frac{11}{15}$	$\frac{13}{15}$	$\frac{13}{15}$	$\frac{14}{15}$
$30A$	$15A$	$2^3 3^3 5^3 30^3 1^{-3} 6^{-3} 10^{-3} 15^{-3}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{7}{30}$	$\frac{11}{30}$	$\frac{13}{30}$	$\frac{17}{30}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{19}{30}$	$\frac{23}{30}$	$\frac{23}{30}$	$\frac{29}{30}$
$15B$	$15B$	$3^2 15^2 1^{-2} 5^{-2}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{15}$	$\frac{7}{15}$
$30B$	$15B$	$1^2 5^2 6^2 30^2 2^{-2} 3^{-2} 10^{-2} 15^{-2}$	$\frac{1}{30}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{7}{30}$	$\frac{11}{30}$	$\frac{13}{30}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{23}{30}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{29}{30}$
$15C$	$15C$	$15^2 3^{-2}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{7}{15}$	$\frac{7}{15}$
$30C$	$15C$	$3^2 30^2 6^{-2} 15^{-2}$	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{7}{30}$	$\frac{3}{10}$	$\frac{11}{30}$	$\frac{13}{30}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{7}{10}$	$\frac{23}{30}$	$\frac{9}{10}$	$\frac{29}{30}$

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$15D$	$15D$	$1^1 3^1 5^1 15^1$	0	0	$\frac{1}{15}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{11}{15}$	$\frac{4}{5}$	$\frac{13}{15}$
$30D$	$15D$	$2^1 6^1 10^1 30^1 1^{-1} 3^{-1} 5^{-1} 15^{-1}$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{7}{10}$	$\frac{23}{30}$	$\frac{5}{6}$	$\frac{9}{10}$	$\frac{29}{30}$
$15E$	$15E$	$1^2 15^2 3^{-1} 5^{-1}$	0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{11}{15}$	$\frac{13}{15}$	$\frac{14}{15}$
$30E$	$15E$	$2^2 3^1 5^1 30^2 1^{-2} 6^{-1} 10^{-1} 15^{-2}$	$\frac{1}{30}$	$\frac{7}{30}$	$\frac{11}{30}$	$\frac{13}{30}$	$\frac{1}{2}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{7}{10}$	$\frac{23}{30}$	$\frac{5}{6}$	$\frac{9}{10}$	$\frac{29}{30}$
$16A$	$16A$	$2^2 16^2 4^{-1} 8^{-1}$	0	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$
$16B$	$16B$	$1^2 16^2 2^{-1} 8^{-1}$	0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{15}{16}$
$16C$	$16B$	$2^1 16^2 1^{-2} 8^{-1}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$
$18D$	$18A$	$9^1 18^1 1^{-1} 2^{-1}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{11}{18}$	$\frac{2}{3}$	$\frac{13}{18}$	$\frac{7}{9}$	$\frac{5}{6}$	$\frac{8}{9}$	$\frac{17}{18}$
$18E$	$18A$	$1^1 18^2 2^{-2} 9^{-1}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{5}{9}$	$\frac{11}{18}$	$\frac{2}{3}$	$\frac{13}{18}$	$\frac{7}{9}$	$\frac{5}{6}$	$\frac{8}{9}$	$\frac{17}{18}$
$18F$	$18B$	$1^2 9^1 18^1 2^{-1} 3^{-1}$	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{11}{18}$	$\frac{2}{3}$	$\frac{13}{18}$	$\frac{7}{9}$	$\frac{5}{6}$	$\frac{8}{9}$	$\frac{17}{18}$
$18G$	$18B$	$2^1 3^1 18^2 1^{-2} 6^{-1} 9^{-1}$	$\frac{1}{18}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{1}{2}$	$\frac{5}{9}$	$\frac{11}{18}$	$\frac{2}{3}$	$\frac{13}{18}$	$\frac{7}{9}$	$\frac{5}{6}$	$\frac{8}{9}$	$\frac{17}{18}$
$18H$	$18C$	$2^2 9^1 18^1 1^{-1} 6^{-1}$	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{2}$	$\frac{5}{9}$	$\frac{11}{18}$	$\frac{2}{3}$	$\frac{13}{18}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{17}{18}$
$18I$	$18C$	$1^1 2^1 18^2 6^{-1} 9^{-1}$	0	$\frac{1}{18}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{1}{2}$	$\frac{5}{9}$	$\frac{11}{18}$	$\frac{13}{18}$	$\frac{7}{9}$	$\frac{5}{6}$	$\frac{8}{9}$	$\frac{17}{18}$
$20C$	$20A$	$2^2 5^2 20^2 1^{-2} 4^{-2} 10^{-2}$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{1}{5}$	$\frac{7}{20}$	$\frac{2}{5}$	$\frac{9}{20}$	$\frac{11}{20}$	$\frac{3}{5}$	$\frac{13}{20}$	$\frac{4}{5}$	$\frac{17}{20}$	$\frac{19}{20}$
$20D$	$20A$	$1^2 20^2 4^{-2} 5^{-2}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{3}{10}$	$\frac{7}{20}$	$\frac{9}{20}$	$\frac{11}{20}$	$\frac{13}{20}$	$\frac{7}{10}$	$\frac{17}{20}$	$\frac{9}{10}$	$\frac{19}{20}$
$20E$	$20B$	$4^1 20^1$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{11}{20}$	$\frac{3}{5}$	$\frac{13}{20}$	$\frac{7}{10}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{17}{20}$	$\frac{9}{10}$	$\frac{19}{20}$
$20F$	$20C$	$2^2 5^1 20^1 1^{-1} 4^{-1}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{11}{20}$	$\frac{3}{5}$	$\frac{13}{20}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{17}{20}$	$\frac{9}{10}$	$\frac{19}{20}$
$20G$	$20C$	$1^1 2^1 10^1 20^1 4^{-1} 5^{-1}$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{11}{20}$	$\frac{3}{5}$	$\frac{13}{20}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{17}{20}$	$\frac{9}{10}$	$\frac{19}{20}$
$21A$	$21A$	$1^2 21^2 3^{-2} 7^{-2}$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{8}{21}$	$\frac{10}{21}$	$\frac{11}{21}$	$\frac{13}{21}$	$\frac{16}{21}$	$\frac{17}{21}$	$\frac{19}{21}$	$\frac{20}{21}$
$42A$	$21A$	$2^2 3^2 7^2 42^2 1^{-2} 6^{-2} 14^{-2} 21^{-2}$	$\frac{1}{42}$	$\frac{5}{42}$	$\frac{11}{42}$	$\frac{13}{42}$	$\frac{17}{42}$	$\frac{19}{42}$	$\frac{23}{42}$	$\frac{25}{42}$	$\frac{29}{42}$	$\frac{31}{42}$	$\frac{37}{42}$	$\frac{41}{42}$
$21B$	$21B$	$7^1 21^1 1^{-1} 3^{-1}$	$\frac{1}{21}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{11}{21}$	$\frac{4}{7}$	$\frac{13}{21}$	$\frac{5}{7}$	$\frac{16}{21}$	$\frac{17}{21}$	$\frac{6}{7}$	$\frac{19}{21}$

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$42B$	$21B$	$1^1 3^1 14^1 42^1 2^{-1} 6^{-1} 7^{-1} 21^{-1}$	$\frac{1}{42}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{23}{42}$	$\frac{25}{42}$	$\frac{9}{14}$	$\frac{29}{42}$	$\frac{31}{42}$	$\frac{11}{14}$	$\frac{37}{42}$	$\frac{13}{14}$
$21C$	$21C$	$3^1 21^1$	0	$\frac{1}{3}$	$\frac{11}{21}$	$\frac{4}{7}$	$\frac{13}{21}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{16}{21}$	$\frac{17}{21}$	$\frac{6}{7}$	$\frac{19}{21}$	$\frac{20}{21}$
$42C$	$21C$	$6^1 42^1 3^{-1} 21^{-1}$	$\frac{1}{2}$	$\frac{23}{42}$	$\frac{25}{42}$	$\frac{9}{14}$	$\frac{29}{42}$	$\frac{31}{42}$	$\frac{11}{14}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{37}{42}$	$\frac{13}{14}$	$\frac{41}{42}$
$22BC$	$22A$	$2^1 22^1$	0	$\frac{1}{2}$	$\frac{6}{11}$	$\frac{13}{22}$	$\frac{7}{11}$	$\frac{15}{22}$	$\frac{8}{11}$	$\frac{17}{22}$	$\frac{9}{11}$	$\frac{19}{22}$	$\frac{10}{11}$	$\frac{21}{22}$
$23AB$	$23AB$	$1^1 23^1$	0	$\frac{1}{23}$	$\frac{12}{23}$	$\frac{13}{23}$	$\frac{14}{23}$	$\frac{15}{23}$	$\frac{16}{23}$	$\frac{17}{23}$	$\frac{18}{23}$	$\frac{19}{23}$	$\frac{20}{23}$	$\frac{21}{23}$
$46AB$	$23AB$	$2^1 46^1 1^{-1} 23^{-1}$	$\frac{1}{2}$	$\frac{25}{46}$	$\frac{27}{46}$	$\frac{29}{46}$	$\frac{31}{46}$	$\frac{33}{46}$	$\frac{35}{46}$	$\frac{37}{46}$	$\frac{39}{46}$	$\frac{41}{46}$	$\frac{43}{46}$	$\frac{45}{46}$
$24C$	$24A$	$2^2 24^2 6^{-2} 8^{-2}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{5}{24}$	$\frac{5}{24}$	$\frac{7}{24}$	$\frac{7}{24}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{11}{24}$	$\frac{11}{24}$
$24D$	$24B$	$2^1 3^2 4^1 24^2 1^{-2} 6^{-1} 8^{-2} 12^{-1}$	$\frac{1}{24}$	$\frac{5}{24}$	$\frac{7}{24}$	$\frac{1}{3}$	$\frac{11}{24}$	$\frac{13}{24}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{17}{24}$	$\frac{19}{24}$	$\frac{11}{12}$	$\frac{23}{24}$
$24E$	$24B$	$1^2 4^1 6^1 24^2 2^{-1} 3^{-2} 8^{-2} 12^{-1}$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{5}{24}$	$\frac{7}{24}$	$\frac{11}{24}$	$\frac{13}{24}$	$\frac{7}{12}$	$\frac{17}{24}$	$\frac{19}{24}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{23}{24}$
$24F$	$24C$	$8^1 24^1 2^{-1} 6^{-1}$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{13}{24}$	$\frac{7}{12}$	$\frac{5}{8}$	$\frac{17}{24}$	$\frac{3}{4}$	$\frac{19}{24}$	$\frac{7}{8}$	$\frac{11}{12}$
$14G$	$24D$	$12^1 24^1 4^{-1} 8^{-1}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{13}{24}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{17}{24}$	$\frac{19}{24}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{23}{24}$
$24H$	$24E$	$2^1 6^1 8^1 24^1 4^{-1} 12^{-1}$	0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{13}{24}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{17}{24}$	$\frac{19}{24}$	$\frac{5}{6}$	$\frac{7}{8}$	$\frac{23}{24}$
$24I$	$24F$	$2^1 3^1 4^1 24^1 1^{-1} 8^{-1}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{13}{24}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{17}{24}$	$\frac{3}{4}$	$\frac{19}{24}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{23}{24}$
$24J$	$24F$	$1^1 4^1 6^1 24^1 3^{-1} 8^{-1}$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{13}{24}$	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{17}{24}$	$\frac{3}{4}$	$\frac{19}{24}$	$\frac{5}{6}$	$\frac{11}{12}$	$\frac{23}{24}$
$52A$	$26A$	$2^1 52^1 4^{-1} 26^{-1}$	$\frac{27}{52}$	$\frac{29}{52}$	$\frac{31}{52}$	$\frac{33}{52}$	$\frac{35}{52}$	$\frac{37}{52}$	$\frac{41}{52}$	$\frac{43}{52}$	$\frac{45}{52}$	$\frac{47}{52}$	$\frac{49}{52}$	$\frac{51}{52}$
$28B$	$28A$	$1^1 4^1 7^1 28^1 2^{-1} 14^{-1}$	0	$\frac{1}{4}$	$\frac{15}{28}$	$\frac{4}{7}$	$\frac{17}{28}$	$\frac{19}{28}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{23}{28}$	$\frac{6}{7}$	$\frac{25}{28}$	$\frac{27}{28}$
$28C$	$28A$	$4^1 28^1 1^{-1} 7^{-1}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{15}{28}$	$\frac{17}{28}$	$\frac{9}{14}$	$\frac{19}{28}$	$\frac{3}{4}$	$\frac{11}{14}$	$\frac{23}{28}$	$\frac{25}{28}$	$\frac{13}{14}$	$\frac{27}{28}$
$56AB$	$28B$	$4^1 56^1 8^{-1} 28^{-1}$	$\frac{29}{56}$	$\frac{31}{56}$	$\frac{33}{56}$	$\frac{37}{56}$	$\frac{39}{56}$	$\frac{41}{56}$	$\frac{43}{56}$	$\frac{45}{56}$	$\frac{47}{56}$	$\frac{51}{56}$	$\frac{53}{56}$	$\frac{55}{56}$
$30F$	$30A$	$1^1 2^1 15^1 30^1 3^{-1} 5^{-1} 6^{-1} 10^{-1}$	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{11}{15}$	$\frac{23}{30}$	$\frac{13}{15}$	$\frac{14}{15}$
$30G$	$30A$	$2^2 3^1 5^1 30^2 1^{-1} 6^{-2} 10^{-2} 15^{-1}$	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{7}{30}$	$\frac{11}{30}$	$\frac{13}{30}$	$\frac{8}{15}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{11}{15}$	$\frac{23}{30}$	$\frac{13}{15}$	$\frac{29}{30}$
$60A$	$30B$	$2^1 10^1 12^1 60^1 4^{-1} 6^{-1} 20^{-1} 30^{-1}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{31}{60}$	$\frac{7}{12}$	$\frac{37}{60}$	$\frac{41}{60}$	$\frac{43}{60}$	$\frac{47}{60}$	$\frac{49}{60}$	$\frac{53}{60}$	$\frac{11}{12}$	$\frac{59}{60}$

Co_0	Co_1	π_g	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$60B$	$30C$	$6^1 60^1 12^{-1} 30^{-1}$	$\frac{31}{60}$	$\frac{11}{20}$	$\frac{37}{60}$	$\frac{13}{20}$	$\frac{41}{60}$	$\frac{43}{60}$	$\frac{47}{60}$	$\frac{49}{60}$	$\frac{17}{20}$	$\frac{53}{60}$	$\frac{19}{20}$	$\frac{59}{60}$
$30H$	$30D$	$1^1 6^1 10^1 15^1 3^{-1} 5^{-1}$	0	$\frac{1}{2}$	$\frac{8}{15}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{7}{10}$	$\frac{11}{15}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{13}{15}$	$\frac{9}{10}$	$\frac{14}{15}$
$30I$	$30D$	$2^1 3^1 5^1 30^1 1^{-1} 15^{-1}$	0	$\frac{1}{2}$	$\frac{17}{30}$	$\frac{3}{5}$	$\frac{19}{30}$	$\frac{2}{3}$	$\frac{7}{10}$	$\frac{23}{30}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{9}{10}$	$\frac{29}{30}$
$30J$	$30E$	$2^1 3^1 5^1 30^1 6^{-1} 10^{-1}$	0	$\frac{1}{30}$	$\frac{8}{15}$	$\frac{17}{30}$	$\frac{3}{5}$	$\frac{19}{30}$	$\frac{2}{3}$	$\frac{11}{15}$	$\frac{23}{30}$	$\frac{4}{5}$	$\frac{13}{15}$	$\frac{14}{15}$
$30K$	$30E$	$2^1 30^1 3^{-1} 5^{-1}$	$\frac{1}{2}$	$\frac{8}{15}$	$\frac{17}{30}$	$\frac{19}{30}$	$\frac{7}{10}$	$\frac{11}{15}$	$\frac{23}{30}$	$\frac{5}{6}$	$\frac{13}{15}$	$\frac{9}{10}$	$\frac{14}{15}$	$\frac{29}{30}$
$33A$	$33A$	$3^1 33^1 1^{-1} 11^{-1}$	$\frac{1}{33}$	$\frac{1}{3}$	$\frac{17}{33}$	$\frac{19}{33}$	$\frac{20}{33}$	$\frac{2}{3}$	$\frac{23}{33}$	$\frac{25}{33}$	$\frac{26}{33}$	$\frac{28}{33}$	$\frac{29}{33}$	$\frac{31}{33}$
$66A$	$33A$	$1^1 6^1 11^1 66^1 2^{-1} 3^{-1} 22^{-1} 33^{-1}$	$\frac{1}{66}$	$\frac{1}{6}$	$\frac{35}{66}$	$\frac{37}{66}$	$\frac{41}{66}$	$\frac{43}{66}$	$\frac{47}{66}$	$\frac{49}{66}$	$\frac{53}{66}$	$\frac{5}{6}$	$\frac{59}{66}$	$\frac{61}{66}$
$35A$	$35A$	$1^1 35^1 5^{-1} 7^{-1}$	$\frac{1}{35}$	$\frac{18}{35}$	$\frac{19}{35}$	$\frac{22}{35}$	$\frac{23}{35}$	$\frac{24}{35}$	$\frac{26}{35}$	$\frac{27}{35}$	$\frac{29}{35}$	$\frac{31}{35}$	$\frac{32}{35}$	$\frac{33}{35}$
$70A$	$35A$	$2^1 5^1 7^1 70^1 1^{-1} 10^{-1} 14^{-1} 35^{-1}$	$\frac{1}{70}$	$\frac{37}{70}$	$\frac{39}{70}$	$\frac{41}{70}$	$\frac{43}{70}$	$\frac{47}{70}$	$\frac{51}{70}$	$\frac{53}{70}$	$\frac{57}{70}$	$\frac{59}{70}$	$\frac{61}{70}$	$\frac{67}{70}$
$36A$	$36A$	$2^1 9^1 36^1 1^{-1} 4^{-1} 18^{-1}$	$\frac{19}{36}$	$\frac{5}{9}$	$\frac{7}{12}$	$\frac{23}{36}$	$\frac{2}{3}$	$\frac{25}{36}$	$\frac{7}{9}$	$\frac{29}{36}$	$\frac{31}{36}$	$\frac{8}{9}$	$\frac{11}{12}$	$\frac{35}{36}$
$36B$	$36A$	$1^1 36^1 4^{-1} 9^{-1}$	$\frac{19}{36}$	$\frac{7}{12}$	$\frac{11}{18}$	$\frac{23}{36}$	$\frac{25}{36}$	$\frac{13}{18}$	$\frac{29}{36}$	$\frac{5}{6}$	$\frac{31}{36}$	$\frac{11}{12}$	$\frac{17}{18}$	$\frac{35}{36}$
$39AB$	$39AB$	$1^1 39^1 3^{-1} 13^{-1}$	$\frac{20}{39}$	$\frac{22}{39}$	$\frac{23}{39}$	$\frac{25}{39}$	$\frac{28}{39}$	$\frac{29}{39}$	$\frac{31}{39}$	$\frac{32}{39}$	$\frac{34}{39}$	$\frac{35}{39}$	$\frac{37}{39}$	$\frac{38}{39}$
$78AB$	$39AB$	$2^1 3^1 13^1 78^1 1^{-1} 6^{-1} 26^{-1} 39^{-1}$	$\frac{41}{78}$	$\frac{43}{78}$	$\frac{47}{78}$	$\frac{49}{78}$	$\frac{53}{78}$	$\frac{55}{78}$	$\frac{59}{78}$	$\frac{61}{78}$	$\frac{67}{78}$	$\frac{71}{78}$	$\frac{73}{78}$	$\frac{77}{78}$
$40AB$	$40A$	$2^1 40^1 8^{-1} 10^{-1}$	$\frac{21}{40}$	$\frac{11}{20}$	$\frac{23}{40}$	$\frac{13}{20}$	$\frac{27}{40}$	$\frac{29}{40}$	$\frac{31}{40}$	$\frac{33}{40}$	$\frac{17}{20}$	$\frac{37}{40}$	$\frac{19}{20}$	$\frac{39}{40}$
$84A$	$42A$	$4^1 6^1 14^1 84^1 2^{-1} 12^{-1} 28^{-1} 42^{-1}$	$\frac{43}{84}$	$\frac{47}{84}$	$\frac{53}{84}$	$\frac{55}{84}$	$\frac{59}{84}$	$\frac{61}{84}$	$\frac{65}{84}$	$\frac{67}{84}$	$\frac{71}{84}$	$\frac{73}{84}$	$\frac{79}{84}$	$\frac{83}{84}$
$60C$	$60A$	$1^1 4^1 6^1 10^1 15^1 60^1 2^{-1} 3^{-1} 5^{-1} 12^{-1} 20^{-1} 30^{-1}$	$\frac{1}{60}$	$\frac{31}{60}$	$\frac{8}{15}$	$\frac{37}{60}$	$\frac{41}{60}$	$\frac{43}{60}$	$\frac{11}{15}$	$\frac{47}{60}$	$\frac{49}{60}$	$\frac{13}{15}$	$\frac{53}{60}$	$\frac{14}{15}$
$60D$	$60A$	$3^1 4^1 5^1 60^1 1^{-1} 12^{-1} 15^{-1} 20^{-1}$	$\frac{1}{60}$	$\frac{31}{60}$	$\frac{17}{30}$	$\frac{37}{60}$	$\frac{19}{30}$	$\frac{41}{60}$	$\frac{43}{60}$	$\frac{23}{30}$	$\frac{47}{60}$	$\frac{49}{60}$	$\frac{53}{60}$	$\frac{29}{30}$

A.2 Neveu-Schwarz and Ramond twisted characters

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$1A$	$1A$	24	$24+4096q+98304q^2+1228800q^3+10747904q^4+74244096q^5+432144096q^6+...$
$2A$	$1A$	-24	$24+4096q+98304q^2+1228800q^3+10747904q^4+74244096q^5+432144096q^6+...$
$2B$	$2A$	8	$8+256q^{1/2}+2048q+11264q^{3/2}+49152q^2+183808q^{5/2}+614400q^3+1882112q^{7/2}+...$
$2C$	$2A$	-8	$8+256q^{1/2}+2048q+11264q^{3/2}+49152q^2+183808q^{5/2}+614400q^3+1882112q^{7/2}+...$
$4A$	$2B$	0	$0+24q^{1/8}+464q^{5/8}+3192q^{9/8}+16656q^{13/8}+69040q^{17/8}+251136q^{21/8}+...$
$2D$	$2C$	0	$0+64q^{1/4}+768q^{3/4}+4992q^{5/4}+24064q^{7/4}+96576q^{9/4}+340224q^{11/4}+...$
$3A$	$3A$	-12	$12+24q^{1/3}+440q^{2/3}+1608q+3192q^{4/3}+13904q^{5/3}+35688q^2+68160q^{7/3}+...$
$6A$	$3A$	12	$12+24q^{1/3}+440q^{2/3}+1608q+3192q^{4/3}+13904q^{5/3}+35688q^2+68160q^{7/3}+...$
$3B$	$3B$	6	$6+64q^{1/3}+384q^{2/3}+1344q+4352q^{4/3}+12672q^{5/3}+32640q^2+79872q^{7/3}+...$
$6B$	$3B$	-6	$6+64q^{1/3}+384q^{2/3}+1344q+4352q^{4/3}+12672q^{5/3}+32640q^2+79872q^{7/3}+...$
3C	**3C**	**-3**	$\mathbf{5}+72q^{1/3}+360q^{2/3}+1368q+4392q^{4/3}+12528q^{5/3}+32760q^2+80064q^{7/3}+...$
6C	**3C**	**3**	$\mathbf{5}+72q^{1/3}+360q^{2/3}+1368q+4392q^{4/3}+12528q^{5/3}+32760q^2+80064q^{7/3}+...$
$3D$	$3D$	0	$0+16q^{1/9}+128q^{4/9}+576q^{7/9}+2048q^{10/9}+6304q^{13/9}+17408q^{16/9}+44416q^{19/9}+...$
$6D$	$3D$	0	$0+16q^{1/9}+128q^{4/9}+576q^{7/9}+2048q^{10/9}+6304q^{13/9}+17408q^{16/9}+44416q^{19/9}+...$
$4B$	$4A$	8	$8+16q^{1/4}+128q^{1/2}+448q^{3/4}+1024q+2272q^{5/4}+5632q^{3/2}+12672q^{7/4}+24576q^2+...$
$4C$	$4A$	-8	$8+16q^{1/4}+128q^{1/2}+448q^{3/4}+1024q+2272q^{5/4}+5632q^{3/2}+12672q^{7/4}+24576q^2+...$
4D	**4B**	**0**	$\mathbf{4}+32q^{1/4}+128q^{1/2}+384q^{3/4}+1024q+2496q^{5/4}+5632q^{3/2}+12032q^{7/4}+24576q^2+...$
$4E$	$4C$	4	$4+32q^{1/4}+128q^{1/2}+384q^{3/4}+1024q+2496q^{5/4}+5632q^{3/2}+12032q^{7/4}+24576q^2+...$
$4F$	$4C$	-4	$4+32q^{1/4}+128q^{1/2}+384q^{3/4}+1024q+2496q^{5/4}+5632q^{3/2}+12032q^{7/4}+24576q^2+...$

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$4G$	$4D$	0	$0+16q^{1/8}+64q^{3/8}+224q^{5/8}+640q^{7/8}+1616q^{9/8}+3776q^{11/8}+8288q^{13/8}+...$
$8A$	$4E$	0	$0+12q^{3/32}+52q^{11/32}+204q^{19/32}+552q^{27/32}+1456q^{35/32}+3396q^{43/32}+7560q^{51/32}+...$
$4H$	$4F$	0	$0+8q^{1/16}+48q^{5/16}+168q^{9/16}+496q^{13/16}+1296q^{17/16}+3072q^{21/16}+6840q^{25/16}+...$
$5A$	$5A$	-6	$6+12q^{1/5}+52q^{2/5}+192q^{3/5}+372q^{4/5}+844q+1584q^{6/5}+3216q^{7/5}+6388q^{8/5}+...$
$10A$	$5A$	6	$6+12q^{1/5}+52q^{2/5}+192q^{3/5}+372q^{4/5}+844q+1584q^{6/5}+3216q^{7/5}+6388q^{8/5}+...$
$5B$	$5B$	4	$4+16q^{1/5}+64q^{2/5}+160q^{3/5}+384q^{4/5}+816q+1664q^{6/5}+3296q^{7/5}+6144q^{8/5}+1...$
$10B$	$5B$	-4	$4+16q^{1/5}+64q^{2/5}+160q^{3/5}+384q^{4/5}+816q+1664q^{6/5}+3296q^{7/5}+6144q^{8/5}+...$
5C	**5C**	**-1**	$\mathbf{3}+20q^{1/5}+60q^{2/5}+160q^{3/5}+380q^{4/5}+820q+1680q^{6/5}+3280q^{7/5}+6140q^{8/5}+...$
10C	**5C**	**1**	$\mathbf{3}+20q^{1/5}+60q^{2/5}+160q^{3/5}+380q^{4/5}+820q+1680q^{6/5}+3280q^{7/5}+6140q^{8/5}+...$
$6E$	$6A$	-4	$4+16q^{1/6}+8q^{1/3}+112q^{1/2}+232q^{2/3}+224q^{5/6}+792q+1408q^{7/6}+1576q^{4/3}+...$
$6F$	$6A$	4	$4+16q^{1/6}+8q^{1/3}+112q^{1/2}+232q^{2/3}+224q^{5/6}+792q+1408q^{7/6}+1576q^{4/3}+...$
$12A$	$6B$	0	$0+12q^{1/8}+40q^{7/24}+24q^{11/24}+192q^{5/8}+360q^{19/24}+384q^{23/24}+1236q^{9/8}+...$
$6G$	$6C$	-4	$4+8q^{1/6}+40q^{1/3}+88q^{1/2}+168q^{2/3}+368q^{5/6}+696q+1216q^{7/6}+2216q^{4/3}+...$
$6H$	$6C$	4	$4+8q^{1/6}+40q^{1/3}+88q^{1/2}+168q^{2/3}+368q^{5/6}+696q+1216q^{7/6}+2216q^{4/3}+...$
$6I$	$6D$	5	$5+8q^{1/6}+32q^{1/3}+88q^{1/2}+192q^{2/3}+368q^{5/6}+672q+1216q^{7/6}+2176q^{4/3}+...$
$6J$	$6D$	-5	$5+8q^{1/6}+32q^{1/3}+88q^{1/2}+192q^{2/3}+368q^{5/6}+672q+1216q^{7/6}+2176q^{4/3}+...$
$6K$	$6E$	2	$2+16q^{1/6}+32q^{1/3}+80q^{1/2}+192q^{2/3}+352q^{5/6}+672q+1280q^{7/6}+2176q^{4/3}+...$
$6L$	$6E$	-2	$2+16q^{1/6}+32q^{1/3}+80q^{1/2}+192q^{2/3}+352q^{5/6}+672q+1280q^{7/6}+2176q^{4/3}+...$
6M	**6F**	**-1**	$\mathbf{3}+12q^{1/6}+36q^{1/3}+84q^{1/2}+180q^{2/3}+360q^{5/6}+684q+1248q^{7/6}+2196q^{4/3}+...$
6N	**6F**	**1**	$\mathbf{3}+12q^{1/6}+36q^{1/3}+84q^{1/2}+180q^{2/3}+360q^{5/6}+684q+1248q^{7/6}+2196q^{4/3}+...$
$6O$	$6G$	0	$0+8q^{1/12}+24q^{1/4}+48q^{5/12}+128q^{7/12}+264q^{3/4}+480q^{11/12}+944q^{13/12}+1680q^{5/4}+...$

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$12B$	$6H$	0	$0+8q^{5/72}+16q^{17/72}+56q^{29/72}+112q^{41/72}+248q^{53/72}+464q^{65/72}+896q^{77/72}+...$
$6P$	$6I$	0	$0+4q^{1/36}+16q^{7/36}+40q^{13/36}+96q^{19/36}+204q^{25/36}+400q^{31/36}+760q^{37/36}+...$
$7A$	$7A$	-4	$4+8q^{1/7}+16q^{2/7}+56q^{3/7}+104q^{4/7}+200q^{5/7}+328q^{6/7}+592q+976q^{8/7}+1560q^{9/7}+...$
$14A$	$7A$	4	$4+8q^{1/7}+16q^{2/7}+56q^{3/7}+104q^{4/7}+200q^{5/7}+328q^{6/7}+592q+976q^{8/7}+1560q^{9/7}+...$
$7B$	$7B$	3	$3+8q^{1/7}+24q^{2/7}+48q^{3/7}+104q^{4/7}+192q^{5/7}+336q^{6/7}+584q+984q^{8/7}+1608q^{9/7}+...$
$14B$	$7B$	-3	$3+8q^{1/7}+24q^{2/7}+48q^{3/7}+104q^{4/7}+192q^{5/7}+336q^{6/7}+584q+984q^{8/7}+1608q^{9/7}+...$
$8B$	$8A$	0	$0+8q^{1/16}+8q^{3/16}+16q^{5/16}+48q^{7/16}+104q^{9/16}+152q^{11/16}+208q^{13/16}+400q^{15/16}+...$
$8C$	$8B$	0	$0+4q^{1/16}+16q^{3/16}+24q^{5/16}+32q^{7/16}+84q^{9/16}+176q^{11/16}+248q^{13/16}+352q^{15/16}+...$
$8D$	$8C$	4	$4+4q^{1/8}+16q^{1/4}+32q^{3/8}+64q^{1/2}+120q^{5/8}+192q^{3/4}+320q^{7/8}+512q+788q^{9/8}+...$
$8E$	$8C$	-4	$4+4q^{1/8}+16q^{1/4}+32q^{3/8}+64q^{1/2}+120q^{5/8}+192q^{3/4}+320q^{7/8}+512q+788q^{9/8}+...$
8F	**8D**	**0**	$\mathbf{2}+8q^{1/8}+16q^{1/4}+32q^{3/8}+64q^{1/2}+112q^{5/8}+192q^{3/4}+320q^{7/8}+512q+808q^{9/8}+...$
$8G$	$8E$	2	$2+8q^{1/8}+16q^{1/4}+32q^{3/8}+64q^{1/2}+112q^{5/8}+192q^{3/4}+320q^{7/8}+512q+808q^{9/8}+...$
$8H$	$8E$	-2	$2+8q^{1/8}+16q^{1/4}+32q^{3/8}+64q^{1/2}+112q^{5/8}+192q^{3/4}+320q^{7/8}+512q+808q^{9/8}+...$
$8I$	$8F$	0	$0+4q^{1/32}+8q^{5/32}+20q^{9/32}+40q^{13/32}+72q^{17/32}+128q^{21/32}+220q^{25/32}+360q^{29/32}+...$
$9A$	$9A$	-3	$3+6q^{1/9}+8q^{2/9}+24q^{1/3}+42q^{4/9}+80q^{5/9}+120q^{2/3}+192q^{7/9}+296q^{8/9}+456q+...$
$18A$	$9A$	3	$3+6q^{1/9}+8q^{2/9}+24q^{1/3}+42q^{4/9}+80q^{5/9}+120q^{2/3}+192q^{7/9}+296q^{8/9}+456q+...$
9B	**9B**	**0**	$\mathbf{2}+6q^{1/9}+12q^{2/9}+24q^{1/3}+42q^{4/9}+72q^{5/9}+120q^{2/3}+192q^{7/9}+300q^{8/9}+456q+...$
18B	**9B**	**0**	$\mathbf{2}+6q^{1/9}+12q^{2/9}+24q^{1/3}+42q^{4/9}+72q^{5/9}+120q^{2/3}+192q^{7/9}+300q^{8/9}+456q+...$
$9C$	$9C$	3	$3+4q^{1/9}+12q^{2/9}+24q^{1/3}+44q^{4/9}+72q^{5/9}+120q^{2/3}+192q^{7/9}+300q^{8/9}+456q+...$
$18C$	$9C$	-3	$3+4q^{1/9}+12q^{2/9}+24q^{1/3}+44q^{4/9}+72q^{5/9}+120q^{2/3}+192q^{7/9}+300q^{8/9}+456q+...$

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$10D$	$10A$	-2	$2+8q^{1/10}+4q^{1/5}+16q^{3/10}+28q^{2/5}+56q^{1/2}+96q^{3/5}+112q^{7/10}+188q^{4/5}+...$
$10E$	$10A$	2	$2+8q^{1/10}+4q^{1/5}+16q^{3/10}+28q^{2/5}+56q^{1/2}+96q^{3/5}+112q^{7/10}+188q^{4/5}+...$
$20A$	$10B$	0	$0+6q^{3/40}+8q^{7/40}+18q^{11/40}+24q^{3/8}+32q^{19/40}+84q^{23/40}+114q^{27/40}+180q^{31/40}+...$
$20B$	$10C$	0	$0+4q^{1/40}+4q^{1/8}+12q^{9/40}+16q^{13/40}+40q^{17/40}+56q^{21/40}+92q^{5/8}+136q^{29/40}+...$
$10F$	$10D$	-2	$2+4q^{1/10}+12q^{1/5}+16q^{3/10}+28q^{2/5}+52q^{1/2}+80q^{3/5}+128q^{7/10}+188q^{4/5}+...$
$10G$	$10D$	2	$2+4q^{1/10}+12q^{1/5}+16q^{3/10}+28q^{2/5}+52q^{1/2}+80q^{3/5}+128q^{7/10}+188q^{4/5}+...$
$10H$	$10E$	3	$3+4q^{1/10}+8q^{1/5}+16q^{3/10}+32q^{2/5}+52q^{1/2}+80q^{3/5}+128q^{7/10}+192q^{4/5}+...$
$10I$	$10E$	-3	$3+4q^{1/10}+8q^{1/5}+16q^{3/10}+32q^{2/5}+52q^{1/2}+80q^{3/5}+128q^{7/10}+192q^{4/5}+...$
$10J$	$10F$	0	$0+4q^{1/20}+8q^{3/20}+12q^{1/4}+24q^{7/20}+36q^{9/20}+64q^{11/20}+104q^{13/20}+152q^{3/4}+...$
$11A$	$11A$	2	$2+4q^{1/11}+8q^{2/11}+12q^{3/11}+24q^{4/11}+36q^{5/11}+56q^{6/11}+88q^{7/11}+128q^{8/11}+...$
$22A$	$11A$	0	$0+2q^{1/44}+2q^{3/44}+2q^{5/44}+4q^{7/44}+4q^{9/44}+6q^{1/4}+8q^{13/44}+10q^{15/44}+12q^{17/44}+...$
$12C$	$12A$	-4	$4+8q^{1/6}+8q^{1/4}+8q^{1/3}+48q^{5/12}+56q^{1/2}+16q^{7/12}+104q^{2/3}+176q^{3/4}+112q^{5/6}+...$
$12D$	$12A$	4	$4+8q^{1/6}+8q^{1/4}+8q^{1/3}+48q^{5/12}+56q^{1/2}+16q^{7/12}+104q^{2/3}+176q^{3/4}+112q^{5/6}+...$
$12E$	$12B$	0	$0+8q^{1/12}+4q^{1/6}+8q^{1/4}+28q^{1/3}+16q^{5/12}+28q^{1/2}+96q^{7/12}+76q^{2/3}+...$
$12F$	$12C$	0	$0+4q^{1/24}+4q^{1/8}+8q^{5/24}+16q^{7/24}+20q^{3/8}+32q^{11/24}+56q^{13/24}+72q^{5/8}+...$
$12G$	$12D$	-1	$1+6q^{1/12}+6q^{1/6}+6q^{1/4}+18q^{1/3}+36q^{5/12}+42q^{1/2}+48q^{7/12}+90q^{2/3}+150q^{3/4}+...$
$12H$	$12D$	1	$1+6q^{1/12}+6q^{1/6}+6q^{1/4}+18q^{1/3}+36q^{5/12}+42q^{1/2}+48q^{7/12}+90q^{2/3}+150q^{3/4}+...$
$12I$	$12E$	2	$2+4q^{1/12}+8q^{1/6}+4q^{1/4}+16q^{1/3}+40q^{5/12}+40q^{1/2}+48q^{7/12}+96q^{2/3}+148q^{3/4}+...$
$12J$	$12E$	-2	$2+4q^{1/12}+8q^{1/6}+4q^{1/4}+16q^{1/3}+40q^{5/12}+40q^{1/2}+48q^{7/12}+96q^{2/3}+148q^{3/4}+...$
$24A$	$12F$	0	$0+6q^{5/96}+2q^{13/96}+6q^{7/32}+24q^{29/96}+18q^{37/96}+24q^{15/32}+84q^{53/96}+56q^{61/96}+...$
$12K$	$12G$	0	$0+2q^{1/144}+4q^{13/144}+6q^{25/144}+12q^{37/144}+18q^{49/144}+28q^{61/144}+44q^{73/144}+...$

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$12I$	$12H$	1	$1+4q^{1/12}+8q^{1/6}+12q^{1/4}+16q^{1/3}+24q^{5/12}+40q^{1/2}+64q^{7/12}+96q^{2/3}+132q^{3/4}+...$
$12M$	$12H$	-1	$1+4q^{1/12}+8q^{1/6}+12q^{1/4}+16q^{1/3}+24q^{5/12}+40q^{1/2}+64q^{7/12}+96q^{2/3}+132q^{3/4}+...$
$12N$	$12I$	-2	$2+4q^{1/12}+4q^{1/6}+12q^{1/4}+20q^{1/3}+24q^{5/12}+44q^{1/2}+64q^{7/12}+84q^{2/3}+132q^{3/4}+...$
$12O$	$12I$	2	$2+4q^{1/12}+4q^{1/6}+12q^{1/4}+20q^{1/3}+24q^{5/12}+44q^{1/2}+64q^{7/12}+84q^{2/3}+132q^{3/4}+...$
$12P$	$12J$	0	$0+4q^{1/24}+4q^{1/8}+8q^{5/24}+16q^{7/24}+20q^{3/8}+32q^{11/24}+56q^{13/24}+72q^{5/8}+...$
$12Q$	$12K$	3	$3+2q^{1/12}+6q^{1/6}+10q^{1/4}+18q^{1/3}+28q^{5/12}+42q^{1/2}+64q^{7/12}+90q^{2/3}+130q^{3/4}+...$
$12R$	$12K$	-3	$3+2q^{1/12}+6q^{1/6}+10q^{1/4}+18q^{1/3}+28q^{5/12}+42q^{1/2}+64q^{7/12}+90q^{2/3}+130q^{3/4}+...$
$24B$	$12L$	0	$0+4q^{11/288}+4q^{35/288}+8q^{59/288}+12q^{83/288}+24q^{107/288}+32q^{131/288}+52q^{155/288}+...$
$12S$	$12M$	0	$0+2q^{1/144}+4q^{13/144}+6q^{25/144}+12q^{37/144}+18q^{49/144}+28q^{61/144}+44q^{73/144}+...$
$13A$	$13A$	-2	$2+4q^{1/13}+4q^{2/13}+8q^{3/13}+12q^{4/13}+24q^{5/13}+32q^{6/13}+48q^{7/13}+68q^{8/13}+...$
$26A$	$13A$	2	$2+4q^{1/13}+4q^{2/13}+8q^{3/13}+12q^{4/13}+24q^{5/13}+32q^{6/13}+48q^{7/13}+68q^{8/13}+...$
$28A$	$14A$	0	$0+4q^{3/56}+4q^{1/8}+8q^{11/56}+8q^{15/56}+20q^{19/56}+16q^{23/56}+32q^{27/56}+48q^{31/56}+...$
$14C$	$14B$	1	$1+4q^{1/14}+4q^{1/7}+8q^{3/14}+12q^{2/7}+16q^{5/14}+24q^{3/7}+36q^{1/2}+52q^{4/7}+68q^{9/14}+...$
$14D$	$14B$	-1	$1+4q^{1/14}+4q^{1/7}+8q^{3/14}+12q^{2/7}+16q^{5/14}+24q^{3/7}+36q^{1/2}+52q^{4/7}+68q^{9/14}+...$
$15A$	$15A$	3	$3+2q^{1/15}+6q^{1/5}+18q^{4/15}+6q^{1/3}+24q^{2/5}+36q^{7/15}+12q^{8/15}+78q^{3/5}+92q^{2/3}+...$
$30A$	$15A$	-3	$3+2q^{1/15}+6q^{1/5}+18q^{4/15}+6q^{1/3}+24q^{2/5}+36q^{7/15}+12q^{8/15}+78q^{3/5}+92q^{2/3}+...$
$15B$	$15B$	-2	$2+4q^{1/15}+8q^{1/5}+12q^{4/15}+4q^{1/3}+28q^{2/5}+40q^{7/15}+16q^{8/15}+68q^{3/5}+88q^{2/3}+...$
$30B$	$15B$	2	$2+4q^{1/15}+8q^{1/5}+12q^{4/15}+4q^{1/3}+28q^{2/5}+40q^{7/15}+16q^{8/15}+68q^{3/5}+88q^{2/3}+...$
$15C$	$15C$	0	$0+4q^{2/45}+4q^{1/9}+4q^{8/45}+8q^{11/45}+8q^{14/45}+24q^{17/45}+28q^{4/9}+32q^{23/45}+...$
$30C$	$15C$	0	$0+4q^{2/45}+4q^{1/9}+4q^{8/45}+8q^{11/45}+8q^{14/45}+24q^{17/45}+28q^{4/9}+32q^{23/45}+...$
$15D$	$15D$	1	$1+4q^{1/15}+4q^{2/15}+4q^{1/5}+12q^{4/15}+12q^{1/3}+20q^{2/5}+32q^{7/15}+36q^{8/15}+52q^{3/5}+...$

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$30D$	$15D$	-1	$1 + 4q^{1/15} + 4q^{2/15} + 4q^{1/5} + 12q^{4/15} + 12q^{1/3} + 20q^{2/5} + 32q^{7/15} + 36q^{8/15} + 52q^{3/5} + ...$
$15E$	$15E$	2	$2 + 2q^{1/15} + 4q^{2/15} + 6q^{1/5} + 10q^{4/15} + 14q^{1/3} + 20q^{2/5} + 28q^{7/15} + 40q^{8/15} + 54q^{3/5} + ...$
$30E$	$15E$	-2	$2 + 2q^{1/15} + 4q^{2/15} + 6q^{1/5} + 10q^{4/15} + 14q^{1/3} + 20q^{2/5} + 28q^{7/15} + 40q^{8/15} + 54q^{3/5} + ...$
$16A$	$16A$	0	$0 + 2q^{1/32} + 4q^{3/32} + 4q^{5/32} + 8q^{7/32} + 10q^{9/32} + 12q^{11/32} + 20q^{13/32} + 24q^{15/32} + ...$
$16B$	$16B$	2	$2 + 2q^{1/16} + 4q^{1/8} + 4q^{3/16} + 8q^{1/4} + 12q^{5/16} + 16q^{3/8} + 24q^{7/16} + 32q^{1/2} + 42q^{9/16} + ...$
$16C$	$16B$	-2	$2 + 2q^{1/16} + 4q^{1/8} + 4q^{3/16} + 8q^{1/4} + 12q^{5/16} + 16q^{3/8} + 24q^{7/16} + 32q^{1/2} + 42q^{9/16} + ...$
$18D$	$18A$	-1	$1 + 4q^{1/18} + 2q^{1/9} + 4q^{1/6} + 4q^{2/9} + 8q^{5/18} + 12q^{1/3} + 16q^{7/18} + 22q^{4/9} + 28q^{1/2} + ...$
$18E$	$18A$	1	$1 + 4q^{1/18} + 2q^{1/9} + 4q^{1/6} + 4q^{2/9} + 8q^{5/18} + 12q^{1/3} + 16q^{7/18} + 22q^{4/9} + 28q^{1/2} + ...$
$18F$	$18B$	2	$2 + 2q^{1/18} + 2q^{1/9} + 4q^{1/6} + 6q^{2/9} + 8q^{5/18} + 12q^{1/3} + 16q^{7/18} + 22q^{4/9} + 28q^{1/2} + ...$
$18G$	$18B$	-2	$2 + 2q^{1/18} + 2q^{1/9} + 4q^{1/6} + 6q^{2/9} + 8q^{5/18} + 12q^{1/3} + 16q^{7/18} + 22q^{4/9} + 28q^{1/2} + ...$
$18H$	$18C$	-1	$1 + 2q^{1/18} + 4q^{1/9} + 4q^{1/6} + 6q^{2/9} + 8q^{5/18} + 12q^{1/3} + 16q^{7/18} + 20q^{4/9} + 28q^{1/2} + ...$
$18I$	$18C$	1	$1 + 2q^{1/18} + 4q^{1/9} + 4q^{1/6} + 6q^{2/9} + 8q^{5/18} + 12q^{1/3} + 16q^{7/18} + 20q^{4/9} + 28q^{1/2} + ...$
$20C$	$20A$	-2	$2 + 4q^{1/10} + 4q^{3/20} + 4q^{1/5} + 4q^{1/4} + 8q^{3/10} + 20q^{7/20} + 12q^{2/5} + 8q^{9/20} + 28q^{1/2} + ...$
$20D$	$20A$	2	$2 + 4q^{1/10} + 4q^{3/20} + 4q^{1/5} + 4q^{1/4} + 8q^{3/10} + 20q^{7/20} + 12q^{2/5} + 8q^{9/20} + 28q^{1/2} + ...$
$20E$	$20B$	0	$0 + 2q^{1/80} + 2q^{1/16} + 2q^{9/80} + 4q^{13/80} + 4q^{17/80} + 8q^{21/80} + 10q^{5/16} + 12q^{29/80} + ...$
$20F$	$20C$	-1	$1 + 2q^{1/20} + 2q^{1/10} + 4q^{3/20} + 6q^{1/5} + 6q^{1/4} + 8q^{3/10} + 12q^{7/20} + 14q^{2/5} + 18q^{9/20} + ...$
$20G$	$20C$	1	$1 + 2q^{1/20} + 2q^{1/10} + 4q^{3/20} + 6q^{1/5} + 6q^{1/4} + 8q^{3/10} + 12q^{7/20} + 14q^{2/5} + 18q^{9/20} + ...$
$21A$	$21A$	2	$2 + 4q^{2/21} + 4q^{1/7} + 8q^{5/21} + 8q^{2/7} + 4q^{1/3} + 20q^{8/21} + 24q^{3/7} + 8q^{10/21} + 32q^{11/21} + ...$
$42A$	$21A$	-2	$2 + 4q^{2/21} + 4q^{1/7} + 8q^{5/21} + 8q^{2/7} + 4q^{1/3} + 20q^{8/21} + 24q^{3/7} + 8q^{10/21} + 32q^{11/21} + ...$
$21B$	$21B$	-1	$1 + 2q^{1/21} + 4q^{2/21} + 2q^{1/7} + 2q^{4/21} + 8q^{5/21} + 4q^{2/7} + 10q^{1/3} + 16q^{8/21} + 18q^{3/7} + ...$
$42B$	$21B$	1	$1 + 2q^{1/21} + 4q^{2/21} + 2q^{1/7} + 2q^{4/21} + 8q^{5/21} + 4q^{2/7} + 10q^{1/3} + 16q^{8/21} + 18q^{3/7} + ...$

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$21C$	$21C$	0	$0+2q^{1/63}+2q^{4/63}+2q^{1/9}+4q^{10/63}+4q^{13/63}+6q^{16/63}+8q^{19/63}+12q^{22/63}++...$
$42C$	$21C$	0	$0+2q^{1/63}+2q^{4/63}+2q^{1/9}+4q^{10/63}+4q^{13/63}+6q^{16/63}+8q^{19/63}+12q^{22/63}++...$
$22BC$	$22A$	0	$0+2q^{1/44}+2q^{3/44}+2q^{5/44}+4q^{7/44}+4q^{9/44}+6q^{1/4}+8q^{13/44}+10q^{15/44}+12q^{17/44}+...$
$23AB$	$23AB$	1	$1+2q^{1/23}+2q^{2/23}+2q^{3/23}+4q^{4/23}+4q^{5/23}+6q^{6/23}+8q^{7/23}+10q^{8/23}+12q^{9/23}+...$
$46AB$	$23AB$	-1	$1+2q^{1/23}+2q^{2/23}+2q^{3/23}+4q^{4/23}+4q^{5/23}+6q^{6/23}+8q^{7/23}+10q^{8/23}+12q^{9/23}+...$
$24C$	$24A$	0	$0+4q^{1/16}+4q^{5/48}+4q^{3/16}+8q^{11/48}+8q^{5/16}+16q^{17/48}+8q^{19/48}+20q^{7/16}+...$
$24D$	$24B$	-2	$2+2q^{1/24}+2q^{1/8}+4q^{1/6}+8q^{1/4}+12q^{7/24}+4q^{1/3}+14q^{3/8}+16q^{5/12}+4q^{11/24}+...$
$24E$	$24B$	2	$2+2q^{1/24}+2q^{1/8}+4q^{1/6}+8q^{1/4}+12q^{7/24}+4q^{1/3}+14q^{3/8}+16q^{5/12}+4q^{11/24}+...$
$24F$	$24C$	0	$0+2q^{1/48}+2q^{1/16}+4q^{5/48}+2q^{3/16}+8q^{11/48}+4q^{13/48}+4q^{5/16}+12q^{17/48}+...$
$14G$	$24D$	0	$0+2q^{1/32}+4q^{7/96}+4q^{5/32}+4q^{19/96}+10q^{9/32}+12q^{31/96}+4q^{35/96}+16q^{13/32}+...$
$24H$	$24E$	0	$0+2q^{1/48}+2q^{1/16}+4q^{7/48}+6q^{3/16}+4q^{11/48}+4q^{13/48}+8q^{5/16}+12q^{17/48}+...$
$24I$	$24F$	-1	$1+2q^{1/24}+2q^{1/12}+2q^{1/8}+2q^{1/6}+4q^{5/24}+6q^{1/4}+8q^{7/24}+10q^{1/3}+10q^{3/8}+...$
$24J$	$24F$	1	$1+2q^{1/24}+2q^{1/12}+2q^{1/8}+2q^{1/6}+4q^{5/24}+6q^{1/4}+8q^{7/24}+10q^{1/3}+10q^{3/8}+...$
$52A$	$26A$	0	$0+2q^{3/104}+2q^{7/104}+2q^{11/104}+2q^{15/104}+4q^{19/104}+4q^{23/104}+4q^{27/104}+8q^{31/104}+...$
$28B$	$28A$	1	$1+2q^{1/28}+2q^{1/14}+2q^{1/7}+4q^{5/28}+4q^{3/14}+2q^{1/4}+6q^{2/7}+10q^{9/28}+8q^{5/14}+...$
$28C$	$28A$	-1	$1+2q^{1/28}+2q^{1/14}+2q^{1/7}+4q^{5/28}+4q^{3/14}+2q^{1/4}+6q^{2/7}+10q^{9/28}+8q^{5/14}+...$
$56AB$	$28B$	0	$0+2q^{1/224}+2q^{17/224}+2q^{25/224}+2q^{33/224}+2q^{41/224}+4q^{7/32}+6q^{57/224}+6q^{65/224}+...$
$30F$	$30A$	1	$1+2q^{1/15}+4q^{1/10}+4q^{1/6}+2q^{1/5}+10q^{4/15}+8q^{3/10}+2q^{1/3}+8q^{11/30}+12q^{2/5}+...$
$30G$	$30A$	-1	$1+2q^{1/15}+4q^{1/10}+4q^{1/6}+2q^{1/5}+10q^{4/15}+8q^{3/10}+2q^{1/3}+8q^{11/30}+12q^{2/5}+...$
$60A$	$30B$	0	$0+2q^{1/40}+4q^{11/120}+2q^{1/8}+4q^{23/120}+6q^{9/40}+8q^{7/24}+8q^{13/40}+4q^{43/120}+...$
$60B$	$30C$	0	$0+2q^{1/360}+2q^{5/72}+2q^{49/360}+4q^{61/360}+4q^{73/360}+4q^{17/72}+4q^{97/360}+4q^{109/360}+...$

Co_0	Co_1	$Z_{R,R}$	$Z_{NS,R}$
$30H$	$30D$	1	$1+2q^{1/30}+2q^{1/10}+2q^{2/15}+2q^{1/6}+4q^{1/5}+4q^{7/30}+4q^{4/15}+6q^{3/10}+8q^{1/3}+...$
$30I$	$30D$	-1	$1+2q^{1/30}+2q^{1/10}+2q^{2/15}+2q^{1/6}+4q^{1/5}+4q^{7/30}+4q^{4/15}+6q^{3/10}+8q^{1/3}+...$
$30J$	$30E$	0	$0+2q^{1/30}+2q^{1/15}+2q^{1/10}+2q^{2/15}+2q^{1/6}+2q^{1/5}+4q^{7/30}+6q^{4/15}+6q^{3/10}+...$
$30K$	$30E$	0	$0+2q^{1/30}+2q^{1/15}+2q^{1/10}+2q^{2/15}+2q^{1/6}+2q^{1/5}+4q^{7/30}+6q^{4/15}+6q^{3/10}+...$
$33A$	$33A$	-1	$1+2q^{1/33}+2q^{1/11}+2q^{4/33}+4q^{2/11}+4q^{7/33}+6q^{3/11}+8q^{10/33}+2q^{1/3}+10q^{4/11}+...$
$66A$	$33A$	1	$1+2q^{1/33}+2q^{1/11}+2q^{4/33}+4q^{2/11}+4q^{7/33}+6q^{3/11}+8q^{10/33}+2q^{1/3}+10q^{4/11}...$
$35A$	$35A$	1	$1+2q^{1/35}+2q^{4/35}+2q^{1/7}+4q^{6/35}+2q^{1/5}+4q^{8/35}+2q^{9/35}+4q^{2/7}+8q^{11/35}+...$
$70A$	$35A$	-1	$1+2q^{1/35}+2q^{4/35}+2q^{1/7}+4q^{6/35}+2q^{1/5}+4q^{8/35}+2q^{9/35}+4q^{2/7}+8q^{11/35}+...$
$36A$	$36A$	-1	$1+2q^{1/18}+2q^{1/12}+2q^{1/9}+2q^{1/6}+4q^{7/36}+2q^{2/9}+2q^{1/4}+4q^{5/18}+8q^{11/36}+...$
$36B$	$36A$	1	$1+2q^{1/18}+2q^{1/12}+2q^{1/9}+2q^{1/6}+4q^{7/36}+2q^{2/9}+2q^{1/4}+4q^{5/18}+8q^{11/36}+...$
$39AB$	$39AB$	1	$1+2q^{2/39}+2q^{1/13}+2q^{5/39}+2q^{2/13}+4q^{8/39}+4q^{3/13}+6q^{11/39}+6q^{4/13}+2q^{1/3}+...$
$78AB$	$39AB$	-1	$1+2q^{2/39}+2q^{1/13}+2q^{5/39}+2q^{2/13}+4q^{8/39}+4q^{3/13}+6q^{11/39}+6q^{4/13}+2q^{1/3}+...$
$40AB$	$40A$	0	$0+2q^{3/80}+2q^{1/16}+2q^{7/80}+4q^{13/80}+2q^{3/16}+4q^{19/80}+4q^{21/80}+6q^{23/80}+4q^{5/16}+...$
$84A$	$42A$	0	$0+2q^{1/168}+2q^{3/56}+2q^{1/8}+2q^{25/168}+4q^{11/56}+4q^{37/168}+4q^{15/56}+6q^{7/24}+...$
$60C$	$60A$	1	$1+2q^{1/60}+2q^{1/10}+2q^{3/20}+2q^{1/6}+2q^{1/5}+4q^{13/60}+2q^{1/4}+4q^{4/15}+4q^{3/10}+...$
$60D$	$60A$	-1	$1+2q^{1/60}+2q^{1/10}+2q^{3/20}+2q^{1/6}+2q^{1/5}+4q^{13/60}+2q^{1/4}+4q^{4/15}+4q^{3/10}+...$

Appendix B

The Monster

The classification of finite simple groups identifies 26 sporadic simple groups, belonging to the naturally occurring families of cyclic, alternating, or Lie groups [2]. The *Monster* group, $\mathbb{M}$, is the largest of the sporadic simple groups. The discovery of the Monster was preceded by a long history of the development of another branch of mathematics, the theory of finite groups [33, 34]. The existence of the Monster was first predicted by Bernd Fischer in 1973, but he did not publish this finding. In 1976, Robert Louis Griess, Jr. made the same discovery from a different direction [37]. In 1981, Griess finally managed to construct the actual Monster itself in an extraordinary tour de force as the group of linear transformations on a vector space of dimension 196883 that preserve a certain commutative but non-associative bilinear product, now called the *Griess product* [6]. The character table of $\mathbb{M}$, a 194×194 array for the 194 irreducible complex representations, was worked out by Fischer, Livingstone, and Thorne in 1978.

The enormous size of the group is responsible for the name, given by Conway.

$$\dim(\mathbb{M}) = 2^{46} \times 3^{20} \times 5^{9} \times 7^{6} \times 11^{2} \times 13^{3} \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$$

$$= 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000$$

$$\sim 10^{54}$$

The Monster contains 20 sporadic groups as subquotients, called the *happy family* by Griess. He calls the remaining 6 sporadic groups, which are not subquotients of $\mathbb{M}$, the *pariahs* [35]. Griess further broke up the happy family into 3 generations. The first generation consists of five maximal subgroups of the Mathieu group, M_{24}. The second generation consists of seven maximal subgroups of Co_1. And finally, the third generation consists of eight maximal subgroups of $\mathbb{M}$ [36]. A diagram of their relations can be found in [51].

The minimal degree of a faithful complex representation of $\mathbb{M}$ is

$$47 \times 59 \times 71 = 196,883,$$

the product of the three largest prime divisors of $\dim(\mathbb{M})$. The smallest faithful linear representation over any field has a dimension of 196,882 over the field with two elements, only one less than the dimension of the smallest faithful complex representation. Additionally, $\mathbb{M}$ has no nontrivial permutation representations with degrees less than

$$97,239,461,142,009,186,000.$$

When ordered by size the 194 irreducible representations in the character table of $\mathbb{M}$ [15] follow

i	$\chi_i(1_{\mathbb{M}})$
1	1
2	196,883
3	21,296,876
4	842,609,326
5	18,538,750,076
6	19,360,062,527
7	293,553,734,298
$\vdots$	$\vdots$
194	258,823,477,531,055,064,045,234,375

The Monster can be realized as a Galois group over the rational numbers [55].

Excluding the Monster, the 19 other groups in the happy family are regarded as the *children* of the Monster. Using the notation from [36], these include,

- Mathieu groups — M_{11}, M_{12}, M_{22}, M_{23}, M_{24}

- Janko group — J_2
- Conway groups — Co_1, Co_2, Co_3
- Fischer groups — Fi_{22}, Fi_{23}, Fi'_{24}
- Highman-Sims group — HiS
- McLaughlin group — McL
- Held group — $Held$
- Suzuki group — Suz
- Harada-Norton group — F_5
- Thompson group — F_3
- Baby Monster group — $\mathbb{B}$

The *Baby Monster* group, $\mathbb{B}$, is the second highest order, sporadic simple group.

$$\dim(\mathbb{B}) = 2^{41} \times 3^{13} \times 5^6 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 47$$

$$= 4,154,781,481,226,426,191,177,580,544,000,000$$

$$\sim 4 \times 10^{33}$$

Shockingly, the Baby Monster is $2^5 \times 3^7 \times 5^3 \times 7^4 \times 11 \times 13^2 \times 29 \times 41 \times 59 \times 71$ times smaller than the Monster [45]. The double cover of $\mathbb{B}$ is the centralizer of an element of order 2 in the monster group. The outer automorphism group is trivial and the Schur multiplier, $M(\mathbb{B})$ has order 2. The group $2\mathbb{B}$ is the largest subgroup of $\mathbb{M}$. Here, (conjugacy classes of) subgroups are the same data as transitive permutation representations. Indeed, the number $2^5 \times \cdots \times 71$ is (up to the factor of 2 between $\mathbb{B}$ and $2\mathbb{B}$) equal to 7,239,461,142,009,186,000.

Appendix C

Monstrous Moonshine

Take $\mathbb{H} = \{\sigma + it | t > 0\} \subset \mathbb{C}$, the upper half plane, together with the *Poincaré metric*,

$$ds^2 = y^{-2}(dx^2 + dy^2),$$

which defines the Poincaré half-plane model of the hyperbolic plane [27]. The quotient of $SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$ by $\{\pm I_2\}$ is the *group of orientation-preserving isometries*, with action by *Möbius transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d} \tag{C.0.0.1}$$

for $\tau \in \mathbb{H}$. We can distinguish the modular group $SL_2(\mathbb{Z})$ as the subgroup of $SL_2(\mathbb{R})$ whose orbits encode the isomorphism types of a complex elliptic curve $E_\tau = \frac{\mathbb{C}}{\mathbb{Z}\tau + \mathbb{Z}}$ for any $\tau \in \mathbb{H}$. In other words, E_τ and $E_{\tau'}$ are isomorphic if and only if $\tau' = \varphi \cdot \tau$ for some $\varphi \in SL_2(\mathbb{Z})$.

Let $\Gamma_0(N)$ be the *Hecke congruence subgroup* of $SL_2(\mathbb{R})$ of level N,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \mid c = 0 \pmod{\mathrm{N}} \right\}$$

The orbits of $\Gamma_0(N)$ correspond to isomorphism pairs (E_τ, C) , where C is a cyclic subgroup of E_τ of order N.

C.1 History

In 1979, McKay famously observed that one of the coefficients in the q-series for the elliptic modular function

$$j = q^{-1} + 744q^0 + 196884q + 21493760q^2 + ... = \sum_{n=-1}^{\infty} a_n q^n$$

is $196883 + 1 = \chi_2(1_{\mathbb{M}}) + 1$, where $\chi_2(1_{\mathbb{M}})$ is the size of the degree 2 irreducible representation of the Monster [16]. Denote by $J(\tau)$ the *normalized elliptic modular invariant*

$$J(\tau) = \frac{1728 g_y(\tau)^3}{g_2(\tau)^3 27 g_3(\tau)^2} - 744 = q^{-1} + 196884q + 21493760q^2 + ...$$

where $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 140G_6(\tau)$ given G_{2k} is the *Einstein series* of weight $2k$,

$$G_{2k} = \sum_{(n,m)\neq(0,0)} (m + n\tau)^{-2k}$$

for $k \geq 2$ [26]. We can also define $J(\tau)$ as

$$J(\tau) = \frac{(1 + 240 \sum_{n>0} \sum_{d|n} d^3 q^n)^3}{q \prod_{n>0}(1 - q^n)^{24}} - 744$$

The first few coefficients of the elliptic modular function are

n	a_n
-1	1
0	744
1	196,884
2	21,493,760
3	864,299,970
4	20,245,856,256
5	333,202,640,600
$\vdots$	$\vdots$

Inspired by McKay's findings, Thompson [54],generalized that

$$\begin{aligned}
196,884 &= 1 + 196,883 \\
21,493,760 &= 1 + 196,883 + 21,296,876 \\
864,299,970 &= 2 \times 1 + 2 \times 196,883 + 21,296,876 + 842,609,326 \\
20,245,856,256 &= 3 \times 1 + 3 \times 196,883 + 21,296,876 + 2 \times 842,609,326 \\
&+ 18538750076 \\
333,202,640,600 &= 4 \times 1 + 5 \times 196,883 + 3 \times 21,296,876 + 2 \times 842,609,326 \\
&+ 18538750076 + 19,360,062,527 + 293,553,734,298
\end{aligned}$$

In other words,

$$a_1 = \chi_1(1_{\mathbb{M}}) + \chi_2(1_{\mathbb{M}})$$

$$a_2 = \chi_1(1_{\mathbb{M}}) + \chi_2(1_{\mathbb{M}}) + \chi_3(1_{\mathbb{M}})$$

$$a_3 = 2 \times \chi_1(1_{\mathbb{M}}) + 2 \times \chi_2(1_{\mathbb{M}}) + \chi_3(1_{\mathbb{M}}) + \chi_4(1_{\mathbb{M}})$$

$$a_4 = 3 \times \chi_1(1_{\mathbb{M}}) + 3 \times \chi_2(1_{\mathbb{M}}) + \chi_3(1_{\mathbb{M}}) + 2 \times \chi_4(1_{\mathbb{M}}) + \chi_5(1_{\mathbb{M}})$$

$$a_5 = 4 \times \chi_1(1_{\mathbb{M}}) + 5 \times \chi_2(1_{\mathbb{M}}) + 3 \times \chi_3(1_{\mathbb{M}}) + 2 \times \chi_4(1_{\mathbb{M}}) + \chi_5(1_{\mathbb{M}}) + \chi_6(1_{\mathbb{M}}) + +\chi_7(1_{\mathbb{M}})$$

So, $a_n = \sum_m x_{nm}\chi_m(1_{\mathbb{M}})$ for $1 \leq n \leq 5$ and $1 \leq m \leq 7$ where the matrix for the x_{nm}'s are

	1	2	3	4	5	6	7
1	1	1	0	0	0	0	0
2	1	1	1	0	0	0	0
3	2	2	1	1	0	0	0
4	3	3	1	2	1	0	0
5	4	5	3	2	1	1	1

These coincidences led Thompson to the following conjecture [54].

Conjecture C.1.0.1. There exists a naturally graded infinite-dimensional monster module $V^\natural = \bigoplus_{n\geq -1} V^\natural_n$ satisfying

$$\dim(V^\natural_n) = a_n$$

for $n \geq -1$ and a_n, coefficients of the q-series for the elliptic modular function.

C.2 Construction

Griess [35], and Conway and Norton [16] each independently hypothesised the existence of an irreducible module with dimension 196883. Griess' masterful development of a monster-invariant algebra on the unique nontrivial 196884-dimensional module was ultimately what established its existence [36].

In C.1.0.1, the generating series of the dimension of the graded parts of $V^\natural$ is the q-expansion of

$$J(\tau) = \sum_{n\geq -1} \dim(V^\natural_n) q^n \tag{C.2.0.1}$$

Since the dimension of a representation is the trace of $\rho(1)$, Thompson analysed what are now known as the *McKay-Thompson series* [53],

$$Z_{1,m}(\tau) = q^{-1} \sum_{n \geq 0} \mathrm{tr}(m, V_n^\natural) q^n \tag{C.2.0.2}$$

where $m \in \mathbb{M}$ and $q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$. Here, m is chosen as a representative of its conjugacy class $[m] = \{gmg^{-1} | g \in \mathbb{M}\}$. Though one would expect 194 versions of the series, we only have 172. This comes from the fact that $[m]$ and $[m^{-1}]$ define the same series. The McKay-Thompson series defines a function of τ which determines four subgroups of $SL_2(\mathbb{R})$ [16] of interest

(fixing group.) $F(m)$ consists of the elements of $SL_2(\mathbb{R})$ that fix $Z_{1,m}$,

(eigengroup.) $E(m)$ consists of the elements of $SL_2(\mathbb{R})$ that multiply $Z_{1,m}$ by m'th roots of 1,

(distended eigengroup.) $D(m)$ consists of the elements of $SL_2(\mathbb{R})$ that multiply $Z_{1,m}$ by any roots of 1,

(converting group.) $C(m)$ consists of the elements of $SL_2(\mathbb{R})$ that convert $Z_{1,m}$ to functions of the form $(AZ_{1,m} + B) \backslash (CZ_{1,m} + D)$.

A discrete group $\Gamma_m \in SL_2(\mathbb{R}), m \in \mathbb{M}$, has *width one at this infinite cusp* if its subgroup of upper-triangular matrices is generated by $\pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Specifically, the subgroup of upper-triangular matrices in Γ_m coincides with $\Gamma_\infty = \left\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | n \in \mathbb{Z}\right\}$.

Two groups G and H are *commensurable* if there exists finite subgroups $G' \subseteq G$ and $H' \subseteq H$ such that $G' \cong H'$. If Γ_m is commensurable with $SL_2(\mathbb{Z})$, meaning Γ_m is isomorphic to a finite subgroup of $SL_2(\mathbb{Z})$, then Γ_m acts naturally on $\mathbb{QP}' = \mathbb{Q} \cup \{\infty\}$. The *cusps* of Γ_m are the orbits of Γ_m on $\mathbb{QP}'$. A cusp $\alpha \in \mathbb{QP}'/\Gamma_m$ is *non-infinite* if $\infty \notin \alpha$.

Conway and Norton conjectured that the $Z_{1,m}(\tau)$'s are directly related to $J(\tau)$.

Conjecture C.2.0.1. [16] For each $m \in \mathbb{M}$, there is a group $\Gamma_m \subset SL_2(\mathbb{R})$ such that $Z_{1,m}$ is the unique normalized principal modulus for Γ_m.

This means that each $Z_{1,m}$ is the unique Γ_m-invariant holomorphic functions on $\mathbb{H}$ which satisfies

$$Z_{1,m}(\tau) = q^{-1} + \mathrm{O}(q) \tag{C.2.0.3}$$

and remains bounded as τ approaches any non-infinite cusps of Γ_m [26], so Γ_m is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ which fixes $Z_{1,m}$. Now, $\Gamma_m = \mathrm{SL}_2(\mathbb{Z})$ when $m = 1_{\mathbb{M}}$, the identity element of $\mathbb{M}$. The corresponding McKay-Thompson series is

$$\begin{aligned} Z_{1,1_{\mathbb{M}}}(\tau) &= q^{-1}\sum_{n\geq 0}\mathrm{tr}(1_{\mathbb{M}}, V_n^\natural)q^n \\ &= q^{-1}\sum_{n\geq 0}\dim(V_n^\natural)q^n \\ &= q^{-1} + 196884q + 21493760q^2 + ... \\ &= J(\tau) \end{aligned} \tag{C.2.0.4}$$

Notice that the McKay Thompson series is equivalent to the graded dimensions and characters we've previously mentioned. For example,

$$Z_{1,1_{\mathbb{M}}}(\tau) = q^{-1}\sum_{n\geq 0}\mathrm{tr}(1_{\mathbb{M}}, V_n^\natural)q^n = Z^{NS}(V^\natural) = \mathrm{tr}_{V^\natural} q^{L_0-1}$$

Because of the presence of symmetry, we can also consider characters that are twined by the action of $\mathbb{M}$. We see that

$$Z_{1,m}(\tau) = q^{-1}\sum_{n\geq 0}\mathrm{tr}(m, V_n^\natural)q^n = Z_m(V^\natural) = \mathrm{tr}_{V^\natural}(mq^{L_0-1})$$

for some $m \in \mathbb{M}$.

Many Hecke congruence subgroups $\Gamma_0(N) \subset SL_2(\mathbb{R})$ appear as Γ_m for elements $m \in \mathbb{M}$.

One example is for $m \in 2B$, the larger of two conjugacy classes of involution in $\mathbb{M}$ [15]. Then, $\Gamma_m = \Gamma_{2B} = \Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \mid c = 0 \pmod 2 \right\}$ [27], [21], and

$$\begin{aligned} Z_{1,m\in 2B} &= q^{-1}\prod_{n>0}(1-q^{2n-1})^{24} + 24 \\ &= q^{-1} + 276q - 2048q^2 + ... \end{aligned}$$

It was verified through calculations by Atkin, Fong, and Smith [52] and further by Borcherds [5] that a graded infinite-dimensional monster module $V^\natural$ exists such that the functions $Z_{1,m}$ behave exactly as predicted by Conway and Norton.

Theorem C.2.1. [52] There exists a graded $\mathbb{M}$-module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ such that, if $Z_{1,m}$ is defined by C.2.0.2, then $Z_{1,m}$ is the unique Γ_m-invariant holomorphic function on $\mathbb{H}$ that satisfies $Z_{1,m}(\tau) = q^{-1} + O(q)$ as τ approaches the infinite cusp and has no poles at any non-infinite cusps of Γ_m, where Γ_m is the discrete subgroup of $SL_2(\mathbb{R})$ as specified by Conway and Norton.

Knowledge about $Z_{1,m}$ translates into knowledge about $\Gamma_m \subseteq SL_2(\mathbb{R})$ [27]. The above characterization of the Mckay-Thompson series $Z_{1,m}$, formulated by Conway and solved by Borcherds are referred to as the *monstrous Moonshine conjectures*.

A definition of what is now called the *Moonshine module*, $V^\natural$ was given and proved in celebrated work by Frenkel, Lepowsky, and Meurman [33, 34]. The Moonshine module $V^\natural$ was built as a $\mathbb{Z}_2$ orbifold of the Leech lattice VOA $V_{\Lambda_{24}}$ [33, 34]. The underlying geometric orbifold it the quotient

$$(\mathbb{R}^{24}/\Lambda_{24}) /\!\!/ \mathbb{Z}_2 \tag{C.2.0.5}$$

This work is one of the earliest examples of an orbifold conformal field theory [26].

The Moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ is a vertex operator algebra of central charge 24 whose graded dimension is given by $J(\tau)$, and whose automorphism group is the Monster group $\mathbb{M}$. The Moonshine module $V^\natural$ is holomorphic and $\deg(v) \neq 1$ for any $v \in V^\natural$ such that $v \neq 0$.

Theorem C.2.2. [5] If $V^\natural$ is the Moonshine module VOA constructed in [33, 34], $Z_{1,m}$ is defined by C.2.0.2 for $m \in \mathbb{M}$ and Γ_m is the discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ as described by C.2.0.1, then $Z_{1,m}$ is the unique normalized principal modulus for Γ_m.

Such a construction is consistent with one of the maximal subgroups on $\mathbb{M}$, of shape $2^{1+24} \cdot \mathrm{Co}_1$. Indeed, Frenkel, Lepowsky, and Meurman found that the graded traces of elements in the centralizer of involution $2^{1+24} \cdot \mathrm{Co}_1 \in \mathbb{M}$ match the functions predicted by Conway and Norton [33, 34]. There are many more ways to construct $V^\natural$ as is described in papers such as [8]. One such is the prime order construction, proposed by Montague [46] and proved in [10], [1].

Appendix D

Code

We share here some significant pieces of the Maple code.

The method *FractionalExponentAdjuster* takes 4 arguments: two polynomials and 2 variable names, say x and X. The first is a polynomial in x with rational exponents which it turns into a polynomial in X with real exponents. It then scales the second polynomial in x into a polynomial in X (for us, $\sqrt{\Delta}$).

```
FractionalExponentAdjuster := proc(P, x :: name, X :: name)
local d := ilcm( (denom@op)~(2, indets(P, identical(x)^rational)) [ ]);
   subs(x=X^d, subsindets(P, identical(x)^rational, p → X^(d * op(2, p)))),
   P → subs(X=x^(1 / d), subsindets(P, identical(X)^posint, p → x^(op(2, p) / d)))
end proc:
```

The method *QuoRem* takes four arguments: takes 4 arguments: two polynomials and 2 variable names, say x and X. The first is a polynomial in x with (some) negative exponents which it turns into a polynomial in X with only positive exponents. It then scales the second polynomial in x into a polynomial in X (for us, $\sqrt{\Delta}$).

```
QuoRem := proc(n, d, x :: name)
local X, ND, R, m, q, r;
   (ND, R) := FractionalExponentAdjuster( [n, d], x, X);
   m := min(0, ldegree~(ND, X) [ ]);
   q := quo(expand(ND/~X^m) [ ], X, r);
   #Remainder gets downshifted, but quotient does not:
   R(q), R(expand(X^m * r))
end proc:
```

The eigenvalues for each twistings are inputed as an array A[i], $i \in \{1, ..., 12\}$.

$$\boxed{Z_{1,g}^{NS,NS}(Fer(24))}$$

$$t1 := sort\left(\sum_{n=-5}^{5}\left((-1)^n q^{\left(\frac{\left(n+\frac{1}{2}+m\right)^2}{2}\right)}\right), q, ascending\right):$$

$t11 := eval(t1, m=A[1]):$
$t12 := eval(t1, m=A[2]):$
$t13 := eval(t1, m=A[3]):$
$t14 := eval(t1, m=A[4]):$
$t15 := eval(t1, m=A[5]):$
$t16 := eval(t1, m=A[6]):$
$t17 := eval(t1, m=A[7]):$
$t18 := eval(t1, m=A[8]):$
$t19 := eval(t1, m=A[9]):$
$t110 := eval(t1, m=A[10]):$
$t111 := eval(t1, m=A[11]):$
$t112 := eval(t1, m=A[12]):$

$$a := convert\left(series\left(expand\left(\frac{1}{2\sqrt{q}}\cdot t11\cdot t12\cdot t13\cdot t14\cdot t15\cdot t16\cdot t17\cdot t18\cdot t19\cdot t110\cdot t111\cdot t112\right), q, 10\right), polynom\right):$$

$$\boxed{Z^{R,NS}_{1,g}(Fer(24))}$$

$$t2 := sort\left(\sum_{n=-5}^{5}\left(q^{\left(\frac{\left(n+\frac{1}{2}+m\right)^2}{2}\right)}\right), q, ascending\right):$$

$t21 := eval(t2, m = A[1]):$
$t22 := eval(t2, m = A[2]):$
$t23 := eval(t2, m = A[3]):$
$t24 := eval(t2, m = A[4]):$
$t25 := eval(t2, m = A[5]):$
$t26 := eval(t2, m = A[6]):$
$t27 := eval(t2, m = A[7]):$
$t28 := eval(t2, m = A[8]):$
$t29 := eval(t2, m = A[9]):$
$t210 := eval(t2, m = A[10]):$
$t211 := eval(t2, m = A[11]):$
$t212 := eval(t2, m = A[12]):$

$$b := convert\left(series\left(expand\left(\frac{1}{2\sqrt{q}} \cdot t21 \cdot t22 \cdot t23 \cdot t24 \cdot t25 \cdot t26 \cdot t27 \cdot t28 \cdot t29 \cdot t210 \cdot t211 \cdot t212\right), q, 10\right), polynom\right):$$

$$\boxed{Z^{NS,R}_{1,g}(Fer(24))}$$

$$t3 := sort\left(\sum_{n=-5}^{5}\left(q^{\left(\frac{(n+m)^2}{2}\right)}\right), q, ascending\right):$$

$t31 := eval(t3, m = A[1]):$
$t32 := eval(t3, m = A[2]):$
$t33 := eval(t3, m = A[3]):$
$t34 := eval(t3, m = A[4]):$
$t35 := eval(t3, m = A[5]):$
$t36 := eval(t3, m = A[6]):$
$t37 := eval(t3, m = A[7]):$
$t38 := eval(t3, m = A[8]):$
$t39 := eval(t3, m = A[9]):$
$t310 := eval(t3, m = A[10]):$
$t311 := eval(t3, m = A[11]):$
$t312 := eval(t3, m = A[12]):$

$$c := convert\left(series\left(expand\left(\frac{1}{2\sqrt{q}} \cdot t31 \cdot t32 \cdot t33 \cdot t34 \cdot t35 \cdot t36 \cdot t37 \cdot t38 \cdot t39 \cdot t310 \cdot t311 \cdot t312\right), q, 10\right), polynom\right):$$

$Z_{1,g}^{R,R}(Fer(24))$

$$t4 := sort\left(\sum_{n=-5}^{5}\left((-1)^n q^{\left(\frac{(n+m)^2}{2}\right)}\right), q, ascending\right):$$

$t41 := eval(t4, m=A[1]):$
$t42 := eval(t4, m=A[2]):$
$t43 := eval(t4, m=A[3]):$
$t44 := eval(t4, m=A[4]):$
$t45 := eval(t4, m=A[5]):$
$t46 := eval(t4, m=A[6]):$
$t47 := eval(t4, m=A[7]):$
$t48 := eval(t4, m=A[8]):$
$t49 := eval(t4, m=A[9]):$
$t410 := eval(t4, m=A[10]):$
$t411 := eval(t4, m=A[11]):$
$t412 := eval(t4, m=A[12]):$

$$d := convert\left(series\left(expand\left(\frac{1}{2\sqrt{q}}\cdot t41\cdot t42\cdot t43\cdot t44\cdot t45\cdot t46\cdot t47\cdot t48\cdot t49\cdot t410\cdot t411\cdot t412\right), q, 10\right), polynom\right):$$

$\sqrt{\Delta}$

$$\mathrm{Delta} := sort\left(expand\left(\prod_{m=1}^{5}(1-q^m)\right), q, ascending\right):$$

$$\mathrm{delta} := sort\left(convert\left(series\left(\Delta^{12}, q, 10\right), polynom\right), q, ascending\right)$$

Initializing the two equations

$left := convert(series(c-d, q, 10), polynom):$

$right := convert(series(b-a, q, 10), polynom):$

$RR := sort(expand(left-right), q, ascending):$

$NSR := sort(expand(left+right), q, ascending):$

Calls to the adjusting methods

```
(qRR, rRR) := QuoRem(RR, delta, q) :
qRR : rRR :
expand(qRR * delta + rRR − RR) : #Test---should get 0

(qNSR, rNSR) := QuoRem(NSR, delta, q) :
qNSR : rNSR :
expand(qNSR * delta + rNSR − NSR) : #Test---should get 0
```

Finding the remainders

$$rem1 := series\left(\frac{rNSR}{\text{delta}}, q=0, 10\right) :$$

$$rem2 := convert(series(rem1, q, 5), polynom) :$$

$$remi1 := series\left(\frac{rRR}{\text{delta}}, q=0, 10\right) :$$

$$remi2 := convert(series(remi1, q, 5), polynom) :$$

$Z^{R,R}_{1,g}(V^{f\natural})$ and $Z^{NS,R}_{1,g}(V^{f\natural})$

$$Zrr := sort(qRR + remi2, q, ascending);\ \ Znsr := sort(qNSR + rem2, q, ascending);$$

Bibliography

[1] Toshiyuki Abe, Ching Hung Lam, and Hiromichi Yamada. On $\mathbb{Z}_p$-orbifold constructions of the moonshine vertex operator algebra. *Math. Z.*, 290(1-2):683–697, 2018.

[2] Michael Aschbacher. The status of the classification of the finite simple groups. *Notices Amer. Math. Soc.*, 51(7):736–740, 2004.

[3] Michael Atiyah. An introduction to topological quantum field theories. *Turkish J. Math.*, 21(1):1–7, 1997.

[4] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986.

[5] Richard E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.*, 109(2):405–444, 1992.

[6] Richard E. Borcherds. What is Moonshine? In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 607–615, 1998.

[7] Ivan M. Burbano, Justin Kulp, and Jonas Neuser. Duality defects in E8. *J. High Energy Phys.*, (10):Paper No. 186, 2022.

[8] Scott Carnahan. 51 constructions of the Moonshine module. *Commun. Number Theory Phys.*, 12(2):305–334, 2018.

[9] Scott Carnahan and Masahiko Miyamoto. Regularity of fixed-point vertex operator subalgebras, 2016.

[10] Hsian-Yang Chen, Ching Hung Lam, and Hiroki Shimakura. $\mathbb{Z}_3$-orbifold construction of the Moonshine vertex operator algebra and some maximal 3-local subgroups of the Monster. *Math. Z.*, 288(1-2):75–100, 2018.

[11] Miranda C. N. Cheng and John F. R. Duncan. The largest Mathieu group and (mock) automorphic forms. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 53–82. Amer. Math. Soc., Providence, RI, 2012.

[12] J. H. Conway. A perfect group of order $8,315,553,613,086,720,000$ and the sporadic simple groups. *Proc. Nat. Acad. Sci. U.S.A.*, 61:398–400, 1968.

[13] J. H. Conway. A characterisation of Leech's lattice. *Invent. Math.*, 7:137–142, 1969.

[14] J. H. Conway. A group of order $8,315,553,613,086,720,000$. *Bull. London Math. Soc.*, 1:79–88, 1969.

[15] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. $\mathbb{ATLAS}$ *of finite groups*. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[16] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bull. London Math. Soc.*, 11(3):308–339, 1979.

[17] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.

[18] Thomas Creutzig, John F. R. Duncan, and Wolfgang Riedler. Self-dual vertex operator superalgebras and superconformal field theory. *J. Phys. A*, 51(3):034001, 29, 2018.

[19] Alberto De Sole and Victor G. Kac. Finite vs affine W-algebras. *Jpn. J. Math.*, 1(1):137–261, 2006.

[20] Chongying Dong. Vertex algebras associated with even lattices. *J. Algebra*, 161(1):245–265, 1993.

[21] Chongying Dong, Haisheng Li, and Geoffrey Mason. Some twisted sectors for the Moonshine module. In *Moonshine, the Monster, and related topics (South Hadley, MA, 1994)*, volume 193 of *Contemp. Math.*, pages 25–43. Amer. Math. Soc., Providence, RI, 1996.

[22] Chongying Dong, Haisheng Li, and Geoffrey Mason. Twisted representations of vertex operator algebras and associative algebras. *Internat. Math. Res. Notices*, (8):389–397, 1998.

[23] Chongying Dong, Haisheng Li, and Geoffrey Mason. Modular-invariance of trace functions in orbifold theory and generalized moonshine. *Communications in Mathematical Physics*, 214(1):1–56, oct 2000.

[24] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[25] John F. Duncan. Super-moonshine for Conway's largest sporadic group. *Duke Math. J.*, 139(2):255–315, 2007.

[26] John F. R. Duncan, Michael J. Griffin, and Ken Ono. Moonshine. *Res. Math. Sci.*, 2:Art. 11, 57, 2015.

[27] John F. R. Duncan and Sander Mack-Crane. The moonshine module for Conway's group. *Forum Math. Sigma*, 3:Paper No. e10, 52, 2015.

[28] John Francis Robert Duncan. *Vertex operators, and three sporadic groups*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–Yale University.

[29] Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the $K3$ surface and the Mathieu group M_{24}. *Exp. Math.*, 20(1):91–96, 2011.

[30] J. S. Frame. The characters of the Weyl group E_8. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 111–130. Pergamon, Oxford, 1970.

[31] P. Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer New York, 2011.

[32] Igor B. Frenkel and V. G. Kac. Basic representations of affine Lie algebras and dual resonance models. *Invent. Math.*, 62(1):23–66, 1980/81.

[33] Igor B. Frenkel, James Lepowsky, and Arne Meurman. A moonshine module for the Monster. In *Vertex operators in mathematics and physics (Berkeley, Calif., 1983)*, volume 3 of *Math. Sci. Res. Inst. Publ.*, pages 231–273. Springer, New York, 1985.

[34] Igor B. Frenkel, James Lepowsky, and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[35] Robert L. Griess, Jr. The structure of the "monster" simple group. In *Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975)*, pages 113–118. Academic Press, New York, 1976.

[36] Robert L. Griess, Jr. The friendly giant. *Invent. Math.*, 69(1):1–102, 1982.

[37] Robert L. Griess, Jr. My life and times with the sporadic simple groups. *ICCM Not.*, 9(1):11–46, 2021.

[38] Theo Johnson-Freyd. The Moonshine anomaly. *Comm. Math. Phys.*, 365(3):943–970, 2019.

[39] Theo Johnson-Freyd and David Treumann. $\mathrm{H}^4(\mathrm{Co}_0; \mathbf{Z}) = \mathbf{Z}/24$. *Int. Math. Res. Not. IMRN*, (21):7873–7907, 2020.

[40] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 1998.

[41] Victor Kac and Weiqiang Wang. Vertex operator superalgebras and their representations. In *Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992)*, volume 175 of *Contemp. Math.*, pages 161–191. Amer. Math. Soc., Providence, RI, 1994.

[42] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[43] Takeshi Kondo. The automorphism group of Leech lattice and elliptic modular functions. *J. Math. Soc. Japan*, 37(2):337–362, 1985.

[44] John Leech. Some sphere packings in higher space. *Canadian J. Math.*, 16:657–682, 1964.

[45] Jeffrey S. Leon and Charles C. Sims. The existence and uniqueness of a simple group generated by $\{3,4\}$-transpositions. *Bull. Amer. Math. Soc.*, 83(5):1039–1040, 1977.

[46] P. S. Montague. Third and higher order NFPA twisted constructions of conformal field theories from lattices. *Nuclear Phys. B*, 441(1-2):337–382, 1995.

[47] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Comm. Math. Phys.*, 123(2):177–254, 1989.

[48] Sven Möller. *A Cyclic Orbifold Theory for Holomorphic Vertex Operator Algebras and Applications*. 2021.

[49] Michael E. Peskin and Daniel V. Schroeder. *An introduction to quantum field theory*. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1995. Edited and with a foreword by David Pines.

[50] Larissa Queen. Modular functions arising from some finite groups. *Math. Comp.*, 37(156):547–580, 1981.

[51] Mark Ronan. *Symmetry and the monster*. Oxford University Press, Oxford, 2006. One of the greatest quests of mathematics.

[52] Stephen D. Smith. On the head characters of the Monster simple group. In *Finite groups—coming of age (Montreal, Que., 1982)*, volume 45 of *Contemp. Math.*, pages 303–313. Amer. Math. Soc., Providence, RI, 1985.

[53] J. G. Thompson. Finite groups and modular functions. *Bull. London Math. Soc.*, 11(3):347–351, 1979.

[54] J. G. Thompson. Some numerology between the Fischer-Griess Monster and the elliptic modular function. *Bull. London Math. Soc.*, 11(3):352–353, 1979.

[55] John G. Thompson. Some finite groups which appear as $\mathrm{Gal}\, L/K$, where $K \subseteq \mathbf{Q}(\mu_n)$. *J. Algebra*, 89(2):437–499, 1984.

[56] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geometry*, 17(4):661–692 (1983), 1982.

[57] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996.