STUDIES IN ALTERNATING AND SIMULTANEOUS COMBINATORIAL GAME THEORY

by

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Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

at

Dalhousie University
Halifax, Nova Scotia
June 2019

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Abstract

Combinatorial game theory has a beautiful algebraic structure. Games form an abelian group under the disjunctive sum and the normal play winning convention. However, not all games can be easily analyzed under this framework. In many cases, we must restrict properties to subclasses of games in order to have any useful analysis.

In this thesis, exact values are either hard to obtain or they are so complicated that they obscure the underlying structure. To aid with the analysis, techniques that approximate the value are used. Tools used for approximations include the reduced canonical form and outcome classes, particularly when values were challenging to calculate. We also present a method to construct game boards for games where initial positions are not naturally defined. Lastly, we develop a framework for simultaneous play combinatorial games, which requires approximation tools from economic game theory. We prove that the profile determines equality under extended normal play and continued conjunctive sum, while the economic game value determines equality for scoring play under the continued conjunctive sum.

Acknowledgements

I would like to thank my supervisors, Dr. Richard Nowakowski and Dr. Paul Ottaway. A special thank you to Richard for encouraging and supporting me in my pursuit of collaborative research experiences, as well as with teaching and outreach experience.

I would like to thank my internal readers, Dr. Jason Brown and Dr. Angela Siegel, and my external examiner, Dr. Matt DeVos, for reading my thesis thoroughly and providing me with insightful questions and very helpful feedback.

Thank you to the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Killam Trusts of Dalhousie University for their financial support.

Thank you to my family for their unconditional love and support throughout this incredible journey. Thank you for always believing in me.

Chapter 1

Introduction

The study of combinatorial games has a well established and elegant theory. Games form an abelian group under the disjunctive sum and the normal play winning convention. Combinatorial game theory, though well structured, has its fair share of very difficult problems. The full analysis of a game, G, involves finding its value. The value of a game determines how G behaves in sums with all other possible games. Finding the value is a hard problem. In this thesis, I examine analysis techniques that stop short of obtaining the value.

Even within the general analysis of combinatorial games, sometimes values are difficult to calculate. Not knowing game values can make it challenging for players to have any reasonable idea of where to move or even which player has an advantage. The *reduced canonical form* is a tool which allows for further simplifications to game options, by dropping the infinitesimals, in order to get a better idea of the overall game behaviour. This very powerful tool is utilized here to analyze the game THINNING THICKETS.

Not all games have opening positions defined by their rulesets. Chapter 4 gives one method for constructing an opening position for such a game: the *conjoined rulesets*. Conjoined rulesets involve two phases of game play: Phase 1 is played under the ruleset of one combinatorial game, which lends itself to set up a board for Phase 2. Phase 2 is played under a second ruleset but is played on the terminal board from Phase 1. Typically of most importance is who moves last in Phase 1; this allows for a focus on outcome classes rather than game values. We explore two case studies. In the first, we were able to determine values of games at the end of Phase 1, while in the second we were able to use outcome classes of the Phase 2 game to determine conditions for the Phase 1 board setup.

Combinatorial game theory (CGT) has expanded by exploring a different winning convention based on scores. Typically for scoring play, a rational player would aim to maximize their score. However, sometimes the structure of the game can be very complex and determining scores is not feasible. This is the case with THE ORTHOGONAL COLOURING GAME. We instead determine a drawing strategy for the second player on a special family of graphs.

Another approach to expanding CGT is to consider simultaneous moves. The main goal of this work is to combine aspects of combinatorial game theory and economic game theory to explore the possibility of a unified algebraic framework for combinatorial games played simultaneously. We quickly noticed that exact values were not attainable. Instead we needed to use tools from economic game theory in order to determine the game value. We have developed methods to test for equality of simultaneous play combinatorial games under certain winning conventions and sums.

This thesis will proceed as follows. In Chapter 3, we will examine a combinatorial game on a graph with in-degree constraints for players' options. The reduced canonical form is the main tool used. In Chapter 4, we take a different approach to ruleset modification, imposing a new way to combine games, called the *conjoined ruleset*, and examine two games within this framework, GO-CUT and SNO-GO.

Next, in Chapter 5 a scoring game called the ORTHOGONAL COLOURING GAME is studied. We show that the second player has a strategy to, in the very least, force a draw for any graphs that admit a strictly matched involution and we characterize graphs with this property. In some special cases, we determine that the outcome is a draw.

In Chapter 6, we define a philosophy for an algebraic framework of simultaneous combinatorial game theory. Imposing simultaneous moves on combinatorial game play, the goal of this chapter is to demonstrate the challenges associated with this ruleset change, and offer a toolkit for future analysis. We follow with Chapter 7 which highlights, using case studies, some results under different sum and winning convention combinations.

This thesis concludes with directions for future work in Chapter 8.

Combinatorial game rulesets can be found in Appendix B and simultaneous combinatorial game rulesets can be found in Appendix C. If a game is the focus of a chapter, the ruleset will appear in-text as well.

Chapter 2

Combinatorial Game Theory: Background

In this chapter, we present the necessary background for combinatorial game theory. Standard references used in this chapter are [2,66]. For other references with slightly different notation, see [12,29]. Often, games can be represented by graphs. Necessary graph theory background can be found in Appendix A.

To avoid repetition of background, the necessary background for Chapters 3, 4, and 5 are combined here. Extra explanations and examples may have appeared in [49], [48] or [4]. Permission to reprint appears in Appendix D.

Definition 2.0.1 [2] A *combinatorial game* is a two player game with perfect information, no chance devices, and players move alternately.

A *ruleset* describes the legal moves, and a *position* is an instance of the game after several (including zero) legal moves [2]. Throughout play, both players have complete knowledge of the board, meaning they know all available moves for themselves and their opponent at all times. We assume perfect (rational) play. The definition of a combinatorial game allows for the possibility of completely solving games with these properties; however, the game structure (size of a game tree) can sometimes prevent this from occurring in any reasonable amount of time.

The two players are called *Left* (L) and *Right* (R). Traditionally Left is female and Right is male, we will use this convention here as well. Due to specific considerations, we will also introduce other characters in Chapter 4 (Alf and Betti), and in Chapter 5 (Alice and Bob). *Short games* are combinatorial games which have finite descent, meaning the game terminates in a finite number of moves. As that is the primary focus of this thesis, we cover only its background here.

Definition 2.0.2 [2] The *Left options* from a game G, denoted by $G^{\mathcal{L}}$, is the set of game positions which arise after a move is made by Left. The *Right options* from a game G, denoted by $G^{\mathcal{R}}$, is the set of game positions which arise after a move is made by Right. All options are games. G^L (G^R) will denote a single Left (Right) option.

Definition 2.0.3 [2] Let G be a combinatorial game. The *game* G is defined in terms of its options. Formally,

$$G = \left\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\right\}.$$

A game G is *impartial* if the options available to Left and Right throughout game play are always the same. Otherwise, the game is said to be *partizan*.

For a partizan game tree, all left slanting arrows indicate Left options from a game G. Similarly, right slanting arrows indicate Right options from a game G. For an example see Figure 2.1, which shows a HACKENBUSH position (first node of the game tree), its options (second depth of the tree), and the full game tree (complete figure).

Definition 2.0.4 [2] Let G be a combinatorial game. The *negative* of G, $-G$, is defined as

$$-G = \left\{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\right\}.$$

The negative of a game G will be important throughout analysis as it reduces the number of positions to study. The concept of the negative of a game will be relevant for Chapter 3, parts of Chapter 6, and will be discussed in Chapter 8.

Definition 2.0.5 [2] The *birthday* of a game $G = \left\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\right\}$ is defined recursively as 1 plus the maximum birthday of any game in $G^{\mathcal{L}} \cup G^{\mathcal{R}}$. For the base case, if $G^{\mathcal{L}} = G^{\mathcal{R}} = \emptyset$, then the birthday of G is 0.

An alternative way to think about a birthday is that a game G is *born on day* k if its game tree has height k [2]. Notably, given a game G, the existence of the options of G with game tree heights $k-1$ or less allows for inductive reasoning about properties of G. In the literature, this is called inducting on the options.

A game is over when a player cannot move on their turn. The winner of the game is determined based on different criteria (predefined at the outset of the game). To date the most well understood winning convention is *normal play*, that is, if you cannot move on your turn you lose. The *misère play* winning convention is that if you cannot move on your turn you win. Lastly, the *scoring play* winning convention assigns scores to game positions based on the ruleset, and when the game is over the player with the higher score wins the game.

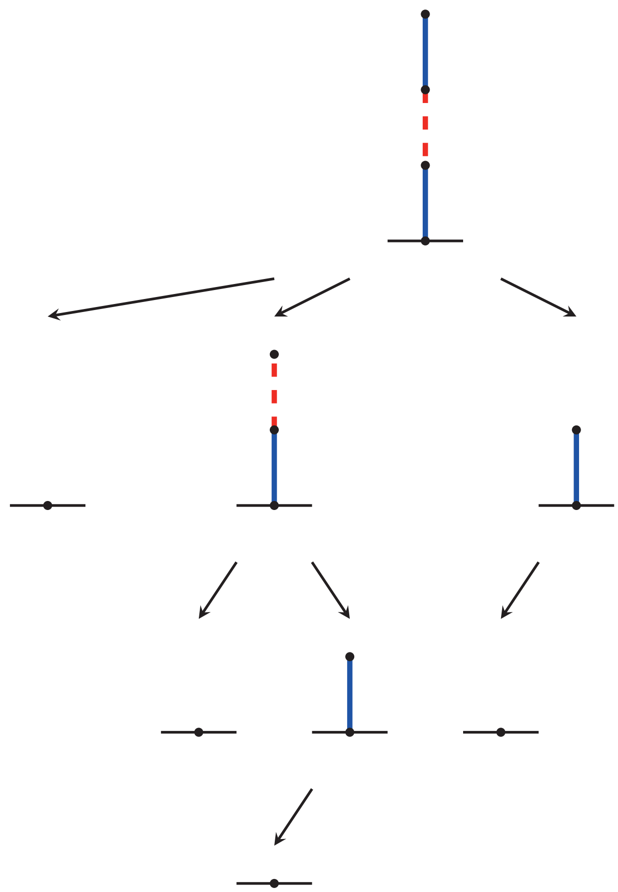


Figure 2.1: A HACKENBUSH position with its full game tree.

We first present the well established background for normal play. This thesis does not investigate game play under the misère convention and so we omit this from the background. For the interested reader, see [66]. Scoring play will be discussed throughout later sections of the thesis as needed.

2.1 Outcomes

Ultimately when studying a combinatorial game, or any game, the goal is to know with certainty which player will win. This is known as the *outcome class* of a game. There are four outcome classes which are partially ordered (see Figure 2.2):

$$o(G) = \begin{cases} \mathcal{L}, \text{Left can force a win regardless of who goes first (positive);} \\ \mathcal{R}, \text{Right can force a win regardless of who goes first (negative);} \\ \mathcal{N}, \text{the Next player can force a win (fuzzy);} \\ \mathcal{P}, \text{the Previous player can force a win (zero).} \end{cases}$$

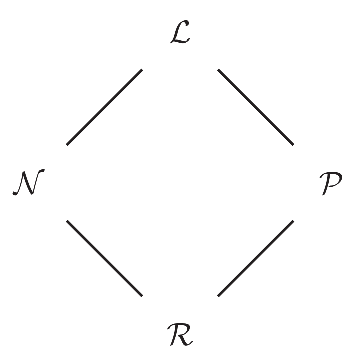


Figure 2.2: The partial order of the normal play outcome classes.

Note that the outcome classes $\mathcal{N}$ and $\mathcal{P}$ are incomparable. That is, a player will not always prefer moving first or moving second over all games; it will instead depend on the game being played. Left will however always prefer a Left win over a Right win, and she will always prefer a Left win over a first or second player win. Similar comparisons hold for Right. Hence, all the other outcome classes are comparable, as shown in Figure 2.2.

The outcome classes for any particular game under normal play can be restricted based on the options available to players. Recall that if both players have all the same options at all times throughout game play, the game is said to be *impartial*. In this case, there are only two possible outcomes: the game is either a first player win (in $\mathcal{N}$) or a second player win (in $\mathcal{P}$). If, however, at any point in game play there are different options available to the players, the game is called *partizan*, and one (or both) of $\mathcal{L}$ and $\mathcal{R}$ could also be possible outcomes.

Games are called *placement games* if tokens are placed on a board and thereafter the tokens cannot be moved or removed [17]. The games COL, DOMINEERING, NOGO, and SNORT are all placement games and appear often throughout later chapters. However, the intuition for placement games does not hold true for all games, and only thinking in terms of placement games can lead the reader astray for general analysis.

2.2 Sums

The sum most commonly used in CGT is the disjunctive sum. Individual games within a sum are often referred to as components.

Definition 2.2.1 [2] Let G and H be combinatorial games. In the *disjunctive sum* of G and H, denoted by $G+H$, on their turn, a player must move in exactly one component. Formally,

$$G+H=\left\{G^{\mathcal{L}}+H,G+H^{\mathcal{L}} \mid G^{\mathcal{R}}+H,G+H^{\mathcal{R}}\right\}.$$

Note that $G^{\mathcal{L}}+H=\left\{A+H : A\in G^{\mathcal{L}}\right\}$.

Example 2.2.2 Figure 2.3 provides an example of the options from a disjunctive sum of two HACKENBUSH positions.

Note that, in the disjunctive sum $A+B$, play no longer has to alternate in each component and the whole game trees of A and B must be considered.

Later in the thesis, we will be examining many ways to extend CGT and thus, it will be useful to define other possibilities for sums of games. We define here the sums we are interested in exploring; for an exhaustive list see [29] or [66].

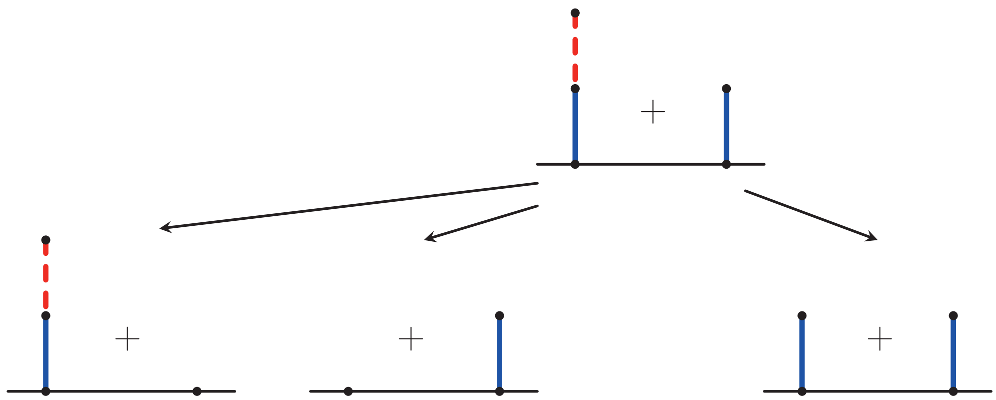


Figure 2.3: The disjunctive sum of two HACKENBUSH positions.

Definition 2.2.3 [66] Let G and H be combinatorial games. In the *conjunctive sum* of G and H, denoted by $G \wedge H$, a player moves in all components. The game ends when any of the components terminate. Formally,

$$G \wedge H = \left\{G^{\mathcal{L}} \wedge H^{\mathcal{L}} \mid G^{\mathcal{R}} \wedge H^{\mathcal{R}}\right\},$$

where $G^{\mathcal{L}} \wedge H^{\mathcal{L}} = \left\{A \wedge B : A \in G^{\mathcal{L}}, B \in H^{\mathcal{L}}\right\}$, and is similarly defined for $G^{\mathcal{R}} \wedge H^{\mathcal{R}}$.

In the conjunctive sum, if a player cannot move in all components on their turn, the game is over. The components in which the player does not have a move are called *terminal* and, based on a specified winning convention, determines the winner.

Example 2.2.4 Let $G = $ SUBTRACTION$(\{1,2\})(3)$ and $H = $ NIM(5). Consider $G \wedge H$ under the normal play winning convention. Then $o(G \wedge H) = \mathcal{N}$ since both G and H are non-terminal games, but after the first turn, H will be terminal if Player 1 removes all tokens from the heap. Hence, in this example, the outcome of $G \wedge H$ is determined by H as H is a first player win and can be terminal in one move by the first player.

Definition 2.2.5 [66] Let G and H be combinatorial games. In the *continued conjunctive sum* of G and H, denoted by $G \bigtriangledown H$, a player moves in all non-terminal components. The game ends when all components are terminal. Formally,

$$G \bigtriangledown H = \begin{cases} G + H & \textit{if } G \textit{ or } H \textit{ have no options;} \\ \left\{G^{\mathcal{L}} \bigtriangledown H^{\mathcal{L}} \mid G^{\mathcal{R}} \bigtriangledown H^{\mathcal{R}}\right\} & \textit{otherwise,} \end{cases}$$

where $G^{\mathcal{L}} \bigtriangledown H^{\mathcal{L}} = \{A \bigtriangledown B : A \in G^{\mathcal{L}}, B \in H^{\mathcal{L}}\}$, and is similarly defined for $G^{\mathcal{R}} \bigtriangledown H^{\mathcal{R}}$.

In the continued conjunctive sum, the last terminal component ends the game and determines the winner.

Example 2.2.6 Let $G = \textsc{subtraction}(\{1,2\})(3)$ and $H = \textsc{nim}(5)$. Then $o(G \bigtriangledown H) = \mathcal{P}$ since both G and H are non-terminal games, G is a second player win and H can either terminate on the first turn or the second (depending on what Player 1 decides to remove). Hence, in this example, the outcome of $G \bigtriangledown H$ is determined by G as G is a second player win, is terminal after the second turn, and the second player can also terminate H, if necessary, on their turn.

To further emphasize the difference between Definition 2.2.3 and Definition 2.2.5, let G be a combinatorial game. Then $G \wedge \emptyset = \emptyset$, while $G \bigtriangledown \emptyset = G$.

Definition 2.2.7 [66] Let G and H be combinatorial games. In the *sequential sum* of G and H, denoted by $G \to H$, players alternate moves in G until it is terminal, then they move in H. Formally,

$$G \to H = \begin{cases} H & \textit{if } G \textit{ has no options;} \\ \{G^{\mathcal{L}} \to H \mid G^{\mathcal{R}} \to H\} & \textit{otherwise,} \end{cases}$$

where $G^{\mathcal{L}} \to H = \{A \to H : A \in G^{\mathcal{L}}\}$, and is similarly defined for $G^{\mathcal{R}} \to H$.

Example 2.2.8 Let $G = \textsc{subtraction}(\{1,2\})(3)$ and $H = \textsc{nim}(5)$. Then $o(G \to H) = \mathcal{N}$ since Player 1 can force Player 2 to be the last player to move in G by removing 2 from 3; thereafter, when in H, Player 1 removes the entire heap and wins.

An interesting part of the analysis of sequential sums (sometimes called the sequential join) is that a player may want to lose in G so that they are the first to play in H, as shown in Example 2.2.8. For more analysis of such games, see [69, 71].

The disjunctive sum is a natural sum to use as it has been successful to work with throughout the study of combinatorial game theory. The sequential sum is very similar to a new sum defined in Chapter 4. The conjunctive and continued conjunctive sums both involve playing in all components at the same time, but game termination in each case follow extremes: first terminal component and last terminal component, respectively. We study these sums in Chapter 6 to explore these extreme cases. Other sums fall somewhere in between these two extremes, where players select a certain number of components to play in, and are left for future work.

For an examination of the twelve sum and winning convention combinations outlined in [29] applied to a specific game, see [42].

2.3 Algebraic Structure

Comparison of games is based on equality and we have the following:

Definition 2.3.1 [2] Let G and H be combinatorial games, and X ranges over all short combinatorial games. The relations *equality* and *greater than* are defined as follows:

- $G = H$ if $(\forall X)$, $o(G+X) = o(H+X)$.
- $G \geq H$ if $(\forall X)$, $o(G+X) \geq o(H+X)$.

From this, several algebraic properties hold.

Theorem 2.3.2 [2] *Equality is an equivalence relation.*

Note that the quotient of games modulo = will be considered after Definition 2.4.1 as more concepts need to be introduced.

Theorem 2.3.3 [2] *The relation $\geq$ is a partial order on games.*

Analyzing games can get cumbersome quickly. The branching factor of a game tree can be very large and can make it difficult to determine the outcome of a game. There are two tools for simplification, called domination and reversibility, which help in reducing the computations to simpler (equivalent) game positions.

Definition 2.3.4 [66] Let G be a game and let G^{L_1} and G^{L_2} be two Left options. If $G^{L_1} \geq G^{L_2}$, we say that G^{L_1} *dominates* G^{L_2} or G^{L_2} is *dominated* by G^{L_1} for Left.

Definition 2.3.5 [66] A Left option, G^L, in G is *reversible* if there exists G^{LR} such that $G^{LR} \leq G$. Then we can replace G^L with the Left options of G^{LR}.

Similar definitions to Definitions 2.3.4 and 2.3.5 hold for Right. For game equivalence associated with implementing domination and reversibility as reductions see Theorem 2.3.6 and Theorem 2.3.7 respectively.

Theorem 2.3.6 [66] *Let G be a combinatorial game and suppose G' is obtained from G by removing some dominated option G^{L_1}. Then $G' = G$.*

Theorem 2.3.7 [2] *Fix a game*

$$G = \{A, B, C, \ldots \mid H, I, J, \ldots\}$$

and suppose that for some Right option of A, call it A^R, $G \geq A^R$. If we denote the Left options of A^R by $\{W, X, Y, \ldots\}$:

$$A^R = \{W, X, Y, \ldots \mid \ldots\}$$

and define the new game

$$G' = \{W, X, Y, \ldots, B, C, \ldots \mid H, I, J, \ldots\},$$

then $G = G'$.

A game which has no dominated or reversible options for either player is said to be in *canonical form*. As equality is an equivalence relation on games, the representative for each equivalence class is given by the canonical form for the games of that class. Thus, the uniqueness of the representative is crucial. The canonical form is chosen as the representative because it has the smallest game tree and tree with least depth within the equivalence class. Within a sum of games, replacing a game H with its canonical form H' allows for the possibility of computational simplification without affecting the game outcome. The next theorem demonstrates that the canonical form is indeed unique.

Theorem 2.3.8 [2] *If G and H are in canonical form and $G = H$, then G is identical to H.*

2.4 Values

Game values are a shorthand description of games and their properties.

Definition 2.4.1 [66] The *game value* of G is its equivalence class modulo =.

Theorem 2.4.2 [66] *The set of game values form an abelian group under addition.*

In partizan play, one set of values for games are characterized by numbers. *Integers* can be thought of as the number of move advantage a player has over their opponent; a positive integer n means that Left has n moves available to her which Right cannot use. Similarly $-n$ means that Right has n moves available to him which Left cannot use. Naturally, Left prefers positive values and Right prefers negative values.

Note that '$\cdot$' will denote when a player does not have any options available.

Definition 2.4.3 [2] Let G be a game. If Left has n free moves available to her, and Right has no moves, then G has *value* n. Formally,

$$G = \{n - 1 \mid \cdot\}.$$

Example 2.4.4 In the game pictured in Figure 2.4, Left has two moves available to her, and Right has no moves available to him. Formally,

$$\begin{aligned} G &= \{1, 0 \mid \cdot\} \\ &= \{1 \mid \cdot\} \quad \textit{removing dominated options} \\ &= 2 \end{aligned}$$

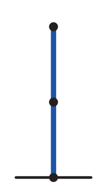


Figure 2.4: A HACKENBUSH position with value 2.

Equivalently, $G = \{\{0 \mid \cdot\} \mid \cdot\} = \{\{\{\cdot \mid \cdot\} \mid \cdot\} \mid \cdot\}$. This notation is not used in practice as using numbers is a more compact way to represent the set theory notation.

Given that we are considering only short games, the only numbers that occur are dyadic rationals.

Definition 2.4.5 [66] A rational number is *dyadic* if its denominator is a power of 2 (in lowest terms).

Moreover, game theoretically, non-integer dyadic rationals can be written as shown in Theorem 2.4.6; which highlights their most simplified form.

Theorem 2.4.6 [66] *For $n \geq 1$ and let m be an odd integer. Then*

$$\frac{m}{2^n} = \left\{ \frac{m-1}{2^n} \,\middle|\, \frac{m+1}{2^n} \right\}$$

is in canonical form.

In other words, let A and B be sets of dyadic rationals where $a < b$ for all $a \in A$ and $b \in B$ then either $\{A \mid B\}$ is the integer, n, closest to 0 which satisfies $a < n < b$ over all $a \in A$, $b \in B$ or, if no such integer exists, then $\{A \mid B\} = \frac{2p+1}{2^q}$ where q is the smallest positive integer such that there is a unique p with $a < \frac{2p+1}{2^q} < b$.

Example 2.4.7 Let G be the HACKENBUSH position pictured in Figure 2.5. In G, Left has an option to move to 0, while Right has an option to move to 1. Hence, $G = \{0 \mid 1\} = \frac{1}{2}$.

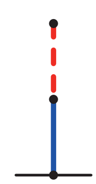


Figure 2.5: A HACKENBUSH position with value $\frac{1}{2}$.

Theorems 2.4.8 and 2.4.9 indicate how to test for equality and comparability respectively. Recall $-G = \left\{-G^{\mathcal{R}} \mid -G^{\mathcal{L}}\right\}$.

Theorem 2.4.8 [2] *$G = H$ if and only if $G - H = 0$.*

Theorem 2.4.9 [2] *$G \geq H$ if and only if Left wins moving second on $G - H$.*

Definition 2.4.10 [2] A *follower* of a game G is any game position H which can be reached from G, including G itself.

Definition 2.4.11 [2] G is *dicotic* (all-small) if every follower H in G has the property that Left can move from H if and only if Right can.

Two important examples of dicotic games are presented in Example 2.4.12.

Example 2.4.12 Let $* = \{0 \mid 0\}$. Each player has exactly one option from $*$, and thereafter the game terminates. Hence, $*$ is a dicotic game.

Now consider $\uparrow = \{0 \mid *\}$ (see Figure 2.6 for its game tree). Each player has an option at every node or the game is terminal. Hence $\uparrow$ is also a dicotic game.

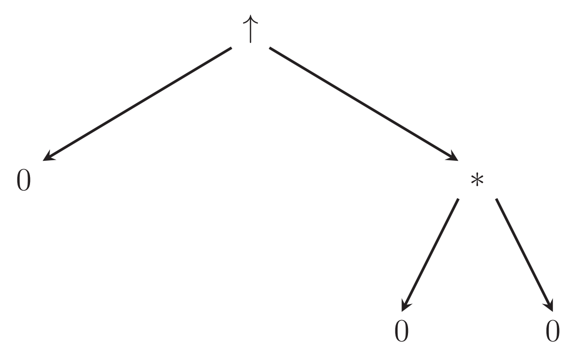


Figure 2.6: Game tree for $\uparrow$.

Definition 2.4.13 [2] The positions $\{y \mid z\}$ for y and z numbers with $y > z$ are called *switches*.

2.5 Sprague-Grundy Theory

Recall *impartial games* are games where both players have the same options available to them at all times. We have a different set of tools available to us to determine the value of an impartial game. NIM is the driving force of the theory of impartial games. Analysis of this game is shown in [2] and originated from [16]. Values that arise in impartial combinatorial game theory are called nimbers.

Definition 2.5.1 [2] The value of a nim-heap of size n, $n > 0$ is the *nimber* $*n$, where

$$*n = \{0, *, *2, \ldots, *(n-1) \mid 0, *, *2, \ldots, *(n-1)\}.$$

Note: the game where neither player has a move is called $0 = \{\cdot \mid \cdot\}$.

Definition 2.5.2 [2] The *minimum excluded value (or mex)* of a set of non-negative integers is the least non-negative integer which does not occur in the set.

Adding nimbers, denoted by $\oplus$, is called the *nim-sum*. Addition of nimbers a and b, $a \oplus b$, is done by writing a and b in binary and adding componentwise without carrying. See Example 2.5.3.

Example 2.5.3 The nim-sum of $*5$ and $*1$, denoted by $*5 \oplus *1$, is calculated as follows:

$$\begin{array}{cc|ccc} & *5 & 1 & 0 & 1 \\ \oplus & *1 & 0 & 0 & 1 \\ \hline & *4 & 1 & 0 & 0 \end{array}$$

Hence, $*5 \oplus *1 = *4$.

Note that these values are called *Grundy values* but more recently called *nimbers* or *nim-values*. Both take on slightly different notation. Grundy values are denoted by $\mathcal{G}(H)$ and omit $*$ notation, whereas nimbers utilize $*$ notation to distinguish nimbers from positive integers. Both may be used, and it will be clear from the context. Important results for impartial games are summarized in the next theorem.

Theorem 2.5.4 [2][Sprague-Grundy Theory]

1. If G is impartial, then $G + G = 0$.

2. For $k > 0$, the canonical form of $*k$ is

$$*k = \{0, *, *2, \ldots, *(k-1) \mid 0, *, *2, \ldots, *(k-1)\}.$$

3. For every impartial game G there is a non-negative integer n such that $G = *n$.

4. For non-negative integers k and j, $*k + *j = *(k \oplus j)$.

5. $\mathcal{G}(G) = \text{mex}\,\{\mathcal{G}(H) : H \textit{ is an option of } G\}$

6. If G, H, and J are impartial games, then $G = H + J$ if and only if $\mathcal{G}(G) = \mathcal{G}(H) \oplus \mathcal{G}(J)$.

Example 2.5.5 Consider a two-pile game of NIM, with piles of size 1 and 2, denoted by NIM$(1, 2)$. Its game tree is shown in Figure 2.7. Its value can be calculated using Theorem 2.5.4, part 5.

$$\begin{aligned} \mathcal{G}(1+2) &= \operatorname{mex}\{\mathcal{G}(1+1), \mathcal{G}(2), \mathcal{G}(1)\} \\ &= \operatorname{mex}\{0, 2, 1\} \\ &= 3. \end{aligned}$$

In order to calculate the value of an impartial game in practice, the idea is as follows. We write out the game tree of a particular game position. All terminal positions are assigned the value zero. Then, backtracking up the game tree, at each node we take the mex of the values of that nodes' children. Note that in a game tree for an impartial game, all options at a given level are available to both players (i.e., left or right slanting arrows are no longer associated with a particular player). For an example, see the game tree for NIM$(1, 2)$ in Figure 2.7. Note: the nodes of Figure 2.7 are generically denoted by $\mathcal{G}(G) = k$ which means the game G has Grundy value k.

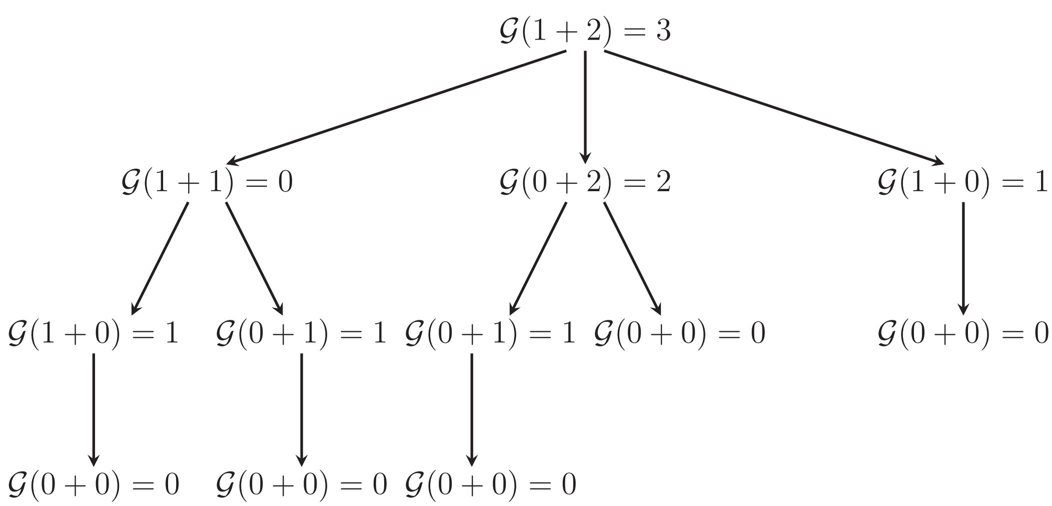


Figure 2.7: Game tree for NIM(1, 2).

A long-standing open question is to determine which nimbers appear as DOMINEERING positions [43]. The generalization of the question is to determine what is the largest n for which $*n$ appears in a game. This is formalized with the following definition.

Definition 2.5.6 [63] The *nim-dimension* of a game H is

$$\max\{k : *2^{k-1} \text{ occurs as a sub-position of } H\}.$$

Note that $0 = *0$ and $* = *1$. A ruleset has nim-dimension 0 if it never has $*$ as a position. Also, if $*2^{k-1}$ occurs, then, using the disjunctive sum, all nimbers up to $*(2^k - 1)$ occur. We consider the nim-dimension of THINNING THICKETS in Chapter 3.

Definition 2.5.7 [66] A game G is an *infinitesimal* if, for every positive number x, we have $-x < G < x$. Let I denote the set of infinitesimals. When $G - H$ is infinitesimal, we say that G and H are *infinitesimally close*, and write $G \equiv_{\mathrm{I}} H$. We will sometimes say that H is G-ish (G Infinitesimally SHifted).

Definition 2.5.8 [2] Let G be a game. The *Left stop* ($LS(G)$) and the *Right stop* ($RS(G)$) are defined recursively by

$$LS(G) = \begin{cases} G, & \textit{if } G \textit{ is equal to a number;} \\ max(RS(G^L)), & \textit{otherwise;} \end{cases}$$

$$RS(G) = \begin{cases} G, & \textit{if } G \textit{ is equal to a number;} \\ min(LS(G^R)), & \textit{otherwise.} \end{cases}$$

Example 2.5.9 Consider the game $G = \{\{1 \mid 0\} \mid \{-1 \mid -2\}\}$. Using Definition 2.5.8 we obtain, $LS(G) = max(RS(\{1 \mid 0\})) = 0$ and $RS(G) = min(LS(\{-1 \mid -2\})) = -1$.

An alternate definition of an infinitesimal is: *G is an infinitesimal if $LS(G) = RS(G) = 0$* [2]. This will be of use in the next chapter.

Chapter 3

THINNING THICKETS

These results first appeared in [49]. Permission to reprint appears in Appendix D.

Sometimes games can appear to have a very complex structure. For example, consider $H = \{\{\{\{1 \mid 0\} \mid 0\} \mid 0\} \mid \{\{1 \mid 0\} \mid 0\}, \{1 \mid 0\}, 1, 0\}$. On the surface, H looks like it could be a difficult game to analyze. Even after typical reductions, we obtain canonical form $H' = \{\{\{\{1 \mid 0\} \mid 0\} \mid 0\} \mid 0\}$. In [20], Calistrate introduces a reduction based on removing inf-dominated and inf-reversible options which allows us to instead study a simpler version of the game. This theory is proven and further developed in [40]. The reduction, called the *reduced canonical form*, denoted by $rcf(G)$, is an approximation tool in combinatorial game theory to allow for game analysis which otherwise wouldn't be possible. We can use the reduced canonical form of the game to obtain values up to, but not including, infinitesimals so this comes at a cost. It keeps the game outcome almost intact: for a game G, if $G \geq g$, where g is a dyadic rational, then the $rcf(G) \geq g$. Similarly, for $G \leq g$, but if $-g \leq G \leq g$ for all positive dyadic rationals g, the outcome is not known. For the above example, $rcf(H) = 0$. Thus far, the reduced canonical form has been used to study a special class of games called *hereditarily transitive*[1], that is, a game G where every G^{LL} (G^{RR}) is also a Left (Right) option of G [66]. Recently, in [52], a variant on PARTIZAN SUBTRACTION has been studied using reduced canonical form.

Games with edge and vertex deletion constraints on undirected graphs have been studied in [45] and [60] respectively. In this chapter, we explore a game on a directed graph, where players' options are based on in-degree of arc colourings. A natural and important question to consider is the following: by implementing the reduced canonical form, is the structure easier to understand than the canonical form structure?

This chapter is organized as follows. In Section 3.1 we provide additional background, specific to the reduced canonical form. Next, in Section 3.2 we develop

[1] Previously called *option-closed*.

motivation for the game and outline the results. In Section 3.3 the concept of nim-dimension is discussed and we prove that the nim-dimension of THINNING THICKETS is infinite. Next, in Section 3.4 we study RED-BLUE CORDONS and RED-BLUE STALKS to demonstrate the power of the reduced canonical form and show that the temperature is not bounded above. The chapter concludes with open questions.

3.1 Background

We first provide additional background required for our analysis of the game THINNING THICKETS. Recall from Chapter 2 that if games G and H are incomparable (confused), then $G \ngeq H$ and $H \ngeq G$; that is, $G - H$ is a first player win [66].

Definition 3.1.1 [66] The *confusion set* of a game G is defined by

$$\mathrm{C}(G) = \{x : x \text{ is a number that is incomparable to } G\}.$$

The next theorem breaks down how to determine what the confusion set is for a game G based on its Left and Right stops.

Theorem 3.1.2 [66] *Let G be a game and x be a number.*

- $LS(-G) = -RS(G)$ *and* $RS(-G) = -LS(G)$.
- *If* $RS(G) > x$*, then* $G > x$*. Similarly, if* $LS(G) < x$*, then* $G < x$.
- *If* $LS(G) > x$*, then* G *is greater than or confused with* x*. Similarly, if* $RS(G) < x$*, then* G *is less than or confused with* x.
- *If* $G \geq x$*, then* $RS(G) \geq x$*. Similarly, if* $G \leq x$*, then* $LS(G) \leq x$.
- $LS(G)$ *and* $RS(G)$ *are the endpoints of* $\mathrm{C}(G)$.

Proposition 3.1.3 [66] *Let G be a short game.*

- $LS(G) \geq RS(G)$
- $RS(G^L) \leq LS(G)$ *for every* G^L *and* $LS(G^R) \geq RS(G)$ *for every* G^R*, even if* G *is equal to a number.*

- $LS(G+x) = LS(G)+x$ *and* $RS(G+x) = RS(G)+x$*, for all dyadic rationals.*

By Theorem 3.1.2, the confusion set is actually an interval with the Left and Right stops of G being the endpoints and is referred to as the *confusion interval*. Now, $\mathrm{LS}(G) \in \mathrm{C}(G)$ if and only if the move that results in reducing the game to the number is a Left move. Left always prefers moving to $\mathrm{LS}(G)$ over letting Right move there. The same is true for the Right stop. Consider $G = \{\{3 \mid 1\} \mid \{-1 \mid -2\}\}$. The Left Stop of G is $LS(G) = 1$ and the Right Stop of G is $RS(G) = -1$. Hence $\mathrm{C}(G) = (-1, 1)$. Determining whether the endpoints are included in the confusion interval requires some effort. The reduced canonical form, however, allows us to ignore the status of the endpoints of the confusion interval.

Definition 3.1.4 [40] $G \geq_I H$ if $G \geq H + \epsilon$ for some infinitesimal ϵ; $G \leq_I H$ is defined similarly.

The *Number Translation Principle* [2] states that if x is a number and G is not, then $G + x = \left\{G^L + x \mid G^R + x\right\}$. Recall from Section 2.5 that G is an infinitesimal if $LS(G) = RS(G) = 0$ [2]. Together, these give the next result, which will be used often in this chapter.

Lemma 3.1.5 *Let G be a game. If $LS(G) = RS(G) = x$, for some number x, then $G \equiv_I x$.*

Proof. By Proposition 3.1.3, if $LS(G) = RS(G) = x$, then $LS(G-x) = RS(G-x) = 0$. By the alternative definition of infinitesimal, this implies G and x are infinitesimally close. Since x is a number, we obtain $G \equiv_I x$. □

Closely related is the *Number Avoidance Theorem* [2] which states: Suppose that x is a number in canonical form with a Left option and that G is not a number. Then, there exists a G^L such that $G^L + x > G + x^L$. That is, in the disjunctive sum of a number and a non-number, the best move is always in the non-number. This result reduces the number of cases to be considered when looking for the best moves.

The next result is new but follows easily from existing results.

Theorem 3.1.6 *Let G and H be games. If $\mathrm{RS}(G) \geqslant \mathrm{LS}(H)$ then $G - H \geqslant \epsilon$ for some infinitesimal ϵ.*

Proof. Case (i): If $\mathrm{RS}(G) > x > y > \mathrm{LS}(H)$, for some numbers x, y, then $G - H > x - y$ and $x - y$ is bigger than any infinitesimal. Case (ii): Assume that $\mathrm{RS}(G) = \mathrm{LS}(H) = x$. By the Number Avoidance Theorem and Proposition 3.1.3, we have $\mathrm{RS}(G - x) = 0$ and $\mathrm{LS}(x - H) = 0$ and therefore, by Lemma 6.4 [66], $G - x \geqslant \epsilon$ and $\delta \geqslant H - x$ for some infinitesimals ϵ, δ. Together they yield $G - H \geqslant \epsilon - \delta$. □

As a consequence, we obtain a result first found in [61].

Corollary 3.1.7 *Let a and b be numbers with $a \geqslant b$ then $a \geqslant_I \{a \mid b\} \geqslant_I b$.*

Definition 3.1.8 [66] Let G be any game.

- A Left option G^L is *Inf-dominated* if $G^L \leq_I G^{L'}$ for some other Left option $G^{L'}$.
- A Left option G^L is *Inf-reversible* if $G^{LR} \leq_I G$ for some G^{LR}.

The definitions for Right options are similar.

For example, let $G = \{1, \{1 \mid 0\} \mid 0\}$. Then $\{1 \mid 0\}$ is an Inf-dominated Left option of G, since $\{1 \mid 0\} \leq 1 + \uparrow$, where $\uparrow = \{0 \mid *\}$. In the reduced canonical form, the Inf-dominated options are removed.

Definition 3.1.9 [66] A game G is said to be in *reduced canonical form* provided that, for every sub-position H of G, either:

- H is a number in canonical form; or
- H is not a number or a number plus an infinitesimal, and contains no Inf-dominated or Inf-reversible options.

Theorem 3.1.10 [66] *For any game G, there is a game G' in reduced canonical form with $G \equiv_{\mathrm{I}} G'$.*

Theorem 3.1.11 [66] *Suppose that G and H are in reduced canonical form. If $G \equiv_{\mathrm{I}} H$, then $G = H$.*

This then shows that the reduced canonical form of a game G is well-defined and unique. In this chapter, we will use the $\equiv_{\mathrm{I}}$ notation instead of defining a function $rcf(G)$. Lastly, the following two results will be used often in the analysis of positions.

Lemma 3.1.12 [66] *If G is not a number and G' is obtained from G by eliminating an Inf-dominated option, then $G' \equiv_{\mathrm{I}} G$.*

Theorem 3.1.13 [40] *If $G = \{G^L \mid G^R\}$ is not a number and $G' = \{G^{L'} \mid G^{R'}\}$ is a game with $G^{L'} \equiv_{\mathrm{I}} G^L$ and $G^{R'} \equiv_{\mathrm{I}} G^R$, then $G' \equiv_{\mathrm{I}} G$.*

The rest of this section is only relevant for Section 3.4.1.

In a sum of games, players would like to have a way of determining which component to play in. The *heat* of a game is a way to measure the urgency of game play.

One way to think about heat is via a taxation to play. How much is a player willing to spend in order to move in a component? If $G^L > G^R$ then subtracting t from G^L and adding t to G^R brings them closer together. Eventually, t becomes too large, and players will no longer want to play in G.

Definition 3.1.14 [66] Let $t \geq -1$. We define G *cooled by* t, denoted by G_t, as follows. If G is equal to an integer n, then simply $G_t = n$. Otherwise, put $\tilde{G}_t = \{G_t^L - t \mid G_t^R + t\}$. Then $G_t = \tilde{G}_t$, unless there is some $t' < t$ such that $\tilde{G}_{t'}$ is infinitesimally close to a number x. In that case, fix the smallest such t' and put $G_t = x$.

Note: The guarantee of the existence of a smallest t' is non-trivial and is shown by Theorem 5.6 of [66].

Definition 3.1.15 [66] The *temperature* of G, denoted by $t(G)$, is the smallest $t \geq -1$ such that G_t is infinitesimally close to a number.

Definition 3.1.16 [66] Let G be a game. The trajectories $\lambda_t(G)$ and $\rho_t(G)$, for $t \geq -1$, are as follows. If G is equal to an integer n, then $\lambda_t(G) = \rho_t(G) = n$. Otherwise, the *scaffolds* $\tilde{\lambda}_t(G)$ (Left scaffold) and $\tilde{\rho}_t(G)$ (Right scaffold) are given by

$$\tilde{\lambda}_t(G) = max_{G^L}(\rho_t(G^L) - t)) \quad \text{and} \quad \tilde{\rho}_t(G) = min_{G^R}(\lambda_t(G^R) + t)).$$

Then $\lambda_t(G) = \tilde{\lambda}_t(G)$ and $\rho_t(G) = \tilde{\rho}_t(G)$, unless there is some $t' < t$ such that $\tilde{\lambda}_{t'}(G) = \tilde{\rho}_{t'}(G)$. In that case, let $x = \tilde{\lambda}_{t'}(G)$ for the smallest such t', and put $\lambda_t(G) = \rho_t(G) = x$.

Temperature can be calculated using the definition, and can be shown graphically using a *thermograph*. Both are shown in the next two examples. For more information about thermographs see [66].

Example 3.1.17 Let $G = \{1 \mid -1\}$. Then $G_t = \{1 - t \mid -1 + t\}$, so if $t = 1$ we obtain

$$\begin{aligned} G_1 &= \{1 - 1 \mid -1 + 1\} \\ &= \{0 \mid 0\} \\ &= * \end{aligned}$$

Since $*$ is infinitesimally close to 0, we obtain $G_t = 0$, for all $t > 1$. Hence, a player is willing to spend $0 \leq t \leq 1$ to play and so $t(G) = 1$. This is shown graphically in Figure 3.1. Note that the Left scaffold corresponds to the left side of the graph and the Right scaffold corresponds to the right side of the graph. The Left and Right scaffolds meet at $t \geq 1$.

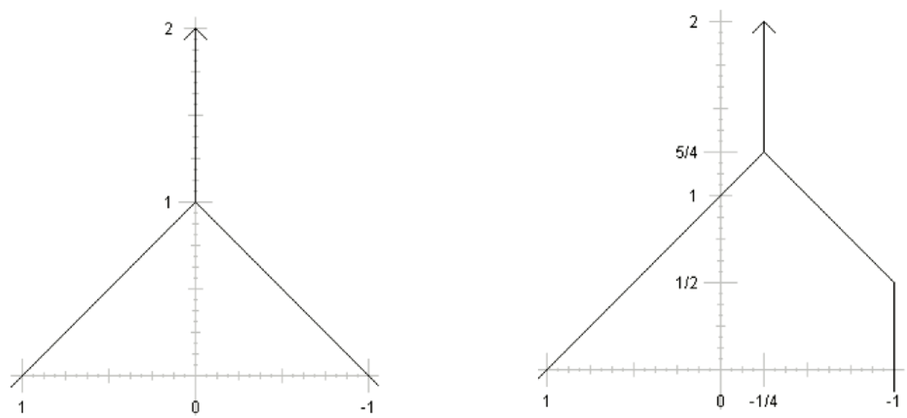


Figure 3.1: Thermographs for $\{1 \mid -1\}$ (left) and $\{1 \mid \{-1 \mid -2\}\}$ (right).

Example 3.1.18 Let $G = \{1 \mid \{-1 \mid -2\}\}$. We need to compute the temperature of G in two stages: first we compute the temperature for $\{-1 \mid -2\}$, then substitute this value into the expression and calculate the temperature of what remains. Let

$H = \{-1 \mid -2\}$. Then $H_t = \{-1 - t \mid -2 + t\}$, so when $t = \frac{1}{2}$ we obtain,

$$H_t = \left\{-\frac{3}{2} \,\middle|\, -\frac{3}{2}\right\}$$
$$= -\frac{3}{2} + *$$

Hence, from here and considering Definition 3.1.14,

$$G_t = \left\{1 - t \,\middle|\, -\frac{3}{2} + t\right\}, \text{for}\, t \geq \frac{1}{2}.$$

When $t = \frac{5}{4}$, this expression becomes

$$= \left\{1 - \frac{5}{4} \,\middle|\, -\frac{3}{2} + \frac{5}{4}\right\}$$
$$= \left\{-\frac{1}{4} \,\middle|\, -\frac{1}{4}\right\}$$
$$= -\frac{1}{4} + *$$

By Definition 3.1.14, we obtain, $G_t = -\frac{1}{4}$, for all $t > \frac{5}{4}$. Hence, $t(G) = \frac{5}{4}$. This is shown graphically in Figure 3.1.

3.2 Motivation, definitions and concepts for THINNING THICKETS

THINNING THICKETS is an offshoot of HACKENBUSH. Both are played on graphs with a set of distinguished vertices (the ground or roots), each arc/edge is coloured red, blue or green. In both, there are rules as to what arcs/edges Left and Right can delete, and also, any vertex not connected to the ground is also deleted.

The worth of a tree, and a game, is in the fruit that it bears. HACKENBUSH has many nice features and interesting analyses as do many of its progeny. For example, every RED-BLUE HACKENBUSH position is *cold*, that is, a number ([12], Chapter 7). A long-standing, and difficult, problem is the analysis of sums of flowers (green stalks with a blue or red flower at the top). Other variants include

- HACKENDOT [74] which has been extended to partially ordered sets in [19, 36];
- TIMBER [62] is an (impartial) variant played on directed graphs;
- TOPPLING DOMINOES [37] in which every number occurs as exactly one position, proved via ordinal sums, and the nimber $*n$ occurs exactly n times.

We were in search of a ruleset which took on many of the properties of HACKENBUSH but that had options which evolved throughout game play. THINNING THICKETS introduces a parity aspect to the game.

Definition 3.2.1 A directed graph G is a *thicket* if there is a subset of vertices $x_1, \ldots, x_k$ called *roots* and every arc is on a directed path to some root.

We will always write an arc as $\overrightarrow{ab}$, where a and b are the initial and terminal vertices respectively. The *in-degree* of an arc $\overrightarrow{ab}$ is the number of arcs with terminal vertex a.

Ruleset for THINNING THICKETS

- Board: A finite thicket in which each arc is coloured blue (single solid), red (dashed) or green (double solid).
- Moves: On a move, each player deletes an arc. Left removes a blue arc or a green arc with even in-degree (including 0) or a red arc with odd in-degree. Right removes a red arc or a green arc with even in-degree (including 0) or a blue arc with odd in-degree. After the arc is deleted, any arc and vertex not on a directed path to a root is also deleted.

In play, when the arc $\overrightarrow{ab}$ is deleted then the in-degree of b changes parity and so the player that can delete the arcs directed out of b also changes. This dynamism exists in only a few analyzed combinatorial games, see [45, 60, 65]. We sought out a game where the dynamism of options would lend itself to interesting analysis. In particular, THINNING THICKETS has hot positions (games where players want to move). Also, THINNING THICKETS is not hereditarily transitive, so from an opening position players are only aware of their current options. For example, in Figure 3.2, if it is Right's turn and he removes the top arc, he opens a move for himself. For clarity, in Figure 3.2 the arcs are labelled with the player who can currently remove it (L for Left and R for Right). The goal is to understand game play of THINNING THICKETS to highlight the game structure introduced by dynamic options.

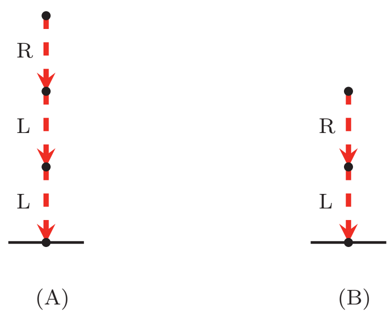


Figure 3.2: Evolving options of THINNING THICKETS game play; (A) the initial position and (B) the position after Right removes the top arc.

We present results for THINNING THICKETS in the cases of cordons, which are tall, thin, singly rooted graphs.

Definition 3.2.2 A *cordon*[2] consists of i) two sets of vertices $V_1 = \{v_0, v_1, \ldots, v_n\}$, where v_0 is the *root*, v_n is the *top vertex* and the others are called *interior* vertices, and $V_2 = \{l_1, l_2, \ldots, l_k\}$, ii) a strictly increasing sequence $\{a(1), a(2), \ldots, a(k)\}$, where $0 < a(1), a(k) \leq n-1$, and iii) the arcs are $\overrightarrow{v_i v_{i-1}}$, $i = 1, 2, \ldots, n$ and $\overrightarrow{l_j v_{a(j)}}$, $j = 1, \ldots, k$. We call the latter *leaf arcs*. The vertex $v_{a(i)}$ is called an *attachment vertex*. If V_2 is empty then we call the cordon a *stalk*, hence $\overrightarrow{v_n v_{n-1}}$ is not considered to be a leaf arc. The *height* of a cordon C, denoted $h(C)$, is the number of arcs in the stalk.

Example 3.2.3 Consider the cordon pictured in Figure 3.3. This is an example of a GREEN CORDONS position, a THINNING THICKETS cordon where all arcs are green. By Definition 3.2.2 we have the following: $V_1 = \{v_0, v_1, v_2, v_3\}$, $V_2 = \{l_1\}$. The interior vertices are v_1 and v_2, the root is v_0 and the top vertex is v_3. Here, $a(1) = 1$ and is the index for the only attachment vertex, v_1, with leaf arc $\overrightarrow{l_1 v_{a(1)}}$. The height of the cordon is 3.

[2]Horticultural definition: A cordon is a tree or shrub, especially a fruit tree, repeatedly pruned and trained to grow on a support as a single rope-like stem.

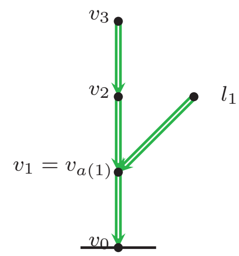


Figure 3.3: A GREEN CORDONS position.

General questions

For a given game, apart from 'Who wins?' and 'How?', it is interesting to know which values can occur and which cannot. In general, this is a very hard question to answer. This can be broken down into sub-questions:

1. What is the greatest number that can occur?
2. What temperatures can occur?
3. For what positive integers n can $*n$ occur? (Nim-dimension.)
4. Are *tinies*, $\{0 \mid \{0 \mid -n\}\}$, and *minies*, $\{\{n \mid 0\} \mid 0\}$, present? These are called *threats* because if a player is allowed to move consecutively in a component they receive an advantage.
5. What infinitesimals can occur?

In the analysis, we find the value $tiny(1) = \{0 \mid \{0 \mid -1\}\}$ occurs (see Figure 3.2 (A)) but were unable to provide any more insight. In addressing these subsidiary questions, it became natural to consider restricted versions of THINNING THICKETS.

To answer (3), we consider GREEN CORDONS (all the arcs are green), which is the impartial version of the game. In Section 3.3, Theorem 3.3.1, we show that the nim-dimension is infinite (i.e., every $*n$ occurs). In Theorems 3.3.2 and 3.3.3 we characterize the GREEN CORDONS positions with values 0 and $*$, and also show a Fibonacci recurrence (Theorem 3.3.4).

Returning to questions (1) and (2), we consider RED-BLUE CORDONS. Unlike HACKENBUSH, the values are not just numbers, indeed some are hot, and the canonical forms can be quite complicated. However, if the infinitesimal values are ignored then

Theorem 3.4.2 shows that RED-BLUE STALKS positions take on only eight values, specifically 0, 1, -1, $\{1 \mid 0\}$, $\{0 \mid -1\}$, $\{1 \mid -1\}$, $\{\{1 \mid 0\} \mid -1\}$, $\{1 \mid \{0 \mid -1\}\}$. Adding leaves gives a richer set of values. The set of values of BLUE CORDONS positions is infinite but describable if, again, infinitesimal values are ignored. In Theorem 3.4.1 we show that the value is either k or the switch $\{k+1 \mid k\}$ plus an infinitesimal for any non-negative integer k. Their negatives will be found by the corresponding RED CORDONS.

In Theorem 3.4.4, we consider a family of cordons where all the arcs but one are blue and the other is red and show that, for any positive integer n, there is a member of this family with temperature greater than n, thereby answering question (2).

3.3 GREEN CORDONS

Recall that a GREEN CORDONS position is a game on a cordon where all arcs are green and directed towards the ground. On a player's turn, one may remove an arc which has even in-degree. In Figure 3.4 do you want to play first or second?

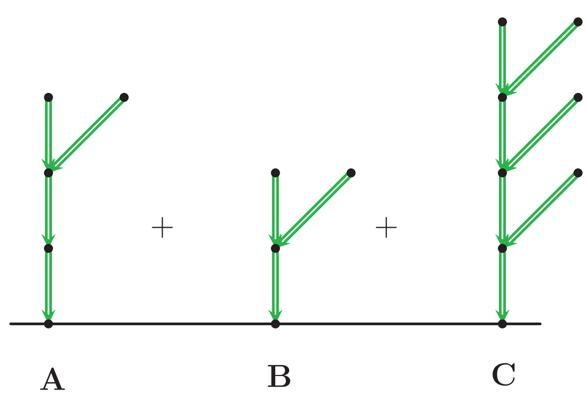


Figure 3.4: The disjunctive sum of three GREEN CORDONS positions.

Theorem 3.3.1 shows that a GREEN CORDONS position can have arbitrarily high nim-values. In Section 3.3.2, we characterize the positions with nim-values 0 and 1 and in Section 3.3.3 we show there is a Fibonacci recurrence associated with each.

3.3.1 Nim-dimension

The value of any GREEN CORDONS positions is a nimber by Theorem 2.5.4, since GREEN CORDONS positions are impartial. Also, recall from Section 2.5, nimbers are calculated recursively using the mex function.

Let $L(n)$ be the GREEN CORDONS position with stalk vertices $\{v_0, v_1, \ldots, v_n\}$ and with a leaf arc at v_i, for $1 \leqslant i \leqslant n-2$. Note that $L(0)$, $L(1)$ and $L(2)$ are the stalks with 0, 1 and 2 arcs, respectively (see Figure 3.5).

Figure 3.5: GREEN CORDONS: $L(1)$ (left) and $L(2)$ (right).

Theorem 3.3.1 *The nim-dimension of* GREEN CORDONS *and* THINNING THICKETS *is infinite.*

Proof. From $L(n)$, cutting $\overrightarrow{v_i v_{i-1}}$, $2 \leqslant i \leqslant n-1$, results in $L(i)$. As cutting stalk arcs is an option from every $L(i)$, $2 \leqslant i \leqslant n-1$, in particular all even height subpositions $L(i)$ appear as subpositions of all cordons of height $i+2$ or greater. Thus, they must have distinct values. Hence, all $L(k)$ where $k = 2n$ for $n = 1, 2, \ldots$ are distinct. Therefore, the nim-dimension of GREEN CORDONS, and hence THINNING THICKETS, is infinite. □

3.3.2 Characterizations for Positions of Value 0 and $*$

Leaves are ordered starting closest to the root. Recall $a(i)$ is the index of the stalk vertex to which the ith leaf arc is attached; that is, leaf l_i is attached to the stalk vertex $v_{a(i)}$. For example, in Figure 3.4, components A, B and C have $a(1)$ equal to 2, 1, and 1 respectively.

Theorem 3.3.2 *Let T be a* GREEN CORDONS *position, of height n. Then $\mathcal{G}(T) = 0$ if and only if either i) T is a stalk and n is even, or, ii) $a(1)$ is even, and all $a(i+1) - a(i)$, $1 \leq i < k$, and $n - a(k)$ are odd, where k is the number of leaf arcs.*

Proof. Let T be a GREEN CORDONS position with stalk vertices $\{v_0, v_1, \ldots, v_n\}$, thus $h(T) = n$. Let A be the set of GREEN CORDONS positions with the following properties: If T is a stalk, then n is even, or, if there are leaves, $a(1)$ is even, and $a(i+1)-a(i)$, $1 \leq i < k$, and $n - a(k)$ are all odd. To prove $\mathcal{G}(T) = 0$ only for $T \in A$, we must show that any option from T is not in A and if $S \notin A$ then S has an option in A (Theorem 2.13 of [2]).

Suppose $T \in A$. If T is a stalk, then the only move is to remove $\overrightarrow{v_n v_{n-1}}$, resulting in a stalk T' and $h(T')$ is odd. If T has a leaf arc, let T' be the resulting tree after a move, then the moves are as follows.

i) Remove $\overrightarrow{v_n v_{n-1}}$ (where $\overrightarrow{l_k v_{n-1}} \notin E(T)$) and thus in T' we have $h(T') - a(k) = n - 1 - a(k)$ is even.

ii) Remove $\overrightarrow{v_n v_{n-1}}$ (where $\overrightarrow{l_k v_{n-1}} \in E(T)$). Then $h(T') = h(T)$ however, $h(T') - a(k-1)$ is even.

iii) Remove $\overrightarrow{v_{a(i)} v_{a(i)-1}}$, where $i > 1$. Now $h(T') - a(i-1)$ is even.

iv) Remove $\overrightarrow{v_{a(1)} v_{a(1)-1}}$. Now T' is a stalk and $h(T')$ is odd.

v) Remove $\overrightarrow{l_i v_{a(i)}}$, where $1 < i < k$. Then $a(i+1) - a(i-1)$ is even.

vi) Remove $\overrightarrow{l_1 v_{a(1)}}$. Then $a(2)$ (which is $a(1)$ for T') is odd.

vii) Remove $\overrightarrow{l_k v_{a(k)}}$. Then $h(T') - a(k-1)$ is even.

Now, consider $S \notin A$. Then,

i) If S is a stalk, then $h(S)$ is odd and we remove $\overrightarrow{v_n v_{n-1}}$ leaving $h(S')$ even, i.e., $S' \in A$.

ii) If $a(1)$ is odd then remove $\overrightarrow{v_{a(1)} v_{a(1)-1}}$ to leave S', a stalk of even height.

iii) If $a(1)$ is even and there exists $a(i+1) - a(i)$ even, where j is the least such i which satisfies this property. Remove $\overrightarrow{v_{a(j+1)} v_{a(j+1)-1}}$, Then S' satisfies: $a(1)$ even, $a(i+1) - a(i)$ odd, for $1 \leq i < j$, and $h(S') - a(j) = a(j+1) - 1 - a(j)$ which is odd.

iv) If $a(1)$ is even, $a(i+1) - a(i)$, for all i, is odd but $n - a(k)$ is even, then delete $\overrightarrow{v_n v_{n-1}}$ to result in $S' \in A$. □

The classification for cordons with nim-value 1 is similar to that for the classification of cordons with nim-value 0 with the roles of nim-values 0 and 1 interchanged. Thus, the details of Theorem 3.3.3 are left to the reader.

Theorem 3.3.3 *Let T be a* GREEN CORDONS *position, of height n. Then $\mathcal{G}(T) = 1$ if and only if either (i) T is a stalk and n is odd; or (ii) all of $a(1)$, $a(i+1) - a(i)$, $1 \leq i < k$, and $n - a(k)$ are odd.*

3.3.3 Fibonacci Connection

Let F_n be the set of GREEN CORDONS positions of height n and nim-value 0. Let $f_n = |F_n|$. Note that $f_0 = 1$, $f_1 = 0$ and $f_2 = 1$.

Theorem 3.3.4 *The value f_n is given by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 1$ and $f_1 = 0$.*

Proof. Let $A_n \subset F_n$ be the subset of positions of F_n with no leaf arc at height 2. Let $C_n \subset F_n$ be the subset of positions of F_n with the first leaf arc at height 2 (i.e., $a(1) = 2$). By Theorem 3.3.2, it follows that $A_n \cap C_n = \emptyset$ and $F_n = A_n \cup C_n$, since a GREEN CORDONS position of nim-value 0 either has its first attachment vertex at height 2 or it does not. Stalks and all positions with $a(1) > 2$ (satisfying all other conditions of Theorem 3.3.2 as well) are in A_n and all remaining positions satisfy $a(1) = 2$ and are in C_n. We will show that there is a bijection between 1) A_n and F_{n-2}, and 2) C_n and F_{n-1}.

1) Consider $T \in F_{n-2}$ of height $n-2$ with $\mathcal{G}(T) = 0$. Add two arcs below the root vertex of T, to get T'. Since $a(1)$ of T was even (or T was a stalk), adding these two arcs below the root will result in the height of T' being n and $a(1)$ for T' is even at an attachment vertex value greater than 2 (or is a stalk of even height). Hence $T' \in A_n$. See Figure 3.6. Hence we have a map $\phi : F_{n-2} \to A_n$ which is injective, based on the characterization of positions with nim-value 0 given in Theorem 3.3.2. Conversely, if $T \in A_n$ then the induced subgraph starting at the stalk vertex of height 2 gives $T' \in F_{n-2}$. Hence we have a map $\rho : A_n \to F_{n-2}$ which is injective, again by Theorem 3.3.2. Moreover, $\rho = \phi^{-1}$ and so together this gives a bijection between A_n and F_{n-2}.

2) Consider $T \in F_{n-1}$ of height $n-1$ with $\mathcal{G}(T) = 0$, $a(2) - a(1)$ is odd, and $a(1)$ is even. Consider T' which we define to be T with an additional arc emanating from the root vertex of T and an arc at the second attachment vertex of T', $\overrightarrow{bv_2'}$. This gives $a(1) = 2$ for T' and $a(2) - a(1)$ is odd. Hence $T' \in C_n$. See Figure 3.7. Hence

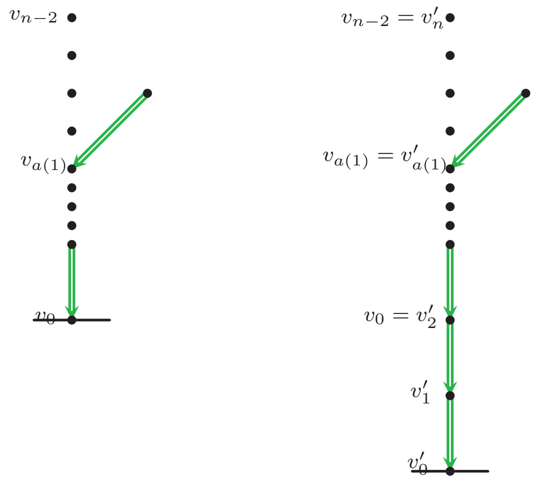


Figure 3.6: $T \in F_{n-2}$ (left) and $T' \in A_n$ (right).

we have a map $\psi : F_{n-1} \to C_n$ which is injective, based on the characterization of positions with nim-value 0 given in Theorem 3.3.2. Conversely, consider $T \in C_n$ with stalk vertices $\{v_0, v_1, \ldots, v_n\}$ and $a(1) = 2$ and let $\overrightarrow{bv_2}$ define the first leaf arc. Then taking the induced subgraph on stalk vertices $\{v_1, \ldots, v_n\}$ (so v_1 is the new root) and all leaves without b we obtain a GREEN CORDONS position T' of height $n-1$ with $a(1)$ even. Hence $T' \in F_{n-1}$. Hence we have a map $\tau : C_n \to F_{n-1}$ which is injective, based on the characterization of positions with nim-value 0 given in Theorem 3.3.2, and $\tau = \psi^{-1}$. Together, we obtain a bijection between C_n and F_{n-1}.

Thus $f_n = |F_n| = |A_n| + |C_n| = |F_{n-2}| + |F_{n-1}| = f_{n-2} + f_{n-1}$, where $|F_0| = 1$ and $|F_1| = 0$. □

As the classification for nim-value 0 and nim-value 1 are symmetric, the following theorem is immediate.

Theorem 3.3.5 *Let H_n be the set of* GREEN CORDONS *positions of height n with nim-value 1. Then $|H_n| = |H_{n-1}| + |H_{n-2}|$, where $|H_0| = 0$, and $|H_1| = 1$.*

Consider Figure 3.4. Using Theorems 3.3.2 and 3.3.3, we see that the nim-values for A, B and C are 0, 1 and 1 respectively. So $\mathcal{G}(A+B+C) = \mathcal{G}(A) \oplus \mathcal{G}(B) \oplus \mathcal{G}(C) = 0 \oplus 1 \oplus 1 = 0$. Hence this is a second player win.

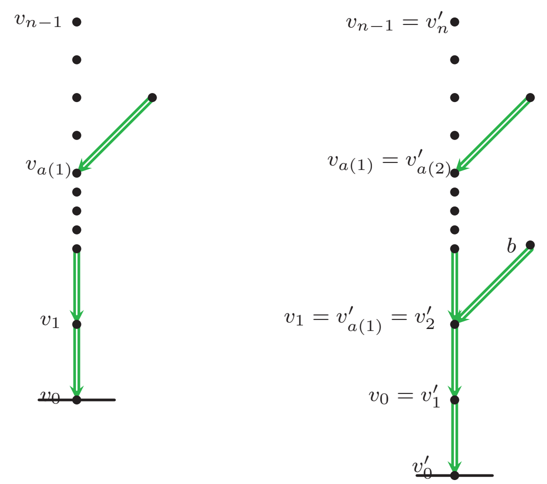


Figure 3.7: $T \in F_{n-1}$ (left) and $T' \in C_n$ (right).

3.4 Multicoloured CORDONS

A BLUE (RED) CORDONS position is a cordon where all the arcs are blue (red). We are only required to analyze BLUE CORDONS positions because RED CORDONS positions are their negatives.

The stalks seem recursively simple. Let $B(n)$ be a stalk with n blue arcs. Recall, Left can only remove $\overrightarrow{v_n v_{n-1}}$ and Right can remove any of the others: for $n > 1$, $B(n) = \{B(n-1) \mid B(0), B(1), \ldots, B(n-2)\}$. The canonical forms are

$$
\begin{aligned}
B(0) &= 0 \\
B(1) &= 1 \\
B(2) &= \{B(1) \mid B(0)\} = \{1 \mid 0\} \\
B(3) &= \{B(2) \mid B(0), B(1)\} = \{\{1 \mid 0\} \mid 0, 1\} = \{\{1 \mid 0\} \mid 0\} \\
B(4) &= \{B(3) \mid B(0), B(1), B(2)\} = \{\{\{1 \mid 0\} \mid 0\} \mid 0\}.
\end{aligned}
$$

It is not too difficult to show that $B(n) = \{B(n-1) \mid 0\}$[3]. The canonical forms will get longer as n increases. However, starting at $n = 3$, the games are infinitesimal

[3]For those conversant with CGSuite [67] notation, $B(n) = \{-_1 \mid 0^{n-2}\}$.

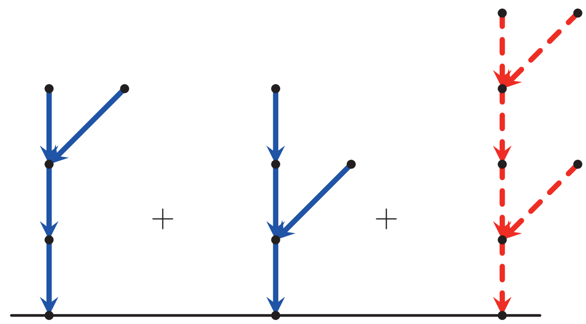


Figure 3.8: Sum of BLUE CORDONS and RED CORDONS positions.

since $LS(B(n)) = RS(B(n)) = 0$. The reduced canonical forms are much simpler: $Rcf(B(n)) = B(n)$ for $n = 0, 1, 2$ and $Rcf(B(n)) = 0$ for $n \geq 3$.

The analysis of BLUE CORDONS, in general, is made simpler by using the reduced canonical forms since there are only a few cases to consider.

First, our notation. A position is denoted by a tuple, each entry of the tuple represents the presence (1), or absence (0), of a leaf arc on v_i, $i > 0$, where left to right in the tuple is top to bottom on the cordon (though difficult, this notation allows for succinct descriptions later on). A position will always start with a 0, since there is no leaf arc at the top vertex (it would be part of the stalk instead) but it is useful to indicate the moves. We don't include the ground vertex so the empty game is $G = [\,]$ and the stalk with one arc is $G = [0]$.

Suppose there is a leaf arc at v_i for some $i > 0$. Our notation would have a 1 in the ith place, as in $[0 \dots 1 \dots]$ or, better, $[\alpha 1 \beta]$ for some arbitrary strings α and β. Around this ith vertex, Left has several cases to consider. Left can move to $[\alpha 0 \beta]$ (note that α starts with 0) by removing the leaf arc at v_i. If the position is $[\alpha 11 \beta]$ Left can remove the stalk arc $v_i v_{i-1}$ to leave $[00\beta]$. Otherwise the position is $[\alpha 10 \beta]$ and Left can remove the stalk arc $v_i v_{i-1}$ to leave $[0\beta]$. For $n > i > 0$, if there is no leaf arc at v_i, the position can be expressed as one of $[\alpha 00 \beta]$, $[\alpha 01 \beta]$ or $[\alpha 0]$, where α has a leading 0. Right has the move to delete the arc below v_i giving the positions, respectively, $[0\beta]$, $[00\beta]$ or $[\,]$. For a simple example, in Figure 3.8, reading from left to right, we have $[010] + [001] - [0101]$.

To emphasize, α indicates a section of the cordon which is arbitrary, and starts with 0 (i.e., non-empty).

Theorem 3.4.1 *The value of a* BLUE CORDONS *position is:*

1. $[0] = 1$, $[00] = \{1 \mid 0\}$, $[\alpha 10] \equiv_I \{1 \mid 0\}$, $[\alpha 00] \equiv_I 0$
2. $[\alpha 01^{2k+1}] \equiv_I k+1, k \geq 0$
3. $[\alpha 001^{2k}] \equiv_I k, k \geq 1$
4. $[\alpha 101^{2k}] \equiv_I \{k+1 \mid k\}, k \geq 1$
5. $[001^{2k}] \equiv_I \{k+1 \mid k\}, k \geq 1$
6. $[01] = 1$, *and in general we have* $[01^{2k}] = [01^{2k+1}] = k+1, k \geq 1$

Proof. From the position $[\alpha\gamma]$, a move to $[\alpha'\gamma]$ is, in our notation, a move in α that modifies, but does not completely eliminate, the string α. For example, let $\alpha = 0000$. Then in $[\alpha 0]$ Right has the options $[000]$, $[00]$ both of which would be combined into one symbol $[\alpha' 0]$. The option to $[0]$ (and to $[\,]$) will be listed separately. β indicates a non-empty string after a move has occurred (could be different from α'). The α' and β strings do not affect the values and so we use them as generic string representations.

If G is the empty cordon, $G = 0$ since neither player has a move. If G is $[0]$ there is only one move, for Left to move to $[\,]$. So $G = \{0 \mid \cdot\} = 1$. If G is $[00]$ then Left has a move to $[0]$ and Right has a move to $[\,]$. Hence $G = \{1 \mid 0\}$. Our template for the typical argument, where we have included the reasons for each step, is

$$\begin{aligned} [000] &= \{[00] \mid [0], [\,]\}\text{—options} \\ &= \{\{1 \mid 0\} \mid 1, 0\}\text{—values} \\ &\equiv_I 0 \quad \text{—reduced canonical form.} \end{aligned}$$

For the rest of the analysis we consider the moves in the order: 1) in α; 2) any moves on the stalk (not in α); then 3) any leaf arc moves. Moves in 1) where at least one such move exists are noted by †. A move which may or may not exist for a player is noted by ‡. Options which do not have either † or ‡ are considered to be guaranteed options.

We go through Case 1 in detail. As Cases 2, 3 and 4 use similar reasoning, so we omit extra explanations in these cases.

Case 1:

$$\begin{aligned}[\alpha 10] &= \{[\alpha' 10]^{\dagger}, [000]^{\dagger}, [0], [\alpha 00] \mid [\alpha' 10]^{\ddagger}, [000]^{\ddagger}, [\,]\} && (3.1)\\ &\equiv_I \{\{1 \mid 0\}^{\dagger}, 0^{\dagger}, 1, 0 \mid \{1 \mid 0\}^{\ddagger}, 0^{\ddagger}, 0\} && (3.2)\\ &\equiv_I \{1 \mid 0\}, \quad \text{by Corollary 3.1.7.} && (3.3)\end{aligned}$$

We obtain (3.1) by determining the options for each player. From here, we use induction on the options to obtain their values in (3.2) up to infinitesimals. Applying Corollary 3.1.7 we obtain the following: for the values of Left's options, $\{1 \mid 0\}$ is Inf-dominated by 1 and 0 is dominated by 1, so the values of Left's options are reduced to 1. Similarly for Right, $\{1 \mid 0\}$ is Inf-dominated by 0 and 1 is dominated by 0, so the values of Right's options are reduced to 0. Hence, $[\alpha 10] \equiv_I \{1 \mid 0\}$. Note also that domination was based on guaranteed options, meaning that if x dominated y, x came from a guaranteed option for that player.

Similarly,

$$\begin{aligned}[\alpha 00] &= \{[\alpha' 00]^{\dagger}, [00]^{\dagger} \mid [\alpha' 00]^{\ddagger}, [00]^{\ddagger}, [0], [\,]\} && (3.4)\\ &\equiv_I \{0^{\dagger}, \{1 \mid 0\}^{\dagger} \mid 0^{\ddagger}, \{1 \mid 0\}^{\ddagger}, 1, 0\} && (3.5)\\ &\equiv_I 0, \quad \text{by Lemma 3.1.5.} && (3.6)\end{aligned}$$

Similarly to the previous case, we obtain (3.4) by determining the options for each player. From here, we use induction on the options to obtain their values in (3.5) up to infinitesimals. For Left, even though it is unknown which of her options will exist, at least one of them exists and in both cases the Left stop of this position is 0; i.e., $LS([\alpha 00]) = 0$. For Right, 1 is dominated by 0 and $\{1 \mid 0\}$ is Inf-dominated by 0. Regardless of whether either of the first two options exist, the Right stop of this position is 0; i.e., $RS([\alpha 00]) = 0$. By Lemma 3.1.5 since $LS([\alpha 00]) = RS([\alpha 00]) = 0$, we obtain $[\alpha 00] \equiv_I 0$.

Case 2:

$$\begin{aligned}
[\alpha 01^{2k+1}] &= \{[\alpha' 01^{2k+1}]^{\dagger}, [01^{2k+1}]^{\dagger}, [001^{2j+1}]_{j=0}^{k-1}, [001^{2j}]_{j=1}^{k-1}, [00], [\,], [\alpha 001^{2k}], \\
&\quad [\beta 01^{2j+1}]_{j=0}^{k-1}, [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 10] \\
&\mid \quad [\alpha' 01^{2k+1}]^{\ddagger}, [01^{2k+1}]^{\ddagger}, [001^{2k}]\} \\
&\equiv_I \{k+1^{\dagger}, k+1^{\dagger}, \{j+1\}_{j=0}^{k-1}, \{j+1 \mid j\}_{j=1}^{k-1}, \{1 \mid 0\}, 0, k, \\
&\quad \{j+1\}_{j=0}^{k-1}, \{j+1 \mid j\}_{j=1}^{k-1}, \{1 \mid 0\} \\
&\mid \quad k+1^{\ddagger}, k+1^{\ddagger}, \{k+1 \mid k\}\} \\
&\equiv_I k+1, \quad \text{by Lemma 3.1.5.}
\end{aligned}$$

Case 3:

$$\begin{aligned}
[\alpha 001^{2k}] &= \{[\alpha' 001^{2k}]^{\dagger}, [001^{2k}]^{\dagger}, [001^{2j}]_{j=1}^{k-1}, [001^{2j+1}]_{j=0}^{k-2}, [00], [\,], \\
&\quad [\alpha 0001^{2k-1}], [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 01^{2j+1}]_{j=0}^{k-1}, [\alpha 001^{2k-1}0] \\
&\mid \quad [\alpha' 001^{2k}]^{\ddagger}, [001^{2k}]^{\ddagger}, [01^{2k}], [001^{2k-1}]\} \\
&\equiv_I \{k^{\dagger}, \{k+1 \mid k\}^{\dagger}, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\}, 0, \\
&\quad k-1, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-1}, \{1 \mid 0\} \\
&\mid \quad k^{\ddagger}, \{k+1 \mid k\}^{\ddagger}, k+1, k\} \\
&\equiv_I k, \quad \text{by Lemma 3.1.5.}
\end{aligned}$$

Case 4:

$$\begin{aligned}
[\alpha 101^{2k}] &= \{[\alpha' 101^{2k}]^{\dagger}, [0001^{2k}]^{\dagger}, [01^{2k}], [001^{2j+1}]_{j=0}^{k-2}, [001^{2j}]_{j=1}^{k-1}, [00], [\,], \\
&\quad [\alpha 001^{2k}], [\alpha 1001^{2k-1}], [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 01^{2j+1}]_{j=0}^{k-2}, [\beta 10] \\
&\mid \quad [\alpha' 101^{2k}]^{\ddagger}, [0001^{2k}]^{\ddagger}, [001^{2k-1}]\} \\
&\equiv_I \{\{k+1 \mid k\}^{\dagger}, k^{\dagger}, k+1, \{j+1\}_{j=0}^{k-2}, \{j+1 \mid j\}_{j=1}^{k-1}, \{1 \mid 0\}, 0, \\
&\quad k, k-1, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\} \\
&\mid \quad \{k+1 \mid k\}^{\ddagger}, k^{\ddagger}, k\} \\
&\equiv_I \{k+1 \mid k\}, \quad \text{by Corollary 3.1.7.}
\end{aligned}$$

Case 5:

$$\begin{aligned}[001^{2k}] &= \{[01^{2k}], [001^{2j}]_{j=1}^{k-1}, [001^{2j+1}]_{j=0}^{k-2}, [00], [\,],\\ &\qquad [0001^{2k-1}], [\beta 101^{2j}]_{j=1}^{k-1}, [\beta 01^{2j+1}]_{j=0}^{k-2}, [\beta 10]\\ &\quad \mid \qquad [001^{2k-1}]\}\\ &\equiv_I \{k+1, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\}, 0,\\ &\qquad k, \{j+1 \mid j\}_{j=1}^{k-1}, \{j+1\}_{j=0}^{k-2}, \{1 \mid 0\}\\ &\quad \mid \qquad k\}\\ &\equiv_I \{k+1 \mid k\}, \quad \text{by Corollary 3.1.7.}\end{aligned}$$

In the next case, we claim that the canonical form is simple.

Case 6: We claim that $[01^{2k}] = k+1$. Consider first the base case: $k = 1$,

$$\begin{aligned}[011] &= \{[001], [00], [\,], [010] \mid \cdot\}\\ &= \{\{1 \mid \{1 \mid 0\}\}, \{1 \mid 0\}, 0, \{1 \mid 0\} \mid \cdot\}\\ &= 2\end{aligned}$$

We need to show that $[01^{2k}] - k - 1 = 0$. If Right moves first, he only has one move which is to $[01^{2k}] - k$. Left's best move is to $[001^{2k-1}] - k$ (removing the top leaf arc). From here, Right has two possibilities. He could move to $[001^{2k-1}] - k + 1$ and Left moves to $[01^{2k-1}] - k + 1$ but $[01^{2k-1}] = k$ (by induction) and so $[01^{2k-1}] - k + 1 = k - k + 1 = 1 > 0$. Otherwise, Right moves to $[001^{2k-2}] - k$ and Left moves to $[01^{2k-2}] - k = 0$ (by induction). So Right loses going first.

Next we check Left moving first in $[01^{2k}] - k - 1$. Left can move to $[\,] - k - 1 < 0$ by taking $\overrightarrow{v_1 v_0}$, and loses. Left could move to $[00] - k - 1$ and Right wins by moving to $-k-1$ (deleting $\overrightarrow{v_1 v_0}$). She could remove a stalk arc which results in a cordon with an even number of leaves: $[001^{2j}] - k - 1 \leq_I \{k \mid k-1\} - k - 1 \leq_I \{-1 \mid -2\} < 0$ ($j \leq k-1$) and loses (by case 3). Or she could remove a stalk arc which results in a cordon with an odd number of leaves, $[001^{2j-1}] - k - 1 \leq_I k - k - 1 \leq_I -1 < 0$ (case 2), and Left loses. What remains to check are the options where Left removes a leaf arc:

1) $[\beta 01^{2j-1}] - k - 1 \leq_I k - k - 1 \leq_I -1 < 0$ ($j \leq k-2$); (case 2)

2) $[\beta 101^{2j}] - k - 1 \leq_I \{k \mid k-1\} - k - 1 < 0$; (case 4)

3) $[\beta 10] - k - 1 \equiv_I \{1 \mid 0\} - k - 1 \leq \{-k \mid -k-1\} < 0$, $k \geq 2$. (case 1)

As all three options result in a negative value, Left loses moving first when removing a leaf arc. Hence $G = [01^{2k}] = k + 1$.

Let $k \geq 1$. We claim $[01^{2k+1}] = k + 1$. Consider first the base case: $k = 1$,

$$\begin{aligned} [0111] &= \{[0011], [001], [00], [\,], [0101], [0110] \mid \cdot\} \\ &= \{\{2 \mid \{1 \mid \{1 \mid 0\}\}\}, \{1 \mid \{1 \mid 0\}\}, \{1 \mid 0\}, 0, \{1 \mid \{1 \mid 0\}\}, \{1, \{1 \mid 0\} \mid 0\} \mid \cdot\} \\ &= 2 \end{aligned}$$

We need to show that $[01^{2k+1}] - k - 1 = 0$. If Right moves first in $[01^{2k+1}] - k - 1$, he only has one move which is to $[01^{2k+1}] - k$ and Left responds to $[001^{2k}] - k$. Right moves to one of the following positions: 1) $[001^{2k-1}] - k$ and Left responds to $[01^{2k-1}] - k = 0$ by induction; 2) $[001^{2k-1}] - k + 1 \equiv_I k - k + 1 > 0$ (case 2). Hence, in both cases Right loses moving first.

Now we consider Left moving first in $[01^{2k+1}] - k - 1$. All moves except $\overrightarrow{v_n v_{n-1}}$ and $\overrightarrow{v_{n-1} v_{n-2}}$ are considered in the previous paragraph. The two extra cases are:

1) $[001^{2k}] - k - 1$ and Right responds to $[001^{2k-1}] - k - 1 \equiv_I k - k - 1 < 0$;

2) $[001^{2k-1}] - k - 1 \equiv_I k - k - 1 < 0$,

both of which follow from case 2.

In all situations, Left loses moving first, so $[01^{2k+1}] = k + 1$. □

In a RED-BLUE STALKS position, the top arc trivially has even in-degree (0) and all other (interior) arcs have odd in-degree. Therefore, in a RED-BLUE STALKS position Left can remove a blue top arc or any red interior arc and Right can remove a red top arc or any blue interior arc.

The notation we use for RED-BLUE STALKS places emphasis on who can move (Left (L) or Right (R)) rather than the colour of the arc. For example, in Figure 3.9, the positions listed from top to bottom, in order from left to right, are RLLR + RRLR + LRLL. Note that removing an arc changes the symbol immediately below (to the right).

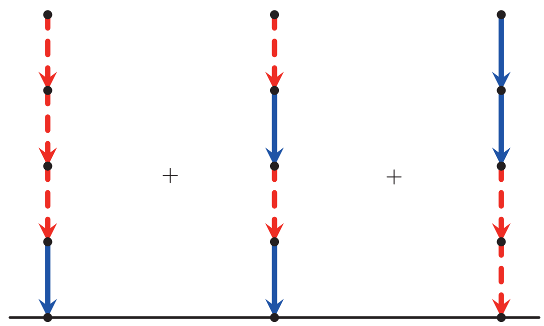


Figure 3.9: A disjunctive sum of RED-BLUE STALKS positions.

Theorem 3.4.2 RED-BLUE STALKS *has the following classification:*

1. *If* $G \in \left\{LR^k, \alpha LLR^k, RL^k, \alpha RRL^k, k \geq 2\right\}$, *then* $G \equiv_I 0$.

2. (a) *If* $G \in \left\{LR, \alpha LLR, \alpha LRL^k, k \geq 3\right\}$, *then* $G \equiv_I \{1 \mid 0\}$.

   (b) *If* $G \in \left\{RL, \alpha RRL, \alpha RLR^k, k \geq 3\right\}$, *then* $G \equiv_I \{0 \mid -1\}$.

3. (a) *If* $G = L^k$, $k \geq 1$, *then* $G = 1$.

   (b) *If* $G = R^k$, $k \geq 1$, *then* $G = -1$.

4. *If* $G \in \{\alpha LRL, \alpha RLR\}$, *then* $G \equiv_I \{1 \mid -1\}$.

5. (a) *If* $G = \alpha RLRR$, *then* $G \equiv_I \{\{1 \mid 0\} \mid -1\}$.

   (b) *If* $G = \alpha LRLL$, *then* $G \equiv_I \{1 \mid \{0 \mid -1\}\}$.

There are no restrictions on α.

Proof. For cases 1 and 4 (cases which encompass a position and its negative), we only show the proof for some of the positions and leave the proof of negatives to the reader. We also show cases 2a, 3a, and 5a. Cases which are not directly shown, namely 2b, 3b, 5b, are negatives of cases 2a, 3a, and 5a, respectively. For the rest of the analysis we consider the moves in the order: 1) in α; and 2) any moves on the stalk (not in α). Options are marked by † if, among all such marked options for a player, at least one such option exists for that player. A move which may or may not exist for a player is marked by ‡.

Case 1:

$$\begin{aligned} LR^k, k \geq 2 \quad &= \quad \{LR^{k-1\dagger}, LR^{\dagger} \quad | \quad \left\{LR^j\right\}_{j=2}^{k-2\ddagger}, LR^{\ddagger}, L, \emptyset\} \\ &\equiv_I \quad \{0^{\dagger}, \{1 \mid 0\}^{\dagger} \quad | \quad 0^{\ddagger}, \{1 \mid 0\}^{\ddagger}, 1, 0\} \\ &\equiv_I \quad 0, \quad \text{by Lemma 3.1.5.} \end{aligned}$$

$$\begin{aligned} \alpha LLR^k, k \geq 3 \quad &= \quad \{\alpha' LLR^{k\ddagger}, RLR^{k\ddagger}, R^{k+1}, LR^{k-1} \\ &\qquad \quad | \quad \alpha' LLR^{k\ddagger}, RLR^{k\ddagger}, \left\{LR^j\right\}_{j=2}^{k-2\ddagger}, LR, L, \emptyset\} \\ &\equiv_I \quad \{0^{\ddagger}, \{0 \mid -1\}^{\ddagger}, -1, 0 \\ &\qquad \quad | \quad 0^{\ddagger}, \{0 \mid -1\}^{\ddagger}, 0^{\ddagger}, \{1 \mid 0\}, 1, 0\} \\ &\equiv_I \quad 0, \quad \text{by Lemma 3.1.5.} \end{aligned}$$

$$\begin{aligned} \alpha LLRR \quad &= \quad \{\alpha' LLRR^{\ddagger}, RLRR^{\ddagger}, RRR, LR \quad | \quad \alpha' LLRR^{\ddagger}, RLRR^{\ddagger}, L, \emptyset\} \\ &\equiv_I \quad \{0^{\ddagger}, \{\{1 \mid 0\} \mid -1\}^{\ddagger}, -1, \{1 \mid 0\} \quad | \quad 0^{\ddagger}, \{\{1 \mid 0\} \mid -1\}^{\ddagger}, 1, 0\} \\ &\equiv_I \quad 0, \quad \text{by Lemma 3.1.5.} \end{aligned}$$

Case 2a: Note that when $\alpha = \emptyset$ we have equality:

$$LR = \{L \mid \emptyset\} = \{1 \mid 0\} \text{ and } LLR = \{RR, L \mid \emptyset\} = \{-1, 1 \mid 0\} = \{1 \mid 0\}$$

When $\alpha \neq \emptyset$ we do not have equality:

$$\begin{aligned}
\alpha LLR &= \{\alpha' LLR^{\ddagger}, RLR^{\ddagger}, RR, L \quad | \quad \alpha' LLR^{\ddagger}, RLR^{\ddagger}, \emptyset\} \\
&\equiv_I \{\{1 \mid 0\}^{\ddagger}, \{1 \mid -1\}^{\ddagger}, -1, 1 \quad | \quad \{1 \mid 0\}^{\ddagger}, \{1 \mid -1\}^{\ddagger}, 0\} \\
&\equiv_I \{1 \mid 0\}, \quad \text{by Corollary 3.1.7.}
\end{aligned}$$

$$\begin{aligned}
\alpha LRL^k, k \geq 3 \quad &= \quad \{\alpha' LRL^{k\ddagger}, RRL^{k\ddagger}, L^{k+1}, \left\{RL^j\right\}_{j=2}^{k-2}, RL, R, \emptyset \\
&\qquad \mid \alpha' LRL^{k\ddagger}, RRL^{k\ddagger}, RL^{k-1}\} \\
&\equiv_I \{\{1 \mid 0\}^{\ddagger}, 0^{\ddagger}, 1, 0, \{0 \mid -1\}, -1, 0 \\
&\qquad \mid \{1 \mid 0\}^{\ddagger}, 0^{\ddagger}, 0\} \\
&\equiv_I \{1 \mid 0\}, \quad \text{by Corollary 3.1.7.}
\end{aligned}$$

Case 3a: To prove that $L^k = 1$, we show that $L^k - 1$ is a second player win, for $k \geq 1$.

Left moving first takes the top arc from L^k leaving $RL^{k-2} - 1$ (if she takes anything lower, she is only eliminating moves for herself). Right responds by playing in RL^{k-2} to $RL^{k-3} - 1$. As long as Left doesn't play $\overrightarrow{v_1 v_0}$, Right will always have a move in RL^j, for $j < k - 2$ and hence can run Left out of moves and saving -1 for his last move, and wins. If Left plays $\overrightarrow{v_1 v_0}$ (the bottom arc), Right responds in -1, and wins the game.

Right moving first, he only has one option which is to move in -1 to 0. Then Left takes the bottom arc on L^k leaving 0, and hence Left wins.

Case 4:

$$\begin{aligned}
\alpha LRL &= \{\alpha' LRL^{\ddagger}, RRL^{\ddagger}, LL, \emptyset \quad | \quad \alpha' LRL^{\ddagger}, RRL^{\ddagger}, R\} \\
&\equiv_I \{\{1 \mid -1\}^{\ddagger}, \{0 \mid -1\}^{\ddagger}, 1, 0 \quad | \quad \{1 \mid -1\}^{\ddagger}, \{0 \mid -1\}^{\ddagger}, -1\} \\
&\equiv_I \{1 \mid -1\}, \quad \text{by Corollary 3.1.7.}
\end{aligned}$$

Case 5a:

$$\begin{aligned}\alpha RLRR &= \{\alpha' RLRR^{\ddagger}, LLRR^{\ddagger}, LR \quad | \quad \alpha' RLRR^{\ddagger}, LLRR^{\ddagger}, RRR, L, \emptyset\}\\ &\equiv_I \{\{\{1 \mid 0\} \mid -1\}^{\ddagger}, 0^{\ddagger}, \{1 \mid 0\} \quad | \quad \{\{1 \mid 0\} \mid -1\}^{\ddagger}, 0^{\ddagger}, -1, 1, 0\}\\ &\equiv_I \{\{1 \mid 0\} \mid -1\}, \quad \text{by Corollary 3.1.7.}\end{aligned}$$

□

3.4.1 Temperature

Let $T(n)$ be the THINNING THICKETS cordon of height $n+1$ where all internal vertices have leaves, and every arc is blue, except for $\overrightarrow{v_1v_0}$ which is red. Right has only one move, he can only delete $\overrightarrow{v_1v_0}$ which deletes the whole cordon. Left has many moves but we only need to consider one.

Let $S(n)$ be the THINNING THICKETS cordon $T(n)$ without a leaf arc on v_1. Let $U(n)$ be defined as $S(n)$ without a leaf arc on v_n. THINNING THICKETS positions $T(n)$, $S(n)$, and $U(n)$ are pictured in Figure 3.10. It follows that $S(n) \geq \{U(n) \mid \cdot\}$ since Right has no move and Left can move to $U(n)$. In $U(n)$, Right's only move is to remove the arc $\overrightarrow{v_nv_{n-1}}$ and Left can move to $S(n-1)$ by removing $\overrightarrow{v_{n+1}v_n}$. Therefore, $U(n) \geq \{S(n-1) \mid U(n-1)\}$.

Lemma 3.4.3 *For $n \geq 1$, both $S(2n)$ and $S(2n+1)$ are greater than or equal to n.*

Proof. Our proof is by induction on the height of the cordon. First, both $S(1), S(2) \geq 1$ since Right has no move and Left can move to 0.

Consider $S(2n+1) - n$, $n > 0$. Right only has one move, that is to $S(2n+1) - (n-1)$. From here, Left responds to $U(2n) - (n-1)$. Right has two options: (i) $U(2n) - (n-2)$ or (ii) $U(2n-1) - (n-1)$. From (i), Left moves to $S(2n-1) - (n-2) \geq 0$, by induction. From (ii), Left responds to $S(2n-2) - (n-1) \geq 0$, by induction. Hence $S(2n+1) - n \geq 0$.

In $S(2n) - n$, Right moves to $S(2n) - (n-1)$ and Left responds to $U(2n) - (n-1)$. From here, either (i) Right moves to $U(2n-1) - (n-1)$ and Left responds to $S(2n-2) - (n-1) \geq 0$, by induction, or (ii) Right moves to $U(2n) - (n-2)$ and Left moves to $S(2n-1) - (n-2) \geq 0$, by induction. □

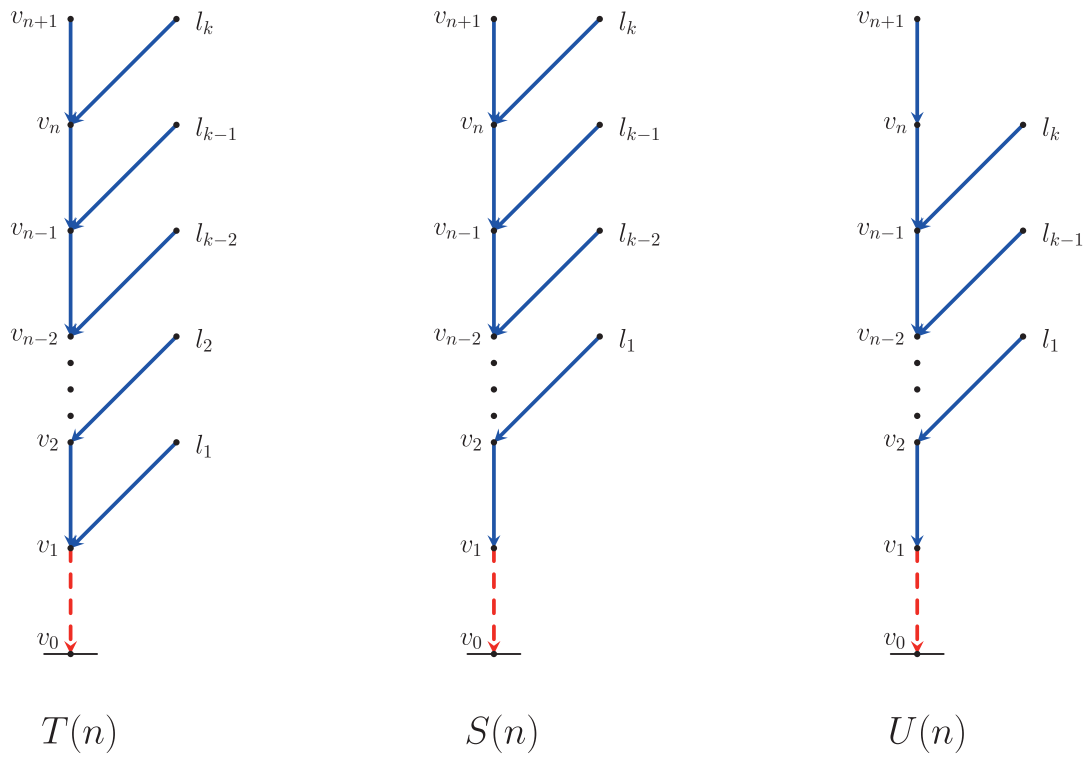


Figure 3.10: Cordons $T(n)$, $S(n)$ and $U(n)$.

The next theorem follows from Lemma 3.4.3. For more information on Left and Right scaffolds see [12].

Theorem 3.4.4 *The temperature of $T(2n)$ is at least n.*

Proof. The Left options of $T(2n)$ include $S(2n)$ and the only Right option is 0. Since $S(2n) \geq n$ then the Right stop of $S(2n) \geq n$. It now follows that the Left scaffold of $T(2n)$ is at least $S(2n)_t - t \geq 2n - t$. The Right scaffold is $0 + t$. The temperature of $T(2n)$ is at least the value of t when these two lines intersect, namely n. □

Corollary 3.4.5 *For any positive integer n there is a* THINNING THICKETS *position with temperature greater than n.*

3.5 Conclusions

Since the nim-dimension is infinite, we know there are positions with value $*n$ for all n. Using CGSuite [67], we were able to find all the positions up to height 10 but no clear patterns for other nim-values emerged. Using similar techniques, to those in Section 3.3.2, we can show:

- A GREEN CORDONS position with one leaf arc has nim-value 2 if and only if $n - a(1)$ is even.
- A GREEN CORDONS position with two leaves has nim-value 2 if and only if $n - a(2)$ is odd and $a(2) - a(1)$ is even.
- A GREEN CORDONS position with two leaves has nim-value 3 if and only if $n - a(2)$ is even.

Question 1 Is there a characterization, similar to that for nim-values 0 and 1, for GREEN CORDONS with nim-value n for $n \geq 2$?

Question 2 Are there THINNING THICKETS positions with values $\{0 \mid \{0 \mid -n\}\}$ for all n?

Chapter 4

Conjoined Games

These results first appeared in [48]. Permission to reprint appears in Appendix D.

This chapter is about games that are played in two phases. For example, the games OVID'S GAME, THREE-, SIX-, and NINE-MEN'S MORRIS [12], and also BUILDING NIM [31] are examples of combinatorial games that have two phases. Specifically, in Phase 1 the board is set up and in Phase 2 the game is played. There are many other games in which Phase 1 is not even defined, but instances of Phase 2 are analyzed. For example, BOXCARS [2], END-NIM [1], HACKENBUSH, PUSH [2], THINNING THICKETS [49], TOPPLING DOMINOES [37] and their variants. There are also recent commercial games with two phases, such as *Fjords* [30] and *Catan* [72]. Though neither game is purely combinatorial, such examples highlight the appeal of games with more than one phase. *Fano330-R-Morris* [59] is a combinatorial game which has an initial placement phase and game play phase, which is played under the misère play winning convention.

Interest in combining games into phase play is not new. Early exploration included the study of the sequential sum: given two games G and H, players play in G until all moves are exhausted and then play H. The winner of H is the winner of the sequential sum of G and H. Under the sequential sum, both game boards, G and H, are known before game play begins. Stromquist and Ullman [71] studied impartial games under this sum; examining the possible outcomes of game combinations and Grundy values. Stewart [69] studied partizan games under the sequential sum, proving that all games under the sequential join form a monoid.

Here, we consider playing Phase 1 as a combinatorial game as well as Phase 2 and analyze two specific games. We were introduced to this concept by Kyle Burke and Urban Larsson (personal communication).

Definition 4.0.1 Let $\mathcal{F}$ and $\mathcal{H}$ be two impartial rulesets. The *conjoined ruleset* $(\mathcal{F} \blacktriangleright \mathcal{H})$ is to play Phase 1 under the $\mathcal{F}$ ruleset and when play is no longer possible to start Phase 2 which is played under the ruleset of $\mathcal{H}$.

Recently, push-button games have been studied [32]. Such games involve two phases: Play the game from ruleset 1 until a player pushes a button to switch to ruleset 2. The button can only be pushed once, at any time during game play, and counts as a turn. The difference is that push-button games can change the rule from ruleset 1 to ruleset 2 at any time during game play under ruleset 1, and if all moves in ruleset 1 are exhausted, then the player whose move it is passes to get to ruleset 2. For conjoined games, there are no pass moves and the main objective is to try to set up the board under ruleset 1 to have a favourable outcome when playing in ruleset 2.

Forming a conjoined game allows for an interesting Phase 1 battle before the 'real' game begins. Since play in Phase 1 sets up the board, it is convenient to have the corresponding game be a placement game [17] (recall, pieces are placed but not moved or removed). The positions at the beginning of Phase 2 will have structure reflecting the Phase 1 rules, and this allows for some partial analysis.

Here we indicate by GAME_{Imp} a partizan game converted into an impartial game by allowing both players to place any of the pieces. This chapter explores the following two games with two-phase play.

$$\text{GO-CUT} = (\text{NOGO}_{Imp} \blacktriangleright \text{CUTTHROAT}_{Imp}), \text{ and}$$
$$\text{SNO-GO} = (\text{SNORT}_{Imp} \blacktriangleright \text{NOGO}_{Imp}).$$

Brief notes about the games: The game NOGO is known as ANTI-ATARI GO[1], but was independently invented by Neil McKay in 2011. (See also [23].) CUTTHROAT was introduced in [57] and the full analysis when played on stars is given in [2]. SNORT is introduced in [12], Vol. 1, and is known as CATS & DOGS in Portugal.

Our results cover playing GO-CUT and SNO-GO on a path, where the first player who cannot move loses. The general questions for CGT games presented in Section 3.2 are very challenging to answer within the context of conjoined games. We focus on values and outcomes. In Section 4.1, we obtain the values for all the possible positions of GO-CUT at the start of Phase 2 but we were not able to find the outcomes of an uncoloured path of length n, as a function of n. By contrast, we find the outcome of an

[1] http://senseis.xmp.net/?AntiAtariGo

uncoloured path for SNO-GO but were not able to find formulas for the corresponding nim-values.

4.1 GO-CUT on a path

Due to a conflict between notation used for games and graphs, within this chapter we use G for graphs and $\mathscr{G}$ for games.

We provide the generalized ruleset.

Ruleset for GO-CUT

- Board: A finite graph, with each vertex either uncoloured or coloured blue or red.
- Moves:
  - *Phase 1:* On a move a player chooses an uncoloured vertex $(\cdot)$ and colours it either red (R) or blue (B) provided every maximal connected monochromatic subgraph is adjacent to an uncoloured vertex.

    When no moves are playable under Phase 1, delete all uncoloured vertices and then delete all monochromatic components. The game is now a disjunctive sum of components each of which contains both red and blue vertices, that is, non-monochromatic components.
  - *Phase 2:* A player chooses a component from the disjunctive sum, deletes one of the vertices then deletes any resulting monochromatic components.

As we restrict our analysis to paths, at the end of Phase 1, for example, we might have the position $[BB \cdot RBB \cdot RB \cdot R \cdot BB]$ which, after deleting the uncoloured vertices leaves $[BB] + [RBB] + [RB] + [R] + [BB]$. Now deleting all monochromatic components, gives the starting position for Phase 2 as $[RBB] + [RB]$.

By the rules of NOGO, at the start of Phase 2, a component will consist of i blue vertices followed by j red vertices (or the reverse) for some $i, j > 0$. Call this an (i, j)-component. To extend the notation, we also refer to $(i, 0)$ and $(0, j)$ components but these correspond to empty components.

Lemma 4.1.1 *The nim-value of an (i, j)-component is $((i-1) \oplus (j-1)) + 1$.*

Proof. Clearly, the nim-value of a $(0, j)$- or $(i, 0)$-component is 0. We will refer to an (i, j)-component by (i, j). If i and j are positive then, by induction,

$$\begin{aligned}\mathcal{G}(i,j) &= \text{mex}\{\mathcal{G}(r,j), \mathcal{G}(i,s), 0 \leq r \leq i-1, 0 \leq s \leq j-1\} \\ &= \text{mex}\{\{(r-1 \oplus j-1)+1, (i-1 \oplus s-1)+1, \\ &\qquad\qquad 1 \leq r \leq i-1, 1 \leq s \leq j-1\} \cup \{0\}\}.\end{aligned}$$

Note that the set $\{(r-1 \oplus j-1), (i-1 \oplus s-1), 1 \leq r \leq i-1, 1 \leq s \leq j-1\}$ is the set of nim-values for NIM played with heaps of size $i-1$ and $j-1$ and hence contains $0, 1, \ldots, (i-1 \oplus j-1)-1$ and does not contain $(i-1 \oplus j-1)$. Adding one to each value gives that both 0 and $(i-1 \oplus j-1)+1$ are missing. Since 0 is an option of (i, j) then

$$\mathcal{G}(i,j) = \text{mex}\{\mathcal{G}(r,j), \mathcal{G}(i,s), 0 \leq r \leq i-1, 0 \leq s \leq j-1\} = (i-1 \oplus j-1)+1.$$

□

4.2 SNO-GO on a path

We were able to obtain winning strategies for the conjoined games of SNORT_{Imp} and NOGO_{Imp} on a path. We give the rules for an arbitrary graph so that a useful general tool, Lemma 4.2.1, can be introduced.

Ruleset for SNO-GO

- Board: A finite graph, with each vertex either uncoloured or coloured blue or red.
- Moves:
  - *Phase 1:* On a move a player chooses an uncoloured vertex ($\cdot$) and colours it red (R) or blue (B) provided that no red vertex is adjacent to a blue vertex.
  - *Phase 2:* When no moves are playable under Phase 1 rules, players can colour an uncoloured vertex red or blue provided that each maximal connected monochromatic subgraph has at least one vertex adjacent to an uncoloured vertex.

Thus, at the end of Phase 1, for example, we might have the position $[BB \cdot RRR \cdot BB \cdot R \cdot BB]$. At the end of the game, the position may look like $[BB \cdot RRRRBB \cdot R \cdot BB]$. Moving into Phase 2 of game play, we are only concerned with the adjacencies to uncoloured vertices. Hence, at the end of Phase 1, if two vertices are coloured the same colour and are adjacent, the edge between them can be contracted. For example, $[BB \cdot RRR \cdot BB \cdot R \cdot BB]$ is equivalent to $[B \cdot RRR \cdot BB \cdot R \cdot BB]$. The next lemma justifies the game theoretic equivalence of the positions after contracting such an edge.

Lemma 4.2.1 *[Reduction.] Let $\mathscr{G}$ be a* SNO-GO *position on a graph G. Suppose x, y are two adjacent vertices that are coloured the same. Let $\mathscr{G}'$ be the position on the board resulting from the contraction of the edge xy, where the vertex resulting from the contraction xy (call it z) has the same colour as x and all other vertices retain their colour. Then the nim-values of $\mathscr{G}$ and $\mathscr{G}'$ are equal.*

Proof. Consider $\mathscr{G} + \mathscr{G}'$ (Note: since the rulesets are impartial this is the same as considering $\mathscr{G} - \mathscr{G}'$). By construction, there is a surjective mapping ϕ from $\mathscr{G}$ to $\mathscr{G}'$ which is the identity mapping for uncoloured vertices and all coloured vertices excluding x and y, and $\phi(x) = \phi(y) = z$, where $x, y \in V(G)$ and $z \in V(G')$. If a player moves in $\mathscr{G}$ (or $\mathscr{G}'$) by colouring a vertex v (or $\phi(v)$, respectively) colour X, where $X \in \{B, R\}$, then their opponent colours $\phi(v)$ (or v, respectively) with X. Thus $\mathscr{G} - \mathscr{G}' = 0$, and hence $\mathscr{G} = \mathscr{G}'$. □

As a guide to intuition, consider a path with n vertices where we label the vertices $x_1, \ldots, x_n$. If two adjacent vertices are coloured the same then we can apply Lemma 4.2.1 so that each monochromatic subpath is reduced to size 1 after Phase 1. For example, the position $[BB \cdot RRR \cdot BB \cdot R \cdot BB]$ becomes $[B \cdot R \cdot B \cdot R \cdot B]$ after applying Lemma 4.2.1 repeatedly. As G is finite, the process terminates in a finite number of steps. This notion is summarized in the next Lemma.

Lemma 4.2.2 *A* SNO-GO *position on a path of n vertices at the beginning of Phase 2 is equal to a path of alternating coloured vertices, each separated by a single uncoloured vertex.*

Proof. At the end of Phase 1, after applying Lemma 4.2.1 repeatedly, all consecutive single coloured vertices (red or blue) get amalgamated into a single representative of that colour. If there is a pair of adjacent uncoloured vertices then either of them can be coloured under Phase 1 rules. Also, if x_1 or x_n is uncoloured then Phase 1 play is still possible. Hence, after all reductions, the position will consist of vertices alternating colours with a single uncoloured vertex between them and the end vertices x_1 and x_n are also coloured. □

At the end of Phase 1, we will call any uncoloured vertex a *hole*. Note that a hole will be adjacent to exactly one red and one blue vertex. We relate this game, using Lemma 4.2.1, to NODE KAYLES [14] or equivalently to DAWSON'S CHESS [12].

Ruleset for NODE KAYLES

- Board: A finite graph.
- Moves: On their turn, a player chooses a vertex and deletes it and all its neighbours.

Lemma 4.2.3 *Given a path on n vertices, let $\mathscr{G}$ be a* SNO-GO *position at the start of Phase 2 play. Furthermore, suppose $\mathscr{G}$ has m uncoloured vertices. If $m \geqslant 2$ then $\mathscr{G}$ is equivalent to* NODE KAYLES *played on a path with $m - 2$ vertices.*

Proof. It is possible that $m = 0$ or $m = 1$. In the first case, all the vertices were coloured the same. In the second, the final position is $B{\cdot}R$. In both cases, there are no moves in Phase 2. If $m \geqslant 2$ playing either of the two holes at the end, without loss of generality $[B{\cdot}R\ldots] \to [BXR\ldots]$, $X \in \{B, R\}$, leaves an illegal Phase 2 position since the leftmost coloured vertex is no longer adjacent to an uncoloured vertex. Playing an interior hole, e.g., $[B{\cdot}R{\cdot}B{\cdot}R{\cdot}B{\cdot}R] \to [B{\cdot}R{\cdot}BXR{\cdot}B{\cdot}R]$, $X \in \{B, R\}$ eliminates playing in the two adjacent holes as legal moves. This is because the coloured vertices adjacent to X are relying on the adjacent uncoloured vertex to satisfy the conditions of NOGO. This mirrors play in NODE KAYLES since players choose a vertex, delete it and its neighbours. This shows that at the beginning of Phase 2, the position is now equivalent to playing NODE KAYLES on a path of length $m - 2$. □

For ease of referencing the players, we call the players Alf and Betti. We assume that Alf plays first on the empty board and Betti plays second.

The outcome class of the sequence of NODE KAYLES on a path is periodic with period length 34 after a pre-period of 52 and the only $\mathcal{P}$ positions are when n is even. For exact values, see the nim-value sequence for DAWSON'S CHESS in Winning Ways [12], volume 1. Our approach is to show that the winning player can ensure to play first at the start at Phase 2, on the equivalent of an odd NODE KAYLES position, or win if the opponent does not allow 3 or more holes.

Before proving the Main Theorem of this section, we need the following lemmas and conventions.

We partition the path into two pieces: the *outer vertices* consisting of vertices x_1, x_2, x_{n-1}, x_n, and the *interior*, consisting of vertices $x_3, \ldots, x_{n-2}$.

Lemma 4.2.4 *Let $\mathscr{G}$ be a Phase 1 SNO-GO position on a path of n vertices. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are the same colour then at the end of Phase 1 there will be an even number of holes. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are opposite colours then at the end of Phase 1 there will be an odd number of holes.*

Proof. Let $\mathscr{G}$ be a Phase 1 SNO-GO position where at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are coloured. At the end of Phase 1, positions will be as in Lemma 4.2.2. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ are the same colour, then given the alternating pattern of the colours, the number of coloured vertices is odd which implies an even number of uncoloured vertices must be separating them. If at least one of $\{x_1, x_2\}$ and one of $\{x_{n-1}, x_n\}$ of the position are different colours, again given the alternating patterns of colours being separated by single uncoloured vertices, this implies an even number of coloured vertices, separated by an odd number of uncoloured vertices. □

Note that Lemma 4.2.5 is referring to the original path before applying Lemma 4.2.1.

Lemma 4.2.5 *Let $\mathscr{G}$ be a* SNO-GO *position on a path of $2k+1$ vertices at the end of Phase 1. Let h be the number of holes. If $h = 0, 2$ or $h \geqslant 3$ and is odd then Alf will win the game.*

Proof. If $h = 0$ then there has been an odd number of moves, the game is over and Alf had the last move.

If $h = 2$ then Phase 2 has no moves but it is Betti's turn to play and so she loses.

If $h \geqslant 3$ and is odd then there has been an even number of moves $(2k + 1 - h)$ in Phase 1 and thus Alf moves first in Phase 2. NODE KAYLES on an odd number of vertices, here $h - 2$, is a first player win [12] and so Alf can win the game. □

Theorem 4.2.6 *Consider* SNO-GO *played on a path of n vertices. The initial position is in $\mathcal{P}$ if n is even and in $\mathcal{N}$ if n is odd.*

Proof. First suppose n is odd. The strategy is for Alf to colour the centre vertex, without loss of generality, blue. Now, until Betti colours an outer vertex (the first outer vertex to be coloured), Alf always plays the same move reflected across the centre vertex. When Betti finally colours an outer vertex there are now several cases to consider. Since Betti's last move was to colour an outer vertex, without loss of generality, suppose Betti colours x_{n-1} or x_n with X. If there are two red interior vertices then Alf colours x_1 with the opposite colour from Betti's choice. By Lemma 4.2.4, at the end of Phase 1 there will be an odd number of holes, at least 3, and by Lemma 4.2.5 Alf will win.

Thus we may suppose that at this point in play, all coloured interior vertices are blue. There are several cases to consider.

1. Suppose every uncoloured interior vertex is adjacent to at least one blue. That is, it is now illegal to colour an interior vertex red. Alf colours x_1 with X. The number of holes will be 0 if X = blue and 2 if X = red. In both cases, by Lemmas 4.2.4 and 4.2.5, Alf can force a win.

2. Suppose there are 4 interior uncoloured vertices with the following properties: (i) any pair of these vertices are at least distance 3 apart and (ii) none of them are adjacent to a blue vertex. Alf colours x_1 with the opposite colour, hence an odd number of holes, and Alf can force at least 3 holes by having two reds separated by a blue vertex.

3. Suppose there is an interior uncoloured vertex that is not adjacent to any blue vertex which has an interior blue vertex between it and the closest outer vertex. By symmetry, the reflected vertex is uncoloured and not adjacent to a blue vertex. Again, Alf colours x_1 with the opposite colour, hence an odd number of holes, and Alf can force at least 3 holes by colouring one of the two uncoloured vertices red.

This leaves the situation where the outermost blue interior vertices are followed by at most 5 uncoloured interior vertices and all other uncoloured interior vertices are adjacent to a blue vertex. Since Alf will play symmetry following any Betti move between the two outer blues we can condense the centre to a single blue vertex. The positions that remain to analyze are one of the following, where it is Betti's turn to move. We suppose Betti will colour one of the leftmost two vertices; a symmetric argument holds for the rightmost vertices.

(i) $[\cdot\cdot B\cdot\cdot]$

(ii) $[\cdot\cdot\cdot B\cdot\cdot\cdot]$

(iii) $[\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot]$

(iv) $[\cdot\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot\cdot]$

(v) $[\cdot\cdot\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot\cdot\cdot]$

(vi) $[\cdot\cdot\cdot\cdot\cdot\cdot\cdot B\cdot\cdot\cdot\cdot\cdot\cdot\cdot]$.

In cases (i) and (ii), Alf forces either 0 or 2 holes by playing symmetrically.

In (iii), if Betti plays x_{n-1}, again Alf forces either 0 or 2 holes by playing symmetrically. Suppose Betti plays to $[B \cdots B \cdots\cdot]$; then Alf plays to $[B \cdots B \cdot\cdot B\cdot]$ forcing 0 or 2 holes. If she plays to $[R \cdots B \cdots\cdot]$, then Alf replies $[R \cdots B \cdot\cdot R\cdot]$ forcing 2 holes.

In (iv) and (v) $[\cdot\cdots\cdot B \cdot\cdots\cdot]$, Alf colours the fourth vertex from the end of the side opposite to that of Betti's last move. Alf can now force 2 or 3 holes.

In (vi), from $[\cdot X \cdots\cdot\cdot B \cdots\cdots\cdot]$ Alf plays to $[\cdot X \cdot\cdot X \cdot\cdot B \cdots\cdots\cdot]$ and can force 0 or 2 holes. From $[X \cdots\cdots B \cdots\cdots\cdot]$ Alf plays to $[X \cdots\cdots B \cdots\cdots Y]$. Suppose without loss of generality that X is red. Regardless of what Betti plays, Alf can colour a vertex red, on the other side of the centre from X. This generates an odd number (> 1) of holes.

If the board is of even length, Betti plays the reflection symmetry and a similar analysis shows that she can force a win. □

4.3 Conclusions

In this chapter we defined a method for combining games, where the termination of the first game sets up the board for a second game. This provides a more structured method of determining opening positions for a game which may not have a standard board. Here we examined two games: GO-CUT and SNO-GO. Though other graphs appear to be hard to study, it may be worthwhile exploring these games on grids or cycles as an extension to this work. Another generalization is to extend this work to partizan games.

Chapter 5

THE ORTHOGONAL COLOURING GAME

After the theory from [53], [54], and [56] was developed, there was a search for a scoring game that could be easily analyzed and highlight the theory. Nowakowski suggested the Latin squares game.

Recall (see Brualdi [18]) that an $n \times n$ square, partially filled with entries taken from $\{1, 2, \ldots, n\}$, has the *Latin property* if each row and column does not contain any repeated entries. A fully filled $n \times n$ square is a *Latin square* if each entry is an integer between 1 and n (inclusive) and each row and each column contains all n integers, which implies that the square has the Latin property. For a (partially filled) $n \times n$ square, X, let $c_X(i, j)$ be the (i, j) entry and $\emptyset$ if (i, j) is unfilled. Let A and B be (partially filled) $n \times n$ squares. Then A and B are *orthogonal* if in the list

$$((c_A(i, j), c_B(i, j)))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

every ordered pair of integers occurs at most once. If A and B are Latin squares, this means that every pair of integers from $\{1, \ldots, n\}^2$ occurs exactly once in the list.

The Latin squares game is played as follows: given two $n \times n$ grids, one owned by each player, and a set of integers $\{1, 2, \ldots, n\}$, players must choose an integer and choose where to place it in a grid such that individual grids maintain the Latin property, and grids together are orthogonal. Players can move in either grid, but their final score is the number of non-empty cells in their board.

After determining that 2×2 grids are second player wins, and speculating that 3×3 grids are draws, this was reintroduced at the *Games and Graphs* 2017 conference in Lyon, France, where Huggan began collaboration on this project with Andres, Mc Inerney and Nowakowski. The work in this chapter is from the ongoing collaboration from that conference and has been accepted (with minor revisions) for publication in Theoretical Computer Science. It is currently available on HAL[1] (see [4]). Co-author

[1] Hyper Article en Ligne (HAL) [open archive] https://hal.archives-ouvertes.fr/ .

permission to reprint can be found in Appendix D. Throughout this chapter, unless stated otherwise, $\mathbb{N}$ will denote the set of non-negative integers.

Consider a graph G where two isomorphic copies of G are labelled G_L and G_R and are partially coloured. A graph has a *proper* colouring if adjacent vertices have distinct colours. *Orthogonality* of two isomorphic, partially coloured graphs, G_L and G_R, means that if v, w are two different vertices in G whose copies v_L, w_L (in G_L) resp. v_R, w_R (in G_R) are coloured, then

$$(c(v_L), c(v_R)) \neq (c(w_L), c(w_R)), \tag{5.1}$$

where $c(x)$ denotes the colour of a vertex x.

The problem has since been generalized as a graph colouring game, called the ORTHOGONAL COLOURING GAME, denoted by $MOC_m(G)$.

Ruleset for the ORTHOGONAL COLOURING GAME, $MOC_m(G)$

- Board: Two initially uncoloured disjoint isomorphic copies G_A and G_B of a given finite graph G.
- Moves: Two players, Alice and Bob, with Alice beginning, alternately choose one of the two graphs G_A or G_B and colour an uncoloured vertex of this graph with a colour from the set $\{1, \ldots, m\}$ such that the colouring is proper and the orthogonality of the graphs is not violated. Alice owns G_A and Bob owns G_B.

When no move is possible any more, the game ends. A player's *score* is the number of coloured vertices in the graph the player owns. If, at the end, the scores of both players are equal, the game result is a *draw*; otherwise, the player with the higher score wins.

The main result of this chapter states that for a special class of graphs, graphs admitting a *strictly matched involution*, the second player, Bob, can achieve at least a draw.

This class of graphs includes many special cases where the game is to create combinatorial objects such as orthogonal Latin rectangles, double diagonal Latin squares, Latin squares, and sudoku squares. The *double diagonal condition* consists

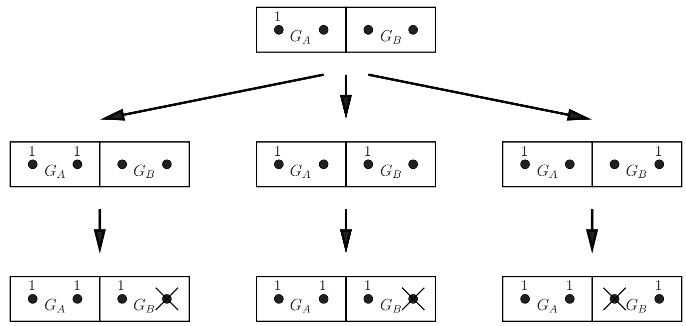


Figure 5.1: Alice's winning strategy for the game $MOC_1(2K_1)$.

in demanding that the coloured entries of each of both diagonals in a square are pairwise different. The *sudoku condition* for an $n \times n$ square with $n = k^2$ and $k \in \mathbb{N}$ forces the coloured entries of each of the k^2 disjoint subsquares of size $k \times k$ to be pairwise different. These combinatorial objects are further considered in Corollary 5.2.3. However, there exist graphs in which optimal play from both players does not result in a draw. The smallest such example of a graph in which Alice wins is the graph $2K_1$ consisting of two isolated vertices with $m = 1$ colour (see Figure 5.1). An example of a case where Bob wins is $MOC_2(C_4)$. He wins as follows: when Alice plays her first move on a C_4, Bob responds in the same C_4 on the non-adjacent vertex, colouring with the opposite colour if it is her C_4 and the same colour if it his C_4. Bob will then win by 2 points. Note that Bob's optimal strategy is not to play just in his graph.

This chapter is structured as follows. In Section 5.1, we motivate our research by the most prominent special case, the game played on the graph associated with Latin squares, and give references to related games and to some results on orthogonal graph colouring. In Section 5.2, we define graphs admitting a strictly matched involution and prove the main result of this chapter. In Section 5.3, we study the graphs in which the game is a draw and prove for the most important special case, orthogonal Latin squares, that the game is a draw if $m = 1$. In Section 5.4, we provide a characterization of the graphs that admit a strictly matched involution which allows us to give an explicit construction for all such graphs. We follow this discussion by a report in Section 5.5 of counting and complexity results from [4] and [3] (including another co-author, Dross), which were primary contributions by co-authors. We conclude with future directions for research.

5.1 Motivation and Observations

The game $MOC_m(G)$ emanates from the overlap of two lines of research: combinatorial and scoring games (specifically, colouring games) and orthogonality of Latin squares or, more generally, of colourings of graphs.

Note that the drawing strategy for the second player in the orthogonal colouring game implies that the second player always has a good response to the first player. If the game is played under the normal play convention, using the drawing strategy from scoring play, the second player can guarantee a win because they will always be the last player to move in the game. Andres et al. [3] proved that it is PSPACE-complete to determine the outcome of the game in the normal play convention when $m \in \mathbb{N}^{\geq 1}$ is the number of colours (even if m is a fixed constant), and an initial partial colouring is given.

The colouring game on graphs in the normal play convention, which was introduced as the ACHIEVEMENT game by Harary and Tuza [44] and called the PROPER k-COLOURING game by Beaulieu et al. [10], is closely related to the orthogonal colouring game on graphs. In the PROPER k-COLOURING game, two players take turns colouring the vertices of a graph, while maintaining that the colouring is proper. Beaulieau et al. [10] showed that this game is PSPACE-complete when $k \in \mathbb{N}^{\geq 1}$ is the number of colours (even if k is a fixed constant), and an initial partial colouring is given. For $k = 1$ colour, the PROPER k-COLOURING game is the well-studied game NODE KAYLES. For specific classes of graphs, it is known which player wins the game NODE KAYLES, *e.g.*, for paths and cycles a complete characterisation was given by Berlekamp et al. [12]. Harary and Tuza [44] characterised the winner in the PROPER k-COLOURING game played with $k = 2$ colours on paths and cycles, and played with any number k of colours on the Petersen graph. Astonishingly, as far as we know, the PROPER k-COLOURING game seems to not have been studied on other classes of graphs for $k \geq 2$ colours.

Here, game-theoretic graph parameters are motivated by trying to get good approximations to graph parameters that are hard to calculate, *e.g.*, chromatic number [13, 35, 39] and domination number [47]. Seo and Slater [64] give generic examples of how such parameters can be defined. Typically, two players choose vertices (or edges or other sub-objects) without violating a given property (*e.g.*, independence). The score is the number of vertices chosen where one player wants to maximise the number and the other to minimise it. Within this context of scoring, since there is no set boundary to determine the winner, it is unclear which player is winning at any time in game play.

In particular, a game-theoretic version of the chromatic number, the *game chromatic number*, introduced by Bodlaender [13], and several of its possible variations have been extensively studied in the last three decades in more than 100 papers (see the partial surveys by Bartnicki et al. [9], Tuza and Zhu [73] or Dunn et al. [33] for some references). Upper bounds for the game chromatic number of many classes of graphs have been determined, *e.g.*, for forests by Faigle et al. [35], outerplanar graphs by Guan and Zhu [41], and planar graphs by Zhu [75]. However, the complexity of determining the game chromatic number of a graph in general is still an open problem.

Larsson et al. [56] extended the Maximiser/Minimiser approach, to use two graphs, G and H, usually, but not necessarily, isomorphic. One player, Left, is the maximiser on G but the minimiser on H and the other player, Right, has the reverse goals. The score is the number of pieces played on G minus the number played in H with Left winning if the score is positive and Right winning if the score is negative, and it is a draw if the score is 0.

Orthogonal colourings of graphs, *i.e.*, proper colourings of two isomorphic copies G_A and G_B of a graph respecting the orthogonality condition (5.1), have been studied as well (*e.g.*, by Archdeacon et al. [5], Ballif [7], or Caro and Yuster [22]). Caro and Yuster [22] studied the minimum number of colours required such that there exist k mutually orthogonal colourings of G. Specifically, the graph versions of combinatorial objects associated with orthogonality were studied by Ballif [7] such as Latin squares and Latin rectangles.

Orthogonal Latin squares are natural combinatorial objects where there are two 'boards' and these form the basis of a specific orthogonal colouring game played on Latin squares. It is known that a Latin square of order n can be regarded as a proper colouring of the cartesian product of K_n with itself. Thus, the concept of orthogonal Latin squares translates easily to graph colourings and the orthogonal colouring game played on Latin squares is equivalent to $MOC_m(K_n \square K_n)$. See Figure 5.3 for an example of play.

5.2 Main Theorem

First, we fix some general notation. For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. We use standard notation from graph theory. The *disjoint union* of two graphs H and H', denoted by $H \cup H'$, is the graph $(V \cup V', E \cup E')$ consisting of an isomophic copy (V, E) of H and an isomorphic copy (V', E') of H' with $V \cap V' = \emptyset$. The disjoint union $H \cup H$ is also denoted by $2H$.

Recall that, for a graph $G = (V, E)$, an *automorphism* is a bijective mapping $\sigma : V \longrightarrow V$ with the property that

$$\forall v, w \in V : \quad (vw \in E \Longleftrightarrow \sigma(v)\sigma(w) \in E).$$

An *involution* of G is an automorphism σ of G with the property

$$\forall v \in V : \quad (\sigma \circ \sigma)(v) = v.$$

We define an involution of G to be *strictly matched* if

(SI 1) the set $F \subseteq V$ of fixed points of σ (*i.e.*, $F = \{v \in V \mid \sigma(v) = v\}$) induces a complete graph (*i.e.*, for every $v, w \in F$ with $v \neq w$ we have $vw \in E$) and

(SI 2) for every $v \in V \setminus F$, we have the (matching) edge $v\sigma(v) \in E$.

If, for a graph G, there exists a strictly matched involution, we say that G admits a strictly matched involution. Before proving the main result of this chapter, let's explore an example of a graph that admits a strictly matched involution.

Example 5.2.1 Consider the graph pictured in Figure 5.2. Let $F = \{x, y\}$ and $V \backslash F = \{v, w\}$. Define σ explicitly as follows: $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(v) = w$ and $\sigma(w) = v$. We need to check that σ is a strictly matched involution. First, we check that σ is a automorphism (i.e., if $uv \in E \iff \sigma(u)\sigma(v) \in E$). We check the edges of E.

- Consider $x, y \in V$ and note $xy \in E$. Then $\sigma(x)\sigma(y) = xy$ so $\sigma(x)\sigma(y) \in E$.

- Consider $x, w \in V$ and note $xw \in E$. Then $\sigma(x)\sigma(w) = xv$ so $\sigma(x)\sigma(w) \in E$.

- Consider $x, v \in V$ and note $xv \in E$. Then $\sigma(x)\sigma(v) = xw$ so $\sigma(x)\sigma(v) \in E$.

- Consider $v, w \in V$ and note $vw \in E$. Then $\sigma(v)\sigma(w) = wv$ so $\sigma(v)\sigma(w) \in E$.

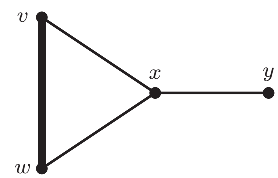


Figure 5.2: A graph which admits a strictly matched involution.

Hence, σ is an automorphism. Next we check that σ is an involution. We consider each vertex v of G, and check that $\sigma \circ \sigma(v) = v$.

- Consider $v \in V$; $\sigma \circ \sigma(v) = \sigma(w) = v$.

- Consider $w \in V$; $\sigma \circ \sigma(w) = \sigma(v) = w$.

- Consider $x \in V$; $\sigma \circ \sigma(x) = \sigma(x) = x$.

- Consider $y \in V$; $\sigma \circ \sigma(y) = \sigma(y) = y$.

Hence, σ is indeed an involution. Lastly, we check that σ is a strictly matched involution. The induced subgraph of G by F is isomorphic to K_2, the complete graph on 2 vertices. Hence, the subgraph of G induced by the vertices from F is a complete graph. We check that $v\sigma(v) \in E$ for all $v \in V \backslash F$. Consider $v \in V \backslash F$: $v\sigma(v) = vw \in E$. Similarly, $w \in V \backslash F$ and $w\sigma(w) = wv \in E$. Both conditions for the involution being strictly matched hold. We have shown that σ is an involutive automorphism which is strictly matched.

Theorem 5.2.2 *Let G be a graph that admits a strictly matched involution and $m \in \mathbb{N}$. Then, the second player has a strategy guaranteeing a draw in the game $MOC_m(G)$.*

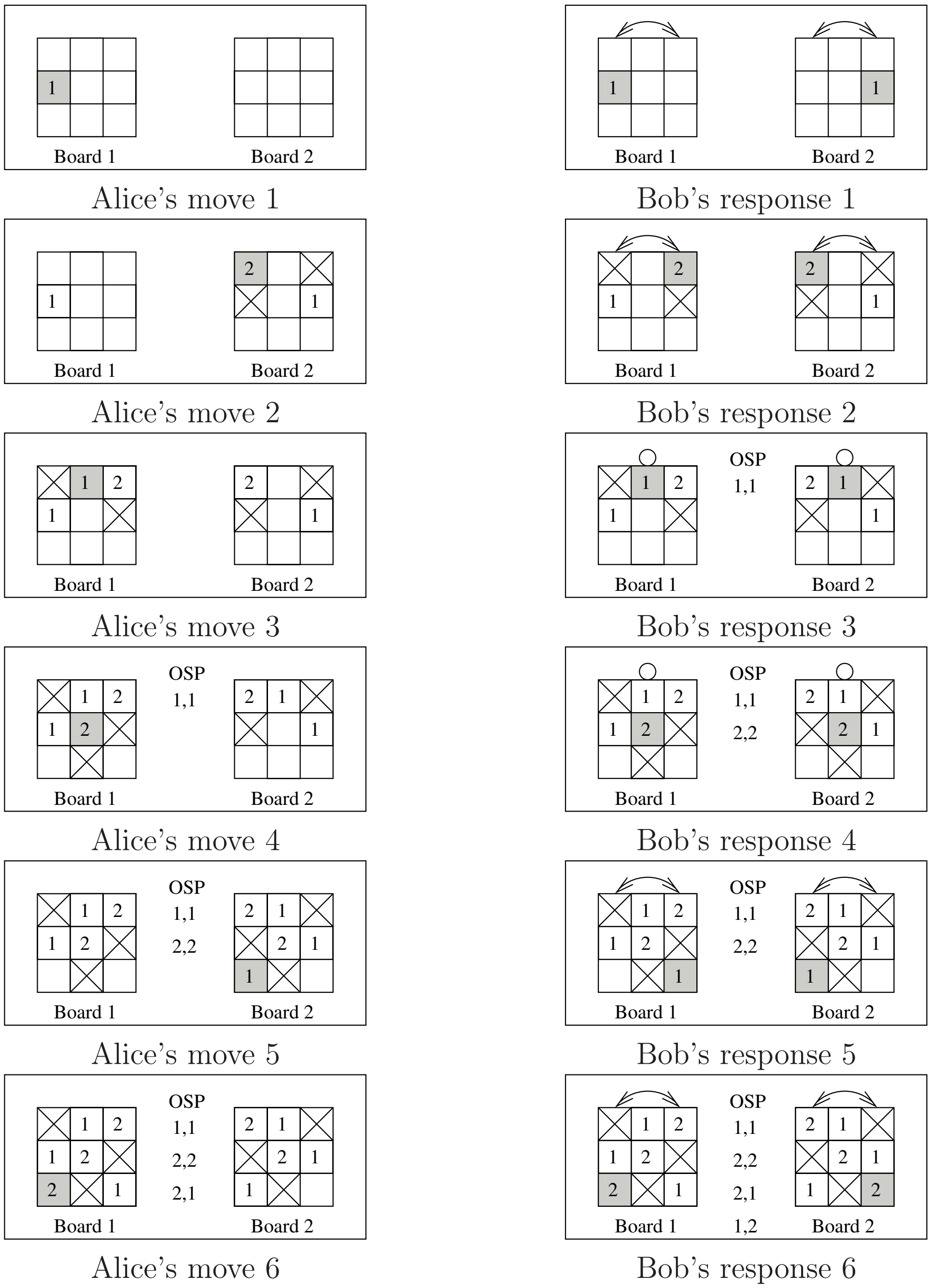


Figure 5.3: Bob's strategy from the proof of Theorem 5.2.2 guarantees a draw in the orthogonal Latin squares colouring game: an example played on 3×3 squares with 2 colours.

In Figure 5.3, we illustrate Bob's strategy given in the following proof of Theorem 5.2.2 on $K_3 \square K_3$, where $K_3 \square K_3$ is represented by a 3×3 board and the involution is given by the mirror symmetry around the middle column of each board. We prove the result for a general graph.

Proof of Theorem 5.2.2. Let G_1 and G_2 be the two copies of $G = (V, E)$. For $k \in \{1, 2\}$, we denote by $c_k(v)$ the colour of the vertex $v \in V$ in G_k. In case the vertex v is uncoloured in G_k, we write $c_k(v) = \emptyset$. To simplify notation and differentiate between the colour of a vertex in a certain copy of G and an actual colour, we refer to the colours as symbols.

Let OSP be the set of *orthogonal symbol pairs*, *i.e.*, the set of those pairs (s_1, s_2) of symbols $s_1, s_2 \in [m]$, such that there exists a vertex $v \in V$ with

$$c_1(v) = s_1 \quad \text{and} \quad c_2(v) = s_2.$$

Let σ be a strictly matched involution, which exists by precondition. For $m = 0$ or $|V| = 0$, the theorem is trivially true. Thus, assume $m, |V| \geq 1$. The strategy of the second player, Bob, is to copy (in a certain sense) Alice's moves in the other copy of the graph. Copying the symbols using the same positions would, in many cases, not be feasible because of orthogonality. Therefore, Bob couples the vertices of a graph with its image under σ of the other graph. Bob always plays the same symbol (=colour) as Alice just previously played.

For $c \in \{c_1, c_2\}$ we define $\overline{c}$ to be the other partial colouring from $\{c_1, c_2\}$ distinct from c.

Consider the case that Alice assigns $c(v) := s$ for some $c \in \{c_1, c_2\}$, some $v \in V$, and some symbol $s \in [m]$. Then, the copying strategy of Bob consists of assigning

$$\overline{c}(\sigma(v)) := s.$$

We will prove that Bob will force a draw with this strategy.

We observe, as a key of our analysis, the following invariants which hold for every $c \in \{c_1, c_2\}$, every $v \in V$, and every $s, s_1, s_2 \in [m]$ after each of Bob's moves:

1. Whenever $c(v) = s$, then $\overline{c}(\sigma(v)) = s$.

2. Whenever $c(v) = \emptyset$, then $\overline{c}(\sigma(v)) = \emptyset$.

3. Whenever $(s_1, s_2) \in OSP$, then $(s_2, s_1) \in OSP$.

4. Whenever $(s_1, s_2) \notin OSP$, then $(s_2, s_1) \notin OSP$.

We will prove by induction on the number of moves that after each move of Alice, Bob's move assigning $\overline{c}(\sigma(v)) = s$ according to his strategy is possible, *i.e.*,

a) the vertex $\sigma(v)$ is uncoloured in the colouring $\overline{c}$;

b) the move keeps the partial colourings being proper;

c) the move does not contradict the orthogonality of the colourings c_1 of G_1 and c_2 of G_2;

and that after each move of Bob, the invariants hold again.

At the beginning of the game, all invariants obviously hold.

Now consider a situation after a move of Alice, where she assigns $c(v) = s$. Therefore, before the move, vertex v was uncoloured, *i.e.*, we had $c(v) = \emptyset$. By invariant 2, we had $\overline{c}(\sigma(v)) = \emptyset$, thus, a) holds.

To prove b), assume to the contrary that the move of Bob would violate the properness of the partial colouring $\overline{c}$, *i.e.*, assume that there exists $w \in V$ with $w \neq \sigma(v)$ and $w\sigma(v) \in E$ such that

$$\overline{c}(w) = s = \overline{c}(\sigma(v)).$$

As Alice played on the vertex v with the partial colouring c, the assignment $\overline{c}(w) = s$ must have been made before her move. Then, by invariant 1, we have $c(\sigma(w)) = s$. But, since $w\sigma(v) \in E$ and σ is an involutive automorphism, we have

$$\sigma(w)v = \sigma(w)\sigma(\sigma(v)) \in E,$$

which contradicts $c(v) = s$, since from $w \neq \sigma(v)$ follows $v \neq \sigma(w)$ by the properties of σ.

To prove c), we remark the following. In the case Alice has created a new element $(s, x) \in OSP$, then by invariant 4, $(x, s) \notin OSP$ before Alice's move. Before proceeding with the proof, we will observe the following two key results.

We claim that for all $x \in [m]$, at any time in the game, it is not possible for Alice to create a new element $(x, x) \in OSP$. To prove this claim assume Alice created a new element $(x, x) \in OSP$. Then, by the definition of OSP, at some point in the game, a player has assigned $c_1(v_1) = x$ and, at some other point in the game, a player has assigned $c_2(v_1) = x$. At least one of these assignments was not performed in the last move with respect to the turn we consider. Without loss of generality the assignment $c_1(v_1) = x$ was performed before the last move (the other case being symmetrical by interchanging the roles of G_1 and G_2). Then, by the invariants, the other player must have assigned in the same pair of moves

$$c_2(\sigma(v_1)) = x. \tag{5.2}$$

If $v_1 \in F$ (*i.e.*, v_1 is a fixed point of σ), then Bob created the new element $(x, x) \in OSP$.

Otherwise, *i.e.*, if $v_1 \in V \setminus F$, as already mentioned above, by the definition of OSP, at some point in the game a player has assigned

$$c_2(v_1) = x. \tag{5.3}$$

But (5.2) and (5.3) contradict the facts that, by the definition (SI 2) of a strictly matched involution, there is a matching edge $v_1\sigma(v_1)$ and, since c_2 is a proper partial colouring, v_1 and $\sigma(v_1)$ cannot be coloured the same colour.

Next we claim that in case Bob must create a new element $(x, x) \in OSP$ by his strategy, he is able to do so without violating orthogonality (*i.e.*, $(x, x) \notin OSP$ before Alice's move). To prove this claim assume $(x, x) \in OSP$ before Alice's move. Let v' be the vertex with $c_1(v') = x = c_2(v')$. By invariant 1 and by orthogonality, v' must be a fixed point of σ (*i.e.*, $v' \in F$). If Bob would have to create $(x, x) \in OSP$ for the second time, by his strategy, this would only be possible if Alice coloured a vertex $v'' \in F$, $v'' \neq v'$, with colour x. But this is impossible, since, by (SI 1), F induces a complete graph, thus, there is an edge $v'v'' \in E$, so that Alice could not have coloured v'' with the same colour as v'. Thus, the assumption is wrong. Therefore, Bob must create a new element (x, x) at most once.

We continue with the proof of c). By the first claim, after Alice's turn, we have

$$(s, x) \neq (x, s).$$

Therefore, also after Alice's move, $(x, s) \notin OSP$. Thus, the assignment

$$c_2(\sigma(v)) = s$$

of Bob is allowed (does not contradict orthogonality) and, since we have $c_1(\sigma(v)) = c_2(v) = x$, it will create a new element $(x, s) \in OSP$, satisfying invariant 3 and invariant 4.

In case Alice has created a new element $(x, s) \in OSP$, the arguments are the same (just interchange the roles of c_1 and c_2).

In case Alice does not create a new element in OSP on her move, Bob will have a feasible move by invariant 3 and invariant 4. Also, by reasons of symmetry, Bob will not create a new element in OSP unless Alice played some symbol s on a vertex in F, in which case, Bob creates the new element $(s, s) \in OSP$, which maintains invariant 3 and invariant 4. The latter move of player 2 is feasible because of the second claim. This proves c).

Now consider a situation after the move of Bob and assume a), b), and c) to be true before his move. We have to prove that the 4 invariants hold again.

Within the proof of c), we have shown that after Bob's move, invariant 3 and invariant 4 hold again. Invariants 1 and 2 follow from the definition of the assignment in Bob's move and the induction hypothesis.

This concludes the inductive step. We have shown that Bob's strategy always allows a reaction to Alice's move. Therefore, the game will end before a move of the first player. In such a situation, Bob's copying strategy results in two partial colourings c_1 and c_2 with exactly the same number of coloured vertices. Thus, the game ends in a draw. □

Corollary 5.2.3 *The second player has a strategy to guarantee a draw in orthogonal colouring games played on $n \times n$ squares satisfying the Latin property (and possibly the double diagonal condition or the sudoku condition) or $n_1 \times n_2$ rectangles.*

Proof. For the graphs associated with such game boards, the assignment $(i, j) \longmapsto$

$(i, n_2 + 1 - j)$, which describes a vertical mirror symmetry, is easily seen to be a strictly matched involution. □

5.3 When the Game is a Draw

In this section, we look at graphs in general and also, at the special case of the graphs associated with Latin squares. We show that both players trivially have a strategy to draw if m is large enough. For the game $MOC_m(K_n \Box K_n)$, we show that Alice has a strategy to draw if $m = 1$, thereby, showing that there exist graphs that admit a strictly matched involution where the optimal result for both players is a draw for some values of m.

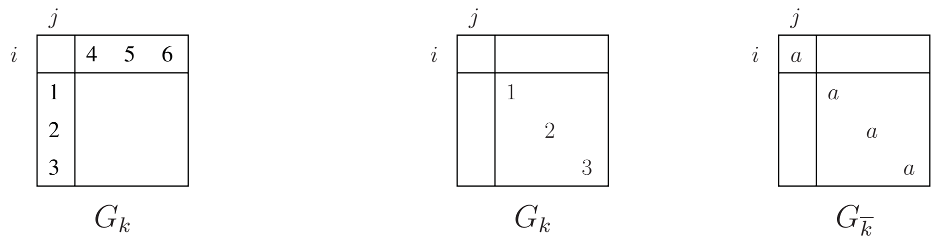
First we note that, if m is large enough, both players have a strategy to force a draw. In the following lemma, for a graph G, the number $\Delta(G)$ is the maximum degree of a vertex in G and $\alpha(G)$ is the independence number (size of a maximum independent set) of G.

Lemma 5.3.1 *For any graph G and all $m \in \mathbb{N}$ with $m \geq \Delta(G) + \alpha(G)$, both players have a strategy to draw in the $MOC_m(G)$ game.*

Proof. We simply show that each of the players' copies, G_1 and G_2, of the graph G will be completely filled at the end of the game. To show this, we consider the worst possible case scenario for an uncoloured vertex v that needs to be coloured with some colour s in some copy G_k of G and show that it is possible to colour vertex v with s in G_k. For $k \in \{1, 2\}$, consider an uncoloured vertex v in some copy G_k of the graph G. Also, let $\overline{k} = 3 - k$. Thus, $G_{\overline{k}}$ is the other copy of the graph G that is not G_k.

In the case of the proper colouring property, the worst case is that every other vertex adjacent to v in G_k has been coloured with a distinct colour. Then, $\Delta(G)$ colours are unavailable to be played on v in G_k by the proper colouring property. See Figure 5.4 (a) for an example in the case $G = K_n \Box K_n$.

By the orthogonality conditions, the worst case is that vertex v in $G_{\overline{k}}$ is coloured with some colour a and the forbidden pairs, (s_1, s_2) with $b = s_k$ and $a = s_{\overline{k}}$, exist for some $a \in [m]$ and $\alpha(G) - 1$ values of b, where $b \in [m]$.



(a) Latin property and (b) orthogonality restrictions

Figure 5.4: In the case the graph G is the graph $K_n \square K_n$: the worst case in the proof of Lemma 5.3.1 due to (a) the Latin property; (b) orthogonality, respectively.

Note that there cannot be more than $\alpha(G) - 1$ colours unavailable for v by the orthogonality conditions. This is because the colour a that appears in v of $G_{\overline{k}}$, may only appear at most $\alpha(G)$ times in $G_{\overline{k}}$ and hence, may only generate at most $\alpha(G)-1$ forbidden pairs with G_k, since v is not coloured yet in G_k. See Figure 5.4 (b) for an example in the case $G = K_n \square K_n$.

To be precise, the worst case is that each of the $\alpha(G) - 1$ unavailable colours by the orthogonality conditions for v in G_k, differ from each of the $\Delta(G)$ colours that are unavailable by the proper colouring property. Then, $\Delta(G) + \alpha(G) - 1$ colours are unavailable for v in G_k in the worst case. Therefore, since $m > \Delta(G) + \alpha(G) - 1$, the vertex v of G_k may always be coloured. □

Corollary 5.3.2 *For all $m, n \in \mathbb{N}$ with $m \geq 3n - 2$, both players have a strategy to draw in the $MOC_m(K_n \square K_n)$ game.*

Proof. By Lemma 5.3.1, the result follows from the facts that $\alpha(K_n \square K_n) = n$ and $\Delta(K_n \square K_n) = 2n - 2$. □

Lemma 5.3.3 *For all $n \in \mathbb{N}$, both players have a strategy to guarantee a draw in the $MOC_1(K_n \square K_n)$ game.*

Proof. By Theorem 5.2.2, Bob has a strategy to force a draw.

Now, we show that Alice has a strategy to force a draw. Let G be $K_n \square K_n$ and let G_k, $k \in \{1, 2\}$, be one copy of G and $G_{\overline{k}}$ be the other copy of G that is not G_k. We identify the graphs G_1 and G_2 with their underlying square boards. As we play with only $m = 1$ colour, it is easy to see that the only possible scores of both players are n and $n - 1$, regardless of strategy. This is due to the fact that the orthogonality condition can only block at most one vertex in a row or column from being coloured, as otherwise, it would violate the Latin property. Therefore, as long as more than one possible vertex exists for a row or column in G_k, then one of the vertices can be coloured. Thus, the vertices of both G_k and $G_{\overline{k}}$ can be coloured until the point is reached where two rows and two columns have no coloured vertices in them and at least one of these vertices can be coloured, guaranteeing a score of at least $n-1$.

Alice, who owns copy G_1 of G, colours a vertex in Bob's copy, G_2 of G, initially. Then, on every subsequent turn until there are $n-2$ coloured vertices in G_2, Alice colours a vertex in $G_{\overline{k}}$ when Bob colours a vertex in G_k. Now, since Alice coloured a vertex in G_2 initially, eventually it is Bob's turn and there are $n-3$ coloured vertices in G_1 and $n-2$ coloured vertices in G_2. We show that Alice can force a draw from here. There are 3 cases based on the next move for Bob.

Case 1 Bob colours a vertex in G_2 and there are no possible moves left in G_2.

In this case, Bob achieved a score of $n-1$ and so Alice can at least draw if not win.

Case 2 Bob colours a vertex in G_2 and there is still a possible move left in G_2.

In this case, Alice colours the last colourable vertex in G_2 and Bob achieves a score of n. Bob is then forced to colour a vertex in G_1 and now it is Alice's turn. There are two rows and columns in G_1 with no coloured vertices in them, and no more vertices may be coloured in G_2. If none of the 4 colourable vertices remaining in G_1 are already coloured in G_2, then Alice will clearly achieve a score of n. Otherwise, at most 2 of the 4 colourable vertices remaining in Alice's board are already coloured in G_2 and, by the Latin property, they are not in the same row or column. Alice colours one of the 4 remaining colourable vertices in G_1 that is in the same row or column (but not the exact same position) as one of those at most 2 already coloured vertices in G_2. Now it is not possible to stop Alice getting a score of n since the last colourable vertex in G_1 is not coloured in G_2.

Case 3 Bob colours a vertex in G_1.

Both G_1 and G_2 have two remaining rows and columns with no coloured vertices in them. There are several cases of the possible situation. Let U_1 (U_2, respectively) be the set of the four possible remaining colourable vertices in G_1 (G_2, respectively). Let $[U_1]$ and $[U_2]$ be the preimage of U_1 and U_2 in G, respectively. Note that at most two of the copies of the vertices in U_1 may already be coloured in G_2 by the Latin property.

Subcase 3.1 1 or 2 of the vertices in U_1 have the property that their copies are already coloured in G_2.

If it is the case that two of the vertices in U_1 have this property, then these two vertices must be in different rows and columns since otherwise, the Latin property would have been violated. Alice colours a vertex in U_1 that is in the same row or column (but not the exact same position) as one of these at most two already coloured vertices in G_2. It is clearly not possible to stop the last colourable vertex in G_1 from being coloured eventually which results in a score of n for Alice.

Subcase 3.2 None of the copies of the vertices in U_1 have already been coloured in G_2.

- If $[U_1] \cap [U_2] = \emptyset$, then clearly both players achieve a score of n.
- If $|[U_1] \cap [U_2]| \in \{1, 2\}$, then clearly Alice has a strategy to get a score of n by playing on a vertex in a position in $[U_1] \cap [U_2]$.
- The case $|[U_1] \cap [U_2]| = 3$ is not possible.
- If $[U_1] = [U_2]$, then Alice colours one of the vertices in U_1. If Bob colours a vertex in U_1, then Alice achieves a score of n and so at least draws the game if not wins. If Bob colours a vertex in U_2 in the same position or same column or row as the vertex Alice just coloured, then Alice can colour a vertex in U_1 and achieves a score of n and again, at least draws the game if not wins. Lastly, if Bob colours a vertex in U_2 but not in the same position nor the same column or row as the vertex Alice just coloured, then either Alice may still colour a vertex in U_1 if there are no forbidden pairs due to orthogonality yet, in which case Alice wins since Bob cannot colour a vertex in U_2 on the next turn by the orthogonality condition, or there is a forbidden pair due to orthogonality, in which case they draw with scores of $n - 1$ each since no vertices in U_1 nor U_2 can be coloured by the orthogonality condition.

□

5.4 Graphs that Admit a Strictly Matched Involution

We denote by $\mathcal{MI}$ the class of graphs that admit a strictly matched involution. See Figure 5.5 for a list of all graphs with at most 5 vertices that admit a strictly matched involution. This construction gives the order of the number of graphs on n vertices that admit a strictly matched involution (including isomorphisms).

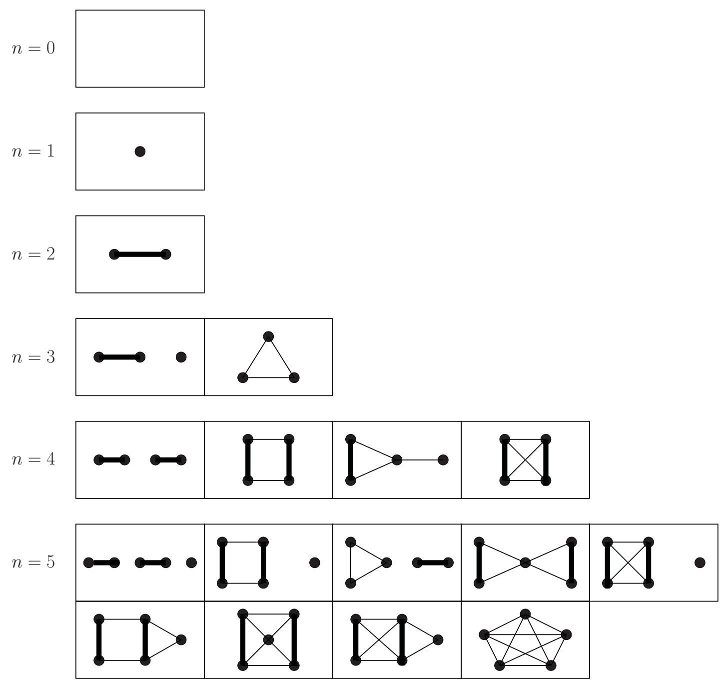


Figure 5.5: List of all graphs with ≤ 5 vertices that admit a strictly matched involution.

5.4.1 Characterising Graphs that Admit a Strictly Matched Involution

We first give an explicit characterization of all graphs $G \in \mathcal{MI}$. We then use this characterization to give an explicit construction for any graph $G \in \mathcal{MI}$. We note however that Andres et al. [3] proved that it is NP-complete to determine whether a graph admits a strictly matched involution.

Theorem 5.4.1 *A graph G admits a strictly matched involution if and only if its vertex set V can be partitioned into a clique C and a set inducing a graph that has a perfect matching M such that:*

1. *for any two edges $vw, xy \in M$, the graph induced by v, w, x, y is isomorphic to*
   - *(a) a $2K_2$ (2 disjoint copies of K_2) or*
   - *(b) a C_4 (there are two possibilities for this) or*
   - *(c) a K_4;*
2. *for any edge $vw \in M$ and any vertex $z \in C$, the graph induced by the vertices v, w, z is isomorphic to*
   - *(a) a $K_1 \cup K_2$ or*
   - *(b) a K_3.*

Proof. First, we prove the forward implication of the theorem, that is, if a graph G admits a strictly matched involution, then the vertices V can be partitioned into a clique C and a matching M such that the properties (1.) and (2.) hold.

Thus, assume $G \in \mathcal{MI}$. Recall from the definition of a graph that admits a strictly matched involution, that (SI 1) and (SI 2) imply that the vertices V can be partitioned into a clique C and a matching M. Now, for any two edges $vw, xy \in M$, the graph induced by $v, w, x, y \in V$ is isomorphic to either:

- a $2K_2$ if no additional edges exist and note that this does not violate any conditions in the definition of a graph that admits a strictly matched involution.

- or a C_4 if vx and wy (vy and wx resp.) are edges in E or a K_4 if $vx, wy, vy, wx \in E$. Indeed, we prove that $vx \in E$ if and only if $wy \in E$ and $vy \in E$ if and only if $wx \in E(G)$, thereby proving that a C_4 or a K_4 are the only two possibilities if additional edges exist. We prove the first case as the other is analogous. Since $G \in \mathcal{MI}$ by assumption, and therefore, by (SI 2) and since σ is an involution, $\sigma(v) = w$ and $\sigma(x) = y$, and since σ is an automorphism, $vx \in E \Leftrightarrow \sigma(v)\sigma(x) = wy \in E$.

For any edge $vw \in M$ and any vertex $z \in C$, the graph induced by $v, w, z \in V$ is isomorphic to either:

- a $K_1 \cup K_2$ if no additional edges exist and note that this does not violate any conditions in the definition of a graph that admits a strictly matched involution.
- or a K_3 if $vz, wz \in E$. Indeed, we prove that $vz \in E$ if and only if $wz \in E$, thereby proving that a K_3 is the only possibility if additional edges exist. The proof is analogous to the second case above and therefore, is omitted.

For the other implication, assume the vertices V can be partitioned into a clique C and a set inducing a graph that has a perfect matching M such that the properties (1.) and (2.) hold. We define a mapping σ as follows. For all vertices $z \in C$, let $\sigma(z) = z$ and for all edges $vw \in M$, let $\sigma(v) = w$ and $\sigma(w) = v$. We will prove that σ is a strictly matched involution.

Clearly, σ is involutive (*i.e.*, $\sigma(\sigma(v)) = v$ for every vertex $v \in V$) and (SI 1) and (SI 2) are satisfied. Now all that remains to show is that σ is a graph homomorphism. That is, it remains to be proven that

$$vw \in E \Longleftrightarrow \sigma(v)\sigma(w) \in E. \tag{5.4}$$

First, the forward direction of (5.4) is proven. Let $vw \in E$. If $vw \in M$, then by our mapping, $\sigma(v) = w$ and $\sigma(w) = v$ and we are done. So, assume $vw \notin M$. If $v, w \in C$, then $\sigma(v) = v$ and $\sigma(w) = w$ and we are done. So, without loss of generality, assume that $v \notin C$ and let $vx \in M$. Then, $\sigma(v) = x$.

If $w \in C$, then $\sigma(w) = w$. Then, by property (2.), the graph induced by the vertices v, x, w is isomorphic to K_3 (since $vw \in E$ and $w \in C$) and hence, $xw = \sigma(v)\sigma(w) \in E$.

If $w \notin C$, then let $wz \in M$. Then, $\sigma(w) = z$. Since M is a matching, $z \notin \{v, x\}$. By property (1.), the graph induced by the vertices v, w, x, z is isomorphic to C_4 or K_4 (since $vw \in E$ and $vw \notin M$) and in either case, $\sigma(v)\sigma(w) = xz \in E$.

Using the forward direction and the fact that σ is involutive, we immediately get the backward direction of (5.4)

$$\sigma(v)\sigma(w) \in E \Longrightarrow vw = \sigma(\sigma(v))\sigma(\sigma(w)) \in E.$$

Thus, σ is strictly matched, *i.e.*, $G \in \mathcal{MI}$. □

Theorem 5.4.1 immediately implies the following structural result.

Corollary 5.4.2 *Any graph G on n vertices admitting a strictly matched involution has a partition of its vertex set into three (possibly empty) vertex subsets inducing a clique C of size $n-2k$ and two isomorphic graphs H and H', each of size k, for some $k \in \mathbb{N}$ with $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, respectively. Moreover,*

- *for any pair (v, v') of corresponding vertices $v \in V(H)$ and $v' \in V(H')$ and any vertex $w \in C$, either both vw and $v'w$ exist or none of them exist;*
- *for any pair (v, v') of corresponding vertices $v \in V(H)$ and $v' \in V(H')$, we have the existence of the matching edge $vv' \in E(G)$;*
- *for any two pairs (v, v') and (w, w') of corresponding vertices with $v, w \in V(H)$ and $v', w' \in V(H')$, either both vw' and $v'w$ exist or none of them exist.*

See Figure 5.6 for a sketch of the structure.

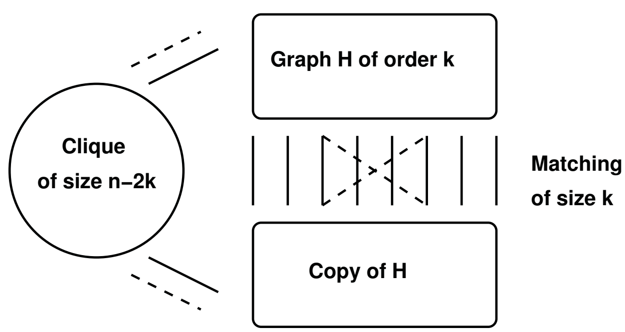


Figure 5.6: The structure of graphs admitting a strictly matched involution.

According to Corollary 5.4.2, we can generate every graph on n vertices admitting a strictly matched involution if we fix some integer $k \leq \frac{n}{2}$ and take two copies of an arbitrary graph on k vertices which are matched by an isomorphism and add possible edges according to the rules given implicitly in Theorem 5.4.1 and explicitly in Corollary 5.4.2. Note that this construction may create isomorphic and even identical graphs.

5.5 Reporting on Other Results

This section reports on results tangential to the ORTHOGONAL COLOURING GAME. In particular, another question a graph theorist may be interested in is the number of graphs which admit a strictly matched involution. A question of concern to computer scientists is the complexity of the game. Both questions will be reported on here. The counting arguments appear in [4] while the complexity results appear in [3].

5.5.1 Counting Results

Let $g(n)$ be the number of isomorphism classes of graphs on n vertices. Let $A(n)$ be the number of isomorphism classes of graphs admitting a strictly matched involution on n vertices.

We use the following well-known fact.

Fact 5.5.1 For any $n \in \mathbb{N}$,

$$\frac{2^{\binom{n}{2}}}{n!} \leq g(n) \leq 2^{\binom{n}{2}}.$$

Theorem 5.5.2 *For any $n \in \mathbb{N}$,*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} 2^{\left(n-\frac{1}{2}\right)k-\frac{3}{2}k^2} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} \leq A(n) \leq 1 + \left\lfloor \frac{n}{2} \right\rfloor \cdot 2^{\left(\frac{n-1}{2}\right)^2}.$$

Corollary 5.5.3 $A(n) = O\left(c(n)\sqrt{g(n)}\right)$ *with* $\log_2(c(n)) = o\left(\log_2 \sqrt[3]{g(n)}\right)$.

Corollary 5.5.4 $A(n) = \Omega\left(d(n)\sqrt[3]{g(n)}\right)$ *with* $\log_2\left(\frac{1}{d(n)}\right) = o\left(\log_2 \sqrt[3]{g(n)}\right)$.

5.5.2 Complexity Results

Theorem 5.5.5 *The problem of deciding whether a graph is in $\mathcal{MI}$ is NP-complete.*

Theorem 5.5.6 *Given an instance $NorMOC_m(G)$ of the orthogonal colouring game that includes a partial colouring, the problem of determining the outcome of $NorMOC_m(G)$ under optimal play is PSPACE-complete for all $m \geq 1$.*

Theorem 5.5.7 *Given an instance $MOC_m(G)$ of the orthogonal colouring game that includes a partial colouring, the problem of determining the outcome of $MOC_m(G)$ under optimal play is PSPACE-complete for all $m \geq 3$.*

5.6 Conclusions

Research within the context of the ORTHOGONAL COLOURING GAME allows for a multitude of directions to pursue for future work. From a game theoretic point of view, we are interested in the following:

Question 3 Determine the outcome for the ORTHOGONAL COLOURING GAME for other classes of graphs.

Beyond the scope of this thesis, but there are game theorists interested in the following:

Question 4 What is the strategy for the multiplayer version of the ORTHOGONAL COLOURING GAME (3 or more players)?

Chapter 6

Simultaneous Combinatorial Game Theory

Research in the area of simultaneous combinatorial game theory began in 2007 under the description of *synchronized games*[1]. There have been many variants of placement games studied under simultaneous moves (see [21, 24–28]). Cincotti et al. studied SYNCHRONIZED CUTCAKE, SYNCHRONIZED DOMINEERING and several variants of SYNCHRONIZED DOMINEERING, determining outcomes for certain grid sizes. They also claimed values but do not provide any supporting theory for those claims. Bahri and Kruskal [6] presented a new method for considering SYNCHRONIZED DOMINEERING which bounds the outcomes using combinatorial game theory techniques. More recently, in 2016, some work has been done with regards to SIMULTANEOUS COPS AND ROBBERS [51]. In particular, in [51] it was proven that the simultaneous cop number is equal to the cop number under alternating play for all graphs, and optimal strategies under simultaneous play are given. To date, no framework for general rulesets or game values has been developed.

We begin our discussion of simultaneous combinatorial game theory with some general definitions. The notation is adapted from a combination of combinatorial game theory notation and economic game theory notation. Note that *round* and *turn* will be used interchangeably throughout this chapter to mean that both players have made a move in the game.

Definition 6.0.1 (Ruleset for simultaneous combinatorial games) Given a set of game positions Ω, a *ruleset* over Ω consists of three functions $L, R, S : \Omega \to 2^{\Omega}$. For $G \in \Omega$, $L(G)$ is the set of *Left options*, which we will denote as $G^{\mathcal{L}}$, $R(G)$ is the set of *Right options*, denoted $G^{\mathcal{R}}$, and $S(G)$ is the set of *simultaneous options*, denoted $G^{\mathcal{S}}$.

[1]Cincotti et al. used synchronized to describe games where the moves were of a particular type. Our scope is more general so we use the term *simultaneous*.

We will represent the options from a combinatorial game G as,

$$G = \left\{G^{\mathcal{L}} \mid G^{\mathcal{S}} \mid G^{\mathcal{R}}\right\}.$$

The elements of $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ in Definition 6.0.1 are the Left and Right options from the underlying combinatorial game. We need both $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ because in a sum, $G \odot H$, players may be able to move in different components on a round. Our guiding philosophy for simultaneous combinatorial games asserts that players know the moves available to them. More importantly, based on this philosophy,

$$G^{\mathcal{L}} \neq \emptyset \quad \text{and} \quad G^{\mathcal{R}} \neq \emptyset \iff G^{\mathcal{S}} \neq \emptyset.$$

That is, if both Left and Right have a move, we insist there is a simultaneous move. Conversely, if there is a simultaneous move, it must arise from a Left and a Right option. In many games, a position obtained by a Left move followed by a Right move can be reached by interchanging the moves. In these cases the moves can be played simultaneously without further clarification. Sometimes however, G^L and G^R played simultaneously may lead to a game board which is not attainable in a combinatorial game under alternating play and so the game rulesets need to have an extra rule to account for the possibility of player interference leading to an illegal CGT position. We define the *interference rule* of a game in terms of the illegal CGT positions. The simultaneous ruleset must deal with the possibility of illegal positions resulting from Left and Right's simultaneous choice of options, in such a way that the spirit of combinatorial game theory is maintained. That is, players must know when they have a move available to them, retain *perfect information*, and the game terminates in a finite number of moves. This means that for the games in this chapter a position G cannot occur as a follower of G because of the interference rules. A case where a position may reoccur, yet still be analyzable, is presented in Section 7.2 where the expected number of moves is finite. We adjust Definition 6.0.1 to follow our philosophy.

Definition 6.0.2 (Ruleset for simultaneous combinatorial games following our philosophy) Given a set of game positions Ω, a *ruleset* over Ω consists of three functions

$L, R, S : \Omega \to 2^{\Omega}$. For $G \in \Omega$, $L(G)$ is the set of *Left options*, which we will denote as $G^{\mathcal{L}}$, $R(G)$ is the set of *Right options*, denoted $G^{\mathcal{R}}$, and $S(G)$ is the set of *simultaneous options*, denoted $G^{\mathcal{S}}$. Moreover, the game can be described in terms of a *game matrix*, denoted $\mathrm{M}(G)$. Left's options, the elements of $G^{\mathcal{L}}$, label the rows, and Right's options, the elements of $G^{\mathcal{R}}$, label the columns. An entry $G^{S_{i,j}}$ will be the result of Left playing option i and Right playing option j in G. A game G is called a *terminal position* if $G^{\mathcal{S}} = \emptyset$.

Hence if Left has m options and Right has n options we have,

$$\mathrm{M}(G) = \begin{array}{c} \\ G^{L_1} \\ G^{L_2} \\ \vdots \\ G^{L_m} \end{array} \begin{array}{c} \begin{array}{cccc} G^{R_1} & G^{R_2} & \dots & G^{R_n} \end{array} \\ \begin{bmatrix} G^{S_{1,1}} & G^{S_{1,2}} & \dots & G^{S_{1,n}} \\ G^{S_{2,1}} & G^{S_{2,2}} & \dots & G^{S_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ G^{S_{m,1}} & G^{S_{m,2}} & \dots & G^{S_{m,n}} \end{bmatrix} \end{array}. \tag{6.1}$$

Noteworthy, is that Definition 6.0.2 does not define the rule associated with an interference of a simultaneous Left and Right option for all games, just that the move is legal. An interference rule must be introduced within each game ruleset. In addition to defining conditions on interference within a game ruleset, we also need to be careful in how the interference is defined in order to avoid problematic game properties. For example, if the interference rule is defined to be that each time interference occurs, the players simply ignore that turn and resume the game with the same game options, then the game can be *loopy*, meaning that it could involve infinite play. A comprehensive overview of loopy games in CGT can be found in [66]. We will not explore loopy games in detail here, but we briefly examine a case study with the possibility of infinite play, in Chapter 7, Section 7.2.

Depending on the type of game, the game interference can have a very intuitive interpretation. For example, in [27] the interference rule in SYNCHRONIZED DOMINEERING was defined as follows: if G^L and G^R are legal, then both dominoes played simultaneously is also legal (i.e., overlapping dominoes on that turn is permitted). Allowing interference could seem like an ideal method for determining a general definition for dealing with the interference problem. However, for games like SNORT, COL, and NOGO this interpretation is not sufficient to know how to play the game.

Consider the game SNORT. Played simultaneously, players could interfere by both colouring vertex v on any given turn, but allowing overlap does not determine the consequence to the vertex colouring of the neighbours of v. Thus, this must be considered as additional rules within individual game play rulesets, rather than an overarching condition which applies to all games.

When defining the interference rule, we also need to ensure finite descent will hold true and that other CGT rule violations are addressed within the interference rule. An example of a combinatorial game where we need to take care in defining the interference rule is NOGO. Possible interference in SIMULTANEOUS NOGO includes (i) players choosing the same vertex, and (ii) players choosing two different vertices which leads to a maximal monochromatic connected subgraph not being adjacent to an uncoloured vertex. The interference rule for SIMULTANEOUS NOGO is as follows: if the resulting simultaneous move produced a legal NOGO position, then the simultaneous move is permitted. Otherwise, the vertices which players tried to colour on that round, leading to an illegal NOGO position, are converted to uncoloured, unplayable vertices for the remainder of the game. The game tree of simultaneous moves can exceed the depth of that of either player on their own (see Example 6.0.3), though based on the interference rule defined above, the game will terminate in a finite number of moves since the graph is finite and at least one vertex is no longer playable after each round.

Example 6.0.3 Consider the following position of SIMULTANEOUS NOGO. A star S_4 where each leaf of the star has itself a new leaf attached and all leaves are coloured blue, and black vertices are uncoloured (see Figure 6.1).

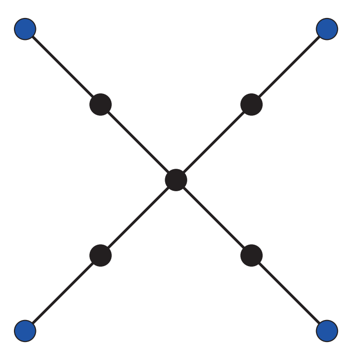


Figure 6.1: A SIMULTANEOUS NOGO position.

If Left were to move alone in this position, regardless of where she moves, she would have three moves remaining thereafter. If Right moves alone in this position, he only has one option, which is to colour the centre vertex red. However, if both players move in this component and choose the centre vertex, Left will have four moves remaining in the component and Right will have none. Note that Left choosing a different vertex on a simultaneous move is dominated, as both pieces would get removed and she would have three moves remaining rather than four. Hence $G = \{3 \mid 4 \mid 0\}$.

A game we will refer to often is called SUBTRACTION SQUARES. The ruleset is as follows.

Ruleset for SUBTRACTION SQUARES, $\mathrm{SQ}(S_L,\, S_R)$ on a strip of squares of length n, denoted $\mathrm{SQ}(S_L,\, S_R)(\underline{n})$

- Board: Let S_L and S_R be sets of positive integers. The board is a strip of n squares, denoted $\underline{n}$.
- Moves: For any $p \in S_L$, $p \leq n$, Left can remove p squares from the left or right side of the strip. Similarly, if $q \in S_R$, $q \leq n$, then Right can remove q squares from the left or right side of the strip.
- Interference rule: If they both take from the same side then $\max\{p, q\}$ squares are removed. If they take from opposite sides then the move is to $n - p - q$ except if $\max\{p, q\} \leq n \leq p + q$ then the move is to 0.

Note that if the interference rule is always to remove $p + q$, without any reference to the side played, then in $\mathrm{SQ}(\{1, 10\}, \{2, 10\})(\underline{12})$ neither player knows if subtracting 10 is legal. The corresponding matrix is

$$\begin{array}{cc} & \begin{array}{cc} 2 & 10 \end{array} \\ \begin{array}{c} 1 \\ 10 \end{array} & \begin{bmatrix} \underline{3} & \underline{1} \\ \underline{0} & ? \end{bmatrix} \end{array}$$

and has a non-entry. On the other hand, if the interference rule is to remove $|p - q|$ then the same game could last forever.

Ottaway's card interpretation: illegal positions are never allowed

An alternative method to deal with interference is to not allow them, and the Left and Right options which led to the interference get removed from game play. A way to model this in actual game play is as follows: Consider the game board, and each position of play is numbered. Then (a) Players receive cards with numbers corresponding to their options. (b) Players each choose a card and place it face up on the table. (c) If no interference occurs, players each take the action they chose. (d) If there is interference, players choose a new card and place it on the table. (e) This continues until either (i) players run out of cards and the game is over; or (ii) a legal simultaneous move occurs and then players continue play starting from step (a).

Under this interpretation, G^L and G^R could exist but G^S might not exist. Practically speaking, this is an easy way to play games simultaneously. However, in terms of analysis, this does not simplify matters.

The focus of this chapter is to develop a general algebraic framework for simultaneous play in combinatorial games. Foundational questions originating from CGT are examined here under simultaneous play, such as

1. What reductions exist for game analysis: within individual games and within game sums?
2. Which game sum is natural for this framework?
3. Which winning convention is natural for this framework?
4. How do we test for game equality?
5. How are values defined?

The goal of this chapter is to establish the basic definitions and algebraic framework for simultaneous combinatorial game theory by answering the questions above. After basic definitions are established, other questions arise such as: (i) as in CGT, is there a group structure? (ii) if not, what algebraic structure does exist in simultaneous combinatorial game theory?

Basic definitions from CGT (see Definition 2.3.1), that we adapt for simultaneous games are as follows: Let G and H be games,

- *Equality of games:* $G = H$ if $(\forall X)\ \alpha(G \odot X) = \alpha(H \odot X)$.

- *Greater than:* $G \geq H$ if $(\forall X)$, $\alpha(G \odot X) \geq \alpha(H \odot X)$.

where $\odot$ is a generic sum and α represents a generic measure of winning and if $\alpha(G) > \alpha(H)$ then Left has a greater chance of winning G than H.

To answer the questions above we need to be able to develop a test for equality, similar to Theorem 2.4.8, which does not involve considering all games. We also assume the players are rational.

This chapter will proceed as follows. First, in Section 6.1 we present tools from economic game theory which we will need to analyze simultaneous combinatorial games. In Section 6.2 three sums from CGT (disjunctive, conjunctive and continued conjunctive) are redefined for simultaneous play, and several examples for game play are examined. In Section 6.3, three winning conventions (extended normal, scoring, and majority win) are defined under each of the three sums. Challenges and proposed solutions for each combination of sum and winning convention are discussed. Case studies for three games are explored in more detail in Chapter 7.

The rulesets for simultaneous combinatorial games can be found in Appendix C. For some other approaches to combining combinatorial game theory and economic game theory see [11, 34, 46].

6.1 Economic Game Theory: Background

Here we provide the basic background for two player zero sum games. See [8] for a standard reference in economic game theory; for a light read, with many practical applications, see [70]. To be consistent with players' names between combinatorial game theory and economic game theory, in later sections we will adopt the combinatorial game theory convention by letting

$$\text{Row Player} = \text{Rose} = \text{Left}$$

and

$$\text{Column Player} = \text{Colin} = \text{Right}.$$

A game matrix or *payoff matrix* is a matrix which summarizes the results of Rose and Colin playing strategies available to them in a game G. *Zero sum* games are games where a gain for one player is a loss for the other player. As we are only considering zero sum games, the matrix will be in terms of the payoffs to Rose, while the payoffs to Colin will be the negatives of the game matrix entries. For a running example, let's consider the classical game of *Rock-Paper-Scissors* (see Example 6.1.1).

Example 6.1.1 *Rock-Paper-Scissors* is a simultaneous game, G, where each player, on a round, chooses one of the three options {Rock, Paper, Scissors}. The rules are that paper beats rock, rock beats scissors and scissors beats paper. The game matrix, A, is as shown in (6.2).

$$A = \begin{array}{c} Rose \backslash Colin \\ \text{Rock} \\ \text{Paper} \\ \text{Scissors} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Rock} & \text{Paper} & \text{Scissors} \end{array} \\ \left[\begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array} \right] \end{array} \tag{6.2}$$

Definition 6.1.2 [8] The *pure strategies* for Rose are the rows of the payoff matrix. The pure strategies for Colin are the columns of the payoff matrix.

In Example 6.1.1, there are three pure strategies for each player and their pure strategies are the same: Rock, Paper, and Scissors.

Definition 6.1.3 [8] Let $A_{n \times m} = (a_{i,j})$ be a matrix game. Row i strictly dominates row k if $a_{ij} > a_{kj}$ for all $j = 1, 2, \ldots, m$. Column j strictly dominates column k if $a_{ij} < a_{ik}$ for all $i = 1, 2, \ldots, n$.

Recall that Rose is trying to win more of the time in matrix A, that is, get an outcome of 1 more often than -1. While Colin is trying to do the reverse (get -1 more often than 1). In Definition 6.1.3 this is demonstrated by players having different applications of domination.

A rational player will not choose a row or column that is strictly dominated, hence we can delete it from the matrix. If equality holds on one or more of the rows (columns, respectively), the domination is said to be weak. If concerned with determining only the value of a game, weakly dominated rows (or columns) can be deleted.

Definition 6.1.4 [8] If there exists a constant $\lambda \in [0, 1]$ so that

$$a_{kj} \leq \lambda a_{pj} + (1-\lambda)a_{qj},\ j = 1, \ldots, m.$$

then row k is dominated by a *convex combination* of rows p and q and row k can be dropped.

Similarly, if there exists a constant $\lambda \in [0, 1]$ so that

$$a_{ik} \geq \lambda a_{ip} + (1-\lambda)a_{iq},\ i = 1, \ldots, n.$$

then column k is dominated by a convex combination of columns p and q and column k can be dropped.

Definition 6.1.5 [8] A *mixed strategy* is a vector $X = (x_1, x_2, \ldots, x_n)$ for Rose and $Y = (y_1, y_2, \ldots, y_m)$ for Colin, where

$$x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad y_j \geq 0, \sum_{j=1}^{m} y_j = 1$$

and denote the set of mixed strategies with k components by

$$S_k = \{(z_1, z_2, \ldots, z_k) \quad | \quad z_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} z_i = 1\}.$$

Definition 6.1.6 [8] Let $A_{n\times m} = (a_{i,j})$ be a matrix game. Given a choice of mixed strategy $X \in S_n$ for Rose and $Y \in S_m$ for Colin, chosen independently, the *expected payoff* of the game to Rose is

$$\begin{aligned}
\text{Ex}(X, Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\text{Prob(Rose uses i and Colin uses j)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\text{Prob(Rose uses i)Prob(Colin uses j)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j = XAY^T
\end{aligned}$$

Definition 6.1.7 [8] A matrix game with matrix $A_{n\times m} = (a_{i,j})$ has *upper and lower values* of the mixed game as

$$v^+ = \min_{Y\in S_m}\max_{X\in S_n} XAY^T \quad \text{and} \quad v^- = \max_{X\in S_n}\min_{Y\in S_m} XAY^T$$

Theorem 6.1.8 [8] *For any $n \times m$ matrix A, we have*

$$\begin{aligned} v^+ &= \min_{Y \in S_m} \max_{X \in S_n} XAY^T \\ &= \max_{X \in S_n} \min_{Y \in S_m} XAY^T \\ &= v^-. \end{aligned}$$

The common value is denoted $v(A)$, or value(A), and that is the value of the game*.* *In addition, there are mixed strategies $X^* \in S_n$, $Y^* \in S_m$ so that*

$$\text{Ex}(X, Y^*) \leq \text{Ex}(X^*, Y^*) = v(A) \leq \text{Ex}(X^*, Y), \forall (X \in S_n, Y \in S_m).$$

Important Note: This chapter and the next chapter uses both CGT values and economic game values. Hence to avoid ambiguity we will denote the value of a game as calculated using Theorem 6.1.8 by EV(G).

In Example 6.1.1, there is no domination of any type. If a player always chose the same pure strategy, over repeated plays their opponent would figure this out and would be able to win. Thus players will instead use mixed strategies to play this game. Using mixed strategies makes it difficult for players to take advantage of their opponents' strategy because there is randomness involved. In particular, if both players choose their optimal mixed strategies neither player can do better by changing their strategies. Based on the symmetries of the pure strategies, players will play each pure strategy an equal number of times and based on the symmetries of the matrix, the players have the same optimal mixed strategy. In particular, Rose will use $X^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and Colin will use $Y^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. This means that over many plays, players will play each pure strategy a third of the time. By Theorem 6.1.8 we obtain EV$(G) = 0$.

Note about notation: In sections to follow, we will omit the '$*$' from optimal strategy notation. It will be clear from the context which strategies are being discussed.

6.2 Sums

Naturally, to extend combinatorial game theory, we are interested in simultaneous games that have components, how these components are played, and extending the concepts of equality and inequality. Here we will consider three sums defined by Conway [29]: disjunctive, conjunctive and continued conjunctive. In CGT, the disjunctive sum leads to rich algebraic structure, and thus is a natural starting point for analysis of simultaneous combinatorial game theory. In conjunctive and continued conjunctive sums, since both have a notion of moving simultaneously in components, they appeared as natural choices for further exploration among the many sums defined in [29]. For other literature exploring other sums, see [38], [50], and [68]. Before redefining the sums for simultaneous play, we discuss some general properties of sums and outcomes under simultaneous play.

First, there is an issue with determining what constitutes a component. A component is defined at the outset of a game. At the beginning of each section, we will discuss what we consider to be a component.

The notions of equality and greater than are defined similarly for the outcome classes of any generic sum, $\odot$. Associativity and commutativity are also defined similarly to CGT play since each game within a sum is considered to be played on its own board and the placement of the board is irrelevant. First mentioned in the introduction, we examine *equality* and *greater than* as to follow suit with CGT. We formally restate the definitions here.

Definition 6.2.1 Let G and H be games,

- *Equality of games:* $G = H$ if $(\forall X)\ \alpha(G \odot X) = \alpha(H \odot X)$.
- *Greater than:* $G \geq H$ if $(\forall X)$, $\alpha(G \odot X) \geq \alpha(H \odot X)$.

where $\odot$ represented any generic sum and α represents a generic measure of winning and if $\alpha(G) > \alpha(H)$ then Left has a greater chance of winning G than H.

In the sums that we consider, it may be possible for the two players to play in different components. Thus in a sum $G \odot H$, the definition of $G \odot H$ may require all of $G^{\mathcal{L}}$, $G^{\mathcal{R}}$, $G^{\mathcal{S}}$, $H^{\mathcal{L}}$, $H^{\mathcal{R}}$, and $H^{\mathcal{S}}$. All the sums have two common properties.

Theorem 6.2.2 *Simultaneous combinatorial games, under a sum, form an equivalence relation and the quotient is a partial order.*

Proof. From the definition of equality, it is clear that: (i) $G = G$ for all G; (ii) if $G = H$ then $H = G$; and (iii) if $G = H$ and $H = K$ then $G = K$. Therefore equality is an equivalence relation.

Equal games are identified to obtain the quotient by '=', that is, the objects are now the equivalence classes. The proof for a partial order is now similar to that for equality. □

An open question is what properties, if any, does the partial order have. In alternating play CGT, the order is a distributive lattice [2]. Here we only know about the continued conjunctive sum with the scoring winning convention, see Corollary 6.3.30.

	$\underline{1}$	$\underline{2}$	$\underline{3}$
$G^{\mathcal{L}}$	$\{\underline{0}\}$	$\{\underline{1}\}$	$\{\underline{2}\}$
$G^{\mathcal{R}}$	–	$\{\underline{0}\}$	$\{\underline{1}\}$
$G^{\mathcal{S}}$	–	$\{\underline{0}\}$	$\{\underline{0}, \underline{1}\}$

Table 6.1: Summary of options for positions of SQ($\{1\}, \{2\}$)($\underline{1} \odot \underline{2} \odot \underline{3}$).

Table 6.1 illustrates the differences between the sums defined in the next three sections. For examples pertaining to SIMULTANEOUS HACKENBUSH, see Section 7.1. We focus on three main sums: disjunctive, conjunctive and continued conjunctive. In later sections, the winning conventions will be defined. Here, we consider their definitions in terms of options. For all definitions, consider G and H to be two simultaneous combinatorial games. This can be extended to larger sums (games with multiple components) by induction.

We need to introduce some notation. Let X and Y be sets of positions and $\odot$ be a sum. Define $X \odot Y = \{x \odot y : x \in X, y \in Y\}$ and if z is a position, then $X \odot z = \{x \odot z : x \in X\}$.

Disjunctive Sum

The *disjunctive sum* of two combinatorial games being played under simultaneous moves, denoted by $G + H$, means that each player chooses a component and plays a legal move in that component. Within the context of disjunctive sum, a component is any board which has moves remaining.

Definition 6.2.3 The set of options from $G+H$ is

$$\left\{G^{\mathcal{L}}+H, G+H^{\mathcal{L}} \mid G^{\mathcal{S}}+H, G+H^{\mathcal{S}}, G^{\mathcal{L}}+H^{\mathcal{R}}, G^{\mathcal{R}}+H^{\mathcal{L}} \mid G^{\mathcal{R}}+H, G+H^{\mathcal{R}}\right\}.$$

Using Table 6.1, consider $G = \mathrm{SQ}(\{1\},\{2\})(\underline{1}+\underline{2}+\underline{3})$. The game is not terminal because Left has moves in $\underline{1}$, $\underline{2}$, and $\underline{3}$, while Right has moves in $\underline{2}$ and $\underline{3}$.

Let G be a game with m Left options and n Right options. Let H be a game with k Left options and ℓ Right options. The matrix of a game sum $G+H$ is as in (6.3).

$$\begin{array}{c} \\ G^{L_1} \\ \vdots \\ G^{L_m} \\ H^{L_1} \\ \vdots \\ H^{L_k} \end{array}
\begin{array}{c} \begin{array}{cccccc} G^{R_1} & \dots & G^{R_n} & H^{R_1} & \dots & H^{R_\ell} \end{array} \\
\begin{bmatrix} G^{S_{1,1}}+H & \dots & G^{S_{1,n}}+H & G^{L_1}+H^{R_1} & \dots & G^{L_1}+H^{R_\ell} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G^{S_{m,1}}+H & \dots & G^{S_{m,n}}+H & G^{L_m}+H^{R_1} & \dots & G^{L_m}+H^{R_\ell} \\ G^{R_1}+H^{L_1} & \dots & G^{R_n}+H^{L_1} & G+H^{S_{1,1}} & \dots & G+H^{S_{1,\ell}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G^{R_1}+H^{L_k} & \dots & G^{R_n}+H^{L_k} & G+H^{S_{k,1}} & \dots & G+H^{S_{k,\ell}} \end{bmatrix} \end{array}. \tag{6.3}$$

To get a better understanding of the representation of $\mathrm{M}(G+H)$ let's look at an example.

Example 6.2.4 Let $G = \mathrm{SQ}(\{1\}\{2\})(\underline{4})$ and $H = \mathrm{SQ}(\{1\}\{2\})(\underline{3})$. The matrix of the disjunctive sum $G+H$ is shown in (6.4). Note that $\underline{X}_t$ means that the player removed from side t (either left (ℓ) or right (r)) from game $\underline{X}$ (either $\underline{4}$ or $\underline{3}$).

$$\mathrm{M}(G+H) = \begin{array}{c} \\ \underline{4}_\ell \\ \underline{4}_r \\ \underline{3}_\ell \\ \underline{3}_r \end{array}
\begin{array}{c} \begin{array}{cccc} \underline{4}_\ell & \underline{4}_r & \underline{3}_\ell & \underline{3}_r \end{array} \\
\begin{bmatrix} \underline{2}+\underline{3} & \underline{1}+\underline{3} & \underline{3}+\underline{1} & \underline{3}+\underline{1} \\ \underline{1}+\underline{3} & \underline{2}+\underline{3} & \underline{3}+\underline{1} & \underline{3}+\underline{1} \\ \underline{2}+\underline{2} & \underline{2}+\underline{2} & \underline{4}+\underline{1} & \underline{4}+\underline{0} \\ \underline{2}+\underline{2} & \underline{2}+\underline{2} & \underline{4}+\underline{0} & \underline{4}+\underline{1} \end{bmatrix} \end{array} \tag{6.4}$$

Note that no matrix entries correspond to terminal games.

Conjunctive Sum

The *conjunctive sum* of two simultaneous combinatorial games, denoted by $G \wedge H$, means that each player plays a legal move in all components. In the conjunctive sum, unlike the disjunctive sum, the Left and Right options are only used for deciding the winner, and are never used to determine the actual moves in the play of the game. We only need to define the options coming from simultaneous moves. Also, the components within a conjunctive sum is the initial set of boards within the sum. Hence, if $G = K_1 \wedge K_2$ then K_1 and K_2 are the components of the sum G, and will always constitute the boards in play. For example, if K_1 breaks up into several smaller games, it remains a single component.

Definition 6.2.5 The set of simultaneous options from $G \wedge H$ are:

$$G \wedge H = \left\{ G^{\mathcal{S}} \wedge H^{\mathcal{S}} \right\}.$$

Using Table 6.1, consider now $G = \mathrm{SQ}(\{1\},\{2\})(\underline{1} \wedge \underline{2} \wedge \underline{3})$. The game is over because Right does not have a move in $\underline{1}$.

For an example of a matrix for this sum, see Example 6.2.6.

Example 6.2.6 Let $G = \mathrm{SQ}(\{1\}\{2\})(\underline{4})$ and $H = \mathrm{SQ}(\{1\}\{2\})(\underline{3})$. The matrix of the conjunctive sum $G \wedge H$ is shown in (6.5), where ℓ/r means that a player chose left in G and right in H.

$$\mathrm{M}(G \wedge H) = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} \underline{2} \wedge \underline{1} & \underline{2} \wedge \underline{0} & \underline{1} \wedge \underline{1} & \underline{1} \wedge \underline{0} \\ \underline{2} \wedge \underline{0} & \underline{2} \wedge \underline{1} & \underline{1} \wedge \underline{0} & \underline{1} \wedge \underline{1} \\ \underline{1} \wedge \underline{1} & \underline{1} \wedge \underline{0} & \underline{2} \wedge \underline{1} & \underline{2} \wedge \underline{0} \\ \underline{1} \wedge \underline{0} & \underline{1} \wedge \underline{1} & \underline{2} \wedge \underline{0} & \underline{2} \wedge \underline{1} \end{bmatrix} \end{array} \tag{6.5}$$

Note that every matrix entry corresponds to terminal games.

Continued Conjunctive Sum

The *continued conjunctive sum* of two simultaneous combinatorial games, denoted by $G \bigtriangledown H$, means that each player plays a legal move in each component where they both have a move. In the continued conjunctive sum, similar to the conjunctive sum, the Left and Right options are only used for deciding the winner, and are never used to determine the actual moves in the play of the game. We only need to define the options coming from simultaneous moves. Also, a component within a conjunctive sum, is the initial set of boards within the sum. Hence, if $G = K_1 \bigtriangledown K_2$ then K_1 and K_2 are the components of the sum G, and will always constitute the boards in play. For example, if K_1 breaks up into several smaller games, it remains as a single component.

Definition 6.2.7 The set of simultaneous options from $G \bigtriangledown H$ is:

$$G \bigtriangledown H = \begin{cases} \{G^{\mathcal{S}} \bigtriangledown H\}, \text{ if } G^{\mathcal{S}} \neq \emptyset \text{ and } H^{\mathcal{S}} = \emptyset; \\ \{G \bigtriangledown H^{\mathcal{S}}\}, \text{ if } H^{\mathcal{S}} \neq \emptyset \text{ and } G^{\mathcal{S}} = \emptyset; \\ \{G^{\mathcal{S}} \bigtriangledown H^{\mathcal{S}}\}, \text{ if } G^{\mathcal{S}} \textit{ and } H^{\mathcal{S}} \text{ are non-empty;} \\ \emptyset, \text{ no move otherwise.} \end{cases}$$

Using Table 6.1, consider $G = \mathrm{SQ}(\{1\},\{2\})(\underline{1} \bigtriangledown \underline{2} \bigtriangledown \underline{3})$. The game is not over because both players have moves in $\underline{2}$ and $\underline{3}$.

To get a better handle on how the matrix of a game sum would be, let's explore an example.

Example 6.2.8 Consider the games $G = \mathrm{SQ}(\{1\}\{2\})(\underline{4})$ and $H = \mathrm{SQ}(\{1\}\{2\})(\underline{3})$. The matrix of the continued conjunctive sum of $G \bigtriangledown H$ is shown in (6.6), where ℓ/r means that a player chose left in G and right in H.

$$\mathrm{M}(G \bigtriangledown H) = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} \underline{2} \bigtriangledown \underline{1} & \underline{2} \bigtriangledown \underline{0} & \underline{1} \bigtriangledown \underline{1} & \underline{1} \bigtriangledown \underline{0} \\ \underline{2} \bigtriangledown \underline{0} & \underline{2} \bigtriangledown \underline{1} & \underline{1} \bigtriangledown \underline{0} & \underline{1} \bigtriangledown \underline{1} \\ \underline{1} \bigtriangledown \underline{1} & \underline{1} \bigtriangledown \underline{0} & \underline{2} \bigtriangledown \underline{1} & \underline{2} \bigtriangledown \underline{0} \\ \underline{1} \bigtriangledown \underline{0} & \underline{1} \bigtriangledown \underline{1} & \underline{2} \bigtriangledown \underline{0} & \underline{2} \bigtriangledown \underline{1} \end{bmatrix} \end{array} \tag{6.6}$$

Note that only entries in red correspond to terminal games.

6.3 Winning Conventions

In simultaneous games, since the players move at the same time, the winning conventions cannot be based purely on who moves last, as in CGT. We define three conventions. The first depends on a *non-losing condition*: if a player has a move in a component C then that player cannot lose the whole game because of C. If, in a sum, a player has a move in each component then that player cannot lose on that turn. This convention is called the *extended normal play*. The second winning convention is determined by a *score* which is assigned at the end of the game, i.e., at a *terminal* position. Lastly, we define a winning convention based on the number of components a player wins, called the *majority winning convention*. All conventions allow a *Draw* as an outcome. Each winning convention will need to be defined separately based on the sum considered. Throughout the remainder of the chapter, we primarily focus on extended normal play and scoring play winning conventions, as these are well studied in combinatorial game theory.

6.3.1 Extended Normal Play

There are three possible outcomes of a terminal position of a simultaneous combinatorial game, denoted by $o_S(G)$, played under extended normal play:

$$o_S(G) = \begin{cases} \mathcal{L}, \text{Left wins } G \text{ if } G^{\mathcal{L}} \neq \emptyset \text{ and } G^{\mathcal{R}} = \emptyset; \\ \mathcal{D}, \text{Draw if } G^{\mathcal{L}} = G^{\mathcal{R}} = \emptyset; \\ \mathcal{R}, \text{Right wins } G \text{ if } G^{\mathcal{R}} \neq \emptyset \text{ and } G^{\mathcal{L}} = \emptyset. \end{cases}$$

In keeping with the two-player, zero sum conventions, the outcomes are partially ordered $\mathcal{L} > \mathcal{D} > \mathcal{R}$. Outcomes need to be extended because not all game positions are $\mathcal{L}$, $\mathcal{D}$ or $\mathcal{R}$. Sometimes a game is not terminal but the combination of options leading to a terminal game is non-deterministic. This was illustrated in Example 6.2.6; regardless of the Left and Right moves, the game terminates immediately. Optimal mixed strategies will depend on the particular game matrix. In many cases within this chapter the optimal mixed strategies will correspond to pure strategies having equal probability of being played, however this is due to the structure of the games we consider and is not true in all games. First we note that by definition of the ruleset, the options of a game exist and have values. This allows for the next concept to be well-defined.

Let $\overline{\mathrm{M}(G)}$ be the matrix $\mathrm{M}(G)$ with each entry $G^{S_{i,j}}$ replaced by $\mathrm{EV}(\overline{\mathrm{M}(G^{S_{i,j}})})$. That is, each simultaneous option of G will be replaced by its value. As each $G^{S_{i,j}}$ is an option of G, they exist and have values.

Definition 6.3.1 Let G be a simultaneous combinatorial game, played under the extended normal winning convention. The *value* of G, $\mathrm{EV}(G)$, is given recursively

$$\mathrm{EV}(G) = \begin{cases} 1, & \text{if } G \text{ is terminal and a Left win,} \\ 0, & \text{if } G \text{ is terminal and a Draw,} \\ -1, & \text{if } G \text{ is terminal and a Right win,} \\ \mathrm{EV}(\overline{\mathrm{M}(G)}), & \text{otherwise.} \end{cases}$$

This means that if $\mathrm{EV}(G) = 1$ then Left can force a win in the game G. Similarly, if $\mathrm{EV}(G) = -1$ then Right can force a win in the game G. If $\mathrm{EV}(G) \in (0, 1)$ then Left has a higher probability of winning G. Similarly, if $\mathrm{EV}(G) \in (-1, 0)$ then Right has a higher probability of winning G. There are subtleties to consider when examining draws, as well as with any $\mathrm{EV}(G) \in (-1, 1)$, and we will explore both in time.

If we assume that a player wants to maximize their expectation of winning, which is precisely the goal of zero sum matrix games in EGT, then we obtain the following.

Theorem 6.3.2 *Let G be a simultaneous combinatorial game. Then $\overline{\mathrm{M}(G)}$ can be reduced by eliminating dominated options (convex and strict).*

Proof. The matrix $\overline{\mathrm{M}(G)}$ is constructed recursively from the options of G. Thus it has real valued entries, one player's gain is the other player's loss, and players want to choose their strategies to maximize their overall return (which here is their probability of winning). □

Eliminating those dominated options also translates back to $\mathrm{M}(G)$ where the corresponding options can also be eliminated. This is called the *reduced* game, $Re(G)$. Note that this is not a canonical form. As we will see shortly, we can't replace G with $Re(G)$ in all situations (see Section 7.2 for an example).

Let $G = \mathrm{SQ}(\{2\},\{5\})(\underline{8})$. The options, outcomes, and values of the options from G are given in (6.7). From $\overline{\mathrm{M}(G)}$ we calculate the value of the game as follows. The symmetry of the matrix implies that both Left and Right will have the same optimal strategy. Suppose Left will use strategy $X = (x, 1-x)$, where $0 \leq x \leq 1$. Then we can use the method of equalizing expectation (see [8], pp.35) across columns to calculate the value of x, as shown in (6.8)-(6.10). Now that we have Left's optimal strategy ($X = (\frac{1}{2}, \frac{1}{2})$, and recall in this case $X = Y$), we can use it to determine the value of the game. The value of the game can be calculated by matrix multiplication as discussed in Section 6.1. Hence $\mathrm{EV}(\underline{8}) = X \cdot \overline{\mathrm{M}(G)} \cdot Y^T = \frac{1}{2}$. In k plays of $\underline{8}$, since Right can never win, Left will expect to win half of the games and the other half will be Draws.

$$\mathrm{M}(G) = \begin{array}{c} \\ l \\ r \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} \underline{3} & \underline{1} \\ \underline{1} & \underline{3} \end{bmatrix} \end{array}, \quad o_S(\mathrm{M}(G)) = \begin{array}{c} \\ l \\ r \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} \mathrm{L} & \mathrm{D} \\ \mathrm{D} & \mathrm{L} \end{bmatrix} \end{array}, \quad \overline{\mathrm{M}(G)} = \begin{array}{c} \\ l \\ r \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \tag{6.7}$$

$$\begin{array}{c} x \\ 1-x \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(1)(x) + (0)(1-x) = (0)(x) + (1)(1-x) \tag{6.8}$$

$$x = 1 - x \tag{6.9}$$

$$x = \frac{1}{2} \tag{6.10}$$

The challenge with this approach is, if the game is a sum of two other games, say G is the sum of H and K, what reductions can be first applied to H and K individually before considering their sum? This is a similar problem previously considered in CGT. Even though games are played under alternating play, one cannot restrict the study of components and insist that play alternates in each component.

Recall from Section 2.3, given a game G, in CGT, there are operations that, when applied repeatedly, result in a game H where (i) $G = H$; (ii) H is in canonical form; and (iii) $o(G + X) = o(H + X)$. One of these operations is to eliminate dominated strategies, or one of two equal strategies[2]. However, eliminating one of two equal

[2]The other, reversing reversible options, is particular to alternating play, and has no analogue in simultaneous play.

strategies can cause problems in sums of simultaneous games. Our goal in the next several sections, as we define the winning convention under each sum, is to determine how to test whether two games G and H are equal and which (if any) elimination techniques of dominated options can be applied to $\overline{\mathrm{M}(G)}$ to simplify calculations in a sum.

Disjunctive sum

Players are moving at the same time, and thus all play combinations across components must be considered when analyzing any particular game under disjunctive sum. By the above value assignments, the value will enable us to determine a local expectation for a player to win a particular game. As in CGT, Left prefers positive values, and Right prefers negative values. However, if the value is zero, it does not imply a Draw. Similarly, if an value is positive it does not guarantee that Left will win.

Definition 6.3.3 The *extended normal winning convention for the disjunctive sum* of simultaneous combinatorial games $G + H$ is:

- Left wins if $G^{\mathcal{L}} \cup H^{\mathcal{L}} \neq \emptyset$ but $G^{\mathcal{R}} \cup H^{\mathcal{R}} = \emptyset$.
- Right wins if $G^{\mathcal{R}} \cup H^{\mathcal{R}} \neq \emptyset$ but $G^{\mathcal{L}} \cup H^{\mathcal{L}} = \emptyset$.
- Otherwise the game is a Draw.

Mimicking the additive nature of CGT, the hope for simultaneous play would be that $\mathrm{EV}(G + H) = \mathrm{EV}(G) + \mathrm{EV}(H)$ or something equally simple. However, the presence of $G^{\mathcal{L}} + H^{\mathcal{R}}$ and $G^{\mathcal{R}} + H^{\mathcal{L}}$ in the options of $G + H$ makes this unlikely for all but a few games as the next example shows.

Example 6.3.4 Consider the game $\mathrm{SQ}(\{1\},\{2\})\ (\underline{n})$. Note, we will use $\mathrm{EV}(\underline{n})$ as shorthand for $\mathrm{EV}(\overline{\mathrm{M}(\mathrm{SQ}(\{1\},\{2\})(\underline{n}))})$. The position $\underline{2}$ is a Draw and $\mathrm{EV}(\underline{2}) = 0$.

Now consider $\underline{2}+\underline{2}$. Playing in the same component always results in $\underline{0}$, regardless, so we do not need to specify which side players are choosing within a component, only which component they are moving in. In the analysis, $l\,\underline{2}$ and $r\,\underline{2}$ denotes playing in the left or right component in $\underline{2}+\underline{2}$. Right can never win so $\mathrm{EV}(\underline{2}+\underline{2}) \geq 0$. See (6.11).

From $\overline{\mathrm{M}(\underline{2}+\underline{2})}$ we calculate the value, $\mathrm{EV}(\underline{2}+\underline{2}) = 1/2$, which is not $\mathrm{EV}(\underline{2})+\mathrm{EV}(\underline{2})$ or $\mathrm{EV}(\underline{2})\mathrm{EV}(\underline{2})$.

$$\mathrm{M}(\underline{2}+\underline{2}) = \begin{array}{c} \\ l\,\underline{2} \\ r\,\underline{2} \end{array} \begin{array}{c} \begin{array}{cc} l\,\underline{2} & r\,\underline{2} \end{array} \\ \begin{bmatrix} \underline{2} & \underline{1} \\ \underline{1} & \underline{2} \end{bmatrix} \end{array}, \qquad \overline{\mathrm{M}(\underline{2}+\underline{2})} = \begin{array}{c} \\ l\,\underline{2} \\ r\,\underline{2} \end{array} \begin{array}{c} \begin{array}{cc} l\,\underline{2} & r\,\underline{2} \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \tag{6.11}$$

Example 6.3.4 shows that the obvious test for $G = H$ in simultaneous, extended normal play cannot be just $\mathrm{EV}(G) = \mathrm{EV}(H)$ even though this is a necessary condition.

Question 5 Is there a set of conditions which only involve options or possibly followers of G and H to prove that $G \geq H$ in simultaneous, extended normal play with disjunctive sum?

We also note that if $G^{\mathcal{L}} = \emptyset$ and $G^{\mathcal{R}} = \emptyset$ then $G = 0$ and the game is a draw. The converse is not true for disjunctive sum.

Example 6.3.4 also shows that under extended normal play and disjunctive sum there are games which are draws, but the sum of draw games is not necessarily a draw. This suggests that knowing how to play individual components does not help determine how to play in a sum. Given these challenges, we concluded that disjunctive sum was not an ideal sum to use in simultaneous combinatorial game theory.

Conjunctive Sum

Consider the conjunctive sum of n components $K_1, K_2, \ldots, K_n$. If $K_1 \wedge K_2 \wedge \ldots \wedge K_n = \emptyset$, then this means that at least one of the components is terminal.

Definition 6.3.5 The *extended normal play winning convention for the conjunctive sum* of simultaneous combinatorial games is:

- Left wins if she has an option in every terminal component, but Right does not (i.e., $K_i^{\mathcal{L}} \neq \emptyset$ for all i and there exists K_i such that $K_i^{\mathcal{R}} = \emptyset$).
- Right wins if he has an option in every terminal component, but Left does not (i.e., $K_i^{\mathcal{R}} \neq \emptyset$ for all i and there exists K_i such that $K_i^{\mathcal{L}} = \emptyset$).
- Otherwise the game is a Draw (i.e., there exist $1 \leq i, j, \leq n$ such that $K_i^{\mathcal{L}} = \emptyset$ and $K_j^{\mathcal{R}} = \emptyset$).

Since players are moving simultaneously, we needed to adjust from the CGT interpretation of who wins. More than one component can become terminal simultaneously, and the individual components can be won by different players. Definition 6.3.5 accounts for this by stating that in order for a player to win, they must have a move remaining in all components at game termination.

Timing Issues

The game may terminate and some components may still have simultaneous moves available. The game $A \wedge B$ finishes when at least one component finishes. This brings in a *timing issue*. A component could be the cause of game termination in one sum, but not another. Hence, one cannot expect $\mathrm{EV}(A \wedge B)$ to be a simple combination of $\mathrm{EV}(A)$ and $\mathrm{EV}(B)$.

A useful game for further examples is $\mathrm{SQ}'(\{1\},\{2\})(\underline{n})$, which is $\mathrm{SQ}(\{1\},\{2\})(\underline{n})$, except Left is not allowed to move in $\underline{2}$. As in $\mathrm{SQ}(\{1\},\{2\})(\underline{n})$, $\underline{0}$ is a Draw and $\underline{1}$ is a Left win but now $\underline{2}$ is a Right win. We use $\mathrm{M}(\underline{a}\wedge\underline{b})$ to denote $\mathrm{M}(\mathrm{SQ}'(\{1\}\,,\{2\})(\underline{a}\wedge\underline{b}))$.

Example 6.3.6 Consider $\mathrm{SQ}'(\{1\},\{2\})(\underline{5}\wedge\underline{6})$. See Figure 6.2 for its game tree, and $\mathrm{M}(\underline{5}\wedge\underline{6})$ is shown in (6.15) where all red entries are terminal.

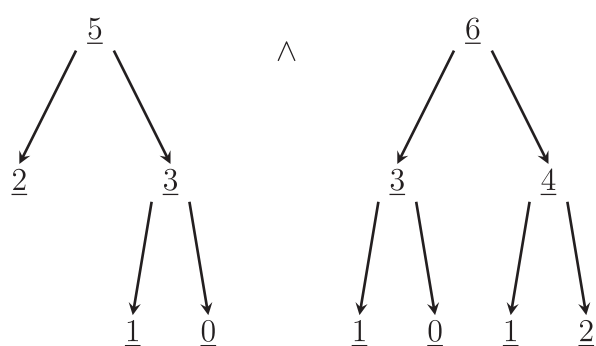


Figure 6.2: Game tree for $\mathrm{SQ}'(\{1\}\,,\{2\})(\underline{5}\wedge\underline{6})$.

If Right's move does not interfere with Left's in $\underline{5}$, this guarantees that Right will win the game $\underline{5}\wedge\underline{6}$, hence $\mathrm{EV}(\underline{2}\wedge\underline{3}) = -1$ and $\mathrm{EV}(\underline{2}\wedge\underline{4}) = -1$. However, Right cannot control this. The matrix for $\mathrm{M}(\underline{3}\wedge\underline{3})$ is given in (6.12), and $\mathrm{M}(\underline{3}\wedge\underline{4})$ is given in (6.13). First note that every entry in both matrices are terminal. Also, any position which has a $\underline{0}$ within the sum is a Draw. Both components must be $\underline{1}$ for a Left win, and for a Right win, both components must be a $\underline{2}$ (this never occurs here).

All other combinations lead to Draws and have value 0. From these facts, we were able to completely fill in the matrices for $\overline{\mathrm{M}(\underline{3} \wedge \underline{3})}$ and $\overline{\mathrm{M}(\underline{3} \wedge \underline{4})}$, given in (6.14). In both cases, the symmetry of the matrix tells us that both players will have the same strategies and all pure strategies will be played equally. Hence, $X = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = Y$ for both games $\underline{3} \wedge \underline{3}$ and $\underline{3} \wedge \underline{4}$. Calculating the value of each game, we obtain $\mathrm{EV}(\underline{3} \wedge \underline{3}) = \frac{1}{4}$ and $\mathrm{EV}(\underline{3} \wedge \underline{4}) = \frac{1}{4}$. These calculations allowed us to determine the values of the matrix entries from $\mathrm{M}(\underline{5} \wedge \underline{6})$ given by (6.15), as shown in $\overline{\mathrm{M}(\underline{5} \wedge \underline{6})}$, given by (6.16).

$$\mathrm{M}(\underline{3} \wedge \underline{3}) = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} \underline{1} \wedge \underline{1} & \underline{1} \wedge \underline{0} & \underline{0} \wedge \underline{1} & \underline{0} \wedge \underline{0} \\ \underline{1} \wedge \underline{0} & \underline{1} \wedge \underline{1} & \underline{0} \wedge \underline{0} & \underline{0} \wedge \underline{1} \\ \underline{0} \wedge \underline{1} & \underline{0} \wedge \underline{0} & \underline{1} \wedge \underline{1} & \underline{1} \wedge \underline{0} \\ \underline{0} \wedge \underline{0} & \underline{0} \wedge \underline{1} & \underline{1} \wedge \underline{0} & \underline{1} \wedge \underline{1} \end{bmatrix} \end{array} \tag{6.12}$$

$$\mathrm{M}(\underline{3} \wedge \underline{4}) = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} \underline{1} \wedge \underline{2} & \underline{1} \wedge \underline{1} & \underline{0} \wedge \underline{2} & \underline{0} \wedge \underline{1} \\ \underline{1} \wedge \underline{1} & \underline{1} \wedge \underline{2} & \underline{0} \wedge \underline{1} & \underline{0} \wedge \underline{2} \\ \underline{0} \wedge \underline{2} & \underline{0} \wedge \underline{1} & \underline{1} \wedge \underline{2} & \underline{1} \wedge \underline{1} \\ \underline{0} \wedge \underline{1} & \underline{0} \wedge \underline{2} & \underline{1} \wedge \underline{1} & \underline{1} \wedge \underline{2} \end{bmatrix} \end{array} \tag{6.13}$$

$$\overline{\mathrm{M}(\underline{3} \wedge \underline{3})} = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \quad \overline{\mathrm{M}(\underline{3} \wedge \underline{4})} = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \tag{6.14}$$

$$\mathrm{M}(\underline{5}\wedge\underline{6}) = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} \underline{3}\wedge\underline{4} & \underline{3}\wedge\underline{3} & \underline{2}\wedge\underline{4} & \underline{2}\wedge\underline{3} \\ \underline{3}\wedge\underline{3} & \underline{3}\wedge\underline{4} & \underline{2}\wedge\underline{3} & \underline{2}\wedge\underline{4} \\ \underline{2}\wedge\underline{4} & \underline{2}\wedge\underline{3} & \underline{3}\wedge\underline{4} & \underline{3}\wedge\underline{3} \\ \underline{2}\wedge\underline{3} & \underline{2}\wedge\underline{4} & \underline{3}\wedge\underline{3} & \underline{3}\wedge\underline{4} \end{bmatrix} \end{array} \tag{6.15}$$

$$\overline{\mathrm{M}(\underline{5}\wedge\underline{6})} = \begin{array}{c} \\ \ell/\ell \\ \ell/r \\ r/\ell \\ r/r \end{array} \begin{array}{c} \begin{array}{cccc} \ell/\ell & \ell/r & r/\ell & r/r \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -1 & -1 \\ \frac{1}{4} & \frac{1}{4} & -1 & -1 \\ -1 & -1 & \frac{1}{4} & \frac{1}{4} \\ -1 & -1 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \end{array} \tag{6.16}$$

From (6.16) we can determine the optimal strategies for each player. Again, by the symmetry of the matrix, both players have the same optimal strategies. In this case, there are several which will give the same value for the game, we consider one such strategy. From Left's perspective, we consider the optimal strategy where Left uses all pure strategies equally likely, i.e., $X = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Suppose Right does the same (using symmetry), $Y = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Then we can determine the value of the game as $\mathrm{EV}(\underline{5}\wedge\underline{6}) = X \cdot \overline{\mathrm{M}(\underline{5}\wedge\underline{6})} \cdot Y^T = -\frac{3}{8}$. Alternatively, since players are using strategies with non-zero probability, one can also consider Left's strategy against any pure strategy of Right and the result will also equal the value of the game. For example, Left playing X against column 1 gives the following: $\frac{1}{4}(\frac{1}{4})+\frac{1}{4}(\frac{1}{4})+\frac{1}{4}(-1)+\frac{1}{4}(-1) = -\frac{3}{8}$.

Now we compare this result to playing $\underline{5}$ and $\underline{6}$ in isolation. We require the matrices for both games. These are given in (6.17). Recursively calculating the values of these positions, we obtain the values for their entries as shown in (6.18).

$$\mathrm{M}(\underline{5}) = \begin{array}{c} \\ \ell \\ r \end{array} \begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} \underline{3} & \underline{2} \\ \underline{2} & \underline{3} \end{bmatrix} \end{array}, \quad \mathrm{M}(\underline{6}) = \begin{array}{c} \\ \ell \\ r \end{array} \begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} \underline{4} & \underline{3} \\ \underline{3} & \underline{4} \end{bmatrix} \end{array}. \tag{6.17}$$

$$\overline{\mathrm{M}(\underline{5})} = \begin{array}{c} \\ \ell \\ r \end{array} \begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \end{array}, \quad \overline{\mathrm{M}(\underline{6})} = \begin{array}{c} \\ \ell \\ r \end{array} \begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \end{array}. \tag{6.18}$$

In both cases, based again on the symmetry of the matrices, players will choose each pure strategy with equal probability. Hence $X = (\frac{1}{2}, \frac{1}{2}) = Y$ for both games, and we obtain $\mathrm{EV}(\overline{\mathrm{M}(\underline{5})}) = -\frac{1}{4}$, $\mathrm{EV}(\overline{\mathrm{M}(\underline{6})}) = \frac{1}{4}$. Comparing these values to the value of their sum, $\mathrm{EV}(\overline{\mathrm{M}(\underline{5} \wedge \underline{6})}) = -\frac{3}{8}$ we notice that the individual components do not tell us anything about how the play will occur in the sum. Hence, in general, $\mathrm{EV}(A \wedge B) \neq \mathrm{EV}(A)\mathrm{EV}(B)$ and $\mathrm{EV}(A \wedge B) \neq \mathrm{EV}(A) + \mathrm{EV}(B)$.

The empty game under conjunctive sum is a zero, acts multiplicatively, and means that the game is a draw. However, even if $\mathrm{EV}(G) = 0$ it does not follow that $\mathrm{EV}(G \wedge H) = 0$ (see Example 6.3.7).

Example 6.3.7 Consider $\mathrm{SQ}(\{1\}, \{2\})(\underline{1} \wedge \underline{2})$. Even though $\underline{2}$ is a draw under simultaneous play, the game $\underline{1} \wedge \underline{2}$ is actually a Left win since $\underline{1}$ is terminal which forces $\underline{1} \wedge \underline{2}$ to be terminal as well.

Recall for CGT, $o(0 + X) = o(X)$, for all X and zero forms an important equivalence class. Example 6.3.7 highlights an important distinction for zeros under simultaneous play, conjunctive sum and extended normal play.

Definition 6.3.8 A game G is a *multiplicative zero* if $o_S(G \wedge X) = o_S(G)$ for all games X.

In fact, this implies the following:

Proposition 6.3.9 *The only multiplicative zero for simultaneous games, played under conjunctive sum and extended normal play is the empty game.*

Proof. By way of contradiction, suppose there exists another game H such that $H = 0$. A necessary condition on H is that $H^{\mathcal{L}} \neq \emptyset$ and $H^{\mathcal{R}} \neq \emptyset$ since otherwise, if only one of the sets were non-empty, H would be a Left or Right win, respectively. Then $o_S(0 \wedge X) = o_S(H \wedge X)$, $\forall X$. Let X be a Left win where $X^{\mathcal{R}} = \emptyset$. Then $0 \wedge X$ is terminal and a Draw (because 0 is terminal and a Draw), but $H \wedge X$ is terminal and a Left win (since X is terminal and Left has options in both H and X). Hence H is not equivalent to the empty game and thus there are no other games which are multiplicative zeros under this sum and winning convention. □

The problems which arose here (zero being unique and timing issues) leads us away from this sum. As we are in search of an algebraic structure which preserves as much CGT structure as possible and remains general, we move on to another sum.

Continued Conjunctive Sum

Definition 6.3.10 The *extended normal play winning convention for the continued conjunctive sum* of $K_1 \triangledown K_2 \triangledown \ldots \triangledown K_n$ is:

- Left wins if she has an option in every terminal component, but Right does not (i.e., $K_i^{\mathcal{L}} \neq \emptyset$ for all i and there exists K_i such that $K_i^{\mathcal{R}} = \emptyset$).

- Right wins if he has an option in every terminal component, but Left does not (i.e., $K_i^{\mathcal{R}} \neq \emptyset$ for all i and there exists K_i such that $K_i^{\mathcal{L}} = \emptyset$).

- Otherwise the game is a Draw (i.e., there exist $1 \leq i, j, \leq n$ such that for $K_i^{\mathcal{L}} = \emptyset$ and $K_j^{\mathcal{R}} = \emptyset$).

Note that this is defined slightly differently from the CGT definition. For a player to win, they must have a move remaining in all terminal components and the game sum only terminates after all components are terminal (as opposed to the CGT version where the last terminal component determines the winner). It is defined in this way to avoid further problems with the timing of component termination as discussed for the conjunctive sum.

For example, $\mathrm{SQ}(\{1\},\{2\})(\underline{1} \triangledown \underline{2})$ is not terminal as there is still game play remaining in $\underline{2}$. To generalize, we obtain the following theorem.

Theorem 6.3.11 *Let $G = G_1 \triangledown G_2 \triangledown \ldots \triangledown G_n$. If any of the components, G_i, is a Draw, then G is also a Draw.*

Proof. This result follows immediately from the definition of continued conjunctive sum under extended normal play. □

The values of G and H are not enough to determine the value of $G \bigtriangledown H$. Several different problems occur. We demonstrate each in sequence.

First, two games can have the same value, but not act the same in a sum.

Example 6.3.12 Consider the games G, H, and X which have the matrices $\overline{\mathrm{M}(G)}$, $\overline{\mathrm{M}(H)}$ and $\overline{\mathrm{M}(X)}$ respectively, as shown in (6.19).

The matrices $\overline{\mathrm{M}(G \bigtriangledown X)}$ and $\overline{\mathrm{M}(H \bigtriangledown X)}$ are shown in (6.20) and (6.21), respectively, where a row indexed by r/s means that Left played r in K and s in Y of $K \bigtriangledown Y$ (similarly for Right).

$$\overline{\mathrm{M}(G)} = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad \overline{\mathrm{M}(H)} = \begin{array}{c} \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cc} z & w \end{array} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \end{array}, \quad \overline{\mathrm{M}(X)} = \begin{array}{c} \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \tag{6.19}$$

$$\overline{\mathrm{M}(G \bigtriangledown X)} = \begin{array}{c} \\ a/e \\ a/f \\ b/e \\ b/f \end{array} \begin{array}{c} \begin{array}{cccc} x/s & x/t & y/s & y/t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{6.20}$$

$$\overline{\mathrm{M}(H \bigtriangledown X)} = \begin{array}{c} \\ c/e \\ c/f \\ d/e \\ d/f \end{array} \begin{array}{c} \begin{array}{cccc} z/s & z/t & w/s & w/t \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{6.21}$$

If knowing the values of $\overline{\mathrm{M}(G)}$ and $\overline{\mathrm{M}(H)}$ were enough to understand their combination in a sum, then we would obtain $\mathrm{EV}(\overline{\mathrm{M}(G \bigtriangledown X)}) = \mathrm{EV}(\overline{\mathrm{M}(H \bigtriangledown X)})$, since $\mathrm{EV}(\overline{\mathrm{M}(G)}) = \mathrm{EV}(\overline{\mathrm{M}(H)}) = 0$. However, this is not the case. Instead we obtain $\mathrm{EV}(\overline{\mathrm{M}(G \bigtriangledown X)}) = 0$ and $\mathrm{EV}(\overline{\mathrm{M}(H \bigtriangledown X)}) = \frac{1}{4}$, and thus G and H do not act the same in a sum and so $G \neq H$, even though they have the same values as individual games. Hence knowing the values of components is not enough to know what will be the result of a sum, and thus we cannot simply use value as a test for equality.

Even worse, dominated strategies for a game G may be the best for $G \bigtriangledown H$. For example, let H be a terminal game in which $H^{\mathcal{L}} = \emptyset$ and $H^{\mathcal{R}} \neq \emptyset$ and let G have the values

$$\overline{\mathrm{M}(G)} = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & \text{-}1 & \text{-}1/2 & 1/4 \\ \text{-}1 & 1 & 1/4 & \text{-}1/2 \end{bmatrix} \end{array} \tag{6.22}$$

where, further, Right has no chance of winning in the positive options, and Left has no chance of of winning in the negative options. Clearly Left plays $(x, y) = (1/2, 1/2)$. The value of G is $-1/4$ and is achieved when $(a, b, c, d) = (0, 0, 1/2, 1/2)$. Right playing $(a, b, c, d) = (1/2, 1/2, 0, 0)$ has value 0. Therefore $(0, 0, 1/2, 1/2)$ strictly dominates $(1/2, 1/2, 0, 0)$ when G is played in isolation.

However, in $G \bigtriangledown H$, Left can never win since Left always loses in H. Therefore, Right achieves the value $-1/2$ with $(a, b, c, d) = (1/2, 1/2, 0, 0)$, as he would only achieve value $-1/4$ with $(a, b, c, d) = (0, 0, 1/2, 1/2)$.

This example shows that in evaluating $G \bigtriangledown H$, we cannot replace G and H by their reduced forms, that is, $\mathrm{EV}(G \bigtriangledown H) \neq \mathrm{EV}(Re(G) \bigtriangledown Re(H))$.

Testing for Equality: Extended Normal Play and Continued Conjunctive Sum

Testing for equality is non-trivial and needs to be examined with care. In particular, for extended normal play, we need to know more about the probabilities of winning, drawing and losing rather than simply the values of G and H.

To capture this additional information, we define the *profile* of a game. The profile describes the details about of a player's probability of winning and their control over the resulting game sum.

Definition 6.3.13 Let G be a game. The *profile* of G, written $\mathrm{Pro}(G) = [\ell_G, r_G]$ describes players' probabilities of winning the game G, ℓ_G for Left, and r_G for Right.

Note that d_G is the probability of a draw, which can be determined from ℓ_G and r_G since $\ell_G + d_G + r_G = 1$. Also, $\mathrm{EV}(G) = 1(\ell_G) + 0(d_G) + (-1)(r_G) = \ell_G - r_G$.

Given the terminal position value assignments for Left win, Right win, and Draw, and the fact that within continued conjunctive sum the components are played independently, we obtain the following general results.

Lemma 6.3.14 *Let G, H be games then*

(i) $0 \le r_G, l_G \le 1$ *and* $0 \le \ell_G + r_G \le 1$;

(ii) $\mathrm{Pro}(G \bigtriangledown H) = [\ell_G \ell_H, r_G r_H]$.

Proof. For any game, G, since ℓ_G and r_G are probabilities of mutually exclusive outcomes then $0 \le r_G, \ell_G \le 1$ and $0 \le \ell_G + r_G \le 1$.

Play in G and play in H are independent which gives $\ell_{G \bigtriangledown H} = \ell_G \ell_H$. Similarly for Right, $r_{G \bigtriangledown H} = r_G r_H$. Hence, $\mathrm{Pro}(G \bigtriangledown H) = [\ell_G \ell_H, r_G r_H]$. □

In many cases, the profile will be comprised of functions, which could be controlled by Left, Right, or both players. Next we examine an example, then proceed to discuss some more general theory. Let $\mathrm{Pro}(\overline{\mathrm{M}(G)})$ be the matrix $\overline{\mathrm{M}(G)}$ with entries replaced by their respective profiles.

Example 6.3.15 Consider the game $G = \mathrm{SQ}'(\{1,4\}, \{2\})(\underline{4})$. The matrix of terminal game values and the matrix with profiles and strategies is shown in (6.23).

$$\overline{\mathrm{M}(G)} = \begin{array}{c} \\ 1_l \\ 1_r \\ 4_l \\ 4_r \end{array} \begin{array}{c} \begin{array}{cc} 2_l & 2_r \end{array} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \qquad \mathrm{Pro}(\overline{\mathrm{M}(G)}) = \begin{array}{c} \\ x \\ w \\ z \\ t \end{array} \begin{array}{c} \begin{array}{cc} y & 1-y \end{array} \\ \begin{bmatrix} [0,1] & [1,0] \\ [1,0] & [0,1] \\ [0,0] & [0,0] \\ [0,0] & [0,0] \end{bmatrix} \end{array} \tag{6.23}$$

where $0 \le x, w, z, t, y \le 1$ and $x + w + z + t = 1$. Then the calculations for ℓ_G and r_G are as follows:

$$\ell_G = x(1-y) + wy = x - xy + wy$$

and

$$r_G = xy + w(1-y) = xy + w - wy$$

With the goal of maximizing the value for Left and minimizing for Right, we calculate:

$$\begin{aligned}
\mathrm{EV}(G) &= \ell_G - r_G \\
&= x - xy + wy - (xy + w - wy) \\
&= x + 2wy - 2xy - w \\
&= x(1-2y) + w(2y-1) \\
&= (2y-1)(w-x)
\end{aligned}$$

Following an EGT approach, since the calculation of the value is dependent on both the choices of Left and Right, it is important to note that neither player can be taken advantage of if their respective factors equal zero. Hence we obtain $w = x$ and $y = \frac{1}{2}$. Now, we can simplify the profile to the following:

$$\mathrm{Pro}(G) = [\ell_G, r_G] = [x, x], \text{ for } 0 \leq x \leq \tfrac{1}{2}, \text{ where Left controls } x.$$

The last example highlights the fact that the profile can be a challenging tool to implement. If a game G is being played on its own, then the game value is computed as usual from EGT. However, if the game is comprised of a sum of two or more components, it is not straightforward. We cannot optimize a profile until we know what other games are within the sum. Consider the game G from Example 6.3.15 and consider games H and K which have matrices $\overline{\mathrm{M}(H)}$ and $\overline{\mathrm{M}(K)}$ as shown in (6.24).

$$\overline{\mathrm{M}(H)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{\mathrm{M}(K)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.24}$$

$$\mathrm{Pro}(G) = [x, x]\,, \text{where}\, 0 \leq x \leq 1/2, \quad \mathrm{Pro}(H) = [1/2, 0]\,, \quad \mathrm{Pro}(K) = [0, 1/2]\,. \tag{6.25}$$

Suppose further that the profiles corresponding to each game are as shown in (6.25). By Lemma 6.3.14, we obtain the profiles of the sums $G \bigtriangledown H$ and $G \bigtriangledown K$ as shown in (6.26).

$$\text{Pro}(G \bigtriangledown H) = [x/2, 0]\,, \quad \text{Pro}(G \bigtriangledown K) = [0, x/2]\,. \tag{6.26}$$

Since $\text{EV}(G \bigtriangledown H) = \frac{x}{2} - 0$, Left controls the value of x, and Left wishes to maximize the resulting value, she assigns $x = \frac{1}{2}$ to give $\text{EV}(G \bigtriangledown H) = \frac{1}{4}$. While for $G \bigtriangledown K$ we have $\text{EV}(G \bigtriangledown K) = 0 - \frac{x}{2}$ where Left controls x. In order to maximize this expression, Left chooses $x = 0$ which gives $\text{EV}(G \bigtriangledown K) = 0$. In conclusion, Left decides the value of x after knowing which games are within the sum.

Depending on the games within a sum, the profile can involve functions of several variables, controlled by either or both players. An important direction for future work is to determine techniques to simplify the profile in such cases.

Using the profile, we developed a test for equality of games under continued conjunctive sum and extended normal play. Note that when the profiles involve functions, equality is determined based on the profiles involving exactly the same functions, up to permutation of the variables, and the ranges of corresponding variables in the functions of G and H must be equal.

Theorem 6.3.16 *Let G and H be simultaneous games played under continued conjunctive sum with the extended normal play winning convention. Then*

(1) $G = H$ if and only if $\text{Pro}(G) = \text{Pro}(H)$; and

(2) $G \geq H$ if and only if $\ell_G \geq \ell_H$ and $r_G \leq r_H$.

Proof. Proof of (1)

$$G = H \tag{6.27}$$

$$\iff \quad \text{EV}(G \bigtriangledown X) = \text{EV}(H \bigtriangledown X), \text{for all } X \tag{6.28}$$

$$\iff \ell_{G \bigtriangledown X} - r_{G \bigtriangledown X} = \ell_{H \bigtriangledown X} - r_{H \bigtriangledown X} \tag{6.29}$$

$$\iff \quad (\ell_G - \ell_H)\ell_X = (r_G - r_H)r_X \tag{6.30}$$

As (6.30) holds for all games X, we consider two specific cases. (i) Let X be a Left win. Then $\text{Pro}(X) = [1, 0]$ and so (6.30) gives $\ell_G - \ell_H = 0$ which implies $\ell_G = \ell_H$. (ii) Let X be a Right win. Then $\text{Pro}(X) = [0, 1]$ and (6.30) gives $r_G - r_H = 0$ which implies $r_G = r_H$. Together this means that the expression (6.30) holds for all X if and only if $\text{Pro}(G) = \text{Pro}(H)$.

Proof of (2)

$$G \geq H \tag{6.31}$$

$$\iff \text{EV}(G \triangledown X) \geq \text{EV}(H \triangledown X), \text{for all}\, X \tag{6.32}$$

$$\iff \ell_{G \triangledown X} - r_{G \triangledown X} \geq \ell_{H \triangledown X} - r_{H \triangledown X} \tag{6.33}$$

$$\iff (\ell_G - \ell_H)\ell_X \geq (r_G - r_H)r_X \tag{6.34}$$

As (6.34) holds for all games X, we consider two specific cases. (i) Let X be a Left win. Then $\text{Pro}(X) = [1, 0]$ and so (6.34) gives $\ell_G - \ell_H \geq 0$ implying that $\ell_G \geq \ell_H$. (ii) Let X be a Right win. Then $\text{Pro}(X) = [0, 1]$ and (6.34) gives $r_G - r_H \leq 0$ which implies $r_G \leq r_H$. Together this means that the expression (6.34) holds for all X if and only if $\ell_G \geq \ell_H$ and $r_G \leq r_H$. □

Corollary 6.3.17 *Let G and H be simultaneous games played under continued conjunctive sum with the extended normal play winning convention. Then*

$$\text{Pro}(G) = \text{Pro}(H) \iff \text{EV}(G \triangledown X) = \text{EV}(H \triangledown X), \textit{ for all games } X.$$

Proof. First assume $\text{Pro}(G) = \text{Pro}(H)$ then we have the following implications

$$\begin{aligned}
&\Rightarrow \text{Pro}(G \triangledown X) = \text{Pro}(H \triangledown X), \text{for all}\, X \\
&\Rightarrow \quad \ell_{G \triangledown X} = \ell_{H \triangledown X} \quad \text{and} \quad r_{G \triangledown X} = r_{H \triangledown X} \\
&\Rightarrow \qquad \ell_G \ell_X - \ell_H \ell_X = r_G r_X - r_H r_X \\
&\Rightarrow \qquad \ell_G \ell_X - r_G r_X = \ell_H \ell_X - r_H r_X \\
&\Rightarrow \qquad \ell_{G \triangledown X} - r_{G \triangledown X} = \ell_{H \triangledown X} - r_{H \triangledown X} \\
&\Rightarrow \qquad \text{EV}(G \triangledown X) = \text{EV}(H \triangledown X)
\end{aligned}$$

Next assume $\text{EV}(G \triangledown X) = \text{EV}(H \triangledown X), \text{for all}\, X$ then we have the following implications

$$\Rightarrow \ell_{G \triangledown X} - r_{G \triangledown X} = \ell_{H \triangledown X} - r_{H \triangledown X}, \text{for all}\, X \tag{6.35}$$

$$\Rightarrow \quad \ell_G \ell_X - r_G r_X = \ell_H \ell_X - r_H r_X, \text{for all}\, X \tag{6.36}$$

As (6.36) holds for all games X, in particular consider two cases. (i) Let X be a Left win. Then $\text{Pro}(X) = [1, 0]$ and so (6.36) gives $\ell_G = \ell_H$. (ii) Let X be a Right win. Then $\text{Pro}(X) = [0, 1]$ and (6.36) gives $r_G = r_H$. Together this means that the expression (6.36) holds for all X if and only if $\ell_G = \ell_H$ and $r_G = r_H$. This means that the expression holds only when $\text{Pro}(G) = \text{Pro}(H)$. □

Even though the description of the profile is relatively straightforward, the strategy of a player cannot always be determined by comparing the profiles. For example, consider the games G and H where $\overline{\text{M}(G)}$ and $\overline{\text{M}(H)}$ are given in (6.37) and their matrices of profiles are given in (6.38).

$$\overline{\text{M}(G)} = \begin{bmatrix} -\frac{1}{4} \end{bmatrix}, \quad \overline{\text{M}(H)} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{64} \end{bmatrix} \tag{6.37}$$

$$\text{Pro}(\overline{\text{M}(G)}) = \begin{bmatrix} \left[\frac{1}{4}, \frac{1}{2}\right] \end{bmatrix}, \quad \text{Pro}(\overline{\text{M}(H)}) = \begin{bmatrix} \left[\frac{5}{8}, \frac{3}{8}\right] & \left[\frac{4}{64}, \frac{5}{64}\right] \end{bmatrix} \tag{6.38}$$

G is completely determined, while Right has choice in H. Let Right's probability vector for $\overline{\text{M}(H)}$ be $Y = (y, 1-y)$. There is no domination in H, so we are interested in determining what Right will do and whether we can say anything about how he will play a game sum, $G \bigtriangledown H$. In this example, the value of G is $-\frac{1}{4}$, and hence in favour of Right. Will he then choose a component in H which also has negative value? Or will he always be concerned with his highest probability of winning? We now show that neither are true. Consider $\text{EV}(G \bigtriangledown H) = \ell_G \ell_H - r_G r_H$. Right wants to minimize this expression. First, we need to calculate $\text{Pro}(H)$. To do so, we consider the mixed strategy for Right, $Y = (y, 1-y)$, and multiply across the Left and Right probabilities respectively in $\text{Pro}(\overline{\text{M}(H)})$ as follows.

$$\text{Pro}(H) = \left[\frac{5}{8}y + \frac{4}{64}(1-y), \frac{3}{8}y + \frac{5}{64}(1-y)\right] \tag{6.39}$$
$$= \left[\frac{40-4}{64}y + \frac{4}{64}, \frac{24-5}{64}y + \frac{5}{64}\right] \tag{6.40}$$
$$= \left[\frac{36}{64}y + \frac{4}{64}, \frac{19}{64}y + \frac{5}{64}\right] \tag{6.41}$$

$$\begin{aligned}\text{EV}(G \bigtriangledown H) &= \ell_G\ell_H - r_G r_H &(6.42)\\ &= \frac{1}{4}\left(\frac{36}{64}y + \frac{4}{64}\right) - \frac{1}{2}\left(\frac{19}{64}y + \frac{5}{64}\right) &(6.43)\\ &= \frac{1}{256}\left(-2y - 6\right) &(6.44)\end{aligned}$$

To minimize the expression (6.44) Right chooses $y = 1$, which could seem counter-intuitive since the value of that component in $\overline{\text{M}(H)}$ is positive. One may argue that the reason for this choice is to maximize his overall chances of winning. However, if we let $\text{Pro}(\overline{\text{M}(H)}) = \left[\left[\frac{5}{8}, \frac{3}{8}\right], \left[\frac{1}{16}, \frac{2}{16}\right]\right]$, let Right's strategy be $Y = (y, 1-y)$, and G be as before we obtain the following:

$$\begin{aligned}\text{Pro}(H) &= \left[\frac{5}{8}y + \frac{1}{16}(1-y), \frac{3}{8}y + \frac{2}{16}(1-y)\right] &(6.45)\\ &= \left[\frac{10-1}{16}y + \frac{1}{16}, \frac{6-2}{16}y + \frac{2}{16}\right] &(6.46)\\ &= \left[\frac{9}{16}y + \frac{1}{16}, \frac{1}{4}y + \frac{1}{8}\right] &(6.47)\end{aligned}$$

$$\begin{aligned}\text{EV}(G \bigtriangledown H) &= \ell_G\ell_H - r_G r_H &(6.48)\\ &= \frac{1}{4}\left(\frac{9}{16}y + \frac{1}{16}\right) - \frac{1}{2}\left(\frac{1}{4}y + \frac{1}{8}\right) &(6.49)\\ &= \frac{1}{64}\left(y - 3\right) &(6.50)\end{aligned}$$

In order for Right to minimize (6.50), he lets $y = 0$. In this case, he chooses the option in H where he has a lower probability of winning, but it minimizes the overall value in the sum.

The above example demonstrates the challenges with the profile. Even though we have proven that game equality under continued conjunctive sum and extended normal play can be determined by the game profiles, working with the profile is a non-trivial task. Further exploring the properties of the profile to gain better insight into potential simplifications is an important continuation of this research.

This work can be modified to account for a draw game being just as good as a win; in other words, players just don't want to lose. As the analysis for the goal of winning has been convoluted, we leave the change in goals as a direction for future work. We believe that the profile can be changed to include the probability of a draw for each player and, with a rescaling, most analysis would still hold. However, this means that both players could win under the assumption that both are happy with a draw. This means that the game would be a nonzero sum game and different tools from economic game theory would likely need to be used.

6.3.2 Scoring Play

A different winning convention to consider is *scoring play*. Under scoring play, the player with the larger score at the end of the game wins. If the scores are the same, the game is a draw.

We can think of the values as in the economic game theory interpretation since players are always trying to maximize their score throughout game play, thus typical domination outlined in Section 6.1 holds here. However, for terminal positions, scores can be assigned in many possible ways. This is determined by the ruleset.

Definition 6.3.18 Let G be a game, played under the scoring play winning convention. The *value*, $\mathrm{EV}(G)$, is given recursively

$$\mathrm{EV}(G) = \begin{cases} \mathit{score}(G), & \text{if } G \text{ is terminal;} \\ \mathrm{EV}(\overline{\mathrm{M}(G)}), & \text{otherwise; where } \overline{\mathrm{M}(G)} \text{ is } \mathrm{M}(G) \text{ in which each } G^{S_{i,j}} \text{ is} \\ & \qquad \text{replaced by } \mathrm{EV}(\overline{\mathrm{M}(G^{S_{i,j}})}). \end{cases}$$

From Definition 6.3.18 we see that if the score is an expectation rather than a guarantee, there is really a range on the possible values based on the matrix entries. See Example 6.3.19 to better understand how positions can lead to non-deterministic scores (values).

Ruleset for SIMULTANEOUS CLOBBER

- Board: A finite graph, with each vertex unoccupied or occupied by an X or an O (typically played on an $m \times n$ grid, alternating X's and O's, where $m, n \in \mathbb{N}$).
- Moves: Left moves an X onto an adjacent O. The O is removed, and is said to be 'clobbered' by the X. Right move an O onto an adjacent X. The X is removed, and is said to be 'clobbered' by the O.
- Interference Rule: If players choose to clobber their opponent's piece which their opponent is also using to clobber theirs, both pieces disappear. If the position is OXO, labelled as ABC, and Left moves to C and Right moves the O in A to B, then Left is said to have clobbered the O in C, but Right merely occupies B (did not clobber the X).

Example 6.3.19 Consider the game G of SIMULTANEOUS CLOBBER on $[OXO]$ with the scoring rule that the number of pieces players clobber is their score. The matrix for this scoring game is shown in (6.51), where ℓ means that a player moved a piece left, and r means a player moved a piece right. There is exactly one possibility of each, for each player, in this example.

$$\begin{array}{c} \\ \ell \\ r \end{array} \begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \tag{6.51}$$

From this, we can see that the $\mathrm{EV}(G) = \frac{1}{2}$ and hence games under the scoring winning convention can have values stemming from mixed strategies.

For score assignment, we follow a CGT approach.

Definition 6.3.20 Let G be a terminal game. The *score* of G is the value of G in the underlying CGT rules.

Note that in a terminal game, at most one player has a move, thus the value of G in the underlying CGT rules is an integer, non-negative if Right cannot move and non-positive if Left cannot move (this follows from Definition 2.4.3). Intuitively, the value of G equals the maximum number of moves one player can make before opening new moves for their opponent or finishing the game. For example, in Figure 6.3 the HACKENBUSH position has a score of 2.

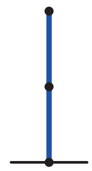


Figure 6.3: A HACKENBUSH position with a score of 2.

When G is re-interpreted as a simultaneous game, under a particular sum, then G is terminal and each terminal component will have a CGT value associated with it. The score of G will depend on the termination rules for that sum. Following the CGT approach, we obtain the following for terminal positions.

Definition 6.3.21 For the scoring play winning convention, the *outcome* of a terminal component, G, is:

$$o_{SS}(G) = \begin{cases} \mathcal{L}, \text{ if } score(G) > 0 \\ \mathcal{R}, \text{ if } score(G) < 0 \\ \mathcal{D}, \text{Draw otherwise.} \end{cases}$$

Disjunctive Sum

The *scoring play winning convention for the disjunctive sum* of $G+H$ is the number of moves remaining when the game has terminated, denoted by $score(G+H)$. The score of $G+H$ is positive if the game is in favour of Left, negative if the game is in favour of Right and zero if the game is a Draw.

Example 6.3.22 Consider the games $A = \{-5 \mid \cdot\}$ and $B = \{\cdot \mid 7\}$. Under simultaneous play, A is a Left win, since she has an option but Right does not. Similarly, B is a Right win since Right has an option but Left does not. So $A \geq B$. Now consider $A+B$ under disjunctive sum. On the first turn, Left plays in A to -5 and Right plays in B to 7. From here, Right will run out of moves before Left, and thus, the game $A + B$ is a Left win. Hence, individual components do not tell us what will happen in a disjunctive sum.

If $G^{\mathcal{L}} = \emptyset$ and $G^{\mathcal{R}} = \emptyset$, then $G = 0$ and the game is a Draw. However, the converse is not true. For example, let $G = \mathrm{SQ}(\{1\}, \{2\})(\underline{2})$ then $\mathrm{EV}(G) = 0$ and the game is a Draw, but both Left and Right have moves.

Conjunctive Sum

Definition 6.3.23 The *scoring play winning convention for conjunctive sum*, $G \wedge H$, of a terminal game is

- $score(G)$, if $G^{\mathcal{L}}$ or $G^{\mathcal{R}} = \emptyset$ and $H^{\mathcal{S}}$ is non-empty
- $score(H)$, if $H^{\mathcal{L}}$ or $H^{\mathcal{R}} = \emptyset$ and $G^{\mathcal{S}}$ is non-empty
- $score(G) + score(H)$, otherwise.

Similar timing problems occur under scoring play as we saw in extended normal play with conjunctive sum.

Continued Conjunctive Sum

Definition 6.3.24 The *scoring play winning convention for the continued conjunctive sum*, $G \bigtriangledown H$, of a terminal position is $score(G) + score(H)$, where Left wins if the score is positive, Right wins if the score is negative and a Draw if the score is zero.

Certain conventions need to be adapted here, as we consider the sum of games. We focus on continued conjunctive sum because playing in components is independent, so the score of each component is independent. Also by definition of continued conjunctive sum, timing is no longer an issue. Thus the overall score of a sum can be determined by the sum of the component scores. Hence, the convention of long term play of components from EGT applies here.

Testing for Equality: Scoring Play and Continued Conjunctive Sum

Theorem 6.3.25 *Let G and H be simultaneous combinatorial games. Then*

1. $\mathrm{EV}(G \bigtriangledown H) = \mathrm{EV}(G) + \mathrm{EV}(H)$

2. $G \geq H \iff \mathrm{EV}(G) \geq \mathrm{EV}(H)$.

Proof. 1. In $G \bigtriangledown H$, play in the two games is independent and the final score in each is counted for the sum. Hence $\mathrm{EV}(G \bigtriangledown H) = \mathrm{EV}(G) + \mathrm{EV}(H)$.

2. Suppose $G \geq H$, then for all X, we have

$$\begin{aligned} & G \geq H \\ \iff & \mathrm{EV}(G \bigtriangledown X) \geq \mathrm{EV}(H \bigtriangledown X), \text{for all } X \\ \iff & \mathrm{EV}(G) + \mathrm{EV}(X) \geq \mathrm{EV}(H) + \mathrm{EV}(X), \text{for all } X \\ \iff & \mathrm{EV}(G) \geq \mathrm{EV}(H). \end{aligned}$$

□

Definition 6.3.26 A game G is an *additive zero* if $\mathrm{EV}(G \bigtriangledown X) = \mathrm{EV}(X)$ for all games X.

Immediately, by definition, we obtain the following result.

Theorem 6.3.27 *A game G is an additive zero under continued conjunctive sum and scoring play winning convention if and only if G is a draw.*

Sticking with the interpretation of CGT scores, maximizing the score is the only concern for the players. Hence, EGT domination applies here. Also, we obtain a simplification tool using the negative of a game G. In CGT, the negative is defined in terms of switching the positions for Left and Right, and negating them. We take this approach here as well.

Definition 6.3.28 Let G be a simultaneous combinatorial game. Then

$$-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{S}} \mid -G^{\mathcal{L}}\},$$

where $-G^{\mathcal{S}}$ means all Left and Right options are interchanged.

The effect on the game matrix is as follows: since all Left options become Right options, and vice versa, the row indices become column indices and vice versa. Since the matrix is zero sum, the effect is to take the transpose of matrix and negate the entries. Remember, what was originally favourable to Left will now be favourable to Right, which is why we negate the entries.

By Theorem 6.3.25 and the definition of continued conjunctive sum, we obtain the following:

Theorem 6.3.29 *Simultaneous combinatorial games under continued conjunctive sum and scoring winning convention form an abelian group.*

Proof. We need to show that the following hold: (i) commutativity, (ii) associativity, (iii) additive zero, (iv) inverses, (v) closure. Some properties are immediate but, for completeness, the proofs are included here. Note first that equivalence classes can be represented as follows: $[x] = \{G : \mathrm{EV}(G) = x\}$, and $[x]$ is in canonical form.

i. We need to show that $G \bigtriangledown H = H \bigtriangledown G$. As games are identified with their value, we can show the result holds based on whether their values are equal.

$$\begin{aligned}\mathrm{EV}(G \bigtriangledown H) &= \mathrm{EV}(G) + \mathrm{EV}(H)\\ &= \mathrm{EV}(H) + \mathrm{EV}(G)\\ &= \mathrm{EV}(H \bigtriangledown G).\end{aligned}$$

ii. We need to show that $G \bigtriangledown (H \bigtriangledown K) = (G \bigtriangledown H) \bigtriangledown K$. Again, as games are identified with their value, we can show the result holds based on whether their values are equal.

$$\begin{aligned}
\mathrm{EV}(G \bigtriangledown (H \bigtriangledown K)) &= \mathrm{EV}(G) + \mathrm{EV}(H \bigtriangledown K) \\
&= \mathrm{EV}(G) + (\mathrm{EV}(H) + \mathrm{EV}(K)) \\
&= \mathrm{EV}(G) + \mathrm{EV}(H) + \mathrm{EV}(K) \\
&= (\mathrm{EV}(G) + \mathrm{EV}(H)) + \mathrm{EV}(K) \\
&= \mathrm{EV}(G \bigtriangledown H) + \mathrm{EV}(K) \\
&= \mathrm{EV}((G \bigtriangledown H) \bigtriangledown K)).
\end{aligned}$$

iii. If $G = \emptyset$, then $\mathrm{EV}(G) = 0$ and since this is a score, this does not change a value in a sum of games. As game equivalence under this sum and winning convention is by value, we note that $G = \emptyset$ is the canonical form of games with value zero. Thus zero is the additive identity of this group. In particular, for any game G in $[0] = \{G : \mathrm{EV}(G) = 0\}$, we obtain the following for all games H,

$$\begin{aligned}
\mathrm{EV}(G \bigtriangledown H) &= \mathrm{EV}(G) + \mathrm{EV}(H) \\
&= 0 + \mathrm{EV}(H) \\
&= \mathrm{EV}(H).
\end{aligned}$$

iv. The inverse of G is $-G$ as

$$\begin{aligned}
\mathrm{EV}(G \bigtriangledown (-G)) &= \mathrm{EV}(G) + \mathrm{EV}(-G) \\
&= \mathrm{EV}(G) - \mathrm{EV}(G) \\
&= 0.
\end{aligned}$$

v. By Theorem 6.3.25, game play is independent and each game produces a rational score. Thus their sum will also be rational and games are closed under summation.

□

Since terminal games are integer valued, the value of any game matrix will be a rational number. Indeed we obtain all rational numbers as shown in the next result.

Corollary 6.3.30 *The quotient of simultaneous games played with the continued conjunctive sum and the scoring winning convention is a total order isomorphic to the rationals.*

Proof. By Theorem 6.3.25, two games are equal if they have the same value; therefore the equivalence classes are indexed by the common value. A terminal game has a CGT integer value and thus the value of any game is a rational number. Moreover, $G \geq H$ if $\mathrm{EV}(G) \geq \mathrm{EV}(H)$ thus the quotient forms a total order.

To obtain any rational number, $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q > 0$, as a game value consider the game matrix to be the identity matrix of size $q \times q$, I_q (or $-I_q$ if p is negative). The value of this matrix is $\frac{1}{q}$ (or $-\frac{1}{q}$ respectively) and adding $|p|$ copies of this game will yield a score of $\frac{p}{q}$. Thus, using identity matrices and sums, we can obtain any rational number. □

Even though we can abstractly construct a matrix to produce any rational number, it is of interest to determine which values can be obtained within specific rulesets.

6.3.3 Majority Play

An alternative winning convention, not studied in combinatorial game theory, is called the *majority* winning convention. It doesn't have a natural analogue under disjunctive sum, but it does have an interpretation under restrictions to games for conjunctive and continued conjunctive sum.

Definition 6.3.31 Consider a conjunctive or continued conjunctive sum of games. The *majority* winning convention states that the overall score of a game is the number of components Left wins minus the number of components Right wins, at the time the game is terminal. Thus, if the final score is positive then Left wins the overall game, negative then Right wins the overall game, and a score of zero is a Draw.

As this winning convention is not rooted in CGT, and does not show to have any immediate benefits over scoring play, we decided to only pursue further exploration in a case study (see Section 7.1.4). It appears to be a hybrid winning convention, in between extended normal play and scoring play. We decided to focus on the extremes (extended normal play and scoring play), and perhaps once those are better understood, the majority play winning convention will be an avenue for future research.

6.4 Conclusions

The goal of this chapter was to present our philosophy for simultaneous combinatorial game theory and its implications. Players were assumed to be rational and have the goal of maximizing their resulting game outcome or score. From this discussion, there are several key points to recall moving forward. The ruleset must carefully define interference rules which take into consideration CGT game constraints, as well as illegal positions stemming from simultaneous moves. This approach guarantees that players know exactly what their legal moves are at all times and imposes constraints on the games to have the finite descent property.

Returning to our original questions, we summarize the results. Domination will depend on the sum and winning convention. Within single games, as players are concerned with maximizing the value, EGT domination applied. However, in a sum under the extended normal play winning convention, we can no longer simply look at individual components. We need to consider all components at the same time and based on the play rules within the sum, develop a game matrix, and then consider how the players will play. In the disjunctive sum, any G and H under consideration must be calculated from scratch, and thus disjunctive sum is an impractical sum for this framework. In the conjunctive sum, there was a timing issue and thus games do not act the same in all sums. In continued conjunctive sum, we defined a notion of game equality based on the profile. Similarly for scoring play, the continued conjunctive sum allowed for more analysis that any other sum. Simultaneous games under continued conjunctive sum and scoring play form an abelian group, and thus have the clearest structure as compared to all other combinations of sums and winning conventions studied here. Play under the continued conjunctive sum and extended normal play winning convention is a worthwhile direction for future work. In particular, better understanding simplifications of the profile and determining whether it is always optimal to choose an extreme of each function range are directions of interest.

Chapter 7

Case Studies

Throughout this chapter, we present several case studies to demonstrate different combinations of sums and winning conventions defined in Chapter 6. In particular, the following table details the section in which a case study is examined. A '−' indicates that the combination was discussed in Chapter 6, but was deemed an unnatural approach to develop algebraic foundations for simultaneous play. A '†' beside a section number indicates that it is simply a discussion regarding particular problems associated with the sum. An 'X' indicates that it was not explored within the case studies.

	Extended Normal	*Scoring*	*Majority*
$+$	7.1.1†	7.2	−
$\wedge$	7.1.2	X	X
$\triangledown$	7.1.2	7.1.3	7.1.4

Sometimes, we are able to obtain more results by restricting analysis based on properties or to certain game classes/rulesets. In CGT, a *universe* has been used to describe the space in which the restriction is considered; however, there is no standard definition for universe. It has been defined under scoring play [55] and examined under misère play [58]. Here, we define it in terms of simultaneous games adapting most notation from [55].

Definition 7.0.1 A *universe*, U, is a subset of games which satisfy the following properties:

- Closed under addition $\odot$: if A and B are in U then $A \odot B \in U$.

- Closed under options: if A in U, and B is an option of A then $B \in U$.

- Closed under inverses: if $A \in U$ then $-A \in U$.

Often players are interested in the universe created by a single game. The next sections focus on specific universes, which allowed us to obtain more results.

First, we look at SIMULTANEOUS HACKENBUSH, which has properties that allow for easy computation of $score(G)$ for particular restricted graph classes. We examine this game under conjunctive sum and continued conjunctive sum with extended normal play and scoring play winning conventions respectively. We also explore the majority play winning convention here under continued conjunctive sum.

Secondly, we look at SIMULTANEOUS CLOBBER, a dicot game. Now, all simultaneous dicot games studied under extended normal play, under any of the three sums are draws and are therefore trivial. However, if we consider a different metric, we can continue to study dicot games under simultaneous moves. One interpretation is to assign a value to one player's actions, as exemplified in this study. Here, we explore small positions under the disjunctive sum and scoring winning convention.

Lastly, we analyze SUBTRACTION SQUARES, specifically $\text{SQ}(\{a\}, \{b\})(\underline{n})$ on general strips. A recurrence relation is given for their values.

7.1 SIMULTANEOUS HACKENBUSH

The game SIMULTANEOUS HACKENBUSH is a game on a graph in which the interference rule is easily describable, useful for demonstrating many concepts, and we were able to prove some results while restricting the universe.

Ruleset for SIMULTANEOUS HACKENBUSH

- Board: A finite graph with edges coloured blue (single solid), red (dashed) or green (double solid). There is a ground, usually depicted by a thicker horizontal line, and a set of special vertices called roots (connected to the ground).

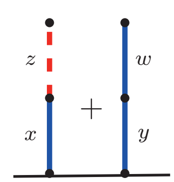
- Moves: Left cuts a blue edge or a green edge. If it is a cut-edge then the component which is not connected to the ground is also deleted. Right cuts a red edge or a green edge. If it is a cut-edge then the component which is not connected to the ground is also deleted.

- Interference Rule: Both Left and Right's options are removed and any connected component no longer connected to the ground is also removed.

All SIMULTANEOUS HACKENBUSH positions described within this chapter will have a *standard* root, that is, a graph rooted at a single vertex.

7.1.1 Examples

Example 7.1.1 For H (in Figure 7.1) the outcomes, extended normal play value assignments, and scores are as shown in Figures 7.2, 7.3, and 7.4, respectively.



$H = H_1 + H_2$

Figure 7.1: A sum of SIMULTANEOUS RED-BLUE HACKENBUSH STALKS positions.

One observation regarding the game H is that, if playing under extended normal play, Left is indifferent to her choices, as she will always win. However, for the scoring version of the game, H^y is less favourable than both other options.

Example 7.1.2 Consider $G_1 \wedge G_2$ pictured in Figure 7.5. On the first turn, Right can guarantee a win in G_1 by playing G_1^c. Left knows this and hence rather than

$$\begin{array}{cc} & H^z \\ H^x & \\ H^y & \\ H^w & \end{array} \left[\begin{array}{c} L \\ L \\ L \end{array}\right]$$

Figure 7.2: Outcomes.

$$\begin{array}{cc} & H^z \\ H^x & \\ H^y & \\ H^w & \end{array} \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

Figure 7.3: Values.

$$\begin{array}{cc} & H^z \\ H^x & \\ H^y & \\ H^w & \end{array} \left[\begin{array}{c} 2 \\ 1 \\ 2 \end{array}\right]$$

Figure 7.4: Scores.

losing the game by playing G_2^f, she will play G_2^d and force the overall game to be a Draw rather than a Right win.

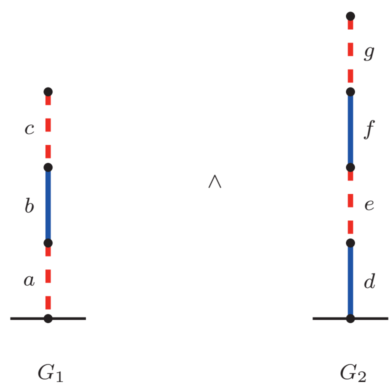


Figure 7.5: Timing Issues.

7.1.2 Results for Extended Normal Play

We call a SIMULTANEOUS HACKENBUSH position which has one blue edge incident with the root followed by anything else a *blue-based* position. We call a position a *blue* based position* if it is blue-based with at least one other edge somewhere else in the position (see Figure 7.7). A *two-blue based* position starts with two consecutive blue edges followed by anything above it, and no additional edge at v_1 (see Figure 7.6). In both figures, A is a generic completion to the SIMULTANEOUS HACKENBUSH position.

Lemma 7.1.3 *Consider a* SIMULTANEOUS HACKENBUSH *position, G. If G is a two-blue based position, then Left has a winning strategy.*

Proof. Either all the edges are blue and Right has no move, or there are moves for Right, but Left can always remove the second edge from the bottom which eliminates all of Right's moves. Then Left wins.

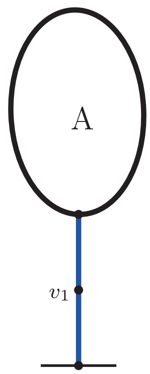


Figure 7.6: Two-blue based position.

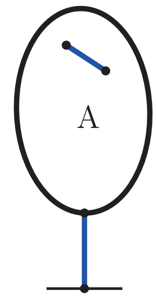


Figure 7.7: Blue* based position.

□

Based on Lemma 7.1.3 we conclude with the following results:

Proposition 7.1.4 *In a conjunctive sum of* SIMULTANEOUS RED-BLUE HACKENBUSH *positions, where at least one component is two-blue based and all other components are blue* based, then Left wins.*

Proof. Consider the first move within the conjunctive sum of SIMULTANEOUS RED-BLUE HACKENBUSH positions satisfying the given properties. Within a two-blue based position (see Figure 7.6), Left will remove the second blue edge from the root. This move guarantees that she will win this component since Right does not have a move on the next round and she does. Now, we need to ensure that Left has a move in all other components as well. Given the properties, there are at least two Left options in all other components: the bottom blue edge, and another edge somewhere else in the connected component. She chooses the latter option, to ensure that the components don't terminate (with her as the loser). In the conjunctive sum, she ends the overall game on the first turn and is the winner. □

If we consider only SIMULTANEOUS RED-BLUE HACKENBUSH STALKS we have a combination of the following positions to consider: (i) two-blue based, (ii) two-red based, (iii) alternating starting with blue, ending alternation with blue, (iv) alternating starting with blue ending with red, (v) alternating starting with red, ending with red, and (vi) alternating starting with red, ending with blue.

In a continued conjunctive sum, we claim that if comprised of only components from (i) and (iii) then the game is a Left win. If the components are from (ii) and

(v) then the game is a Right win. Otherwise the game is a Draw. The following propositions outline the cases (iii) - (vi). Cases (i) and (ii) are restrictions (and the negative respectively) of Proposition 7.1.4 restricted to stalks.

Proposition 7.1.5 *For* SIMULTANEOUS RED-BLUE HACKENBUSH STALKS *which are purely alternating, starting with a blue edge and ending with a red edge, Right cannot lose.*

Proof. For every blue edge, there is a red edge directly above it. Right's strategy is to play the highest red edge available. When Left removes the last blue edge (rooted), Right will have an option to remove the red edge directly above it, the game is over and it is a Draw. □

Proposition 7.1.6 *For* SIMULTANEOUS RED-BLUE HACKENBUSH STALKS *which are purely alternating, starting with a blue edge and ending with a blue edge, Right cannot use this component to force a Draw.*

Proof. Consider the induced subgraph on the vertices $\{v_1, \ldots, v_n\}$. This is the negative of the position described in Proposition 7.1.5. Left will have one edge remaining after the game on the subgraph has terminated (Right has no move) and thus she will win this component. □

Proposition 7.1.7 *For* SIMULTANEOUS RED-BLUE HACKENBUSH STALKS *which starts by alternating and after alternation ends in two red edges, Right cannot lose this component.*

Proof. Right can guarantee a Draw in this component by choosing the second red edge after alternation. Even if Left has chosen an edge above Right's choice on this round, the resulting position is either (i) as in Proposition 7.1.5, or (ii) starts and ends with red edges. In (i) by Proposition 7.1.5 Right cannot lose. In (ii), consider the induced subgraph on $\{v_1, \ldots, v_n\}$, this is as in Proposition 7.1.5. If Right ignores the edge connected to the ground v_0v_1, after simultaneous play ends in the subgraph, Right still has a move in the game (namely v_0v_1), and Left does not, and hence Right cannot lose. □

7.1.3 Results for Scoring Play

Definition 7.1.8 The *score* of a SIMULTANEOUS RED-BLUE HACKENBUSH position is defined as the number of blue or red edges remaining after simultaneous play has ended. If there are n blue edges remaining, the score of the position is n. If there are n red edges remaining, the score is $-n$.

Lemma 7.1.9 *Consider a* SIMULTANEOUS HACKENBUSH STALK *with alternating blue and red edges. An optimal play has players moving furthest away from the ground.*

Proof. We prove this claim for alternating blue and red edges, starting with a blue edge. Symmetric proofs hold true if the stalk started with a red edge. There are two cases to consider: 1) ending with a blue edge; 2) ending with a red edge. Label the edges l_1, ..., l_{n+1} for Left's options which l_1 being the edge closest to the ground, and r_1, ..., r_n for Right's options. In both cases, consider their pure strategies as the labels of the following matrix rows and columns respectively.
Case 1: Applying the simultaneous moves recursively, the final matrix is the following $(n+1) \times n$ matrix:

$$M = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

All pure strategies for Left are dominated by the final row (l_{n+1}) and hence the score of the game is 1 and thus a Left win.
Case 2: Applying the simultaneous moves recursively, the final matrix will be the following $n \times n$ matrix:

$$M = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}$$

All pure strategies for Right are dominated by the final column (r_n) and hence the score of the game is 0 and thus is a Draw.

□

Note: Lemma 7.1.9 does not hold in all sums, as demonstrated in Example 7.1.2.

Theorem 7.1.10 *The score of a* SIMULTANEOUS RED-BLUE HACKENBUSH STALK *is the number, n, of blue (or red, respectively) edges before the first alternation between red and blue edges. If the alternation begins and ends with the same colour, then the score is n (or $-n$ respectively). If the alternation begins and ends with different colours, then the score is $n-1$ (or $-n+1$ respectively).*

Proof. We show the proof for the stalk of score n and $n-1$. It is a similar proof for $-n$ and $-n+1$.

Consider a stalk where there are n blue edges followed by a series of alternating red and blue edges, ending in two blue edges, followed by a string of α edges. There are six cases to consider:

- Case 1: Both players move in α. By induction, this game has value n.

- Case 2: Left moves in α, Right moves in the first alternating part. We are left with the position, of n blue edges, and an alternating red-blue stalk above that, ending in blue. By Lemma 7.1.9, both players will play their furthest edges and hence, in each turn the top two edges will be chosen. Right will run out of moves and Left will have n edges remaining.

- Case 3: Right moves in α and Left moves in the first alternating part. The remaining stalk will have n blue edges followed by alternating red-blue stalk above, ending in red. Again by Lemma 7.1.9, both players will choose the furthest edges from the ground. This will result in $n-1$ blue edges at the end of simultaneous game play and thus is a dominated option (Case 1 and 2 are better options for Left).

- Case 4: Both players move in the first alternating part. By Lemma 7.1.9, both players will play at the top of this section of the stalk. Hence we are left with the position, of n blue edges, and an alternating red-blue stalk above that, ending in blue. Thus this falls into Case 2, and ends with a score of n.

- Case 5 and 6: Left moving in the all blue string while Right moves in either α or the first alternating part. These options are dominated because it will result in a value less than n.

□

Under alternating play, the CGT values of alternating RED-BLUE HACKENBUSH STALKS (starting with a blue edge) are approaching 2/3 as the height of the stalk approaches infinity. Alternating RED-BLUE HACKENBUSH STALKS (starting with a blue edge) with value less than 2/3 are Draws in SIMULTANEOUS HACKENBUSH, while positions with values greater than 2/3 are Left wins in SIMULTANEOUS HACKENBUSH. This fact, and Lemma 7.1.9, lead us to the following conjecture.

Conjecture 7.1.11 Let G be a HACKENBUSH tree. If the CGT value of G is greater than 2/3, then $o_S(G)$ is a Left win.

Similar results hold for Right if the roles of red and blue edges are interchanged.

Next we present a theorem which highlights a specific example of a universe which forms a subgroup under continued conjunctive sum and the scoring play winning convention. The scores within this subclass are special as they are in fact CGT scores.

Theorem 7.1.12 SIMULTANEOUS RED-BLUE HACKENBUSH STALKS *form a subgroup under continued conjunctive sum and scoring play.*

Proof. The identity element is the game of score zero. Equivalence classes are by CGT score and given the types of positions considered, all scores are integer valued. The inverse of G is $-G$. Associativity and commutativity follows immediately. □

Consider cordons (see Definition 3.2.2) which are undirected. The leaf arcs correspond to *leaf edges* in the undirected graph.

Theorem 7.1.13 *A* SIMULTANEOUS RED-BLUE HACKENBUSH CORDON *of height n with all stalk edges blue and a blue leaf edges and b red leaf edges has score $n+a-b > 0$.*

Proof. On a stalk, by Lemma 7.1.9, we know players will play furthest away from the ground. Consider now a cordon, where all stalk edges are blue and there are a blue leaf edges and b red leaf edges. First let's consider Right's strategy. He only has leaf edges to play. If he chooses a leaf edge closer to the ground, and Left cuts a stalk edge below some red leaf edges, Right loses options. Hence Right will play leaf edges furthest away from the ground, to have minimal interference with Left. Left on the other hand, can either match Right's option by taking the stalk edge corresponding to the edge incident to a red leaf edge, or she could take a blue leaf edge. She is guaranteed all moves (since Right cannot interfere with any of Left's options), hence it is in her best interest to play leaf edges one at a time (including the nth stalk edge), and thus Left has a total of $n+a$ moves, and, if Left plays optimally, Right will have b moves. Hence the value of the cordon is the number of extra moves Left has when Right runs out of moves, which is precisely $n+a-b$. □

The negatives of the positions in Theorem 7.1.13, obtained by interchanging the roles of red and blue edges, result in a score of $-n-a+b<0$.

7.1.4 Results for Majority Play

The classification of SIMULTANEOUS HACKENBUSH STALKS under continued conjunctive sum and majority play winning convention is quite simple. An empty game is a draw.

Theorem 7.1.14 *The score of a sum of* SIMULTANEOUS HACKENBUSH STALKS *under continued conjunctive sum and majority play winning convention is as follows:*

$$B-R+AB-AR$$

where B is the number of stalks within the sum that start with at least two blue edges plus the number of blue stalks of height 1*, R is the number of stalks within the sum that start with at least two red edges plus the number of red stalks of height* 1*, AB is the number of stalks starting with a blue edge and the alternation ends with a blue edge, and AR is the number of stalks starting with a red edge and the alternation ends with a red edge.*

Proof. The proof is immediate from scoring play results. Rather than a score of a component, say for Left, contributing 2 or more to the overall score, the component contributes to the sum of the number of component wins for Left. Similarly for Right. Then the tally is added (or subtracted) accordingly. □

7.2 SIMULTANEOUS CLOBBER

Recall the ruleset for SIMULTANEOUS CLOBBER.

Ruleset for SIMULTANEOUS CLOBBER

- Board: A finite graph, with each vertex unoccupied or occupied by an X or an O (typically played on an $m \times n$ grid, alternating X's and O's, where $m, n \in \mathbb{N}$).
- Moves: Left moves an X onto an adjacent O. The O is removed, and is said to be 'clobbered' by the X. Right moves an O onto an adjacent X. The X is removed, and is said to be 'clobbered' by the O.
- Interference Rule: If players choose to clobber their opponent's piece which their opponent is also using to clobber theirs, both pieces disappear. If the position is OXO, labelled as ABC, and Left moves to C and Right moves the O in A to B, then Left is said to have clobbered the O in C, but Right merely occupies B (did not clobber the X).

For example, $[OX]$ played simultaneously, after one move becomes $[\ ]$. If it was defined simply as a placement swap then the game would be *loopy* (both players could insist on only choosing that move and the game would never end).

The scoring variant that we consider here is the number of O's clobbered. This is an asymmetric game since the best Right can do is hope for a Draw. So we know that the outcome classes are restricted to Left wins and Draws.

First, we looked at SIMULTANEOUS CLOBBER played on the complete graph on n vertices, K_n, where each vertex has an O except for one which has an X. There are two possibilities: Left and Right choose matching vertices, and hence the game goes to zero. This can happen in $(n-1)$ ways. Or they don't match in their choices (i.e., Left clobbers one of Right's pieces and a different piece of Right takes the place of Left's piece which moved). This can happen in $(n-1) \times (n-2)$ ways. Hence

Theorem 7.2.1 *The value of* SIMULTANEOUS CLOBBER *with one piece for Left on* K_n *is defined by the following recurrence relation with initial value* $\mathrm{EV}(K_2) = 0$*:*

$$\mathrm{EV}(K_n) = \frac{1}{n-1}(0) + \frac{n-2}{n-1}(1 + \mathrm{EV}(K_{n-1})) \tag{7.1}$$

which implies for $n \geq 2$*,*

$$\mathrm{EV}(K_n) = \frac{n}{2} - 1. \tag{7.2}$$

Proof. We prove by induction that the closed-form solution (7.2) is indeed correct given the recurrence relation (7.1). First let $a_n = \mathrm{EV}(K_n)$ and we rewrite both expressions and simplify as follows.

$$a_n = \frac{n-2}{n-1}(1 + a_{n-1}) \tag{7.3}$$

where $a_2 = 0$ which implies for $n \geq 2$,

$$a_n = \frac{n}{2} - 1. \tag{7.4}$$

To prove the claim, we first check the base case: Let $n = 2$ then substituting in (7.4) gives

$$a_2 = \frac{2}{2} - 1 = 0 \tag{7.5}$$

The base case holds.

The induction hypothesis is that the expression (7.4) holds for $n = k$, for some $k > 2$. Now we will show that the expression (7.4) holds for $n = k + 1$.

Let $n = k + 1$. We substitute this into the expression (7.3) and obtain

$$a_{k+1} = \frac{(k+1)-2}{(k+1)-1}\left(1 + a_{(k+1)-1}\right) \tag{7.6}$$

$$= \frac{k-1}{k}\left(1 + a_k\right) \tag{7.7}$$

Furthermore, by the induction hypothesis, we can also substitute an expression in for a_k to obtain

$$a_{k+1} = \frac{k-1}{k}\left(1 + \left(\frac{k}{2} - 1\right)\right) \tag{7.8}$$

We simplify (7.8) as follows

$$a_{k+1} = \frac{k-1}{k}\left(1 + \left(\frac{k}{2} - 1\right)\right) \tag{7.9}$$

$$= \frac{k-1}{k}\left(\frac{k}{2}\right) \tag{7.10}$$

$$= \frac{k-1}{2} \tag{7.11}$$

$$= \frac{k-1}{2} + 1 - 1 \tag{7.12}$$

$$= \frac{k+1}{2} - 1 \tag{7.13}$$

The expression (7.13) is precisely the result for $n = k+1$ in (7.4). Given the recurrence relation, we showed by induction that $a_{k+1} = \frac{k+1}{2} - 1$. Hence, equation (7.2) is the closed-form solution for the recurrence relation (7.1), with the given initial condition. □

Next, we look at this game on an infinite path, starting with one piece for Left $H = [\ldots OOXOO \ldots]$. Then we place two Left pieces adjacent to one another, $J = [\ldots OXXO \ldots]$. The analysis of subsequent positions which involve increasing the distance between Left pieces and where she only has two pieces on the infinite path, are left to the interested reader. Preliminary values are shown in Table 7.1, calculations follow immediately afterward.

Positions	Values
$E = [\ldots OOO]$	0
$F = [\ldots OOX]$	0
$G = [\ldots OOXO]$	$(-1+\sqrt{5})/2$
$H = [\ldots OOXOO\ldots]$	$(1+\sqrt{5})/4$
$J = [\ldots OOXXOO\ldots]$	$1/2$

Table 7.1: Values for some SIMULTANEOUS CLOBBER positions.

Case 1: Let $E = [\ldots OOO]$. There are no moves for Left, and under this scoring ruleset, the value of this game is 0.

Case 2: Let $F = [\ldots OOX]$. From this position, each player only has one option: Left can move left and Right can move right. Hence, no O is clobbered and the value is 0.

Case 3: Let $G = [\ldots OOXO]$. From G both players have two options available to them, involving moving a piece left (ℓ) or right (r). The matrices of options and scores are given in (7.14).

$$\mathrm{M}(G) = \begin{array}{c} \\ \ell \\ r \end{array}\!\!\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} G{+}1 & E \\ E & [XO]+1 \end{bmatrix} \end{array}, \quad \overline{\mathrm{M}(G)} = \begin{array}{c} \\ \ell \\ r \end{array}\!\!\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} \mathrm{EV}(G)+1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \tag{7.14}$$

Note that the only play remaining in $[XO]+1$ is in $[XO]$ because 1 is a score. So $\mathrm{EV}([XO]+1) = 1$. Let $\mathrm{EV}(G) = q$ and suppose Left's strategy is $X = (x, 1-x)$ then we can determine the mixed strategy for Left by equalizing expectations across Right's pure strategies (note: he will use each strategy with non-zero probability since there is no domination). Hence we obtain the following:

$$(q+1)(x)+(0)(1-x)=(0)(x)+(1)(1-x) \tag{7.15}$$
$$qx+x=1-x \tag{7.16}$$
$$qx+2x=1 \tag{7.17}$$
$$(q+2)x=1 \tag{7.18}$$
$$x=\frac{1}{q+2} \tag{7.19}$$

Now, this gives $X=(\frac{1}{q+2},\frac{q+1}{q+2})$. Again, we can consider any pure strategy used with non-zero probability in Right's mixed strategy to determine the value of the game. $\mathrm{EV}(G)=q=(\frac{1}{q+2})(q+1)$. This gives a quadratic in q which we solve as follows:

$$q=\left(\frac{1}{q+2}\right)(q+1) \tag{7.20}$$
$$q\,(q+2)=q+1 \tag{7.21}$$
$$q^2+2q=q+1 \tag{7.22}$$
$$q^2+q-1=0 \tag{7.23}$$
$$\tag{7.24}$$

By the quadratic formula, we obtain that $\mathrm{EV}(G)=q=\frac{-1+\sqrt{5}}{2}$ (taking the positive value since the values are non-negative under this scoring ruleset).

Case 4: Let $H=[\ldots OOXOO\ldots]$. From H both players have two options available to them, involving moving a piece left (ℓ) or right (r). The matrices $\mathrm{M}(H)$ and $\overline{\mathrm{M}(H)}$ is given in (7.25).

$$\mathrm{M}(H)=\begin{array}{c} \\ \ell \\ r \end{array}\!\!\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} G+1 & E \\ E & G+1 \end{bmatrix} \end{array},\quad \overline{\mathrm{M}(H)}=\begin{array}{c} \\ \ell \\ r \end{array}\!\!\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} \frac{-1+\sqrt{5}}{2}+1 & 0 \\ 0 & \frac{-1+\sqrt{5}}{2}+1 \end{bmatrix} \end{array}. \tag{7.25}$$

The strategies to play in $\mathrm{M}(H)$ are the same for both Left and Right because of the matrix symmetry. Also, players will play each strategy equally. Hence $X = (\frac{1}{2}, \frac{1}{2})$ and the value of the game is $\mathrm{EV}(H) = \left(\frac{1}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{4}$.

Case 5: Let $J = [\ldots OOXXOO\ldots]$. From J both players have two options available to them, involving moving a piece left (ℓ) or right (r). The matrices $\mathrm{M}(J)$ and $\overline{\mathrm{M}(J)}$ is given in (7.26).

$$\mathrm{M}(J) = \begin{array}{c} \\ \ell \\ r \end{array}\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} F+1 & F \\ F & F+1 \end{bmatrix} \end{array}, \quad \overline{\mathrm{M}(J)} = \begin{array}{c} \\ \ell \\ r \end{array}\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}. \tag{7.26}$$

As before, by symmetry of the matrix, both players will play each strategy with equal probability, hence $X = (\frac{1}{2}, \frac{1}{2}) = Y$ and $\mathrm{EV}(J) = \frac{1}{2}(1) + \frac{1}{2}(0) = \frac{1}{2}$.

$$\begin{array}{c} \\ \ell \\ r \end{array}\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Figure 7.8: $\overline{\mathrm{M}([\ldots OX] + [XO\ldots])}$.

Consider the position $G = [\ldots OX] + [XO\ldots]$. If we calculate $\mathrm{EV}(G)$ by using the reduced values (see Table 7.1) we find the expected value is 0. But actually calculating the expected value of the disjunctive sum, we obtain the value $\frac{1}{2}$ (see Figure 7.8, applying the same strategies discussed in Case 5); i.e.,

$$\mathrm{EV}(G+H) \neq \mathrm{EV}(Re(G) + Re(H)). \tag{7.27}$$

This exemplifies once again that under disjunctive sum, we encounter problems with using previously defined values in a different sum.

7.3 SUBTRACTION SQUARES $\mathrm{SQ}(\{a\},\{b\})(\underline{n})$

SUBTRACTION SQUARES $\mathrm{SQ}(\{a\},\{b\})(\underline{n})$ is a special case of $\mathrm{SQ}(S_L, S_R)(\underline{n})$ introduced in Chapter 6. In this section, Left and Right each respectively only have one element in their subtraction set, $S_L = \{a\}$ and $S_R = \{b\}$.

Theorem 7.3.1 *In the game* $\mathrm{SQ}(\{a\},\{b\})(\underline{n})$*, where* $a < b$*, under extended normal play, the value of* $\underline{n}$ *is given by*

$$\mathrm{EV}(\underline{n}) = \begin{cases} 0, \textit{ if } n < a; \\ 1, \textit{ if } a \le n < b; \\ (\mathrm{EV}(\underline{n-b}) + \mathrm{EV}(\underline{n-b-a}))/2, \textit{ if } a+b \le n. \end{cases}$$

Proof. If $n < a$ then neither player has a move and the game is a draw. If $a \le n < b$ then only Left has a move and the game is a Left win. Suppose $n \ge b$. With equal probability the players play on the same side, leaving $n-b$ squares, or play on opposite sides leaving $\max\{0, n-a-b\}$ giving $\mathrm{EV}(\underline{n}) = (\mathrm{EV}(\underline{n-b}) + \mathrm{EV}(\underline{n-b-a}))/2$. □

For given a, b, solving the game $\mathrm{SQ}(\{a\},\{b\})(\underline{n})$ means solving the recurrence $2\mathrm{EV}(\underline{n}) = \mathrm{EV}(\underline{n-b}) + \mathrm{EV}(\underline{n-b-a})$ with the initial conditions. For example, in $\mathrm{SQ}(\{1\},\{2\})(\underline{n})$, we have $2\mathrm{EV}(\underline{n}) = \mathrm{EV}(\underline{n-2}) + \mathrm{EV}(\underline{n-3})$. Letting $a_n = \mathrm{EV}(\underline{n})$, we can rewrite this equation as $a_n = \frac{1}{2}a_{n-2} + \frac{1}{2}a_{n-3}$ and solve it using the characteristic equation as follows:

$$\lambda^n = \left(\frac{1}{2}\right)\lambda^{n-2} + \left(\frac{1}{2}\right)\lambda^{n-3} \tag{7.28}$$

$$\lambda^3 = \left(\frac{1}{2}\right)\lambda + \frac{1}{2} \tag{7.29}$$

$$\lambda^3 - \left(\frac{1}{2}\right)\lambda - \frac{1}{2} = 0 \tag{7.30}$$

Solving for λ will give us the roots of the characteristic equation and help us determine a generalized closed-form for $\mathrm{EV}(\underline{n})$. Hence, we first recognize that $\lambda = 1$ is a root of the equation (7.30). Then, we factor the expression $\lambda^3 - \left(\frac{1}{2}\right)\lambda - \frac{1}{2} = (\lambda - 1)(\lambda^2 +$

$\lambda + \frac{1}{2}) = 0$ and futhermore, using the quadratic formula we obtain the following:

$$(\lambda - 1)\left(\lambda - \frac{-1+i}{2}\right)\left(\lambda - \frac{-1-i}{2}\right) = 0.$$

Hence, a general solution for a_n is as follows:

$$a_n = C_1(1)^n + C_2\left(\frac{-1+i}{2}\right)^n + C_3\left(\frac{-1-i}{2}\right)^n.$$

As we want a closed-form based on the initial conditions, we now solve for C_1, C_2, and C_3. Using initial conditions $a_0 = 0$, $a_1 = 1$, $a_2 = 0$ (these correspond to an empty board ($\underline{0}$) being a Draw, a single square ($\underline{1}$) being a Left win and two squares ($\underline{2}$) being a Draw, respectively), we obtain the following linear equations respectively.

$$\begin{aligned} C_1 + C_2 + C_3 &= 0 \\ C_1 + \left(\frac{-1+i}{2}\right)C_2 + \left(\frac{-1-i}{2}\right)C_3 &= 2 \\ C_1 + \left(\frac{-i}{2}\right)C_2 + \left(\frac{i}{2}\right)C_3 &= 0 \end{aligned}$$

Solving the system of linear equations allows us to solve for the constants of the closed-form for $\text{EV}(\underline{n})$, with given initial conditions. We obtain $C_1 = \frac{2}{5}$, $C_2 = -\frac{1}{5} - \frac{2}{5}i$ and $C_3 = -\frac{1}{5} + \frac{2}{5}i$. Putting everything together, we solve the recurrence relation:

$$\text{EV}(\underline{n}) = \frac{1}{5}\left(2 - (1+2i)\left(-\frac{1}{2} + \frac{i}{2}\right)^n - (1-2i)\left(-\frac{1}{2} - \frac{i}{2}\right)^n\right).$$

Since all the terms which are raised to the power n are less than 1 in modulus, then $lim_{n\to\infty}\text{EV}(\underline{n}) = 2/5$.

With regards to the profile (see Section 6.3.1), we have the following.

Corollary 7.3.2 *For the subtraction game* $\text{SQ}(\{a\},\{b\})(\underline{n})$, *where* $a < b$, $l_{\underline{n}} = \text{EV}(\underline{n})$.

Proof. Since Right cannot win, $r_{\underline{n}} = 0$. □

7.4 Conclusions

One of the most important results within the theory of combinatorial games is that we understand how to sum games under different rulesets. Naturally, as we extend the theory to simultaneous play, we would like to have a similar theory developed here. For example, let

$$G = OXO \qquad \odot \qquad \mathrm{SQ}'(\{1\}\,,\{2\})(\underline{4}) \qquad \odot$$

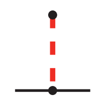


Figure 7.9: A sum of three games.

Where do players want to move in G under different sums and models? Table 7.2 gives the game results. Initially, however, it is unclear which option is best given a particular sum and model. Ultimately, we would like to determine a method for combining sums of different rulesets to know the overall result for simultaneous play.

	$+$	$\wedge$	$\triangledown$
Extended Normal Play	-1/2	R	D
Scoring	-1/2	-1	-1/2

Table 7.2: Outcomes and values for G from Figure 7.9 based on different sums.

Chapter 8

Conclusion

8.1 Summary

First, we exemplified the importance of the simplification tool, reduced canonical form, in the analysis of an otherwise complex game, THINNING THICKETS. Next we introduced a method of combining games, using conjoined rulesets, to develop a method for constructing an opening position for games which otherwise wouldn't have one. This highlighted a unique challenge for players: the need to play the first game in a way which sets up the second game board favourably. Both Chapters 3 and 4 focused on games under the normal play winning convention. In Chapter 5, we shifted gears to study the ORTHOGONAL COLOURING GAME, a scoring game, and demonstrated a strategy for the second player to guarantee a draw when the game is played on a graph which admits a strictly matched involution.

In Chapter 6 we provided a philosophy for simultaneous combinatorial game theory, the initial framework for combinatorial games where players move at the same time. This was followed by Chapter 7 which gave insight into analysis of simultaneous games using case studies. From this work, we concluded that the most natural approach to studying simultaneous combinatorial games is under continued conjunctive sum and the scoring play winning convention. Under this framework games form an abelian group, using values as a measure of game equality.

We conclude this thesis by considering future directions for research.

8.2 Future Directions

Potential for future work within the area of combinatorial game theory is abundant. Directions to extend upon the philosophies and results presented within this thesis is the focus of this section. We reiterate previous problems mentioned earlier, as well as connect across sections, when appropriate, to get a broader view of future research for this body of work.

One may be curious about the potential for simultaneous versions of games presented in Chapters 3, 4, and 5. First, we would need a concept of simultaneous versions of the games. SIMULTANEOUS THINNING THICKETS takes on a similar interference rule to SIMULTANEOUS HACKENBUSH: Both players remove their respective arcs and any arc no longer on a directed path to a root gets removed. It isn't hard to classify stalks based on their colouring, prove that all stalks have deterministic outcomes, and to determine the outcome of sums of stalks under conjunctive and continued conjunctive sums.

Now consider scoring play. In order for a stalk to have non-zero score, the position needs to only have options for one player (called a *mono-player component*). The score of a stalk can be at most 1 since a move in a mono-player component opens moves for their opponent. Under conjunctive sum, the timing issue described in Section 6.3.1 still applies here. With regards to continued conjunctive sum, the score will simply be the number of mono-player Left components minus mono-player Right components. If the score is positive the game is a Left win, negative it is a Right win, and a Draw otherwise.

The study of cordons is not nearly as straightforward. Unless the cordon is a mono-player position, the game could involve mixed strategies. Thus, we end this analysis here and leave the rest of the analysis of SIMULTANEOUS THINNING THICKETS as an open question.

Question 6 What general analysis can be done for SIMULTANEOUS THINNING THICKETS?

Another natural question, within the context of this thesis, is how to play simultaneous conjoined games and what that analysis would entail. The analysis of conjoined games demonstrated that determining how to play the first phase, in order to best set up the second phase, is a non-trivial task. Under simultaneous play, depending on the game, not only is it non-trivial, but it can also be non-deterministic. Thus, at this time, we believe that understanding the basic properties of conjoined games under purely combinatorial analysis would be more fruitful for future research. Pushing forward with game classifications (for example, placement games for the first phase) is a natural next step, which leads to the following question.

Question 7 Which game classes lend themselves to well structured conjoined game combinations?

We extended combinatorial game theory with alternating play to allow simultaneous moves and develop the basic concepts required to analyze these games. We then introduced and investigated three types of sums and two winning conventions. In the disjunctive sum, under alternating play, the outcome of $G + H$ can be found by first reducing G and H then considering the sum of the resulting games. However, for simultaneous play, we have shown that this only holds in the continued conjunctive sum under the scoring winning convention.

In the case studies, we examined a weaker form of equality and inequality, where positions from the same game (for example) are compared. We formalize that approach. Given a sum $\odot$, let an $\odot$-*system*, $\mathbb{S}_{\odot}$ be a set of positions closed under options and sums. That is, if $G \in \mathbb{S}_{\odot}$ then (i) every position obtainable from G are also in $\mathbb{S}_{\odot}$; (ii) also, if $H \in \mathbb{S}_{\odot}$ then $G \odot H \in \mathbb{S}_{\odot}$. Equality and inequality in $\mathbb{S}_{\odot}$, are given as in Definition 6.2.1, except now $X \in \mathbb{S}_{\odot}$. Recall that $Re(G)$ is the reduced game of G.

Question 8 What $\mathbb{S}_{\odot}$ have reductions so that $\mathrm{EV}(G \odot H) = \mathrm{EV}(Re(G) \odot Re(H))$?

An interesting and important class of CGT games are the *dead-ending* games, $\mathbb{D}$, which are defined by the property that if a player has no moves in a particular position then there is no sequence of moves that the opponent may make that will allow the player to move again, see [58]. For example, in DOMINEERING, if there is no space for Left to place a vertical domino then allowing Right to place any number of horizontal dominoes will not create space for a vertical domino. TRIDOMINEERING and QUADROMINEERING belong to this class.

Question 9 For each sum, investigate $\mathbb{D}_{\odot}$.

In the study of simultaneous combinatorial games, we encountered the need for mixed strategies to determine game values, which changes how we can interpret the results, and restricts our analysis to a certain extent. An approach to mitigate the non-determinism is to consider one player to be superior in the sense that they can see their opponents move and react to it on the simultaneous move. This was first outlined within the context of placement games by Bahri and Kruskal [6], as the *relaxation principle*. Here, as a future direction for analysis, we suggest a general version of this, not restricted to placement games, which we call the *Cheating Robot* approach. We consider Right to be the 'superior being' who can see Left's move and respond to it on the simultaneous turn.

The Cheating Robot approach is deterministic which makes it an intriguing future direction for research in simultaneous combinatorial game theory. We do however have evidence that the theory will not be straightforward to develop.

Definition 8.2.1 Let $G = \{G^{\mathcal{L}} \mid G^{\mathcal{S}} \mid G^{\mathcal{R}}\}$ be a combinatorial game played under the Cheating Robot approach. Then $-G = \{-G^{\mathcal{R}} \mid -G^{\mathcal{S}} \mid -G^{\mathcal{L}}\}$, where the cheating robot is always Right.

In misère play, an important question regarding inverses is: when is $-G$ the inverse of G? In most cases, $G-G \neq 0$. We considered the same question under the Cheating Robot framework, and noticed that here too $G - G \neq 0$ for all G. We demonstrate this with the game SIMULTANEOUS TOPPLING DOMINOES.

Ruleset for SIMULTANEOUS TOPPLING DOMINOES

- Board: A row of black (B), white (W) and gray (G) dominoes.
- Moves: Left topples a black or gray domino either to the left or the right. Right topples a white or gray domino either to the left or the right. All dominoes in the direction of the toppling also gets removed.
- Interference Rule: If players topple in the same direction, all dominoes in the same direction of the toppling also gets removed. If players topple towards one another, all dominoes between the two chosen dominoes are removed (including the ends).

Consider a game G under the Cheating Robot approach. Consider the game sum to be disjunctive sum. Let G be a TOPPLING DOMINOES position $G = BWB$ and consider $G-G$, where the negative of G is switching the colours of the dominoes and Right is the cheating robot. We can't change the role of the robot for $-G$ because then we arrive at an impasse where no player could ever make a move because the reasoning is circular. Hence $G - G = BWB + WBW$ and Right is the cheating robot. If $-G$ were the inverse of G, then the game would always be a draw. However, one can check that this isn't the case under the Cheating Robot approach. This means that no matter what Left does, Right has a good response. Let's check. Left has six options from the position $G - G$. We list them with overarrows demonstrating the domino choice and direction of toppling along with a best response for Right. In all cases, Right's response will force a Right win. The cases are as follows:

(i) $\overleftarrow{B}\overrightarrow{W}B + WBW$

(ii) $\overrightarrow{B}\overrightarrow{W}B + WBW$

(iii) $B\overleftarrow{W}\overleftarrow{B} + WBW$

(iv) $B\overleftarrow{W}\overrightarrow{B} + WBW$

(v) $BWB + \overleftarrow{W}\overleftarrow{B}W$

(vi) $BWB + W\overrightarrow{B}\overrightarrow{W}$.

Notice that in cases (i-iv) the first component will be terminal after the first turn. For the second turn, Right can play a similar strategy to what was played in cases (v-vi) to guarantee that there will be one of his pieces remaining, while Left will not have any dominoes remaining. For cases (v-vi) Right has a move remaining in the second component, and thus on the second round follows Left into the first component, playing one of four strategies as seen in the first components of (i-iv).

Cheating Robot is a novel way to look at a difficult problem of non-deterministic outcomes for simultaneous combinatorial game theory, to approximate results using deterministic methods. This is a general future direction for simultaneous combinatorial game theory.

Question 10 Is there a theory of reductions and canonical forms for the Cheating Robot model similar to that of CGT under disjunctive sum and normal play?

Question 11 How good of an approximation is the Cheating Robot version of simultaneous CGT games?

Sometimes, game players are more interested in exploring specific universes. This interest may come out of necessity (the entire space doesn't allow for algebraic structure), or the interest in knowing complete information about a single game of study is their only concern. In either case, the question is the same:

Question 12 When does restricting to a specific game universe under simultaneous play allow for complete analysis of a game?

Overall, the most interesting (and promising) future direction is the Cheating Robot approach to simultaneous combinatorial game theory. It generalizes the concept of the 'relaxation principle' from [6] and opens the doors to a deterministic approach for the study of simultaneous combinatorial games. It is much closer to the original CGT analysis and allows for the possibility of expanding the scope of simultaneous games to pursuit-evasion games, as well as studying simultaneous combinatorial games in more depth.

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Appendix A

Graph Theory

Here we present basic graph theory definitions required for this thesis. For a standard reference for graph theory, see [15].

Definition A.0.1 [15] A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence* function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

Definition A.0.2 [15] The ends of an edge are called *incident* with the edge. Two vertices which are incident to a common edge are called *adjacent*, and two distinct adjacent vertices are called *neighbours*.

Definition A.0.3 [15] An edge with identical ends is called a *loop*. Two or more edges with the same pair of ends are called *parallel edges*. A graph is *simple* if it has no loops or parallel edges.

Definition A.0.4 [15] A *complete graph* on n vertices, denoted K_n, is a graph in which any two vertices are adjacent.

Definition A.0.5 [15] A *path* on n vertices, denoted P_n, is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise.

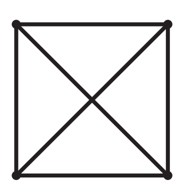

Figure A.1: K_4.

Figure A.2: P_3.

Definition A.0.6 [15] A *cycle* on n vertices, denoted by C_n, is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and nonadjacent otherwise.

Definition A.0.7 [15] An acyclic graph is a graph that contains no cycles, also called a *forest*. A connected acyclic graph is called a *tree*. A *rooted tree* is a tree with a specified root vertex, x.

Throughout the thesis we do not specify the root by name; it will be clear from the context based on the game and not necessary to associate a label to the root.

Definition A.0.8 [15] A *directed graph* D, is an ordered pair $(V(D), A(D))$, consisting of a set $V(D)$ of vertices and a set $A(D)$ of *arcs*, together with an incidence function ψ_D that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D. If a is an arc and $\psi_G(a) = (u, v)$, then u is the *tail* of a and v is the *head* of a.

If u and v are vertices of a graph G, we will write an arc $\psi_G(a) = (u, v)$ as $\overrightarrow{uv}$, where u and v are the tail and head vertices respectively. The *in-degree* of an arc $\overrightarrow{uv}$ is the number of arcs with head vertex u.

Definition A.0.9 [15] To *identify* nonadjacent vertices x and y of a graph G is to replace these vertices by a single vertex incident to all the edges which were incident in G to either x or y. To *contract* an edge e of a graph G is to delete the edge and then identify its ends.

Definition A.0.10 [15] A *matching*, denoted by M, in a graph is a set of pairwise nonadjacent edges. A vertex incident with an edge of M is said to be *covered* by M. A *perfect matching* is one which covers every vertex of the graph.

Definition A.0.11 [15] A *k-colouring* is an assignment of k colours to its vertices. A colouring is *proper* if no two adjacent vertices are assigned the same colour.

Definition A.0.12 [15] An *independent (stable)* set is a set of vertices no two of which are adjacent. An independent set in a graph is *maximum* if the graph contains no larger independent set.

Appendix B

Combinatorial Game Rulesets

CLOBBER

- Board: A finite graph, with each vertex unoccupied or occupied by an X or an O (typically played on an $m \times n$ grid, alternating X's and O's, where $m, n \in \mathbb{N}$).
- Moves: Left moves an X onto an adjacent O. The O is removed, and is said to be 'clobbered' by the X. Right move an O onto an adjacent X. The X is removed, and is said to be 'clobbered' by the O.

COL

- Board: A finite graph, with each vertex either uncoloured or coloured blue or red.
- Moves: Left colours an uncoloured vertex, with no blue neighbours, blue. Right colours an uncoloured vertex, with no red neighbours, red.

CUTTHROAT

- Board: A finite graph, where each vertex is coloured red or blue.
- Moves: Left removes a blue vertex. Right removes a red vertex. All incident edges and any monochromatic connected components are also removed.

The impartial version, where a player can remove a vertex of either colour on their turn, is denoted by CUTTHROAT$_{Imp}$.

DOMINEERING

- Board: A subset of the squares of a finite grid.
- Moves: Left places a 2×1 domino on the grid in two unoccupied vertically adjacent squares. Right places a 1×2 domino on the grid in two unoccupied horizontally adjacent squares.

GO-CUT

- Board: A finite graph, with each vertex either uncoloured or coloured blue or red.
- Moves:
  - *Phase 1:* On a move a player chooses an uncoloured vertex $(\cdot)$ and colours it either red (R) or blue (B) provided every maximal connected monochromatic subgraph is adjacent to an uncoloured vertex.

    When no moves are playable under Phase 1, delete all uncoloured vertices and then delete all monochromatic components. The game is now a disjunctive sum of components each of which contains both red and blue vertices, that is, non-monochromatic components.
  - *Phase 2:* A player chooses a component from the disjunctive sum, deletes one of the vertices then deletes any resulting monochromatic components.

HACKENBUSH

- Board: A finite graph with edges coloured blue (single solid), red (dashed) or green (double solid). There is a ground, usually depicted by a thicker horizontal line, and a set of special vertices called roots (connected to the ground).
- Moves: Left cuts a blue edge or a green edge. If it is a cut-edge then the component which is not connected to the ground is also deleted. Right cuts a red edge or a green edge. If it is a cut-edge then the component which is not connected to the ground is also deleted.

NIM

- Board: n heaps of counters of size k_i, $i = 1, \ldots, n$, where $k_i \geq 1$.
- Moves: A player can choose a heap, say heap i, and the player removes up to k_i counters from heap i.

NODE KAYLES

- Board: A finite graph.
- Moves: On their turn, a player chooses a vertex and deletes it and all its neighbours.

NOGO

- Board: A finite graph, with each vertex either uncoloured or coloured blue or red.
- Moves: Left colours an uncoloured vertex, $(\cdot)$, blue provided that each maximal connected monochromatic subgraph has at least one vertex adjacent to an uncoloured vertex. Right colours an uncoloured vertex, $(\cdot)$, red provided that each maximal connected monochromatic subgraph has at least one vertex adjacent to an uncoloured vertex.

The impartial version, where a player can colour a vertex red or blue on their turn, is denoted by NOGO$_{Imp}$.

ORTHOGONAL COLOURING GAME, $MOC_m(G)$

- Board: Two initially uncoloured disjoint isomorphic copies G_L and G_R of a given finite graph G.
- Moves: Two players, Left and Right, with Left beginning, alternately choose one of the two graphs G_L or G_R and colour an uncoloured vertex of this graph with a colour from the set $\{1, \ldots, m\}$ such that the colouring is proper and the orthogonality of the graphs is not violated. Left owns G_L and Right owns G_R.

SNO-GO

- Board: A finite graph, with each vertex either uncoloured or coloured blue or red.
- Moves:
  - *Phase 1:* On a move a player chooses an uncoloured vertex $(\cdot)$ and colours it red (R) or blue (B) provided that no red vertex is adjacent to a blue vertex.
  - *Phase 2:* When no moves are playable under Phase 1 rules, players can colour an uncoloured vertex red or blue provided that each maximal connected monochromatic subgraph has at least one vertex adjacent to an uncoloured vertex.

SNORT (also known as CATS & DOGS)

- Board: A finite graph, with each vertex either uncoloured or coloured blue or red.
- Moves: Left colours an uncoloured vertex, with no red neighbours, blue. Right colours an uncoloured vertex, with no blue neighbours, red.

The impartial version, where a player can colour a vertex red or blue on their turn, is denoted by SNORT$_{Imp}$.

SUBTRACTION$(S_L, S_R)(n)$

- Board: A non-negative integer n.
- Moves: Left subtracts some element of S_L from n. Right subtracts some element of S_R from n. The result of the subtraction must be non-negative.

The impartial version is SUBTRACTION$(S)(n)$, where both players are choosing from the same subtraction set, S.

THINNING THICKETS

- Board: A finite directed graph where each arc is coloured blue (single solid), red (dashed) or green (double solid). The graph has a subset of vertices $x_1, \ldots, x_k$ called roots and every arc is on a directed path to some root.
- Moves: On a move, each player deletes an arc. Left removes a blue arc or a green arc with even in-degree (including 0) or a red arc with odd in-degree. Right removes a red arc or a green arc with even in-degree (including 0) or a blue arc with odd in-degree. After the arc is deleted, any arc and vertex not on a directed path to a root is also deleted.

Appendix C

Simultaneous Combinatorial Game Rulesets

SIMULTANEOUS CLOBBER

- Board: A finite graph, with each vertex unoccupied or occupied by an X or an O (typically played on an $m \times n$ grid, alternating X's and O's, where $m, n \in \mathbb{N}$).
- Moves: Left moves an X onto an adjacent O. The O is removed, and is said to be 'clobbered' by the X. Right moves an O onto an adjacent X. The X is removed, and is said to be 'clobbered' by the O.
- Interference Rule: If players choose to clobber their opponent's piece which their opponent is also using to clobber theirs, both pieces disappear. If the position is OXO, labelled as ABC, and Left moves to C and Right moves the O in A to B, then Left is said to have clobbered the O in C, but Right merely occupies B (did not clobber the X).

SIMULTANEOUS DOMINEERING

- Board: A subset of the squares of a finite grid.
- Moves: Left places a 2×1 domino on the grid in two unoccupied vertically adjacent squares. Right places a 1×2 domino on the grid in two unoccupied horizontally adjacent squares.
- Interference Rule: If players overlap on a turn, the overlap is allowed. Players cannot overlap previously placed pieces.

SIMULTANEOUS HACKENBUSH

- Board: A finite graph with edges coloured blue (single solid), red (dashed) or green (double solid). There is a ground, usually depicted by a thicker horizontal line, and a set of special vertices called roots (connected to the ground).
- Moves: Left cuts a blue edge or a green edge. If it is a cut-edge then the component which is not connected to the ground is also deleted. Right cuts a red edge or a green edge. If it is a cut-edge then the component which is not connected to the ground is also deleted.
- Interference Rule: Both Left and Right's options are removed and any connected component no longer connected to the ground is also removed.

SIMULTANEOUS THINNING THICKETS

- Board: A finite directed graph where each arc is coloured blue (single solid), red (dashed) or green (double solid). The graph has a subset of vertices $x_1, \ldots, x_k$ called roots and every arc is on a directed path to some root.
- Moves: On a move, each player deletes an arc. Left may delete a blue arc with an even in-degree (including 0) or a red arc with an odd in-degree. Similarly, Right may delete a red arc with an even in-degree (including 0) or a blue arc with an odd in-degree. Both may delete a green arc with even in-degree. After the arc is deleted, any arc and vertex not on a directed path to a root is also removed.
- Interference Rule: Both players remove their respective arcs. Any arc no longer on a directed path to a root gets removed.

SIMULTANEOUS TOPPLING DOMINOES

- Board: A row of black (B), white (W) and gray (G) dominoes.
- Moves: Left topples a black or gray domino either to the left or the right. Right topples a white or gray domino either to the left or the right. All dominoes in the direction of the toppling also gets removed.
- Interference Rule: If players topple in the same direction, all dominoes in the same direction of the toppling also gets removed. If players topple towards one another, all dominoes between the two chosen dominoes are removed (including the ends).

SUBTRACTION SQUARES, $\mathrm{SQ}(S_L, S_R)$ on a strip of squares of length n, denoted $\mathrm{SQ}(S_L, S_R)(\underline{n})$

- Board: Let S_L and S_R be sets of positive integers. The board is a strip of n squares, denoted $\underline{n}$.
- Moves: For any $p \in S_L$, $p \leq n$, Left can remove p squares from the left or right side of the strip. Similarly, if $q \in S_R$, $q \leq n$, then Right can remove q squares from the left or right side of the strip.
- Interference Rule: If they both take from the same side then $\max\{p, q\}$ squares are removed. If they take from opposite sides then the move is to $n - p - q$ except if $\max\{p, q\} \leq n \leq p + q$ then the move is to 0.

Appendix D

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