Improved Stefan equation correction factors to accommodate sensible heat storage during soil freezing or thawing

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Abstract

In permafrost regions, the thaw depth strongly controls shallow subsurface hydrologic processes that in turn dominate catchment runoff. In seasonally freezing soils, the maximum expected frost depth is an important geotechnical engineering design parameter. Thus, accurately calculating the depth of soil freezing or thawing is an important challenge in cold regions engineering and hydrology.

The Stefan equation is a common approach for predicting the frost or thaw depth, but this equation assumes negligible soil heat capacity and thus exaggerates the rate of freezing or thawing. The Neumann equation, which accommodates sensible heat, is an alternative implicit equation for calculating freeze-thaw penetration. This study details the development of correction factors to improve the Stefan equation by accounting for the influence of the soil heat capacity and non-zero initial temperatures. The correction factors are first derived analytically via comparison to the Neumann solution, but the resultant equations are complex and implicit. Thus, explicit equations are obtained by fitting polynomial functions to the analytical results. These simple correction factors are shown to significantly improve the performance of the Stefan equation for several hypothetical soil freezing and thawing scenarios.

Keywords

Cold regions hydrology, freeze-thaw, cryogenic soils, permafrost, frost depth, active layer

1. Introduction

The seasonal penetration of the frost or thaw front is an important consideration in cold regions hydrology (French, 2007; Woo, 2012). Pore ice reduces the hydraulic conductivity of soils (Kurylyk and Watanabe, 2013), and thus the upper surface of frozen, saturated soil in permafrost regions acts as a relatively impermeable unit that restricts subsurface flow to the perched seasonally thawed active layer (Carey and Woo, 2001). Lateral subsurface runoff can decrease during concomitant lowering of the thawing front and saturated zone due to the inverse relationship between the depth of organic soils and saturated hydraulic conductivity (Carey and Woo, 1999; Quinton et al., 2000). Hence, the location of the thawing front exerts a strong control on shallow subsurface flow (Carey and Woo, 2005; Metcalfe and Buttle, 1999; Wright et al., 2009), which is the dominant runoff mechanism in many northern catchments (Quinton and Marsh, 1999; Tetzlaff et al., 2014).

From a geotechnical engineering perspective, predicting the maximum frost depth is essential for determining appropriate foundation depths to minimize the influence of frost heave (Andersland and Ladanyi, 2004). Furthermore, cyclical freeze-thaw action has been known to effect the physical and mechanical properties of soils including hydraulic conductivity (Konrad, 2000), consolidation (Edwards, 2013), and strength (Qi et al., 2008). Consequently, geotechnical engineers have developed most of the theory and methodology for determining the frost depth (e.g., Andersland and Ladanyi, 1994; Aldrich and Paynter, 1953; Jumikis, 1977), and these approaches have been adopted and modified by permafrost hydrologists and geomorphologists (e.g., French 2007; Woo, 2012).

As reviewed by Kurylyk et al. (2014a), there has been recent intensified interest in the quantitative study of shallow subsurface thermal regimes due to the potential influence of changing climatic conditions. For example, warming air temperatures and changing precipitations regimes can lead to a reduction in the insulating winter snowpack, which can paradoxically lead to a decrease in winter surface temperatures (Groffman et al., 2001; Kurylyk et al., 2013) and an increase in the maximum frost depth. Also, recent climate warming has already produced measurable increases in the active layer thickness in many regions of the world (e.g., Harris et al., 2009; Romanovsky and Osterkamp, 1997). Alterations to atmospheric conditions and subsurface thermal regimes may both contribute to changing surface hydrological processes (Tetzlaff et al., 2013). The influence of changing climate conditions on seasonal soil freezing or thawing can be studied using mechanistic approaches that consider governing heat transfer processes.

The two most common analytical solutions applied to calculate the rate of soil freezing or thawing are the Neumann (ca. 1860) and Stefan (1891) equations (Lunardini, 1981). The Neumann equation accounts for both latent and sensible soil heat, whereas the Stefan equation is an approximate approach that neglects the sensible heat required to change the soil temperature (Kurylyk et al., 2014b). Despite its approximate nature, the Stefan equation has been more widely implemented because the Neumann equation is implicit and requires a constant surface temperature. The simplicity of the Stefan equation also facilitates its incorporation into freeze-thaw algorithms that accommodate more complex conditions such as soil layering (Kurylyk et al., 2015) and changing moisture content (Hayashi et al., 2007). The Stefan equation indicates that the rate of freezing or thawing is proportional to the square root of the cumulative thawing or freezing index (i.e., the product of surface temperature and time). The proportionality constant is a function of the latent heat released or absorbed from the soil during pore water phase change and the soil thermal properties. Several researchers have employed a modified form of the Stefan equation by empirically determining the proportionality constant from measured site conditions (Anisimov et al., 2002; Hinkel and Nicholas, 1995; Woo, 1976). These approaches work well for estimating the active layer thickness where data exists for calibration, but such an approach is not generally transferable to other locations or climates.

The errors arising from the lack of inclusion of sensible heat in the Stefan equation can be considerable, especially for calculation of the frost depth. For example, by not considering the influence of the soil heat capacity, the Stefan equation can potentially overestimate the frost depth by up to 30% (e.g., Aldrich and Paynter, 1953). This is especially true when soil temperatures in the zone subject to freeze-thaw are not close to 0°C at the commencement of freezing or thawing at the surface. The source and magnitude of these errors are seldom explained in studies that incorporate the Stefan equation, or some variant thereof, to calculate the depth to the freezing or thawing front. The accuracy of the Stefan equation can potentially be improved via the inclusion of a quasi-empirical correction factor. The forms and application of previously proposed correction factors can be obtained from cold regions geotechnical engineering texts (Aldrich and Paynter, 1953; Andersland and Ladanyi, 1994; Jumikis, 1977; Lunardini, 1981). However, no mention of these Stefan correction factors are made in recent cold regions hydrology or geomorphology review papers that present variations of the Stefan equation to predict the depth of soil freeze-thaw (Bonnaventure and Lamoureux, 2013; Kurylyk et al., 2014a; Riseborough et al., 2008; Zhang et al., 2008). Recent permafrost hydrology or geomorphology texts (e.g. French, 2007; Woo, 2012) are also void of any discussion of these Stefan correction factors. Thus, it appears that their adoption has been primarily limited to geotechnical engineering applications.

The objectives of the present study are fourfold:

1. Derive analytical expressions for the Stefan correction factor via comparison to the Neumann equation;
2. Obtain reasonable fits to these implicit, analytical equations using simple polynomial functions;
3. Compare the performance of these new correction factors to those previously proposed in literature for a variety of soil freezing or thawing conditions, including non-zero initial temperature; and
4. Demonstrate the utility of these correction factors for simple illustrative examples.

We begin by presenting the Stefan and Neumann equations to highlight the assumptions of the Stefan approach and to derive analytical, implicit equations for correcting the Stefan equation to account for soil sensible heat. The implicit and complex nature of the analytical correction factors prohibits their incorporation into engineering practice or cold regions hydrology or land surface models, and thus we propose simple polynomial equations to reasonably approximate the analytical results. These flexible polynomial equations are functions of dimensionless numbers that are readily calculated. The underlying theory and derivations presented in this study are extensive. However, readers who are only interested in the final results may advance to Table 5, which presents the four alternative correction factor equations developed in this study and the respective settings for their applications.

2. Theory and Methods

2.1 Stefan and Neumann equations for soil thawing

Stefan (1891) proposed an equation for predicting the thawing or freezing of sea ice, and this equation has been modified and applied to calculate the rate of soil freezing and thawing. The governing equation, initial conditions, boundary condition, and derivation are provided in the supplementary material (Appendix S1.1). Figure 1 presents the conceptual model for the Stefan solution for calculating soil thawing. Soil freezing will be discussed later. The Stefan equation for soil thawing in the case of variable surface temperature is:

$$X(t) = \sqrt{\frac{2k_u I(t)}{\phi\rho_w L}} \tag{1}$$

where k_u is the bulk thermal conductivity of the upper unfrozen zone (W m^{-1} $°C^{-1}$), X is the distance (m) between the surface and the interface between the thawed and frozen zones (Fig. 1b), ϕ is the volumetric moisture content (volume of water divided by total soil volume) that has undergone phase change, ρ_w is the density of water (kg m^{-3}), L is the mass based latent heat of fusion for water (3.34×10^5 J kg^{-1}), and $I(t) = \int_0^t T_s(\tau)d\tau$ (T_s = surface temperature, °C) and is known as the surface thawing index (Fig. 1).

The product of the volumetric water content and water density yields the mass of moisture per unit volume of the unfrozen zone. Assuming complete phase change, this product can be

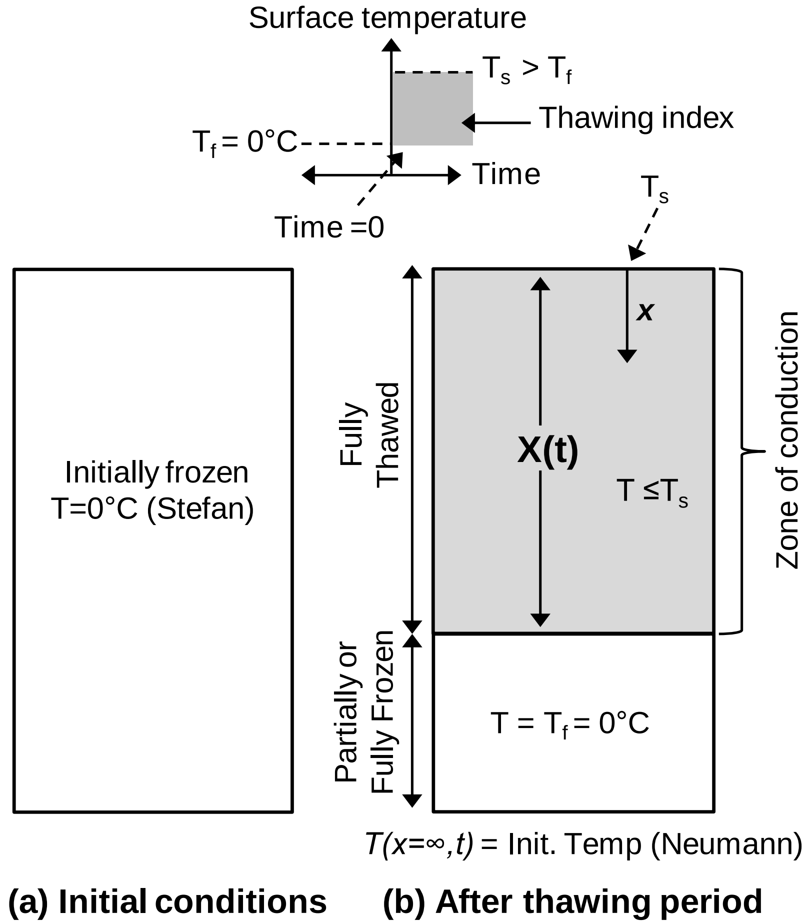

Figure 1: Conditions for the Stefan solution in the case of soil thawing for (a) the initial conditions and (b) after a period of thawing has occurred. This figure also represents the Neumann solution conditions when T_i = 0°C (modified from Kurylyk *et al*., 2014b). Note that if the initial temperature is less than 0°C, as in the case of the Neumann solution, the zone of conduction extends below the thaw front.

interchanged with the product of the volumetric ice content and ice density in the frozen zone. Often air temperature, rather than surface temperature, is used to form the thawing index. In this case the air thawing index is multiplied by the empirical '*n* factor', which is ratio of $I(t)$ to the integral of the air temperature (Klene et al., 2001). This *n* factor crudely accounts for the influence of snowpack, vegetation, and other surface conditions on surface thermal regimes.

If the surface temperature is constant, the Stefan equation becomes:

$$X = \sqrt{\frac{2k_u T_s t}{\phi \rho_w L}} \tag{2}$$

where t is time (s). Due to its simplicity, the Stefan equation has been widely implemented in cold regions engineering practice (Andersland and Ladanyi, 1994; Jumikis, 1977; Lunardini, 1981), cold regions hydrology models (Carey and Woo, 2005; Fox, 1992; Woo et al., 2004), and land surface schemes (Li and Koike, 2003; Yi et al., 2006). The Stefan solution assumes a linear temperature distribution in the upper zone (Appendix S1.1), which implies that the heat capacity of the soil is negligible and the soil is uniform.

The dimensionless Stefan number is proportional to the ratio of sensible heat to latent heat absorbed during thawing (Kurylyk et al., 2014b; Lunardini, 1981). In the case of soil thawing with initial temperatures at 0°C and a constant surface temperature T_s, the Stefan number is:

$$S_{T1} = \frac{c_u \rho_u T_s}{L \phi \rho_w} \tag{3}$$

Although, this form of the Stefan number is commonly defined as the exact ratio of sensible heat to latent heat (Andersland and Anderson, 1978), this is only the case when the entire thawed domain is uniformly at a temperature of T_s. The Stefan number for freezing soils will be discussed later. The derivation of Eq. (2) tacitly assumes that the Stefan number is zero, and thus the Stefan equation error can be shown to be strongly related to the Stefan number (Kurylyk et al., 2014b).

Figure 2 shows the ratio of the Stefan number to the surface temperature versus the moisture saturation for common soils. The relationship between S_{T1}/T_s and the moisture saturation is not linear as the bulk unfrozen heat capacity is also dependent on the moisture saturation (see Eq. 3). Figure 2 indicates that the Stefan number for soils with saturations higher than 0.25 will typically be less than 1 as the average surface temperatures during the thawing period are usually less than 20°C in permafrost regions. Hence, only S_{T1} values ranging between 0 and 1 will be considered in this study.

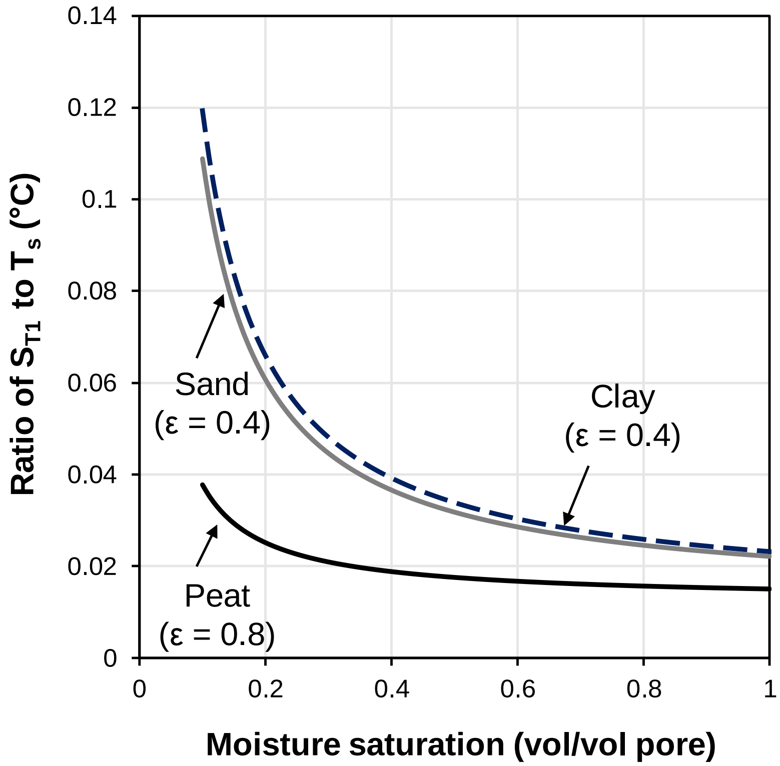

Figure 2: Ratio of the Stefan number during thawing (Equation 3, S_{T1}) to surface temperature (Ts) versus the moisture saturation. The ε symbol represents porosity ($\varepsilon S = \phi$). Thawed zone saturated bulk heat capacities for the different soils were taken from Bonan (2008, p. 134). The relationship between saturation and the bulk heat capacities was obtained via the volumetrically weighted arithmetic mean of the constituent heat capacities.

Neumann (ca. 1860) derived a mathematically exact solution (i.e., the assumption of a linear temperature profile is relaxed) to the propagation of the frost or thaw front, but this equation has not been frequently applied due to its implicit nature, increased complexity, and constant surface temperature requirement. The exactness of the Neumann equation has been verified with numerical methods, and the equation has thus served as a benchmark for cold regions heat transport numerical models (Kurylyk et al, 2014b). The initial conditions, boundary conditions, and assumptions of the Neumann solution are presented in the supplementary material (Appendix S1.2). Of particular note, the Neumann solution assumes a bottom (infinite depth) boundary temperature at the initial temperature, and thus does not account for two-directional freezing in permafrost soils or two-directional thawing in non-permafrost soils (Lunardini, 1981; Woo et al., 2004). In the case of soil thawing, the Neumann equation is:

$$X = m\sqrt{t} \tag{4}$$

where m is the coefficient of proportionality (m $s^{-0.5}$). This can be found via the implicit equation below:

$$0.5 L \phi \rho_w \sqrt{\pi}\, m = \frac{k_u T_s}{\sqrt{\alpha_u}} \exp\left(\frac{-m^2}{4\alpha_u}\right) \Big/ \operatorname{erf}\left(\frac{m}{2\sqrt{\alpha_u}}\right) + \frac{k_f T_i}{\sqrt{\alpha_f}} \exp\left(\frac{-m^2}{4\alpha_f}\right) \Big/ \operatorname{erfc}\left(\frac{m}{2\sqrt{\alpha_f}}\right) \tag{5}$$

where α_f and α_u are the thermal diffusivities of the frozen and unfrozen zones (m^2 s^{-1}), k_f is the bulk frozen zone thermal conductivity (W m^{-1} $°C^{-1}$), T_i is the initial temperature (< 0°C), erf is the error function, and erfc is the complementary error function. Equation (5) is sometimes presented in a slightly different form as T_i is often defined as the number of degrees below zero.

Table 1: Details for thawing/freezing scenarios shown in Figures 3 and 4

Description	Silty Clay[1]
Total water content ϕ	0.4
Unfrozen zone thermal cond. (W m^{-1} $°C^{-1}$)	1.07
Frozen zone thermal cond. (W m^{-1} $°C^{-1}$)	1.75
Unfrozen zone heat capacity (J m^{-3} $°C^{-1}$)	2.88×10^6
Frozen zone heat capacity (J m^{-3} $°C^{-1}$)	2.19×10^6
Boundary and initial conditions during thawing (Figure 3)	
Constant surface temperatures[2], T_s (°C)	15, 10, 5
Uniform initial temperatures[2], T_i (°C)	-2
Boundary and initial conditions during freezing (Figure 4)	
Constant surface temperature[3], T_s (°C)	-3
Uniform initial temperatures[3], T_i (°C)	1, 2, 5

[1]All thermal properties taken from McClymont et al. (2013).

[2]Runs 1-3 (for thawing) had surface temperature of 15, 10, and 5°C and initial temperatures of -2°C.

[3]Runs 1-3 (for freezing) had surface temperature of -3°C and initial temperatures of 1, 2 and 5°C.

2.2 Illustrative soil thawing examples and previous Stefan correction factors

The Neumann and Stefan equations both indicate that the penetration of the thawing front is proportional to the square root of time when the surface temperature is constant. To illustrate the potential inaccuracy of the Stefan equation, the thawed depths for three separate scenarios were calculated via the Stefan and Neumann equations. Table 1 presents the thermal properties, initial conditions, and boundary conditions used for the illustrative calculations. Initial temperatures of only -2°C were selected as the snowpack typically insulates the ground thermal regime during the winter months and maintains soil temperature close to 0°C (Kurylyk et al., 2013; Zhang et al., 2005). Figure 3 demonstrates that, for these examples, the Stefan equation over-predicts the thaw depth and yields normalized errors between 8 and 9%. These errors arise solely because the Stefan equation assumes negligible soil heat capacity. Stefan equation errors can be more pronounced for soils with lower moisture contents (higher Stefan numbers) or for colder soil temperatures at the onset of thaw, such as in wind scoured sites or other areas with snow-free conditions in the winter.

Aldrich and Paynter (1953) and others have proposed a modified Stefan equation to account for the influence of sensible heat:

$$X_{corr} = \lambda \sqrt{\frac{2k_u T_s t}{\phi \rho_w L}} \tag{6}$$

where X_{corr} is the corrected thaw depth (m), λ is a dimensionless correction factor that is less than 1. Figure 3c indicates that if the appropriate value is obtained for λ, the corrected Stefan equation should exactly equal the Neumann equation for any point in time. Sometimes Eq. (6) is known as the 'Modified Berggren Equation'.

Various empirically or analytically obtained correction factors have been proposed in geotechnical engineering literature. Aldrich and Paynter (1953) listed two alternative forms for λ which they obtained from earlier engineering work and/or from comparison to the Neumann solution. These simplify to Eqs. (7a) and (8a) once appropriate substitutions for differing nomenclature are made and further reduce to Eqs. (7b) and (8b) when the initial temperature is assumed to be zero.

$$\lambda_{1a} = \left\{1 + S_{T1}\left(-\frac{T_i}{T_s} + \frac{1}{2}\right)\right\}^{-0.5} \tag{7a}$$

$$\lambda_{1b} = \left(1 + \frac{S_{T1}}{2}\right)^{-0.5} \tag{7b}$$

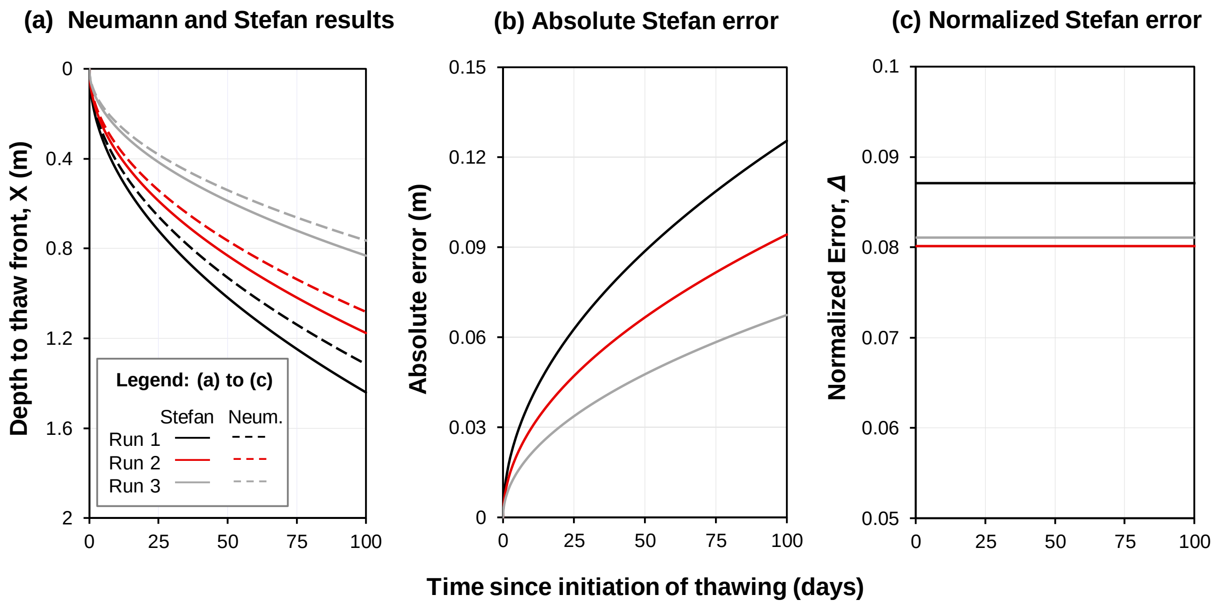

Figure 3: (a) Predicted depths to the thaw front for the Stefan (Eq. 2, solid lines) and Neumann (Eqs. 4-5, dashed lines) equations, (b) absolute error of the Stefan equation, and (c) normalized error of the Stefan equation vs. time for thawing scenarios 1-3. Soil thermal properties, initial conditions, and surface temperatures are presented in Table 1. The absolute errors were obtained via comparison to the Neumann solution, and the normalized errors were calculated as the difference between the Stefan and Neumann equations divided by the Stefan equation.

$$\lambda_{2a} = 0.707\left\{1 + S_{T1}\left(-\frac{T_i}{T_s} + \frac{1}{2}\right)\right\}^{-0.5} \tag{8a}$$

$$\lambda_{2b} = 0.707\left(1 + \frac{S_{T1}}{2}\right)^{-0.5} \tag{8b}$$

Nixon and McRoberts (1973) proposed a modified form of the Stefan equation via a comparison of the Stefan and Neumann solutions. When their expression is rearranged, it can be shown that they were essentially suggesting the following expression for λ:

$$\lambda_3 = \left(1 - \frac{S_{T1}}{8}\right) \tag{9}$$

Lunardini (1981, pp. 384-386) employed the heat balance integral method to obtain various corrections to the Stefan equation based on different assumptions for the temperature distribution. The associated λ values can be extracted by dividing Lunardini's (1981) corrected equations by the simple Stefan equation (Eq. 2). His most accurate λ for a given thawing scenario was shown to be:

$$\lambda_4 = \left(\frac{\sqrt{1+2S_{T1}} - 1}{S_{T1}} \right)^{0.5} \tag{10}$$

This is the first time these previously proposed correction factors have been presented in a single resource. In the following sections, alternative correction factors are proposed, and the performances of these expressions are compared.

2.3 Derivation of an implicit, analytical correction factor for soil thawing

The Neumann equation (Eq. 4) can be compared to the corrected Stefan equation (Eq. 6) to obtain the relationship between λ and m:

$$\lambda \sqrt{\frac{2k_u T_s}{\phi \rho_w L}} = m \tag{11}$$

Eq. (11) can be inserted into Eq. (5) to obtain an implicit mathematical expression for λ that is exact when the assumptions of the Neumann equation are met. This expression reduces to Eq. (12b) when the initial temperatures are 0°C.

$$0.5\ \lambda \sqrt{2\pi k_u T_s L \phi \rho_w} = \frac{k_u T_s}{\sqrt{\alpha_u}} \exp\left(\frac{-\lambda^2 k_u T_s}{2\alpha_u \phi \rho_w L} \right) \Big/ \operatorname{erf}\left(\lambda \sqrt{\frac{k_u T_s}{2\phi \rho_w L \alpha_u}} \right) + \frac{k_f T_i}{\sqrt{\alpha_f}} \exp\left(\frac{-\lambda^2 k_u T_s}{2\alpha_f \phi \rho_w L} \right) \Big/ \operatorname{erfc}\left(\lambda \sqrt{\frac{k_u T_s}{2\phi \rho_w L \alpha_f}} \right) \tag{12a}$$

$$0.5\ \lambda \sqrt{2\pi k_u T_s L \phi \rho_w} = \frac{k_u T_s}{\sqrt{\alpha_u}} \exp\left(\frac{-\lambda^2 k_u T_s}{2\alpha_u \phi \rho_w L} \right) \Big/ \operatorname{erf}\left(\lambda \sqrt{\frac{k_u T_s}{2\phi \rho_w L \alpha_u}} \right) \tag{12b}$$

This implicit form is inconvenient, and charts have been produced to alternatively allow the user to obtain λ from dimensionless parameters (Andersland and Ladanyi, 1994). These charts cannot be integrated directly into models or spreadsheet calculations, and thus the previously noted approximate expressions for λ have been proposed (Eqs. 7-10).

2.3.1 Analytical correction factor for soil thawing when T_i = 0°C

As shown in Eq. (12b) the mathematical expression for λ simplifies if the initial soil temperature is 0°C. If the Stefan number for soil thawing (Eq. 3) is inserted into Eq. (12b), the following simplification can be obtained:

$$\lambda\sqrt{\frac{\pi}{2S_{T1}}} = \exp\left(-\lambda^2\frac{S_{T1}}{2}\right)\Big/\operatorname{erf}\left(\lambda\sqrt{\frac{S_{T1}}{2}}\right) \tag{13}$$

This correction factor, which is only a function of S_{T1}, is not dependent on the frozen zone thermal properties because there is no conductive flux in the frozen zone when the initial temperatures are 0°C. Plots of λ vs S_{T1} will be generated from Eq. (13) to obtain an explicit polynomial expression to estimate λ directly from the Stefan number.

2.3.2 Analytical correction factor for soil thawing when T_i < 0°C

Equation (12a) is the implicit function for determining the Stefan correction for soil thawing when the 0°C initial temperature assumption is relaxed. This expression simplifies to:

$$\begin{aligned}\lambda\sqrt{\frac{\pi}{2S_{T1}}} &= \exp\left(\frac{-\lambda^2 S_{T1}}{2}\right)\Big/\operatorname{erf}\left(\lambda\sqrt{\frac{S_{T1}}{2}}\right) + \\ &\beta\frac{T_i}{T_s}\left\{\exp\left(\frac{-\lambda^2\delta S_{T1}}{2}\right)\Big/\operatorname{erfc}\left(\lambda\sqrt{\frac{S_{T1}\delta}{2}}\right)\right\}\end{aligned} \tag{14}$$

where β and δ are dimensionless parameters that account for the differences in the thermal properties of the frozen and unfrozen zones.

$$\beta = \sqrt{\frac{k_f c_f \rho_f}{k_u c_u \rho_u}} \tag{15}$$

$$\delta = \frac{\alpha_u}{\alpha_f} = \frac{k_u c_f \rho_f}{k_f c_u \rho_u} \tag{16}$$

Thus, when the soil temperatures are less than 0°C at the onset of surface thaw, the λ correction factor is an implicit function of three dimensionless variables (S_{T1}, δ, and $\beta T_i/T_s$). Typical values for β and δ can be explored using conventional approaches for estimating the bulk thermal properties of a soil-water-ice matrix as detailed in Appendix S.1.3.

2.4 Derivation of implicit, analytical correction factors for soil freezing

The Neumann and corrected Stefan equations for soil thawing can be modified for the case of soil freezing, and these can be respectively shown to be (Lunardini, 1981):

$$0.5\,L\,\phi\rho_w\,\sqrt{\pi}\,m = -\frac{k_f T_s}{\sqrt{\alpha_f}}\exp\left(\frac{-m^2}{4\alpha_f}\right)\Big/\operatorname{erf}\left(\frac{m}{2\sqrt{\alpha_f}}\right) - \frac{k_u T_i}{\sqrt{\alpha_u}}\exp\left(\frac{-m^2}{4\alpha_u}\right)\Big/\operatorname{erfc}\left(\frac{m}{2\sqrt{\alpha_u}}\right) \tag{17}$$

$$X_{corr} = \lambda\sqrt{\frac{2k_f(-T_s)t}{\phi\rho_w\,L}} \tag{18}$$

As before, Eq. (18) is the implicit equation from which the m term is obtained for the Neumann solution ($X = m\times t^{\,0.5}$). Note that because the frozen zone is now the upper layer, the locations of the frozen and unfrozen zone thermal properties have been interchanged in comparison to the Stefan and Neumann equations for soil thawing. The negative signs in Eqs. (17) and (18) arise because the relative signs of T_i and T_s have been interchanged in comparison to the case of soil thawing.

The Stefan equation errors can potentially be much greater for soil freezing than for thawing because the magnitude of the T_i/T_s ratios are typically higher. Figure 4 presents illustrative examples of the differences between the approximate Stefan equation (λ = 1, Eq. 18) and the Neumann solution (Eq. 17) for soil freezing. Table 1 presents the thermal properties used for the Stefan and Neumann calculations in Figure 4. Average surface temperatures of -3°C were chosen as these are typical of surface temperatures beneath a snowpack in permafrost regions (Brenning et al., 2005; Hoelzle, 1992). Initial conditions of 1, 2, and 5°C were chosen to represent different conditions for the average soil temperature before the onset of freezing (Table 1).

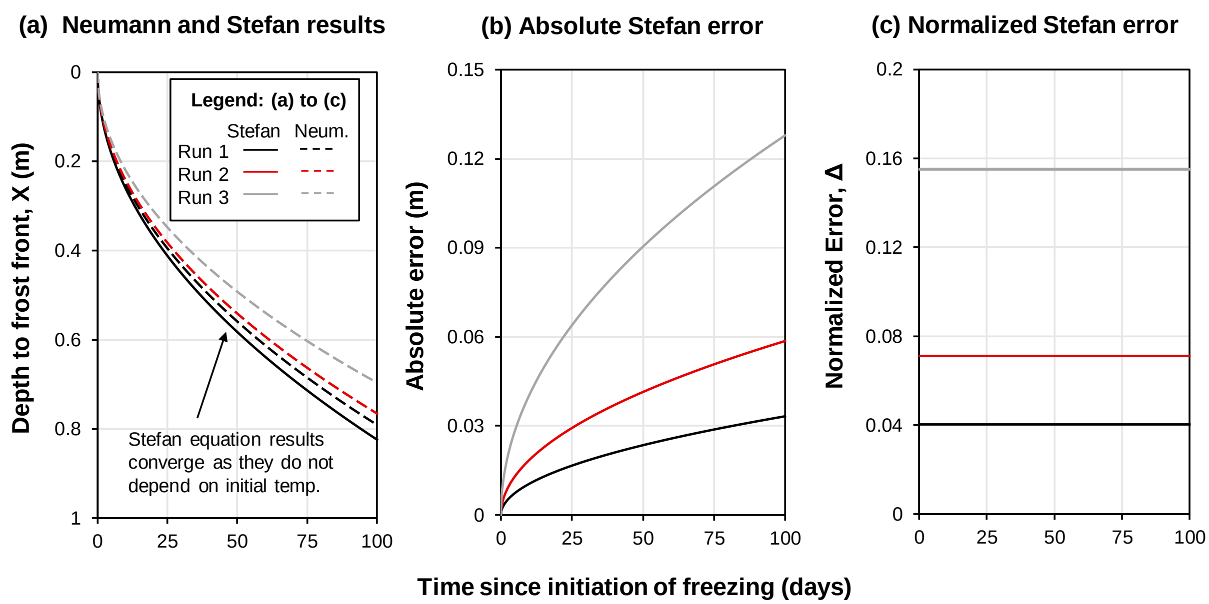

Figure 4: (a) Predicted depths to the frost front for the Stefan (Eq. 18, λ=1, solid lines) and Neumann (Eq. 17, dashed lines) equations, (b) Absolute error of the Stefan equation, and (c) normalized error of the Stefan equation vs. time for thawing scenarios 1-3 (initial temperatures = 1, 2, and 5°C, respectively). Soil thermal properties, initial temperatures, and surface temperatures are presented in Table 1. The Stefan errors were obtained via comparison to the Neumann solution.

Figure 4 demonstrates that, for these examples, the Stefan equation predicts frost depths that are up to 15.5% greater than those obtained from the Neumann equation. The Stefan equation is independent of the initial temperature as indicated by the overlap of the Stefan equation curves in Figure 4a. Thus, the errors of the Stefan equation increase with increasing initial temperature (Figure 4b). Because the error is related to the T_i/T_s ratio, the Stefan errors can be even higher in non-permafrost soils under deep snowpack where surface temperatures during soil freezing may be closer to 0°C. For example, for Run 3, the relative error of the Stefan equation increases to 23% if a surface temperature of -1°C is applied (results not shown), which is more typical of temperature beneath a snowpack in seasonally freezing soils (Kurylyk et al., 2013). Also, as in the case of soil thawing, the normalized errors of the Stefan equation are constant in time (Figure 4c), which suggests that the concept of applying a constant correction factor λ is also valid in the case of freezing.

2.4.1 Analytical correction factor for soil freezing when T_i = 0°C

Similar to before, the λ term can be inserted into Eq. (17) using the relationship between m and λ (see Eq. 11 for thawing).

$$0.5\,L\,\phi\rho_w\,\lambda\sqrt{\frac{2\pi k_f\left(-T_s\right)}{\phi\rho_w\,L}}=-\frac{k_f T_s}{\sqrt{\alpha_f}}\exp\left(-\frac{\lambda^2 k_f\left(-T_s\right)}{2\,\alpha_f\,\phi\rho_w\,L}\right)\Bigg/\operatorname{erf}\left(\lambda\sqrt{\frac{k_f\left(-T_s\right)}{2\alpha_f\,\phi\rho_w\,L}}\right)$$
$$-\frac{k_u T_i}{\sqrt{\alpha_u}}\exp\left(-\frac{\lambda^2 k_f\left(-T_s\right)}{2\,\alpha_u\phi\rho_w\,L}\right)\Bigg/\operatorname{erfc}\left(\lambda\sqrt{\frac{k_f\left(-T_s\right)}{2\alpha_u\phi\rho_w\,L}}\right) \qquad (19)$$

If T_i = 0°C, then Eq. (19) simplifies to:

$$\lambda\sqrt{\frac{\pi}{2S_{T2}}}=\exp\left(\frac{-\lambda^2 S_{T2}}{2}\right)\Bigg/\operatorname{erf}\left(\lambda\sqrt{\frac{S_{T2}}{2}}\right) \qquad (20)$$

where S_{T2} is the Stefan equation during freezing (Eq. 21), and all other terms have been previously defined.

$$S_{T2}=\frac{c_f\,\rho_f\left(-T_s\right)}{L\phi\,\rho_w} \qquad (21)$$

Note that similar to the case of soil thawing, the Stefan number does not depend on the initial temperature.

A comparison of Eqs. (20) and (13) reveals the strong mathematical parallels for the analytical correction factor equations during freezing or thawing when T_i = 0°C.

2.4.2 Analytical correction factor(s) for soil freezing when T_i > 0°C

Even when the T_i = 0°C assumption is relaxed, Eq. (19) can be shown to simplify to:

$$\lambda\sqrt{\frac{\pi}{2S_{T2}}}=\exp\left(\frac{-\lambda^2 S_{T2}}{2}\right)\Bigg/\operatorname{erf}\left(\lambda\sqrt{\frac{S_{T2}}{2}}\right)+\frac{T_i}{T_s}\frac{1}{\beta}\exp\left(\frac{-\lambda^2 S_{T2}}{2\delta}\right)\Bigg/\operatorname{erfc}\left(\lambda\sqrt{\frac{S_{T2}}{2\delta}}\right) \qquad (22)$$

where the definitions of β and δ are the same as in the case of soil thawing (Eqs. 15 and 16).

Comparisons between Eqs. (22) and (14) indicate the similarities between the analytical correction factor equations during freezing or thawing when $T_i \neq 0$°C. The primary differences are that the relative positions of β and δ have changed and the Stefan number for freezing (S_{T2}) is now employed.

3. Results and Discussion

3.1 Correction factors for the Stefan equation during thawing

3.1.1 Correction factors for soil thawing when T_i = 0°C

The simplest Stefan correction factor for soil thawing applies when the initial temperature is zero. In this case, the correction factor is only a function of the Stefan number, and various explicit equations (Eqs. 7-10) have been proposed to approximate the form of the implicit, analytical equation (Eq. 13). Figure 5 presents the results for the correction factors considered in this study versus the typical range of Stefan numbers experienced during soil thawing (0 to 1, see Fig. 2). The approximate Stefan correction factors listed by Aldrich and Paynter (1953) and Lunardini (1981) (Eqs. 7b, 8b, and 10) are inaccurate across the typical range of Stefan numbers. The results from Eq. (8b) are not even presented as this proved to be a very inaccurate expression that would not be visible on the vertical axis range in Figure 5. The correction factor proposed by Nixon and McRoberts (1973) in Eq. (9) is reasonably accurate for the typical range of Stefan numbers experienced for thawing soils, although its accuracy diminishes around S_{T1} values of 0.5 (Figure 5).

An even more accurate expression can be found by fitting a second order polynomial to the analytical results. The best fit (Figure 5) was found to be:

$$\lambda_5 = 1 - 0.16 S_{T1} + 0.038 S_{T1}^{\;2} \tag{23}$$

Table 2 presents the root-mean-square-error (RMSE) between the approximate λ correction factors and the implicit expression for λ (Eq. 13) for when the initial temperature is 0°C. The λ_5 term has the lowest RMSE by over an order of magnitude for Stefan numbers ranging up to 1.

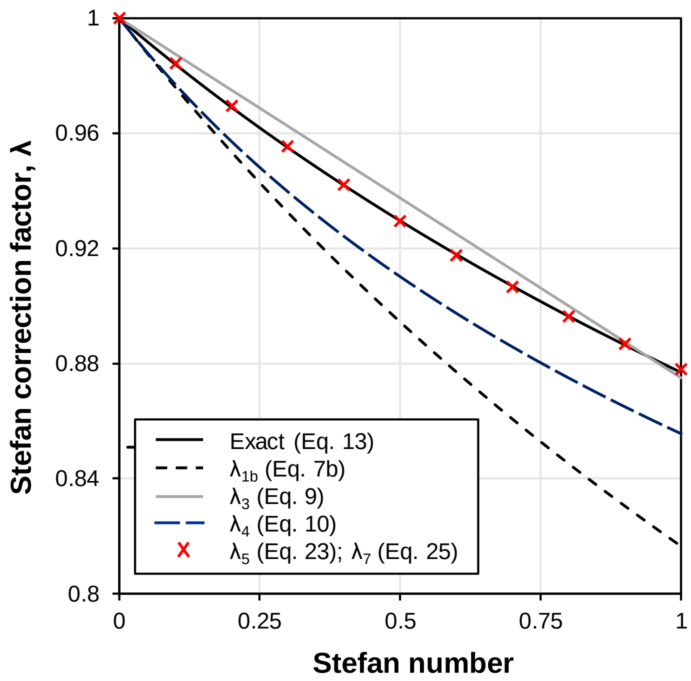

Figure 5: The analytical correction factor expression (Eq. 13), previously proposed approximate correction factors (Eqs. 7b, 9, and 10), and the new polynomial expression (Eq. 23) vs. the Stefan number for when T_i = 0°C.

Table 2: Performance of approximate Stefan correction factors during thawing when T_i = 0°C

Equation reference	Equation number	Symbol	RMSE (S_{T1}: 0 to 1)
Aldrich and Paynter (1953)	7b	λ_{1b}	0.038
Aldrich and Paynter (1953)	8b	λ_{2b}	0.297
Nixon and McRoberts (1973)	9	λ_3	0.006
Lunardini (1981)	10	λ_4	0.018
Present study	23	λ_5	0.0004

3.1.2 Correction factors for soil thawing when T_i < 0°C

It is not generally true that soil temperatures in the zone of seasonal thaw will exactly equal 0°C at the beginning of surface thaw. Typical initial shallow soil temperatures at the onset of surface thaw are -1 to -3°C, depending on the temperature at the bottom of the snowpack (Brenning et al., 2005; Hoelzle, 1992). Thus, approximate Stefan equation correction factors should be obtained for further reducing the predicted thaw rate due to negative initial temperatures.

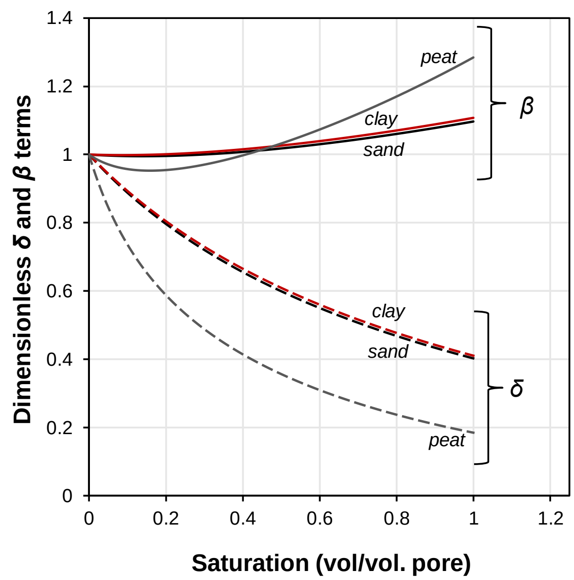

Figure 6: The dimensionless *β* and *δ* terms vs. the moisture (ice or water) saturation for sand, clay, and peat. The solid grain heat capacities were indirectly obtained from the bulk heat capacities presented in Bonan (2008, p. 134) by using the volumetrically weighted arithmetic mean method. The *β* and *δ* terms were calculated according to the method detailed in Appendix S.1.3. Porosities for the sand, clay, and peat were taken as 0.4, 0.4, and 0.8, respectively (Bonan, 2008).

Eq. (14) provides the implicit, analytical equation for obtaining λ during soil thawing when T_i < 0°C. This implicit relationships is difficult to reproduce using simple, polynomial functions given its dependence on three independent dimensionless numbers (S_{T1}, δ, and $\beta T_i/T_s$).

Figure 6 indicates that β (Eq. 15) ranges between 0.95 and 1.3 and that δ (Eq. 16) ranges between 0.2 and 1 and for typical soil types and moisture saturations. Appendix S.1.4 demonstrates that δ does not exert considerable influence on the analytical expression (Eq. 14) for this range of values, and thus hereafter we tacitly assume δ =1 to simplify the resultant equations to be a function of only two dimensionless numbers (S_{T1} and $\beta T_i/T_s$).

The previously proposed approximate correction factors presented in Eqs. (9) and (10) (Lunardini, 1981; Nixon and McRoberts, 1973) do not include a term related to the initial temperatures, and hence these forms cannot accommodate initial temperatures less than 0°C. Thus, only Eqs. (7a) and (8a) can in theory account for the further reduction in the soil thaw rate due to subfreezing initial conditions, and these have already been shown to be very inaccurate even when T_i = 0°C (Figure 5).

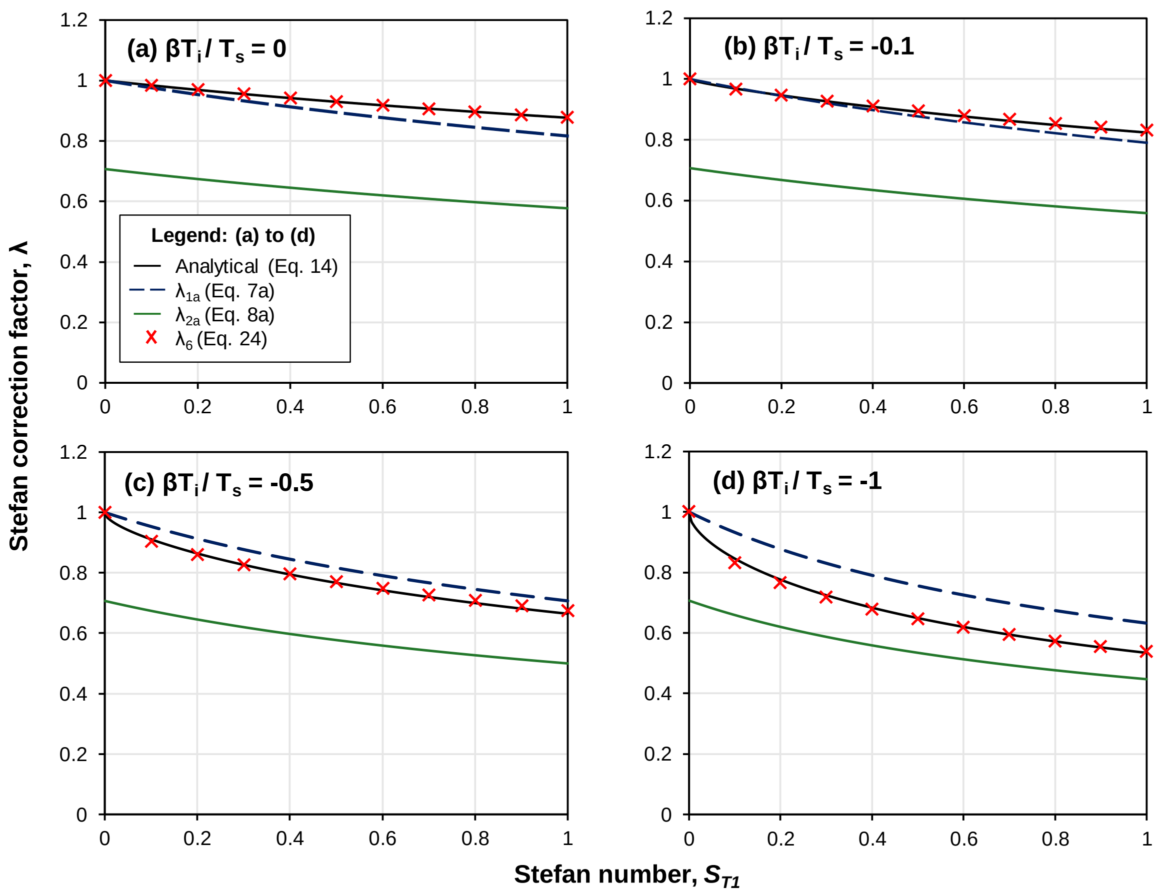

Figure 7: The analytical correction factor expression (Eq. 14), previously proposed approximate correction factors (Eqs. 7a and 8a) and the new polynomial correction factor (Eq. 24) vs. Stefan number for βT_i/T_s ratios: 0, -0.1, -0.5, and -1. These results are valid for soil thawing when δ = 1, which is a reasonable assumption (Appendix S.1.4).

Figure 7 shows the correction factor obtained from the implicit, analytical equation (Eq. 14), and the approximate, explicit equations for four different $\beta T_i/T_s$ ratios. The magnitude of the initial temperatures is typically less than the average surface temperature during thawing, and β typically has a narrow range between 0.95 and 1.3 (Fig. 6). Thus only $\beta T_i/T_s$ ratios up to -1 are considered. The two correction factor equations listed by Aldrich and Paynter (1953) can yield errors up to 25%, and these would translate into associated 25% errors in the calculated thaw depth.

An excellent fit for the λ correction factor for all of the considered $\beta T_i/T_s$ ratios was found to be:

$$\lambda_6 = \left[1 + 0.147 S_{T1}\left(\beta \frac{T_i}{T_s}\right)^2 + 0.535\sqrt{S_{T1}}\beta \frac{T_i}{T_s}\right] \times \lambda_5 \qquad (24)$$

where λ_5 can be obtained from Eq. (23).

Note that the bracketed term in Eq. (24) represents the further reduction of the correction factor due to initial temperatures less than 0°C. The resultant polynomial equation is a readily calculated constant that is only a function of the Stefan number and the product of β and the ratio of the initial temperature to surface temperature. Table 3 presents the RMSE value between the explicit, approximate correction factor equations (Eq. 7a, 8a, and 24) and the implicit, analytical equation (Eq. 14) for each non-zero $\beta T_i/T_s$ ratio considered. In all cases, the RMSE values associated with Eq. (24) are at most 25% of those obtained using Eqs. (7a) or (8a).

Table 3: Performance of approximate correction factor results for thawing when T_i < 0°C

Equation reference	Equation number	Symbol	RMSE (S_{T1}: 0 to 1)
(a) $\beta(T_i / T_s)$ = *-0.1*			
Aldrich and Paynter (1953)	7a	λ_{1a}	0.019
Aldrich and Paynter (1953)	8a	λ_{2a}	0.273
Present study	24	λ_6	0.004
(b) $\beta(T_i / T_s)$ = *-0.5*			
Aldrich and Paynter (1953)	7a	λ_{1a}	0.046
Aldrich and Paynter (1953)	8a	λ_{2a}	0.199
Present study	24	λ_6	0.006
(c) $\beta(T_i / T_s)$ = *-1*			
Aldrich and Paynter (1953)	7a	λ_{1a}	0.100
Aldrich and Paynter (1953)	8a	λ_{2a}	0.134
Present study	24	λ_6	0.007

3.2 Correction factors for the Stefan equation during freezing

As previously noted, there are strong parallels between the implicit, analytical equations for determining λ during freezing or thawing. These parallels suggest that the approximate correction factors obtained for the case of thawing (Eqs. 23 and 24) may, in some cases, also be applied for the case of freezing, provided that appropriate substitutions are made.

3.2.1 Correction factors for soil freezing when T_i = 0°C

When initial temperatures are 0°C, the analytical expressions for the correction factors during freezing or thawing (Eqs. 13 and 20) are exactly identical except that the Stefan numbers for freezing and thawing are interchanged.

Hence, in this case the appropriate correction factor equation during freezing can be obtained by replacing S_{T1} in Eq. (23) with S_{T2}:

$$\lambda_7 = 1 - 0.16 S_{T2} + 0.038 {S_{T2}}^2 \tag{25}$$

The range of Stefan numbers experienced during soil freezing is typically more constrained than in the case of thawing. Thus, this equation form, which was obtained for Stefan numbers ranging up to 1, is still appropriate in the case of soil freezing. The RMSE values in Table 2 and graphical fits in Figure 5 are still valid in the case of freezing.

3.2.2 Correction factors for soil freezing when T_i >0°C

When initial temperatures are above 0°C, the appropriate Stefan equation correction factor for soil freezing should not be taken from the polynomial function that was obtained for thawing (Eq. 24). This equation was only shown to be accurate in the case of thawing for $\beta(T_i/T_s)$ values ranging from 0 to -1 (Figure 7). Negative values were considered in the case of thawing because T_i was negative and T_s was positive. In the case of freezing, the signs of these temperatures are switched, and the ratio remains negative. However, the potential range of this ratio increases in the case of freezing.

The thawing and freezing scenarios also differ because β moves from the numerator to the denominator in the dimensionless number in the case of freezing (Eq. 14 vs. 22). Recall that β

ranges between about 0.95 and 1.3 (Fig. 6). Here we consider ($T_i/\beta T_s$) ratios of -1, -5, and -10. A ratio of -10 would not likely be realized in a permafrost environment, but it may be achieved in a seasonally freezing environment where initial temperature are higher and surface temperature are closer to 0°C (Kurylyk et al., 2013).

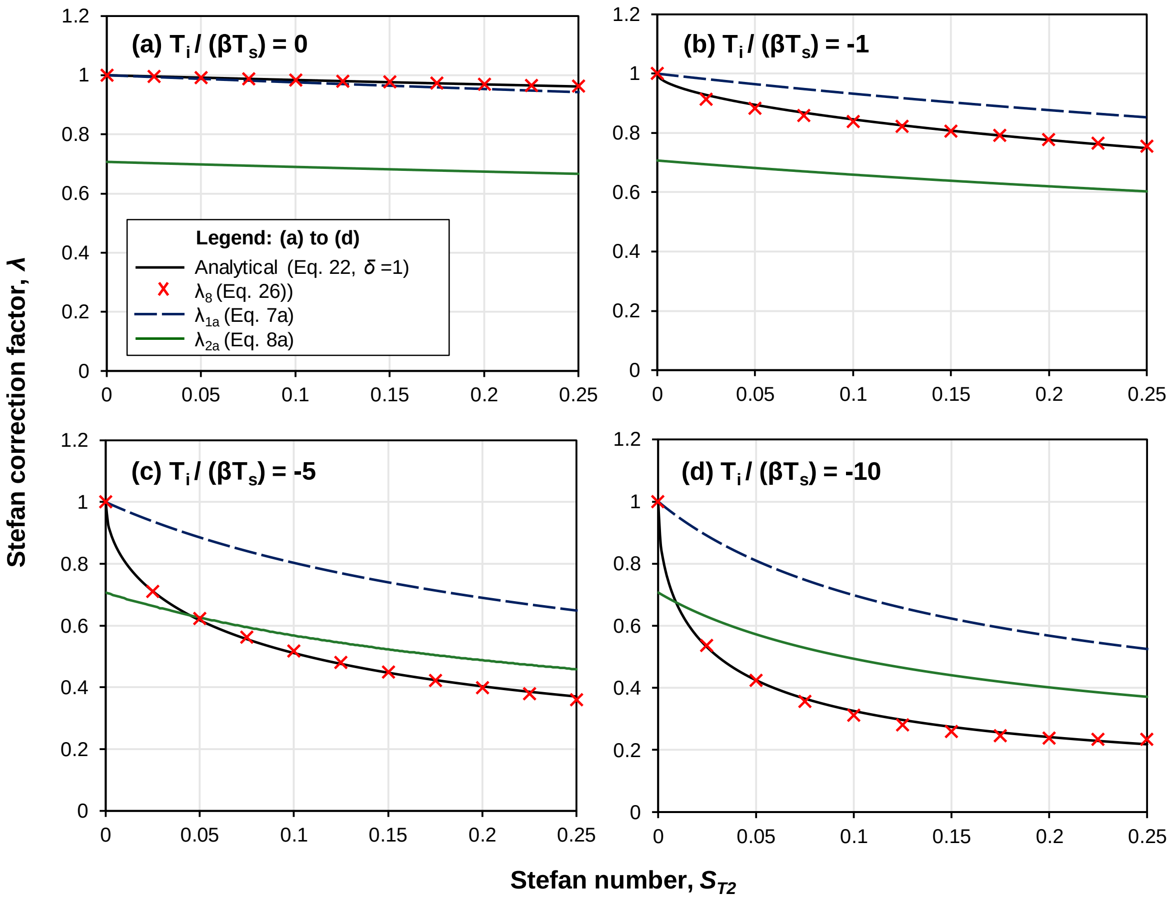

Figure 8: The analytical correction factor expression (black, Eq. 22), previously proposed approximate correction factors (Eqs. 7a and 8a), and new polynomial correction factor (Eq. 26) vs. the Stefan number for soil freezing when T_i >0°C and when δ =1 and $\beta T_i/T_s$= 0, -1, -5, and -10.

The magnitude of the Stefan numbers is typically less than 0.1 in the case of freezing. Thus, to develop the approximate Stefan equation correction factor for soil freezing, we constrain the range of Stefan numbers to be 0 to 0.25. As in the case of soil thawing, δ exerts only minor influence on the λ value obtained from the analytical expression (Eq. 22), and thus δ is assumed to equal 1 (Appendix S.1.4). A reasonable fit to the implicit, analytical equation (Eq. 22) can be obtained by the following function:

$$\lambda_8 = \left[1 + 0.061\left(S_{T2}^{\ 0.88}\left(-\frac{T_i}{\beta T_s}\right)^{1.65}\right) - 0.43\left(S_{T2}^{\ 0.44}\left(-\frac{T_i}{\beta T_s}\right)^{0.825}\right)\right]\lambda_7 \qquad (26)$$

where λ_7 is given in Eq. (25). Figure 8 presents results from the analytical equation and the three approximate equations (Eqs. 7a, 8a, and 26) that accommodate positive initial temperatures at the onset of surface freezing. Table 4 indicates that the RMSE values for Eq. (26) are always at least an order of magnitude smaller than those for the previous equations for the scenarios considered.

Table 4: Performance of approximate correction factor results for freezing when $T_i > 0$°C

Equation reference	Equation number	Symbol	RMSE (S_{T2}: 0 to 0.25)
	(c) $T_i / (\beta T_s) = -1$		
(Aldrich and Paynter, 1953)	7a	λ_{1a}	0.087
(Aldrich and Paynter, 1953)	8a	λ_{2a}	0.188
Present study	26	λ_8	0.008
	(c) $T_i / (\beta T_s) = -5$		
(Aldrich and Paynter, 1953)	7a	λ_{1a}	0.274
(Aldrich and Paynter, 1953)	8a	λ_{2a}	0.077
Present study	26	λ_8	0.006
	(c) $T_i / (\beta T_s) = -10$		
(Aldrich and Paynter, 1953)	7a	λ_{1a}	0.347
(Aldrich and Paynter, 1953)	8a	λ_{2a}	0.153
Present study	26	λ_8	0.010

3.3 Summary of results and practical significance

Table 5 summarizes the four approximate correction factor equations obtained in this study. The appropriate selection of these equations depends only on the initial temperature and the nature of the phase change (i.e., freezing or thawing). The corrected Stefan equation is obtained by the product of the correction factor equation and the standard Stefan equation for freezing or thawing.

Table 5: Summary of approximate equations for estimating the Stefan correction factor

Scenario	Analytical equation	Approximate correction factor equation[1,2,3]
		(a) Thawing
$T_i = 0°C$	Eq. (23)	$\lambda_5 = 1 - 0.16 S_{T1} + 0.038 S_{T1}{}^2$
$T_i < 0°C$	Eq. (24)[2,3]	$\lambda_6 = \left[1 + 0.147 S_{T1}\left(\beta \dfrac{T_i}{T_s}\right)^2 + 0.535\sqrt{S_{T1}}\beta \dfrac{T_i}{T_s}\right] \times \lambda_5$
		(b) Freezing
$T_i = 0°C$	Eq. (25)	$\lambda_7 = 1 - 0.16 S_{T2} + 0.038 S_{T2}{}^2$
$T_i > 0°C$	Eq. (26)[2,3]	$\lambda_8 = \left[1 + 0.061\left(S_{T2}{}^{0.88}\left(-\dfrac{T_i}{\beta T_s}\right)^{1.65}\right) - 0.43\left(S_{T2}{}^{0.44}\left(-\dfrac{T_i}{\beta T_s}\right)^{0.825}\right)\right]\lambda_7$

[1]The expressions for obtaining S_{T1}, S_{T2}, and β are given in Eqs. (3), (21), and (15), respectively.

[2]The λ_5 and λ_7 which are required for Eq. (24) and (26) respectively can be found in Eqs. (23) and (25).

[3]Eqs. (24) and (26) do not include the dimensionless δ term (Eq. 16), and thus they tacitly imply δ =1. Appendix S1.4 demonstrate that δ exerts little control on the correction factor, at least for typical δ ranges (0.2 to 1), and thus the δ =1 assumption is generally justified.

The derivations of the Stefan equation correction factors presented in Sections 3.1 and 3.2 are somewhat onerous, and the forms of the resultant analytical expressions are not conducive to inclusion in land surface schemes, hydrology models, or engineering practice. However, these implicit equations can be reasonably approximated with the simple polynomial equations given in Table 5. When initial temperatures can be assumed to equal 0°C, the appropriate correction factor requires no more information than forms proposed in previous studies (Aldrich and Paynter, 1953; Lunardini, 1981; Nixon and McRoberts, 1973) as only the Stefan number is required. However, the performance of the second order polynomial equation proposed in this study (Eq. 23 or 25) can be shown to perform considerably better than previous equations (Figure 5 and Table 2). When non-zero initial temperatures are accommodated, the resultant polynomial equations (Eq. 24 or 26) also require β and the ratio of the initial to surface temperatures as input parameters. The β parameter, which is merely a measure of the difference in thermal properties between the frozen and unfrozen zones, can be easily obtained via Eq. (15). In the case of lower moisture saturation (e.g., < 40%), β can be assumed to be 1 for most soils (Fig. 6). These polynomial equations (Table 1) can be included in spreadsheet-based programs for calculating ground freezing and thawing in engineering practice or coded into cold regions hydrology models that currently employ the Stefan equation to calculate the rate of freeze-thaw.

To demonstrate the utility of the equations presented in Table 5, we return to the examples given in Figures 3 and 4. These figures compare the depths to the thaw or frost fronts calculated by the Stefan and Neumann equations for a total of six scenarios. In all cases, the Stefan equation overestimates the penetration of the thaw or frost front. Figure 9 presents the corrected Stefan equation results overlaid on the results previously presented in Figures 3 and 4. For the thawing results (Figure 9a), the corrected Stefan equation was calculated as the product of λ_6 (Eq. 24) and the standard Stefan equation for thawing (Eq. 2). For the freezing results (Figure 9b), the corrected Stefan equation was obtained by inserting λ_8 (Eq. 26) into the Stefan equation for freezing (Eq. 18). The corrected Stefan equation results overly the Neumann solution results and thus demonstrate that the Stefan equation has been appropriately modified to account for the influence of sensible heat. These Stefan equation correction factors can also be incorporated into previously proposed Stefan-type algorithms which accommodate soil layering (Jumikis, 1977; Kurylyk, 2015), variable surface temperatures (Andersland and Anderson, 1978), depth-dependent initial temperature (Jumikis, 1977; Kurylyk, 2015), and temporally changing moisture content (Hayashi et al., 2007). Details regarding these algorithms and appropriate modifications can be found in Appendix S1.5.

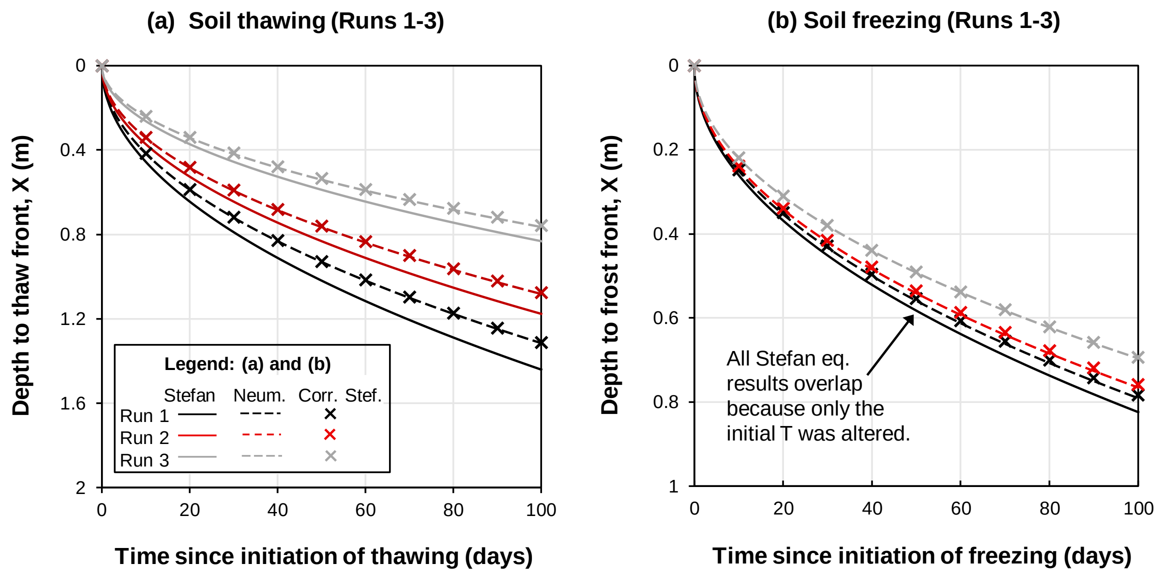

Figure 9: The calculated depths to the thaw front (a) and frost front (b) obtained from the Neumann, Stefan, and corrected Stefan equations. Equation parameters are listed in Table 1. The *λ* correction factors for the thawing and freezing runs were calculated with Eqs. (24) and (26) respectively (Table 5).

4. Limitations

There are limitations associated with the Stefan and Neumann solutions that are not addressed herein. These limitations stem from assumptions in the governing equations including: the infinitesimal freeze-thaw temperature range, strictly one-dimensional heat flow, and negligible heat advection (Kurylyk et al., 2014b). As indicated in Appendix S1.5, other assumptions of the analytical solutions (e.g., homogeneous soil, constant surface temperature, and temporally constant moisture) can be relaxed by incorporating the polynomial correction factors developed in this study into the Stefan-type algorithms proposed by previous researchers. Of particular note, the correction factors presented in Table 5 are obtained from the Neumann solution, which assumes a constant temperature. However, these factors are applied to the Stefan equation for which it is common to replace the product of the constant surface temperature and time with the thawing or freezing index due to a variable surface temperature (see Eqs. 1 and 2).

One practical limitation associated with all analytical solutions for calculating soil freeze-thaw is that the surface temperature must be obtained. In engineering practice, the entirely empirical *n*-

factor (Klene et al., 2001) is often used due to the lack of surface temperature date for many sites. The *n*-factor has been shown to exhibit inter-annual variability in mountainous regions due to inter-annual changes in snowpack (Juliussen and Humlum, 2007). However, alternative quasi-empirical approaches exist to determine surface temperature or the ground heat flux by balancing the surface energy fluxes (e.g., Hwang, 1976; Williams et al., 2015).

The equations proposed in this study should not be utilized if the dimensionless numbers (S_{T1}, S_{T2}, $\beta T_i/T_s$, and $T_i/(\beta T_s)$) are outside of the ranges considered in this study. In such cases, the principles demonstrated herein can be applied to obtain alternative expressions.

In general, our purpose is not to overcome all of the limitations associated with the Stefan equation, but rather to propose an approach for relaxing the specific assumption of negligible heat capacity. This assumption can severely limit the utility and accuracy of the Stefan equation, especially when the initial temperatures prior to freezing or thawing deviate from 0°C. Hence, our proposed Stefan equation correction factors (Table 5) should be a useful contribution to the large community of engineers and cold regions scientists who still apply the Stefan equation to calculate soil freeze-thaw.

5. Conclusions and Summary

It has long been known that the Stefan equation can overestimate the penetration of soil freeze-thaw because the sensible heat required to change the temperature of soil effectively retards the rate of soil freezing or thawing (Aldrich and Payner, 1953). Previous studies have suggested that a correction factor less than 1 can be applied to account for soil heat storage and thus improve the accuracy of Stefan equation predictions. However, most previous proposed correction factors are only valid in the simplest thawing or freezing case when initial temperature are 0°C. The present study has demonstrated that all of these approximate equations, except for the form proposed by Nixon and McRoberts (1973), perform poorly even for this simplest case (Fig. 5 and Table 2). Furthermore, non-zero initial temperatures can further impede the rate of soil freezing or thawing, and the correction factor proposed by Nixon and McRoberts (1973) cannot accommodate this phenomenon. Thus none of the previously proposed correction factors identified in this study are appropriate for correcting the Stefan equation when initial temperatures are not equal to 0°C.

We have proposed four alternative polynomial equations for approximating the Stefan equation correction factor λ (Table 5). These polynomial expressions can be implemented in more flexible Stefan-type algorithms that consider other factors such as multiple soil layers, variable surface temperature via the thawing index, or temporally changing moisture conditions.

Despite its limitations, the Stefan equation is still frequently applied in permafrost settings to calculate the rate of freeze-thaw as indicated by its consideration in several recent review papers (Bonnaventure and Lamoureux, 2013; Kurylyk et al., 2014a; Riseborough et al.; 2008; Zhang et al., 2008). The proposed modifications to this equation improve its fidelity to physical processes and are timely given the emerging concerns regarding the influence of climate change on subsurface thermal regimes in cold regions.

6. Acknowledgements

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7. Supporting Information

All supporting information can be found in the supplementary Appendix S1, which is further divided into five subsections as referenced in this paper.

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