DERIVATIVES AND SPECIAL VALUES OF HIGHER-ORDER TORNHEIM ZETA FUNCTIONS

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Abstract. We study analytic properties of the higher-order Tornheim zeta function, defined by a certain n-fold series ($n \geq 2$) in $n+1$ complex variables. In particular, we consider the function $\omega_{n+1}(s)$, obtained by setting all variables equal to s. Using a free-parameter method due to Crandall, we first give an alternative proof of the trivial zeros of $\omega_{n+1}(s)$ and evaluate $\omega_{n+1}(0)$. Our main result, however, is the evaluation of $\omega'_{n+1}(0)$ for any $n \geq 2$. This is again achieved by using Crandall's method, and it generalizes recent results in the cases $n = 2, 3$. Properties of Bernoulli numbers and of higher-order Bernoulli numbers and polynomials play an important role throughout this paper.

1. Introduction

One of the best-known multiple zeta functions is the double series

$$\mathcal{W}(r,s,t) := \sum_{m,n\geq 1} \frac{1}{m^r}\frac{1}{n^s}\frac{1}{(m+n)^t}, \tag{1.1}$$

which converges for all complex r, s, t with $\mathrm{Re}(r+t) > 1$, $\mathrm{Re}(s+t) > 1$, and $\mathrm{Re}(r+s+t) > 2$. This series was first investigated for positive integers r, s, t by Tornheim [22] in 1950, and independently by Mordell [13] in 1958 for the special case $r = s = t$. It is therefore often called a Tornheim (double) sum or Mordell-Tornheim (double) sum or series. Furthermore, Witten [23] studied a wider class of such series, which Zagier [24] called Witten zeta functions, a name sometimes also attached to (1.1).

Inspired by a preprint of Romik's paper [21], J. M. Borwein and the first author [3] studied the analytic properties of the function $\omega_3(s) := \mathcal{W}(s,s,s)$, with emphasis on the values $\omega_3(0)$ and $\omega'_3(0)$. Some of these results were earlier and independently obtained by Onodera [17, 18].

It is the purpose of this paper to extend these results to a multi-dimensional analogue of the Tornheim zeta function (1.1) which can be defined, for $n \geq 2$, by

$$\mathcal{W}(r_1,\ldots,r_n,t) := \sum_{m_1,\ldots,m_n\geq 1} \frac{1}{m_1^{r_1}\cdots m_n^{r_n}(m_1+\cdots+m_n)^t}, \tag{1.2}$$

where $r_1,\ldots,r_n$ and t are complex variables with (initially) $\mathrm{Re}(r_j) > 1$ for $j = 1,\ldots,n$ and $\mathrm{Re}(t) > 0$. This class of multiple series was studied by several authors;

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see, e.g., [1], [10], or [17]. In analogy to $\omega_3(s)$ we also define

$$\omega_{n+1}(s) := \mathcal{W}(s,\dots,s,s). \tag{1.3}$$

Onodera [17] showed that

$$\omega_{n+1}(0) = \frac{(-1)^n}{n+1} \qquad (n \ge 1). \tag{1.4}$$

In this paper we provide a different proof of this identity, while our main result is an evaluation of $\omega'_{n+1}(0)$ for all $n \ge 2$, generalizing

$$\omega'_3(0) = \log(2\pi), \tag{1.5}$$

which was obtained in [3] and [18]. In fact, we are going to prove the following result, which involves derivatives of the Riemann zeta function $\zeta(z)$ which, as is well know, can be analytically continued to the whole complex plane, with the exception of a simple pole at $z = 1$.

Theorem 1.1. *For any integer $n \ge 2$ we have*

$$\omega'_{n+1}(0) = (-1)^n \log(2\pi) + \frac{2}{(n-1)!} \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} s(n,2j+1)\zeta'(-2j), \tag{1.6}$$

where $s(n,k)$ are the Stirling numbers of the first kind.

We recall that the (signed) Stirling numbers of the first kind can be defined by the relation

$$(x-n+1)_n = \sum_{j=0}^{n} s(n,j)x^j, \tag{1.7}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol; see, e.g., [16, Sect. 26.8].

Theorem 1.1 is not new. Independently of the prsesent paper, Onodera in another, more recent, publication [19] obtained generalizations of (1.4) and (1.6); for details, se the final Section 8 below. However, our method is quite different from Onodera's, and a second purpose of the present paper is to give another application of Crandall's free-parameter method.

Returning to the identity (1.6), we see that for $n = 2$ it reduces to (1.5). Furthermore, since $s(n,n) = 1$ and $s(n,n-1) = -n(n-1)/2$ for all $n \ge 0$, as well as $s(5,3) = 35$ and $s(6,3) = -225$ (see again [16, Sect. 26.8]), the next four cases are

$$\begin{aligned}
\omega'_4(0) &= -\log(2\pi) + \zeta'(-2),\\
\omega'_5(0) &= \log(2\pi) - 2\zeta'(-2),\\
\omega'_6(0) &= -\log(2\pi) + \tfrac{35}{12}\zeta'(-2) + \tfrac{1}{12}\zeta'(-4),\\
\omega'_7(0) &= \log(2\pi) - \tfrac{15}{4}\zeta'(-2) - \tfrac{1}{4}\zeta'(-4).
\end{aligned}$$

All these identities can be slightly rewritten if we use the known evaluations

$$\zeta'(0) = -\frac{1}{2}\log(2\pi) \quad \text{and} \quad \zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}; \tag{1.8}$$

see, e.g., [16, Sect. 25.6(ii)]. It should also be mentioned that Bailey and Borwein [2] obtained experimentally the evaluations of $\omega'_{n+1}(0)$ for all $n \le 18$. Their values are in agreement with Theorem 1.1, which was obtained later.

This paper is structured as follows. We begin with a section on some important special functions that will be required, and derive some general results based on Crandall's free parameter method. In Section 3, Crandall's method is used to evaluate $\omega'_{n+1}(s)$ at all nonpositive integers. The remainder of the paper is then devoted to the proof of Theorem 1.1, beginning with the basic set-up in Section 4. The following two sections then contain a sequence of technical lemmas, and everything is put together in the short Section 7. Finally, Onodera's recent results will be given in Section 8.

2. Some preliminaries

As in the paper [3], which dealt with the case $n = 2$, our main tool will be an expansion of $\mathcal{W}(r_1, \dots, r_n, t)$, applying a free parameter method due to Crandall. For this purpose we use two important special functions, namely polylogarithms and the incomplete gamma function.

The *polylogarithm* of *order* s is defined by

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \tag{2.1}$$

For each fixed $s \in \mathbb{C}$, the series (2.1) defines an analytic function of z for $|z| < 1$; in particular, $\mathrm{Li}_0(z) = z/(1-z)$ and $\mathrm{Li}_1(z) = -\log(1-z)$. The series also converges when $|z| = 1$, provided that $\mathrm{Re}(s) > 1$; for instance, $\mathrm{Li}_s(1) = \zeta(s)$, the Riemann zeta function. While the polylogarithm satisfies numerous other properties (see, e.g., [16, Sect. 25.12]), we mainly require the following representation.

Lemma 2.1. *For any $s \in \mathbb{C}$ not a positive integer, and for $|\log z| < 2\pi$, we have*

$$\mathrm{Li}_s(z) = \sum_{m=0}^{\infty} \zeta(s-m) \frac{\log^m z}{m!} + \Gamma(1-s)(-\log z)^{s-1}. \tag{2.2}$$

This identity can be found in [7, pp. 27–30], in a slightly more general form. The multiple Tornheim zeta function now enters through the following integral representation.

Lemma 2.2. *For $r_1, \dots, r_n > 1$ and $t > 0$ we have*

$$\Gamma(t)\mathcal{W}(r_1, \dots, r_n, t) = \int_0^{\infty} x^{t-1} \prod_{j=1}^{n} \mathrm{Li}_{r_j}(e^{-x}) dx \tag{2.3}$$

For $n = 2$, this identity was first given, without proof, in [4] as identity (6.2), and a proof was later provided in [3]. The proof of (2.3) is similar to that of the special case in [3]; we give it here for the sake of completeness.

Proof of Lemma 2.2. We use Euler's integral for $\Gamma(s)$, namely

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt \qquad (\mathrm{Re(s)} > 0) \tag{2.4}$$

and substitute $t = mx$, so that

$$\Gamma(s) = m^s \int_0^{\infty} e^{-mx} x^{s-1} dx.$$

Replacing s with t and m with $m_1+\cdots+m_n$, we get for $\mathrm{Re}(t)>0$,

$$\frac{1}{(m_1+\cdots+m_n)^t}=\frac{1}{\Gamma(t)}\int_0^\infty e^{-(m_1+\cdots+m_n)x}x^{t-1}dx. \tag{2.5}$$

If we substitute (2.5) into (1.2) and change the order of summation and integration, we get

$$\mathcal{W}(r_1,\ldots,r_n,t)=\frac{1}{\Gamma(t)}\int_0^\infty x^{t-1}\bigg(\sum_{m_1=1}^\infty\frac{e^{-m_1x}}{m_1^{r_1}}\bigg)\cdots\bigg(\sum_{m_n=1}^\infty\frac{e^{-m_nx}}{m_n^{r_n}}\bigg)dx.$$

Finally, with (2.1) this gives (2.3). $\square$

The main result of this section has its origin in a method of Crandall in the case $n=2$, which provided an expansion of $\mathcal{W}(r_1,r_2,t)$ with a free parameter $\theta>0$; see [3] for further details. Here we extend this to arbitrary $n\geq 2$. In what follows, we require the *incomplete Gamma function*, defined by

$$\Gamma(a,z):=\int_z^\infty y^{a-1}e^{-y}\mathrm{d}y. \tag{2.6}$$

Theorem 2.3. *Let $n\geq 2$ be an integer and $r_1,\ldots,r_n,t$ complex variables with $r_j\notin\mathbb{N}$ for $1\leq j\leq n$. Then for any real θ with $0<\theta<2\pi$ we have*

$$\begin{aligned}\Gamma(t)\mathcal{W}(r_1,\ldots,r_n,t)&=\sum_{m_1,\ldots,m_n\geq 1}\frac{\Gamma(t,(m_1+\cdots+m_n)\theta)}{m_1^{r_1}\cdots m_n^{r_n}(m_1+\cdots+m_n)^t}\\&+\sum_{\substack{\{a_1,\ldots,a_k\}\\ \subseteq\{1,\ldots,n\}}}\bigg(\sum_{u_{a_1},\ldots,u_{a_k}\geq 0}\frac{\theta^w}{w}\prod_{i=1}^n\Gamma(1-r_i)\prod_{j=1}^k\frac{(-1)^{u_{a_j}}\zeta(r_{a_j}-u_{a_j})}{(u_{a_j})!\Gamma(1-r_{a_j})}\bigg)\\&+\frac{\theta^w}{w}\prod_{i=1}^n\Gamma(1-r_i),\end{aligned}\tag{2.7}$$

where the final term is considered to be the case $k=0$, and

$$w=t-(n-k)+\sum_{j=1}^k(u_{a_j}-r_{a_j})+\sum_{i=1}^n r_i.$$

Proof. From the definition (2.6) we have, for any $\theta>0$,

$$\Gamma(t,(m_1+\cdots+m_n)\theta)=\int_{(m_1+\cdots+m_n)\theta}^\infty y^{t-1}e^{-y}dy.$$

The substitution $y=(m_1+\cdots+m_n)x$ then yields

$$\int_\theta^\infty x^{t-1}e^{-(m_1+\cdots+m_n)x}dx=\frac{\Gamma(t,(m_1+\cdots+m_n)\theta)}{(m_1+\cdots+m_n)^t}. \tag{2.8}$$

Using (2.5) and the definition (1.2), we get

$$\Gamma(t)\mathcal{W}(r_1,\ldots,r_n,t)=\sum_{m_1,\ldots,m_n\geq 1}\frac{1}{m_1^{r_1}\cdots m_n^{r_n}}\int_0^\infty x^{t-1}e^{-(m_1+\cdots+m_n)x}dx,$$

and splitting the integral into two parts,

$$\begin{aligned}
(2.9)\qquad &\Gamma(t)\mathcal{W}(r_1,\dots,r_n,t)\\
&=\sum_{m_1,\dots,m_n\geq 1}\frac{1}{m_1^{r_1}\cdots m_n^{r_n}}\left(\int_\theta^\infty+\int_0^\theta\right)x^{t-1}e^{-(m_1+\cdots+m_n)x}dx\\
&=\sum_{m_1,\dots,m_n\geq 1}\frac{\Gamma(t,(m_1+\cdots+m_n)\theta)}{n_1^{r_1}\cdots m_n^{r_n}(m_1+\cdots+m_n)^t}+\int_0^\theta x^{t-1}\prod_{j=1}^n\mathrm{Li}_{r_j}(e^{-x})dx,
\end{aligned}$$

where we have used (2.8) and (2.1), respectively. Now we apply the representation (2.2) to obtain

$$\begin{aligned}
x^{t-1}\prod_{j=1}^n\mathrm{Li}_{r_j}(e^{-x}) &= x^{t-1}\prod_{j=1}^n\Big(\sum_{u_j=0}^\infty\zeta(r_j-u_j)\frac{(-x)^{u_j}}{u_j!}+\Gamma(1-r_j)x^{r_j-1}\Big)\\
&=\sum_{\substack{\{a_1,\dots,a_k\}\\ \subseteq\{1,\dots,n\}}}\left(\sum_{u_{a_1},\dots,u_{a_k}\geq 0}x^{w-1}\prod_{i=1}^n\Gamma(1-r_i)\prod_{j=1}^k\frac{(-1)^{u_{a_j}}\zeta(r_{a_j}-u_{a_j})}{(u_{a_j})!\Gamma(1-r_{a_j})}\right)\\
&\quad+x^{w-1}\prod_{i=1}^n\Gamma(1-r_i),
\end{aligned}$$

with w as in the statement of the theorem, and where the final term on the right is considered to be the case $k=0$. Finally, integrating this last identity over x from 0 to θ and substituting into (2.9) leads to the desired identity (2.7). □

3. Special values of $\omega_{n+1}(s)$

As a first application of Theorem 2.3 we provide an alternative proof of (1.4). In fact, we are going to prove the following more general result of Onodera [17].

Theorem 3.1 (Onodera). *For any integer $n\geq 2$ we have*

$$\omega_{n+1}(0)=\frac{(-1)^n}{n+1},\qquad \omega_{n+1}(-\nu)=0,\quad \nu=1,2,3,\dots \tag{3.1}$$

For the proof of this result, and also for later in this paper, we require the Bernoulli numbers, which can be defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^\infty B_n\frac{t^n}{n!}\qquad (|t|<2\pi). \tag{3.2}$$

The first few Bernoulli numbers are $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, and $B_{2k+1}=0$ for $k\geq 1$. The Bernoulli polynomials can be defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}\qquad (|t|<2\pi). \tag{3.3}$$

Clearly, $B_n(0)=B_n$ for all $n\geq 1$. For properties of Bernoulli numbers and polynomials see, e.g., [16, Ch. 24].

Proof of Theorem 3.1. We set $r_1 = \cdots = r_n = t$ and replace this common variable by $s-\nu$, where $\nu \geq 0$ is an arbitrary integer. Then (2.7) simplifies to

$$\begin{aligned}\Gamma(s-\nu)\omega_{n+1}(s-\nu) &= \sum_{m_1,\ldots,m_n\geq 1} \frac{\Gamma(s-\nu,(m_1+\cdots+m_n)\theta)}{(m_1\cdots m_n(m_1+\cdots+m_n))^{s-\nu}} \\ &+ \sum_{k=1}^{n}\binom{n}{k}\Gamma(1-s+\nu)^{n-k}\sum_{u_1,\ldots,u_k\geq 0}\left(\prod_{j=1}^{k}\frac{(-1)^{u_j}\zeta(s-\nu-u_j)}{u_j!}\right)\frac{\theta^w}{w} \\ &+ \Gamma(1-s+\nu)^n\frac{\theta^w}{w},\end{aligned} \tag{3.4}$$

where $w = u_1+\cdots+u_k+(n-k+1)(s-\nu)-(n-k)$, and the final term on the right is once again considered to be the case $k=0$.

Now w is a multiple of s exactly when

$$u_1+\cdots+u_k = (\nu+1)(n-k)+\nu, \tag{3.5}$$

in which case $w=(n-k+1)s$. We also use the well-known evaluation

$$\zeta(-\mu) = (-1)^\mu\frac{B_{\mu+1}}{\mu+1} \qquad (\mu = 0,1,2,\ldots), \tag{3.6}$$

where B_k is the kth Bernoulli number defined in (3.2). Therefore, if we multiply both sides of (3.4) by s and let $s\to 0$, then all terms on the right of (3.4) vanish, with the exception of

$$R_n(\nu) := \sum_{k=1}^{n}\binom{n}{k}\nu!^{n-k}\sideset{}{^*}\sum_{u_1,\ldots,u_k\geq 0}\frac{(-1)^\nu}{n-k+1}\prod_{j=1}^{k}\frac{B_{\nu+1+u_j}}{u_j!(\nu+1+u_j)},$$

where $\sum^*$ indicates that the sum is taken over all $u_1,\ldots,u_k\geq 0$ that satisfy (3.5). Next, it is easy to verify that

$$\binom{n}{k}\frac{1}{n-k+1} = \frac{1}{n+1}\binom{n+1}{k},$$

so that

$$R_n(\nu) = \frac{(-1)^\nu}{n+1}\sum_{k=1}^{n}\binom{n+1}{k}\nu!^{n-k}\sideset{}{^*}\sum_{u_1,\ldots,u_k\geq 0}\prod_{j=1}^{k}\frac{B_{\nu+1+u_j}}{u_j!(\nu+1+u_j)}. \tag{3.7}$$

On the other hand, it is a well-known property of the gamma function that for integers $\nu\geq 0$,

$$\lim_{s\to 0} s\Gamma(s-\nu) = \frac{(-1)^\nu}{\nu!},$$

so that the left-hand side of (3.4) becomes

$$\lim_{s\to 0} s\Gamma(s-\nu)\omega_{n+1}(s-\nu) = \frac{(-1)^\nu}{\nu!}\omega_{n+1}(-\nu). \tag{3.8}$$

Returning to (3.7), we first consider the case $\nu=0$ and note that the condition (3.5) simplifies to $u_1+\cdots+u_k = n-k$. Shifting summation in the k-fold sum on the right of (3.7) gives

$$R_n(0) = \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k}\sum_{\substack{u_1,\ldots,u_k\geq 1\\ u_1+\cdots+u_k=n}}\prod_{j=1}^{k}\frac{B_{u_j}}{u_j!}. \tag{3.9}$$

The sum on the right of (3.9) has been evaluated in [5] as $(-1)^n$; this and (3.8) lead to the first identity in (3.1).

When $\nu \geq 1$, the situation is somewhat different, and as a consequence of another result on Bernoulli polynomials [6], we obtain from (3.7) that $R_n(\nu) = 0$. This, together with (3.8), proves the second identity in (3.1). □

Remark. When ν is odd, then by (3.5) at least one u_j must be odd, so at least one subscript $\nu + 1 + u_j$ is odd and is at least 3. This means that all products on the right of (3.7) vanish, and thus $R_n(\nu) = 0$. Therefore the results in [6] are only needed where ν is even.

4. Proof of Theorem 1.1

1. The main tool for proving Theorem 1.1 is Theorem 2.3, in the special case

$$\begin{aligned}\Gamma(s)\omega_{n+1}(s) &= \sum_{m_1,\dots,m_n\geq 1} \frac{\Gamma(s,(m_1+\cdots+m_n)\theta)}{(m_1\cdots m_n(m_1+\cdots+m_n))^s} \\ &+ \sum_{k=1}^{n}\binom{n}{k}\Gamma(1-s)^{n-k}\sum_{u_1,\dots,u_k\geq 0}\left(\prod_{j=1}^{k}\frac{(-1)^{u_j}\zeta(s-u_j)}{u_j!}\right)\frac{\theta^w}{w} \\ &+ \Gamma(1-s)^n\frac{\theta^{(n+1)s-n}}{(n+1)s-n},\end{aligned} \tag{4.1}$$

where $w = u_1 + \cdots + u_k + (n-k+1)s - (n-k)$; this identity is the same as (3.4) with $\nu = 0$.

We follow the same strategy as in [3], namely multiplying both sides of (4.1) by s, followed by taking the derivative of both sides with respect to s, at $s = 0$. For greater transparency of the proof, we introduce the following notation:

$A(s)$: the left-hand side of (4.1);
$B(s)$: the first term on the right of (4.1);
$C(s)$: the sum of all those terms in the second line of (4.1) that have a pole at $s = 0$;
$D(s)$: the sum of all other terms in the second and third lines of (4.1).

2. We begin with $A(s)$, the easiest term to deal with. We require the first few terms of the Laurent expansion of the Gamma function about the origin, which can be written as

$$s\Gamma(s) = 1 - \gamma s + O(s^2), \tag{4.2}$$

where γ is the Euler-Mascheroni constant. We then have

$$\begin{aligned}\frac{d}{ds}[sA(s)]_{s=0} &= \frac{d}{ds}\left[s\Gamma(s)\omega_{n+1}(s)\right]_{s=0} \\ &= \omega_{n+1}'(0)\cdot\lim_{s\to 0}(s\Gamma(s)) + \frac{d}{ds}[s\Gamma(s)]_{s=0}\cdot\omega_{n+1}(0),\end{aligned}$$

and using (4.2) twice, as well as (3.1), we get

$$\frac{d}{ds}[sA(s)]_{s=0} = \omega_{n+1}'(0) + (-1)^{n+1}\frac{\gamma}{n+1}. \tag{4.3}$$

3. To deal with the term $B(s)$, we first note that

$$\frac{d}{ds}[sB(s)]_{s=0} = sB'(s)\big|_{s=0} + B(s)\big|_{s=0} = B(0). \tag{4.4}$$

With this in mind, we state and prove the following lemma.

Lemma 4.1. *For any $\theta > 0$ we have*

$$\sum_{m_1,\dots,m_n\geq 1} \Gamma(0,(m_1+\cdots+m_n)\theta) = \int_1^\infty \frac{du}{(e^{\theta u}-1)^n u}. \tag{4.5}$$

Proof. We use the identity

$$\Gamma(0,x) = E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt \quad (x>0), \tag{4.6}$$

where $E_1(x)$ is the exponential integral; see, e.g., (6.2.1) and (6.11.1) in [16]. Setting $x = (m_1+\cdots+m_n)\theta$ and then making the substitution $t = (m_1+\cdots+m_n)\theta u$ in the integral on the right of (4.6), we get

$$\Gamma(0,(m_1+\cdots+m_n)\theta) = \int_1^\infty e^{-(m_1+\cdots+m_n)\theta u}\frac{du}{u},$$

which after interchanging summation and integration yields

$$\sum_{m_1,\dots,m_n\geq 1} \Gamma(0,(m_1+\cdots+m_n)\theta) = \int_1^\infty \left(\sum_{m_1,\dots,m_n\geq 1} e^{-(m_1+\cdots+m_n)\theta u}\right)\frac{du}{u}. \tag{4.7}$$

Now, since

$$\begin{aligned}\sum_{m_1,\dots,m_n\geq 1} e^{-(m_1+\cdots+m_n)\theta u} &= \prod_{j=1}^n \sum_{m_j\geq 1} e^{-m_j\theta u}\\ &= \left(\frac{e^{-\theta u}}{1-e^{-\theta u}}\right)^n = \frac{1}{(e^{\theta u}-1)^n},\end{aligned}$$

the identity (4.7) now gives (4.5). □

With (4.4) and (4.5) we therefore get

$$\frac{d}{ds}[sB(s)]_{s=0} = \int_1^\infty \frac{du}{(e^{\theta u}-1)^n u}. \tag{4.8}$$

4. To determine the summands that make up $C(s)$, we consider

$$w = u_1+\cdots+u_k+(n-k+1)s-(n-k)$$

(see (4.1)) and note that w is a multiple of s if and only if

$$u_1+\cdots+u_k = n-k \quad (1\leq k\leq n); \tag{4.9}$$

in this case $w = (n-k+1)s$, thus giving the desired pole at $s=0$. We then get from the second row of (4.1),

$$sC(s) = \sum_{k=1}^n \binom{n}{k}\Gamma(1-s)^{n-k}\frac{(-1)^{n-k}}{n-k+1}\theta^{(n-k+1)s}\sum\nolimits^{*}\prod_{j=1}^k \frac{\zeta(s-u_j)}{u_j!}, \tag{4.10}$$

where $\sum^*$ indicates summation over all nonnegative $u_1, \dots, u_k$ satisfying the condition (4.9). We now consider the terms

$$C_k(s) := \Gamma(1-s)^{n-k}\theta^{(n-k+1)s}\prod_{j=1}^{k}\zeta(s-u_j), \tag{4.11}$$

so that

$$sC(s) = \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{n-k}}{n-k+1}\sum{}^{*}\left(\prod_{j=1}^{k}\frac{1}{u_j!}\right)C_k(s). \tag{4.12}$$

Our goal now is to find the derivative of $sC(s)$ at $s=0$, which means evaluating $C_k'(0)$ for $k=1,2,\dots,n$. We begin with $k=n$. In this case we have $u_j=0$ for all $j=1,\dots,n$, by (4.9). Then by (4.11) we get

$$\begin{aligned}C_n'(0) &= \frac{d}{ds}\left[\theta^s\zeta(s)^n\right]_{s=0}\\ &= \left[\theta^s n\zeta(s)^{n-1}\zeta'(s) + \theta^s\log(\theta)\zeta(s)^n\right]_{s=0}\\ &= \zeta(0)^{n-1}\left[n\zeta'(0)+\log(\theta)\zeta(0)\right],\end{aligned}$$

and with (1.8) and the well-known evaluation $\zeta(0)=-1/2$ we get

$$C_n'(0) = (-\tfrac{1}{2})^n(n\cdot\log(2\pi)+\log(\theta)). \tag{4.13}$$

When $1\le k\le n-1$, then by (4.11) we get

$$\begin{aligned}C_k'(0) = &\left[(n-k)\Gamma(1-s)^{n-k-1}(-\Gamma'(1-s))\theta^{(n-k+1)s}\prod_{j=1}^{k}\zeta(s-u_j)\right]_{s=0}\\ &+\left[\Gamma(1-s)^{n-k}\theta^{(n-k+1)s}(n-k+1)\log\theta\prod_{j=1}^{k}\zeta(s-u_j)\right]_{s=0}\\ &+\left[\Gamma(1-s)^{n-k}\theta^{(n-k+1)s}\sum_{\ell=1}^{k}\left(\prod_{j=1}^{k}\zeta(s-u_j)\frac{\zeta'(s-u_\ell)}{\zeta(s-u_\ell)}\right)\right]_{s=0},\end{aligned}$$

and thus, using $\Gamma'(1)=-\gamma$ (see, e.g., (5.4.11) in [16]), we have

$$\begin{aligned}C_k'(0) = &\big((n-k)\gamma+(n-k+1)\log\theta\big)\prod_{j=1}^{k}\zeta(-u_j)\\ &+\sum_{j=1}^{k}\zeta(-u_1)\cdots\zeta'(-u_j)\cdots\zeta(-u_k),\end{aligned} \tag{4.14}$$

recalling that $u_1,\dots,u_k$ are subject to (4.9). If in (4.14) we set $k=n$, then $u_1=\dots=u_n=0$, and the right-hand side of (4.14) reduces to

$$(\log\theta)\zeta(0)^n + n\zeta'(0)\zeta(0)^{n-1} = (-\tfrac{1}{2})^n(\log\theta + n\log(2\pi)),$$

where we have used the fact that $\zeta(0)=-1/2$ and also the first identity in (1.8). But this means that (4.14) holds for all k, $1\le k\le n$. Before we substitute this into (4.12), we apply (3.6) to get

$$\Big(\prod_{j=1}^{k}\frac{1}{u_j!}\Big)\Big(\prod_{j=1}^{k}\zeta(-u_j)\Big)=(-1)^{n-k}\prod_{j=1}^{k}\frac{B_{u_j+1}}{(u_j+1)!},\tag{4.15}$$

where we have used (4.9) to obtain the power of -1. Similarly,

$$\begin{aligned}\Big(\prod_{j=1}^{k}\frac{1}{u_j!}\Big)&\Big(\sum_{j=1}^{k}\zeta(-u_1)\cdots\zeta'(-u_j)\cdots\zeta(-u_k)\Big)\\ &=(-1)^{n-k}\sum_{j=1}^{k}\frac{B_{u_1+1}}{(u_1+1)!}\cdots\frac{(-1)^{u_j}\zeta'(-u_j)}{u_j!}\cdots\frac{B_{u_k+1}}{(u_k+1)!}.\end{aligned}\tag{4.16}$$

Combining (4.15) and (4.16) with (4.14) and (4.12), we then get

(4.17)

$$\begin{aligned}\frac{d}{ds}[sC(s)]_{s=0}=\sum_{k=1}^{n}\frac{\binom{n}{k}}{n-k+1}{\sum}^*\Big\{&\big((n-k)\gamma+(n-k+1)\log\theta\big)\prod_{j=1}^{k}\frac{B_{u_j+1}}{(u_j+1)!}\\ &+\sum_{j=1}^{k}\frac{B_{u_1+1}}{(u_1+1)!}\cdots\frac{(-1)^{u_j}\zeta'(-u_j)}{u_j!}\cdots\frac{B_{u_k+1}}{(u_k+1)!}\Big\},\end{aligned}$$

where $\sum^*$ indicates the same summation as in (4.10). We'll return to (4.17) later.

5. Finally, to deal with the derivative of $sD(s)$ at $s=0$, we note that, just as in (4.4), we only need to evaluate $D(0)$. With (4.1) and (3.6) we get

$$D(0)=\sum_{k=1}^{n}\binom{n}{k}\sum_{u_1,\dots,u_k\ge 0}\Big(\prod_{j=1}^{k}\frac{B_{u_j+1}}{(u_j+1)!}\Big)\frac{\theta^{\lambda-n}}{\lambda-n}+\frac{\theta^{-n}}{-n},\tag{4.18}$$

where $\lambda:=u_1+\cdots+u_k+k$. Keeping in mind that eventually we wish to take the limit as $\theta\to 0$, we disregard the cases where $\lambda\ge n$ and denote the remaining sum by $\overline{D}(0)$; also note that $\lambda=n$ is equivalent to (4.9), so this case has already been dealt with. Now, collecting the terms in (4.18) that belong to a fixed λ, $1\le\lambda\le n-1$, we get

$$\begin{aligned}\overline{D}(0)&=\sum_{\lambda=1}^{n-1}\Big(\sum_{k=1}^{\lambda}\binom{n}{k}\sum_{\substack{u_1,\dots,u_k\ge 0\\ u_1+\cdots+u_k=\lambda-k}}\prod_{j=1}^{k}\frac{B_{u_j+1}}{(u_j+1)!}\Big)\frac{\theta^{\lambda-n}}{\lambda-n}+\frac{\theta^{-n}}{-n}\\ &=\sum_{\lambda=1}^{n-1}\Big(\sum_{k=1}^{\lambda}\binom{n}{k}\sum_{\substack{u_1,\dots,u_k\ge 1\\ u_1+\cdots+u_k=\lambda}}\prod_{j=1}^{k}\frac{B_{u_j}}{u_j!}\Big)\frac{\theta^{\lambda-n}}{\lambda-n}+\frac{\theta^{-n}}{-n}.\end{aligned}\tag{4.19}$$

In order to simplify this expression, we define for positive integer n and λ,

$$S_1(n,\lambda):=\sum_{k=1}^{\lambda}\binom{n}{k}\sum_{\substack{u_1,\dots,u_k\ge 1\\ u_1+\cdots+u_k=\lambda}}\prod_{j=1}^{k}\frac{B_{u_j}}{u_j!};\tag{4.20}$$

this expression will also occur elsewhere in the proof of Theorem 1.1. The identity (4.18), along with the fact that $D(0) = \overline{D}(0) + O(\theta)$, can now be written as

$$\frac{d}{ds}[sD(s)]_{s=0} = D(0) = \sum_{\lambda=1}^{n-1} S_1(n,\lambda)\frac{\theta^{\lambda-n}}{\lambda-n} + \frac{\theta^{-n}}{-n} + O(\theta). \tag{4.21}$$

5. Evaluating the integral in Lemma 4.1

After the outline of the proof of Theorem 1.1 in the previous section, we now need to further evaluate the expressions in (4.8), (4.17), and (4.21), and combine them with the expression in (4.3). This will be done in this and the following sections, beginning with the evaluation of the integral on the right of (4.8).

For the next lemma, and indeed for much of the remainder of this paper, we require the *Bernoulli polynomials of order* n, defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^n e^{tx} = \sum_{j=0}^{\infty} B_j^{(n)}(x)\frac{t^j}{j!} \qquad (|t| < 2\pi). \tag{5.1}$$

Although this definition makes sense for a wider class of n, here we will restrict n to positive integers. For $n = 1$, the identity (5.1) is the generating function of the ordinary Bernoulli polynomials, and thus $B_j^{(1)}(x) = B(x)$; see (3.3). The Bernoulli *numbers* of order n are defined by $B_j^{(n)} = B_j^{(n)}(0)$. Later in this section we also require the *harmonic numbers*

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \qquad (n \geq 1). \tag{5.2}$$

We now evaluate the integral on the right of (4.8) through a succession of lemmas.

Lemma 5.1. *Let $0 < R < 2\pi$ be fixed. Then for $0 < \theta \leq R$ we have*

$$\begin{aligned}\int_1^{\infty} \frac{du}{(e^{\theta u}-1)^n u} &= \sum_{j=n+1}^{\infty} \frac{B_j^{(n)}}{j!(j-n)} + \int_1^{\infty} \frac{dt}{t(e^t-1)^n} \\ &\quad + \sum_{j=0}^{n-1} \frac{B_j^{(n)}}{j!}\frac{1-\theta^{j-n}}{j-n} - \frac{B_n^{(n)}}{n!}\log\theta + O(\theta).\end{aligned} \tag{5.3}$$

Proof. We denote the integral on the left by I and begin by making the substitution $t = \theta u$, and then split I into two parts:

$$I = \int_\theta^{\infty} \frac{dt}{t(e^t-1)^n} = \int_\theta^1 \frac{1}{t^{n+1}}\left(\frac{t}{e^t-1}\right)^n dt + \int_1^{\infty} \frac{dt}{t(e^t-1)^n}. \tag{5.4}$$

We denote the first integral on the right by $I_1(\theta)$ and use (5.1) with $x = 0$ to obtain

(5.5)

$$\begin{aligned}I_1(\theta) &= \int_\theta^1 \left(\frac{1}{t^{n+1}}\sum_{j=0}^{\infty} B_j^{(n)}\frac{t^j}{j!}\right)dt \\ &= \sum_{j=0}^{n-1} \frac{B_j^{(n)}}{j!}\frac{1-\theta^{j-n}}{j-n} - \frac{B_n^{(n)}}{n!}\log\theta + \sum_{j=n+1}^{\infty} \frac{B_j^{(n)}}{j!(j-n)} - \sum_{j=n+1}^{\infty} \frac{B_j^{(n)}}{j!(j-n)}\theta^{j-n},\end{aligned}$$

where we have interchanged the order of the infinite series and the integral (this is allowed since the series is absolutely and uniformly convergent for $|t| \leq R < 2\pi$), and then integrated t^{j-n-1}. Finally we slightly rewrite the last term on the right of (5.5), and obtain

$$\theta \sum_{j=n+1}^{\infty} \frac{B_j^{(n)}}{j!(j-n)} \theta^{j-n-1} = O(\theta), \tag{5.6}$$

since the series in (5.6) is bounded whenever $|\theta| \leq R < 2\pi$, by (5.1). The desired identity (5.3) now follows immediately from (5.4), (5.5), and (5.6). $\square$

Our next goal is to obtain an expression for the first two terms on the right of (5.3). To do so, we evaluate the integral

$$I_2^{(n)} := \int_0^\infty \left(\frac{e^t}{(e^t-1)^n} - \sum_{j=0}^{n} B_j^{(n)}(1) \frac{t^{j-n}}{j!} \right) \frac{dt}{te^t} \quad (n \geq 1) \tag{5.7}$$

in two different ways. But first we need to evaluate an auxiliary integral.

Lemma 5.2. *For a positive integer n and for $\alpha \in \mathbb{C} \setminus \{n\}$ with $\mathrm{Re}(\alpha) > n-1$ we have*

$$\int_0^\infty \frac{t^{\alpha-1}}{(e^t-1)^n} dt = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{j=1}^{n} s(n,j) \zeta(\alpha+1-j). \tag{5.8}$$

Proof. Let $I(\alpha)$ be the integral in (5.8). As a special case of the identity 2.3.12.1 in [20, p. 333] we obtain

$$I(\alpha) = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{k=0}^{\infty} \frac{(k+1)_{n-1}}{(k+n)^\alpha}.$$

We now multiply numerator and denominator on the right by $k+n$ and note that $(k+1)_{n-1}(k+n) = (k+1)_n$. Shifting the summation by n we then get

$$I(\alpha) = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{k=1}^{\infty} \frac{(k-n+1)_n}{k^{\alpha+1}},$$

noting that the terms for $k = 1, 2, \ldots, n-1$ vanish. Finally we use (1.7) with $x = k$, obtaining

$$I(\alpha) = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{k=1}^{\infty} \sum_{j=1}^{n} s(n,j) \frac{k^j}{k^{\alpha+1}} = \frac{\Gamma(\alpha)}{(n-1)!} \sum_{j=1}^{n} s(n,j) \sum_{k=1}^{\infty} \frac{1}{k^{\alpha+1-j}},$$

where we have used the fact that $s(n,0) = 0$ for all $n \geq 1$. This completes the proof of (5.8). $\square$

The first evaluation of the integral $I_2^{(n)}$ in (5.7) is now given by the following lemma.

Lemma 5.3. *For any integer $n \geq 1$ we have*

$$I_2^{(n)} = \sum_{j=1}^{n} \frac{s(n,j)}{(n-1)!} \zeta'(1-j) - \frac{1}{n!} \sum_{j=1}^{n-2} (-1)^{n-j} \binom{n}{j} B_j^{(n)}(1) H_{n-j}. \tag{5.9}$$

Proof. Introducing a parameter α in (5.7), we define

$$I_2^{(n)}(\alpha) := \int_0^\infty \left(\frac{e^t}{(e^t-1)^n} - \sum_{j=0}^n B_j^{(n)}(1) \frac{t^{j-n}}{j!} \right) \frac{t^{\alpha-1}}{e^t} dt. \tag{5.10}$$

Splitting the integral and using Lemma 5.2 as well as Euler's integral (2.4), we get

$$\begin{aligned} I_2^{(n)}(\alpha) &= \int_0^\infty \frac{t^{\alpha-1}}{(e^t-1)^n} dt - \sum_{j=0}^n \frac{B_j^{(n)}(1)}{j!} \int_0^\infty t^{\alpha+j-n-1} e^{-t} dt \\ &= \frac{\Gamma(\alpha)}{(n-1)!} \sum_{j=1}^n s(n,j)\zeta(\alpha+1-j) - \sum_{j=0}^n \frac{B_j^{(n)}(1)}{j!} \Gamma(\alpha+j-n). \end{aligned}$$

Since $\Gamma(\alpha) = (\alpha-1)(\alpha-2)\cdots(\alpha+j-n)\Gamma(\alpha+j-n)$, we can rewrite this last identity as

$$I_2^{(n)}(\alpha) = \alpha\Gamma(\alpha) \left(\sum_{j=0}^n \frac{s(n,j)}{(n-1)!} \frac{\zeta(\alpha+1-j)}{\alpha} - \frac{1}{\alpha} \sum_{j=1}^n \frac{B_j^{(n)}(1)}{j!(\alpha-1)\cdots(\alpha+j-n)} \right). \tag{5.11}$$

We denote the two sums in (5.11) by S_1 and S_2, respectively, and consider them separately. First, we have

$$S_1 = \sum_{j=0}^n \frac{s(n,j)}{(n-1)!} \cdot \frac{\zeta(\alpha+1-j) - \zeta(1-j)}{\alpha} + \frac{1}{\alpha} \sum_{j=1}^n \frac{s(n,j)}{(n-1)!} \zeta(1-j). \tag{5.12}$$

Next, for reasons that will soon become apparent, we write

$$\frac{-1}{(\alpha-1)\cdots(\alpha+j-n)} = \left(\frac{(-1)^{n-j}}{(n-j)!} - \frac{1}{(\alpha-1)\cdots(\alpha+j-n)} \right) - \frac{(-1)^{n-j}}{(n-j)!}, \tag{5.13}$$

so that

$$\begin{aligned} \frac{-S_2}{\alpha} &= \sum_{j=1}^n \frac{1}{\alpha} \left(\frac{(-1)^{n-j}}{(n-j)!} - \frac{1}{(\alpha-1)\cdots(\alpha+j-n)} \right) \frac{B_j^{(n)}(1)}{j!} \\ &\quad - \frac{1}{\alpha n!} \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} B_j^{(n)}(1). \end{aligned} \tag{5.14}$$

Now we use the known identity

$$\sum_{j=0}^n (-1)^j \binom{n}{j} a^j B_j^{(m)}(x) = (-a)^n B_n^{(m)}(x - \tfrac{1}{a})$$

(see identity (50.8.15) in [9, p. 340]) and set $a = 1$ and $x = 1$. Then we get

$$\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} B_j^{(m)}(1) = B_n^{(m)}(0) = B_n^{(m)}, \tag{5.15}$$

which is also known as the nth Nörlund number.

To deal with the second term in (5.12), we use the known identity

$$B_n^{(k)} = k \binom{n}{k} \sum_{r=0}^{k-1} (-1)^{k-1-r} s(k,k-r) \frac{B_{n-r}}{n-r},$$

where B_{n-r} is an ordinary Bernoulli number; see [14, p. 148]. Setting $k=n$ and changing the order of summation, we get

$$B_n^{(n)} = n\sum_{j=1}^{n}(-1)^{j-1}s(n,j)\frac{B_j}{j}.$$

Then with (3.6) we get

$$\sum_{j=1}^{n}\frac{s(n,j)}{(n-1)!}\zeta(1-j) = \frac{1}{(n-1)!}\sum_{j=1}^{n}s(n,j)(-1)^{j-1}\frac{B_j}{j} = \frac{1}{n!}B_n^{(n)}. \tag{5.16}$$

With (5.15) and (5.16) we now see that the second terms in (5.12) and (5.14) cancel, which means that we may take the limit as $\alpha\to 0$. In doing so, we first note that $\alpha\Gamma(\alpha)\to 1$ by (4.2). Next, the first term in (5.12) converges to the first term in (5.9). Finally, a well-known limit for the harmonic number H_m is given by

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\left(1-\frac{m!}{(\alpha+1)(\alpha+2)\cdots(\alpha+m)}\right) = H_m;$$

see, e.g., [8, p. 281f.]. Replacing α by $-\alpha$ and then multiplying both sides by $(-1)^{m+1}/m!$, we get

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\left(\frac{(-1)^m}{m!}-\frac{1}{(\alpha-1)(\alpha-2)\cdots(\alpha-m)}\right) = -\frac{(-1)^m}{m!}H_m.$$

With this, the first term on the right of (5.14) becomes, as $\alpha\to 0$,

$$-\sum_{j=1}^{n}(-1)^{n-j}\frac{H_{n-j}}{(n-j)!}\frac{B_j^{(n)}(1)}{j!}.$$

But this is the second term in (5.9) if we note that $H_0=0$ and $B_{n-1}^{(n)}(1)=0$, where this last identity follows from $B_{n-1}^{(n)}(x)=(x-1)(x-2)\cdots(x-n+1)$ for $n\geq 2$; see, e.g., [14, Ch. 6]. The proof of Lemma 5.3 is now complete. □

For the second evaluation of the integral $I_2^{(n)}$, we first need to evaluate a certain integral and an infinite series. It turns out that both can be written in terms of the exponential integral defined in (4.6). We also require the alternating sum of consecutive factorials, given by

$$b_n := \sum_{j=0}^{n}(-1)^j(n-j)! \qquad (n\geq 0), \tag{5.17}$$

which satisfies the simple recurrence relation $b_0=1$ and for $n\geq 1$,

$$b_n = n! - b_{n-1}. \tag{5.18}$$

The first few terms, starting with b_0, are 1, 0, 2, 4, 20, 100, 620, …; see A058006 and A153229 in [15]. It will be convenient to also set $b_{-1}=0$, which is consistent with (5.17) and (5.18).

Lemma 5.4. *For any integer $k\geq 1$ we have*

$$\int_1^\infty \frac{e^{-x}}{x^k}dx = \frac{1}{(k-1)!}\left(e^{-1}b_{k-2}-(-1)^kE_1(1)\right). \tag{5.19}$$

Proof. We use induction on k. The base case $k = 1$ is just (4.6). Now we denote the integral in (5.19) by J_k and use integration by parts, which gives

$$J_k = \lim_{t\to\infty}\left[-\frac{e^{-x}}{x^k}\right]_1^t - k\int_1^\infty \frac{e^{-x}}{x^{k+1}}dx = e^{-1} - kJ_{k+1}.$$

Assuming that (5.19) holds for some $k \geq 1$, we then have

$$\begin{aligned} J_{k+1} &= \frac{1}{ke} - \frac{1}{k}J_k \\ &= \frac{1}{ke} - \frac{1}{k!}\left(\frac{1}{e}b_{k-2} - (-1)^k E_1(1)\right) \\ &= \frac{1}{k!}\left(\frac{1}{e}\big((k-1)! - b_{k-2}\big) - (-1)^{k+1}E_1(1)\right). \end{aligned}$$

Using (5.18) with $n = k - 1$, we see that (5.19) holds for $k + 1$, and the proof is complete. □

We will now see that the sequence (b_k) and the exponential integral $E_1(1)$ also occur in the following evaluation.

Lemma 5.5. *For any integer $m \geq 0$ we have*

$$\begin{aligned} \sum_{j=m+1}^{\infty}\frac{(-1)^j}{j!(j-m)} = \frac{(-1)^m}{m!}&\left(H_m - \gamma - E_1(1) + \frac{(-1)^{m-1}}{e}b_{m-1}\right) \\ &+ \sum_{j=0}^{m-1}\frac{(-1)^j}{j!(m-j)}. \end{aligned} \tag{5.20}$$

Proof. This identity is related to the incomplete Gamma function. In fact, the identity (8.4.15) in [16], with $z = 1$, gives

$$\frac{(-1)^m}{m!}\left(E_1(1) - \frac{1}{e}\sum_{j=0}^{m-1}(-1)^j j!\right) = \frac{(-1)^m}{m!}\psi(m+1) - \sum_{\substack{j=0\\ j\neq m}}^{\infty}\frac{(-1)^j}{j!(j-m)}, \tag{5.21}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ $(z \neq 0, -1, -2, \dots)$ is the digamma (or psi) function. Using the identity

$$\psi(m+1) = H_m - \gamma$$

(see, e.g., [16, 5.4.140]) as well as (5.17), the identity (5.21) can easily be rewritten in the form (5.20). □

For greater ease of notation, we denote the final term on the right of (5.20) by d_m, that is,

$$d_m := \sum_{j=0}^{m-1}\frac{(-1)^j}{j!(m-j)}. \tag{5.22}$$

The sequence $(m!d_m)$ can be found as entry A002741 in [15], where the generating function is given as

$$-\frac{\log(1-x)}{e^x} = \sum_{k=0}^{\infty} d_k x^k \qquad (|x| < 1). \tag{5.23}$$

This is in fact easy to see since the Cauchy product of the series for $-\log(1-x)$ and for e^{-x} immediately gives the sum in (5.22).

We are now ready to state and prove the second evaluation of the integral $I_2^{(n)}$ defined in (5.7).

Lemma 5.6. *For any integer $n \geq 1$ we have*

$$I_2^{(n)} = \sum_{k=n+1}^{\infty} \frac{B_k^{(n)}}{k!(k-n)} + \int_1^{\infty} \frac{dt}{t(e^t-1)^n} + \gamma \frac{B_n^{(n)}}{n!} - \frac{1}{n!} \sum_{j=0}^{n-2} (-1)^{n-j} \binom{n}{j} B_j^{(n)}(1) H_{n-j} + \sum_{j=0}^{n-1} \frac{B_j^{(n)}}{j!(j-n)}. \tag{5.24}$$

Proof. We split the defining integral in (5.7) into two parts. The easier second part is the integral from 1 to ∞, which can be rewritten as

$$\begin{aligned} &\int_1^{\infty} \frac{dt}{t(e^t-1)^n} - \sum_{j=0}^{n} \frac{B_j^{(n)}(1)}{j!} \int_1^{\infty} \frac{e^{-t}}{t^{n+1-j}} dt \\ &= \int_1^{\infty} \frac{dt}{t(e^t-1)^n} - \sum_{j=0}^{n} \frac{B_j^{(n)}(1)}{j!} \cdot \frac{1}{(n-j)!} \left(\frac{1}{e} b_{n-j-1} + (-1)^{n-j} E_1(1) \right) \\ &= \int_1^{\infty} \frac{dt}{t(e^t-1)^n} - \frac{e^{-1}}{n!} \sum_{j=0}^{n-1} \binom{n}{j} B_j^{(n)}(1) b_{n-j-1} - E_1(1) \frac{B_n^{(n)}}{n!}, \end{aligned} \tag{5.25}$$

where we have used Lemma 5.4 and the identity (5.15).

To deal with the first part of the integral in (5.7), i.e., the integral from 0 to 1, we begin by rewriting the integrand as

$$\left(\left(\frac{t}{e^t-1} \right)^n - e^{-t} \sum_{j=0}^{n} B_j^{(n)}(1) \frac{t^j}{j!} \right) t^{-n-1}. \tag{5.26}$$

First we note that by (5.1) we can write

$$\left(\frac{t}{e^t-1} \right)^n = e^{-t} \left(\frac{t}{e^t-1} \right)^n e^t = e^{-t} \sum_{j=0}^{\infty} B_j^{(n)}(1) \frac{t^j}{j!}. \tag{5.27}$$

Then, with the convention that $\binom{k}{j} = 0$ when $j > k$, we have the Cauchy product

$$\begin{aligned} e^{-t} \sum_{j=0}^{n} B_j^{(n)}(1) \frac{t^j}{j!} &= \left(\sum_{j=0}^{\infty} (-1)^j \frac{t^j}{j!} \right) \left(\sum_{j=0}^{n} B_j^{(n)}(1) \frac{t^j}{j!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} (-1)^{k-j} \binom{k}{j} B_j^{(n)}(1) \right) \frac{t^k}{k!}. \end{aligned}$$

This, together with (5.27) and (5.15) means, first of all, that the expression in large parentheses in (5.26) is $e^{-t} O(t^{n+1})$, and thus the integral from 0 to 1 converges. It also means that the first part of the integral in (5.7), i.e., the integral from 0 to 1, is

$$\begin{aligned}
(5.28)\qquad &\int_0^1 \left(\left(\frac{t}{e^t-1}\right)^n - e^{-t}\sum_{j=0}^{n} B_j^{(n)}(1)\frac{t^j}{j!}\right) t^{-n-1}dt \\
&= \int_0^1 \left(\sum_{k=n+1}^{\infty} \left(B_k^{(n)} - (-1)^k \sum_{j=0}^{n} (-1)^j \binom{k}{j} B_j^{(n)}(1)\right) \frac{t^{k-n-1}}{k!}\right) dt \\
&= \sum_{k=n+1}^{\infty} \frac{B_k^{(n)}}{k!(k-n)} - \sum_{j=0}^{n} (-1)^j \frac{B_j^{(n)}(1)}{j!} \sum_{k=n+1}^{\infty} \frac{(-1)^k}{(k-j)!(k-n)},
\end{aligned}$$

where we have interchanged the integral and the infinite series, which is allowed in this case. Now we rewrite the second series in the last line of (5.28) as

$$\sum_{k=n+1}^{\infty} \frac{(-1)^k}{(k-j)!(k-n)} = (-1)^j \sum_{k=n+1-j}^{\infty} \frac{(-1)^k}{k!(k+j-n)} = (-1)^j \sum_{k=m+1}^{\infty} \frac{(-1)^k}{k!(k-m)},$$

where we have set $m = n-j$. With (5.20), the second term in the last line of (5.28) now becomes

$$\begin{aligned}
(5.29)\qquad & -\sum_{j=0}^{n} \frac{B_j^{(n)}(1)}{j!} \cdot \frac{(-1)^{n-j}}{(n-j)!} \left(H_{n-j} - \gamma - E_1(1) + \frac{(-1)^{n-j-1}}{e} b_{n-j-1} \right) \\
&\qquad - \sum_{j=0}^{n} \frac{B_j^{(n)}(1)}{j!} d_{n-j},
\end{aligned}$$

where b_n and d_n are defined by (5.17) and (5.22), respectively.

To deal with the final sum in (5.29), we note that by (5.23) and (5.1) with $x = 1$, the generating function of this sum, seen as a convolution, is

$$F(t) := -\frac{\log(1-t)}{e^t} \left(\frac{t}{e^t-1}\right)^n e^t.$$

We can rewrite this as a different convolution, namely

$$\begin{aligned}
F(t) = -\log(1-t)\left(\frac{t}{e^t-1}\right)^n &= \left(\sum_{j=1}^{\infty} \frac{t^j}{j}\right)\left(\sum_{j=0}^{\infty} B_j^{(n)} \frac{t^j}{j!}\right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{j=0}^{n-1} \frac{B_j^{(n)}}{(n-j)j!}\right) t^n.
\end{aligned}$$

Equating coefficients of like powers of t, we then get

$$\sum_{j=0}^{n} \frac{B_j^{(n)}(1)}{j!} d_{n-j} = \sum_{j=0}^{n-1} \frac{B_j^{(n)}}{j!(n-j)}.$$

Substituting this into (5.29), and then (5.29) into (5.28), we see that the first part of the integral $I_2(n)$ of (5.7) is

$$\sum_{k=n+1}^{\infty}\frac{B_k^{(n)}}{k!(k-n)}-\sum_{j=0}^{n}\frac{B_j^{(n)}(1)}{j!}\frac{(-1)^{n-j}}{(n-j)!}\left(H_{n-j}-\gamma-E_1(1)+\frac{(-1)^{n-j-1}}{e}b_{n-j-1}\right)$$
$$+\sum_{j=0}^{n-1}\frac{B_j^{(n)}}{j!(j-n)}.$$

Using (5.15) with $m=n$, this expression becomes

$$\sum_{k=n+1}^{\infty}\frac{B_k^{(n)}}{k!(k-n)}-\frac{1}{n!}\sum_{j=0}^{n-2}(-1)^{n-j}\binom{n}{j}B_j^{(n)}(1)H_{n-j}$$
$$+\frac{\gamma+E_1(1)}{n!}B_n^{(n)}-\frac{e^{-1}}{n!}\sum_{j=0}^{n-1}\binom{n}{j}B_j^{(n)}(1)b_{n-j-1}+\sum_{j=0}^{n-1}\frac{B_j^{(n)}}{j!(j-n)}, \tag{5.30}$$

The second sum in (5.30) may be taken from 0 to $n-2$ since $H_0=0$ by convention, and $B_{n-1}^{(n)}(1)=0$. Similarly, the second-last sum in (5.30) may be taken from 0 to $n-1$ since $b_{-1}=0$ by convention.

If we now add (5.25) and (5.30), we see that a few terms cancel, and the remaining terms give (5.24). This completes the proof of Lemma 5.6. □

Finally, to obtain the desired evaluation of the integral on the right of (4.8), we equate $I_2^{(n)}$ in (5.9) and (5.24) and note that the sums that contain H_{n-j} are the same in both identities, and therefore cancel. We then combine the resulting identity with (5.3), obtaining the following intermediate result.

Lemma 5.7. *Let $0<R<2\pi$ be fixed. Then for $0<\theta\le R$ and for any integer $n\ge 1$ we have*

$$\int_1^{\infty}\frac{du}{(e^{\theta u}-1)^n u}=\sum_{j=1}^{n}\frac{s(n,j)}{(n-1)!}\zeta'(1-j)-\frac{B_n^{(n)}}{n!}(\gamma+\log\theta)$$
$$-\sum_{j=0}^{n-1}\frac{B_j^{(n)}}{j!}\frac{\theta^{j-n}}{j-n}+O(\theta). \tag{5.31}$$

6. Evaluating the derivatives of $sC(s)$ and $sD(s)$

We begin by evaluating the sum $S_1(n,\lambda)$ defined in (4.20). This will be required a few times in this section.

Lemma 6.1. *For all integers $1\le\lambda<n$ we have*

$$S_1(n,\lambda)=\frac{1}{\lambda!}B_\lambda^{(n)}=\frac{(n-1-\lambda)!}{(n-1)!}\,s(n,n-\lambda). \tag{6.1}$$

Proof. In [6] it was shown that

$$\sum_{k=1}^{\lambda}\binom{n}{k}\sum_{\substack{u_1,\ldots,u_k\ge 1\\ u_1+\cdots+u_k=\lambda}}\binom{\lambda}{u_1,\ldots,u_k}B_{u_1}(x)\cdots B_{u_k}(x)=B_\lambda^{(n)}(nx). \tag{6.2}$$

First we set $x = 0$, obtaining the analogue of (6.2) for Bernoulli *numbers*. Dividing both sides of (6.2) by $\lambda!$ and comparing this with (4.20), we get the first equation in (6.1). The second equation follows from the identity

$$B_\lambda^{(n)} = \frac{s(n, n-\lambda)}{\binom{n-1}{\lambda}}; \tag{6.3}$$

see Theorem 2.2 in [12]. □

Remark. In the case $n = \lambda + 1$, we note that

$$B_\lambda^{(\lambda+1)} = (-1)^\lambda \lambda! \tag{6.4}$$

(see, e.g., [11, p. 130]); therefore the right-hand side of (6.1) becomes $(-1)^\lambda$, while the left-hand side is the sum in (3.9), with n in place of λ. Thus the evaluation of $R_n(0)$ in (3.9) is just a special case of Lemma 6.1.

Next we use Lemma 6.1 to evaluate the expression in (4.17).

Lemma 6.2. *Let $C(s)$ be as defined at the beginning of Section 4. Then*

$$\begin{aligned} \frac{d}{ds}[sC(s)]_{s=0} &= \frac{B_n^{(n)}}{n!}(\gamma + \log\theta) + (-1)^{n+1}\frac{\gamma}{n+1} \\ &\quad + \frac{1}{(n-1)!}\sum_{j=0}^{n-1}(-1)^j s(n, j+1)\zeta'(-j). \end{aligned} \tag{6.5}$$

Proof. It is easy to verify that

$$\binom{n}{k}\frac{n-k}{n-k+1} = \binom{n}{k} - \frac{1}{n+1}\binom{n+1}{k}.$$

Hence with (4.17), (4.20) and (6.1) we get

$$\begin{aligned} \frac{d}{ds}[sC(s)]_{s=0} &= \left(\frac{B_n^{(n)}}{n!} - \frac{1}{n+1}\cdot\frac{B_n^{(n+1)}}{n!}\right)\gamma + \frac{B_n^{(n)}}{n!}\log\theta \\ &\quad + \sum_{k=1}^{n}\frac{\binom{n}{k}}{n-k+1}{\sum}^*\sum_{j=1}^{k}\frac{B_{u_1+1}}{(u_1+1)!}\cdots\frac{(-1)^{u_j}\zeta'(-u_j)}{u_j!}\cdots\frac{B_{u_k+1}}{(u_k+1)!}, \end{aligned} \tag{6.6}$$

where ${\sum}^*$ indicates the same summation as in (4.10). Using (6.4), we see that the first term on the right of (6.6) give the first terms on the right of (6.5). It now remains to evaluate the multiple sum in the second line of (6.6), which we denote by S_n.

Using symmetry in the indices u_j, and using the identity

$$\frac{k}{n-k+1}\binom{n}{k} = \binom{n}{k-1},$$

we rewrite S_n as

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{k\binom{n}{k}}{n-k+1} \sum_{\substack{u_1+\cdots+u_k=n-k \\ u_1,\ldots,u_k\geq 0}} \frac{B_{u_1+1}}{(u_1+1)!}\cdots\frac{B_{u_{k-1}+1}}{(u_{k-1}+1)!}(-1)^{u_k}\frac{\zeta'(-u_k)}{u_k!} \\ &= \sum_{k=1}^{n}\binom{n}{k-1}\sum_{j=0}^{n-k}(-1)^j\frac{\zeta'(-j)}{j!}\sum_{\substack{u_1+\cdots+u_{k-1}=n-k-j \\ u_1,\ldots,u_{k-1}\geq 0}}\frac{B_{u_1+1}}{(u_1+1)!}\cdots\frac{B_{u_{k-1}+1}}{(u_{k-1}+1)!} \\ &= \sum_{j=0}^{n-1}(-1)^j\frac{\zeta'(-j)}{j!}\sum_{k=1}^{n-j}\binom{n}{k-1}\sum_{\substack{u_1+\cdots+u_{k-1}=n-k-j \\ u_1,\ldots,u_{k-1}\geq 0}}\prod_{i=1}^{k-1}\frac{B_{u_i+1}}{(u_i+1)!}, \end{aligned}$$

where we first renamed $j=u_k$, and then changed the order of summation over j and k. Next we shift the summations in the multiple sum, obtaining

$$\begin{aligned} S_n &= \sum_{j=0}^{n-1}(-1)^j\frac{\zeta'(-j)}{j!}\sum_{k=1}^{n-j}\binom{n}{k-1}\sum_{\substack{u_1+\cdots+u_{k-1}=n-1-j \\ u_1,\ldots,u_{k-1}\geq 1}}\prod_{i=1}^{k-1}\frac{B_{u_i}}{u_i!} \\ &= \sum_{j=0}^{n-1}(-1)^j\frac{\zeta'(-j)}{j!}\sum_{k=0}^{n-1-j}\binom{n}{k}\sum_{\substack{u_1+\cdots+u_k=n-1-j \\ u_1,\ldots,u_k\geq 1}}\prod_{i=1}^{k}\frac{B_{u_i}}{u_i!}. \end{aligned}$$

Comparing this with (4.20) and noting that there is no contribution from $k=0$, we see that the sum over k is $S_1(n,n-1-j)$. From (6.1) we therefore get

$$S_n = \sum_{j=0}^{n-1}(-1)^j\frac{\zeta'(-j)}{j!}\cdot\frac{j!}{(n-1)!}s(n,j+1),$$

which completes the proof of Lemma 6.2. □

Lemma 6.1 can also be used to simplify the identity (4.21). In fact, we have

$$\frac{d}{ds}[sD(s)]_{s=0} = \sum_{\lambda=0}^{n-1}\frac{B_\lambda^{(n)}}{\lambda!}\cdot\frac{\theta^{\lambda-n}}{\lambda-n}+O(\theta). \tag{6.7}$$

For $\lambda\geq 1$ this is clear from (6.1), while the term for $\lambda=0$ follows from the fact that $B_0^{(n)}=1$ for all n.

7. Completing the Proof of Theorem 1.1

Finally, to complete the proof of Theorem 1.1, it remains to combine the different elements according to the beginning of Section 4, that is, to equate

$$\frac{d}{ds}[sA(s)]_{s=0} = \frac{d}{ds}[sB(s)]_{s=0}+\frac{d}{ds}[sC(s)]_{s=0}+\frac{d}{ds}[sD(s)]_{s=0}, \tag{7.1}$$

and then take the limit as $\theta\to 0$. These four terms were evaluated in (4.3), (5.31), (6.5), and (6.7), respectively.

After substantial cancellations, especially of all terms that contain singularities at $\theta=0$, the identity (7.1) reduces to

$$\omega_{n+1}'(0) = \frac{1}{(n-1)!}\left(\sum_{j=1}^{n}s(n,j)\zeta'(1-j)+\sum_{j=0}^{n-1}(-1)^js(n,j+1)\zeta'(-j)\right)+O(\theta). \tag{7.2}$$

As $\theta \to 0$, the O-term disappears, and by combining the two sums we obtain

$$\omega'_{n+1}(0) = \frac{2}{(n-1)!} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} s(n, 2j+1)\zeta'(-2j). \tag{7.3}$$

This completes the proof of Theorem 1.1 if we recall that by (1.7) we have $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, and also $s(n,1) = (-1)^{n-1}(n-1)!$; see, e.g., [16, Sect. 26.8].

8. Further Remarks

In this section we briefly discuss K. Onoderas's very recent and interesting paper [19] and show that his main result contains our Theorem 1.1 as a special case. For positive integers a, b, Onodera defined the multiple zeta function

$$\zeta_{a,b}(s) = \sum_{\substack{k_1,\dots,k_a \geq 1, l_1,\dots,l_b \geq 1 \\ k_1+\cdots+k_a = l_1+\cdots+l_b}} \frac{1}{(k_1 \cdots k_a l_1 \cdots l_b)^s}. \tag{8.1}$$

Note that $\omega_{n+1}(s) = \zeta_{n,1}(s)$. The main result in [19] is as follows.

Theorem 8.1 (Onodera). *Let a and b be positive integers, and $r = a+b-1$. Then*

$$\zeta_{a,b}(0) = \frac{(-1)^r}{\binom{r+1}{a}}, \tag{8.2}$$

$$\zeta'_{a,b}(0) = (-1)^{a-1}\frac{2ab}{r!} \sum_{\substack{k=1 \\ k \text{ odd}}}^{r} \binom{r}{k} B_{r-k}^{(r+1)}(a)\zeta'(1-k). \tag{8.3}$$

Here $B_j^{(n)}(x)$ is the jth Bernoulli polynomial of order n, defined in (5.1). We now set $b = 1$ and $a = n$ in (8.3); then $r = n$. Using the well-known reflection formula for higher-order Bernoulli polynomials (see, e.g., [11, p. 128]), followed by another relevant identity (see [11, p. 129, (4)]), we have

$$(-1)^{a-1}B_{r-k}^{(r+1)}(a) = (-1)^{n-1}B_{n-k}^{(n+1)}(n) = B_{n-k}^{(n+1)}(1) = \frac{k}{n}B_{n-k}^{(n)}. \tag{8.4}$$

By (6.3), higher-order Bernoulli numbers can be written in terms of Stirling numbers of the first kind. In particular we have, with (8.4) and (6.3),

$$(-1)^{a-1}\binom{r}{k}B_{r-k}^{(r+1)}(a) = \binom{n}{k}\frac{k}{n} \cdot \frac{s(n,k)}{\binom{n-1}{n-k}} = s(n,k).$$

With $k = 2j+1$, we now see that (8.3) with $b = 1$ is the same as (1.6).

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