Nonlinear Identities for Bernoulli and Euler Polynomials

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Dedicated to the memory of my friend and mentor Jonathan M. Borwein

Abstract. It is shown that a certain nonlinear expression for Bernoulli polynomials, related to higher-order convolutions, can be evaluated as a product of simple linear polynomials with integer coefficients. The proof involves higher-order Bernoulli polynomials. A similar result for Euler polynomials is also obtained, and identities for Bernoulli and Euler numbers follow as special cases.

Keywords: Bernoulli polynomials, Bernoulli numbers, Euler polynomials, Euler numbers, convolution identities.

1 Introduction

Various types of multiple zeta functions and Euler sums played an important role in Jonathan Borwein's work in experimental mathematics. A particularly interesting class of such series is the Mordell-Tornheim-Witten zeta function

$$\mathcal{W}(r,s,t) := \sum_{m,n\geq 1} \frac{1}{m^r n^s (m+n)^t}, \tag{1.1}$$

which converges for all complex r,s,t with $\mathrm{Re}(r+t)>1$, $\mathrm{Re}(s+t)>1$, and $\mathrm{Re}(r+s+t)>2$, and can be meromorphically continued to all of $\mathbb{C}$. While Jonathan Borwein and his co-authors studied the series (1.1) (see, e.g., [2], [6], [7]), he also considered multi-dimensional analogues, especially

$$\mathcal{W}(r_1,\dots,r_n,t) := \sum_{m_1,\dots,m_n\geq 1} \frac{1}{m_1^{r_1}\dots m_n^{r_n}(m_1+\dots+m_n)^t}; \tag{1.2}$$

see [2], [3], [4], [5].

An interesting method repeatedly used in the papers cited above, both for theoretical results and high-precision computations, is due to Crandall and is

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based on a free parameter; see, e.g., [6], [7] for some details. As a particular application of this method, the results on (1.1) obtained in [7] were first generalized by H. Tomkins [18] to (1.2) in the case $n = 3$, and then very recently to arbitrary n in [9].

For the main results in this last paper, the following identity is required: For all integers $n \geq 1$ we have

$$\sum_{m=1}^{n} \binom{n+1}{m} \sum_{\substack{j_1,\dots,j_m \geq 1 \\ j_1+\cdots+j_m=n}} \prod_{i=1}^{m} \frac{B_{j_i}(1)}{j_i!} = 1. \tag{1.3}$$

Here $B_k(x)$ is the kth *Bernoulli polynomial*, which can be defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}, \qquad |t| < 2\pi. \tag{1.4}$$

Equivalently it can be defined by

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j} x^j, \tag{1.5}$$

where B_k is the kth *Bernoulli number*, defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \qquad |t| < 2\pi. \tag{1.6}$$

For the first few Bernoulli numbers and polynomials, see Table 1 at the end of this paper.

It is the main purpose of this paper to prove a polynomial analogue of (1.3), namely the following result.

Theorem 1. *For any integer $n \geq 1$ we have*

$$\sum_{m=1}^{n} \binom{n+1}{m} \sum_{\substack{j_1,\dots,j_m \geq 1 \\ j_1+\cdots+j_m=n}} \prod_{i=1}^{m} \frac{B_{j_i}(x)}{j_i!} = \frac{1}{n!} \prod_{j=1}^{n} \big((n+1)x - j\big). \tag{1.7}$$

Setting $x = 1$, we immediately obtain (1.3). Similarly, with $x = 0$ and using the fact that $B_k(0) = B_k$, we have the following identity for Bernoulli numbers.

Corollary 1. *For any integer $n \geq 1$ we have*

$$\sum_{m=1}^{n} \binom{n+1}{m} \sum_{\substack{j_1,\dots,j_m \geq 1 \\ j_1+\cdots+j_m=n}} \prod_{i=1}^{m} \frac{B_{j_i}}{j_i!} = (-1)^n. \tag{1.8}$$

We illustrate Theorem 1 with the first few cases.

Example. For $n = 1, 2, 3$ we have, respectively,

$$\begin{aligned} 2\,B_1(x) &= 2x-1,\\ \tfrac{3}{2}\,B_2(x) + 3\,B_1(x)^2 &= \tfrac{1}{2}(3x-1)(3x-2),\\ \tfrac{2}{3}\,B_3(x) + 6\,B_1(x)B_2(x) + 4\,B_1(x)^3 &= \tfrac{1}{6}(4x-1)(4x-2)(4x-3). \end{aligned}$$

In connection with extending two interesting identities of Matiyasevich [13] and Miki [14], expressions similar in nature to the left-hand side of (1.7) have been studied before (see [1], [10]), but the right-hand side has never been as easy as that of (1.7). We therefore believe that this identity is new.

We conclude this introduction by rewriting (1.7) in terms of the multinomial coefficient defined by

$$\binom{n}{j_1,\dots,j_m} = \frac{n!}{j_1!\cdots j_m!}.$$

Upon multiplying both sides of (1.7) by $n!$, we then get

$$\sum_{m=1}^{n}\binom{n+1}{m}\sum_{\substack{j_1,\dots,j_m\ge 1\\ j_1+\cdots+j_m=n}}\binom{n}{j_1,\dots,j_m}B_{j_1}(x)\cdots B_{j_m}(x) = \prod_{j=1}^{n}\big((n+1)x-j\big). \tag{1.9}$$

It is this identity which we will prove below. We begin with some auxiliary results in Section 2 and complete the proof in Section 3. We conclude this paper with some further remarks in Section 4, including an analogue of Theorem 1 for Euler polynomials.

2 Some Auxiliary Results

The multiple sum on the left of (1.9), namely

$$T_m(n;x) := \sum_{\substack{j_1,\dots,j_m\ge 1\\ j_1+\cdots+j_m=n}}\binom{n}{j_1,\dots,j_m}B_{j_1}(x)\cdots B_{j_m}(x), \tag{2.1}$$

is very similar to the higher-order convolution

$$S_m(n;x) := \sum_{\substack{j_1,\dots,j_m\ge 0\\ j_1+\cdots+j_m=n}}\binom{n}{j_1,\dots,j_m}B_{j_1}(x)\cdots B_{j_m}(x). \tag{2.2}$$

A slightly more general form of this last expression was evaluated by the present author [8], and then by several other authors, including Huang and Huang [12] who used a different method, and Petojević [16] who evaluated the sum in terms of Stirling numbers of the first kind. Both papers, and numerous others, contain evaluations of other related expressions of the type of (2.2).

In what follows, we will use the higher-order Bernoulli polynomials, defined as follows. Given an integer m (not necessarily positive), the kth *Bernoulli polynomial of order* m, denoted $B_k^{(m)}(x)$, is defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^m e^{xt} = \sum_{k=0}^{\infty} B_k^{(m)}(x)\frac{t^k}{k!}, \qquad |t|<2\pi. \tag{2.3}$$

By comparing this with (1.4), we see that $B_k^{(1)}(x) = B_k(x)$. Raising both sides of (1.4) to the power m and using the identities (2.2) and (2.3), we get

$$S_m(n;x) = B_n^{(m)}(mx). \tag{2.4}$$

This fact was earlier used in [8] and [12].

Next we need to connect the sums $S_m(n;x)$ and $T_m(n;x)$ with each other.

Lemma 1. *For any integers $m,n\geq 1$ we have*

$$S_m(n;x) = \sum_{j=1}^{m}\binom{m}{j}T_j(n;x), \tag{2.5}$$

$$T_m(n;x) = \sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}S_j(n;x). \tag{2.6}$$

Proof. To obtain (2.5), we subdivide the sum $S_m(n;x)$ according to the number of indices j_i that are 0. If none of them is 0, we simply have $T_m(n;x)$. If exactly one of them is 0, then we have m copies of $T_{m-1}(n;x)$. If exactly two of them are 0, we get $\binom{m}{2}$ copies of $T_{m-2}(n;x)$, and so on, until we reach the case where exactly $m-1$ of the indices are 0; this happens $\binom{m}{m-1}$ times, giving m copies of $T_1(n;x)$. Adding everything, we get (2.5).

The identity (2.6) can be obtained in different ways: Either directly by an inclusion/exclusion argument, or by solving a linear system that is inherent in (2.5), or, most easily by appealing to a general result on inverting finite sums; see, e.g., [17, p. 43].

Towards the eventual proof of (1.9), we now evaluate the following sum.

Lemma 2. *For any integer $n\geq 1$ we have*

$$\sum_{m=1}^{n}\binom{n+1}{m}T_m(n;x) = \sum_{k=1}^{n}(-1)^{n-k}\binom{n+1}{k}B_n^{(k)}(kx). \tag{2.7}$$

Proof. We use (2.6) and change the order of summation:

$$\begin{aligned}\sum_{m=1}^{n}\binom{n+1}{m}&\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}S_k(n;x)\\ &= \sum_{k=1}^{m}(-1)^k S_k(n;x)\sum_{m=k}^{n}(-1)^m\binom{n+1}{m}\binom{m}{k}.\end{aligned}$$

The inner sum of this last expression is an alternating analogue of the Vandermonde convolution, and can be evaluated as $(-1)^n\binom{n+1}{k}$; see, e.g., [11, (3.119)]. With this and (2.4), we immediately get (2.7).

3 The Proof of Theorem 1

By Lemma 2, in order to finish the proof of (1.9), and thus of Theorem 1, we need to evaluate the right-hand side of (2.7). Using the generating function (2.3), we rewrite

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{n-k}\binom{n+1}{k}B_n^{(k)}(kx) &= \sum_{k=1}^{n}(-1)^{n-k}\binom{n+1}{k}\frac{d^n}{dt^n}\left(\frac{te^{tx}}{e^t-1}\right)^k\Bigg|_{t=0}\\ &= \frac{d^n}{dt^n}\sum_{k=1}^{n}(-1)^{n-k}\binom{n+1}{k}\left(\frac{te^{tx}}{e^t-1}\right)^k\Bigg|_{t=0}.\end{aligned} \tag{3.1}$$

To simplify notation, we set $A(t) := te^{tx}/(e^t-1)$. Using a binomial expansion, we then have

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{n-k}\binom{n+1}{k}A(t)^k\\ &\qquad= -\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}A(t)^k + (-1)^{n+1} + A(t)^{n+1}\\ &\qquad= -(A(t)-1)^{n+1} + (-1)^{n+1} + A(t)^{n+1}.\end{aligned} \tag{3.2}$$

We note that the constant coefficient in the Maclaurin expansion of $A(t)$ as a function of t is 1. Therefore we can write

$$\big(A(t)-1\big)^{n+1} = \big(tB(t)\big)^{n+1} = t^{n+1}B(t)^{n+1},$$

where $B(t)$ is analytic at $t=0$. Hence

$$\frac{d^n}{dt^n}\,\left(t^{n+1}B(t)^{n+1}\right)\big|_{t=0} = 0,$$

while

$$\frac{d^n}{dt^n}\,A(t)^{n+1}\big|_{t=0} = \frac{d^n}{dt^n}\left(\left(\frac{t}{e^t-1}\right)^{n+1}e^{(n+1)xt}\right)\Bigg|_{t=0} = B_n^{(n+1)}((n+1)x),$$

where we have again used (2.3). This, together with (3.2), (3.1) and (2.7) gives the intermediate result

$$\sum_{m=1}^{n}\binom{n+1}{m}T_m(n;x) = B_n^{(n+1)}((n+1)x). \tag{3.3}$$

Finally we use a well-known explicit formula for $B_n^{(n+1)}(x)$ (see, e.g., [15, p. 130]), which immediately gives

$$B_n^{(n+1)}((n+1)x) = \prod_{j=1}^{n} \big((n+1)x - j\big). \tag{3.4}$$

With (3.3), this completes the proof of (1.9) and of Theorem 1.

4 Further Remarks

1. If we set $m = n+1$ in (2.5), we get

$$S_{n+1}(n;x) = \sum_{j=1}^{n} \binom{n+1}{j} T_j(n;x) + T_{n+1}(n;x).$$

From (2.1) it is clear that $T_{n+1}(n;x) = 0$ since it is an empty sum. Therefore (3.3) and (3.4) lead to the following consequence concerning the convolution sum defined in (2.2).

Corollary 2. *For any $n \ge 1$ we have*

$$S_{n+1}(n;x) = \prod_{j=1}^{n} \big((n+1)x - j\big).$$

2. Whenever a result on Bernoulli polynomials is obtained, it is a natural question to ask whether there are analogues for Euler polynomials. The *Euler polynomial of order m and degree k*, $E_k^{(m)}(x)$, is defined by the generating function

$$\left(\frac{2}{e^t+1}\right)^m e^{xt} = \sum_{k=0}^{\infty} E_k^{(m)}(x) \frac{t^k}{k!}, \qquad |t| < \pi,$$

and the (ordinary) *Euler polynomial* of degree k by $E_k(x) := E_k^{(1)}(x)$. Various properties, including recurrence relations, of these polynomials can be found, e.g., in [15, p. 143ff].

If we replace each "B" by "E" in (1.7) and (1.9), then all details of the proof carry through, up to the equivalent of (3.3). We therefore get the following result.

Theorem 2. *For any integer $n \ge 1$ we have*

$$\sum_{m=1}^{n} \binom{n+1}{m} \sum_{\substack{j_1,\ldots,j_m \ge 1 \\ j_1+\cdots+j_m = n}} \prod_{i=1}^{m} \frac{E_{j_i}(x)}{j_i!} = \frac{1}{n!} E_n^{(n+1)}((n+1)x). \tag{4.1}$$

In contrast to Theorem 1, however, the right-hand side of (4.1) does not have an easy evaluation. The first few polynomials $E_n^{(n+1)}(x)$ are listed in Table 1.

n	B_n	$B_n(x)$	$E_n^{(n+1)}(x)$
0	1	1	1
1	$-1/2$	$x - \frac{1}{2}$	$x - 1$
2	$1/6$	$x^2 - x + \frac{1}{6}$	$x^2 - 3x + \frac{3}{2}$
3	0	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^3 - 6x^2 + 9x - 2$
4	$-1/30$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$	$x^4 - 10x^3 + 30x^2 - 25x - \frac{5}{2}$
5	0	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$	$x^5 - 15x^4 + 75x^3 - 135x^2 + \frac{75}{2}x + \frac{99}{2}$
6	$1/42$	$x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$	$x^6 - 21x^5 + \frac{315}{2}x^4 - 490x^3 + \frac{945}{2}x^2$ $+294x - 357$

Table 1: B_n, $B_n(x)$ and $E_n^{(n+1)}(x)$ for $0 \le n \le 6$.

We finish by deriving an analogue of Corollary 1 for Euler numbers. The kth *Euler number of order* n is defined by

$$E_k^{(n)} := 2^k E_k^{(n)}(\tfrac{n}{2});$$

see, e.g., [15, p. 143]. In particular, this implies

$$E_k(\tfrac{1}{2}) = 2^{-k} E_k, \qquad E_n^{(n+1)}(\tfrac{n+1}{2}) = 2^{-n} E_n^{(n+1)},$$

where E_k is the kth (ordinary) *Euler number*. Setting $x = \frac{1}{2}$ in (4.1) and multiplying both sides by $2^n n!$, we get the following identity, written in a form analogous to (1.9).

Corollary 3. *For any integer $n \ge 1$ we have*

$$\sum_{m=1}^{n} \binom{n+1}{m} \sum_{\substack{j_1,\dots,j_m \ge 1 \\ j_1+\cdots+j_m=n}} \binom{n}{j_1,\dots,j_m} E_{j_1} \cdots E_{j_m} = E_n^{(n+1)}.$$

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