NETWORK RELIABILITY, SIMPLICIAL COMPLEXES, AND POLYNOMIAL ROOTS

by

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This dissertation is dedicated to Tannis Stallard, for she is the one who showed me just how beautiful Mathematics can be.

This dissertation is in loving memory of Dr. Susan Roddy. Thank you for reminding me to be true to myself even when others didn't agree.

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Symbols

G, H	Graphs
$E(G)$	Edge set of a Graph G
$V(G)$	Vertex set of a Graph G
$\overline{G}$	Complement of G
P_n	Path on n vertices
$P_n^{\mathbf{k}}$	Path on n vertices with bundle sizes $\mathbf{k} = [k_1, k_2, \ldots, k_n]$
C_n	Cycle on n vertices
K_n	Complete graph on n vertices
RS	Royle-Sokal graph
$\Theta_{i,j,k}$	Theta graph with branches of lengths i, j, and k
$\Theta_{l[k]}$	Generalized theta graph with k branches of length l
$\text{Rep}(G, e, k)$	Replaces edge e in G by P_k
e^k	Replaces edge e by a bundle of k edges
$G - e$	Deletion of edge e in G
$G - v$	Deletion of vertex v in G
$G \bullet e$	Contraction of edge e in G
$G[H]$	Gadget replacement on G with copies of G
$G + P_k$	Path addition on G with P_k
$\text{Rel}(G; q)$	Reliability of the graph G
$\mathcal{C}$	Simplicial Complex
$\mathcal{C}^*$	Dual of a Simplicial Complex
τ	Face of $\mathcal{C}$
σ	Facet of $\mathcal{C}$
$\dim(\tau), \lvert\tau\rvert$	Dimension (or cardinality) of τ
d	Dimension of $\mathcal{C}$
F_i	Number of faces of dimension i

H_i	For shellable complexes, counts the number of monomials of degree i in an order ideal of monomials
F-vector	The vector $\langle F_0, F_1, \ldots, F_d \rangle$
H-vector	The vector $\langle H_0, H_1, \ldots, H_d \rangle$
$\mathcal{C}_1 * \mathcal{C}_2$	Join of two complexes
$\text{del}_x C$	Deletion complex of x in $\mathcal{C}$
$\text{link}_x C$	Link complex of x in $\mathcal{C}$
$\overline{F_i}$	Simplex (or power set) of the face F_i
$\mathcal{BR}(G, \preceq)$	Broken circuit complex of G using ordering $\preceq$
Prob	Probability of an event happening
$\mathcal{PD}_{m,p}$	Pure d-dimensional complexes on $[m]$ with $p \in (0, 1)$
$\mathcal{M}$	Matroid
$\text{Th}(\mathcal{M}, v, k)$	k-Thickening of v in $\mathcal{M}$
$\text{Rep}(\mathcal{M}, v, k)$	k-Replacement of v in $\mathcal{M}$
$\text{Cog}(G)$	Cographic matroid of G
$\mathcal{SP}$	Class of series-parallel graphs
$\text{Rel}(\mathcal{C}, q)$	Reliability of the complex $\mathcal{C}$
$\text{mgen}_{\mathcal{M}}$	Multivariate generating polynomial of $\mathcal{M}$
$\mathcal{RS}$	Royle-Sokal matroid
$\{s, t\}$	Set of terminals
T	Tendril of G
$\widehat{G}$	The graph $G - T$
$C_{s,t}(G)$	Two-terminal complex of G
$\Delta_T(G)$	Simplex on the edges of T in G
$\text{Rel}_2(G, q)$	Two Terminal Reliability of the graph G

Abstract

Assume that the vertices of a graph G are always operational, but the edges of G fail independently with probability $q \in [0, 1]$. The *all-terminal reliability* of G is the probability that the resulting subgraph is connected. The all-terminal reliability is a polynomial in q, and it was conjectured that all the roots of (nonzero) reliability polynomials fall inside the closed unit disk centred at 0. It has since been shown that there exist some connected graphs which have their reliability roots outside the closed unit disk, but these examples seem to be few and far between, and the roots are only barely outside the disk.

In this dissertation we generalize the notion of reliability to simplicial complexes and matroids and investigate when the roots fall inside the closed unit disk. We then shift our attention to discuss a related problem – among all reliability polynomials of graphs on n vertices, which has a root of smallest modulus (that is, the distance from the root to the origin in the complex plane). We also show a mathematical statement that distinguishes the class of simple graphs from the class of all graphs using all-terminal reliability. Finally, we explore two-terminal reliability — in particular, the similarities and differences between two-terminal reliability polynomials and the all-terminal reliability, with a focus on their roots.

Acknowledgements

Thank you to all of my friends and family both near and far. You have always been there for me when I needed you, and it is because of you that I was able to stay happy and healthy.

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Chapter 1

Introduction

A well known model of network robustness is the **(all terminal) reliability** of a graph, in which the vertices and edges can take on two possible states – either they are present (in which case we say that they are **operational**), or they are not present (in which case we say that they have **failed**). For the all terminal reliability, we have that the vertices are always operational, but each edge fails independently with probability $q \in [0, 1]$. The reliability of the undirected, connected graph G, $\mathrm{Rel}(G; q)$, is the probability that the operational edges form a connected spanning subgraph, that is, that the operational edges contain a spanning tree. We will see in Section 1.2.2 that the reliability of a graph is always a polynomial in q (and in $p = 1 - q$, the probability that the edge is operational), and is not identically 0 if and only if G is connected.

Other forms of well-studied reliabilities are **K-terminal** and **two-terminal** reliabilities (see [23] for a survey). These reliabilities relax the condition that all vertices (which we also call **terminals**) need to be able to communicate, but instead either a specific set of terminals K or only 2 specific terminals need to be in the same connected component (for K-terminal and two-terminal, respectively).

Much of the early work on the all-terminal reliability focused on exact calculations, and then, when the problem was found to be intractable (#P-hard) [23, pg. 30] emphasis was placed on efficient methods of bounding the function. Most of these methods centred on the coefficients of the aforementioned polynomial under expansion with a variety of bases (for instance, the Kruskal-Katona bounds and the Ball-Provan bounds – see Sections 5.4.2 and 5.5.5, respectively, in [23]). The location of the roots of polynomials became of interest as they have direct implications for the relationship between the coefficients. For example, Newton's well known theorem (see, for example, [25]) shows that a polynomial $p(x) = \sum a_i x^i$ with positive coefficients having all real roots implies that the coefficients are log concave ($a_i^2 \geq a_{i-1}a_{i+1}$) and hence unimodal (non-decreasing and then non-increasing). Furthermore, a result of Brenti et. al [10] showed that if all of the complex roots z of a polynomial $f(x) \in \mathbb{R}[x]$ lie in the sector $|\arg(x)| < \pi/3$, then the sequence of coefficients of $f(x)$ is strictly concave (and is either all positive or negative). If we are able to understand the location of the roots of reliability polynomials, then we may be able to get new inequalities and better methods for estimating the coefficients of the reliability polynomial which have real-world applications (see, for example, [50] where Pino et. al used reliability to compute the reliability of a gas distribution network in the Netherlands). Moreover, the roots of these polynomials are of interest in and of themselves.

In this dissertation, we will begin in Chapter 2 by extending the exploration of whether the reliability roots of a graph are bounded by generalizing reliability to simplicial complexes and matroids. We show that the reliability roots for matroids of rank 3 and paving matroids of rank 4 do, indeed, fall inside of the closed unit disk. We also prove that the all-terminal reliability roots of shellable complexes are dense in the complex plane, and that the real reliability roots of any matroid lie in $[-1, 0) \cup \{1\}$. Finally, we also show that the all-terminal reliability roots of thickenings of the Fano matroid can lie outside the unit disk.

We then shift our attention in Chapter 3 to discuss a related problem – among all reliability polynomials of graphs on n vertices, which has a root of smallest modulus? We prove that (for $n \geq 2$), the rational roots are $-1, -1/2, -1/3, \ldots, -1/(n-1), 1$. Moreover, we show that for $n \geq 3$, the root of minimum modulus among all graphs of order n is rational, and determine all roots of smallest moduli and the corresponding graphs. To close the chapter, we provide the first nontrivial mathematical property that distinguishes, via reliability, the class of simple graphs (that is, those without loops and multiple edges) from that of graphs in general.

Finally, we explore two-terminal reliability in Chapter 4 — in particular, the similarities and differences between two-terminal reliability polynomials and the all-terminal reliability polynomial. We classify when the underlying two-terminal simplicial complex is a matroid, and the effect it has on the two-terminal reliability polynomial. We then prove that the two-terminal reliability roots are dense in the two unit disks centred at 0 and 1, as well as determine two operations we can use to push roots outside of these two disks. Finally, we explore the real two-terminal reliability roots.

It is important to note that all computations and plots in this dissertation were performed by using Maple$^{\text{TM}}$2015, and all illustrations were created in GeoGebra$^{\text{TM}}$. Please see Appendix A for all Maple code.

1.0.1 Graph Theory Background

A **graph** G (see, for example, [30]) is a pair $(V(G), E(G))$ where $V(G)$ is a finite set of **vertices** and $E(G)$ is a multiset of **edges**. Each edge e has a set of one or two vertices associated with it which call the **endpoints** of e. The **order** of a graph is $|V(G)| = n$, and the **size** of a graph is $|E(G)| = m$. All graphs are considered to be finite (that is, both the order and the size are finite). A **simple** graph is a graph which has no multiple edges nor any loops (edges of size 1). If u and v share an edge, then they are said to be **adjacent**. A pair of edges that share a common endpoint are also called **adjacent**. We say that the **degree** of v, denoted by $\deg(v)$, is the number of vertices adjacent to v. A vertex with degree 0 is called an **isolated vertex**.

For example, consider the graph G in Figure 1.1. We see that $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, v_4 an isolated vertex, and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Since G contains a loop (e_1) and a pair of multiple edges (e_3, e_4), this is not a simple graph. Finally, the degree of v_3 is 3, and the adjacent vertices to v_3 are v_1 and v_2. For simplicity, we shall denote the vertex set of a graph G as V and the edge set as E when the graph is obvious. We may sometimes write an edge e with endpoints u and v by $\{u, v\}$, or simply uv.

We say that the graph $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Furthermore, for all $u, v \in V'$, if u and v are adjacent in H if and only if they are adjacent in G, then we call H an **induced subgraph**. The subtle difference between a subgraph and an induced subgraph is that one can choose the vertices and edges for a subgraph, but one needs only to choose the vertices to determine an induced subgraph. We call a subgraph H of a connected graph G a **spanning subgraph** if it contains all of the vertices of G.

Suppose S is a set of vertices. Then $G \setminus S$, called the **vertex deletion** of S from G, is the induced subgraph of G with all vertices of S, as well as all of their incident edges, removed. Analogously, if S is a set of edges, then $G \setminus S$, called the

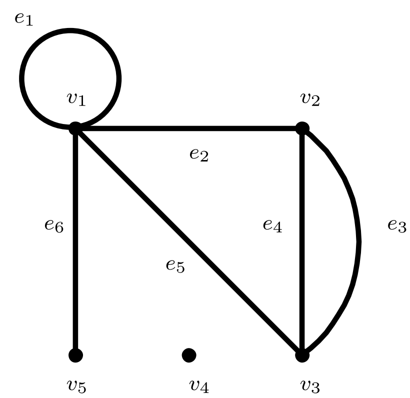


Figure 1.1: An example of a graph that is not simple

edge deletion of S from G, is the subgraph of G with all edges of S removed. For simplicity, if $S = \{v\}$ or $S = \{e\}$, then we shall write $G - v$ or $G - e$ for the deletion of v or e, respectively. The **contraction** of an edge $e = \{u, v\}$ from G, denoted by $G \bullet e$, is the graph where e is removed, u and v are replaced by a new vertex v', and u and v are replaced by v' in every remaining edge in which either appear. This results in a graph with one less edge and one less vertex. If S is a set of edges, then $G \bullet S$ is the graph resulting from repeated contractions of edges in S.

A **path** from u to v is an alternating sequence of vertices and edges $u = v_0, e_1, v_1, \ldots,$ $e_n, v_n = v$ where $e_i = \{v_{i-1}, v_i\}$ for $i = 1, 2, \ldots, n$. A graph is **connected** if there is a path between every pair of vertices, and **disconnected** otherwise. We say that a maximal connected subgraph of G is a **connected component** of G. A vertex v is a **cut vertex**, or an edge e is a **bridge**, if $G \setminus v$ or $G \setminus e$ is disconnected, respectively.

The **complement** of a simple graph $G = (V, E)$, denoted by $\overline{G} = (V, E')$, is a simple graph on the same vertex set, and a pair of vertices are adjacent if and only if they are nonadjacent in G.

Common Graphs

Here we will list a few common graphs that we will be using throughout this dissertation.

Example 1.0.1. *Path Graphs*

The most basic graph that we will consider is called a **path** graph. The path graph on n vertices, denoted by P_n, has $n-1$ edges with vertices ordered sequentially, say $v_1, v_2, \ldots, v_n$ such that v_i and v_{i+1} for $i = 1, 2, \ldots, n-1$ are adjacent. See Figure 1.2.

Figure 1.2: The path graph P_4

Example 1.0.2. *Cycle Graphs*

The **cycle** graph on n vertices, denoted by C_n, is the next step of generalization that we will use. This generalization is of P_n, but now we add the edge connecting the two extreme endpoints of the path. See Figure 1.3.

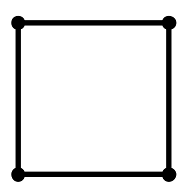


Figure 1.3: The cycle graph C_4

Example 1.0.3. *Tree Graphs*

The next graph we will be considering is another generalization of a path graph, called a **tree** graph. A tree graph on n vertices again has $n-1$ edges, but it has no *cycle* as a subgraph. See Figure 1.4 for an example of a tree on 4 vertices. We call a tree a **spanning tree** of a graph G if if is a subgraph of G and every vertex of G is included in the tree.

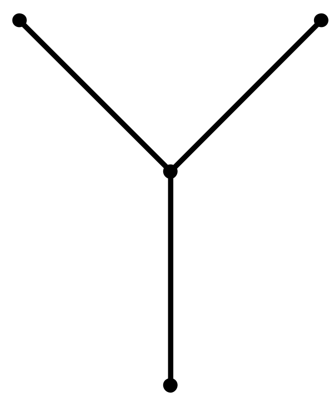


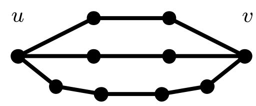
Figure 1.4: An example of a tree on 4 vertices

Example 1.0.4. *(Generalized) Theta Graphs*

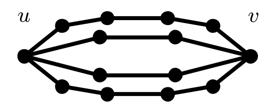
Another very common graph we will be considering is called a **theta** graph. This graph, denoted by Θ_{i_1,i_2,i_3}, is a graph on $i_1+i_2+i_3$ edges constructed by first fixing two vertices, u and v, and then placing three internally disjoint paths of length i_1, i_2, and i_3 between them. An example of $\Theta_{3,3,5}$ can be seen in Figure 1.5a.

We can also generalize this class to **generalized theta** graphs, with $k \geq 3$ internally disjoint paths between u and v. An example of the generalized theta graph $\Theta_{5,3,3,5}$ can be seen in Figure 1.5b.

Example 1.0.5. *Complete Graphs*



(a) The Theta Graph $\Theta_{3,3,5}$



(b) The Generalized Theta Graph $\Theta_{5,3,3,5}$

Figure 1.5: Examples of theta and generalized theta graphs

This next graph is denser than the previous graphs – we require every possible edge to be present. The **complete** graph on n vertices, denoted by K_n, is the simple graph on $\binom{n}{2}$ edges (so that every pair of vertices is adjacent). An example can be found in Figure 1.6.

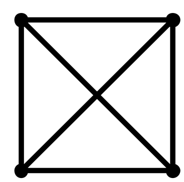


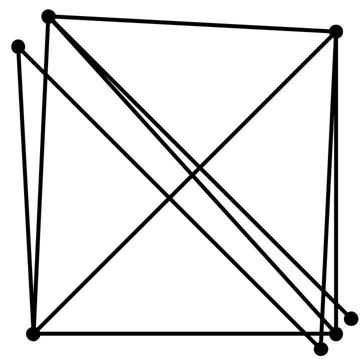
Figure 1.6: The complete graph K_4

Example 1.0.6. *(Complete) k-Partite or Multipartite Graphs*

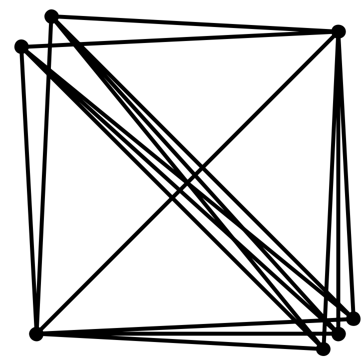
The last graph that we will commonly use will be the **k-partite** or **multipartite** graph. This graph has its vertices partitioned into k disjoint parts or collections such that no two pairs of vertices in the same part are adjacent. If each vertex is adjacent to every vertex in every other part, we call it **complete**, and denote it by $K_{r_1,r_2,\ldots,r_k}$. An example of both instances can be found in Figure 1.7.

Common Operations

Throughout this dissertation, we will also be utilizing two common operations: an *edge path replacement*, and an *edge bundling*.



(a) A 4-partite graph



(b) A complete 4-partite graph $K_{2,1,1,3}$

Figure 1.7: Examples of 4-partite and complete 4-partite graphs

Operation 1.1. *Edge Path Replacement*

This operation, denoted by $\text{Rep}(G, e, k)$ replaces the edge e in G by P_k.

Operation 1.2. *Edge Bundling*

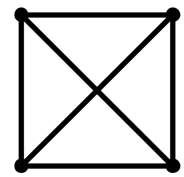
This operation, denoted by e^k, replaces the edge e in G by a bundle of k edges, where a bundle of edges is a collection of multiple edges all of which share common endpoints (for example, edges e_3 and e_4 in Example 1.1 on page 4 is a bundle of two edges).

Both of these operations can be seen in Figure 1.8. Dotted lines represent the edges that have been replaced.

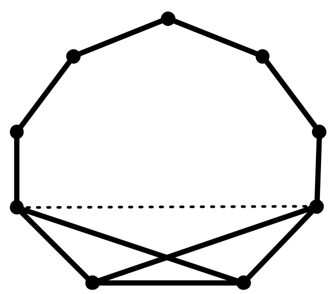
The reader is referred to [58], for example, for any graph theory definitions or examples omitted from this section.

1.2.1 Simplicial Complex and Matroid Background

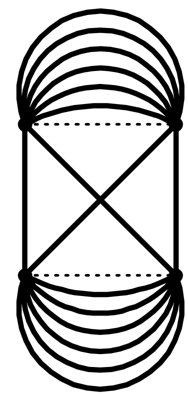
The other major mathematical objects we will be considering are (abstract simplicial) complexes and matroids (see, for instance, [12] or [23]).



(a) Original graph, K_4



(b) Edge path replacement of length 6 on K_4



(c) Edge bundling e^6 on two parallel edges of K_4

Figure 1.8: Examples of an edge path replacement and an edge bundling on K_4

Complexes

An **abstract simplicial complex** (or, for brevity, just simply **complex**) $\mathcal{C}$ on a finite set V (called either the **ground set** or **vertex set**) is a collection of subsets of V such that $\emptyset \in \mathcal{C}$, and for any $tau \in \mathcal{C}$, if $\tau' \subseteq \tau$ then $\tau' \in \mathcal{C}$ (which is called the **inheritance property**). We say that the elements of V are **vertices**, and the elements of $\mathcal{C}$, which we normally denote by τ, are **faces** (the cardinality of V is the **order** of $\mathcal{C}$). Faces of $\mathcal{C}$ that are maximal with respect to inclusion are called **facets**, and are normally denoted by σ. In fact, since complexes are closed under containment, a complex is completely determined by its facets. If τ is a face of $\mathcal{C}$, then the **dimension**[1] of τ, denoted by $\dim(\tau)$, is $|\tau|$, and the dimension of the

[1]For any reader who is familiar with abstract simplicial complexes from the topological lens, our definition of dimension is the combinatorial definition which is one more than the topological

complex is the dimension of one of the facets with respect to their cardinality. For matroids, we shall use the term **rank**, denoted by r, to mean dimension. The complex $\mathcal{C} = (V, \{\emptyset\})$ is called an **empty complex** and has dimension 0. Any complex on set V whose facets are all subsets of $\mathcal{C}$ is called a **subcomplex** of $\mathcal{C}$. Faces of dimension d are called d-faces, and if we restrict a complex to only its d-dimensional faces, then we say that it is the **d-skeleton** of $\mathcal{C}$. We will see how we can "read" a picture of a complex – that is, how we can interpret the connection between the list of faces and its geometric illustration – in the coming subsection.

The **F-vector** of the complex $\mathcal{C}$ is the sequence $\langle F_0, F_1, \ldots, F_d \rangle$, where F_i is the number of faces of cardinality i in the complex. The **F-polynomial** is the generating function of the F-vector, and is given by

$$f_{\mathcal{C}}(x) = \sum_{\sigma \in \mathcal{C}} x^{|\sigma|} = \sum_{i=0}^{d} F_i x^i$$

(the degree of the polynomial is clearly the dimension of the complex). The **H-polynomial** of complex $\mathcal{C}$ of dimension d is given by

$$h_{\mathcal{C}}(x) = (1-x)^d f_{\mathcal{C}}\left(\frac{x}{1-x}\right),$$

and the **H-vector**[2] of the complex is the vector of coefficients $\langle H_0, H_1, \ldots, H_d \rangle$ of the H-polynomial; alternatively,

$$H_k = \sum_{i=0}^{k} F_i (-1)^{k-i} \binom{d-i}{k-i}. \tag{1.1}$$

Given two complexes $\mathcal{C}_X$ and $\mathcal{C}_Y$ on disjoint sets X and Y, respectively, define the **join**, denoted by $\mathcal{C}_X * \mathcal{C}_Y$, to be the complex on $X \cup Y$ whose faces are of the form $\tau_1 \cup \tau_2$ where τ_1 and τ_2 are faces of $\mathcal{C}_X$ and $\mathcal{C}_Y$, respectively (if X and Y are not disjoint, we take isomorphic disjoint copies $\mathcal{C}_X$ and $\mathcal{C}_Y$ to form $\mathcal{C}_X * \mathcal{C}_Y$). The complex $\mathcal{C}$ is said to be **connected** if it cannot be written as the join of two other complexes each of positive order[3] , and if $\mathcal{C} = \mathcal{C}_1 * \mathcal{C}_2 * \cdots * \mathcal{C}_k$ where each $\mathcal{C}_i$ is connected, then $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are the **components** of $\mathcal{C}$.

definition (which is the cardinality of τ less one).

[2]We remark that in the topological or commutative algebra setting, it is common for the F- and H-vectors be written in the lower case f or h instead of F and H, which is more commonly used in combinatorics.

[3]We note that this is a different definition than the graph theoretic definition of being connected in that there could be two disjoint components, but still be connected.

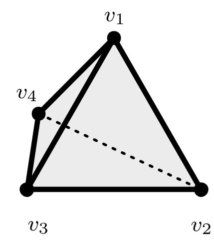


Figure 1.9: Example of a simplex on 4 vertices

For a given vertex v of the underlying set F, the **deletion** and **link** subcomplexes are those on $X\backslash\{v\}$ of order $m-1$, with faces

$$\text{del}_v\mathcal{C} = \{\tau : \tau \in \mathcal{C},\ v \notin \tau\}$$

and

$$\text{link}_v\mathcal{C} = \{\tau\backslash\{v\} : v \in \tau \in \mathcal{C}\}.$$

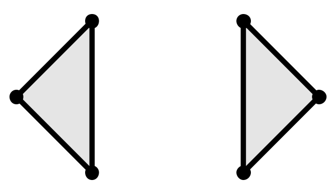
Common Complexes

Example 1.2.1. *Simplices*

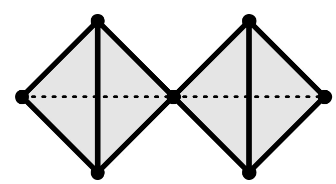
The most basic of complexes we will consider is a *simplex*. For any finite set X, the power set of X, denoted by $\overline{X}$, is a called a **simplex**. It is easy to see that a simplex is always a complex. For instance, consider the simplex $\mathcal{C}$ with facet $\sigma = \{v_1, v_2, v_3, v_4\}$ (see Figure 1.9 – dotted lines in the illustration represents faces in the background). We can calculate its F-vector by counting the number of faces of each dimension. Its 1-faces (i.e., faces of cardinality 1) are represented as vertices; its 2-faces are lines; its 3-faces are triangles; and its 4-faces is the entire tetrahedron. Therefore, since every subset (or d-face) is in the complex, we see that its F-vector is $\langle 1, 4, 6, 4, 1\rangle$. The calculation of the H-vector is slightly more complicated using an illustration, so we use 1.1 to calculate its H-vector to be $\langle 1\rangle$.

Example 1.2.2. *Pure Complexes*

A very common type of complex we will be considering is called a *pure* complex. We say that a complex is **pure** if all facets are of the same dimension, and a complex is **pure k-dimensional** if all facets are of dimension k. In Figure 1.10 we have two examples of pure complexes. The first example, Figure 1.10a, is a connected pure complex with two facets of dimension 3 (it cannot be written as the join of two complexes). Its F-vector is $\langle 1, 6, 6, 2\rangle$, and its H-vector is $\langle 1, 3, -3, 1\rangle$. In the second example, Figure 1.10b, we have a disconnected pure complex (it is the join of the complex in 1.10a with a single vertex) with two facets of dimension 4. Its F-vector is $\langle 1, 7, 12, 8, 2\rangle$, and its H-vector is also $\langle 1, 3, -3, 1\rangle$.



(a) A connected pure complex with two facets of dimension 3



(b) A disconnected pure complex with two facets of dimension 4

Figure 1.10: Examples of pure complexes

Example 1.2.3. *Shellable Complexes*

Another very common type of complex that we will be considering is called a *shellable* complex. If the facets of a pure d-dimensional complex can be ordered $\sigma_1, \sigma_2, \ldots F_d$ with $j = 2, 3, \ldots, d$, and if the subcomplexes

$$\left(\bigcup_{j=1}^{m-1} \overline{\sigma_j}\right) \bigcap \overline{\sigma_m}$$

are pure $(d-1)$-complexes for all $m = 2, \ldots, d$, then the complex is called **(pure) shellable**. If, instead, we relax the requirement that $\mathcal{C}$ is a pure complex, but still have the above condition except only requiring that at each step the subcomplex is a pure $(\dim(\sigma_m) - 1)$-dimensional complex, then the complex is called **non-pure shellable** (this was first introduced by Bjorner and Wachs in [7, 8]). Intuitively, pure shellable complexes can be thought of as building the complex by piecing its shell together, and non-pure shellable complexes can be thought of as piecing its shell together in a similar 'nice' way. Examples of both types of complexes are illustrated in Figure 1.11.

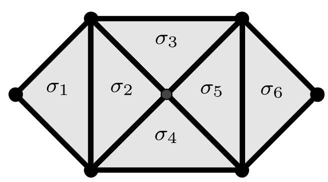
Indeed, in Figure 1.11a, we have a complex with 6 facets, $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$, all of dimension 3, so it is pure. We see that if we start with σ_1 and join it with σ_2, then the intersection between σ_1 and σ_2 is an edge of dimension 2. Continuing this process, we see that at each step the new facet intersects the previous subcomplex in the 2nd dimension (sometimes more than once, which can be seen in the case when we join σ_5 – its intersection with $\cup_{i<5}\overline{\sigma_i}$ occurs at two edges). Therefore, it is shellable. One can calculate its F- and H-vectors to be $\langle 1, 7, 12, 6\rangle$ and $\langle 1, 4, 1\rangle$, respectively.

On the other hand, if we consider the complex in Figure 1.11b, we see that this complex has 3 facets of dimensions $4, 3$, and 2. If we start with the facet σ_1 (of dimension 4) and join it with σ_2 of dimension 3, we see that its intersection occurs at an edge of dimension 2, which is one less than the dimension of σ_2. Then, when we join σ_3, of dimension 2, we see that its intersection occurs in dimension 1 (twice). Therefore, this is non-pure shellable. Again, one can calculate its F- and H-vectors to be $\langle 1, 5, 9, 5, 1\rangle$ and $\langle 1, 1, -2, 1\rangle$, respectively.

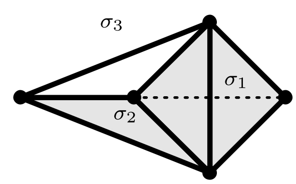
A counter example for both types of shellability can be found in Figure 1.11c. As we can see, this complex only has two faces of dimension 3 whose intersection is at a vertex (or dimension 1) which is one dimension too low.

One observation we can make with regards to shellable 2-dimensional complexes is that they are precisely the connected graphs of size at least 1 without isolated vertices (we start at 2-dimensional shellable complexes since all 1-dimensional complexes are shellable). Indeed, suppose we have a connected graph (which is a 2-dimensional complex). We first choose an ordering of the edges so that it builds a spanning tree of the graph. Then, we can add all remaining edges one by one, as long as the edge we add is connected to the spanning tree. Since the graph is connected (i.e. has only one component), we will never have a case where we have that the intersection with the edge being added is in dimension 0. On the other hand, a disconnected graph with an edge cannot be shellable as shellability gives a connected graph that must encompass the whole graph as there are no isolated vertices.

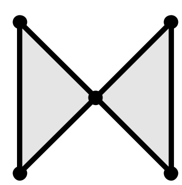
It is important to note that for both shellable and non-pure shellable, the order of the facets matter. For instance, if we started with σ_1 and σ_6 for the pure complex in Figure 1.11a, then their intersection is empty (clearly not dimension 2). Furthermore, if we started with σ_2 and then joined σ_1 in the non-pure complex in Figure 1.11b, we see that its intersection occurs in dimension 2 which is two less than the dimension of σ_1. Therefore, the order of the shelling matters, and for a complex to be non-pure shellable, we must have the facets ordered by non-increasing dimension.



(a) (Pure) shellable complex



(b) Non-pure shellable complex



(c) Example of a non-shellable complex

Figure 1.11: Examples of pure shellabe, non-pure shellable, and non-shellable complexes

Example 1.2.4. *Broken Circuit Complexes*

The last type of complex we will consider is called a **Broken Circuit Complex** of a graph G. This complex, denoted by $\mathcal{BR}(G, \preceq)$, is the complex generated by first labelling all edges of a graph $e_1, \ldots, e_m$, and then fixing an ordering on the edges, $\preceq$

(we tend to use the natural ordering $e_1 < e_2 < \cdots < e_m$). We then list all cycles of G, which we call **circuits**, and remove their $\preceq$-least edge (which we call **broken circuits**). Finally, the faces of $\mathcal{BR}(G, \preceq)$ are all subsets of $E(G)$ that contain no broken circuit (the facets are the spanning trees of G that do not contain a broken circuit – see, for example, [20, Prop. 3.1]).

For example, consider the graph in Figure 1.12a. We see that this graph, K_4, has 6 edges labelled $e_1, \ldots, e_6$ with circuits and broken circuits – with the natural ordering $e_1 < e_2 < \cdots < e_6$ – given in the table below (written multiplicatively).

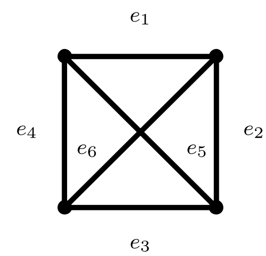
Circuit	**Broken Circuit**
$e_1e_2e_5$	e_2e_5
$e_2e_3e_6$	e_3e_6
$e_1e_4e_6$	e_4e_6
$e_3e_4e_5$	e_4e_5
$e_1e_2e_3e_4$	$e_2e_3e_4$
$e_2e_4e_5e_6$	$e_4e_5e_6$
$e_1e_3e_5e_6$	$e_3e_5e_6$

One can calculate that there are $4^2 = 16$ spanning trees of K_4; four trees isomorphic to $e_1e_4e_5$, and 12 trees isomorphic to $e_1e_2e_3$ (see Figures 1.12b and 1.12c, respectively). After computing all of the spanning trees, and eliminating all trees that contain a broken circuit, we are left with the broken circuit complex $\mathcal{BR}(K_4, \preceq)$ generated by the facets:

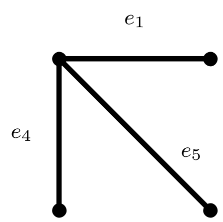
$$\{e_1e_2e_3, e_1e_2e_4, e_1e_2e_6, e_1e_3e_4, e_1e_3e_5, e_1e_5e_6\}.$$

Matroids

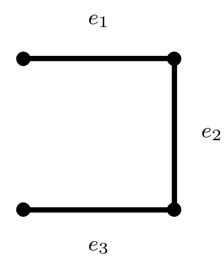
Finally, the last mathematical object we will be considering is a special type of complex called a *matroid*. A **matroid** $\mathcal{M}$ on ground set V is a simplicial complex together with the property that if τ and τ' are both faces in $\mathcal{M}$ with $|\tau| > |\tau'|$, then there exists some element $x \in \tau \setminus \tau'$ such that $\tau' \cup \{x\}$ is a face in $\mathcal{M}$ (this property we call the **exchange property**). It is easy to see that this exchange property forces matroids to be pure (if there were a facet, say σ_1, which had smaller dimension than



(a) The complete graph K_4 with edges labelled



(b) An isomorphic class of subgraphs of K_4



(c) Another isomorphic class of subgraphs of K_4

Figure 1.12: K_4 and its spanning trees

another facet, say σ_2, then we would be able to augment σ_2 with an element from σ_1, thus increasing its dimension. We repeat this process until all facets are of the same dimension). Furthermore, it can also be shown that matroids are shellable (see, for instance, Section 5.5.2 on page 61 in [23] together with Theorem 5.3).

Readers are recommended the book [44] for any matroid definitions or examples omitted in this section.

Example 1.2.5. *Matroids*

For an example of a matroid, let us consider the complex in Figure 1.13. We see that its faces are:

faces of dimension 0: $\emptyset$

faces of dimension 1: $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}$

faces of dimension 2: $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\},$
$\{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_4, v_6\}, \{v_5, v_6\}$

and so it follows that its F- and H-vectors are $\langle 1, 6, 11\rangle$ and $\langle 1, 4, 6\rangle$, respectively.

It is easy to see that this is a complex (as it is closed under containment). One can also confirm that the exchange property holds. For instance, consider the faces $\{v_1\}$ and $\{v_2, v_6\}$. The exchange property says that there is a vertex in $\{v_2, v_6\}$ such that its union with $\{v_1\}$ yields another face that is in $\mathcal{M}$. Indeed, this is true as both $\{v_1, v_2\} \in \mathcal{M}$, and $\{v_1, v_6\} \in \mathcal{M}$.

It is important to note that not all choices of vertices may be used in the exchange property. Indeed, if we consider the faces $\{v_1\}$ and $\{v_2, v_4\}$, we see that $\{v_1, v_2\} \in \mathcal{M}$ but $\{v_1, v_4\} \notin \mathcal{M}$. We only require that at least one choice be present.

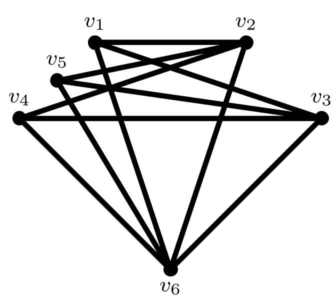


Figure 1.13: Example of a matroid

One can also see from this example that if $\mathcal{M}$ is a d-dimensional matroid, $d \geq 2$, then its 2-skeleton forms a complete multipartite graph. To prove this, we split the claim into two lemmas. We first show that the complex generated by the faces of the k-skeleton of a matroid is a matroid, and then show that a 2-dimensional pure complex is a matroid if and only if its 2-skeleton is a complete multipartite graph.

Lemma 1.2.1. *Suppose $\mathcal{M}$ is a d-dimensional matroid, $2 \leq k \leq d$, and S is the*

k-skeleton of $\mathcal{M}$. Then the complex generated by S, that is,

$$\overline{S} = \bigcup_{\alpha \in S} \overline{\alpha}$$

is a matroid.

Proof. By definition, $\overline{\alpha}$ is a simplex and so $\overline{S}$ is a complex. Suppose $\tau_1, \tau_2 \in \overline{S}$ with $|\tau_1| < |\tau_2|$. We need to show that we can add a vertex $v \in \tau_2 \setminus \tau_1$ such that $\tau_1 \cup \{v\} \in \overline{S}$. Indeed, as $\tau_1, \tau_2 \in \mathcal{M}$ (as $\mathcal{M}$ contains $\overline{S}$), there is an $x \in \tau_2 \setminus \tau_1$ such that $\tau_1 \cup \{x\} \in \mathcal{M}$. We can extend $\tau_1 \cup \{x\}$ to a facet σ of $\mathcal{M}$ of dimension d, and taking any $\sigma' \subset \sigma$ of dimension k that contains $\tau_1 \cup \{x\}$, then we see that $\tau_1 \cup \{x\} \in \overline{\sigma'} \subseteq \overline{S}$. Therefore, $\overline{S}$ is a matroid. □

Lemma 1.2.2. *A 2-dimensional pure complex $\mathcal{M}$ is a matroid if and only if its 2-skeleton is a complete multipartite graph.*

Proof. Suppose $\mathcal{M}$ is a matroid. Then there cannot be distinct vertices u, v, and w such that $\{u, v\}$ is a face but $\{u, w\}$ and $\{v, w\}$ are not (since $\mathcal{M}$ is pure, $\{w\}$ is a face of $\mathcal{M}$). This means that the 2-skeleton G, viewed as a graph, cannot contain an induced $K_2 \cup K_1$. That is, $\overline{G}$, the complement of G, cannot contain a P_3 as an induced subgraph. It follows that that every connected component of $\overline{G}$ must be a complete subgraph. It follows that G is a complete multipartite graph.

Suppose now that the 2-skeleton G of $\mathcal{M}$ is a complete multipartite graph, say with parts $X_1, \ldots, X_k$, ($k \geq 2$ as otherwise there would be no edges). Suppose $\tau_1, \tau_2 \in \mathcal{M}$, with $|\tau_1| < |\tau_2|$. We want to show that for some vertex $v \in \tau_2 \setminus \tau_1, \tau_1 \cup \{v\} \in \mathcal{M}$. If $\tau_1 = \emptyset$, then any $v \in \tau_2$ will do (as $\tau_1 \cup \{v\} = \{v\} \in \mathcal{M}$). So, $|\tau_1| = 1 < |\tau_2| = 2$. Suppose $\tau_2 = \{u, v\}$ with u and v in different parts. Without loss of generality, $u \in X_1$ and $v \in X_2$. Let $\tau_1 = \{w\}$. Either $w \notin X_1$ or $w \notin X_2$. If $w \notin X_1$, then $\tau_1 \cup \{u\} = \{w, u\} \in \mathcal{M}$ as w and u are in different parts of the complete multipartite graph G. Similarly, if $w \notin X_2$, then $\tau_1 \cup \{v\} = \{w, v\} \in \mathcal{M}$. In any event, $\mathcal{M}$ satisfies the augmentation property, and so $\mathcal{M}$ is a matroid. □

For simplicity, for the remainder of this dissertation we will denote a face of a complex $\{v\}$ simply by v, and for faces of higher dimension we will write them as a product of vertices; for example, $\{v_1, v_2\}$ will be denoted by $v_1 v_2$.

1.2.2 Network Reliability Background

As we introduced earlier, the *all-terminal reliability* of a graph G, denoted by $\mathrm{Rel}(G;q)$, is a measure of how reliable a network is if edges of the graph G fail independently with probability $q \in [0,1]$ (but the vertices are always operational). The reliability of a loopless, connected graph G of order n and size m can be expressed in a variety of useful forms, by expanding the polynomial in terms of different bases (see, for example, [23]). The two forms we will be using are the F-Form and H-Form:

$$\mathrm{Rel}(G;q) = \sum_{i=0}^{m-n+1} F_i q^i (1-q)^{m-i} \qquad \text{(F-Form)} \tag{1.2}$$

$$= (1-q)^{n-1} \sum_{i=0}^{m-n+1} H_i q^i \qquad \text{(H-Form)}, \tag{1.3}$$

with the definitions of F_i and H_i being given shortly.

The interpretation of the coefficients of the F-form are quite straightforward. Suppose we have i edges that have failed (which occurs with probability q^i), and therefore have $m-i$ edges operational (which occurs with probability $(1-q)^{m-i}$). For this to be a valid state in our reliability model, then the operational edges must form a connected spanning subgraph (that is, a spanning subgraph H of G for which all vertices of G are present in H, and H is connected). We let F_i denote the number of ways that we can have i edges failing while having the operational edges form a connected spanning subgraph. Since there needs to be at least a spanning tree operational, $F_i = 0$ for $i > d = m - n + 1$.

If we consider the collection of sets whose removal leaves the graph connected, then we see that this forms a complex whose facets are the maximal sets of edges whose removal leaves the graph connected. This complex is a well-studied one, and is, in fact, a matroid; it is called the **cographic matroid**, denoted by $\mathrm{Cog}(G)$ (see Theorem 5.5.1 on page 60 in [23]). Furthermore the coefficients of the F-form and H-form of the reliability polynomial of a graph G are precisely the F- and H-vectors of the cographic matroid of G. We saw earlier that that all matroids are pure and shellable, and it is also known that the H-vector of a shellable complex has all positive entries (see Section 5.5.3 on page 63 of [23]). Therefore, for any graph G, its associated H-polynomial will always have strictly positive coefficients, and furthermore, using both the F-form and the H-form, we are able to show a nice property between F_d and the H-vector. That is, since

$$\sum_{i=0}^{m-n+1} F_i q^i (1-q)^{m-i} = (1-q)^{n-1} \sum_{i=0}^{m-n+1} H_i q^i$$

then

$$\sum_{i=0}^{m-n+1} F_i q^i (1-q)^{m-n+1-i} = \sum_{i=0}^{m-n+1} H_i q^i.$$

When evaluating this at $q = 1$, and noting that all terms of $(1-q)^{m-n+1-i} = 0$ except when $i = m - n + 1 = d$, it follows that

$$F_d = \sum_{i=0}^{d} H_i. \tag{1.4}$$

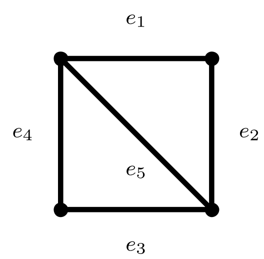
Let us consider the example in Figure 1.14 and compute its reliability. We see that the graph G in question, given in Figure 1.14a, has $n = 4$ vertices and $m = 5$ edges. Therefore, the dimension of the cographic matroid will be $d = 5 - 4 + 1 = 2$. Trivially if we remove no edges then G will remain connected, and so $F_0 = 1$. We can see that since all vertices have degree at least 2 then we can remove any single edge and the remaining subgraph of G will still remain connected. Since there are 5 edges, we have $F_1 = 5$. However, we cannot arbitrarily remove any two edges, as, for example, if we remove both e_1 and e_2, then the resulting subgraph will not be connected. One can check that the only pairs of edges that can be removed while leaving a connected spanning subgraph are $e_1e_3, e_1e_4, e_1e_5, e_2e_3, e_2e_4, e_2e_5, e_3e_5$, and e_4e_5. Therefore, $F_2 = 8$, and so the reliability of G is

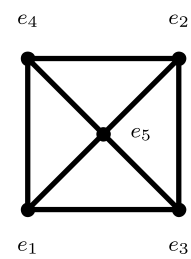
$$\begin{aligned} \mathrm{Rel}(G;q) &= 1q^0(1-q)^5 + 5q^1(1-q)^4 + 8q^2(1-q)^3 \\ &= (1-q)^3(4q^2 + 3q + 1). \end{aligned}$$

The cographic matroid of G, $\mathrm{Cog}(G)$, is illustrated in Figure 1.14b. We remind the reader that the *edges* of G become the *vertices* of $\mathrm{Cog}(G)$.

Let us now calculate the reliabilities for the common graphs we mentioned earlier.

Example 1.0.3 *Tree Graphs*





(a) Example graph G used to calculate its all-terminal reliability

(b) underlying cographic matroid of G, $\mathrm{Cog}(G)$

Figure 1.14: Example of a graph and its underlying cographic matroid

The reliabilities polynomials of trees are identical since the removal of any edge results in disconnecting the graph (this is also true for paths, but every path is a tree and so we need not include a separate example – though paths will become useful later on). Therefore, if G is a tree of size m, then all edges need to be operational and so

$$\mathrm{Rel}(G; q) = (1 - q)^m.$$

Example 1.0.2 *Cycle Graphs*

Cycles are slightly more complicated as to keep the graph operational, we can remove any edge (or no edges), but we can never remove more than 1 edge. Therefore, if G is a cycle C_n then

$$\begin{aligned}\mathrm{Rel}(G; q) &= (1 - q)^n + nq(1 - q)^{n-1} \\ &= (1 - q)^{n-1}((n - 1)q + 1).\end{aligned}$$

Example 1.0.4 *Theta Graphs*

The theta graph is where the calculation for classes of graphs gets to be much more difficult (and interesting!). Let us first calculate the reliabilities for the two examples in Figures 1.5a and 1.5b on page 7, and then calculate the reliability of a general generalized theta graph.

Let us first consider $G = \Theta_{3,3,5}$ as shown in Figure 1.5a. It is easy to see that we can remove either no edges or exactly one edge and not disconnect G, and if we remove any three edges then we will disconnect G. However, when we try to remove exactly 2 edges we see that we can do so only if we remove exactly one edge in two different paths (and so there are $9+15+15$ ways to do so). Therefore, the reliability polynomial of $\Theta_{3,3,5}$ is

$$\begin{aligned}\mathrm{Rel}(\Theta_{3,3,5};q) &= (1-q)^{11} + 11q(1-q)^{10} + (9+15+15)q^2(1-q)^9\\ &= (1-q)^9(29q^2+9q+1).\end{aligned}$$

Similarly, we can compute the reliability polynomial of $\Theta_{5,3,3,5}$ in Figure 1.5b except now we are able to remove 3 edges (as long as no two are in the same path), but we cannot remove any 4. Therefore, we have the reliability polynomial of $\Theta_{5,3,3,5}$ as

$$\begin{aligned}\mathrm{Rel}(\Theta_{5,3,3,5};q) &= (1-q)^{16} + 16q(1-q)^{15} + 94q^2(1-q)^{14} + 240q^3(1-q)^{13}\\ &= (1-q)^{13}(161q^3+65q^2+13q+1).\end{aligned}$$

We can generalize this formula a bit further if we consider the generalized theta graph with k paths, each of length l, which we denote as $\Theta_{l[k]}$. Then its reliability polynomial would be

$$\begin{aligned}\mathrm{Rel}(\Theta_{l[k]};q) =&(1-q)^{kl} + klq(1-q)^{kl-1} + \binom{k}{2}l^2q^2(1-q)^{kl-2} + \cdots +\\ &\binom{k}{k-1}l^{k-1}q^{k-1}(1-q)^{kl-(k-1)}.\end{aligned}$$

However, this formula is a bit cumbersome to analyze, and so we find it more useful to formulate the reliability of $\Theta_{l[k]}$ in the following way:

$$\mathrm{Rel}(\Theta_{l[k]};q) = ((1-q)^l + lq(1-q)^{l-1})^k - (lq(1-q)^{l-1})^k.$$

The idea is that, considering only one branch of the theta graph, we can have either every edge operational (with probability $(1-q)^l$), or exactly one edge failing (with probability $lq(1-q)^{l-1}$). This is true for all k branches. However, we cannot have the scenario where we have one edge down in every branch, and so we need to remove that probability. We note that if $k = 2$, then G becomes a cycle graph.

Example 1.0.5 *Complete Graphs*

We can calculate the reliability polynomial of a particular complete graph, for example K_4 as shown in Figure 1.0.5 on page 6. Doing so, we get the reliability polynomial

$$\begin{aligned}\text{Rel}(G;q) &= (1-q)^6 + 6q(1-q)^5 + 15q^2(1-q)^4 + 16q^3(1-q)^3 \\ &= (1-q)^3(6q^3 + 6q^2 + 3q + 1).\end{aligned}$$

In general for complete graphs, calculating the reliability seems to be hard for there is no known explicit formula for its reliability. However, even though there is no known explicit formula, there does exist a recursive formula (see, for example, [23], page 33) which is

$$\text{Rel}(K_n;q) = 1 - \left(\sum_{i=1}^{n-1} \binom{n-1}{i-1} q^{i(n-1)} \text{Rel}(K_i;q) \right).$$

See below for a table of the reliability polynomials of K_n for n from 2 to 7.

n	$\mathbf{Rel(K_n;q)}$
2	$1-q$
3	$q^3 - 2q^2 + 1$
4	$-2q^6 + 5q^5 - 2q^4 - 2q^3 + 1$
5	$6q^{10} - 18q^9 + 12q^8 + 7q^7 - 6q^6 - 2q^4 + 1$
6	$-24q^{15} + 84q^{14} - 78q^{13} - 20q^{12} + 44q^{11} + 3q^9 - 8q^8 - 2q^5 + 1$
7	$120q^{21} - 480q^{20} + 570q^{19} - 340q^{17} + 80q^{16} + 70q^{14} - 20q^{12} + 11q^{11} - 10q^{10} - 2q^6 + 1$

Now that we have introduced reliability polynomials, we are in a position where we can start to generalize them.

Chapter 2

Reliability Roots of Simplicial Complexes and Matroids

2.1 Reliability Polynomials and Their Roots

A natural question when studying certain types of polynomials is to examine their analytic and algebraic properties. For instance, recall that the reliability of the generalized theta graph $\Theta_{5,3,3,5}$ is:

$$\mathrm{Rel}(\Theta_{5,3,3,5}; q) = (1-q)^{13}(161q^3 + 65q^2 + 13q + 1).$$

If we plot the polynomial on the interval $[0, 1]$, then it is easy to see that the function is decreasing on $[0, 1]$ (see Figure 2.1). Moreover, it was also shown in [5] that there is at most one fixed-point (as a function of $p = 1 - q$), and there is an inverted S-shape to the curve.

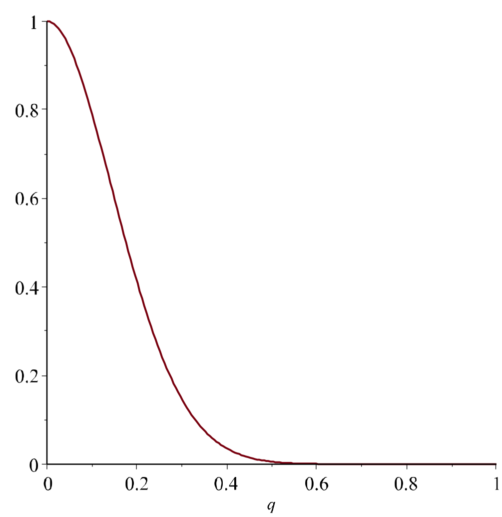


Figure 2.1: Plot of $\mathrm{Rel}(\Theta_{5,3,3,5}; q)$ with $q \in [0, 1]$

Other analytical questions can be asked about reliability polynomials, such as where the inflection points of the curve occur, or, which has been of greater interest, where roots of the reliability polynomial, called the **reliability roots**, are located (in this chapter and the next, we assume all graphs are finite, connected, and loopless, unless otherwise stated). For instance, one can see that the only real root in $[0, \infty)$ of $\mathrm{Rel}(\Theta_{5,3,3,5}; q)$, or for any reliability polynomial of a connected graph of order at least 2, is when $q = 1$. Indeed, recall that the H-form of the reliability polynomial shows that we have a factor of $(1 - q)$ (up to some multiplicity) together with the roots of the H-polynomial (which is a polynomial with strictly positive coefficients). Therefore, the only root for $q \geq 0$ can occur at $q = 1$.

What about for $q < 0$? Or q rational? Brown and Colbourn were able to show in [14] that the real reliability roots of graphs are always in $[-1, 0) \cup \{1\}$, and we will further talk about the latter question in Chapter 3.

Finally, what about if we consider the complex reliability roots? Are there any patterns that we can see from plotting these? Let us return to our examples, and see where their complex roots lie.

Examples - *Plotting Complex Reliability Roots*

Recall that we have found the following formulas:

- $\mathrm{Rel}(P_n; q) = (1 - q)^n$
- $\mathrm{Rel}(C_n; q) = (1 - q)^{n-1}((n - 1)q + 1)$
- $\mathrm{Rel}(\Theta_{3,3,5}; q) = (1 - q)^9(29q^2 + 9q + 1)$
- $\mathrm{Rel}(\Theta_{5,3,3,5}; q) = (1 - q)^{13}(161q^3 + 65q^2 + 13q + 1)$
- $\mathrm{Rel}(\Theta_{l[k]}; q) = ((1 - q)^l + lq(1 - q)^{l-1})^k - (lq(1 - q)^{l-1})^k$
- $\mathrm{Rel}(K_4; q) = (1 - q)^3(6q^3 + 6q^2 + 3q + 1)$
- $\mathrm{Rel}(K_n; q) = 1 - \left(\sum_{i=1}^{n-1} \binom{n-1}{i-1} q^{i(n-1)} \mathrm{Rel}(K_i; q) \right)$

One can clearly see that the reliability polynomials for P_n and C_n have all of their roots inside of the closed unit disk centred at 0 (which will henceforth just be referred to as the "unit disk" unless otherwise stated). We can also calculate the complex roots of $\mathrm{Rel}(\Theta_{3,3,5}; q), \mathrm{Rel}(\Theta_{5,3,3,5}; q)$ and $\mathrm{Rel}(K_4; q)$ to be $1, -\frac{9}{58} \pm \frac{1}{58} i\sqrt{35}$, and $1, -\frac{1}{7}$, $-\frac{3}{23} \pm \frac{1}{23} i\sqrt{14}$, for $\mathrm{Rel}(\Theta_{3,3,5}; q)$ and $\mathrm{Rel}(\Theta_{5,3,3,5}; q)$, respectively, and approximately

$-0.6265382933, -0.1867308534 \pm 0.4807738846i$ for $\mathrm{Rel}(K_4; q)$. Again, these are all inside of the unit disk. Since we cannot solve $\mathrm{Rel}(\Theta_{l[k]}; q)$ and $\mathrm{Rel}(K_n; q)$ exactly, by running through k and l between 2 and 30 for $\mathrm{Rel}(\Theta_{l[k]}; q)$, and running through n between 2 and 20 for $\mathrm{Rel}(K_n; q)$, we see that their roots are, again, inside of the closed unit disk (see Figures 2.2a and 2.2b for plots of those roots).

Does this trend continue? By finding the complex roots for all simple graphs up to order 7, it does, indeed, seem to be true (see Figure 2.2c). This lead Brown and Colbourn to make the following conjecture [14]:

Conjecture 2.1.1. Brown-Colbourn (1992)
Let G be any connected graph. Then all the roots of Rel$(G; q)$ lie in the closed unit disk.

In support of the conjecture, it was shown in [14] that

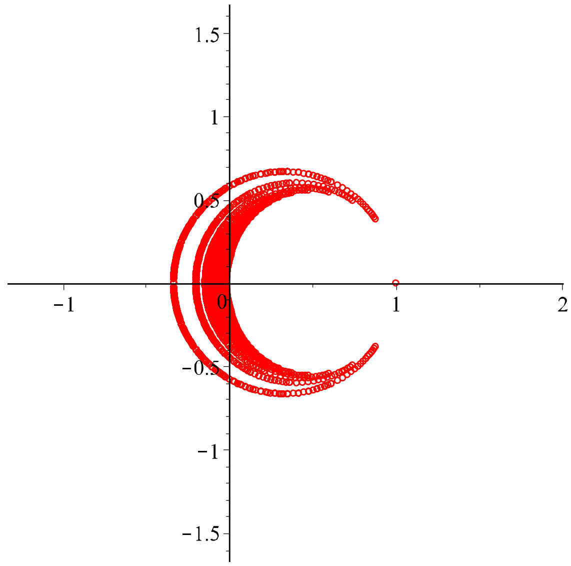
- the real reliability roots of graphs are always in $[-1, 0) \cup \{1\}$ (and hence in the unit disk), and

- every graph has a subdivision for which the roots lay in the unit disk.

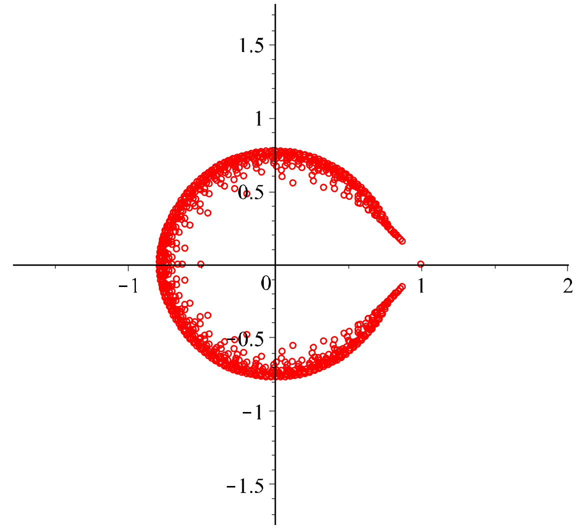
As well, they also proved that the closure of the (complex) reliability roots contains the unit disk.

This conjecture was thought to be true, and indeed was proven for the class of *series-parallel* graphs [57]. The class of **series-parallel** graphs, denoted by $\mathcal{SP}$, is defined recursively. Every (possibly non-simple) graph G in the class has a pair of unordered vertices, $\{s, t\}$, which are called **terminals**. If there exists an edge between s and t, then $G \in \mathcal{SP}$. Let G and G' be two graphs in $\mathcal{SP}$ with terminals $\{s, t\}$ and $\{s', t'\}$, respectively. If G and G' have no edges in common, and only the vertex in common is $t = s'$, then $G \cup G' \in \mathcal{SP}$ which we call a **series** connection (see Figure 2.3b). If, instead, we have that G and G' have no edges in common, with only the vertices $s = s'$ and $t = t'$ in common, then $G \cup G'$ is a **parallel** connection (see Figure 2.3c).

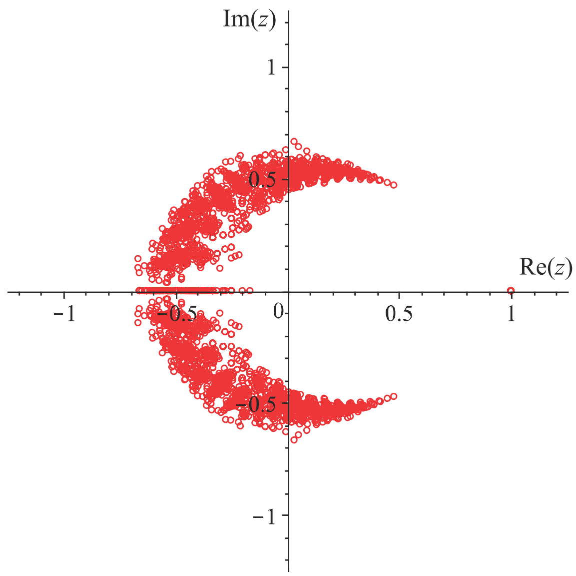
However, in 2005 Royle and Sokal [46] found a counterexample:



(a) Reliability roots of generalized theta graphs with k paths of length l, for l and k between 2 and 30.



(b) Reliability roots of complete graphs on up to 20 vertices

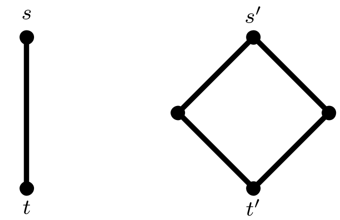


(c) The roots of reliability polynomials of all simple graphs of order 7

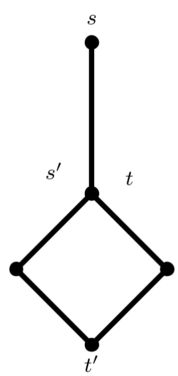
Figure 2.2: Plots of reliability roots of graphs

Example 2.1.1. *Royle-Sokal Graph*

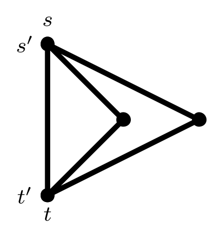
To get the Royle-Sokal Graph, we take $G = K_4$ and replace any pair of parallel edges by a bundle of 6. We saw this originally as an example of the edge bundle



(a) Initial graphs



(b) An example of a series connection



(c) An example of a parallel connection

Figure 2.3: Examples of series-parallel graphs

operation (Figure 1.2 on page 8), but we will also include it here as its stand-alone example (see Figure 2.4). The reliability polynomial for the Royle-Sokal graph can be calculated to be

$$\begin{aligned}\mathrm{Rel}(G;q) &= (1-q)^3(6q^{13}+10q^{12}+14q^{11}+18q^{10}+22q^9+26q^8+ \\ &\quad 26q^7+22q^6+18q^5+14q^4+10q^3+6q^2+3q+1)\end{aligned}$$

which has root with modulus (or distance from the origin in the complex plane) approximately 1.0017.

An open problem remains whether the roots are still bounded (perhaps by a slightly larger disk), or, indeed, if there are reliability roots that have modulus tending to infinity. Some work has been done in finding roots of largest modulus, with Brown-Mol [19] pushing the roots out to approximately 1.113, but much is still left as a

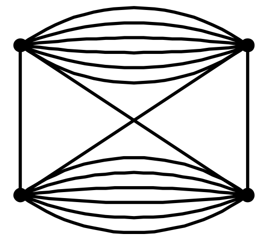


Figure 2.4: The Royle - Sokal graph

mystery. Our hope is that if we can generalize the notion of reliability to more abstract objects, then maybe we can gain an insight into whether the roots are, indeed, bounded, or even under what abstract properties boundedness of reliability roots might be guaranteed.

2.2 Generalizing Reliability to Complexes

In order to better explore reliability roots in relation to the unit disk, we extend the notion of reliability to more abstract structures (we remark that [57] extends to matroids, and [13] to general set systems). We start not with a graph but with a general complex $\mathcal{C}$.

To extend the notion of reliability to complexes, we consider a complex $\mathcal{C}$ on finite set X of cardinality m, and define the **reliability polynomial of** $\mathcal{C}$[1] as

$$\mathrm{Rel}(\mathcal{C};q) = \sum_{F\in\mathcal{C}} q^{|F|}(1-q)^{|X\setminus F|},$$

with the choice for the variable q becoming apparent shortly. The idea is that we *choose* each element of X independently with probability q (in contrast to q being the probability of an edge failing in the all-terminal reliability), and ask the probability that the chosen vertices form a face of the complex. Grouping the terms by their exponent of q, we can express the reliability in terms of the F-vector (and F-polynomial)

[1] It is important to note that, even though graphs can also be thought of as 2-dimensional complexes, the reliability of a simplicial complex $\mathrm{Rel}(\mathcal{C};q)$ and the reliability of a graph $\mathrm{Rel}(G;q)$ are very different, even though they share very similar notation.

of the complex:

$$\mathrm{Rel}(\mathcal{C};q) = \sum_{i=0}^{d} F_i q^i (1-q)^{m-i} = (1-q)^m f_{\mathcal{C}}\left(\frac{q}{1-q}\right),$$

How does this notion of reliability of complexes relate to all-terminal reliability? Recall that the set of subsets of the edge set of a connected graph G whose removal leaves the graph connected is the cographic matroid of G. Recall further that the sequence $\langle F_0, F_1, \ldots, F_{m-n+1}\rangle$ from the F-form (1.2) of the reliability polynomial is in fact the F-vector of the cographic matroid. Therefore, edges *failing* in G with probability q correspond to vertices in $\mathrm{Cog}(G)$ being *chosen* with probability q. Thus the reliability of the cographic matroid of a graph G is precisely the (all-terminal) reliability of G:

$$\mathrm{Rel}(G;q) = \mathrm{Rel}(\mathrm{Cog}(G);q).$$

An example of this is re-visiting the example first seen in Figure 1.14 on page 21. Recall that the reliability polynomial of G is

$$\mathrm{Rel}(G;q) = 1q^0(1-q)^5 + 5q^1(1-q)^4 + 8q^2(1-q)^3, \tag{2.1}$$

and the facets of $\mathrm{Cog}(G)$, written multiplicatively, are $e_1e_3, e_1e_4, e_1e_5, e_2e_3, e_2e_4, e_2e_5, e_3e_5,$ and e_4e_5. Calculating $\mathrm{Rel}(\mathrm{Cog}(G);q)$, we first start with the probability of choosing none of the vertices. This can be done in exactly one way (as $F_0 = 1$), and this happens with probability $q^0(1-q)^5$. Next, there are $F_1 = 5$ ways of choosing 1 vertex that is a face in $\mathrm{Cog}(G)$, and that happens with probability $q^1(1-q)^4$. Finally, there are $F_2 = 8$ ways of choosing 2 vertices that form a face in $\mathrm{Cog}(G)$, and that happens with probability $q^2(1-q)^3$. Summing these states together results in the identical polynomial as that in (2.1).

For a complex $\mathcal{C}$ of dimension d on a set of cardinality m,

$$\begin{aligned}\mathrm{Rel}(\mathcal{C};q) &= (1-q)^m f_{\mathcal{C}}(q/(1-q)) \\ &= (1-q)^{m-d}(1-q)^d f_{\mathcal{C}}(q/(1-q)) \\ &= (1-q)^{m-d} h_{\mathcal{C}}(q)\end{aligned}$$

Thus the roots of $\mathrm{Rel}(\mathcal{C};q)$ and $h_{\mathcal{C}}(q)$ coincide, except for some roots at $q = 1$. We will make use of the following lemma.

Lemma 2.2.1. *Let $\mathcal{C}, \mathcal{C}_1$, and $\mathcal{C}_2$ be complexes with the ground set of $\mathcal{C}$ being V. Then:*

1. *Rel*$(\mathcal{C}; q) \neq 0$.

2. *Rel*$(\mathcal{C}_1 * \mathcal{C}_2; q) =$ *Rel*$(\mathcal{C}_1; q) \cdot$ *Rel*$(\mathcal{C}_2; q)$, *and*

3. *For any element $v \in V$,*

$$\mathit{Rel}(\mathcal{C}; q) = q \cdot \mathit{Rel}(\mathit{link}_v\mathcal{C}; q) + (1-q) \cdot \mathit{Rel}(\mathit{del}_v\mathcal{C}; q).$$

Proof. Let m be the cardinality of the ground set V of $\mathcal{C}$, and let m_1 and m_2 be the cardinalities of the ground sets of $\mathcal{C}_1$ and C_2, respectively. For (1), since we always have that $\emptyset \in \mathcal{C}$, we have a term $(1-q)^m$ in the reliability. As all other terms will include powers of q, this is the only term that will contribute to the constant term (which is 1). Therefore, the polynomial cannot be identically 0.

The equality in (2) follows as the faces of $\mathcal{C}_1 * \mathcal{C}_2$ are $\{\tau_1 \cup \tau_2 | \tau_1 \in \mathcal{C}_1, \tau_2 \in \mathcal{C}_2\}$, and so the f-polynomial is

$$\begin{aligned}
f_{\mathcal{C}_1 * \mathcal{C}_2}(x) &= \sum_{\tau \in \mathcal{C}_1 * \mathcal{C}_2} x^{|\tau|} \\
&= \sum_{\tau_1 \in \mathcal{C}_1, \tau_2 \in \tau_2} x^{|\tau_1 \cup \tau_2|} \\
&= \sum_{\tau_1 \in \mathcal{C}_1, \tau_2 \in \tau_2} x^{|\tau_1|} x^{|\tau_2|} \\
&= \sum_{\tau_1 \in \mathcal{C}_1} x^{|\tau_1|} \sum_{\tau_2 \in \tau_2} x^{|\tau_2|} \\
&= f_{\mathcal{C}_1}(x) f_{\mathcal{C}_2}(x).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{Rel}(\mathcal{C}_1 * \mathcal{C}_2; q) &= (1-q)^m f_{\mathcal{C}_1 * \mathcal{C}_2}\left(\frac{q}{1-q}\right) \\
&= (1-q)^{m_1+m_2} f_{\mathcal{C}_1}\left(\frac{q}{1-q}\right) f_{\mathcal{C}_2}\left(\frac{q}{1-q}\right) \\
&= (1-q)^{m_1} f_{\mathcal{C}_1}\left(\frac{q}{1-q}\right) (1-q)^{m_2} f_{\mathcal{C}_2}\left(\frac{q}{1-q}\right) \\
&= \mathrm{Rel}(\mathcal{C}_1; q) \cdot \mathrm{Rel}(\mathcal{C}_2; q)
\end{aligned}$$

Finally, we show (3).

$$\begin{aligned}
f_{\mathcal{C}}(x) &= \sum x^{|\tau|} \\
&= \sum_{\tau\in\mathcal{C}, v\in\tau} x^{|\tau|} + \sum_{\tau\in\mathcal{C}, v\notin\tau} x^{|\tau|} \\
&= \sum_{\alpha\in\mathrm{link}_v(\mathcal{C})} x^{|\alpha|+1} + \sum_{\alpha\in\mathrm{del}_v(\mathcal{C})} x^{|\alpha|} \\
&= x f_{\mathrm{link}_v(\mathcal{C})}(x) + f_{\mathrm{del}_v(\mathcal{C})}(x)
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Rel}(\mathcal{C};q) &= (1-q)^m f_{\mathcal{C}}\left(\frac{q}{1-q}\right) \\
&= (1-q)^m \left[\frac{q}{1-q} f_{\mathrm{link}_v(\mathcal{C})}\left(\frac{q}{1-q}\right) + f_{\mathrm{del}_v(\mathcal{C})}\left(\frac{q}{1-q}\right)\right] \\
&= q(1-q)^{m-1} f_{\mathrm{link}_v(\mathcal{C})}\left(\frac{q}{1-q}\right) + (1-q)(1-q)^{m-1} f_{\mathrm{del}_v(\mathcal{C})}\left(\frac{q}{1-q}\right) \\
&= q\mathrm{Rel}(\mathrm{link}_v(\mathcal{C});q) + (1-q)\mathrm{Rel}(\mathrm{del}_v(\mathcal{C});q).
\end{aligned}$$

□

From this lemma, it follows that if $\mathcal{C}$ has a **loop** x (an element of V that belong to no face) then the removal of x from the underlying set corresponds to division of $\mathcal{C}$ by $1-q$, and moreover, if $\mathcal{C}$ has a **coloop** x (that is, a vertex in every facet of $\mathcal{C}$), then $\mathrm{del}_x\mathcal{C} = \mathrm{link}_x\mathcal{C}$, and hence $\mathcal{C}$ and $\mathrm{link}_x\mathcal{C}$ have the same reliability. Hence we will often assume that the complex under question has no loops or coloops.[2]

Let us calculate the reliabilities of our example complexes from Chapter 1.

Example 2.2.1. *Reliability Polynomials of Complexes*

We first recall the simplex, pure, and shellable complexes, as well as our matroid from Examples 1.2.1, 1.2.2, 1.2.3, and 1.2.5, respectively, together with their F- and H-vectors.

[2]We remark that in the literature sometimes reliability of complexes has been studied in the guise of *coherent systems* (see, for example, [42]), which are collections of subsets of a finite set X closed under *superset*. For a coherent system $\mathcal{S}$ on set X, its reliability is

$$\mathrm{CSRel}(\mathcal{S};p) = \sum_{S\in\mathcal{S}} p^{|S|}(1-p)^{|X\setminus S|},$$

which is the same as $\mathrm{Rel}(\mathcal{S};q)$, where $q = 1-p$ and $\mathcal{S}$ is the complex with faces $X\backslash\tau$ for $\tau\in\mathcal{S}$.

Complex	**Label**	**Figure**	**Page**	**F − vector**	**H − vector**
Simplex	$\mathcal{C}_{1234}$	1.9	11	$\langle 1,4,6,4,1\rangle$	$\langle 1\rangle$
Pure (Connected)	$\mathcal{C}_{1234,4567}$	1.10a	12	$\langle 1,7,12,8,2\rangle$	$\langle 1,3,-3,1\rangle$
Pure (Disconnected)	$\mathcal{C}_{123,456}$	1.10b	12	$\langle 1,6,6,2\rangle$	$\langle 1,3,-3,1\rangle$
Shellable	$\mathcal{C}_{123,234,245,346,456}$	1.11a	14	$\langle 1,7,12,6\rangle$	$\langle 1,4,1\rangle$
Non-Pure Shellable	$\mathcal{C}_{1234,125,35}$	1.11b	14	$\langle 1,5,9,5,1\rangle$	$\langle 1,1,-2,1\rangle$
Matroid	$\mathcal{M}$	1.13	17	$\langle 1,6,11\rangle$	$\langle 1,4,6\rangle$

It follows that the reliabilities of these complexes are:

$$\begin{aligned}
\mathrm{Rel}(\mathcal{C}_{1234};q) &= (1-q)^4 + 4q(1-q)^3 + 6q^2(1-q)^2 + 4q^3(1-q) + q^4 \\
&= 1 \\
\mathrm{Rel}(\mathcal{C}_{1234,4567};q) &= (1-q)^7 + 7q(1-q)^6 + 12q^2(1-q)^5 + 8q^3(1-q)^4 + 2q(1-q)^3 \\
&= (1-q)^3(1+3q-3q^2+q^3) \\
\mathrm{Rel}(\mathcal{C}_{123,456};q) &= (1-q)^7 + 6q(1-q)^6 + 6q^2(1-q)^5 + 2q^3(1-q)^4 \\
&= (1-q)^3(1+3q-3q^2+q^3) \\
\mathrm{Rel}(\mathcal{C}_{123,234,245,346,456};q) &= (1-q)^6 + 7q(1-q)^5 + 12q^2(1-q)^4 + 6q^3(1-q)^3 \\
&= (1-q)^3(1+4q+q^2) \\
\mathrm{Rel}(\mathcal{C}_{1234,125,35};q) &= (1-q)^5 + 5q(1-q)^4 + 9q^2(1-q)^3 + 5q^3(1-q)^2 + q^4(1-q) \\
&= (1-q)(1+q-2q^2+q^3) \\
\mathrm{Rel}(\mathcal{M};q) &= (1-q)^6 + 6q(1-q)^5 + 11q^2(1-q)^4 \\
&= (1-q)^4(1+4q+6q^2)
\end{aligned}$$

Let's consider the reliability roots of the various complexes listed. We summarize the roots in the table below, rounding to 10 decimal places.

Reliability	**Non-Unity Roots**	**Maximum Modulus**
$\mathrm{Rel}(\mathcal{C}_{12345}; q)$	No roots	
$\mathrm{Rel}(\mathcal{C}_{1234,4567}; q)$	$-0.2599210499, 1.6299605249 \pm 1.0911236360i$	1.9614591767
$\mathrm{Rel}(\mathcal{C}_{123,456}; q)$	$-0.2599210499, 1.6299605249 \pm 1.0911236360i$	1.9614591767
$\mathrm{Rel}(\mathcal{C}_{123,234,245,346,456}; q)$	$-3.7320508080, -0.2679491924$	3.7320508080
$\mathrm{Rel}(\mathcal{C}_{1234,125,35}; q)$	$-0.4655712319, 1.2327856160 \pm 0.7925519925i$	1.4655712319
$\mathrm{Rel}(\mathcal{M}; q)$	$-0.3333333333 \pm 0.2357022604i$	0.4082482904

As we can see, we have already found roots outside of the unit disk. However, are these bounded, or can their moduli tend off to infinity? Let us instead consider general complexes of various dimensions and classes.

For an empty complex of dimension 0, the reliability is 1, which obviously has no roots outside the unit disk. For dimension 1, the reliability is of the form $(1-q)+mq = 1+(m-1)q$, which has its root in the unit disk (and real). The situation changes dramatically when the dimension grows larger. For example, let $m \geq 4$ and consider the complex $\mathcal{P}_m$ on set $X = \{1, 2, \ldots, m\}$ with faces

$$\emptyset, \{1\}, \{2\}, \ldots, \{m\}, \{1, 2\}, \{3, 4\}, \{4, 5\}, \ldots, \{m-1, m\}.$$

The reliability of the 2-dimensional complex $\mathcal{P}_m$ can be calculated to be

$$\begin{aligned}\mathrm{Rel}(\mathcal{P}_m; q) &= (1-q)^m + mq(1-q)^{m-1} + (m-2)q^2(1-q)^{m-2} \\ &= (1-q)^{m-2}((1-q)^2 + mq(1-q) + (m-2)q^2) \\ &= (1-q)^{m-2}(-q^2 + (m-2)q + 1),\end{aligned}$$

which has a root at 1 and

$$\frac{m-2+\sqrt{(m-2)^2+4}}{2} \sim m-2,$$

which grows arbitrarily large (and positive) as m tends to infinity. Thus we should insist on some properties (even beyond pureness, as the complex above is pure), shared by cographic matroids, of our complexes if we hope to have roots always in (or close to) the unit disk.

2.2.1 Restricting Reliability to Shellable Complexes

Recall that a pure d-dimensional complex is shellable if its facets can be ordered $\sigma_1, \sigma_2, \ldots \sigma_d$ with $j = 2, 3, \ldots, d$, with the subcomplexes

$$\left(\bigcup_{j=1}^{m-1} \overline{\sigma_j} \right) \bigcap \overline{\sigma_m}$$

being all pure $(d-1)$-complexes for all $m = 2, \ldots, d$. Shellable complexes arise in a variety of combinatorial and topological settings, and have some very nice properties that provide inequalities amongst the components of the H-vector (see, for example, [23]). We are particularly interested in this class of complexes as it is a highly structured class (compared to, say, pure complexes), that properly contains the class of matroids (and in particular, cographic matroids). For a shellable complex, we saw earlier that the H-vector is known to consist of nonnegative integers, and indeed has a variety of interpretations:

- An **interval partition** is a collection of disjoint intervals $[L, U] = \{S : L \subseteq S \subseteq U\}$ such that every face in the complex belongs to precisely one interval. Simplicial complexes that have a partitioning $\{[L_i, U_i] | 1 \leq i \leq b\}$ with U_i a facet for each i are called **partitionable**. It is known that all shellable complexes are partitionable [23, pg. 63 - 64], and moreover, H_i is the number of lower sets L_j of cardinality i.

- As shown in [11], an **order ideal of monomials** $\mathcal{N}$ is a set of monomials closed under divisibility. For every shellable complex, there is an order ideal of monomials $\mathcal{N}$ such that H_i counts the number of monomials of degree i in the set.

It follows that the sequence $H_0, H_1, \ldots, H_d$ consists of nonnegative integers with no internal zeros (and in fact, if $\mathcal{C}$ has no coloops, then H_d is nonzero).

Many complexes that arise in combinatorial and other settings turn out to be shellable (in particular, matroids are always shellable [23]). One might hope that, since matroids are shellable and their reliability roots seem to be bounded, the extra condition of shellability on a complex might force the reliability roots inside the unit disk, but such is not the case, even in dimension 2. In fact, the following is true:

Theorem 2.2.1. *A pure 2-dimensional complex $\mathcal{C}$ with F-vector $\langle 1, m, F_2\rangle$ has a reliability root outside the unit disk if and only if $F_2 \in [\frac{m}{2}, m-2] \cup [m, 2m-5]$. If, moreover, $\mathcal{C}$ is shellable, then it has a reliability root outside the unit disk if and only if $F_2 \in [m, 2m-5]$.*

Proof. Assume to begin with that $\mathcal{C}$ is a pure 2-dimensional complex; its reliability is given by

$$\begin{aligned} \mathrm{Rel}(\mathcal{C}; q) &= F_2 \cdot q^2(1-q)^{m-2} + m \cdot q(1-q)^{m-1} + (1-q)^m \\ &= (1-q)^{m-2}\left((F_2 - m + 1)q^2 + (m-2)q + 1\right) \end{aligned}$$

so it suffices to consider the roots of

$$r(q) = (F_2 - m + 1)q^2 + (m-2)q + 1.$$

As the removal of loops cannot introduce any roots different than $q = 1$, we can assume that $\mathcal{C}$ has no loops. The complex has no isolated vertices (that is, maximal faces of cardinality 1), as otherwise the complex would not be pure. Therefore, we have that $F_2 \geq m/2$. We split the argument up into two cases, depending on whether F_2 is less than m or not.

Note that if $F_2 = m-1$ then $r(q) = (m-2)q + 1$ which has all of its roots in the unit disk centred at 0, so we can assume that $F_2 \neq m-1$, and hence we can focus on the roots of

$$r_1(q) = q^2 + \frac{m-2}{F_2 - m + 1}q + \frac{1}{F_2 - m + 1}.$$

The Hurwitz Criterion (see [53]), states that a real polynomial $z^2 + bz + c$ has all of its roots in the unit disk if and only if $|c| \leq 1$ and $|b| \leq c + 1$. Thus we set

$$b = \frac{m-2}{F_2 - m + 1} \text{ and } c = \frac{1}{F_2 - m + 1}.$$

Clearly the first condition, $|c| \leq 1$, holds as F_2 is an integer different from $m-1$. So everything hinges on whether

$$|b| = \left|\frac{m-2}{F_2 - m + 1}\right| \leq \frac{1}{F_2 - m + 1} + 1 = c + 1. \tag{2.2}$$

First consider the case that $F_2 \leq m-1$; as $F_2 \neq m-1$, we have that $F_2 < m-1$ and so $F_2 - m + 1 < 0$. Assume that $m \geq 2$ (since $\mathcal{C}$ is a 2-dimensional pure complex, and so must have a facet of dimension 2). Now,

$$|b| = \left|\frac{m-2}{F_2-m+1}\right| = \frac{m-2}{m-F_2-1} \leq \frac{1}{F_2-m+1} + 1 = \frac{m-F_2-2}{m-F_2-1}.$$

This is equivalent to

$$m-2 \leq m-F_2-2,$$

and so $F_2 \leq 0$. This can't happen as $F_2 \geq 1$ (since $\mathcal{C}$ is a 2-dimensional pure complex, it must have at least one face of dimension 2). Therefore, there is always a reliability root outside the closed unit disk when $F_2 < m-1$.

Now assume that $F_2 \geq m$.Then a simple calculation shows that (2.2) holds if and only if $F_2 \geq 2m-4$. Therefore, in this case, $\mathrm{Rel}(\mathcal{C};q)$ has a root outside the unit disk if and only if $m \leq F_2 \leq 2m-5$.

Thus we conclude that $\mathcal{C}$ has a reliability root outside the unit disk if and only if $F_2 \in [\frac{m}{2}, m-2] \cup [m, 2m-5]$. As we saw earlier, a 2-dimensional complex is shellable if and only if it is connected as a graph, so shellability implies that $F_2 \geq m-1$, and hence if $\mathcal{C}$ is shellable, then it has a reliability root outside the unit disk if and only if $F_2 \in [m, 2m-5]$. $\square$

Furthermore, when $F_2 = m \geq 2$, (corresponding to a 2-dimensional shellable complex whose facets form a unicyclic connected graph), then the F-vector is $\langle 1, m, m\rangle$ and so the reliability is

$$(1-q)^2 + mq(1-q) + mq^2 = q^2 + (m-2)q + 1.$$

This has a root at

$$-\frac{m}{2} + 1 - \frac{\sqrt{m^2-4}}{2},$$

and this root can grow arbitrarily large in absolute value. A plot of the reliability roots of all shellable complexes of order 5 can be found in Figure 2.5.

A very interesting question following this theorem is asking whether there exist any 2-dimensional cographic matroids whose F_2 falls inside of the interval $[m, 2m-5]$ (as then that class of cographic matroids would lie outside of the closed unit disk, and hence contradict the Brown-Colbourn Conjecture. Let us examine this question further, and answer it in the negative.

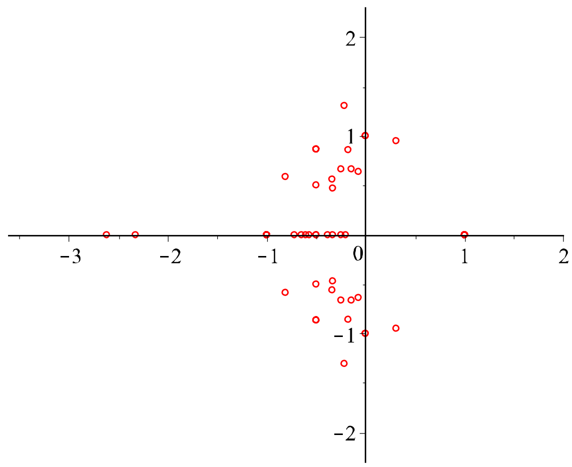


Figure 2.5: A plot of the complex roots of all shellable complexes of order 5

Suppose we have a cographic matroid Cog(G) being of dimension two (that is, one can remove at most two edges from G). This corresponds to only two graphs – namely the bond of two cycles together at one vertex (we can remove exactly one edge from one cycle and one from the other), or a theta graph (we can remove exactly one edge from any two of the three branches).

Suppose G is the bond of two cycles together at a vertex. If we have m edges in total, with say m_1 edges in the first cycle, then the number of ways we can remove two edges would be $m_1(m - m_1)$. However, since we need at least 2 edges in each cycle (as otherwise we would have loops which we don't allow), this is equivalent to having $F_2 \geq 2(m-2) = 2m-4 \geq 2m-5$, and so it is outside of the aforementioned interval.

Suppose now that G is a theta graph with m edges with, say m_1, m_2, and m_3 edges in each of the branches, respectively. Then, since we cannot have any two edges being removed from the same branch, the number of ways we can remove two edges of G is precisely

$$\begin{aligned} F_2 &= \binom{m}{2} - \binom{m_1}{2} - \binom{m_2}{2} - \binom{m_3}{2} \\ &= \frac{m^2 - m - (m_1^2 - m_1 + m_2^2 - m_2 + m_3^2 - m_3)}{2}. \end{aligned}$$

Since $m_3 = m - m_1 - m_2$,

$$\begin{aligned} F_2 &= \frac{m^2 - m - (m_1^2 - m_1 + m_2^2 - m_2 + (m - m_1 - m_2)^2 - (m - m_1 - m_2)))}{2} \\ &= \frac{m^2 - (m_1^2 + m_2^2 + m^2 + m_1^2 + m_2^2 - 2mm_1 - 2mm_2 + 2m_1m_2)}{2} \\ &= mm_1 + mm_2 - m_1^2 - m_1m_2 - m_2^2. \end{aligned} \tag{2.3}$$

Replacing m by $m_1 + m_2 + m_3$ yields

$$\begin{aligned} F_2 &= (m_1 + m_2 + m_3)m_1 + (m_1 + m_2 + m_3)m_2 - m_1^2 - m_1m_2 - m_2^2 \\ &= m_1m_2 + m_1m_3 + m_2m_3 \end{aligned}$$

It can be seen that this is smallest when two of the m_i's are precisely 1. Without loss of generality, suppose $m_1 = m_2 = 1$. Then, from (2.3) we see that $F_2 \geq 2m-3 \geq 2m-5$. Therefore, F_2 is not in the aforementioned interval, and hence there are no cographic matroids of dimension 2 which have $F_2 \in [m, 2m-5]$.

We consider now other shellable complexes of any dimension, and we see that the roots can be more than unbounded. That is, it is not simply one root whose modulus is unbounded, but in fact, they are dense in the entire complex plane.

Theorem 2.2.2. *The reliability roots of shellable complexes are dense in the complex plane.*

Proof. Recall that the broken circuit complex of a graph G (see [3, 6], for example) is formed by fixing a linear order $\preceq$ on the edges of G and declaring any circuit minus its $\preceq$-least edge to be a broken circuit. The broken circuit complex $\mathcal{BR}(G, \preceq)$ is the complex on the edge set of G whose faces are those subsets that do not contain a broken circuit.

Every broken circuit complex is shellable [48], and the dimension of the complex, for a graph with n vertices and c connected components, is $n - c$. The interest in broken circuit complexes arises from the surprising fact that if G is a graph of order n and $\langle a_0, a_1, \ldots, a_{n-1} \rangle$ is the F-vector of $\mathcal{BR}(G, \preceq)$, then the well-known **chromatic polynomial** of G, $\pi(G, x)$, can be expressed as

$$\pi(G, x) = \sum_{i=0}^{n-c} (-1)^i a_i x^{n-i}.$$

Sokal [52] has proven that the roots of chromatic polynomials are dense in the complex plane. Now

$$\begin{aligned} (-q)^n \pi\left(G, \frac{q-1}{q}\right) &= (-q)^n \sum_{i=0}^{n-c} (-1)^i a_i \left(\frac{q-1}{q}\right)^{n-i} \\ &= (1-q)^n \sum_{i=0}^{n-c} a_i \left(\frac{q}{1-q}\right)^{i} \\ &= (1-q)^c \cdot \mathrm{Rel}(\mathcal{BR}(G, \preceq); q). \end{aligned}$$

As it is easy to prove that the image (and preimage) of a dense set under a linear fractional transformation is again dense, we see that the roots of the reliability polynomials of broken circuit complexes are also dense in the complex plane. □

A plot of the roots of reliability polynomials of broken circuit complexes for all graphs on 8 vertices is shown in Figure 2.6.

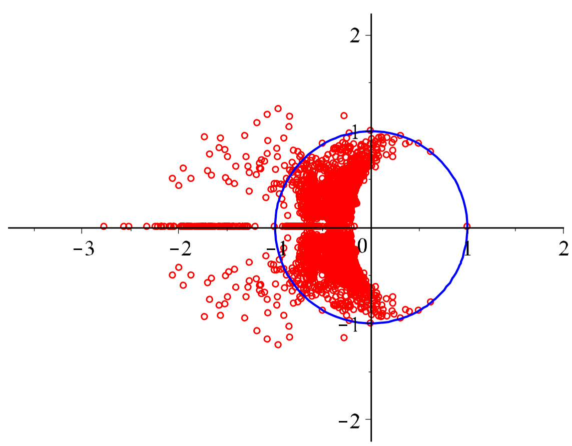


Figure 2.6: A plot of reliability roots of all broken circuit complexes for graphs of order 8 (red) with unit circle (blue).

Turning our attention back to the Brown-Colbourn Conjecture, one may be wondering why it is so hard to find reliability roots of graphs outside the disk centred at 0. The answer must be that there is a more restrictive property than shellability at play.

2.2.2 Restricting Reliability to Matroids

We have mentioned that every matroid is shellable, but matroids are indeed a proper subclass of shellable complexes (see Figure 2.7 for an example of a shellable complex that is not a matroid). Indeed, we have shown in the discussion regarding shellable complexes in Example 1.2.3 that the 2-skeleton of a matroid must be a complete k-partite graph, which this is not). Further, we have seen that reliability of graphs is the same as the reliability of a certain family of matroids (namely cographic matroids). Therefore, it is reasonable to see what happens for the reliability roots of general matroids – do they behave as they do for graphs (and cographic matroids), or does the extension past cographic matroids allow for the kind of extreme behaviour we have seen for other shellable complexes? It is easy to observe that the join of two matroids is again a matroid, so as reliability is multiplicative over connected components, the reliability roots of a sum of matroids is the union of the reliability roots of the components. Thus we can assume that matroids under consideration are connected.

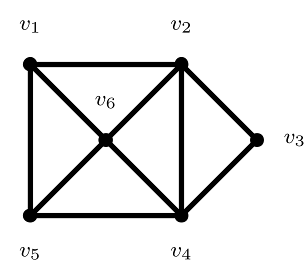


Figure 2.7: A 2-dimensional shellable complex that is not a matroid

All complexes of dimension at most 1 are matroids, and we have already seen that such complexes have their reliability roots in the unit disk. In [57] it was shown that uniform matroids $U_{n,r}$ (those on an set X of size n whose facets are all r-subsets of X) have their reliability roots in the unit disk. Moreover, it was also shown there that the same is true of reliability roots of cographic matroids of series-parallel graphs (which we recall are graphs that can be built up from a single vertex by series and parallel operations). As well, we can prove that the real reliability roots of all matroids lie in the unit disk.

Theorem 2.2.3. *The real reliability roots of any matroid $\mathcal{M}$ lie in $[-1, 0) \cup \{1\}$, and* *hence lie in the unit disk.*

Proof. Let $\mathcal{M}$ have order m and rank r. We can assume that $\mathcal{M}$ is connected and has no loops or coloops. Note that as

$$\mathrm{Rel}(\mathcal{M};q) = (1-q)^{m-r}\sum_{i=0}^{r} H_i q^i$$

and all the H_i are positive, there are clearly no non-negative roots except 1. Moreover, in [14] it was shown that for any connected matroid $\mathcal{M}$ of rank r with H-vector $\langle H_0, H_1, \ldots, H_r\rangle$, any $b \geq 1$ and any $j \in \{0, 1, \ldots, r\}$, we have that

$$(-1)^j \sum_{i=0}^{j} (-b)^i H_i \;\; \geq \;\; 0 \tag{2.4}$$

with equality possible only if $b = 1$. In particular by setting $b = 1$ and $j = r$ we have,

$$\sum_{i=0}^{r} (-1)^{r-i} H_i \;\; \geq \;\; 0, \tag{2.5}$$

and we shall often make use of this inequality throughout this dissertation. Taking $j = r$ above, we find that

$$(-1)^r h_{\mathcal{M}}(-b) \;\; > \;\; 0$$

for $b > 1$, and hence $h_{\mathcal{M}}(q)$ is nonzero for $q \in (1, \infty)$. It follows that $\mathrm{Rel}(\mathcal{M};q)$ has no roots less than -1, and so all of the real roots of $\mathrm{Rel}(\mathcal{M};q)$ lie in $[-1, 0) \cup \{1\}$. $\square$

Thus we see that the usual techniques from real analysis (such as the Intermediate Value Theorem) that one might use to try to locate roots outside the unit disk are going to fail for matroids, as then any such root must necessarily be nonreal. From [19] and [46] we know that there are (nonreal) reliability roots of some matroids (namely cographic matroids) that lie a little bit outside the disk centred at 0. These matroids have dimension 13 or higher. Might there be matroids of small dimension with reliability roots outside the unit disk?

2.3 Reliability Roots of Matroids of Small Dimension

Because of earlier remarks, we can assume all matroids (and complexes) have no loops or coloops. From the previous section, any matroid of rank 0 or 1, being a complex, has its roots in the unit disk (in fact, any complex of dimension at most 1 is trivially a matroid). The next case is rank 2. We have already seen that for general shellable 2-dimensional complexes, the reliability roots can be unbounded. However, what about if we insist on the complex being a matroid? Then the situation becomes markedly different.

Theorem 2.3.1. *Let $\mathcal{M}$ be a matroid of rank* 2*. Then the roots of Rel*$(\mathcal{M}; q)$ *are all inside the closed unit disk.*

Proof. Let $\mathcal{M}$ be of order m. We saw in the previous chapter that the graph determined by the 2-skeleton G of $\mathcal{M}$ of rank $r \geq 2$ must be a complete multipartite graph. We will show that the number of edges of G (i.e. F_2 of $\mathcal{M}$) is at least $2m-4$; by Theorem 2.2.1, $\mathcal{M}$ will have all of its reliability roots in the unit disk.

Let the parts of the complete multipartite graph G be $V_1, V_2, \ldots, V_k$. Clearly $k \geq 2$ as $\mathcal{M}$, being of rank 2, has a face of cardinality 2. If $k \geq 3$, one can combine parts and decrease the number of edges (i.e. decreases the value of F_2), so we can assume that $k = 2$. Thus for some $j \in \{2, 3, \ldots, \lfloor m/2 \rfloor\}$, $G = K_{j,m-j}$ ($j \neq 1$ as otherwise $\mathcal{M}$ has a coloop, and the proof reverts to the dimension 1 case). It is easy to see that G has $j(m-j) \geq 2(m-2) = 2m-4$ as the function $g(x) = x(m-x)$ is increasing on $[2, m/2]$. Thus G has at least $2m-4$ edges, and hence $F_2 \geq 2m-4$. From Theorem 2.2.1, we conclude that all the roots of $\mathcal{M}$ lie in the unit disk. □

We now turn to rank 3 matroids, where again we can prove that the reliability roots are always in the unit disk (along the way, we prove a new nonlinear inequality among the terms in the H-vector of matroids of rank 3).

Theorem 2.3.2. *Let $\mathcal{M}$ be a rank* 3 *matroid. Then all the roots of Rel*$(\mathcal{M}; q)$ *lie inside the unit disk.*

Proof. Let $\mathcal{M}$ have order m. Clearly we can assume that $\mathcal{M}$ is connected, as otherwise its reliability is the product of the reliabilities of matroids of smaller rank, and we are done. It can be shown that if a matroid (without loops or coloops) of order m and rank r is connected, then $H_r \geq m - r$ (see, for example, [6, pg. 244]). Therefore, if $m > r$ (which we can always assume, as otherwise the matroid is a simplex with reliability 1), $H_r \geq 1$. We shall make use of this here and throughout this section.

Farebrother [28] proved that the roots of a real cubic polynomial $x^3+a_2x^2+a_1x+a_0=0$ lie in the *open* unit disk $\{z : |z| < 1\}$ if and only if the following conditions all hold:

$$\begin{aligned} 1+a_0+a_1+a_2 &> 0 \\ 1-a_0+a_1-a_2 &> 0 \\ 3+a_2-a_1-3a_0 &> 0 \\ 1-a_0^2+a_0a_2-a_1 &> 0 \end{aligned}$$

For our purposes, it will be better to rewrite these conditions for the cubic $a_3x^3+a_2x^2+a_1x+a_0=0$, where all a_i's are positive:

$$\begin{aligned} a_0+a_1+a_2+a_3 &> 0 \\ a_3-a_2+a_1-a_0 &> 0 \\ 3a_3+a_2-a_1-3a_0 &> 0 \\ a_3^2-a_0^2+a_0a_2-a_1a_3 &> 0 \end{aligned}$$

It is well known that the roots of a (complex) polynomial depend continuously on the coefficients [32, 33]. It follows that if a_0, a_1, a_2 and a_3 are all positive with $a_3 \geq a_1$, then the roots of $a_3x^3+a_2x^2+a_1x+a_0=0$ are in the *closed* unit disk $\{z : |z| \leq 1\}$ if

$$a_0+a_1+a_2+a_3 \geq 0 \tag{2.6}$$

$$a_3-a_2+a_1-a_0 \geq 0 \tag{2.7}$$

$$3a_3+a_2-a_1-3a_0 \geq 0 \tag{2.8}$$

$$a_3^2-a_0^2+a_0a_2-a_1a_3 \geq 0. \tag{2.9}$$

(If not, there would be a root outside the disk centred at 0. By increasing a_3 to $a_3+\varepsilon$ and a_2 to $a_2+\varepsilon/2$, then provided ε is sufficiently small but positive, we could keep the root outside the disk centred at 0, but have strict inequality hold in all four conditions, a contradiction to Farebrother's result. Indeed, the first three inequalities are straightforward to show, and the fourth follows from showing that $4a_3-2a_1+a_0 \geq 0$ is satisfied when $a_3 \geq a_1$ (which will become apparent in a moment – that is, $H_3 \geq H_1$).

Consider the reliability polynomial of $\mathcal{M}$:

$$\mathrm{Rel}(\mathcal{M};q) = (1-q)^{m-r}(1+H_1q+H_2q^2+H_3q^3)$$

Clearly this has $m - r$ roots at 1, and so we are only interested in the roots of

$$h_{\mathcal{M}}(q) = 1 + H_1 q + H_2 q^2 + H_3 q^3.$$

We set $a_3 = H_3,\ a_2 = H_2,\ a_1 = H_1$ and $a_0 = 1$. Then rephrasing conditions (2.6), (2.7), (2.8) and (2.9), we need to show

$$\begin{aligned} H_3 + H_2 + H_1 + 1 &\geq 0 \quad (2.10)\\ H_3 - H_2 + H_1 - 1 &\geq 0 \quad (2.11)\\ 3H_3 + H_2 - H_1 - 3 &\geq 0 \quad (2.12)\\ H_3^2 - 1 + H_2 - H_3 H_1 &\geq 0. \quad (2.13)\end{aligned}$$

Conditions (2.10), (2.11), and (2.12) are quite simple to show. Since all coefficients of the H-vector are nonnegative, (2.10) follows immediately. For (2.11), we use (2.5), which implies for $r = 3$ that

$$-1 + H_1 - H_2 + H_3 \geq 0,$$

that is, condition (2.11).

For (2.12), we will use a result of Há et al [31] that showed that matroids of rank 3 satisfy Stanley's Conjecture, that is, that their H-vectors are pure O-sequences (the H-vectors of pure order ideals of monomials). In particular, it follows that $H_0 \leq H_1 \leq \cdots \leq H_{\lfloor \frac{r}{2} \rfloor}$, and $H_i \leq H_{r-i}$ for $0 \leq i \leq \lfloor \frac{r}{2} \rfloor$ (see [34]). So $H_3 \geq H_0$ and $H_2 \geq H_1$. We conclude that

$$3H_3 + H_2 - H_1 - 3 \geq 0,$$

so condition (2.12) is true.

The last inequality to check is (2.13). We start by observing that the 2-skeleton of a rank-3 matroid must be a complete k-partite graph G with $k \geq 3$ (as mentioned earlier, for $k \leq r$, the k-skeleton of a rank r matroid also generates a matroid). Indeed, if it were just a complete bipartite graph, then the dimension of $\mathcal{M}$ would be 2. Furthermore, if $k = 3$, then each part of the complete k-partite graph must have cardinality at least 2, as otherwise there would be a coloop in the matroid. Therefore, the smallest possible number of edges in G would be a complete 3-partite graph with independent sets having $2, 2$ and $m-4$ elements, and from this we can determine that $F_2 \geq 4m-12$.

Using this, the fact that $H_1 = m - r = m - 3$ and by considering the number of faces of cardinality 2 covered in an interval partition of $\mathcal{M}$, we find that

$$H_2 = F_2 - 2H_1 - 3H_0 \geq 2m - 9 \tag{2.14}$$

Finally, from this and (2.5) we get

$$H_3 \geq H_2 - H_1 + H_0 \geq m - 5.$$

Now if $H_3 \geq m - 3 = H_1$ then (2.13) holds (strictly), as $H_2 \geq 3$. Moreover, if $H_3 = m - 5$ then

$$\begin{aligned} H_3^2 - 1 + H_2 - H_3H_1 &= H_2 + (-2)(m-5) - 1 \\ &= H_2 - (2m-9) \\ &\geq 0 \end{aligned}$$

by (2.14). Our final case is that $H_3 = m - 4$. Here

$$\begin{aligned} H_3^2 - 1 + H_2 - H_3H_1 &= H_2 + (-(m-4)) - 1 \\ &= H_2 - (m-3) \\ &\geq 0 \end{aligned}$$

provided $2m - 9 \geq m - 3$, that is, provided that $m \geq 6$. An exhaustive check of all matroids on less than 6 vertices (whether of rank 3 or otherwise) can confirm that no root falls outside of the closed unit disk. Therefore, all matroids of rank 3 have all of their roots inside of the closed unit disk. □

Unfortunately, we were unable to prove that all matroids of rank 4 have their roots inside the disk centred at 0 since we cannot describe their H-vector accurately enough for our methods, and so we shall focus on paving matroids of rank 4. A **paving matroid** of rank r on a set X of cardinality m has a complete $(r-1)$-skeleton, that is, all $(r-1)$-subsets of X are faces, so that its H-vector has the form $\langle 1, m-r, \ldots, \binom{m-r+i-1}{i}, \ldots, \binom{m-2}{r-1}, H_r \rangle$, with $H_r = F_r - \binom{m-1}{r-1}$ [40, Proposition 3.1]. Every uniform matroid is obviously a paving matroid, and it has been conjectured

[27, p. 3.17] that almost every matroid of order m is a paving matroid. We remark that there has been some progress on this conjecture (for instance in [47] where it was shown that the ratio of the logs of the number of matroids to paving matroids tend to 1), but the conjecture in its entirety is still open.

Theorem 2.3.3. *The roots of a paving matroid $\mathcal{M}$ of rank 4 all fall inside the closed unit disk.*

Proof. Again, we can assume that $\mathcal{M}$ is connected and has no loops or coloops, and hence $m \geq 5$. If $m = 5$ then the H-polynomial of $\mathcal{M}$ has the form $H_4 q^4 + q^3 + q^2 + q + 1$, while for $m = 6$, it has the form

$$H_4 q^4 + 4q^3 + 3q^2 + 2q + 1, \tag{2.15}$$

where in either case $H_4 \geq m - 4$. We use the well-known *Eneström-Kakeya* Theorem (see, for example, [1] or [49]), which states that if $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial with $0 < a_0 \leq a_1 \leq \cdots \leq a_n$ then the roots of $f(x)$ lie in the (closed) unit disk. It follows that the roots of polynomials of the form (2.15) are in the unit disk, except possibly for the polynomials $2q^4 + 4q^3 + 3q^2 + 2q + 1$ and $3q^4 + 4q^3 + 3q^2 + 2q + 1$. However, direct calculations in this case show that roots lie in the unit disk, so we can now assume that $m \geq 7$.

For this proof, we will be using Farebrother's [28] necessary and sufficient conditions for the moduli of all roots of quartic polynomials to fall inside the unit disk: for a real quartic polynomial $x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$, the roots fall inside the open unit disk if and only if

$$\begin{aligned}
1 &> a_0 \\
3 + 3a_0 &> a_2 \\
1 + a_3 + a_2 + a_1 + a_0 &> 0 \\
1 - a_3 + a_2 - a_1 + a_0 &> 0 \\
(1 - a_0)(1 - a_0^2) - a_2(1 - a_0)^2 + (a_3 - a_1)(a_1 - a_0 a_3) &> 0.
\end{aligned}$$

We can use a similar argument that we used in Theorem 2.3.2 regarding adjusting coefficients slightly to force roots to be inside of the disk centred at 0 in the case where there is an equality. It follows that the roots are in the (closed) unit disk if

$$1 > a_0 \tag{2.16}$$
$$3+3a_0 \geq a_2 \tag{2.17}$$
$$1+a_3+a_2+a_1+a_0 \geq 0 \tag{2.18}$$
$$1-a_3+a_2-a_1+a_0 \geq 0 \tag{2.19}$$
$$(1-a_0)(1-a_0^2)-a_2(1-a_0)^2+(a_3-a_1)(a_1-a_0a_3) > 0. \tag{2.20}$$

Indeed, let us assume we have equality as stated above, and assume that there is a root outside of the closed unit disk. Then if we replace a_0 by $a_0+\epsilon$, with ϵ sufficiently small, (2.17), (2.18), and (2.19) will have a strict inequality. We see that for (2.20), the difference when a_0 becomes $a_0+\epsilon$ is a polynomial in ϵ with no constant term, and so it goes to 0 as ϵ does. Therefore, if we replace a_0 by $a_0+\epsilon$, we'd be able to, for small enough ϵ, keep a root outside of the disk while having the five strict inequalities holding – a contradiction.

The H-polynomial of $\mathcal{M}$ (whose roots we are interested in) is given by

$$\begin{aligned} h_{\mathcal{M}}(q) &= H_0+H_1q+H_2q^2+H_3q^3+H_4q^4 \\ &= 1+(m-4)q+\binom{m-3}{2}q^2+\binom{m-2}{3}q^3+H_4q^4 \\ &= H_4\left(\frac{1}{H_4}+\frac{m-4}{H_4}q+\frac{\binom{m-3}{2}}{H_4}q^2+\frac{\binom{m-2}{3}}{H_4}q^3+q^4\right). \end{aligned}$$

By (2.16)–(2.20), we need only show that

$$H_4 > 1 \tag{2.21}$$
$$3H_4+3 \geq H_2 \tag{2.22}$$
$$H_4+H_3+H_2+H_1+1 \geq 0 \tag{2.23}$$
$$H_4-H_3+H_2-H_1+1 \geq 0 \tag{2.24}$$
$$(H_4-1)(H_4^2-1)-H_2(H_4-1)^2+(H_3-H_1)(H_4H_1-H_3) > 0. \tag{2.25}$$

Clearly condition (2.21) holds as $H_4 \geq m-4 \geq 3$. Condition (2.23) holds since all of the H_i are positive. Condition (2.24) follows directly from inequality (2.5), and implies

$$H_4 \geq H_3-H_2+H_1-H_0 = \frac{1}{6}m^3-2m^2+\frac{53}{6}m-15. \tag{2.26}$$

It follows that

$$3H_4 + 3 \geq \frac{1}{2}m^3 - 6m^2 + \frac{53}{2}m - 42 > \frac{1}{2}m^2 - \frac{7}{2}m + 6 = H_2$$

as $m \geq 7$, so condition (2.22) holds.

All that remains is condition (2.25). We set $l = H_1 = m - 4$ (which is at least 3). We substitute into the left-hand side of (2.25) the values for $H_i = \binom{l+i-1}{i}$ for $i \leq 3$ and set $z = H_4$. We need to show that

$$\begin{aligned}(z-1)(z^2-1) - \binom{l+1}{2}(z-1)^2 + \\ \left(\binom{l+2}{3} - l\right)\left(zl - \binom{l+2}{3}\right) \quad > \quad 0. \qquad (2.27)\end{aligned}$$

Denote the left-hand side of (2.27) by $f = f(z)$. The derivative $f'(z)$ is a quadratic in z that opens up, and we can verify that its discriminant is $-l^4 - 4l^3 + l^2 - 8l + 16$, which is negative for $l \geq 2$. Therefore, there are no real roots and so the derivative is positive everywhere. Thus f is increasing on $[2, \infty)$, so that as $l \geq 3$, by (2.26) we have $z \geq \frac{1}{6}m^3 - 2m^2 + \frac{53}{6}m - 15 = \frac{1}{6}l^3 + \frac{5}{6}l - 1 > 2$

$$\begin{aligned} f(z) &\geq f\left(\frac{1}{6}l^3 + \frac{5}{6}l - 1\right) \\ &= \frac{l(l-2)(l-1)(l^2-l+12)(l^4+l^3+5l^2-l+12)}{216} \\ &> 0 \end{aligned}$$

(both the polynomials $l^2 - l + 12$ and $l^4 + l^3 + 5l^2 - l + 12$ have only complex roots, and hence are always positive). Therefore, (2.25) holds, and we are done. □

2.4 Reliability Roots of Matroids Outside the Disk Centred at 0

In the previous section, we showed that the roots of matroids of rank at most 3 and paving matroids of rank 4 have their roots inside of the unit disk. There are, to be sure, reliability roots of matroids outside the disk centred at 0 – we know from network reliability that there are graphs whose all terminal reliability roots have modulus greater than 1, and hence there are cographic matroids that have roots outside of the disk centred at 0. Are there other matroids with reliability roots outside the unit disk? There exist, of course, operations on matroids that would yield roots outside of

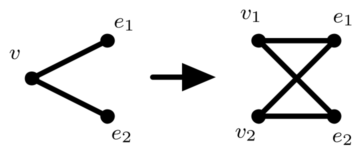


Figure 2.8: Example of a k-thickening on v of a matroid with $k = 2$

the unit disk. As noted earlier, the reliability roots of $\mathcal{M}_1 * \mathcal{M}_2$ is simply the union of the reliability roots of $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore, if either $\mathcal{M}_1$ or $\mathcal{M}_2$ have roots outside of the disk centred at 0, then $\mathcal{M}_1 * \mathcal{M}_2$ will have roots outside of the disk centred at 0. It follows that we can embed any matroid in another that has a root outside of the unit disk. However, this seems somewhat artificial.

Two other operations, though, yield other matroids with reliability roots outside the disk centred at 0. We generalize the graph operations, path replacement and edge bundling (Operations 1.1 and 1.2, respectively, on page 8) to matroids.

Operation 2.5. *k-thickening*

The first operation, which is a generalization of a path replacement, we define to be a **k-thickening** of a matroid at vertex v, denoted by $\mathrm{Th}(\mathcal{M}, v, k)$. This is defined to be the matroid such that

- if v is a vertex in a face $\tau \in \mathcal{M}$ then

$$(\tau \setminus \{v\}) \cup \{v_1\}, (\tau \setminus \{v\}) \cup \{v_2\}, \ldots, (\tau \setminus \{v\}) \cup \{v_k\} \in \mathrm{Th}(\mathcal{M}, v, k),$$

- if v is not an element in a face $\tau \in \mathcal{M}$, then $\tau \in \mathrm{Th}(\mathcal{M}, v, k)$.

In essence, we place $k-1$ new elements in parallel to v (which corresponds to replacing an edge by a path). For example, see Figure 2.8. If we consider the faces in the original matroid $\mathcal{M}$, we have $\emptyset, v, e_1, e_2, ve_1$, and ve_2. Then $\mathrm{Th}(\mathcal{M}, v, 2)$ would be the matroid with faces $\emptyset, v_1, v_2, e_1, e_2, v_1e_1, v_2e_1, v_1e_2, v_2e_2$.

A useful tool for us will be to calculate the H-polynomial for this type of operation. We can do this by using the F-polynomial, together with the deletion and link complexes.

It is easy to see that

$$f_{\mathrm{Th}(\mathcal{M},v,k)}(q) = f_{\mathrm{del}_v \mathcal{M}}(q) + kqf_{\mathrm{link}_v \mathcal{M}}(q)$$

since if v is a vertex in a face $\tau \in \mathcal{M}$, then we remove v (which gives the link of v) and we add in k other vertices, and we just take the deletion of v if v is not in a face $\tau \in \mathcal{M}$. We then translate the F-polynomial into the H-polynomial by multiplying by $(1-q)^d$ and making the change of variable $q \to q/(1-q)$. That is,

$$(1-q)^d f_{\mathrm{Th}(\mathcal{M},v,k)}\left(\frac{q}{1-q}\right) = (1-q)^d f_{\mathrm{del}_v \mathcal{M}}\left(\frac{q}{1-q}\right) + (1-q)^d kqf_{\mathrm{link}_v \mathcal{M}}\left(\frac{q}{1-q}\right)$$

Therefore, the H-polynomial of $\mathrm{Th}(\mathcal{M}, v, k)$ is given by

$$h_{\mathrm{Th}(\mathcal{M},v,k)}(q) = h_{\mathrm{del}_v \mathcal{M}}(q) + kq(1-q) \cdot h_{\mathrm{link}_v \mathcal{M}}(q).$$

As k grows large $kq(1-q) \cdot h_{\mathrm{link}_v \mathcal{M}}(q)$ will dominate over $h_{\mathrm{del}_v \mathcal{M}}(q)$, that is, if we consider $(1/k)h_{\mathrm{Th}(\mathcal{M},v,k)}(q)$, which has the same roots as $h_{\mathrm{Th}(\mathcal{M},v,k)}(q)$, it approaches coefficient-wise to $q(1-q)h_{\mathrm{link}_v \mathcal{M}}(q)$ which leads to the roots of $h_{\mathrm{Th}(\mathcal{M},v,k)}(q)$ being close to those of $h_{\mathrm{link}_v \mathcal{M}}(q)$ (and 0). Therefore, if $h_{\mathrm{link}_v \mathcal{M}}(q)$ has a root outside of the disk centred at 0, then $h_{\mathrm{Th}(\mathcal{M},v,k)}(q)$ will also have a root outside of the disk centred at 0 provided k is large enough.

Operation 2.6. *k-replacement*

Another operation that we will focus on is a generalization of replacing an edge of a graph with k edges in parallel (an operation which Royle and Sokal used on a pair of opposite edges to construct their graph). We define the **k-replacement** at vertex v, denoted by $\mathrm{Rep}(\mathcal{M}, v, k)$, to be the matroid such that

- if v is an element of a face $\tau \in \mathcal{M}$ then $(\tau \backslash \{v\}) \cup \{v_1, v_2, \ldots, v_k\} \in \mathrm{Rep}(\mathcal{M}, v, k)$; and
- if v is not an element of a face $\tau \in \mathcal{M}$ then for any $\alpha \subsetneq \{v_1, v_2, \ldots, v_k\}, \tau \cup \alpha \in \mathrm{Rep}(\mathcal{M}, v, k)$.

See Figure 2.9 for an example. This time, we will only calculate the F-polynomial, which will become apparent in a moment. If we use a k-replacement on a single element v, then we get the F-polynomial

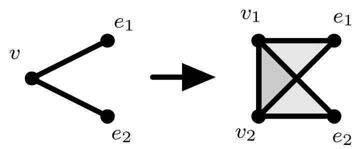


Figure 2.9: Example of a k-replacement on v of a matroid with $k = 2$

$$f_{\text{Rep}(\mathcal{M},v,k)}(q) = ((1+q)^k - q^k) f_{\text{del}_{\mathcal{M}}(v)} + q^k f_{\text{link}_{\mathcal{M}}(v)} \tag{2.28}$$

Indeed, we can use the deletion complex $\text{del}_{\mathcal{M}}(v)$ to consider all faces which do not have v as a face (and so by our definition, we would add in every possible proper subset of $\{v_1, \ldots, v_k\}$ which has a generating polynomial $(1+q)^k - q^k)$, as well as the link complex $\text{link}_{\mathcal{M}}(v)$ to consider all faces which do have v as a face (and so by our definition, we would add in the entire simplex $\{v_1, \ldots, v_k\}$ which has q^k as its generating polynomial).

However, it would be more beneficial to be able to use a k-replacement on multiple vertices simultaneously, with different values of k, and so we will introduce a multivariate generating polynomial of the matroid. Given a matroid $\mathcal{M}$ on elements $v_1, v_2, \ldots v_m$, for each face $\sigma \in \mathcal{M}$ we introduce the variable q_i if $v_i \in \tau$ and p_i if $v_i \notin \tau$, and define the **multivariate generating polynomial** by

$$\text{mgen}_{\mathcal{M}}(q_1, p_1, \ldots, q_m, p_m) \quad = \quad \sum_{\tau \in \mathcal{M}} \prod_{v_i \in \tau} q_i \prod_{v_j \notin \tau} p_j.$$

Let us see another, more interesting example of this operation.

Example 2.6.1. For example, consider the matroid $\mathcal{M}$ with facets

$$\mathcal{M} = \{\{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}. \tag{2.29}$$

The faces of $\mathcal{M}$ are therefore

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,4\},$$
$$\{2,3,4\}, \{1,2,3\}\}.$$

The multivariate generating polynomial for $\mathcal{M}$ is given by

$$\begin{aligned}\text{mgen}_{\mathcal{M}}(q_1,p_1,\ldots,q_4,p_4) &= p_1p_2p_3p_4+p_1p_2p_3q_4+p_1p_2p_4q_3+p_1p_2q_3q_4+\\&\quad p_1p_3p_4q_2+p_1p_3q_2q_4+p_1p_4q_2q_3+p_1q_2q_3q_4+\\&\quad p_2p_3p_4q_1+p_2p_3q_1q_4+p_2p_4q_1q_3+p_3p_4q_1q_2+\\&\quad p_3q_1q_2q_4+p_4q_1q_2q_3.\end{aligned}$$

Using $\text{mgen}_{\mathcal{M}}(q_1,p_1,\ldots,q_m,p_m)$, we are able to generate the F-polynomial of the matroid (2.29) by simply setting every $q_i = q^k$ and every $p_i = (1+q)^k - q^k$. Though this is a valid method to finding the reliability roots (we would still need to translate the F-polynomial into the H-polynomial), we can generate the reliability polynomial directly by setting $q_i = q_i^k$, and every $p_i = 1 - q_i^k$. Indeed, if for each element v_i we replace it by $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{k_i}}\}$, then a face that contained v_i is present with probability q_i^k (all of the vertices need to be chosen), and a face that didn't contain v_i is present with probability $1 - q_i^k$ (any subset of $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{k_i}}\}$ can be chosen except for the entire set). Putting these together, we have the probability being

$$\prod_{v_i\in\tau} q^{k_i} \prod_{v_i\notin\tau} 1 - q^{k_i}.$$

Therefore, for $\mathbf{k} = (k_1, k_2, \ldots, k_m)$, the reliability polynomial of the $\mathbf{k}$-replacement on all vertices, obtained by sequentially carrying out a k_i-replacement at vertex v_i, has reliability given by

$$\text{mgen}_{\mathcal{M}}(q^{k_1}, 1-q^{k_1}, q^{k_2}, 1-q^{k_2}, \ldots, q^{k_m}, 1-q^{k_m}). \tag{2.30}$$

Let us now revisit the Royle-Sokal Example (first stated as Example 2.1.1 on page 27), but this time through the lens of reliability of complexes.

Example 2.1.1 Revisited *Royal-Sokal Cographic Matroid*

To build the Royal-Sokal cographic matroid, we start with $\text{Cog}(K_4)$, and carry out two 6-replacements of a pair of parallel edges in the K_4 (which are vertices in its cographic matroid) to get matroid $\mathcal{RS}$. Then

$$\begin{aligned}\text{Rel}(\text{Cog}(RS);q) &= \text{mgen}_{\mathcal{M}}(q^6, 1-q^6, q, 1-q, q^6, 1-q^6, q, 1-q, q, 1-q, q, 1-q) \\ &= (1-q)^3\left(6q^{13}+10q^{12}+14q^{11}+18q^{10}+22q^9+26q^8+\right. \\ &\quad \left. 26q^7+22q^6+18q^5+14q^4+10q^3+6q^2+3q+1\right)\end{aligned}$$

which is indeed the reliability of the Royle-Sokal graph, and has a root outside the disk centred at 0.

The upshot is that we can start with any matroid and carry out a **k**-replacement in the hope of finding other reliability roots outside the unit disk. Using this approach we can indeed find a connected matroid that is not a cographic matroid with a reliability root outside the disk centred at 0, as follows.

Consider the well-known *Fano Matroid*, F_7, which is the matroid of order 7 whose facets are all 3-tuples of the set $\{1,2,3,4,5,6,7\}$ except for the 3-tuples whose vertices form a line or circle in the Fano Plane (illustrated in Figure 2.10). This matroid is connected and is known to be non-cographic (and non-graphic) [44, pg. 643-644].

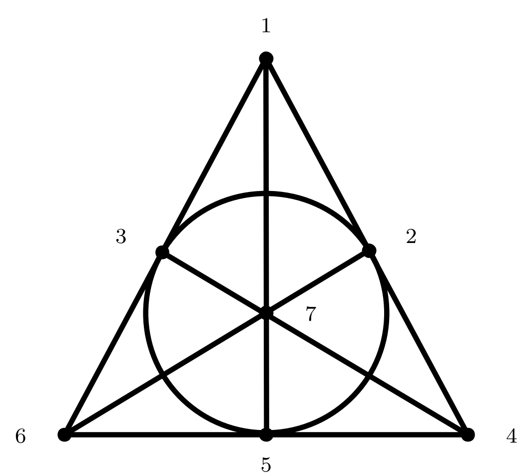


Figure 2.10: Fano Plane of order 7

We can calculate its multivariate generating polynomial, and then carry out the

k-replacements using $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5, k_6, k_7)$ to obtain the polynomial

$$\begin{aligned}
\mathrm{Rel}(\mathrm{Th}_{F_7,\mathbf{k}}, q) \quad = \quad & 1 - q^{k_2+k_7+k_6} - q^{k_2+k_3+k_5} - q^{k_2+k_4+k_3+k_6} - q^{k_2+k_4+k_7+k_5} \\
& +2q^{k_2+k_4+k_3+k_5+k_7} + 2q^{k_2+k_4+k_3+k_7+k_6} - q^{k_7+k_5+k_6+k_3} \\
& -q^{k_5+k_6+k_4} + 2q^{k_5+k_6+k_2+k_7+k_3} + 2q^{k_5+k_6+k_2+k_7+k_4} \\
& +2q^{k_5+k_6+k_2+k_3+k_4} + 2q^{k_7+k_5+k_6+k_3+k_4} - 6q^{k_5+k_6+k_2+k_7+k_3+k_4} \\
& -q^{k_7+k_3+k_4} - q^{k_1+k_3+k_2+k_7} + 2q^{k_1+k_4+k_2+k_3+k_7} - q^{k_1+k_4+k_2} \\
& -q^{k_1+k_5+k_7} - q^{k_1+k_5+k_3+k_4} + 2q^{k_1+k_5+k_2+k_3+k_7} + 2q^{k_1+k_5+k_2+k_3+k_4} \\
& +2q^{k_1+k_5+k_2+k_7+k_4} + 2q^{k_1+k_5+k_3+k_7+k_4} - 6q^{k_1+k_5+k_2+k_3+k_7+k_4} \\
& +2q^{k_1+k_6+k_2+k_3+k_7} + 2q^{k_1+k_6+k_2+k_3+k_4} + 2q^{k_1+k_6+k_2+k_3+k_5} \\
& +2q^{k_1+k_6+k_2+k_7+k_4} + 2q^{k_1+k_6+k_2+k_7+k_5} + 2q^{k_1+k_6+k_2+k_4+k_5} \\
& +2q^{k_1+k_6+k_3+k_7+k_4} + 2q^{k_1+k_6+k_3+k_7+k_5} + 2q^{k_1+k_6+k_3+k_4+k_5} \\
& +2q^{k_1+k_6+k_7+k_4+k_5} + 13q^{k_1+k_6+k_2+k_3+k_7+k_4+k_5} - 6q^{k_1+k_6+k_2+k_3+k_7+k_4} \\
& -6q^{k_1+k_6+k_2+k_3+k_7+k_5} - 6q^{k_1+k_6+k_2+k_3+k_4+k_5} - 6q^{k_1+k_6+k_2+k_7+k_4+k_5} \\
& -6q^{k_1+k_6+k_3+k_7+k_4+k_5} - q^{k_1+k_6+k_2+k_5} - q^{k_1+k_6+k_7+k_4} - q^{k_1+k_6+k_3}
\end{aligned} \tag{2.31}$$

There are, we have found, many choices for **k**-replacements that result in roots outside of the unit disk. For example, consider $\mathbf{k} = (1, 4, 4, 4, 5, 4, 5)$. Then we obtain that the reliability of the matroid formed is

$$\begin{aligned}
\mathrm{Rel}(\mathrm{Th}_{F_7,(1,4,4,4,5,4,5)}, q) \quad = \quad & (1-q)^4(q^3+q^2+q+1)(13q^{20}+33q^{19}+60q^{18}+ \\
& 94q^{17}+124q^{16}+140q^{15}+146q^{14}+142q^{13}+ \\
& 129q^{12}+111q^{11}+94q^{10}+77q^9+61q^8+46q^7+ \\
& 34q^6+24q^5+16q^4+10q^3+6q^2+3q+1)
\end{aligned}$$

which has maximum modulus approximately 1.0018475452. We can iterate through all possible combinations of the k_i into (2.31) between 1 and 5, which yields a number of polynomials with roots that are outside of the unit disk. After iterating through all of the combinations of k_i, we found that there were six reoccurring maximum moduli greater than 1. We have included these moduli as well as the first instance of **k** that produced the root in the table in Figure 2.11, plus a plot of all roots of those six polynomials in Figure 2.12. The root with largest modulus occurs at $\mathbf{k} = (4, 4, 5, 4, 5, 5, 5)$, which has the reliability polynomial

$$\begin{aligned}\mathrm{Rel}(\mathrm{Th}_{F_7,(4,4,5,4,5,5,5)}, q) \quad = \quad & (q-1)^4(q+1)(q^2+1)(q^4+q^3+q^2+q+1) \\ & (13q^{21}+26q^{20}+39q^{19}+52q^{18}+60q^{17}+57q^{16}+ \\ & 54q^{15}+51q^{14}+49q^{13}+42q^{12}+36q^{11}+30q^{10}+ \\ & 24q^9+19q^8+15q^7+12q^6+9q^5+6q^4+4q^3+ \\ & 3q^2+2q+1)\end{aligned}$$

$\mathbf{k}$	Approximate Root of Maximum Modulus	Maximum Modulus
$(1, 4, 4, 4, 5, 4, 5)$	$0.1440344331 \pm 0.9914396533i$	1.0018475453
$(2, 2, 5, 2, 5, 5, 5)$	$-0.8767135845 \pm 0.4887049107i$	1.0037226704
$(3, 3, 3, 5, 5, 5, 3)$	$-0.6684629266 \pm 0.7458897760i$	1.0015958477
$(3, 3, 5, 3, 5, 5, 5)$	$-0.7454201308 \pm 0.6771760394i$	1.0070841871
$(4, 4, 4, 5, 5, 5, 4)$	$0.1330903394 \pm 0.9988306129i$	1.0076584896
$(4, 4, 5, 4, 5, 5, 5)$	$0.2142892928 \pm 0.9857044786i$	1.0087285165

Figure 2.11: Roots of maximum modulus for $\mathrm{Rel}(\mathrm{Th}_{F_7,\mathbf{k}}, q)$ with $\mathbf{k} \leq (5, 5, 5, 5, 5, 5, 5)$.

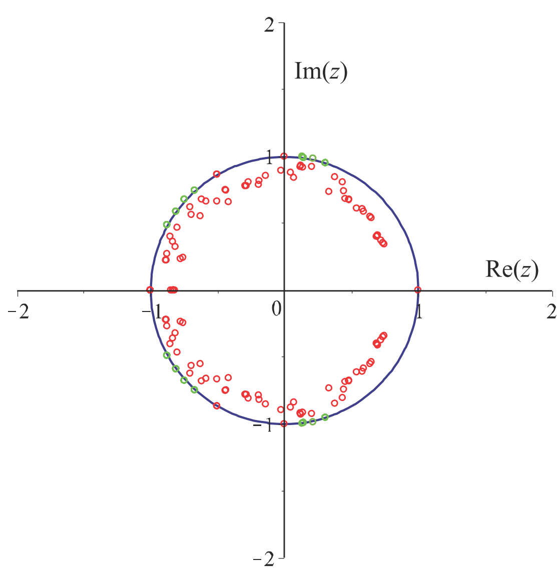


Figure 2.12: Plot of all roots of the six polynomials attained by the **k**-thickenings in Figure 2.10, with the roots of modulus larger than 1 highlighted in green, and the unit circle in blue.

It seems unclear why $(4, 4, 5, 4, 5, 5, 5)$ yields a root of maximum modulus (among the ones considered), and how the maximum moduli of the roots might depend on the components of $\mathbf{k}$ (one could try substituting various values for $\mathbf{k}$, for instance, $\mathbf{k}$ could be of the form $[l, l, l+1, l, l+1, l+1, l+1]$ which produces a root with modulus approximately 1.0095560967 for $l = 5$, but all subsequent values of l for this particular substitution seem to have roots of maximum modulus decreasing). Even though the largest known modulus of a reliability root is approximately 1.13 (via Brown-Mol), this is a method to generate non-cographic matroids which have reliability roots outside of the closed unit disk. In other words, there are matroids that are not cographic that do not satisfy the Brown-Colbourn Conjecture.

2.7 Random Pure Complexes and the Brown-Colbourn Conjecture

One question that naturally arises from the aforementioned results is asking whether almost all matroids have their reliability roots inside the unit disk. The difficulty on approaching this question is that, unlike graphs (which have the Erdös–Rényi model), there is no generative probabilistic model for matroids (it is this issue that has hampered attempts to show that almost all matroids are paving).

However, we will show that while we do not know whether almost all matroids have their reliability roots in the unit disk, for almost all pure d-dimensional complexes, the reliability roots are all in the unit disk (see Figure 2.13 for a plot of all the reliability roots for all pure d dimensional complexes for d from 1 to 6). As the pure d-dimensional complexes on $[m] = \{1, 2, \ldots, m\}$ are in a $1-1$ correspondence with the collections of d-subsets of $[m]$, we can form a generative probabilistic model $\mathcal{PD}_{m,1/2}$ for pure d-dimensional complexes on $[m]$ by randomly choosing facets, that is, each d-subset of $[m]$ independently with probability $1/2$ (under this model, each pure d-dimensional complex on $[m]$ occurs with equal probability); we extend the model (as is done for graphs) to $\mathcal{PD}_{m,p}$ by fixing any $p \in (0, 1)$ and choosing each d-subset of $[m]$ independently with probability p.

Theorem 2.7.1. *For fixed positive integer d and fixed real number $p \in (0, 1)$, the reliability roots of almost all pure d-dimensional complexes in $\mathcal{PD}_{m,p}$ lie inside the unit disk.*

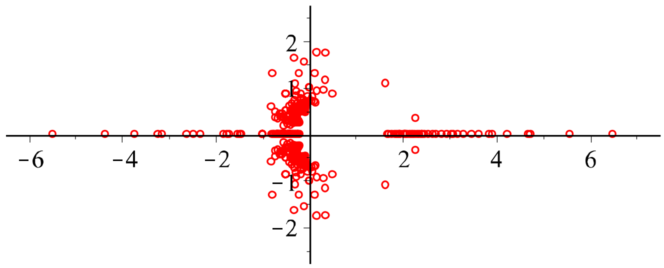


Figure 2.13: Plot of the reliability roots for all pure d-dimensional complexes for d from 1 to 6.

Proof. Clearly we can assume $d \geq 2$ as all complexes of dimension at most 1 have their reliability roots in the unit disk. Let $\varepsilon > 0$. Consider the following events:

- E_1 is the event that there are no loops.
- E_2 is the event that every $d-1$ subset of $[m]$ is a subset of a facet.
- E_3 is the event that the number of facets is greater than $(1-\varepsilon)p\binom{m}{d}$.

Now the probability that a fixed vertex v is a loop is $(1-p)^{\binom{m-1}{d-1}}$, so as m goes to infinity,

$$\mathrm{Prob}(\overline{E_1}) \quad \leq \quad m(1-p)^{\binom{m-1}{d-1}} = o(1).] \tag{2.32}$$

Similarly, as the probability that a fixed subset S of $[m]$ of size $d-1$ is not a subset of any facet is $(1-p)^{m-(d-1)}$, it follows that

$$\mathrm{Prob}(\overline{E_2}) \quad \leq \quad \binom{m}{d-1}(1-p)^{m-(d-1)} = o(1). \tag{2.33}$$

Moreover, the number of facets is modeled by a binomial distribution on $M = \binom{m}{d}$ trials each occurring with probability p (the mean of the distribution is pM). The Chernoff lower tail bounds [22] implies that for independent random variables $X_1, \ldots, X_M$, with each X_i always lying in $[0,1]$, if we set $X = \sum X_i$ and $\mu = E(X)$, then

$$\mathrm{Prob}(X \leq (1-\varepsilon)\mu) \leq e^{-\mu\varepsilon^2/2}.$$

It immediately follows that

$$\mathrm{Prob}(\overline{E_3}) \quad \leq \quad e^{-pM\varepsilon^2/2} = o(1). \tag{2.34}$$

From (2.32), (2.33) and (2.34) we see that

$$\mathrm{Prob}(\overline{E_1 \cap E_2 \cap E_3}) = \mathrm{Prob}(\overline{E_1} \cup \overline{E_2} \cup \overline{E_3}) = o(1),$$

so that

$$\lim_{m\to\infty} \mathrm{Prob}(E_1 \cap E_2 \cap E_3) = 1.$$

Consider any pure d-dimensional complex $\mathcal{C}$ that lies in $E_1 \cap E_2 \cap E_3$. From E_1 and E_2, we see that the k-skeletons are full for all $k < d$, so that $F_i = \binom{m}{i}$ for $i < d$, and E_3 implies that $F_d > (1-\varepsilon)p\binom{m}{d}$.

It is not hard to verify (see [57, Proposition 6.3]) that the H-vector of the uniform matroid $U(m,d)$ is $\langle \binom{m-d-1}{0}, \binom{m-d}{1}, \ldots, \binom{m-d+i-1}{i}, \ldots, \binom{m-1}{d} \rangle$. The uniform matroid $U(m,d)$ and pure complex $\mathcal{C}$ have the same F-vector vector except for possibly F_d, and H_i of an H-vector only depends on F_j for $j \leq i$, and so we conclude that for $\mathcal{C}$,

$$H_i = \binom{m-d+i-1}{i}$$

for $i = 0, 1, \ldots, d-1$. Now from a binomial identity, we find that

$$\sum_{i=0}^{d-1} \binom{m-d+i-1}{i} = \binom{m-1}{d-1},$$

and so for $\mathcal{C}$, from (1.4) we find that

$$H_d = F_d - \sum_{i=0}^{d-1} \binom{m-d+i-1}{i} > (1-\varepsilon)p\binom{m}{d} - \binom{m-1}{d-1}.$$

It is trivial to check that

$$H_i = \binom{m-d+i-1}{i} \leq H_{i+1} = \binom{m-d+i}{i+1}$$

provided $m \geq d+1$ (which we can assume, as we are interested in the limit as $m \to \infty$). Moreover,

$$H_d = \binom{m}{d} - \binom{m-1}{d-1} \geq H_{d-1} = \binom{m-2}{d-1}$$

provided

$$(1-\varepsilon)pm(m-1) \geq d(m+d) + (m-1)d,$$

which clearly holds if m is sufficiently large (as d and p are fixed). Thus the (positive) coefficients of the H-polynomial of $\mathcal{C}$ are nondecreasing, so we conclude by the Eneström-Kakeya Theorem that all of the roots of the polynomial have modulus at most 1, and hence the same is true of the roots of the reliability polynomial of $\mathcal{C}$. It follows that for almost all complexes in $\mathcal{PD}_{m,p}$, their reliability roots lie inside the unit disk. $\square$

Are almost all pure d-dimensional complexes on $[m]$ shellable? It is true for $d = 2$ (as almost all graphs are connected), but it seems unlikely if $d \geq 3$. But if so, then almost all shellable complexes would have their roots in the unit disk. As well, while every matroid is a pure complex, we do not know whether the H-vector of almost all matroids is nondecreasing (which would be sufficient to proving that the roots of almost all matroids are in the unit disk).

Our interest in the maximum moduli of all-terminal reliability roots led us to explore reliability roots in a more general setting. We now turn back to all-terminal reliability, but focus on roots of minimum modulus.

Chapter 3

All Terminal Reliability Roots of Smallest Modulus

3.1 Introduction

As we have seen, much of the work on reliability has not only been focused on efficient ways of estimation (see, for example, the Ball-Povan Bounds on page 68 of [23]), but also analytic properties of the functions such as the location of the roots. Most of this work has been to analyze the location of the root with maximum modulus. However, much regarding roots of maximum modulus is still left open. For example, we know that the only root for any (possibly non-simple) graph on 2 vertices is 1, and for (possibly non-simple) graphs on 3 vertices, we have that the root of maximum modulus is also 1. Indeed, all graphs on 2 or 3 vertices are series-parallel graphs, and so the claim follows by Wagner's result [57]. However, on 4 vertices we do not know the maximum modulus of a reliability root, nor the extremal graph. The Sokal-Royle graph has a reliability root outside of the unit disk, but it is not known if a different bundling of edges will push a reliability root out further.

If we cannot characterize the roots of maximum modulus, what about roots of *minimum* modulus? If we consider other graph polynomials, the question isn't very interesting. In the case for chromatic polynomials, we know that 0 is always a root and so clearly it must be the root with smallest modulus. On the other hand, if we consider the independence polynomial, though there is no known smallest root, Brown, Dilcher, and Nowakowski showed in [17] that the root of smallest modulus is always real.

How about for reliability polynomials? We know that 0 can never be a root, but how close can we get to having a modulus of 0 for a graph on n vertices?

Another natural (and seemingly unrelated) question to ask is whether we can characterize the *rational* roots of reliability polynomials. Let us recall some of the reliability polynomials listed in Chapter 1. For instance, the reliability polynomial of tree graphs on n vertices is $(1-q)^{n-1}$, and the reliability polynomial of cycle graphs on n vertices is $(1-q)^{n-1}((n-1)q+1)$. Clearly, all of the roots of these polynomials are rational, and in the case of cycle graphs, is of the form $-1/(n-1)$. Which graphs have rational reliability roots, and, even further, of what form can they be? Is the set of rational reliability roots dense in, say, $[-1,0)$, or does it form a special subset?

In this chapter we will show that the roots of smallest modulus are extremal in the sense that for all graphs of order $n \geq 3$, a root of smallest modulus is unique and is rational. We contrast this with the fact that it is known that there are graphs with reliability roots outside the unit disk [19, 46], and it follows that for all $n \geq 4$, a reliability root of largest modulus among all those for graphs of order n is necessarily not rational. Also, we will characterize the rational reliability roots for graphs of order n, as well as simple graphs of order n, and thereby find the first nontrivial mathematical property that distinguishes, via reliability roots, the class of simple graphs from that of all graphs.

3.2 Rational Reliability Roots

We recall that we can expand the reliability of a graph G of order n and size m (that is, with m edges) in terms of different bases (see, for example, [23]). The two useful expansions that we have used are the F- and H-forms:

$$\begin{aligned} \mathrm{Rel}(G;q) &= \sum_{i=0}^{m-n+1} F_i q^i (1-q)^{m-i} \qquad \text{(F-Form)} \qquad (3.1)\\ &= (1-q)^{n-1} \sum_{i=0}^{m-n+1} H_i q^i \qquad \text{(H-Form)} \qquad (3.2) \end{aligned}$$

We saw that each F_i counts the number of subsets of i edges whose deletion leaves G still connected; the collection of such subsets is the cographic matroid of G, $\mathrm{Cog}(G)$. Furthermore, we recall that the dimension of the complex, d, is the common cardinality of any maximal set, and when the graph is loopless (as we shall assume), it is $d = m - n + 1$, the corank of graph G.

We have also seen that the H_i have a number of interesting and useful interpretations:

- There is a partition of the faces of $\mathrm{Cog}(G)$ into intervals $[\tau,\sigma] = \{\alpha \in \mathrm{Cog}(G) : \tau \subseteq \alpha \subseteq \sigma\}$, where τ and σ are faces of $\mathrm{Cog}(G)$, $\tau \subseteq \sigma$ and σ is a facet, which necessarily has cardinality d. Then H_i counts the number of lower sets τ that have cardinality i.

- There is an order ideal of monomials, that is, a set of monomials $\mathcal{M}(G)$ closed under divisibility, such that H_i counts the number of monomials in the set with degree i. (The construction of such a set of monomials $\mathcal{M}(G)$ can be achieved through connections to commutative algebra – see, for example, [4].)

We are now ready to find all rational reliability roots of graphs of order n, and draw a connection to reliability roots of smallest modulus.

Theorem 3.2.1. *Let $n \geq 2$. Then the rational numbers that are reliability roots of graphs of order n are $-1, -1/2, \ldots, -1/(n-1)$ and 1.*

Proof. First we observe that the reliability of the cycle of order $n \geq 2$, C_n, is given by

$$\mathrm{Rel}(C_n; q) = (1-q)^n + nq(1-q)^{n-1} = (1-q)^n(1+(n-1)q),$$

which has roots at $q = 1$ and $-1/(n-1)$. As adjoining a new vertex to a single vertex of a graph multiplies the reliability polynomial by $p = 1-q$, we can take any cycle of order at most n and extend it to a graph of order n with the same reliability roots (only the multiplicity of 1 changes). It follows that $1, -1, -1/2, \ldots, -1/(n-1)$ are reliability roots of graphs of order n. We need to show that there are no other rational roots of a graph of order n.

From (1.3), the rational reliability roots of a graph are 1 together with those of its H-polynomial,

$$\sum_{i=0}^{m-n+1} H_i q^i.$$

The connection between the H_i and interval partitions of the cographic matroid implies that $H_0 = 1$, as the empty set is always a face (the unique face of cardinality 0), and hence is the lower set in exactly one interval of an interval partition of $\mathrm{Cog}(G)$. As all of the H_i are positive integers and the constant term is 1, the well known Rational Root Theorem implies that the only *possible* rational roots are of the form $1/k$ for some non-zero integer k. As clearly the positivity of the H_i's implies that no real root of the H-polynomial is positive, we are left only with $-1/k$, with k a *positive* integer, as possible rational reliability roots of graphs. For graphs of order n, we show that $k \leq n-1$.

We proceed by induction to show that if z is a reliability root of a graph G of order n, then $|z| \geq 1/(n-1)$ (this also follows from Theorem 4.6 of [14], but we shall include a shorter, more elegant proof). This claim is trivial for $n = 2$, as the only possible graph of this order consists of a bundle of edges between the two points, and thus its reliability has the form $1 - q^m$, which clearly has all its roots on the disk centred at 0 of radius $1 = 1/(n-1)$. We assume now that $n \geq 3$ and proceed by induction. We can assume that the graph has no cut vertices, for the following reason. If a graph G of order n had blocks $B_1, \ldots, B_l$ with $l \geq 2$, then

$$\mathrm{Rel}(G; q) = \prod_{i=1}^{l} \mathrm{Rel}(B_i; q),$$

which implies that the reliability roots of G are the union of those for the B_i. However, if B_i has order n_i, then $n_i < n$, then by induction, the reliability roots of B_i has modulus at least $1/(n_i - 1) > 1/(n-1)$, and thus G has all of its roots of modulus at least $1/(n-1)$.

We need some more information on the coefficients of the H-polynomial. Now as G has no cut vertices, it has no bridges, and so $F_1 = m$, that is, each of the m edges is a face of $\mathrm{Cog}(G)$. However, only $d = m - n + 1$ of them appear in the interval whose lower set is $\emptyset$ (as the upper sets always have cardinality d). It follows that $H_1 = m - (m - n + 1) = n - 1$. Moreover, it is easy to see from the connection to order ideals of monomials that for $0 \leq i \leq d-1$,

$$\frac{H_i}{H_{i+1}} \geq \frac{H_0}{H_1} = \frac{1}{n-1}.$$

This inequality is equivalent to $(n-1)H_i \geq H_{i+1}$, which holds as in any associated order ideal of monomials $\mathcal{M}(G)$, multiplying each monomial in $\mathcal{M}(G)$ of degree i by each variable x certainly covers all monomials of degree H_{i+1} at least once. Alternatively, one can make reference to a result from Huh [35], where it was shown that the H-vector $\langle H_0, H_1, \ldots, H_d \rangle$ of any representable matroid – and in particular, any cographic matroid – is *log concave*, that is, for $1 \leq i \leq d-1$,

$$H_{i-1}H_{i+1} \leq H_i^2.$$

From this it follows that for $1 \leq i \leq d-1$,

$$\frac{H_{i-1}}{H_i} \leq \frac{H_i}{H_{i+1}},$$

and so

$$\frac{H_0}{H_1} \leq \frac{H_1}{H_2} \leq \cdots \leq \frac{H_{d-1}}{H_d}. \tag{3.3}$$

We now turn to the Eneström-Kakeya theorem (see, for example, [45, pg. 255]), which states that if a real polynomial $g(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ has positive coefficients, then all the roots of g lie in the annulus $r \leq |x| \leq R$ where $r = \min_{0 \leq j \leq k-1}\{a_j/a_{j+1}\}$ and $R = \max_{0 \leq j \leq k-1}\{a_j/a_{j+1}\}$. It suffices to show that the H-polynomial of G has no root of modulus smaller than $1/(n-1)$. However, from (3.3), the minimum value of the ratios of successive H_i's is

$$r = \frac{H_0}{H_1} = \frac{1}{n-1}.$$

We deduce from the Eneström-Kakeya Theorem that the reliability polynomial of G has no root with modulus less than $1/(n-1)$, and hence has no rational (nor real or complex!) root of absolute value less than $1/(n-1)$. It follows that the collection of rational reliability roots of graphs of order n is precisely $\{1, -1, -1/2, \ldots, -1/(n-1)\}$. □

Now that we have characterized the rational reliability roots, we show that for $n \geq 3$, the unique reliability root of a graph of order n of smallest modulus is rational, while such is not the case for $n = 2$: for $n = 2$, we have seen that the reliability polynomial has the form $1 - q^m$, and hence all the m-th roots of unity are reliability roots of smallest modulus.

Theorem 3.2.2. *For graphs of order $n \geq 3$, the minimum modulus of a reliability root is $1/(n-1)$, the only reliability root of this modulus is $-1/(n-1)$, and only occurs for the cycle C_n.*

Proof. As seen in the proof of the previous theorem, no reliability root of a graph of order n has a modulus less than $1/(n-1)$, and C_n has a root, $-1/(n-1)$, of this modulus. Let G be a graph of order n with a reliability root of modulus $1/(n-1)$; from the previous theorem, G cannot have a cut vertex. A result of [1] states that a polynomial $g(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ has a root of modulus $r = \min_{0 \le j \le k-1}\{a_j/a_{j+1}\}$ only if

$$\gcd\left(\left\{i \in \{1, 2, \ldots, k\} : \frac{a_{i-1}}{a_i} > r\right\}\right) > 1.$$

Thus we need to consider when

$$\frac{H_{i-1}}{H_i} > \frac{H_0}{H_1} = \frac{1}{n-1},$$

that is, when

$$(n-1)H_{i-1} \quad > \quad H_i. \tag{3.4}$$

As noted in [14], the set of indices where (3.4) holds is $\{2, 3, \ldots, d-1\}$ (which was proved by using a result by Stanley [54] on canonical forms), but we provide an alternate proof here. From the interpretation of H_i's as counting the number of monomials of degree i in an order ideal of monomials $\mathcal{M} = \mathcal{M}_G$, we see that $\mathcal{M}$ contains $n-1$ variables (i.e. monomials of degree 1). If for some $i \ge 2$ (3.4) fails to hold, that is, $(n-1)H_{i-1} = H_i$, then if x is any variable and m any monomial of degree $i-1$ in $\mathcal{M}$, xm must be a monomial (of degree i) in $\mathcal{M}$ and every monomial of degree i in $\mathcal{M}$ must arise uniquely in this way. As $n \ge 3$, $\mathcal{M}$ has at least $n-1 \ge 2$ variables. It follows that some monomial of degree i in $\mathcal{M}$ must have the form xym' for some monomial m' of degree $i-2$ (if some monomial of degree $i-1$ contains two distinct variables, add any variable to it, and if x^{i-1} is a monomial of degree $i-1$ in $\mathcal{M}$, then for any other variable y, yx^{i-1} must be a monomial of degree i in $\mathcal{M}$). However, then the monomial xym' arises by adding variable x to the monomial ym' of degree $i-1$ in $\mathcal{M}$, while xym' arises also by adding variable y to the monomial xm' of degree $i-1$ in $\mathcal{M}$. This contradicts the fact that every monomial of degree i in $\mathcal{M}$ arises uniquely by adding a variable to a monomial of degree $i-1$ in $\mathcal{M}$.

It follows that

$$\gcd\left(\left\{i \in \{1, 2, \ldots, d\}) : \frac{H_{i-1}}{H_i} > r = \frac{H_0}{H_1}\right\}\right) = \gcd(\{2, 3, \ldots, d = m-n+1\}),$$

and so, if $d = m-n+1$, the corank of G, is at least 3, then the gcd of the set is 1, and we conclude that there is no root of modulus $r = 1/(n-1)$. When $d = 0$, G is a tree of order at least 3, and hence has a cut vertex, a contradiction. If $d = 1$, then as G has no cut vertices and is unicyclic, $G = C_n$, and its only root of modulus $1/(n-1)$ is indeed $-1/(n-1)$, as we have already seen.

The final remaining open case is when $d = 2$, that is, when $m = n + 1$, and our argument corrects a slip in [14]. As in the proof of Theorem 3.2.1, we can assume that G has no cut vertices (and hence no bridges), as otherwise, the minimum modulus of a reliability root must be larger than $1/(n-1)$. One can characterize all bridgeless graphs G of corank 2 as follows. As G has no bridges, every vertex has degree at least 2. If we have a vertex x of degree 2, with neighbours y and z, we remove x and add in an edge from y to z; this operation deletes a vertex and an edge, and hence leaves the corank the same. We repeat this procedure until we can no longer do so, to arrive at a graph G' (possibly with loops and/or multiple edges), of corank 2, where each vertex has degree at least 3 (in general, we can do this with any fixed corank to derive a finite list of graphs for which every graph of that corank is a *subdivision* of one of these graphs). If G' has order n' and size m', as every vertex has degree at least 3 and the sum of the vertices is twice the number of edges, we have $2m = 2(n'+1) \geq 3n'$, which implies that $n' \leq 2$. The only graphs G' of order at most 2 with corank 2 and all vertices of degree at least 3 are (i) two loops bonded at a vertex, or (ii) two vertices joined by 3 edges. This implies that G must either be (i) two cycles bonded at a vertex, or (ii) a *theta graph* consisting of two vertices x and y joined by three internally disjoint paths, say of lengths l_1, l_2 and l_3, each of cardinality at least 1. We can ignore the first case as it has a cut vertex, so we only focus on the remaining case, (ii).

As there are no bridges, $H_1 = n - 1$. As the only subsets of two edges whose removal leaves G disconnected are two edges in one of the three internally disjoint paths, we see that

$$F_2 = \binom{n+1}{2} - \binom{l_1}{2} - \binom{l_2}{2} - \binom{l_3}{2}.$$

It follows (by considering an interval partition of $\mathrm{Cog}(G)$) that $F_2 = H_2 + H_1 + H_0 = H_2 + n$, so

$$\begin{aligned} H_2 &= F_2 - n \\ &= \binom{n+1}{2} - \binom{l_1}{2} - \binom{l_2}{2} - \binom{l_3}{2} - n \\ &= \binom{n}{2} - \binom{l_1}{2} - \binom{l_2}{2} - \binom{l_3}{2}. \end{aligned}$$

As some $l_i \geq 2$, we have that $H_2 \leq \binom{n}{2} - 1$. Now since the corank is 2, the H-polynomial will be

$$h(G, q) = H_2 q^2 + (n-1)q + 1.$$

By the quadratic formula, the roots of this are

$$\frac{-(n-1) \pm \sqrt{(n-1)^2 - 4H_2}}{2H_2}.$$

We have two cases, depending on whether the roots are real or not.

First, if the roots are real, then $(n-1)^2 - 4H_2 \geq 0$, that is, $H_2 \leq (n-1)^2/4$. The root of smallest modulus is $\frac{-(n-1)+\sqrt{(n-1)^2-4H_2}}{2H_2}$, and this has modulus greater than $1/(n-1)$ if and only if

$$\frac{-(n-1) + \sqrt{(n-1)^2 - 4H_2}}{2H_2} < \frac{-1}{n-1}.$$

This holds if and only if $H_2^2 + (n-1)H_2 > 0$, which is clearly true as $H_2 > 0$. Secondly, if the roots of (3.5) are non-real, then $H_2 > (n-1)^2/4$. Both roots have the same modulus, and

$$\frac{\sqrt{(n-1)^2 + 4H_2 - (n-1)^2}}{2H_2} > \frac{1}{n-1}$$

is true provided that

$$H_2 < (n-1)^2.$$

However, from above, $H_2 \leq \binom{n}{2} - 1$, and $\binom{n}{2} - 1 < (n-1)^2$ as $n \geq 3$. Thus in this case there is no root of modulus $1/(n-1)$.

Thus, in conclusion, for connected graphs of order at least 3, the minimum modulus of a reliability root is $1/(n-1)$, and only occurs for a cycle of order n. Moreover, the only reliability root of this modulus is $-1/(n-1)$. □

3.3 Distinguishing the Class of Simple Graphs via Reliability

Loops in graphs clearly have no effect on reliability, but multiple edges do. However, throughout the literature, whatever nontrivial mathematical properties have been found to hold for reliability polynomials of graphs in general have also been shown to hold for simple graphs, and vice versa. For example:

- A graph G of order n and size m is said to be *(uniformly) optimal* for a family $\mathcal{F}_{n,m}$ of graphs if for any other graph H of the same order and size, $\mathrm{Rel}(G;q) \geq \mathrm{Rel}(H;q)$ for *all* $q \in [0,1]$. Let $\mathcal{G}_{n,m}$ and $\mathcal{S}_{n,m}$ denote the classes of all graphs of order n and size m, and the simple such graphs, respectively. Then the known values of n and m for which no optimal graphs have been discovered for $\mathcal{G}_{n,m}$ and $\mathcal{S}_{n,m}$ coincide [16].

- Reliability roots of small simple graphs, as well as for various families of graphs, were noted to be in the unit disk centered at the origin, leading to a well known conjecture that reliability roots always lie in this disk [14]. Settling the conjecture in the negative, Royle and Sokal [46] proved first that there are reliability roots of graphs with multiple edges outside the unit disk, and then that the same is true even for simple graphs.

So the question remains – does the allowance of multiple edges add anything to reliability of mathematical consequence (that is, does multiple edges change the behaviour of the roots)? Can we distinguish the class of simple graphs in an interesting way via reliability from the class of all graphs?

Our work in the previous section can help us to do so. We have seen that the rational reliability roots of graphs of order $n \geq 2$ is the set $\{-1, -1/2, \ldots, -1/(n-1), 1\}$. However, the examples provided that achieve these consists of a cycle of length l $(2 \leq l \leq n)$, followed by the sequential attachment of leaves – such a cycle has rational reliability root at $-1/(l-1)$ (and 1); these are simple graphs, except for when the cycle has length 2, corresponding to a root at -1.

There are indeed many ways to introduce multiple edges to achieve -1 as a reliability root. For example, take any connected graph G of order at least 2, simple or otherwise. For any positive integer k, we can form the graph G^k by replacing each edge by a bundle of k parallel edges. Then it is easy to see that

$$\mathrm{Rel}(G^k;q) = \mathrm{Rel}(G;q^k),$$

and it follows that for k even,

$$\mathrm{Rel}(G^k;-1) = \mathrm{Rel}(G;1) = 0.$$

A calculation of the rational reliability roots of small graphs (up to order 8) does not turn up -1 as a root. So is it possible that -1, while a reliability root of many graphs, is never a reliability root of a simple graph? So, if a graph has multiple edges, is it the only case where -1 can be a reliability root? In fact, this is exactly what we shall prove – the set of rational reliability roots distinguishes the class of simple graphs.

Theorem 3.3.1. *Suppose G is a connected graph. If every pair of adjacent vertices of G are joined by an odd number of edges, then -1 is not a root of the reliability polynomial of G.*

Proof. We first determine the sign of a reliability polynomial at -1 (that is, whether the polynomial at -1 is positive or negative). Let G be a graph, possibly with multiple edges (but without loops). Then as noted earlier, the H-form of $\mathrm{Rel}(G;q)$ is given by

$$\mathrm{Rel}(G;q) \quad = \quad (1-q)^{n-1}\sum_{i=0}^{m-n+1} H_i q^i, \tag{3.5}$$

where each H_i is a positive integer. As noted earlier, Brown and Colbourn [14] proved that the real reliability roots always lie in $[-1,0)\cup\{1\}$, so the sign of $\mathrm{Rel}(G;q)$ is constant on $(-\infty,-1)$. It follows from the H-form that the sign of $\mathrm{Rel}(G;q)$ to the left of -1 is $(-1)^{m-n+1}$, and by continuity, the sign at -1 is either $(-1)^{m-n+1}$ or $\mathrm{Rel}(G;-1)=0$.

We now proceed by induction (on the size m) to show that if every edge of a connected graph G of order n and size m is a bundle of odd size, then $\mathrm{Rel}(G;-1)\neq 0$. When $m=0$ or 1, the only choice for G is K_1 or K_2, with reliability polynomials 1 and $1-q$, respectively, and clearly -1 is not a root of either polynomial. So we assume $n\geq 3$, and the result holds for smaller size graphs.

If G has no cycles, then G, having only bundled edges of odd size, is a bundled tree, with reliability polynomial of the form $\prod_{i=1}^{n-1}(1-q^{k_i})$ where all the k_i are odd, and so $\mathrm{Rel}(G;-1)=2^{n-1}\neq 0$. Otherwise, G has an edge e (possibly bundled of odd size) that is in a cycle. Let $\overline{e}$ be the set of all edges parallel to e (that is, all edges with the same endpoints and set $k=|\overline{e}|$). Then, by the well known Factor Theorem for reliability (see, for example [23, page 13]),

$$\mathrm{Rel}(G;q) \quad = \quad q^k\cdot\mathrm{Rel}(G-\overline{e};q)+(1-q^k)\cdot\mathrm{Rel}(G\bullet\overline{e};q), \tag{3.6}$$

where we recall that $G - \overline{e}$ and $G \bullet \overline{e}$ are the graphs formed from G by deleting (in sequence) edges in $\overline{e}$, and contracting edges in $\overline{e}$. Clearly $G-\overline{e}$ is connected, has order n and size $m-k$, while $G \bullet \overline{e}$ is connected and has size $m-k$ as well, but has order $n-1$. Moreover, every pair of adjacent vertices in G are joined by an odd number of edges, and the same is true for $G - \overline{e}$. By the Factor Theorem (3.6),

$$\mathrm{Rel}(G;-1) \quad = \quad (-1)\cdot \mathrm{Rel}(G-\overline{e};-1) + 2\cdot \mathrm{Rel}(G\bullet \overline{e};-1), \tag{3.7}$$

From earlier, either the sign of $\mathrm{Rel}(G-\overline{e};-1)$ is $(-1)^{(m-k)-n+1}$ or $\mathrm{Rel}(G-\overline{e};-1)=0$; similarly, either the sign of $\mathrm{Rel}(G\bullet\overline{e};-1)$ is $(-1)^{(m-k)-(n-1)+1}$ or $\mathrm{Rel}(G\bullet e;-1)=0$. In other words, the signs $\mathrm{Rel}(G-\overline{e};-1)$ and $\mathrm{Rel}(G\bullet\overline{e};-1)$ will be opposite of each other, and so $(-1)\cdot\mathrm{Rel}(G-\overline{e};-1)$ and $2\cdot\mathrm{Rel}(G\bullet\overline{e};-1)$ have the same sign. It follows from (3.7) that

$$|\mathrm{Rel}(G;-1)| \quad = \quad |\mathrm{Rel}(G-\overline{e};-1)| + 2\cdot|\mathrm{Rel}(G\bullet\overline{e};-1)|,$$

and hence

$$|\mathrm{Rel}(G;-1)| \quad \geq \quad |\mathrm{Rel}(G-\overline{e};-1)|.$$

However, by induction on m, $\mathrm{Rel}(G-\overline{e};-1)\neq 0$. We conclude that

$$|\mathrm{Rel}(G;-1)| \quad > \quad 0,$$

and hence $\mathrm{Rel}(G;-1)\neq 0$, and we are done. □

This theorem has a very useful corollary which helps distinguish the class of simple graphs from the class of all graphs.

Corollary 3.3.1. *If -1 is a root of a reliability polynomial of a connected graph G, then G is not simple.*

There are still many open problems for rational reliability roots which we will address in Chapter 5. However, let us switch our focus to another type of reliability – the *two-terminal reliability.* We will see what the similarities and differences are between this other type of reliability and the all-terminal reliability.

Chapter 4

On Two Terminal Reliability

We have been focussing on the all terminal reliability, but there are two other main types of reliabilities: two-terminal reliability and K-terminal reliability. In this section, we will focus on two-terminal reliability of a graph G, which we denote by $\text{Rel}_2(G, s, t; q)$, which has been well studied (for instance, see [23, 36, 38, 41, 43]). In this setting, as was the case for the all terminal reliability, we have that vertices are always operational and that edges fail with probability $q \in [0, 1]$. However, we fix two vertices, s and t (which we call terminals), and only require that s and t remain connected when edges fail – not all vertices being connected.

As is the case with the all-terminal reliability, the two-terminal reliability is also a polynomial that can be given in various forms, the F-form and the H-form:

$$\text{Rel}_2(G, s, t; q) = \sum_{i=0}^{m-\mu} F_i q^i (1-q)^{m-i} \qquad \text{(F-Form)} \tag{4.1}$$

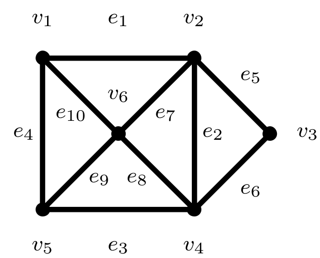
$$= (1-q)^{\mu} \sum_{i=0}^{m-\mu} H_i q^i \qquad \text{(H-Form).} \tag{4.2}$$

where μ is the length of the shortest (s, t)-path. Furthermore, we recall that we can also use the Factor Theorem that allows us to split the reliability into two parts via deletion and contraction of an edge e. That is,

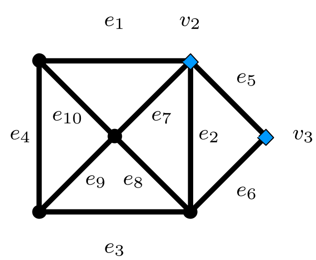
$$\text{Rel}_2(G, s, t; q) = q \cdot \text{Rel}_2(G - e, s, t; q) + (1-q) \cdot \text{Rel}_2(G \bullet e, s, t; q) \tag{4.3}$$

where $G - e$ and $G \bullet e$ are the graphs formed by deleting the edge e, and contracting edge e, respectively. We note that if the endpoints of e are s and t, then $\text{Rel}_2(G \bullet e, s, t; q) = 1$.

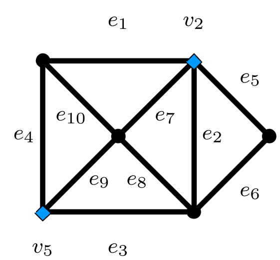
Indeed, just as we saw with the all-terminal reliability, the two-terminal reliability has an underlying simplicial complex. However, instead of F_i counting the number of ways one can remove i edges which leaves the graph connected, F_i counts the number of ways the removal of i edges leaves s and t in the same connected component. For example, consider the graph in Figure 4.1a. Let us compute its two-terminal reliability with various choices of terminals.



(a) Example graph for two-terminal reliability



(b) Two-terminal reliability with terminals v_2, v_3



(c) Two-terminal reliability with terminals v_2, v_5

Figure 4.1: Graph with two choices of terminals

Our first choice is setting $\{s,t\} = \{v_2, v_3\}$ (see Figure 4.1b, terminals illustrated as blue squares). The facets of the two-terminal complex (which are the complements of the (s,t)-paths) are given in the table below.

Path	**Facet (written multiplicatively)**
e_5	$e_1e_2e_3e_4e_6e_7e_8e_9e_{10}$
e_2e_6	$e_1e_3e_4e_5e_7e_8e_9e_{10}$
$e_6e_7e_8$	$e_1e_2e_3e_4e_5e_9e_{10}$
$e_3e_6e_7e_9$	$e_1e_2e_4e_5e_8e_{10}$
$e_1e_3e_4e_6$	$e_2e_5e_7e_8e_9e_{10}$
$e_1e_6e_8e_{10}$	$e_2e_3e_4e_5e_7e_9$
$e_1e_3e_6e_9e_{10}$	$e_2e_4e_5e_7e_8$
$e_1e_4e_6e_8e_9$	$e_2e_3e_5e_7e_{10}$
$e_3e_4e_6e_7e_{10}$	$e_1e_2e_5e_8e_9$

From here, one can compute its F-vector and H-vector to be

$$\langle 1, 10, 44, 112, 180, 184, 113, 44, 10, 1\rangle$$

and

$$\langle 1, 1, 0, 0, -2, -2, 1, 9, -10, 3\rangle,$$

respectively. From the example, we make the observation that, in general, the coefficients of the H-vector are not always all non-negative (as was the case for the all-terminal reliability). As a result, some methods that we've used before – for instance the Eneström-Kakeya Theorem – are unavailable for two-terminal reliabilities in the same manner that we have used it before (that is, we may still use the theorem with respect to the F-polynomial as the F-vector is non-negative, and then use the linear transformation q goes to $q/(1-q)$, but we are not able to use it directly on the H-polynomial). Returning back to our example, its two-terminal reliability polynomial is:

$$\begin{aligned}\mathrm{Rel}_2(G, v_2, v_3; q) &= (1-q)^{10} + 10q(1-q)^9 + 44q^2(1-q)^8 + 112q^3(1-q)^7 + \\ &\quad 180q^4(1-q)^6 + 184q^5(1-q)^5 + 113q^6(1-q)^4 + \\ &\quad 44q^7(1-q)^3 + 10q^8(1-q)^2 + q^9(1-q) \\ &= (1-q)(1 + q - 2q^4 - 2q^5 + q^6 + 9q^7 - 10q^8 + 3q^9).\end{aligned}$$

However, the behaviour of the polynomial changes if we choose a different set of terminals. For example, suppose we choose $\{s,t\} = \{v_2, v_5\}$ instead (see Figure 4.1c). Then the facets of the two-terminal complex are:

Path	**Facet (written multiplicatively)**
e_1e_4	$e_2e_3e_5e_6e_7e_8e_9e_{10}$
e_2e_3	$e_1e_4e_5e_6e_7e_8e_9e_{10}$
e_7e_9	$e_1e_2e_3e_4e_5e_6e_8e_{10}$
$e_1e_9e_{10}$	$e_2e_3e_4e_5e_6e_7e_8$
$e_2e_8e_9$	$e_1e_3e_4e_5e_6e_7e_{10}$
$e_3e_5e_6$	$e_1e_2e_4e_7e_8e_9e_{10}$
$e_3e_7e_8$	$e_1e_2e_4e_5e_6e_9e_{10}$
$e_4e_7e_{10}$	$e_1e_2e_3e_5e_6e_8e_9$
$e_1e_3e_8e_{10}$	$e_2e_4e_5e_6e_7e_9$
$e_2e_4e_8e_{10}$	$e_1e_3e_5e_6e_7e_9$
$e_5e_6e_8e_9$	$e_1e_2e_3e_4e_7e_{10}$
$e_4e_5e_6e_8e_{10}$	$e_1e_2e_3e_7e_9$

Once again, one can compute its F-vector and H-vector to be

$$\langle 1, 10, 45, 119, 199, 204, 110, 29, 3\rangle$$

and

$$\langle 1, 2, 3, 3, -1, -8, -5, 12, -4\rangle,$$

respectively. Therefore, its two-terminal reliability polynomial is

$$\begin{aligned}
\mathrm{Rel}_2(G, v_2, v_5; q) &= (1-q)^{10} + 10q(1-q)^9 + 45q^2(1-q)^8 + 119q^3(1-q)^7 + \\
&\quad 199q^4(1-q)^6 + 204q^5(1-q)^5 + 110q^6(1-q)^4 + \\
&\quad 29q^7(1-q)^3 + 3q^8(1-q)^2 \\
&= (1-q)^2(1 + 2q + 3q^2 + 3q^3 - q^4 - 8q^5 - 5q^6 + 12q^7 - 4q^8).
\end{aligned}$$

We can see some similarities between these two two-terminal reliabilities if we plot them both on the interval $[0, 1]$ (see Figure 4.2). First, we can see that they both have an inverted S-shape to their plots. However, $\mathrm{Rel}_2(G, v_2, v_3; q)$ seems to be less reliable when q is smaller than approximately 0.5493773811, and more reliable when q is greater. We can also see that there are no real roots in the interval $[0, 1)$. However, in comparison to the all-terminal reliability, there may be irrelevant edges. We say that an edge not on any (s, t)-path is called an **irrelevant edge**, and the collection of all irrelevant edges we call the **tendril** of G, which we denote by T (see Figure 4.3 for an example of a graph with a tendril). Unlike the all-terminal reliability where all vertices need to be able to communicate with each other, it is easy to see that a tendril does not affect the two-terminal reliability (since, if an edge $e \in T$ is not on a path between s and t, then its operability does not affect the two-terminal reliability). Therefore, we can remove the edges of the tendril, and any resulting isolated vertex.

Beyond the behaviour of the two-terminal reliability in $[0, 1]$, we can also study other analytic properties (such as was done in Cox's PhD Thesis [26] for inflection points, fixed points, and average reliability), as has been done for the all-terminal reliability. In particular, what can we say about the roots of the two-terminal reliability polynomial? For the previous two examples, the roots of largest moduli are approximately the complex numbers $1.4217024525 \pm 0.1830922276i$ with moduli 1.4334436254, and $(1 + \sqrt{5})/2 \approx 1.6180339887$, respectively (see Figure 4.4 for the plots of all their reliability roots). Our task now will be on the location of the roots of two-terminal reliabilities.

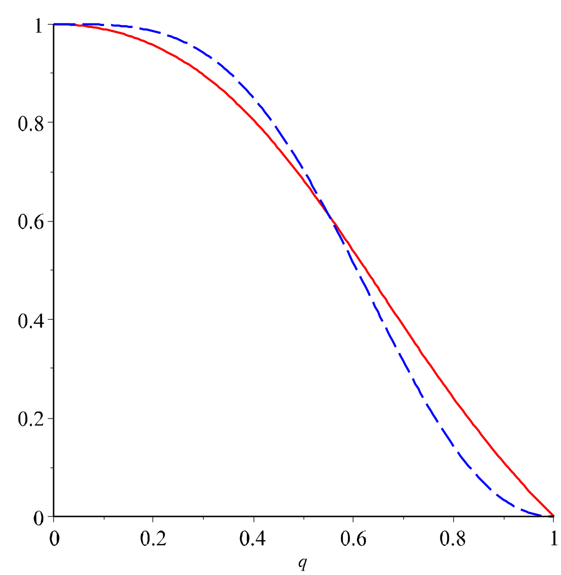


Figure 4.2: Plot of both $\mathrm{Rel}_2(G, v_2, v_3; q)$ (red solid line) and $\mathrm{Rel}_2(G, v_2, v_5; q)$ (blue dashed line) with $q \in [0, 1]$

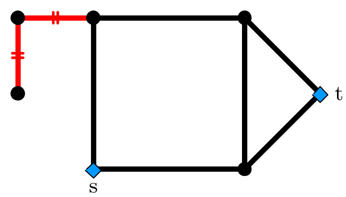


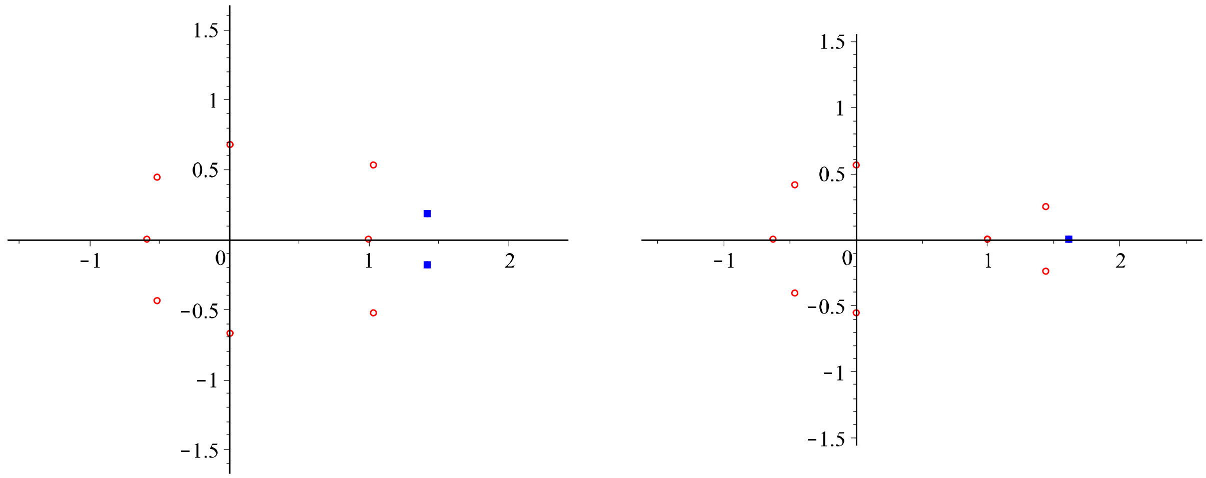
Figure 4.3: Example of a graph with terminals s and t (blue squares) with a tendril (red lines with two bars)

4.1 Roots of Two-Terminal Reliability

The area of studying the roots of two-terminal reliability was pioneered by Tanguy [55, 56] who was able to calculate exact roots for certain families of graphs including recursive families of graphs whose underlying graphs are undirected, which were then extended to recursive graphs whose underlying graphs are directed. However, much is still left unknown, including any potential Brown-Colbourn-like conjecture about a disk that contains the roots.

The first area we would like to explore is whether a root of two-terminal reliability polynomials can have modulus larger than 1. In [56], Tanguy showed that there are roots for the double-fan of 150 vertices with modulus approximately 1.4 (a *fan* on n vertices is a graph for which there is a path on n vertices, all vertices of which are connected to an extreme vertex. A *double fan* on n vertices is the graph where we take two fans on n vertices and identify the paths – see Figure 4.5 for an example of a double fan on 4 vertices). So, we can get roots outside of the disk centred at 0, but how far can we go?

Recall that the two-terminal reliability roots with maximum modulus for the examples in Figure 4.1, were approximately $1.4217024525 \pm 0.1830922276i$ with moduli 1.4334436254, and $(1+\sqrt{5})/2 \approx 1.6180339887$, respectively. One observation we can see is that by changing the choice of terminals, we obtained a root with larger



(a) Two-terminal reliability roots of $\mathrm{Rel}_2(G, v_2, v_3; q)$.

(b) Two-terminal reliability roots of $\mathrm{Rel}_2(G, v_2, v_5; q)$.

Figure 4.4: Plot of the two-terminal reliability roots of the examples from Figure 4.1. Blue squares indicate roots of largest moduli.

modulus. So, the choice of terminals is important.

To examine this further, let us revisit the examples we saw in Chapter 1 with the various families of graphs, but this time we can choose the terminals.

Example 1.0.3 *Tree Graphs*

To calculate $\mathrm{Rel}_2(G, s, t; q)$ for G a tree on n vertices, we first need to choose our terminals. We note that since there will only ever be one path between s and t (as there are no cycles), this will be equivalent to studying the two-terminal reliability of a path (since every edge of that path must be up, and the collection of edges that are not on that path is the tendril, and it has no effect on the probability). Therefore, if there is a path on μ edges between s and t, then its two-terminal reliability has the form

$$\mathrm{Rel}_2(G, s, t; q) = (1 - q)^{\mu},$$

which clearly has all of its roots at 1 (and hence in the closed unit disk).

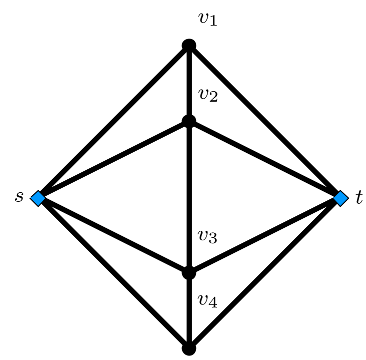


Figure 4.5: Double-fan on four vertices with s and t the blue squares

Example 1.0.2 *Cycle Graphs*

Our next example is the cycle graph. Here, regardless of the terminals chosen, there will always be two paths between s and t. Suppose we have m edges, with m_1 edges on one path P_1 between s and t, and m_2 edges on the other path P_2 (and so $m_1 + m_2 = m$). Then we have the two-terminal reliability polynomial

$$\begin{aligned}
\mathrm{Rel}_2(C_m, s, t; q) &= \mathrm{Prob}(P_1 \text{ or } P_2 \text{ operational}) \\
&= \mathrm{Prob}(P_1 \text{ is operational}) + \mathrm{Prob}(P_2 \text{ is operational}) - \\
&\quad\ \mathrm{Prob}(P_1 \text{ and } P_2 \text{ are operational}) \\
&= (1-q)^{m_1} + (1-q)^{m_2} - (1-q)^{m_1+m_2}.
\end{aligned}$$

For m_1 and m_2 running between 1 and 30 each, we find that the two-terminal reliability polynomial with the root of largest modulus – $1 + \sqrt{2} \approx 2.4142135623$ – occurs for C_4 with antipodal terminals (a plot of the roots for all the aforementioned choices of m_1 and m_2 can be found in Figure 4.6). This is a two-terminal reliability root that's quite far out from the unit disk centred at 0 (compared to the reliability roots we have seen outside of the unit disk in the all-terminal case).

Example 1.0.4 *(Generalized) Theta Graphs*

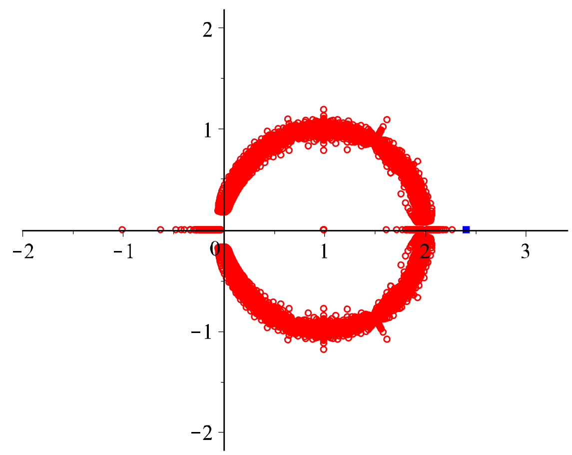


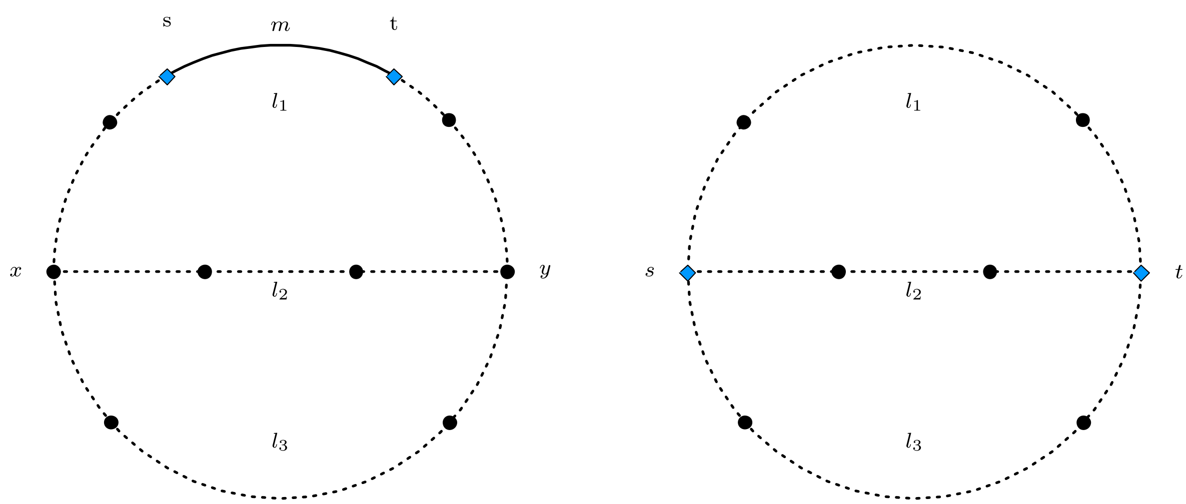
Figure 4.6: Plot of all $\text{Rel}_2(C_m, s, t; q)$ with root of maximum modulus highlighted (blue square) with $m = m_1 + m_2$ where m_1 and m_2 are between 1 and 30

With theta graphs there are even more choices for the terminals. For instance, suppose we have the theta graph Θ_{l_1,l_2,l_3}. Then, if we choose s and t to be in the same branch, say of length l_1, with a path of length m between s and t (see Figure 4.7a), then

$$\begin{aligned}\text{Rel}_2(\Theta_{l_1,l_2,l_3}, s, t; q) &= (1-q)^m + \\ &\quad (1-(1-q)^m)\big[(1-q)^{l_2+(l_1-m)} + (1-q)^{l_3+(l_1-m)} - \\ &\quad (1-q)^{l_1+l_2+l_3-m}\big].\end{aligned}$$

That is, either the path in the branch between s and t is operational (with probability $(1-q)^m$), or it has failed (with probability $1-(1-q)^m$), and then either the path from s to x, x to y along the path of length l_2, and then y to t is operational (with probability $(1-q)^{l_2+(l_1-m)}$), or the path from s to x, x to y along the path of length l_3, and then y to t is operational (with probability $(1-q)^{l_3+(l_1-m)}$). Finally, we need to exclude the probability that both the paths from x to y are operational.

Running l_1, l_2, and l_3 between 1 and 10, and letting m be any distance between 1 and l_1, we can get a two-terminal reliability root with modulus 2.5630958996 with $l_1 = 3$, $l_2 = l_3 = 2$, and $m = 2$ (i.e., s and t are vertices of distance 2 apart in the



(a) Θ_{l_1,l_2,l_3} with terminals in the same independent branch

(b) Θ_{l_1,l_2,l_3} with terminals the extreme points

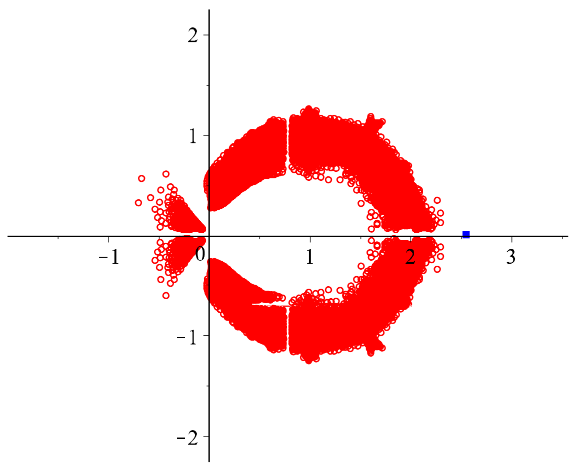
Figure 4.7: Example of two choices of terminals for $\text{Rel}_2(\Theta_{l_1,l_2,l_3}, s, t; q)$.

first branch of length 3, with one being an extreme point). See a plot of the roots for all aforementioned choices of l_1, l_2, l_3, and m in Figure 4.8a.

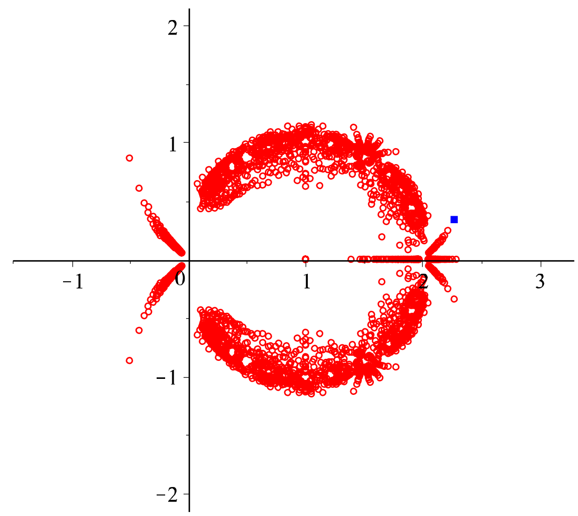
If we restrict s and t to be the extreme points, then the formula can be written as

$$\begin{aligned}\text{Rel}_2(\Theta_{l_1,l_2,l_3}, s, t; q) &= (1-q)^{l_1} + (1-q)^{l_2} + (1-q)^{l_3} \\ &- ((1-q)^{l_1+l_2} + (1-q)^{l_1+l_3} + (1-q)^{l_2+l_3}) \\ &+ (1-q)^{l_1+l_2+l_3}\end{aligned}$$

This is calculated by inclusion/exclusion on each path being operational; that is, either one path is up, excluding the cases where exactly two paths are up, and then including the case where all three paths are up. Letting l_1, l_2, and l_3 run between 1 and 10 again results in a root approximately $2.2712298784 + 0.3406250193i$ with maximum modulus 2.2966302629 with $l_1 = l_2 = l_3 = 2$ (see a plot of all roots of these choices in Figure 4.8b). Though this is a smaller maximum modulus than before, this form is easier to study. It is also interesting to see that, in this case, the root with maximum modulus is not real.



(a) Plot of complex roots for terminal choices in the same branch



(b) Plot of complex roots for terminals the extreme points

Figure 4.8: Plot of roots for two choices of terminals, with the length of the branches ranging from 1 to 10.

Since the best case in this example seems to appear when $l_1 = l_2 = l_3$, perhaps we should restrict our class of theta graphs to the case where $l_1 = l_2 = l_3 = l$. If we do this, then the above becomes

$$\text{Rel}_2(\Theta_{l,l,l}, s, t; q) = 3(1-q)^l - 3(1-q)^{2l} + (1-q)^{3l}$$

which can be simplified to

$$\text{Rel}_2(\Theta_{l,l,l}, s, t; q) = 1 - (1 - (1-q)^l)^3.$$

The idea is that $(1-q)^l$ is the probability that every edge along one path is operational, so $1-(1-q)^l$ is the probability that at least one edge in a particular path has failed. This is true for all three branches. Finally, if we take this probability away from 1, then we get the probability that there is at least one branch operational which is equivalent to s and t being able to communicate. By once again running this for l at least 1, we get that the maximum modulus occurs when $l = 2$. Can we get better by considering generalized theta graphs?

Extending our formula to $\Theta_{l[k]}$ is straightforward; i.e.,

$$\mathrm{Rel}_2(\Theta_{l[k]}, s, t; q) = 1 - (1 - (1 - q)^l)^k.$$

Letting l and k run from 1 to 10, we find that the root with maximum modulus of $1 + \sqrt{2} \approx 2.4142135624$ occurs when $l = 2$ and $k = 6$. The roots are shown in Figure 4.9.

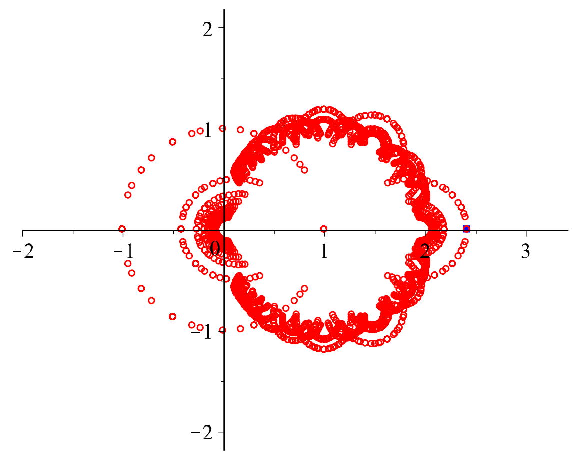


Figure 4.9: Plot of the roots of $\Theta_{l[k]}$ with l and k ranging between 1 and 10.

Example 1.0.5 *Complete Graphs*

Just like in the all-terminal case, we do not have an explicit general formula for the two-terminal reliability of a graph, but we do have a recursive formula (see [23], page 33) which utilizes the all-terminal reliability:

$$\mathrm{Rel}_2(K_n; q) = \sum_{j=2}^{n} \binom{n-2}{j-2} \mathrm{Rel}(K_n; q) q^{j(n-j)}.$$

As we can see, we do not need to specify the terminals for the two-terminal reliability of a complete graph. That is because the choice of s and t are not important as every choice yields an isomorphic two-terminal graph; that is, there is a graph automorphism that carries any two terminals to any other two terminals. Below is a table of the two-terminal reliability polynomials for the complete graphs on 2 to 6 vertices, followed by Figure 4.10 which plots the roots of the complete graphs of orders 2 to 25 (with the largest root a blue square). We find that the root with the largest modulus occurs at approximately 1.6180339887, for K_3.

n	$\mathbf{Rel}_2(K_n; q)$
2	$1 - q$
3	$(1-q)(-q^2+q+1)$
4	$(1-q)(2q^5-3q^4-q^3+q^2+q+1)$
5	$(1-q)(-6q^9+12q^8-7q^6-q^5-q^4+q^3+q^2+q+1)$
6	$(1-q)(24q^{14}-60q^{13}+18q^{12}+38q^{11}-6q^{10}-$ $6q^9-9q^8-q^7-q^6-q^5+q^4+q^3+q^2+q+1$

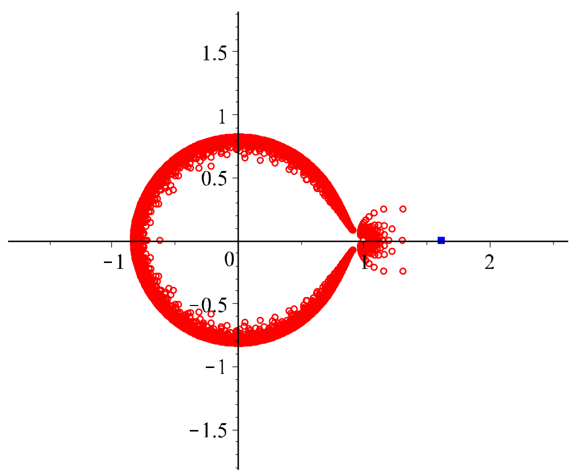


Figure 4.10: Root of two-terminal reliabilities of complete graphs of order 2 to 25.

As we can see, it is quite easy to find roots outside of the closed unit disk for two-terminal reliability polynomials. This begs the question about when all two-terminal reliability roots fall inside of the unit disk.

4.1.1 Connection to Matroids

One of the most important tactics that we have at our disposal when trying to analyze the location of the roots of the all-terminal reliability polynomial is knowing the structure of the underlying simplicial complex, or more specifically, the cographic matroid. This has allowed us to obtain information about the H-polynomial which we could then use to help in determining where the roots lie. However, there is very little known about the structure of the underlying simplicial complex for two-terminal reliabilities, and, unfortunately, it seems very rare that the simplicial complex is indeed a matroid.

We say that, for a graph G with terminals s and t connected by a path (as we shall assume henceforth), the **two-terminal complex of G**, which we denote by $\mathcal{C}_{s,t}(G)$, is the complex whose faces are the sets of edges whose removal leaves s and t connected. We note that G may contain some irrelevant edges which can always be removed, and in the complex they form a simplex and thus $\mathcal{C}_{s,t}(G)$ can be written as a join

$$\mathcal{C}_{s,t}(G) = \Delta_T(G) * \mathcal{C}_{s,t}(G)',$$

where $\Delta_T(G)$ is the simplex generated by the tendril T of G, and $\mathcal{C}_{s,t}(G')$ is the two-terminal complex on $G' = G - T$.

In general, $\mathcal{C}_{s,t}(G)$ is just a complex and not a matroid. For instance, consider $G = C_n$ with $n \geq 3$ and s and t adjacent. Then $\mathcal{C}_{s,t}(C_n)$ is a connected (it cannot be written as a join of two complexes), non-pure complex (and hence is not a matroid). However, when is $\mathcal{C}_{s,t}(G)$ a matroid? Moreover, if $\mathcal{C}_{s,t}(G)$ is a matroid, is it a different class of matroids than the cographic matroid? In order to answer this, we will first need a very useful lemma.

Lemma 4.1.1. (*Join of Matroids)*
The complex $\mathcal{C} = \mathcal{C}_1 * \mathcal{C}_2$ is a matroid if and only if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are matroids.

Proof. The implication that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are matroids then $C = \mathcal{C}_1 * \mathcal{C}_2$ is a matroid follows from a result by Oxley [44, Prop. 4.2.8, p. 124]. Therefore, we just need to show that if $\mathcal{C}$ is a matroid, then both $\mathcal{C}_1$ and $\mathcal{C}_2$ must be as well.

For a contradiction, let us suppose that, without loss of generality, $\mathcal{C}_1$ is not a matroid but instead just a complex. We want to show that $C = \mathcal{C}_1 * \mathcal{C}_2$ is not a matroid (regardless if $\mathcal{C}_2$ is a matroid or not). As $\mathcal{C}_1$ is not a matroid, there exist faces τ_1 and τ_2 of C_1 such that $|\tau_1| > |\tau_2|$ but there is no $x \in \tau_1 \setminus \tau_2$ such that $\tau_2 \cup \{x\} \in \mathcal{C}_1$. But then $\alpha_1 = \tau_1 \cup \emptyset$ and $\alpha_2 = \tau_2 \cup \emptyset$ are faces of $\mathcal{C}_1 * \mathcal{C}_2$. It follows that

$$|\alpha_1| = |\tau_1| > |\tau_2| = |\alpha_2|$$

and there is no $x \in \alpha_1 \setminus \alpha_2 = \tau_1 \setminus \tau_2$ such that $\alpha_2 \cup \{x\} = \tau_2 \cup \{x\} \in \mathcal{C}_1 * \mathcal{C}_2$ (as the vertices of τ_2 and τ_1 are in $\mathcal{C}_1$, and hence $\tau_2 \cup \{x\}$, for any $x \in \alpha_1 \setminus \alpha_2 = \tau_1 \setminus \tau_1$, would have to be a face of $\mathcal{C}_1$ to be a face of $\mathcal{C}_1 * \mathcal{C}_2$). Therefore, $\mathcal{C}$ is not a matroid. □

Before we move onto the main theorem of the section, we first need to state a very important property of matroids, which we formulate into a lemma (see, for instance, [44, Lemma 1.2.2, p. 15]).

Lemma 4.1.2. (Basis Exchange Axiom)
Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be the collection of facets of a complex $\mathcal{C}$. Then $\mathcal{C}$ is a matroid if and only if whenever σ_i and σ_j are members of σ and $x \in \sigma_i \setminus \sigma_j$, there is an element $y \in \sigma_j \setminus \sigma_i$ such that $(\sigma_i \setminus \{x\}) \cup \{y\} \in \sigma$ (if $\mathcal{C}$ is a complex, then we call each σ_i a ***basis****).*

From here we can prove the main theorem of this section.

Theorem 4.1.1. *The two-terminal reliability complex of a graph $\mathcal{C}_{s,t}(G)$ is a matroid if and only if it is a bundled (s,t)-path, possibly with irrelevant edges.*

Proof. Let $P_n^{\mathbf{k}}$ be a path on n vertices, possibly with additional irrelevant edges, with edges bundled by $\mathbf{k} = [k_1, k_2, \ldots k_{n-1}]$ edges (see an example of $P_n^{\mathbf{k}}$ for $n = 5$ and $\mathbf{k} = [1, 2, 3, 1]$ in Figure 4.11).

As we saw earlier, we can split $\mathcal{C}_{s,t}(G)$ into a direct sum of two complexes: $\Delta_T(G)$, the complex generated by the tendril T of G, and $\mathcal{C}_{s,t}(\widehat{G})$, the two-terminal complex on $\widehat{G} = G - T$. We also saw earlier that $\Delta_T(G)$ is a simplex (and hence a matroid). Moreover, it is easy to see that $\mathcal{C}_{s,t}(P_n^{\mathbf{k}})$ =Cog$(P_n^{\mathbf{k}})$, as they consist of the same faces (the removal of edges of $P_n^{\mathbf{k}}$ leaves an (s,t)-path if and only if they leave a spanning connected subgraph). Therefore, by Lemma 4.1.1, $\mathcal{C}_{s,t}(G)$ is a matroid.

Now, we will consider the case when the two-terminal complex is a matroid. We shall rely on the notion of the dual $\mathcal{M}^*$ of a matroid M, whose bases are the complements (with respect to the ground set) of the bases of $\mathcal{M}$ – see, for example, Chapter 2, page 64 in Oxley's textbook [44]. We shall also make use of Lemma 4.1.2, the Basis Exchange Axiom, of matroids. $\mathcal{C}_{s,t}(G)$ can be split as $\mathcal{C}_{s,t}(G) = \Delta_T(G) * \mathcal{C}_{s,t}(\widehat{G})$,

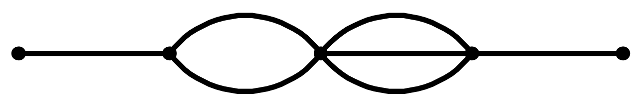


Figure 4.11: Example of the bundled path $P_5^{\mathbf{k}}$ with $\mathbf{k} = [1, 2, 3, 1]$

where T is the tendril of G, and $\widehat{G}$ is the subgraph $G - T$. Δ_T is a simplex as it corresponds to irrelevant edges. From Lemma 4.1.1, $\mathcal{C}_{s,t}(\widehat{G})$ is a matroid, so without loss of generality, we shall assume that $G = \widehat{G}$ and $\mathcal{C}_{s,t}(G) = \mathcal{C}_{s,t}(\widehat{G})$.

Since the dual of a matroid is a matroid, $\mathcal{C}^*_{s,t}(G)$ is a matroid. Recall that the facets of $\mathcal{C}_{s,t}(G)$ are edges whose removal leaves an (s,t)-path, and so the facets of $\mathcal{C}^*_{s,t}(G)$ are precisely the edges of those paths between s and t. Let us fix a path P_1 such that P_1 is the path $s = v_1, v_2, \ldots, v_k = t$. Since $\mathcal{C}^*_{s,t}(G)$ is a matroid, all (s,t)-paths contain exactly k vertices (i.e. exactly $k-1$ edges). Suppose we have a different (s,t)-path P_2 such that P_2 is the path $s = u_1, u_2, \ldots, u_k = t$. We will show that $u_i = v_i$ for all i.

Suppose – to reach a contradiction – that $u_i \neq v_i$ for some i. Then consider the first i such that u_i is different from v_i (clearly $2 \leq i < k$). Consider the next u_j that is again on the path P_1 (there must be one as $u_k = v_k = t$), say $u_j = v_r$ (so $j > i$). Clearly $r > i$. Moreover, $j = r$ as otherwise we can shorten one of the paths to find an (s,t)-path that is shorter than P_1 and P_2, a contradiction. Now, consider the (s,t)-path P_3 given by

$$s = v_1, v_2, \ldots, v_{i-1}, u_i, u_{i+1}, \ldots, u_j = v_j, v_{j+1}, \ldots, v_k = t.$$

Then take the edge $x = v_{i-1}v_i$, which belongs to P_1 but not P_3. Since $\mathcal{C}^*_{s,t}(G)$ is a matroid, by the Basis Exchange Axiom, there must be a $y \in P_3$ with $y \notin P_1$ such that

$(P_1 \setminus \{x\}) \cup \{y\}$ is an (s,t)-path. However, the only possible way for this to occur is if both x and y share the same endpoints; i.e., is a part of a bundled edge. Therefore, the ends of y must be $v_{i-1} = u_{i-1}$ and v_i, a contradiction, as the only possible edge for y in P_3 is $u_{i-1}u_i$, but u_i is not in P_1 (and hence cannot be v_i). Thus the vertices of P_2, in order, must also be $v_1, v_2, ..., v_k$, and this is for any path different from P_1. It follows that the edges in the (s,t)-paths are in bundles with the edges of P_1. i.e. the relevant edges (that are in some (s,t)-path) form a bundled path between s and t. □

With this theorem, we are not only able to classify all graphs whose two-terminal reliability complex is a matroid, but we can also show that they satisfy the essence of the Brown-Colbourn conjecture – having reliability roots falling inside of the closed unit disk.

Corollary 4.1.1. *All two-terminal reliability roots of a graph whose two-terminal reliability complex is a matroid falls inside of the closed unit disk.*

Proof. Since $C_{s,t}(G)$ is a matroid, then we know that G is a bundled path with s and t being the endpoints of the path, possibly with additional irrelevant edges. Suppose the bundled path is of length n with bundle sizes $\mathbf{k} = [k_1, k_2, \ldots, k_n]$. As the tendril does not affect the reliability, the two-terminal reliability polynomial is

$$\mathrm{Rel}_2(G, s, t; q) = (1 - q^{k_1})(1 - q^{k_2}) \ldots (1 - q^{k_n})$$

which has all of its roots inside of the closed unit disk. □

Therefore, we have been able to classify graphs whose two-terminal complexes are a matroid, and in fact, all such matroids are cographic matroids. Indeed, recall that if $\mathcal{C}_{s,t}(G)$ is a matroid, then it can be written as $\Delta_T(G) * \mathcal{C}_{s,t}(\widehat{G})$ where $\Delta_T(G)$ is the simplex of tendrils, and $\mathcal{C}_{s,t}(\widehat{G})$ is the two-terminal complex of $\widehat{G} = G - T$. One can clearly see that $\mathcal{C}_{s,t}(\widehat{G})$ =Cog($\widehat{G}$). What about $\Delta_T(G)$? It is easy to see that $\Delta_T(G)$ is the cographic matroid of the graph G_0, where G_0 has only a single vertex with $|T|$ loops (since they are loops, one can remove any subset of them and the graph is still "connected"). Therefore, $\Delta_T(G) * \mathcal{C}_{s,t}(\widehat{G})$ is a direct sum of two cographic matroids, which is itself a cographic matroid (in fact, it is the cographic matroid of $\widehat{G}$ with some loops attached).

Therefore, if the two-terminal complex of a graph G is a matroid, then its matroid is a cographic matroid, and all of their two-terminal reliability roots are inside of the closed unit disk. However, this is a small family of graphs (and terminals), so can we determine a larger region of the complex plane that contain two-terminal reliability roots?

4.1.2 Regions of Density in the Complex Plane

With regards to the all-terminal reliability, we may not have that every root is inside of the closed unit disk, but we do have that the roots are dense in the closed unit disk. Is there a similar region in the complex plane for density of two-terminal reliability roots? Surprisingly, there are *two* disks (at least) in the closure of the two-terminal reliability roots.

Theorem 4.1.2. *The closure of two-terminal reliability roots contain the closed unit disks centred at* 0 *and* 1*.*

Proof. Let us revisit our formula for the two-terminal reliability polynomial of $\Theta_{l[k]}$:

$$\mathrm{Rel}_2(\Theta_{l[k]}; q) = 1 - (1 - (1 - q)^l)^k.$$

Let us compute the roots.

$$\begin{aligned}
&\mathrm{Rel}_2(\Theta_{l[k]}; q) = 1 - (1 - (1 - q)^l)^k = 0 \\
\iff &(1 - (1 - q)^l)^k = 1 \\
\iff &1 - (1 - q)^l = \omega \text{ (for } \omega \text{ some } k\text{th root of unity)} \\
\iff &(1 - q)^l = 1 - \omega \\
\iff &q = 1 - \nu \text{ (for } \nu \text{ some } l\text{th root of } 1 - \omega)
\end{aligned}$$

Let r and θ satisfy $0 < r < 1$ and $0 < \theta < 2\pi$, and let $\epsilon > 0$. such that $0 < r - \epsilon, r + \epsilon < 1, 0 < \theta - \epsilon$, and $\theta + \epsilon < 2\pi$. We will show that there is a ω, a kth root of unity, so that some ν, an lth root of $1 - \omega$, is in the small pie-shaped piece $\{Re^{i\gamma} | r - \epsilon < R < r + \epsilon \text{ and } \theta - \epsilon < \gamma < \theta + \epsilon\}$; this will show that the closure of $q = 1 - \nu$ is the unit disk centred at 1.

First, as the l arguments of the l-th roots of a number are equally spaced out, for all sufficiently large $l \geq L$, we can ensure that the argument γ of some l-th root of any nonzero number is in $[\theta - \epsilon, \theta + \epsilon]$.

Second, the k-th roots of 1, running over all k, fill up the boundary of the unit circle. So, if we consider 1 minus these values, the resulting complex numbers take on values whose moduli are close to every number in $[0, 2]$ as the unit circle gets shifted right by one unit.

We can choose a k-th root of ω such that $(r-\epsilon)^L < |1-\omega| < (r+\epsilon)^L$. Then there is an L-th root, say ν, of $1-\omega$ such that $r-\epsilon < |\nu| < r+\epsilon$, and the argument γ of ν lies in $[\theta-\epsilon, \theta+\epsilon]$. As noted earlier, this implies that the closure of the roots of two-terminal reliabilities of $\Theta_{l[k]}$ contain the closed unit disk centred at 1.

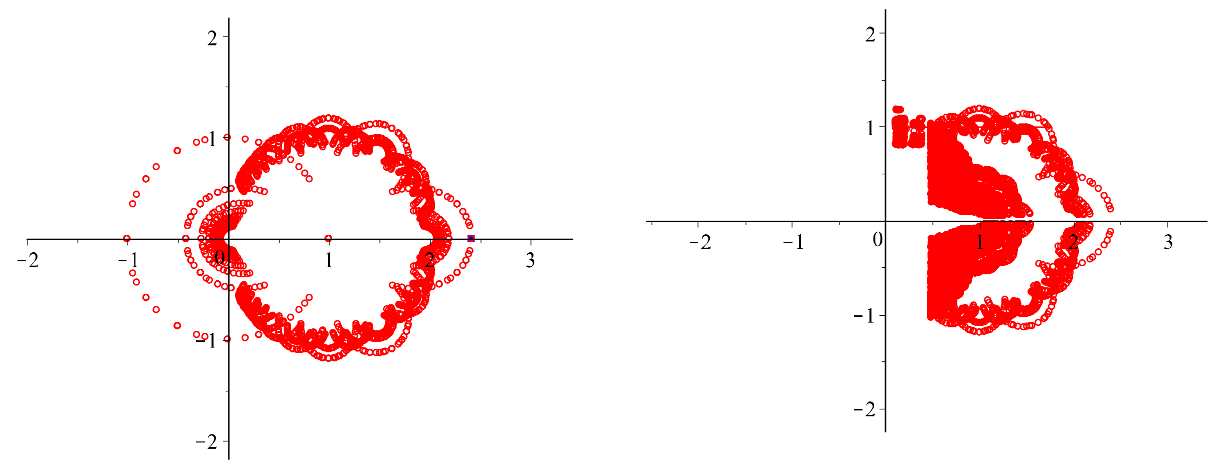
What about for the closed unit disk centred at 0? Let us consider the graph $G = \Theta_{l[k]}$ and replace every edge in G by a bundle of size m. The two-terminal roots of G are the mth roots of the two-terminal roots of $\Theta_{l[k]}$. Since we know that the closure of the two-terminal reliability roots of $\Theta_{l[k]}$ are dense in the unit disk centred at 1, we can choose roots whose moduli are close to every number in $[0, 1]$. Then, we repeat a process like the one above to get roots whose closure contains the closed unit disk centred at 0. □

To see a visual of this, see Figure 4.12 which shows the roots of $\Theta_{l[k]}$ for l and k between 1 and 10, and then shows the mth roots of those for m between 1 and 10.

4.1.3 Roots Outside of the Disks

We have just seen that the closure of roots of two-terminal reliability polynomials contains the unit disks centred at 0 and 1. We have also seen that there are graphs that have reliability roots outside the two unit disks. For example, $\mathrm{Rel}_2(C_4, s, t; q)$ with s and t antipodal (see Example 1.0.2 on page 79) which has a root at $1 + \sqrt{2} \approx 2.4142135624$. Various theta graphs (Example 1.0.4 on page 79) have roots 2.2966392629 and $1 + \sqrt{2}$.

There also exist trivial operations we can use to produce graphs whose roots are just as far out of the disks. For example, adjoining two graphs by identifying a terminal from one with the terminal of the other. If one has a root outside of the two unit disks, then since the resulting two-terminal reliability would just be the product of the individual two-terminal reliabilities, this new graph would also have a root outside of the close unit disks. However, are there any non-trivial operations we can use to produce other such roots?



(a) Plot of the two-terminal roots of $\Theta_{l[k]}$ with l and k from 1 to 10

(b) Plot of the mth roots of the two-terminal roots of $\Theta_{l[k]}$ with l and k from 1 to 10

Figure 4.12: Illustration of the closure of the roots inside the two unit disks centred at 0 and 1 by taking the mth roots of $\Theta_{l[k]}$

Path Addition

Consider the operation of taking a graph G with terminals s and t and adding a path of length k between s and t (the following process is illustrated in a sequence of figures in Figure 4.13). We would like to study the *limit of roots* of G. In [2], Beraha, Kahane, and Weiss defined a **limit of roots** of a family of polynomials $\{P_n\}$ to be a complex number z for which there exists a subsequence of integers (n_k) and complex numbers (z_{n_k}) such that z_{n_k}, is a root of P_{n_k}, and $z_{n_k} \to z$ as $k \to \infty$. The well known Beraha-Kahane-Weiss (BKW) Theorem states that if the sequence of polynomials (f_n) are of the form

$$f_n = \alpha_0 \lambda_0^n + \alpha_1 \lambda_1^n + \cdots + \alpha_m \lambda_l^n$$

with the α_i being polynomials and the λ_j being polynomials, then the limit of roots of f are precisely the complex numbers z such that

- one of the $|\lambda_i(z)|$ exceeds the others and $\alpha_i(z) = 0$, or
- at least two of $|\lambda_i(z)|$ are equal and bigger than the rest.

The BKW Theorem also requires two non-degeneracy conditions: no α_i is identically 0, and $\lambda_i \neq \omega\lambda_j$ for any $i \neq j$ and any root of unity ω.

Let us begin our sequence of functions by first taking $\text{Rel}_2(G, s, t; q)$ where G is any graph. Now, let's add a path between s and t of length 2 (we denote this by $G + P_2$). Using the Factor Theorem (3.6) on the first edge in the new path, we can calculate the two-terminal reliability as

$$\text{Rel}_2(G + P_2, s, t; q) = q\text{Rel}_2(G, s, t; q) + (1 - q)((1 - q) + q\text{Rel}_2(G, s, t; q)).$$

Indeed, if the first edge of the path fails, then the two-terminal reliability is the same as if there were no path (as the remaining edge becomes a tendril and so does not contribute to the reliability). If, however, that edge is operational, then we can contract that edge. We are now in the case where the remaining edge is either operational (and hence there is a path between s and t so it is always reliable), or the edge fails (in which case the reliability is the same as before). Let us try to simplify this formula so that we can use induction later.

$$\begin{aligned}\text{Rel}_2(G + P_2, s, t; q) &= q\text{Rel}_2(G, s, t; q) + (1 - q)((1 - q) + q\text{Rel}_2(G, s, t; q)) \\ &= q\text{Rel}_2(G, s, t; q)(1 + (1 - q)) + (1 - q)^2\end{aligned}$$

Let us now consider what happens if we add a path of length 3. Then

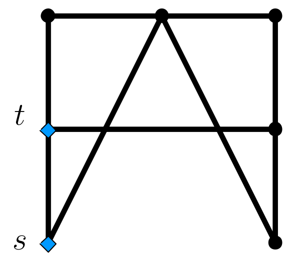
$$\begin{aligned}\text{Rel}_2(G + P_3, s, t; q) &= q\text{Rel}_2(G, s, t; q) + (1 - q)\text{Rel}_2(G + P_2, s, t; q) \\ &= q\text{Rel}_2(G, s, t; q) + (1 - q)\left[q\text{Rel}_2(G, s, t; q)(1 + (1 - q)) + (1 - q)^2\right] \\ &= q\text{Rel}_2(G, s, t; q)(1 + (1 - q) + (1 - q)^2) + (1 - q)^3 \\ &= q\text{Rel}_2(G, s, t; q)\frac{(1 - q)^3 - 1}{(1 - q) - 1} + (1 - q)^3 \\ &= \text{Rel}_2(G, s, t; q)(1 - (1 - q)^3) + (1 - q)^3\end{aligned}$$

Proceeding by induction, we can formulate the reliability of adding a path of length k ($k \geq 3$) as

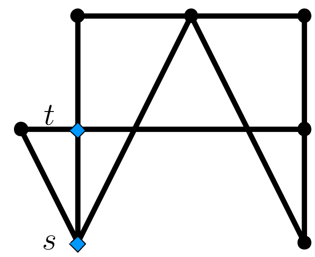
$$\begin{aligned}\text{Rel}_2(G + P_k, s, t; q) &= \text{Rel}_2(G, s, t; q)(1 - (1 - q)^{k+1}) + (1 - q)^{k+1} \\ &= \text{Rel}_2(G, s, t; q) \cdot 1^{k+1} + (1 - \text{Rel}_2(G, s, t; q)) \cdot (1 - q)^{k+1}.\end{aligned}$$

Now, let us set $\alpha_1 = \text{Rel}_2(G, s, t; q)$ and $\alpha_2 = 1 - \text{Rel}_2(G, s, t; q)$, as well as $\lambda_1 = 1$ and $\lambda_2 = (1 - q)$. Using the first part of the BKW Theorem, if for some q, $f_0(q) =$

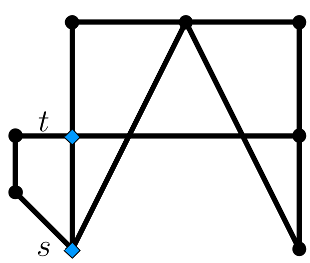
$\text{Rel}_2(G, s, t; q) = 1$, and $|1 - q| > 1$, then q is a limit of roots of $\text{Rel}_2(G, s, t; q)$. From searching all simple graphs of order 7, we are able to find a graph (see Figure 4.13a) which has a two-terminal reliability root at approximately 2.587039432. Not only this, but we have infinitely many graphs with roots tending to this (real) number.



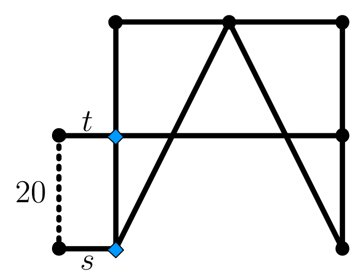
(a) Original graph G with a two-terminal reliability root approximately 2.5783146807



(b) Graph $G + P_2$ with with a two-terminal reliability root approximately 2.5923756643



(c) Graph $G + P_3$ with with a two-terminal reliability root approximately 2.5853431258



(d) Graph $G + P_{20}$ with with a two-terminal reliability root approximately 2.5870402488

Figure 4.13: Example of a sequence of simple graphs with a two-terminal reliability root outside of the closed unit disk centred at 1 using a path addition. Terminals are in blue squares.

Gadget Replacement

In [26, p. 56], and later expanded on in [19], Cox introduced the notion of a *gadget replacement*. The **gadget replacement** on a graph G with gadget H, denoted by $G[H]$, is the graph constructed by replacing every edge of G not by a bundle or a path, but instead by an entire new graph, H. This type of replacement requires us to fix two vertices in H, x and y, and then considers what happens if one replaces an edge $e = \{v_1, v_2\}$ in G with a copy of H, where we identify $v_1 = x$ and $v_2 = y$ (an example is illustrated in Figure 4.14). In the all-terminal case, edge e failing corresponds to x and y not being able to communicate, but every vertex in the copy of H being able to be connected to x or y but not both. Edge e being operational corresponds to all vertices in the copy of H being connected in H.

However, when we move to two-terminal reliability, things become simpler. Consider what happens if we replace not only one edge in G with another two-terminal graph H, but we replace *every* edge – how does the two-terminal reliability get affected? Operational edges in G correspond to their endpoints being able to communicate in H, and edges failing correspond to the endpoints of these edges not being able to communicate in H. Therefore, we can translate this by $(1-q) \mapsto \mathrm{Rel}_2(H, x, y, q)$ and $q \mapsto 1 - \mathrm{Rel}_2(H, x, y, q)$. Thus,

$$\mathrm{Rel}_2(G[H], s, t; q) = \mathrm{Rel}_2(G, s, t, (1 - \mathrm{Rel}_2(H, x, y, q))).$$

So, if r is a root of $\mathrm{Rel}_2(G, s, t; q)$, then any solution to

$$1 - \mathrm{Rel}_2(H, x, y, q) = r$$

or

$$\mathrm{Rel}_2(H, x, y, q) = 1 - r \tag{4.4}$$

is a two-terminal reliability root of $G[H]$ with terminals s and t.

Let us consider now what happens if we use repeated gadget replacements on all edges of a graph, say $G = C_4$, with gadget, say $H = C_4$ (see Figure 4.15 for an illustration of this process, with terminals in each antipodal). We can calculate the two-terminal reliability polynomial of C_4, with s and t antipodal, to be

$$\mathrm{Rel}_2(C_4, s, t; q) = 2(1-q)^2 - (1-q)^4.$$

Using (4.4) for any two-terminal reliability root r of G, we get two-terminal roots of $G[H]$ by solving

$$2(1-q)^2 - (1-q)^4 = 1 - r.$$

Solving for q we get as one of the roots of $C_4[C_4]$ with terminals s and t

$$q = 1 + \sqrt{1+\sqrt{r}}. \tag{4.5}$$

So if $r = 1+\sqrt{2}$, then $q = 1 + \sqrt{1+\sqrt{r}} \approx 2.5980531825$ is a root of $C_4[C_4]$ with antipodal vertices.

What happens if we iterate? We consider what happens in the limit by setting $f(r) = 1+\sqrt{1+\sqrt{r}}$ and considering only fixed points. Solving $f(r) = r+\sqrt{1+\sqrt{r}} = r$ results in the fixed point:

$$\begin{aligned}
& r = 1 + \sqrt{1+\sqrt{r}} \\
\iff & (r-1)^2 = 1 + \sqrt{r} \\
\iff & ((r-1)^2 - 1)^2 = r \\
\iff & (r-1)^4 - 2(r-1)^2 - (r-1) = 0 \\
\iff & (r-1)\left[(r-1)^3 - 2(r-1) - 1)\right] = 0
\end{aligned}$$

One of the roots of this is

$$r = \frac{3+\sqrt{5}}{2} \approx 2.6180339887.$$

Now, is this an attracting fixed point? By taking the derivative of $f(r)$ we get

$$f'(r) = \frac{1}{4\sqrt{\sqrt{r+1}}\sqrt{r}}.$$

We know that x is an attractive fixed point of f if $|f'(x)| < 1$ and is a repelling fixed point if $|f'(x)| > 1$ (see [51, pages 83-84] for a general discussion on fixed points, as well as proof of this result). Evaluating we get

$$|f'((3+\sqrt{5})/2)| \approx 0.0954915028 < 1.$$

Therefore, $r = (3+\sqrt{5})/2$ is an attractive fixed point. We can do some calculations to see that if we start with a root at least $r \approx 0.06$, then after iterating through (4.5) we will have a root tending to $(3+\sqrt{5})/2 \approx 2.6180339887$.

As an illustration, let us carry out some of the calculations directly. We start with $\text{Rel}_2(C_4, 1, 3; q) = -q^4 + 4q^3 - 4q^2 + 1$ which has a root at $1 + \sqrt{2}$. If we set $r = 1+\sqrt{2}$ and then solve $\text{Rel}_2(C_4, 1, 3; q) = 1-r$ (which corresponds to doing a gadget replacement with copies of C_4 on every edge), then we get a root at approximately 2.5980531824. A summary of the first 10 iterations is shown in the table in Figure 4.16 (rounded to 10 decimal places). As we can see, it only takes about four iterations to be accurate to within 4 decimal places. After 10 iterations, we are precise to 10 decimal places. For a full illustration of the plot of many possible roots of many choices of r (not just taking the largest root, but considering all roots, stopping at the first $10,000$), see Figure 4.17. One can see that this type of operation seems to produce a fractal! We are uncertain as to whether the fractal is connected or not, or what other properties of these fractals might have.

4.1.4 Real Roots of Two-Terminal Reliability Polynomials

We can also study the location of the real roots of two-terminal reliability polynomials. We know that in the all-terminal case, we have that the real roots are dense in $[-1, 0) \cup \{1\}$ (see [14]). However, what can be said about two-terminal roots? We know from Theorem 4.1.2 that the closure of the *complex* two-terminal roots contain $[-1, 2]$, but what about limits of *real* two-terminal roots? Certainly, such limits cannot contain any real number in $(0, 1)$ as the two-terminal reliability (of a connected graph) is positive in this interval.

Can 0 be a limit of real two-terminal roots? Let us consider the theta graph $\Theta_{l[k]}$ once more with terminals s and t extreme points. We have seen that its two-terminal reliability polynomial is

$$\text{Rel}_2(\Theta_{l[k]}, s, t; q) = 1 - (1 - (1 - q)^l)^k. \tag{4.6}$$

It is easy to calculate that at $q = 0$ we have $\text{Rel}_2(\Theta_{l[k]}, s, t, 0) = 1$. However, what if we take a number slightly smaller than 0, say $-\epsilon$? Then

$$\lim_{l\to\infty} (1 - (-\epsilon))^l = \infty.$$

Therefore,

$$\lim_{l\to\infty} 1 - (1 - (-\epsilon))^l = -\infty.$$

Let us fix k to be even. Then

$$\lim_{l\to\infty}(1-(1-(-\epsilon))^l)^k = \infty.$$

Finally, we get that

$$\lim_{l\to\infty} 1-(1-(1-(-\epsilon))^l)^k = -\infty.$$

As clearly $\mathrm{Rel}_2(\Theta_{l[k]}, s, t, 0) = 1 > 0$, by the Intermediate Value Theorem, there must be a real root in $(-\epsilon, 0)$. It follows that 0 is a limit of real two-terminal roots.

The number 0 being a limit of real roots is not all that surprising as it happens for the all-terminal case as well. However, we can also show that there is a real root to the right of 2 that is a limit of real two-terminal roots (which is very different than the all-terminal case).

Let us once again consider $\mathrm{Rel}_2(\Theta_{l[k]}, s, t; q)$. It is easy to show that, if l is even and fixed, and k is even, say $k = 2K$, we have $\mathrm{Rel}_2(\Theta_{l[2K]}, s, t; 2) = 1 > 0$. Now, what happens if we take a number slightly larger than 2, say $2+\epsilon$? Let us choose l even and large enough so that $\epsilon > 2^{\frac{1}{l}} - 1$. Then

$$(1-(2+\epsilon))^l = (-1-\epsilon)^l > 2.$$

Then, let us take the limit of (4.6) as K tends to infinity:

$$\lim_{K\to\infty}(1-(-1-\epsilon)^l)^{2K} = \infty.$$

And so

$$\lim_{K\to\infty} 1-(1-(1-(2+\epsilon))^l)^{2K} = -\infty$$

Therefore, by the Intermediate Value Theorem, there must be a real root in $(2, 2+\epsilon)$. We draw the plot of the roots for various values of l and k in Figure 4.18, which was first seen in Figure 4.9 on page 83, but this time we let l and k run between 1 and 10, and only include the real roots.

Two Terminal Inflection Points

Another property that we can study regarding two-terminal reliability polynomials is the location of the inflection points in $[0, 1]$. In [15], Brown and Cox were able to use a family of threshold graphs to show that the inflection points of all-terminal reliability polynomials are dense in this interval. What about for two-terminal reliability polynomials? Cox showed in her PhD thesis [26] that they are, indeed, dense in $[0,1]$. However, all of the graphs used had multiple edges and were without cut vertices. We will show that the inflection points of two-terminal reliability polynomials are dense in $[0,1]$ is true when restricted to simple graphs as well.

Theorem 4.1.3. *Inflection points of the two-terminal reliability of simple, connected graphs (without cut vertices) are dense in* $[0,1]$.

Proof. Let us once again consider a generalized theta graph $\Theta_{l[k]}$ with k paths of length l. We know that its two-terminal reliability polynomial is given by

$$\mathrm{Rel}_2(\Theta_{l[k]}; q) = 1 - (1-(1-q)^l)^k.$$

It will be convenient for us to do this proof in terms of $p = 1-q$, and we consider

$$f(p) = \mathrm{Rel}_2(\Theta_{l[k]}, s, t; p) = 1 - (1-p^l)^k.$$

By the chain rule, $f'(p) = -\mathrm{Rel}_2'(\Theta_{l[k]}, s, t, 1-p)$ and $f'' = \mathrm{Rel}_2''(\Theta_{l[k]}, s, t, 1-p)$, so it suffices to show that the points of inflection of $f(p)$ are dense in $[0,1]$. Therefore, its inflection points occur when

$$f''(p) = lkp^{l-2}(1-p^l)^{k-2}\left[(l-1)(1-p^l) - l(k-1)p^l\right]$$

changes sign. This occurs when $(l-1)(1-p^l) - l(k-1)p^l = 0$ and so when

$$p = \sqrt[l]{\frac{l-1}{lk-1}}.$$

Therefore, we need only show that for any $x \in (0,1)$ and any $\epsilon > 0$ with $\epsilon < x$ and $x + \epsilon < 1$ there exists an l and k such that

$$(x-\epsilon) \leq \sqrt[l]{\frac{l-1}{lk-1}} \leq (x+\epsilon). \tag{4.7}$$

Suppose we do have such an x and ϵ. Then (4.7) can be re-written in terms of k. That is,

$$k \geq \frac{(l-1)+(x+\epsilon)^l}{l(x+\epsilon)^l}$$

and

$$k \leq \frac{(l-1)+(x-\epsilon)^l}{l(x-\epsilon)^l}$$

Therefore, it suffices to show that

$$\frac{(l-1)+(x-\epsilon)^l}{l(x-\epsilon)^l} - \frac{(l-1)+(x+\epsilon)^l}{l(x+\epsilon)^l} \geq 1. \tag{4.8}$$

for some l and k. Indeed, the left side is equal to

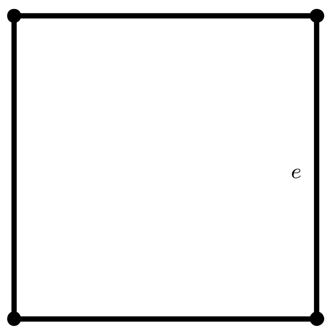
$$\begin{aligned}
&\frac{(l-1)(x+\epsilon)^l-(l-1)(x-\epsilon)^l}{l(x-\epsilon)^l(x+\epsilon)^l}\\
=&\frac{(l-1)}{l}\left[\frac{(x+\epsilon)^l-(x-\epsilon)^l}{(x-\epsilon)^l(x+\epsilon)^l}\right]\\
=&\frac{(l-1)2\epsilon}{l}\left[\frac{(x+\epsilon)^{l-1}+(x+\epsilon)^{l-2}(x-\epsilon)+\cdots+(x-\epsilon)^{l-1}}{(x-\epsilon)^l(x+\epsilon)^l}\right]\\
\geq&\frac{(l-1)2\epsilon}{l}\frac{l(x-\epsilon)^{l-1}}{(x+\epsilon)^l(x-\epsilon)^l}\\
=&\frac{(l-1)2\epsilon}{(x-\epsilon)(x+\epsilon)^l}.
\end{aligned}$$

By taking the natural log of this, we get

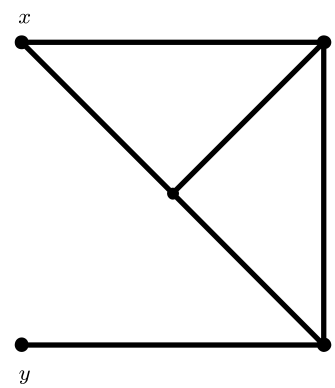
$$\ln(2\epsilon)+\ln(l-1)-\ln(x-\epsilon)-l\ln(x+\epsilon).$$

Now, since x and ϵ are fixed, $\ln(2\epsilon), \ln(x-\epsilon)$ and $\ln(x+\epsilon)$ are as well. As $\ln(x+\epsilon)<0$, it follows that this expression tends to infinity, so that the left hand side of (4.8) is eventually greater than 1. In other words, for sufficiently large l, we can choose an integer k such that (4.7) holds, and we are done. □

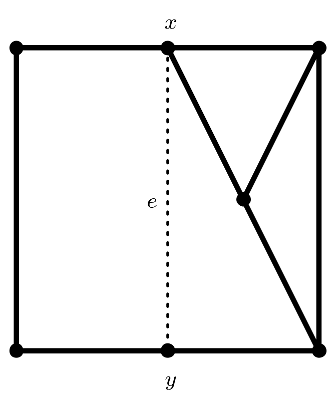
As we can see, there are many similarities and differences between the all-terminal reliability and the two-terminal reliability of a connected graph G. Though both have a very similar shape when plotted on the interval $[0,1]$, their roots behave very differently. Moreover, the underlying simplicial complexes are extremely different (and only in one class of graphs do they coincide). This leads us to many open problems, which we discuss in the following chapter.



(a) An initial graph G

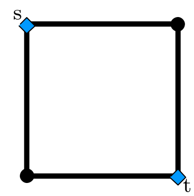


(b) A gadget graph H

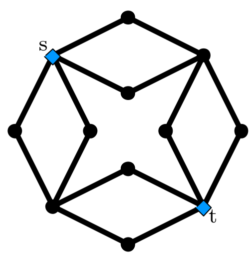


(c) A gadget replacement $G[H]$. The dotted line indicates the edge that has been replaced

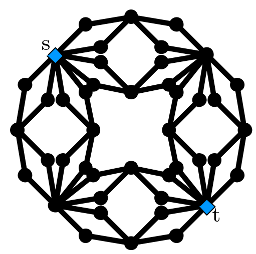
Figure 4.14: An example of a gadget replacement



(a) Initial graph C_4 with antipodal terminals



(b) First iteration of the gadget replacement, $C_4[C_4]$



(c) Second iteration of the gadget replacement, $C_4[C_4[C_4]]$

Figure 4.15: Example of a sequence of gadget replacements $G[H]$ with G and H both copies of C_4.

Iteration	**Choice of r**	**Root with Largest Modulus**
0		2.4142135624
1	2.4142135624	2.5980531824
1	2.5980531824	2.6161212065
3	2.6161212065	2.6178512906
4	2.6178512906	2.6180165422
5	2.6180165422	2.6180323228
6	2.6180323228	2.6180338297
7	2.6180338297	2.6180339736
8	2.6180339736	2.6180339873
9	2.6180339873	2.6180339886
10	2.6180339886	2.6180339887

Figure 4.16: An iteration of the first 10 gadget replacements of $G[H]$ with $G = C_4$ and $H = C_4$, both with antipodal terminals, as well as the chosen root r at each step together with the two-terminal reliability root with largest modulus – all two-terminal reliability roots shown are real numbers.

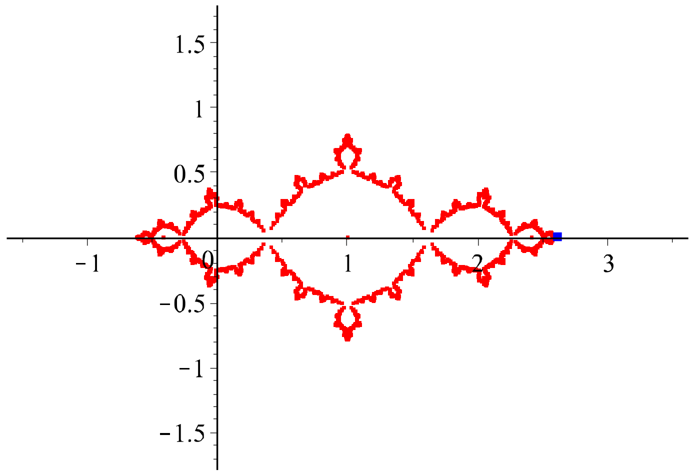


Figure 4.17: Plot of the two-terminal roots of repeated gadget replacement on C_4. The limit root, $\dfrac{3+\sqrt{5}}{2}$, is a blue square.

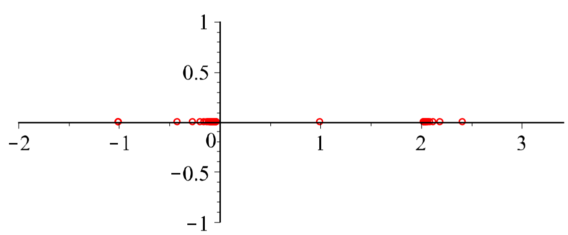


Figure 4.18: Plot of the real two-terminal roots of $\Theta_{l[k]}$ with l and k ranging between 1 and 10.

Chapter 5

Conclusion

Throughout this dissertation we have explored two types of network reliabilities – the all-terminal reliability and the two-terminal reliability – with particular interest in the location of their roots and their underlying simplicial complexes. In this chapter, we will discuss which salient open problems have arisen from our work.

5.1 Open Problems In Reliability of Complexes and Matroids

There are still many questions open on reliability roots for complexes in general, and matroids in particular. We have seen that the reliability roots of all matroids of small rank (rank at most 3) are in the unit disk.

Problem 5.1.1. What is the smallest rank (or order) of a matroid with a reliability root outside the unit disk?

The cographic matroids corresponding to the Royle-Sokal graph has rank 13 and order 16, and is the smallest one we know.

It would be of interest to find other constructions (other than those raised in Section 2.4) that produce reliability roots outside the unit disk, both for matroids and other complexes. The most salient open question is how large in moduli can a reliability root of a matroid be?

Problem 5.1.2. Are the reliability roots of matroids bounded?

It seems likely that they are bounded, perhaps even by 2, but of course there may be some extremal families that have roots far outside the disk centred at 0.

We have seen that the paving matroids of rank 4 (and smaller rank) have roots inside the unit disk. Almost all of the coefficients of the H-polynomial of paving matroids are completely described – only the leading coefficient varies from one paving matroid of order m and rank r.

Problem 5.1.3. Are the reliability roots of paving matroids always in the unit disk?

Paving matroids are widely believed to dominate all matroids (see [9, 27] for example) – it has been conjectured [39, Conjecture 1.6] that almost all matroids of order m are paving matroids. We have found throughout our work that the reliability roots of matroids are rarely outside the disk centred at 0 (see Figure 5.1 for a plot of all all-terminal reliability roots of all paving matroids up to order 8).

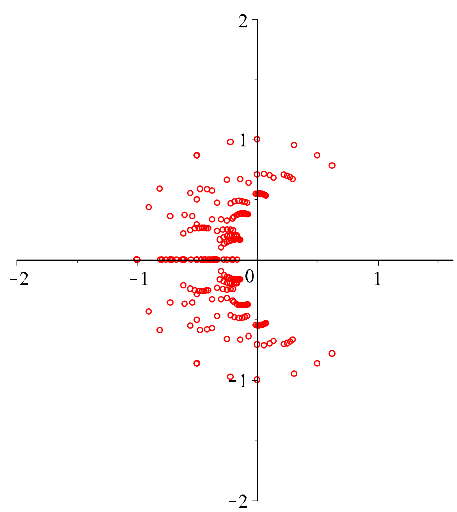


Figure 5.1: Plot of the all-terminal roots of all paving matroids up to order 8.

5.2 Open Problems on Rational Roots of Reliability Polynomials

Throughout Chapter 3 we saw that the rational roots of the all-terminal reliability of connected graphs G is the set $\{-1, -1/2, \ldots, -1/(n-1), 1\}$, and moreover, if -1 is a reliability root, then G is not simple. However, there are still some open problems that one could study.

Problem 5.2.1. Can we characterize which rational numbers can be reliability roots of 2-edge connected or 2-connected graphs?

The construction of graphs of order n with rational reliability roots in $\{-1/k : 1 \leq k \leq n-1\} \cup \{1\}$ requires the introduction of bridges to the graph, and hence the examples are not 2-edge-connected. What if we restrict to 2-edge-connected graphs? If we allow multiple edges, we can still attain the same rational roots by attaching new vertices not by a single edge but by a bundle of at least two edges. However, what if we insist on *simple* 2-edge-connected graphs? In this case, there may be rational numbers missing from the reliability root set. For example, among all such graphs of order 8, the rational reliability roots are $1, -1/2, -1/3, -1/4, -1/5$ and $-1/7$.

The question is even more interesting for 2-connected graphs, that is, those without cut vertices. We do not know whether all of $-1, -1/2, \ldots, -1/(n-1)$ can be roots. Among simple 2-connected graphs of order n, the rational reliability roots may be even sparser – for order 8, the rational roots are only $1, -1/2, -1/3, -1/4$ and $-1/7$.

Problem 5.2.2. What are the rational reliability roots of other forms of reliability? In particular, the rational roots of two-terminal reliability?

For other forms of reliability, it seems much more difficult to determine the rational reliability roots. Consider, for example, the two-terminal reliability for a graph G with terminals s and t. From calculations on small graphs, it seems that the only rational two-terminal reliability roots are 1 and -1, but an argument seems elusive.

The example of two-terminal reliability shows that for reliabilities other than all-terminal, it may be the case that the roots of the smallest modulus are not rational. In particular, for $n = 3$, one can verify that the only two-terminal reliability polynomials are of the form

$$1 - q^m,$$

$$(1 - q^{m_1})(1 - q^{m_2}),$$

and

$$1 - q^{m_1} + q^{m_1}(1 - q^{m_2})(1 - q^{m_3}) = q^{m_1+m_2+m_3} - q^{m_1+m_2} - q^{m_1+m_3} + 1,$$

where m, m_1, m_2 and m_3 are any positive integers. In all cases, the only rational roots are 1 and -1, but for $m_1 = m_2 = m_3$, the third polynomial has a root at $(1 - \sqrt{5})/2 \approx -0.6180$.

Problem 5.2.3. If the set of rational number roots for the all-terminal reliability polynomials of all connected graphs of order n is only $\{-1, -1/2, \ldots, -1/(n-1), 1\}$, then what would the set of Gaussian rational numbers be?

Beyond the rational numbers, it might be interesting to ask what other complex numbers might be reliability roots (in the all-terminal case). One could focus on *Gaussian rational numbers*, that is, complex numbers of the form $a + ib$ where a and b are rational numbers. We have found a variety of Gaussian rationals (that are not rationals) that are reliability roots; among simple graphs of order at most 8, the following are all reliability roots:

$$\pm\frac{1}{2}i,\ \frac{-1}{4} \pm \frac{1}{4}i,\ \frac{-1}{5} \pm \frac{2}{5}i,\ \frac{-2}{5} \pm \frac{1}{5}i,\ \frac{-1}{5} \pm \frac{1}{10}i,\ \frac{-3}{10} \pm \frac{1}{10}i,\ \frac{-3}{13} \pm \frac{2}{13}i.$$

If we ask only about those Gaussian rationals that are purely imaginary (i.e. $a = 0$), then we can say that the set of such reliability roots contains

$$\left\{\pm\frac{1}{k}i : k \text{ is a positive integer}\right\},$$

since, by replacing each edge in a graph G by a pair of parallel edges, the effect on the reliability polynomial is to replace q by q^2, so each root of the form $-1/k^2$ (say of cycle C_{k^2+1}) yields two reliability roots, $\pm\frac{1}{k}i$. (We remark that the purely imaginary reliability roots, with no conditions on the imaginary part, are in fact dense in the interval between $-i$ and i, as it was shown in [14] that the real reliability roots are dense in $[-1, 0]$.) Whether the purely imaginary Gaussian reliability roots of graphs of order n is the set

$$\left\{\pm\frac{1}{k}i : 1 \leq k \leq \lfloor\sqrt{n-1}\rfloor\right\}$$

remains open.

Finally, for rational roots of the all-terminal reliability, we can consider what the "second-best" root is. That is, if we are considering the root of maximum modulus, then we know that 1 is always a root (and so it is very likely that it is the root with largest modulus). On the other hand, if we are considering the root of minimum modulus, then we know that occurs for the cycle graph C_n with the root $-1/(n-1)$. However, if we remove these roots from consideration, what's the next best choice?

Problem 5.2.4. Other than 1 and $-1/(n-1)$, what is the all-terminal reliability root with largest modulus and smallest modulus, respectively? For which graphs can these occur?

The former is quite difficult to answer as we have seen that there can be roots with modulus larger than 1, and we do not know if there is a maximum. However, the latter question seems to be quite interesting. Consider, for example, the table in Figure 5.2 that lists the 6 graphs of order 8 with the smallest all-terminal reliability roots. We see that the graph with the smallest all-terminal reliability root is C_8, which we expect. However, what is a bit surprising is that all other graphs are either cycle graphs with a leaf (that is, a vertex of degree 1), or a theta graph. Is this always true? That is, for fixed n, is the second-best root of smallest modulus always C_{n-1} with a leaf?

We can also ask many of these questions with regards to the reliability of complexes and matroids.

Problem 5.2.5. What are the rational reliability roots of complexes or matroids? When can -1 be a reliability root of these? Is there any insight into the structure of certain complexes or matroids if they have -1 as a reliability root?

As we saw in Chapter 3, we were able to determine if a graph was not simple if it had -1 as a reliability root. Right now, we do not know what the connection to complexes or matroids would be.

5.3 Open Questions on Two Terminal Reliability

Roots and Limiting Curves

The most salient open problem in two-terminal reliability is whether or not there is a class of graphs which have two-terminal reliability roots with modulus tending to infinity. Here, we will find a class of graphs whose two-terminal roots do, indeed, tend to infinity – sort of.

To begin, we recall the Beraha-Kahane-Weiss (BKW) theorem which states that if a polynomial f is of the form

$$f = \alpha_0\lambda_0^n + \alpha_1\lambda_1^n + \cdots + \alpha_m\lambda_l^n$$

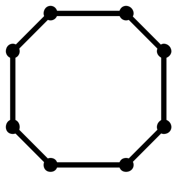
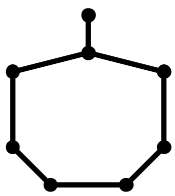
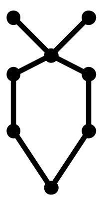
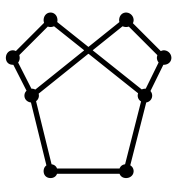
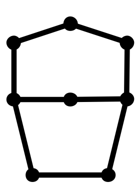
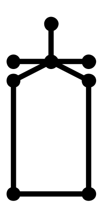
Graph	**Reliability Polynomial**	**Root**	**Modulus**
	$(1-q)^7(7q+1)$	-0.1428571429	0.1428571429
	$(1-q)^7(6q+1)$	-0.1666666667	0.1666666667
	$(1-q)^7(5q+1)$	-0.2	0.2
	$(1-q)^7(19q^2+7q+1)$	$-0.1842105263-0.1367408532i$	0.2294157338
	$(1-q)^7(18q^2+7q+1)$	$-0.1944444444-0.1332175423i$	0.2357022604
	$(1-q)^7(4q+1)$	-0.25	0.25

Figure 5.2: The 6 graphs of order 8 with the smallest all-terminal reliability roots

then the limit of roots of f are precisely the complex numbers z such that

- one of the $|\lambda_i(z)|$ exceeds the others and $\alpha_i(z) = 0$, or
- at least two of $|\lambda_{i_1}(z)| = |\lambda_{i_2}(z)| = \cdots = |\lambda_{i_{n-1}}| > |\lambda_{i_n}(z)|$ including the case where all λ_i's are equal.

Consider now the complete graph $K_{1,1,n}$ with s and t in the single bundles (see Figure 5.3). Then the two-terminal reliability (in p) is

$$\begin{aligned}\text{Rel}_2(s,t,p) &= p + (1-p)(1-(1-p^2)^n)\\ &= 1\cdot 1^n + (p-1)\cdot(1-p^2)^n \qquad (5.1)\end{aligned}$$

Indeed, either the edge between s and t is operational, or it has failed and then a path on two edges between s and t is operational. We can see that this satisfies the form for the BKW Theorem with $\alpha_1 = 1, \alpha_2 = (p-1), \lambda_1 = 1$, and $\lambda_2 = (1-p^2)$. Using the first part of the BKW Theorem does not result in any limit points. Therefore, we only need to consider the case where both λ_1 and λ_2 are equal.

Suppose $|p^2-1| = 1$. We know that the solutions to this will be the same solutions as the squaring of both sides. That is,

$$|p^2-1|^2 = 1$$

Let us write $p = a+bi$. Then the modulus is

$$\begin{aligned}|p^2-1|^2 = 1 &\implies |(a+bi)^2-1|^2 = 1\\ &\implies |(a^2-b^2-1)+(2abi)|^2 = 1\\ &\implies (a^2-b^2-1)^2+(2ab)^2 = 1\\ &\implies a^4+b^4+1-2a^2b^2-2a^2+2b^2+4a^2b^2 = 1\\ &\implies a^4+b^4+2a^2b^2-2a^2+2b^2 = 0\\ &\implies (a^2+b^2)^2-2a^2+2b^2 = 0\end{aligned}$$

This is in the form of the Lemniscate of Booth (also known as a hippopede, see [37]) which is $(x^2+y^2)^2 - cx^2 + dy^2 = 0$ where c and d are positive. Plotting this limit curve, we see that the shape of the roots tend to the infinity symbol (see Figure 5.4

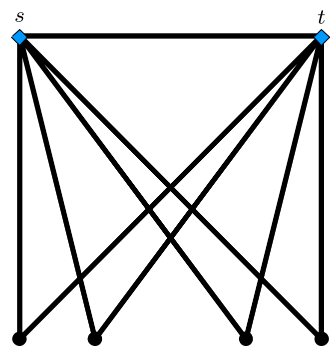


Figure 5.3: Example of the complete 3-partite graph $K_{1,1,4}$

Pure or Shellable Two Terminal Simplicial Complexes

We saw in Chapter Four that we were able to classify when the two-terminal complex $\mathcal{C}_{s,t}(G)$ is a matroid (i.e., is a bundled path between s and t, possibly with irrelevant edges called a tendril, T). Furthermore, we know that a matroid is both pure and shellable. However, other than this one class of graphs, we do not know the structure of $\mathcal{C}_{s,t}(G)$. It seems natural to generalize this to both pure complexes, as well as (possibly non-pure) shellable complexes. We shall start with pure complexes.

Problem 5.3.1. When is the underlying two-terminal simplicial complex of a connected graph G with terminals s and t a pure complex?

This question in and of itself is interesting as, since the facets of $\mathcal{C}_{s,t}(G)$ are sets of edges whose removal leaves s and t connected, each facet is in a one-to-one correspondence with paths between s and t. Therefore, $\mathcal{C}_{s,t}(G)$ is pure if and only if every path between s and t is of the same length. Let us try to determine when this is the case.

Let $P_{s,t}$ be the collection of paths between s and t, and consider any two paths

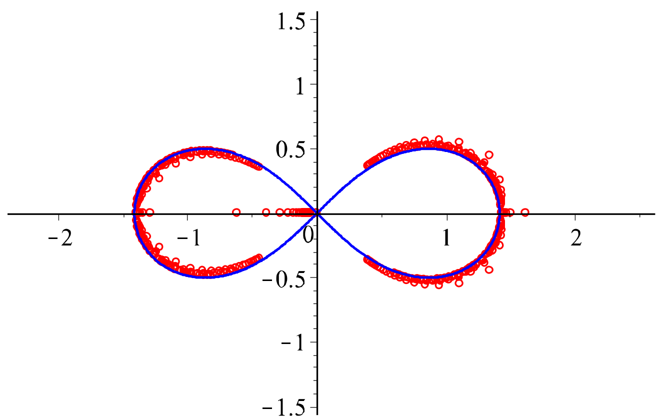


Figure 5.4: Plot of the roots of $K_{1,1,n}$ for n from 1 to 20 (red), and the Lemniscate of Booth $(a^2+b^2)^2-2a^2+2b^2=0$ (blue)

$p_1, p_2 \in P_{s,t}$ which connect s to t. We will show that there are no other edges with one endpoint strictly in p_1 and one endpoint in p_2 (though we do allow edges that are common to both paths, or bundling edges). We first consider the symmetric difference of p_1 and p_2 (thus, removing all edges that p_1 and p_2 have in common). One observation is that this results in (potentially disjoint) cycles where one path of the cycle is strictly from p_1, and the other path of the cycle is strictly from p_2. Therefore, let us consider one of these cycles, call it C_1, and denote the aforementioned paths c_1 and c_2 (so $c_1 \subset p_1$ and $c_2 \subset p_2$), and let v_s and v_t be the vertices where c_1 and c_2 meet (see Figure 5.5).

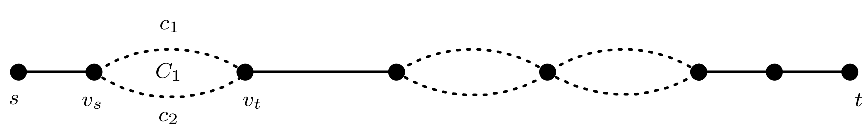


Figure 5.5: Example cycle decomposition of p_1 and p_2

We first note that both c_1 and c_2 must be of the same length. Indeed, suppose the contrary. Then since p_1 and p_2 are of the same length, the sum of all of the paths

$c_{1,i}$ (that is, the edges in all cycles C_i that are strictly in p_1) and the sum of all of the paths $c_{2,i}$ must be equal. However, if, say, $c_{1,i}$ and $c_{2,i}$ are of different lengths, (without loss of generality suppose that $c_{1,i}$ is shorter), then if we first traverse along p_1 up until $v_{s,i}$, then instead traverse across $c_{2,i}$ to $v_{t,i}$, and then continue along p_1 to t, then this would be another path in $P_{s,t}$ but of strictly longer length; a clear contradiction. Therefore, c_1 and c_2 must be of the same length, call it k.

Now we would like to show that there cannot be any other edges from any other path in $P_{s,t}$ that can strictly connect c_1 to c_2 (i.e., connects a vertex that is strictly in c_1 to a vertex that is strictly in c_2). First, let us label the vertices of c_1 $v_i, 1 \leq i \leq k-1$, and label the vertices of c_2 $u_i, 1 \leq i \leq k-1$. Suppose that there does exist an edge, call it e, between c_1 and c_2. By a similar argument as above, the edge must only create an alternate path between v_s and v_t of length k. Suppose that one endpoint of e connects to vertex v_i, $1 \leq i \leq k-2$ ($i \neq k-1$ as otherwise the only vertex it can be connected to would be v_t which isn't strictly contained in c_2). Since e clearly counts as an edge in this new path, the other vertex must be connected at u_j, where $i+1+(k-j) = k$ and it follows that $i+1 = j$. This creates a path of length k by first travelling along c_1 until we arrive at v_i, then travel across e to u_j, and then travel to t (see Figure 5.6 with $k = 6, i = 3$, and so $j = 4$). However, if we first travel to u_j, then along e to v_i, then to v_t, call this path p', then this has length $j+1+(k-(j-1)) = k+2 > k$. Another contradiction.

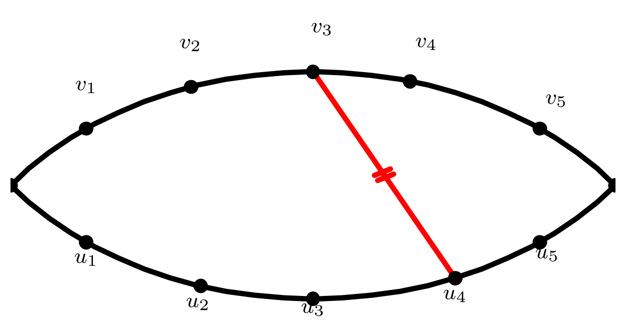


Figure 5.6: Example of an edge e (red with two dashes) intersecting two paths of length $k = 6$ at v_3 and v_4.

Therefore, if we take any pair of paths between s and t and take their cycle-decompositions, then there cannot be any alternate edges between these cycles.

This also leads us to another open problem.

Problem 5.3.2. Consider the following two operations:

1. Replace an edge by a bundle (thus creating a generalized theta graph with each branch length 1);

2. Replace an edge by a path. If the edge is in a branch of a generalized theta graph, then one edge in all parallel branches must be replaced by a path of the same length.

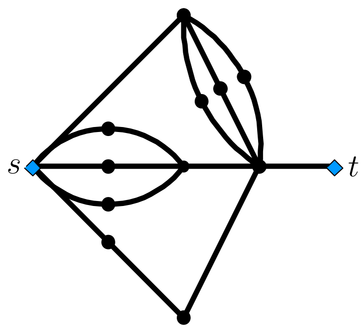
Is it true that every graph whose two terminal reliability complex is pure will appear after repeated application of the above two operations, starting with K_2?

It is easy to see that both of these operations maintain purity. However, are these the only two operations needed to generate all graphs with a pure $\mathcal{C}_{s,t}(G)$ complex? For instance, let us consider the example in Figure 5.7. We start by setting $G_0 = K_2$ (Figure 5.7b). Next, let us perform operation 2 on the red edge, thus creating a path of length 2 – call this new graph G_1 (Figure 5.7c). Next, we perform operation 1 on the blue edge to create a bundle of 3 edges (thus creating a generalized theta graph) – call this graph G_2 (Figure 5.7d. Then, performing operation 2 on all green edges (as, since each edge is in a generalized theta graph, we need to do the same operation to one of the edges in each branch) – call this graph G_3 (Figure 5.7e). Performing operation 1 on the purple edges yields G_4 (Figure 5.7f), and then finally performing operation 2 on the orange edges yields the desired graph whose two-terminal complex is pure (Figure 5.7a).

If we are able to show that all graphs whose two-terminal complex can be built up in this way, then perhaps we can use the structure of these graphs to produce bounds on the F- or H-vectors of $\mathcal{C}_{s,t}(G)$. Doing so would allow us to study the two-terminal reliability roots of this class of graphs.

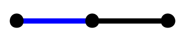
We can also apply the same reasoning to study (non-pure) shellable complexes.

Problem 5.3.3. When is the underlying two-terminal simplicial complex of a connected graph G with terminals s and t a shellable complex?

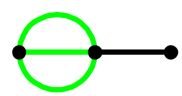


(a) Graph with a pure two-terminal complex

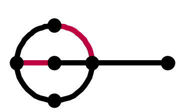
(b) Start with a K_2 and call it G_0



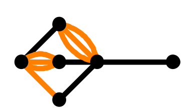
(c) Replace red edge in G_0 by a path of length 2, and call this graph G_1



(d) Replace blue edge in G_1 by a bundle of 3 edges – creating a generalized theta graph. Call this graph G_2



(e) Replace all edges in the green theta graph of G_2 by paths of length 2. Call this graph G_3.



(f) Replace purple edges in G_3 by a bundle of 3 edges. Call it G_4.

Figure 5.7: Example of an iteration through the two pure-complex operations

Since the facets of $\mathcal{C}_{s,t}(G)$ are sets of edges whose removal leaves s and t connected, each facet is in a one-to-one correspondence with paths between s and t. Therefore, perhaps we can find sufficient conditions on the paths of a graph to force a shellable $\mathcal{C}_{s,t}(G)$. Unfortunately, a proof eludes us at the moment, but we do have the following conjecture

Conjecture 5.3.1. *$\mathcal{C}_{s,t}(G)$ is shellable if each path between s and t can be ordered $p_1, \ldots, p_n$ such that $p_i = (p_{i-1} \setminus e) \cup \epsilon_i$ for $e \in E(p_{i-1}), \epsilon_i \subset E(G), e \notin \epsilon_i$, and $|\epsilon_i| = 1$ or 2 for $i \geq 2$.*

In other words, every path can be ordered such that every path p_i is the previous path p_i with exactly one edge removed, e, and then either a path of length one or two, ϵ_i, replacing e. An example of this, see Figure 5.8, is taking the wheel graph W_3 (which is the cycle graph C_3 with every vertex connected to a central fourth vertex), setting s as one vertex in the cycle and t as the center vertex, and then removing one edge adjacent to s (see Figure 5.8a). Then, if we take the order as shown in the table in Figure 5.8b, it can be seen that its two-terminal complex (see Figure 5.8c) is precisely the (non-pure) shellable complex we saw earlier in Example 1.2.3 (see Figure 1.11b on page 14).

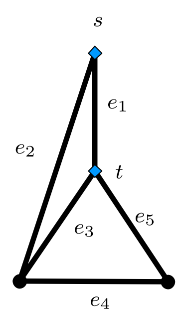
We note that this is, indeed, a proper subset of the class of graphs whose two-terminal complex is shellable, as a cycle graph on n vertices with adjacent terminals will always be shellable regardless of the length of the path (which is $n-1$). Indeed, since the facets, say σ_1 and σ_2 with $\dim(\sigma_1) = 1$ and $\dim(\sigma_2) = n-1$, have intersection $\sigma_1 \cap \sigma_2 = \emptyset$ which is of dimension 0 $=\dim(\sigma_1) - 1$, this would be proper shelling for a non-pure shellable complex.

Some observations we can make with regards to shellable two-terminal complexes are:

- Since we would require the facets of $\mathcal{C}_{s,t}(G)$ to be ordered $\sigma_1, \ldots, \sigma_d$ such that

$$\left(\bigcup_{j=1}^{m-1} \overline{\sigma_j}\right) \bigcap \overline{\sigma_m}$$

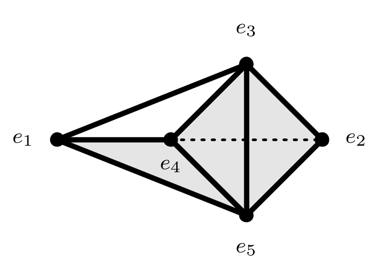
  are all pure ($\dim\sigma_m - 1$)-complexes for all $m = 2, \ldots, d$ with σ_i in *descending*



(a) The graph W_3 less an edge with terminals s and t whose two-terminal complex is shellable

Path	Corresponding Facet
e_1	$e_2e_3e_4e_5$
e_2e_3	$e_1e_4e_5$
$e_2e_4e_5$	e_1e_3

(b) The (s,t)-paths of the graph in (a) listed in ascending order, and their corresponding facets (listed in descending order)



(c) (Non-pure) shellable two-terminal complex of the graph in (a)

Figure 5.8: Example of a graph whose two-terminal complex is (non-pure) shellable

(a) Example of a graph whose two-terminal complex is shellable with two paths that differ by a 3-path replacement.

(b) Example of a graph whose two-terminal complex is not shellable with two paths, p_2 and p_3, that differ by a 3-path replacement.

Figure 5.9: Example of two graphs with a 3-path replacement. (a) is shellable, whereas (b) is not

order, and σ_i are in one-to-one correspondence with the paths p_i (in fact, they are complements of each other), then p_i would need to be in *ascending* order.

- If p_i is p_{i-1} with one edge replaced by three edges instead of one or two, then σ_i, σ_{i-1} may not be a valid ordering of facets for a shelling. For instance, consider the graphs in Figure 5.9. We can see that the paths from s to t in Figure 5.9a are $p_1 : e_1, e_2$ and $p_2 : e_1, e_3, e_4, e_5$. Therefore, the facets (written multiplicatively) of $\mathcal{C}_{s,t}(G)$ are $\sigma_1 = e_3e_4e_5$ and $\sigma_2 = e_2$. Since $\sigma_1 \cap \sigma_2 = \emptyset$ is of dimension 0, and σ_2 is of dimension 1, this is a (non-pure) shellable complex. However, considering that the paths of the graph in Figure 5.9b are $p_1 : e_1, p_2 : e_2, e_3$, and $p_3 : e_4, e_5, e_6$, the facets of $\mathcal{C}_{s,t}(G)$ are $e_2e_3e_4e_5e_6, e_1e_4e_5e_6$, and $e_1e_2e_3$. One can verify that $F_1 \cap \sigma_2$ satisfies the shelling property, but $(\sigma_1 \cup \sigma_2) \cap \sigma_3$ does not (as it intersects F_1 in dimension 2 – one lower than its dimension – but it intersects σ_2 in dimension 1). Thus, $(\sigma_1 \cup \sigma_2) \cap \sigma_3$ is not a pure 2-dimensional complex (as any other shelling order would not be in descending order with respect to dimension).

What is nice about this conjecture is, if it is true, then restricting the size of ϵ_i to only being 1 results in a two-terminal complex that is both shellable *and* pure. We know that if $\mathcal{C}$ is a matroid then it is both pure and shellable, but in general not all pure and shellable complexes are matroids. However, in our case, this seems to be precisely the same class of complexes.

Conjecture 5.3.2. *If $\mathcal{C}_{s,t}(G)$ is pure and shellable, then it is a matroid.*

If Conjecture 5.3.1 is true, then $\mathcal{C}_{s,t}(G)$ being shellable would imply that it would satisfy the path conditions therein. Furthermore, if it is pure, then every path needs to be of the same length. The only way this can happen is if $|\epsilon_i| = 1$ for all i (as otherwise, if there is a collection of edges ϵ_i of size 2, then there would be two paths where we can replace one edge of the first path by two edges of the second path; a clear contradiction). Therefore, if every path between s and t differs by exactly one edge, then G must be a bundled path (possibly with irrelevant edges). By Theorem 4.1.1, the underlying simplicial complex would be a matroid.

We have seen that, while all-terminal and two-terminal reliabilities lead to what may seem to be elementary mathematical functions – polynomials with integer coefficients – their roots and underlying simplicial complexes are nontrivial and intriguing. There are many easily stated open problems that will undoubtedly be the source of future work for those inside combinatorics and outside as well.

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Appendix A

Required Packages

```
with(GraphTheory) :
with(combinat) :
with(plots) :
```

Complex Procedures

Input: a set of faces (as sets)
Output: a list of faces

```
complx := proc(lst)
 local S, i :
 S := { } :
 for i from 1 to nops(lst) do
   S := S union powerset(lst[i]) :
 od:
 convert(S, list);
 end:

 card := l → nops(l) :
```

Input : a list of faces of a complex
Output : a list of faces organized by dimension.

```
stratify := proc(c)
 local d, faces, i, flist, j :
 flist := NULL :
 d := max(seq(card(c[i]), i = 1 ..nops(c)));
 for i from 0 to d do
     faces := NULL :
      for j from 1 to nops(c) do
          if card(c[j]) = i then
             faces := faces, c[j]
          fi:
      od:
 flist := flist, [faces] :
od:
[flist];
 end:
```

Input : a list of facets of a complex
Output : a boolean true if the complex is pure

```
isPure := proc(c)
 local d, i :
 d := nops(c[1]) :
 for i from 2 to nops(c) do
   if nops(c[i]) ≠ d then
     RETURN(false);
   fi:
 od:
 d;
 end:
```

Input : a list of faces of a complex
Output : a boolean true if the H − vector is all nonnegative

```
testHvector := proc(C)
 local h, i :
  h := hpoly(C) :
 for i from 0 to degree(h, x) do
    if coeff(h, x, i) < 0 then
       RETURN(false);
    fi:
 od:
 true;
 end:
```

Input : a list of facets of a complex
Output : a boolean true if the complex is shellable

```
isShellable := proc(lst)
 local d, S, i, T, F, C :
 C := complx(lst) :
 if testHvector(C) = false then
   RETURN(false) :
 fi:
 d := nops(convert(lst[1], list)) :
 S := convert(complx([lst[1]]), set) :
 for i from 2 to nops(lst) do
   T := convert(convert(complx([lst[i]]), set) intersect S, list) :
   F := fcts(T) :
   if isPure(F) = false or isPure(F) ≠ d − 1 then
     RETURN(false);
   fi:
   S := S union convert(complx([lst[i]]), set) :
 od:
 true;
 end:
```

Input : a set of vertices and a permutation
Output : a set of sets that have been permuted

```
permuteSet := proc(S, prm)
 local newS, s :
 newS := NULL :
 for s in S do
    newS := newS, prm[s] :
 od:
 {newS};
 end:
```

Input : a set of vertices **and** a permutation
Output : a set of sets that have been permuted

```
permuteSets := proc(F, prm)
```

```
local newF, S :
newF := NULL :
for S in F do
   newF := newF, permuteSet(S, prm) :
od:
{newF};
end:
```

Input : a set of vertices
Output : the number of vertices to permute

```
allPermuteSets := proc(F, n)
local p, T, i, prm :
p := permute(n) :
T := { } :
for i from 1 to nops(p) do
  prm := p[i] :
  T := T union {permuteSets(F, prm)} :
od:
T;
end:
```

Input : a list of faces
Output : the facets of the complex

```
ComplexFacets := proc(fList)
   local i, j, k, l, FacetList, FaceList;
   FaceList := fList;
   FacetList := NULL;
   for i from 1 to nops(FaceList[nops(FaceList)]) do
      FacetList := FacetList, FaceList[nops(FaceList)][i];
    od;
   for j from 0 to nops(FaceList) - 2 do
     for k from 1 to nops(FaceList[(nops(FaceList) - 1) - j]) do
        for l from 1 to nops(FaceList[((nops(FaceList) - 1) - j) + 1]) do
          if `subset`(FaceList[(nops(FaceList) - 1) - j][k], FaceList[((nops(FaceList) - 1) - j)
     + 1][l]) then
            break;
          fi;
          if l = nops(FaceList[((nops(FaceList) - 1) - j) + 1]) then
            FacetList := FacetList, FaceList[(nops(FaceList) - 1) - j][k];
          fi;
        od;
     od;
    od;
   return [FacetList];
end:
```

Input : a graph G
Output : the faces of the two − terminal complex of G

```
TTComplex := proc(G, u, v)
local V, n, E, S, s, H, i, faces, EdgesLabelled, EdgesLabelledSet, SLabelled, EdgeSet, tempS :
V := Vertices(G) :
n := nops(V) :
E := Edges(G) :
print(DrawGraph(Graph( {seq([Edges(G)[i], i], i = 1..nops(Edges(G)))}), style = spring));
EdgesLabelled := [seq(i, i = 1..nops(E))];
EdgesLabelledSet := convert(EdgesLabelled, set);
SLabelled := powerset(EdgesLabelled);
faces := {{ }};
for s in SLabelled do
   EdgeSet := [ ];
   for i from 1 to nops(s) do
    EdgeSet := [EdgeSet[ ], E[s[i]]] :
  od:
  H := Graph(V, convert(EdgeSet, set)) :
  if Distance(H, u, v) < infinity then
   tempS := convert(s, set) :
   faces := faces[ ], EdgesLabelledSet minus tempS :
  fi:
od:
return [ComplexFacets(stratify({faces}), false), stratify({faces})];
end:
```

Matroid Procedures

Input : a set of facets
Output : a boolean true if the facets form a matroid

```
checkMatroid := proc(s)
local s1, j, k, x, y, B1, B2, B, flg;
s1 := convert(s, list) :
 for j from 1 to nops(s1) do
   for k from 1 to nops(s1) do
     if j ≠ k then
      B1 := s1[j] :
      B2 := s1[k] :
      for x in B1 minus B2 do
        flg := false :
        for y in B2 minus B1 do
          B := (B1 minus {x}) union {y} :
          if member(B, s) = true then
             flg := true :
             break:
          fi:
        od:
        if flg = false then
          print(B1, B2, x);
         RETURN(false) :
       fi:
```

```
    od:
   fi:
 od:
od:
true :
end:
```

Input : a list of facets of a matroid
Output : a set of facets of a matroid

```
convertMatroidToSets := proc(matroid)
local i;
[seq(convert(matroid[i], set), i = 1 ..nops(matroid))];
end:
```

Input : a set of facets of a matroid
Output : a boolean true **if** the matroid is paving

```
checkPaving := proc(M)
local c, N, k, i :
c := stratify(complx(M));
N := nops(c[2]) :
k := nops(c) :
for i from 3 to k − 1 do
  if nops(c[i]) ≠ binomial(N, i − 1) then
    RETURN(false);
 fi:
od:
return true;
end:
```

Plotting Procedures

Input : a list of roots (possibly complex)
Output : a plot of the roots

```
PlotRoots := proc(rts)
local q, realMin, realMax, imMin, imMax, i, r, im, p :
q := NULL :
realMin := ∞ :
realMax := - ∞ :
imMin := ∞ :
imMax := - ∞ :
for i to nops(rts) do
 r := ℜ(rts[i]) :
 im := ℑ(rts[i]) :
 q := q, [r, im] :
 if r < realMin then realMin := r fi:
 if r > realMax then realMax := r fi:
 if im < imMin then imMin := im fi:
 if im > imMax then imMax := im fi:
od:
p := plot([q], realMin-1 ..realMax + 1, imMin-1 ..imMax + 1, style = point, symbol = circle, scaling
```

```
    = constrained, color = red) :
 end:
```

Polynomial Procedures

Input : a set of facets
Output : a multivariate version of the reliability polynomial

```
mgen := proc(s)
 local c, r, face, l, v, V, prd, lst :
 c := complx(s) :
 r := 0 :
 V := union(seq(s[k], k = 1 ..nops(s))) :
 for face in c do
  prd := 1 :
  for v in V do
   if member(v, face) then
    prd := prd·q||v :
   else:
    prd := prd·p||v :
  fi:
  od:
 r := r + prd:
 od:
 lst := NULL :
for v in V do
   lst := lst, p||v, q||v :
 od:
 lst := [lst]:
 unapply(r, lst);
 end:
```

Input : a (weighted) graph with two terminals
Output : the two − terminal reliability polynomial (in p)

```
TwoTerminalReliability := proc(G1, u, v)
 local V, E, n, k, Gdel, Gcon, w, Vcon, u1, e, sm, x, G :
 G := CopyGraph(G1) :
 V := Vertices(G) :
 E := Edges(G) :
 n := nops(V) :
 if nops(E) = 0 then
  RETURN(0);
 fi:
 sm := 0 :
 for x in V do
  if x ≠ u and x ≠ v then
    sm := sm + Degree(G, x) :
  fi:
 od:
 if sm = 0 then
```

```
   if not member( {u, v}, E) then
     RETURN(0);
   else
     k := GetEdgeWeight(G, {u, v}) :
    RETURN(1 − (1 − p)^k);
  fi:
fi:
if Distance(G, u, v) = infinity then
 RETURN(0);
fi:
for e in E do
    if nops(e intersect {u, v}) = 0 then
      k := GetEdgeWeight(G, e) :
      Gdel := DeleteEdge(G, e, inplace = false) :
      Gcon := Contract(G, e) :
     RETURN((1 − p)^k·TwoTerminalReliability(Gdel, u, v) + (1 − (1 − p)^k)
    ·TwoTerminalReliability(Gcon, u, v));
    else
       if nops(e intersect {u, v}) = 1 then
         w := op(1, {u, v} minus e) :
        k := GetEdgeWeight(G, e) :
       Gdel := DeleteEdge(G, e, inplace = false) :
       Gcon := Contract(G, e) :
        Vcon := Vertices(Gcon) :
        u1 := op(1, e intersect convert(Vcon, set)) :
         RETURN((1 − p)^k·TwoTerminalReliability(Gdel, u, v) + (1 − (1 − p)^k)
    ·TwoTerminalReliability(Gcon, u1, w));
      fi:
    fi:
od:
end:
```

Input : a two − terminal polynomial in p
Output : the corresponding two − terminal polynomial in q

```
TwoTerminalReliabilityq := proc(G1, u, v)
  expand(subs(p = 1 − q, TwoTerminalReliability(G1, u, v)));
end;
```

Input : a number of vertices
Output : the corresponding all − terminal polynomial in q

```
alltermcomprel := proc(n) options remember;
local j, s :
if n = 1 then
 RETURN(1) :
fi:
if n = 2 then
```

```
RETURN(1 - q):
fi:
s := 0:
for j from 1 to n - 1 do
  s := s + binomial(n - 1, j - 1)·alltermcomprel(j)·q^(j·(n - j)):
od:
expand(1 - s);
end;
```

Input : a number of vertices and the number of terminals
Output : the corresponding K − terminal polynomial in q

```
ktermcomprel := proc(n, k) options remember;
local s, j:
s := 0:
for j from k to n do
  s := s + binomial(n - k, j-k)·alltermcomprel(j)·q^(j·(n - j)):
od:
expand(s);
end;
```

Input : a stratified list of faces of a complex
Output : the corresponding F − polynomial

```
FVectorComplx := proc(C)
  local i, j, indexList := [ ], fPoly;
  for i from 1 to nops(C) do
    indexList := [indexList[ ], nops(C[i])];
  od;
  fPoly := 0;
  for j from 1 to nops(C) do
    fPoly := fPoly + indexList[j]·(x)^(j - 1);
  od;
  return fPoly;
end:
```

Input : an F − polynomial
Output : the corresponding H − polynomial

```
HVectorfromF := proc(FPoly)
  local d := degree(FPoly, x), fpoly;
  fpoly := simplify((1 - x)^d·subs(x = x/(1 - x), FPoly));
  return fpoly;
end:
```