

# ENTROPY AND THE MONITORING PROBLEM

by

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# Abstract

Entropy is a well-known quantity that is used to describe various phenomena in physics and information theory. Like energy or information, entropy cannot be measured directly and traditionally is used to describe the state of other physical quantities. Recently, a Russian physicist Anatoly Panchenkov introduced a new, more general, notion of entropy.

In view of the principle of maximum  $P$ -entropy, a system evolves in the direction of its maximum lifespan, e.g., the life expectancy of humans, or business structures increases. An important feature of the differential equations that follow from the principle of maximum  $P$ -entropy is that they can be used to describe not only evolution, for example, as the equation of classical mechanics, but also events.

In this thesis we will investigate how to employ the  $P$ -entropy to construct mathematical models that can be used in the theory of monitoring.

# List of Abbreviations and Symbols Used

VCM Virtual Continuous Medium

$\mathbb{Z}$  integer numbers

$\mathbb{R}$  real numbers

$H_f$  total entropy

$H_q$  structural entropy

$H_p$  impulse entropy

$\Theta$  potential of acceleration

$\Psi$  potential of scalar impulse

$m$  total mass

$V$  total volume

$\rho$  density of the virtual continuous medium

$A$   $\left\{ \begin{array}{l} \text{flow velocity in the virtual continuous medium} \\ \text{tangent space} \end{array} \right.$

$\Omega$  the phase space

$\Lambda$  the matrix in the Riccati equation

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# Chapter 1

## Introduction

Mathematical modelling is one of the most challenging scientific undertakings. It has been used to describe and study the evolution of different phenomena in the real world in terms of mathematical laws that also availed us of new mathematical theories [18]. For example, calculus was developed by Newton and Leibnitz to describe the laws of mechanics, but its development has also created a new field of mathematics.

The use of mathematical models is instrumental in understanding the behaviour of complex systems. Ideally, we wish to have a model that can be used to describe not only the dynamics of such an evolution (that is its motion, or a continuous change in time), but also any special events, such as, for instance, emergence or collapse. The latter is especially difficult and is a relatively underdeveloped area of mathematical modelling.

The importance of such problems is well-documented, and they are studied by scientists in social sciences as well. For example, an American historian J. Tainter, who is a professor at the University of Utah, in his famous book "Collapse of Complex Societies" [23] has investigated the evolution and collapse of complex societies, using mainly the tools of social sciences.

In this thesis we will study the methods of mathematical modelling developed recently by a Russian physicist A. Panchenkov to see how they can be applied to monitoring-type problems in economics and elsewhere. Panchencov's theory is based on a new, extended notion of entropy and utilises various techniques and notions of geometry, physics, fluid dynamics and asymptotic theory.

### 1.1 Background

Throughout the thesis we will call the new, extended notion of entropy from A. Panchenkov *the P-entropy*. The development of the *P-entropy* is rooted in earlier works by Panchenkov, in which he developed certain asymptotic methods applied to the problems of fluid dynamics. More specifically, he developed mathematical models that were used to design

the so-called “ground effect vehicles (GEV)” in the former USSR. “A ground effect vehicle is one that attains level flight near the surface of the Earth, making use of the aerodynamic interaction between the wings and the surface known as ground effect. Best known are the Soviet ekranoplanes, but names like wing-in-ground-effect (WIG), flarecraft, sea skimmer, or wing-in-surface-effect ship (WISE) are also used. In recent years a large number of different GEV types have been developed for both civilian and military use. However, these craft have yet to enter widespread use. The German Tandem Airfoil Flairboats or Skimmerfoils constructed by Gunther Jörg differ in their use of self-stabilizing wings. Although they might look similar and/or have related technical characteristics, ground effect vehicles are not aircraft, seaplanes, hovercraft, or hydrofoilsground effect is a separate technology altogether” (by International Maritime Organization). A. Panchenkov has 12 patented inventions that were implemented during the course of the design and construction of Soviet ekranoplanes, in particular, the ekranoplanes of the series ADP. Panchenkov developed an asymptotic method, called *the method of limiting correctness* which was used to solve the boundary value problems arising in fluid dynamics that were instrumental in describing the unique vortices created in the wake of ekranoplanes.

The  $P$  entropy introduced by Panchenkov is an independent quantity, which unlike the conventional entropy of information theory or thermodynamics, is not linked to other physical quantities. It is a conserved quantity under that arises from the variational principle of maximum ( $P$ -) entropy, introduced by Panchenkov to derive the underlying differential equations. According to this principle, the evolution of an object of study occurs in a new medium, called *the virtual continuous medium*.

In what follows, we will describe how the behaviour of complex systems can be studied from this viewpoint in the framework of the monitoring-type problems.

## 1.2 Literature Review

There is a vast literature on mathematical modelling. While working on this thesis, I used various works on entropy in information theory and thermodynamics, for example, those by Clausius (see [5]), who developed the modern terminology, Boltzmann (see [3]), Gibbs (see [9]), Shannon (see [21]) and others.

In spite of the many achievements in the fields of thermodynamics and information theory related to entropy, these results are still rather limited, especially because they are linked to specific areas of science. A more general notion of entropy that generalizes known results in the entropy theory was proposed by Panchenkov, who has been studying entropy since the 1960s, subsequently developing in recent years a theory based on the new notion of entropy [15]. It must be mentioned that his theory was influenced by the results of I. Prigogine on the entropy of open systems [17]



Based on what has been done, it is conceivable that there is a substantial potential of application of the new theory based on the notion of the  $P$ -entropy to the problems arising in social sciences, particularly, in economics.

### 1.3 Outline

The thesis is organized as follows. Chapters 2 and 3 provide necessary mathematical background. Specifically, in Chapter 2 I review basic notions of calculus of variations. Chapter 3 is devoted to Hamiltonian mechanics and related topics of differential geometry. In Chapter 4, I introduce and describe the properties of the  $P$ -entropy, as well as the new variational principle - the principle of maximum entropy due to Panchenkov. Chapter 5 deals with geometric structures necessary for the implementation of Panchenkov's method. In Chapter 6 I implement Panchenkov's results and apply them to a specific model, that results in a system of differential equations. In Chapter 7 I make the conclusions.

## Chapter 2

# Calculus of Variations

### 2.1 The Fundamental Problem of Calculus of Variations

The calculus of variations is concerned with the maxima or minima of functionals, which are collectively called extrema. The notion of a functional, which can be described as “function of a function”, plays an important role in applied mathematics, physics, mechanics and other sciences. More precisely, functional is a correspondence which assigns a determined real or complex number to each function or curve belonging to some class. [19]. This chapter is based on the Chapter 3 in the *Mathematical Methods of Classical Mechanics* by Arnold [2].

Let us see the following example to get a better understanding.

**Example 2.1. (*Circular motion*)** Consider a weight swung in a circle with constant radius  $r$  and constant speed  $v$ . Then we want to find uniform circular motion in this plane. The motion of the system can be described in terms of a single parameter  $q = q(t)$  via the map  $\mathbf{x}(q) = (r \cos q, r \sin q)$ . Then the motion is parametrized by  $(t, q(t))$  for  $(t_0, t_1)$ , where  $q(t_0) = q_0$  and  $q(t_1) = q_1$ . The quantity  $I$ , the action of the weight, we want to minimize depends on the specific evolution being followed, and thus is a function of  $q$ ,  $I[q]$ . This  $I[q]$  is a functional.

#### 2.1.1 Variations

**Definition 2.1.** Let  $I[y]$  be a functional defined on some normed linear space and let

$$\Delta I[y, h] = I[y + h] - I[y]$$

be its *increment*, corresponding to the increment  $h = h(x)$  of the independent variable  $y = y(x)$ . If  $y$  is fixed,  $\Delta I[h]$  is a functional of  $h$ , in general a non-linear functional. Suppose that

$$\Delta I[y, h] = \delta I[y, h] + \epsilon \|h\|,$$

where  $\phi[h]$  is a linear functional and  $\epsilon \rightarrow 0$  as  $\|h\| \rightarrow 0$ . Then the functional  $I[y]$  is said to be differentiable, and the *principal linear part* of the increment  $\Delta I[h]$ , i.e., the linear functional  $\phi[h]$  which differs from  $\Delta I[h]$  by an infinitesimal of order higher than 1 relative to  $\|h\|$ , is called the *variation* (or *differential*) of  $I[y]$  and is denoted by  $\delta I[h]$ .

### 2.1.2 Extrema

As we previously mentioned, the calculus of functionals is mostly concerned with the minima or maxima of functionals, which are collectively called extrema.

**Definition 2.2.** A functional  $I[y]$  is said to have an *extremum* at the function  $y_0$  if  $\Delta y = I[y] - I[y_0]$  has the same sign for all  $y$  in an arbitrarily small neighbourhood of  $y_0$ . The function  $y_0$  is called an *extremal function*. The extremum  $I[y_0]$  is called a *maximum* if  $\Delta I \leq 0$  everywhere in an arbitrarily small neighbourhood of  $y_0$ , and a *minimum* if  $\Delta I \geq 0$  therein.

### 2.1.3 Function spaces

Functionals are defined on spaces of functions.

**Definition 2.3.** A *function space* is a Banach space made of functions. Each function in the space can be thought of as a point.

We regard each function  $y = y(x)$  in this framework as a point in some space and the space whose elements are functions will be called a *function space*.

- Example 2.2.**
1.  $C^0(a, b)$  is the space of all continuous functions in  $[a, b] \subset \mathbb{R}$
  2.  $C^1(a, b)$  is the space of all functions  $y(x)$  defined in  $[a, b]$  which are continuous and have continuous first derivatives.
  3.  $C^n(a, b)$  is the the space of all functions  $y(x)$  in  $[a, b]$  that are continuous and have continuous derivatives up to the order  $n$ , where  $n \in \mathbb{N}$ .

**Theorem 2.4.** [8] A necessary condition for a differentiable functional  $I[y]$  to have an extremum at  $y = y^*$  is that its variation vanish at  $y = y^*$ , i.e.  $\delta I[y, h] = 0$  at  $y = y^*$  and for all admissible  $h$ .

*Proof.* Suppose  $I[y]$  has a minimum at  $y = y^*$ . Hence  $\Delta I[y, h] = \delta I[y] + \epsilon \|h\|$ , where  $\epsilon \rightarrow 0$  as  $\|h\| \rightarrow 0$ . Thus, for a sufficiently small  $\|h\|$  the sign of  $\Delta I[y, h]$  will be the same as the sign of  $\delta I[y, h]$ . Suppose  $\delta I[y, h_0] \neq 0$  for some  $h_0$ . Then  $\forall \alpha \in \mathbb{R}$ , we have  $\delta I[y, \alpha h_0] = \alpha \delta I[y, h_0]$ . Hence  $\Delta I[y, h] = \delta I[y, h] + \epsilon \|h\|$  can be made to have either sign for an arbitrary small  $\|h\|$ . But this is impossible since by the hypothesis  $I[y]$  has a minimum at  $y = y^*(x)$ , that is

$$\Delta I[y^*, h] = I[y^* + h] - I[y^*] \geq 0$$

for sufficient small  $\|h\|$ . □

**Remark 2.1.1.** The theorem 2.4 does not always work in many cases the functional is not even differentiable. In practice we usually employ variational principle to achieve for the extrema.

#### 2.1.4 The fundamental problem of calculus of variations

The fundamental problem of calculus of variations can be formulated as follows: given a function  $F = F(t, q, \dot{q})$ , find the functions  $q(t)$  corresponding to the extremal points of the integral

$$I = \int_{t_1}^{t_2} F(t, q, \dot{q}) dx,$$

subject to the boundary conditions

$$\begin{cases} q(t_1) = q_1, \\ q(t_2) = q_2. \end{cases} \quad (2.1)$$

How do we identify the functions  $y(x)$  that maximize or minimize the integral  $I$ ? Like the vanishing derivative for variables, the Euler-Lagrange condition represents precisely these functions.

## 2.2 Euler-Lagrange Equation

**Definition 2.5.** The equation

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0$$

is called the *Euler-Lagrange equation* for the functional

$$I[q] = \int_{t_1}^{t_2} F(t, q, \dot{q}) dx.$$

where  $[t_1, t_2] \subset \mathbb{R}$ ,  $q = q(t)$ ,  $\dot{q}$  is the derivative of  $q$ .

**Theorem 2.6.** [2] *A necessary condition for the integral*

$$I = \int_{t_1}^{t_2} f(t, q, \dot{q}) dx$$

*to attain an extremum subject to the boundary conditions*

$$\begin{cases} q(t_1) = q_1, \\ q(t_2) = q_2. \end{cases} \quad (2.2)$$

*is that the function  $y=y(x)$  satisfy the Euler-Lagrange equation*

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{q}} - \frac{\partial F}{\partial q} = 0.$$

Now let  $\mathbf{q}$  be a vector in the  $n$ -dimensional coordinate space  $\mathbb{R}^n$ , and  $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  as a function of  $2n + 1$ . We can generalize the Euler-Lagrange equation as follows:

The equation

$$\frac{d}{dt} \left( \frac{\partial F}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial F}{\partial \mathbf{q}} = 0.$$

is the *generalized Euler-Lagrange equation* for the functional

$$I[\mathbf{q}] = \int_{t_1}^{t_2} F(t, \mathbf{q}, \dot{\mathbf{q}}) dx.$$

## 2.3 Hamilton's Principle of Least Action

Here we present Newton's equations of dynamics

$$\frac{d}{dt}(m_i \dot{q}_i) + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \dots, n$$

where  $U = U(q_i)$  is potential energy,  $t$  is time,  $m_i$  are masses,  $q_i$  are positions and  $\dot{q}_i$  are momenta.

**Theorem 2.7.** [2] *Motions of the mechanical system above coincide with extrema of the functional*

$$I[\mathbf{q}] = \int_{t_1}^{t_2} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt,$$

where  $L = T - U$  is the difference between the kinetic and potential energy, called the Lagrangian. It is subject to the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = 0.$$

*Proof.* Since  $U = U(\mathbf{q})$  and  $T = \sum \frac{m_i \dot{q}_i^2}{2}$ , where  $\mathbf{q} = (q_1, q_2, \dots, q_n)$  We have  $\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = m_i \dot{q}_i$  and  $\frac{\partial L}{\partial q_i} = -\frac{\partial U}{\partial q_i}$ . It satisfies the Euler-Lagrange equations. □

**Definition 2.8.**  $L(t, q, \dot{q}) = T - U$  is the Lagrangian function or Lagrangian,  $q_i$  are the generalized coordinates,  $\dot{q}_i$  are generalized velocities,  $\frac{\partial L}{\partial \dot{q}_i} = p_i$  are generalized momenta,  $\frac{\partial L}{\partial q_i}$  are generalized forces,  $\int_{t_1}^{t_2} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt$  is the action.

Now let us see how Euler-Lagrange equations are applied.

**Example 2.3.** *Consider the example 2.1 again. Since the system has no external forces acting on it, there is no potential energy, i.e.  $U = 0$ . But it has kinetic energy:*

$$T = \frac{1}{2} r^2 \dot{q}^2.$$

Hence, the Lagrangian function for this system is given by

$$L(q, v) = \frac{1}{2} r^2 v^2,$$

where  $v = \dot{q}$ . Hence  $\frac{\partial L}{\partial q} = 0$  and  $\frac{\partial L}{\partial v} = r^2 v$ , the Euler-Lagrangian equation according to Theorem 2.7 is

$$\frac{d}{dt} (r^2 v) = 0.$$

## Chapter 3

# Hamiltonian Mechanics

This chapter is a review of Hamiltonian mechanics compatible with Chapter 7 in *First Steps in Differential Geometry* [11].

### 3.1 Hamiltonian Function and Hamilton's Equations

We start from the Lagrangian equations  $\dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{q}}$ , where  $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}$ , with a given Lagrangian  $L: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ , that is  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ , where  $\mathbf{q}$  is the position,  $\dot{\mathbf{q}}$  is the generalized velocity. Using these functions, we can define a new function  $H: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  by *Legendre transformation* [2] as follows:

$$H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t).$$

The new function is called *Hamiltonian function* for the system.

**Theorem 3.1.** [2] *The system of Lagrange equations is equivalent to the system of  $2n$  first order equations (canonical Hamilton's equations)*

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (3.1)$$

where  $H(\mathbf{p}, \mathbf{q}, t) = \mathbf{p}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t)$ ,  $\mathbf{q} = (q_1, q_2, \dots, q_n)$  is position,  $\mathbf{p} = (p_1, p_2, \dots, p_n)$  is momentum.

*Proof.* The total differential of the Hamiltonian

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

is equal to the total differential of RHS

$$dH = \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt.$$

Therefore,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t},$$

where  $i = 1, \dots, n$ . □

**Example 3.1.** Returning to Example 2.3. We had  $L(q, v) = \frac{1}{2}r^2v^2$ . The momentum coordinate is given by  $p = \frac{\partial L}{\partial v} = r^2v$ , and then we can solve for  $v$  to obtain  $v(p, q) = \frac{p}{r^2}$ . Hence

$$H(q, p) = pv - L = p \left( \frac{p}{r^2} \right) - \frac{1}{2}r^2 \left( \frac{p}{r^2} \right)^2 = \frac{p^2}{2r^2}.$$

Hamilton's equations for this system, then are,

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{r^2}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = 0$$

Now back to the Hamilton's system, the time evolution of  $H$  system is

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}.$$

The state  $r(p_1, p_2, \dots, p_n; q_1, q_2, \dots, q_n)$  of the system at a given time is determined by  $2n$  numbers. The space of positions and momentum is called the *phase space* [2].

Now consider  $(p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n)$ , which are coordinates on the cotangent bundle  $T^*\mathbb{R}^n$ . Hence,

$$H : T^*\mathbb{R}^n \rightarrow \mathbb{R}, \quad H \in C^\infty(U), \quad U \subset \mathbb{R}^{2n}$$

**Definition 3.2.** Let  $\alpha$  be a  $k$ -form on  $\mathbb{R}^n$ . Then  $\alpha$  is *closed* if  $d\alpha = 0$ , where  $d$  denotes the exterior derivative[4].

**Definition 3.3.** Let  $M^{2n}$  be an even-dimensional differentiable manifold. A *symplectic structure* on  $M^{2n}$  is a closed non-degenerate differential 2-form  $\alpha$  on  $M^{2n}$  :

$$d\alpha = 0; \quad \alpha_{ij} = -\alpha_{ji} \quad \text{and} \quad \forall \epsilon \neq 0, \exists \eta \text{ such that } \alpha(\epsilon, \eta) \neq 0 \quad (\epsilon, \eta \in M^{2n})$$



**Theorem 3.4.** [2] *The cotangent bundle  $T^*\mathbb{R}^n$  has a natural symplectic structure:*

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i. \quad (3.2)$$

It follows  $\omega$  is closed since  $d\omega = 0$ .

And  $\omega$  is *non-degenerate* since  $\forall X_H \neq 0, \exists X_f$ , such that  $\omega(X_H, X_f) \neq 0$ , where  $X_H, X_f \in T^*\mathbb{R}^n$ . Also,  $\omega^{-1}$  is a contravariant since  $\omega$  is non-degenerate.

In this case, starting with a Hamiltonian function  $H \in C^\infty(\mathbb{R}^n)$ , it produces a vector field  $X_H$  :

$$H \rightarrow X_H : \omega(X_H, \cdot) = i(X_H)\omega,$$

where  $i(X_H)\omega$  is the interior product [4] of  $\omega$  with  $X_H$  and is an 1-form.

Thus,  $i(X_H)\omega = -dH$ , according to which we can verify

$$dH(X_H) = -\omega(X_H, X_H) = 0.$$

Since

$$\omega(X, Y) = -\omega(Y, X),$$

substitute

$$X = Y = X_H,$$

yields

$$\omega(X_H, X_H) = 0.$$

The  $i(X_H)\omega = -dH$  defines the vector field  $X_H$  uniquely.

It follows that

$$-dH = -\sum_{i=1}^n \frac{\partial H}{\partial q_i} dq_i - \sum_{i=1}^n \frac{\partial H}{\partial p_i} dp_i,$$

and

$$i(X_H)\omega = \sum_{i=1}^n dp_i \wedge dq_i.$$

Therefore,

$$X_H = -\sum_{i=1}^n \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}.$$

## 3.2 Hamiltonian Vector Field

First we note that the Cartan's Lemma [11],

$$\begin{aligned}\mathcal{L}_{X_H}\omega &= i(X_H)d\omega + d(i(X_H)\omega) \\ &= d(i(X_H)\omega) \\ &= d(-dH) = 0,\end{aligned}$$

where  $\mathcal{L}$  denotes the *Lie derivative* [12]. Thus,  $\mathcal{L}_{X_H}\omega = 0$

**Definition 3.5.** Let  $X$  be a smooth vector field on a domain  $U \subset \mathbb{R}^n$ . A *first integral* of  $X$  is a smooth function  $f : U \rightarrow \mathbb{R}$  such that  $\mathcal{L}_X f = X[f] = df(X) = 0$ .

It follows the Hamiltonian function  $H$  and its corresponding vector field  $X_H$  that  $\mathcal{L}_{X_H}H = X_H(H) = dH(X_H) = -\omega(X_H, X_H) = 0$ .

Naturally, the Hamiltonian function  $H$  is a first integral of the vector field with Hamiltonian function  $H$ ,  $X_H$  which is defined in the following.

**Definition 3.6.** A vector field  $X$  that satisfies  $\mathcal{L}_X H = 0$  is called a *Hamiltonian vector field* and is denoted by  $X_H = \omega^{-1}dH$ .

Since  $dH = 0$  along  $X_H$ , we have

**Proposition 3.7.** Let  $f, g : T^*\mathbb{R}^n \rightarrow \mathbb{R}$  be smooth functions with Hamiltonian vector fields  $X_f$  and  $X_g$ , respectively. Then

$$dg(\cdot) = -\omega(X_g, \cdot),$$

$$dg(X_f) = -\omega(X_g, X_f) = \omega(X_f, X_g) = \{f, g\},$$

where  $\{f, g\}$  is the *Poisson bracket* defined below.

A coordinate transformation is called *canonical* if it preserves the canonical Hamiltonian equations (3.1).

## 3.3 Poisson Bracket

The symplectic structure  $\omega$  (3.2) gives rise to the *Poisson bracket* of two smooth functions on  $\mathbb{R}^{2n}$  :

$$\{f, g\} = \omega(X_f, X_g),$$

where  $f, g \in C^\infty(U)$ ,  $U \subset \mathbb{R}^{2n}$  are any smooth Hamiltonian functions and  $X_f, X_g \in T^*\mathbb{R}^n$  are corresponding Hamiltonian vector fields, respectively. Specially, *Poisson bracket* in *canonical coordinates* can be defined as follows:

**Definition 3.8.** In canonical coordinates  $(p_i, q_i)$  on the phase space, given two functions  $f(p_i, q_i), g(p_i, q_i)$ , the *Poisson bracket* is denoted by  $\{f, g\}$  and defined as follows:

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

More generally,

$$\{f, g\} = \omega^{-1}(df, dg),$$

where  $d$  denotes the exterior derivative and thus  $df, dg$  are one-form, accordingly.

The Poisson bracket satisfies:

$$\{f, g\} = -\{g, f\},$$

$$X_{\{f, g\}} = [X_f, X_g],$$

and

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \Leftrightarrow d\omega = 0.$$

## Chapter 4

# The P-entropy

In 1865, Rudolf Clausius for the first time described entropy as the “transformation-content” [5] on the basis of Carnot’s theorem, which begun its study. Later, scientists such as Ludwig Boltzmann, Erwin Schrodinger and James Maxwell described entropy from a statistical point of view. In 1877 Boltzmann employed a probabilistic method to measure the entropy of an ensemble of ideal gas particles, in which he defined entropy to be the measurement of chaos. In 1948, Shannon introduced the concept of the entropy into information theory and defined entropy to measure the probability distribution of events. Recently, Panchenkov presented a new notion of entropy, in which he separated the entropy from the energy. As a result, Panchenkov obtained entropy as an independent quantity.

### 4.1 Thermodynamical Entropy

#### 4.1.1 Clausius entropy

Recall that Rudolf Clausius discovered that the algebraic sum of all the transformations occurring in a reversible cyclical process can only be zero. In other words,

$$\oint \frac{\delta Q_R}{T} = \sum \frac{\delta Q_R}{T} = 0,$$

where  $\delta Q_R$  is energy flow into the system due to heating and  $T$  is the absolute temperature of the body when that energy is absorbed in any process that is cyclical and reversible.

Moreover,  $\oint \frac{\delta Q_R}{T}$  is independent of path and only determined by the initial and final states. In a way,  $\oint \frac{\delta Q_R}{T}$  is an increment of a state function. Thus, Clausius [22] defined

the state function to be *entropy*, that is,

$$dS = \frac{\delta Q_R}{T},$$

or

$$\Delta S = S_a - S_b = \int_a^b \frac{\delta Q_R}{T},$$

where  $S$  is entropy,  $dS$  is the infinitesimal change of entropy  $S$  and  $\Delta S$  is the increment of entropy  $S$  from  $a$  to  $b$ .

For any irreversible cyclic process, Clausius proved:

$$\oint \frac{\delta Q_I}{T} = \sum \frac{\delta Q_I}{T} < 0,$$

where  $\delta Q_I$  is the energy flow into the system due to heating and  $T$  is the absolute temperature of the body when that energy is absorbed in any process that is cyclical and irreversible.

Therefore, for any cyclic process, such inequality, which is called Clausius inequality, must hold:

$$\oint \frac{\delta Q}{T} = \sum \frac{\delta Q}{T} \leq 0,$$

where  $\delta Q$  is the energy flow into the system due to heating and  $T$  is the absolute temperature of the body when that energy is absorbed in the cyclic process.

Now, if we have a cyclic process which contains both a reversible process and a process which can be either reversible or irreversible, we have

$$\oint \frac{\delta Q}{T} = \int_a^b \frac{\delta Q}{T} + \int_b^a \frac{\delta Q_R}{T} \leq 0,$$

which yields

$$\int_a^b \frac{\delta Q}{T} \leq \int_a^b \frac{\delta Q_R}{T},$$

that is

$$\Delta S \geq \int_a^b \frac{\delta Q}{T},$$

or

$$dS \geq \frac{\delta Q}{T},$$

where equality holds true only for the reversible process.

And for an isolated system, since  $\delta Q = 0$ , we have

$$dS_{iso} \geq 0, \tag{4.1}$$

where  $S_{iso}$  is the entropy of an isolated system and  $dS_{iso}$  is the infinitesimal change of entropy  $S_{iso}$ . *The second law of thermodynamics* [3] states: the entropy of an isolated system never decreases, i.e. such a system will spontaneously proceed towards thermodynamic equilibrium.

#### 4.1.2 Boltzmann entropy

Clausius entropy describes the irreversibility of thermodynamics in a macroscopic way. Later, Ludwig Boltzmann re-defined the entropy microscopically and statistically. Boltzmann, for the first time, gave the definition of a *microstate*, which is a specific microscopic configuration of a thermodynamic system that the system may occupy with a certain probability in the course of its thermal fluctuations [3]. He defined the thermodynamic probability  $W$ , which he also defined to be the density in phase space, to be the number of ways the atoms or molecules of a thermodynamic system (also called the amount of microstates) are arranged. The value of  $W$  is proportional to the probability of a macroscopic state under certain probability distribution. Thus, thermodynamic probability can describe the amount of chaos in a thermodynamic system.

**Example 4.1.** *Let us toss a coin. If it comes up a head, it will be labelled 1. Otherwise, it will be labelled 0. Now toss the coin 10 times, we have a series  $s = (s_1, s_2, \dots, s_{10})$ , where  $1 \leq i \leq 10$ ,  $s_i = 0$  or 1. If the head appears  $i$  times, we call it “case- $i$ ”. And it can be proved that the number of microstates, which is the number of all possible head-upward outcomes, is  $C_{10}^i$  in the system. Apparently, case-0 and case-10 are most ordered two cases. The case-5, which has  $C_{10}^5 = 252$  different microstates, is the most chaotic one since the number of microstates is the greatest among all the cases. Thus, the greater the number of microstates in a system is, the more chaotic the system is and vice versa.*

Eventually, Boltzmann made a conclusion about the relation between the entropy  $S$  and thermodynamic probability  $W$ , that is  $S \propto \ln W$ . Hence,

$$S = k_B \ln W, \tag{4.2}$$

where  $S$  is the Boltzmann entropy,  $W$  is the number of microstates to a given macrostate, and  $k_B$  is the Boltzmann constant, which is equal to  $1.38065 \times 10^{23} \text{ J/K}$ . The equation (4.2) is called the *Boltzmann principle* and acts as a link between the microscopic and the macroscopic worlds.

**Remark 4.1.1.** According to either macroscopic or microscopic definition of entropy, it is always a measure of chaos or disorder. The Boltzmann principle can be generalized

to Gibbs entropy [9]

$$S = -k_B \sum \rho \ln \rho, \quad (4.3)$$

where  $\rho$  is the density in phase space, under the condition that microstates have non-equal probabilities of a thermodynamic system.

In addition from (4.2), the entropy as a function of the thermodynamic probability  $W$  can also describe the amount of chaos in a thermodynamic system. In other words, entropy is a measure of disorder for a certain thermodynamic system. The more chaotic the system is, the larger the entropy is and vice versa. Boltzmann's re-explanation of entropy gives opportunity for people to understand the theory of entropy deeply and widely. As a result, the explanation of entropy from then has more applications outside of thermodynamics, and can be extensively applied to other scientific fields.

## 4.2 Information Entropy

By the end of 1940s, Shannon published his epochal paper *A Mathematical Theory of Communications* [21], in which he for the first time introduced the Boltzmann entropy into information theory. He then defined it to be a measure of unpredictability of information content and presented the information entropy theory.

Now consider a discrete random variable  $X$  with possible values  $x_1, x_2, \dots, x_n$  ( $n \in \mathbb{N}$ ), the probability of  $i$ th value is  $p_i$ , and  $p_i \geq 0$ ,  $\sum_{i=1}^n p_i = 1$ . To better characterize the unpredictability  $H$  of the  $X$ ,  $H$  has to satisfy the following properties[6]:

- $H(p_1, p_2, \dots, p_n)$  is a continuous function of  $p_i$ , where  $i = 1, 2, \dots, n$  and  $n \in \mathbb{N}$ ;
- $H(\frac{1}{n}, \dots, \frac{1}{n})$  is a monotone increasing function of  $n$ , where  $n \in \mathbb{N}$ ;
- $H(p_1, p_2, \dots, p_n)$  ( $n \in \mathbb{N}$ ) is subadditive, namely:

$$H(p_1, p_2, \dots, p_n) = H\left(\sum_{i=1}^k p_i, p_{k+1}, \dots, p_n\right) + \left(\sum_{i=1}^k\right) H(p'_1, p'_2, \dots, p'_k),$$

where  $p'_i = \frac{p_i}{\sum_{i=1}^k p_i}$ ,  $k < n$ ,  $n \in \mathbb{N}$ .

**Note:** The subadditivity here means that if we treat part of  $X$  to be a whole new subject, then the unpredictability of  $X$  is equal to the sum of the unpredictability of the new  $X$ , in which part of it is a new subject, and the unpredictability within the whole new subject. Therefore, the subadditivity states the relation between the unity  $H$  and

its parts. For example, we want to choose 5 students from a math division, suppose there are 5 classes of the division. Then the unpredictability that we choose a student from the division is the same the unpredictability that we firstly choose the student's class plus the unpredictability that we then choose the student from the class.

Thus, follows the function  $H$  satisfying the above three properties:

$$H(p_1, p_2, \dots, p_n) = - \sum_{i=1}^n p_i \log_c p_i,$$

where  $n \in \mathbb{N}$ ,  $c > 0$  and we regulate  $0 \log_c 0 = 0$ . And  $H(X) = H(p_1, p_2, \dots, p_n)$  is the information entropy of the discrete random variable  $X$ . The new definition of entropy identifies two important properties:

$$H(X) \geq 0;$$

[6]

$$H(X) \leq \ln(n),$$

where the second equality holds only for  $X$  that is uniformly distributed, that is  $\forall 1 \leq i \leq n$ ,  $p_i = \frac{1}{n}$ ,  $H(x) = \ln(n)$ ,  $n \in \mathbb{N}$ , which is stated as the *principle of maximum entropy*. Hence, Shannon entropy is related to negative entropy (often noted as *neg-entropy*), which is a term introduced by physicist Erwin Schrodinger in his 1944 article *What is Life*, [20] to explain how living systems export entropy to their environment to maintain themselves to exist as long as possible, in other words, it is the negative of entropy, described by a physicist Léon Brillouin, who found out an equation to state that the changing bit value information requires at least  $k_B T \ln(2)$  energy, where  $k_B$  is Boltzmann constant,  $T$  is temperature. As a result, he defined the *principle of negentropy* [10], that is, information is the increase of the neg-entropy and is also the decrease of the entropy. There is no violation of the Second Law of Thermodynamics involved, since a reduction in any local system's entropy results in an increase of entropy elsewhere.

## 4.3 Panchenkov's Entropy

### 4.3.1 Preliminary

After Shannon introduced his new version of entropy, the subject received a lot attention in the scientific world and many scientists contributed to the theory of entropy. Basically, they had the following goals:

(1) to extend the entropy theory from a closed system to an open system;



(2) to introduce the neg-entropy theory to describe how a system can evolve to an ordered system under the influence of the outside world.

In 1969 Ilya Prigogine, whose focused study on the self-organization, introduced and developed the dissipative system theory. In his famous paper entitled *Time, structure and fluctuations* he wrote that “a *dissipative structure* was an open thermodynamic system which was operating out of, and often far from, thermodynamic equilibrium in an environment with which it exchanges energy and matter, and moreover, in such a system, the non-equilibrium may become a source of order” [17].

In particular, he has pointed out that there is no exchange of energy and matter between an isolated system and the outside world. On the other hand, for an open system there exists an exchange of entropy between the local system and its outside environment. Hence, the entropy of the open system consists of the following two parts: the entropy produced within the system and the entropy that system gets from the outside world. The dual presentation of the total entropy is formed as follows:

$$dS = d_iS + d_eS, \quad (4.4)$$

where  $dS$  is the total entropy of an open system,  $d_iS$  is the entropy produced within the system, or, to be more specific, it is only produced by the irreversible process within the system;  $d_eS$  is the entropy flow generated by the exchange between the system and its outside environment.

Moreover, it follows from the equation (4.1) that the second law of thermodynamics assumes that the entropy produced within the system is always non-negative, that is  $d_iS \geq 0$ . But for  $dS$  and  $d_eS$ , there is no such law bounding them. Thus, from (4.4) it can be concluded that  $d_eS \leq -d_iS$  quantitatively is a necessary condition to maintain the total entropy  $dS$  at a low level, that is to keep the system evolving in the direction of becoming more and more ordered.

Thus, Prigogine drew a conclusion that a thermodynamically open system, moving from thermodynamic equilibrium, can reduce the entropy to a low level and then keep the system ordered in a time, space or a structural sense if it can extract sufficient energy flow, matter flow or information flow from its outside world. He also mentioned fluctuations are very essential to the order of an open system, in such a way that it can be interpreted that the fluctuations lead to the order of the open system. As for an open system away from thermodynamic equilibrium, a stochastic small fluctuation can grow into a large one, while an ordered system can emerge from such a process even if it is originally unstable. This phenomenon offers opportunities for the system to become a dissipative one. It is natural to call such changes events.

However, the completion of the modern entropy theory has never come to an end. We still want to

(1) describe events of any systems and then monitor them;

(2) extend the entropy theory so that it is based on a more general notion of entropy.

This motivated Panchenkov to introduce a new notion of entropy. We will call it the *P-entropy* in what follows.

The *P-entropy* can be interpreted in a number of ways. For instance, the *P-entropy* can be described, roughly speaking, as a measure of order of a complex system. In his theory, a complex system is modelled by a virtual continuous medium, which contains certain geometric structures, evolving along the direction in which the system can maximize its time of existence.

We will explain the conceptions of *P-entropy* in what follows. This part is compatible with *Entropy*, a book by Panchenkov [14].

### 4.3.2 Virtual continuous medium

The world that we live in is physically “a random distribution of mass and energy in time and in space” [7]. To describe such a complex and various world, it is necessary to simplify and match it to such a virtual continuous medium (which is abbreviated by VCM in what follows) where we can rebuild the construction. This idea can be obtained from Albert Einstein, who wrote in *General relativity* [7] that “according to the general theory of relativity, space is endowed with physical qualities” [7]. On the basis of such a hypothesis, we can view the world as a virtual medium of certain geometrical structures that evolves continuously with time.

The virtual continuous medium is an abstract continuum, characterized according to Panchenkov by several important properties, namely [14]:

- 1) The virtual continuous medium is embedded in a bounded area of the phase space; so it is determined by the coordinates  $(\mathbf{q}, \mathbf{p})$ , where  $\mathbf{q} \in \Omega_q \subset \mathbb{R}^n$  is the generalized position coordinate and  $\mathbf{p} \in \Omega_p \subset \mathbf{T}^*\mathbb{R}^n$  is the generalized momenta coordinate;
- 2) Motion in the phase space is characterized by  $\mathbf{q}$  and  $\mathbf{p}$ ;
- 3) The virtual continuous medium evolves over time  $t \in \mathbb{R}$ , which is a continuous variable, such that one can perform a continuous motion with same direction of  $x$ ;
- 4) The virtual continuous medium has a continuous density, the state function of which is  $\rho = \rho(\mathbf{q}, \mathbf{p}, t)$ ;
- 5) The mass of the virtual continuous medium is preserved;
- 6) Entropy is defined in the virtual continuous medium;
- 7) The underlying variational principle of the virtual continuous medium is the principle of maximum entropy;
- 8) The entropy of the virtual continuous medium satisfies a dual representation.

### 4.3.3 Rationale

The  $P$ -entropy, macroscopically, is defined to be a measure of “perfectness” of a system rather than disorder or chaos. The idea comes from the neg-entropy theory. There is no contradiction with Clausius’ definition, in which the entropy is a state function of how a thermodynamic system extracts work from its outside environment. Now, if you define it from the point of view of how a system tries to keep itself ordered and exists as long as possible, that is to extract the “entropy” of the system outside, you will find this new entropy has a “positive” meaning. It is compatible with Prigogine’s dissipative structure theory. However, it is clear, according to the second thermodynamic law, that we are referring here to an open system, only, which can make exchanges with the outside. Since the time of Arthur Eddington, the entropy has been treated as the “arrow of time”. The concept of  $P$ -entropy leads to a further generalization of this, and furthermore, by its definition, the  $P$ -entropy measures the largest possible existing time of a system. Thus, as a matter of fact, the  $P$ -entropy is the representation of the longest possible existing time of a system.

Microscopically, the Boltzmann entropy offers us opportunities to reconsider the notion of entropy under various conditions. From the equation (4.3), we can derive a formula in the corresponding VCM. Firstly, let us describe how one can pass from a stochastic environment to the VCM.

**Definition 4.1.** Define  $\Omega$  to be the phase space, where the statistical ensemble is located.

$$\Omega \subset \mathbb{R}^n \oplus \mathbb{R}_n,$$

where  $\mathbb{R}^n$  is vector space and  $\mathbb{R}_n$  is dual vector space.

It can be represented as:

$$\Omega = \Omega_q \times \Omega_p,$$

where  $\Omega_q \subset \mathbb{R}^n$  is the *configuration space* and  $\Omega_p \subset \mathbb{R}_n$  is the *momentum space*.

**Remark 4.3.1.** Note that here and throughout this thesis, following Panchenkov’s notations, we use the brackets  $\{ , \}$  to define a space with structures and coordinates, rather than a set.

**Definition 4.2.** The *stochastic environment* is as follows :

$$S = \{f, t | f = f(\mathbf{q}, \mathbf{p}, t), \mathbf{q} \in \Omega_q, \mathbf{p} \in \Omega_p, \Omega = \Omega_q \times \Omega_p; \Omega \subset \mathbb{R}^n \oplus \mathbb{R}_n; t \in [0, T]\},$$

where the  $f = f(\mathbf{q}, \mathbf{p}, t)$  is the density function of the virtual continuous medium.

Now, in *stochastic environment* we rewrite equation (4.3) to be :

$$B_f = - \int_{\Omega} f \ln f \, d\Omega,$$

where  $B_f$  is the total entropy. Recall what we saw in the Shannon entropy, the total sum of probability is unity. Here density function  $f$  acts like probability in the stochastic environment, such that the total sum of density is unity, that is,

$$m = \int_{\Omega} f \, d\Omega = 1. \quad (4.5)$$

Applying the fact that  $P$ -entropy represents the longest possible existing time of a system:

$$H_f = \max B_f.$$

To conclude, Panchenkov proposed the fundamental theorem of Panchenkov's entropy.

**Theorem 4.3.** *Panchenkov's entropy always obtains the maximum value in the evolution process and is a constant.*

$$H_f = \text{constant}$$

The theorem is proven in[14] (see p. 46) by treating the integral  $\int_{\Omega} f \ln f \, d\Omega$  as the distribution  $\langle \ln f, f \rangle$  with the constraint  $\langle 1, f \rangle = 1$  and deriving the conditions  $\langle U_f, \varphi \rangle = 0$ ,  $\langle 1, \varphi \rangle = 0$ , where  $\varphi$  is a test function and  $U_f = \ln f_0 + \lambda$  and  $f_0$  is the extremal value of  $f$ . It follows then  $f_0 = \text{const}$ , or  $H_f = \text{const}$ . The fact that  $f_0$  is a maximum is verified directly.

As an extension of Prigogine's theory, Panchenkov's theory completely separates entropy flow from energy flow. It is a fundamental principle of Panchenkov's theory of entropy, under this hypothesis, that the  $P$ -entropy becomes an quantity independent of energy, for which the energy exchange between the system and its outside can not change the entropy of a system since it is a constant.

#### 4.3.4 Continuity of P-entropy

Continuity is a necessary property for entropy no matter how it is defined. Let us now explore the continuity of the  $P$ -entropy.

The  $P$ -entropy is a continuous function since we have proven  $H_f = \text{constant}$  always in the stochastic environment by equation (??), which proves  $f$  is a constant function,  $f(\mathbf{q}, \mathbf{p}, t) = \text{constant}$ . This implies,

$$\frac{\partial f}{\partial t} = 0 \text{ and } \nabla f = 0. \quad (4.6)$$

The equation (4.5) has proved the conservation of mass, which is guaranteed by the condition that the time derivative of the mass inside a control volume is equal to the mass flux  $\mathbf{F}$ , across its boundaries. Mathematically,

$$\frac{\partial m}{\partial t} = \int_S \mathbf{F} \cdot \mathbf{n} \, dS,$$

where  $m$  is the unity mass,  $S = \partial\Omega$  is the surface of  $\Omega$ ,  $\mathbf{n}$  is the unit outward normal vector on  $S$ .

Here, without loss of generality, we take  $\mathbf{n} = -\mathbf{1}$ , which also ensures that outward flow results in a decrease in the mass with respect to time. Then use the divergence theorem,

$$\int_S \mathbf{F} \, dS = - \int_{\Omega} \nabla \cdot \mathbf{F} \, d\Omega.$$

Also take partial derivative to equation (4.5) with respect to time,

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \int_{\Omega} f \, d\Omega = \int_{\Omega} \frac{\partial f}{\partial t} \, d\Omega.$$

Therefore,

$$\int_{\Omega} \frac{\partial f}{\partial t} = - \int_{\Omega} \nabla \cdot \mathbf{F} \, d\Omega.$$

Now take  $\mathbf{F} = f \cdot \mathbf{A}$ , where  $\mathbf{A} = \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{p}}{\partial t}$  is the flow velocity in the stochastic environment.

Hence,

$$\frac{\partial f}{\partial t} = -\nabla \cdot (f \cdot \mathbf{A})$$

yields,

$$\frac{\partial f}{\partial t} + \nabla f \cdot \mathbf{A} + f \cdot (\nabla \cdot \mathbf{A}) = 0. \quad (4.7)$$

We know  $f = f(\mathbf{q}, \mathbf{p}, t)$ , thus the total derivative of it with respect to time is:

$$\frac{df}{dt} = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial t} + \frac{\partial f}{\partial t} = \nabla f \cdot \mathbf{A} + \frac{\partial f}{\partial t},$$

Since  $f = \text{Const}$ , we get the continuity equation in the stochastic environment:

$$\frac{\partial f}{\partial t} + \nabla f \cdot \mathbf{A} = 0.$$

Besides, under the condition of equation (4.6), we can get an important property from the equation (4.7), which is physically the subject of incompressibility, as follows:

$$\nabla \cdot \mathbf{A} = 0.$$

### 4.3.5 Duality of P-entropy

We have already known the additivity of entropy in microscopic condition from the Boltzmann's Principle, and also learnt that an important condition for Shannon entropy. Now for the  $P$ -entropy, we will find that the additivity of entropy also exists macroscopically.

Let us explore the volume of the stochastic environment. Physically,

$$m = f \cdot V, \quad (4.8)$$

where  $m$  is the mass of the stochastic environment,  $f$  is the density function in the stochastic environment and  $V$  is the volume of the stochastic environment. Then from the equation (4.5),

$$\int_{\Omega} f d\Omega = f \cdot V,$$

$$V = \frac{1}{f} \int_{\Omega} f d\Omega$$

Taking reciprocal of  $f$  inside the integral we obtain  $V = \int_{\Omega} d\Omega$ . Because of the incompressibility,  $\nabla \cdot \mathbf{A} = 0$ , hence  $div \mathbf{A} = 0$ , where  $\mathbf{A} = \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{p}}{\partial t}$ , we get the following equation by Liouville theorem:

$$\frac{dV}{dt} = \int_{\Omega} \nabla \cdot \mathbf{A} d\Omega,$$

hence,

$$\frac{dV}{dt} = 0.$$

Therefore,  $V$  is a constant and is preserved in any time transformation.

Since  $H_f$  and  $V$  are both constant, we can represent  $H_f$  in terms of  $V$  following from the equation (4.5) and (4.8), namely:

$$f \cdot V = 1 \quad \text{and} \quad \int_{\Omega} f d\Omega = 1,$$

where mass is unity in the stochastic environment.

Also,

$$H_f = - \int_{\Omega} f \ln f d\Omega$$

We know  $f$  is constant, hence,

$$\begin{aligned}
H_f &= -\ln f \int_{\Omega} f \, d\Omega \\
&= -\ln f \\
&= \ln \frac{1}{f} \\
&= \ln V.
\end{aligned} \tag{4.9}$$

Now look back to the definition of phase space  $\Omega = \Omega_q \times \Omega_p$ , the  $P$ -entropy must inherit the separability from the dual definition of the phase space, in which we can be allowed to define two kinds of entropy  $H_q$  and  $H_p$  located in  $\Omega_q$  and  $\Omega_p$ , respectively. The former one is called structural entropy while the other is called impulse entropy.

Since  $P$ -entropy is microscopically additive, there must exist such a property in macro-state. Thus, we can say

$$H_f = H_q + H_p,$$

where  $H_q \in \Omega_q$  and  $H_p \in \Omega_p$ . Hence,

**Theorem 4.4.**  *$P$ -entropy has a dual representation, that is  $H_f = H_q + H_p$ , where  $H_q \in \Omega_q$  and  $H_p \in \Omega_p$ , which is also called the duality of Panchenkov's entropy.*

*Proof.* We will prove the duality from the point of view of phase volume. Define  $f_q = f(\mathbf{q}, t)$  as the stochastic density in the configuration space and  $f_p = f(\mathbf{p}, t)$  as the stochastic density in the momentum space. Then  $f_q$  and  $f_p$  are independent since  $\Omega_q \cap \Omega_p = \emptyset$ . Hence, the total density is multiplicative, that is  $f = f_q \cdot f_p$ . The conservation of mass in the two different spaces are

$$\int_{\Omega_q} f_q \, d\mathbf{q} = 1 \quad \text{and} \quad \int_{\Omega_p} f_p \, d\mathbf{p} = 1.$$

The relations  $f_q \cdot V_q = 1$  and  $f_p \cdot V_p = 1$  hold true as well. The structural entropy  $H_q$  and the impulse entropy  $H_p$  has similar representations with total entropy in their corresponding spaces, that is:

$$H_q = - \int_{\Omega_q} f_q \ln f_q \, d\mathbf{q} \quad \text{and} \quad H_p = - \int_{\Omega_p} f_p \ln f_p \, d\mathbf{p}.$$

Similar to the proof of the relation between the phase volume and the total entropy, we can obtain:

$$H_q = \ln V_q \quad \text{and} \quad H_p = \ln V_p,$$

where  $V_q$  is the volume of  $\Omega_q$  and  $V_p$  is the volume of  $\Omega_p$ . Then,

$$\begin{aligned}
H_f &= - \int_{\Omega} f \ln f \, d\Omega \\
&= - \ln f \int_{\Omega} f \, d\Omega \\
&= - \ln(f_q \cdot f_p) \\
&= -(\ln f_q + \ln f_p) \\
&= \ln V_q + \ln V_p.
\end{aligned}$$

Therefore,

$$H_f = H_q + H_p,$$

which is the duality of  $P$ -entropy. □

The duality is a very important property for  $P$ -entropy and a key principle for the model involving  $P$ -entropy.

**Remark 4.3.2.** Recall that the total entropy  $H_f = H_q + H_p$ ,  $H_f = \text{const}$ , where  $H_q$  is the structural entropy (measures the quality of the structure of the system), and  $H_p$  is the impulse entropy (measures the resources of the system). From this viewpoint all processes in the universe can be described as the transformation of  $H_p$  into  $H_q$  and vice versa. In a conservative system we have a complete transformation of one component of the total entropy into the other. In a dissipative system the situation is different: dissipation conserves a part of the impulse entropy  $H_p$  and prevent it from transforming it into the structure. The same phenomena in this case destroys a part of the structure entropy  $H_q$ , and so we have [15]

$$H_q := H_q^0 - H_\epsilon, \quad H_p := H_p^0 + H_\epsilon,$$

where  $H_q^0$  and  $H_p^0$  are the structural and impulse entropies respectively and  $H_\epsilon$  is the “frozen” entropy. We can show that  $\frac{dH_\epsilon}{dt} \geq 0$  and so it is natural to connect  $H_\epsilon$  with the thermodynamical entropy  $H_T$ . On the other hand, if we identify  $p$  with a vector  $v$  in the Boltzman formula  $H_T = - \int_{\Omega_p} f(u, t) \ln f(u, t) du$  we can identify in this way  $H_p$  with  $H_T$ . We can assume that in a dissipative system the conservative component  $H_p^0 = 0$  we can identify  $H_p$  with  $H_\epsilon$ , which means that in a dissipative system we have a transformation of structural entropy into the impulse entropy. Since this process



irreversible, the impulse entropy in this case is “passive”, i.e., it becomes the “frozen” entropy  $H_\epsilon$ .

#### 4.3.6 Variational principle

As we discussed in Chapter 2, a variational problem is constrained by some variational principle. The whole model is evolving along the direction that maximizes the existing time of the system, which is precisely the variational principle of the  $P$ -entropy model—the *principle of maximum  $P$ -entropy*. In addition, it is also the maximum principle of  $P$ -entropy, as mentioned, that  $P$ -entropy actually represent the existing time of a system. As detailed by Schrodinger, all systems follow the principle that they will try to extend their existing time, through which the maximum principle of  $P$ -entropy is employed.

Based on this variational principle, an evolution can be described as the transformation:  $H_q \rightleftharpoons H_p$ , which assumes different forms according to the models in question.

## Chapter 5

# Geometric Structures of the VCM

In the previous chapter we introduced following the notion of the  $P$ -entropy, and described how it could be used to describe a model for a complex system, called “virtual continuous medium”. In this chapter we will develop the ideas by introducing more structures within the VCM.

### 5.1 From a Stochastic Environment to a VCM

To pass from a stochastic environment to a VCM, Pachenkov employed the following identification:  $f \rightarrow \rho$ , where  $\rho$  is a continuous physical density of the VCM, such that the property in the stochastic environment could be preserved in the process. As a result, the  $P$ -entropy model could be generalized and applied to the VCM.

Mathematically, we can think of the stochastic environment and the VCM as two different sets with certain structures. As we know, there always exist such morphisms between them. Pachenkov assumed that a special one satisfying certain requirement can be established among all of such morphisms.

In this view the  $P$ -entropy model can be generalized from the stochastic environment to the whole VCM.

Thus, the VCM can be defined as follows:

$$M = \{\mathbf{q}, \mathbf{p} \mid \rho = \rho(\mathbf{q}, \mathbf{p}, t); \mathbf{q} \in \Omega_q, \mathbf{p} \in \Omega_p, \Omega = \Omega_q \times \Omega_p; \Omega \subset \mathbb{R}^n \oplus \mathbb{R}_n; t \in [0, T]\}.$$

### 5.2 Entropy Manifold

We obtain the first reduction of the phase space  $\Omega$  by introducing the invariant  $H_f$ . It is the *entropy manifold*:

$$\mathcal{E} = \{\mathbf{q}, \mathbf{p} \mid \mathcal{E} \subset \Omega, H_f = \text{constant}\},$$

where  $H_f$  is the  $P$ -entropy. By the theorem 4.4, we have the following representation of entropy manifold as a direct product:

$$\mathcal{E} = \mathcal{E}_q \times \mathcal{E}_p,$$

where  $\mathcal{E}_q = \{\mathbf{q} \mid \mathcal{E}_q \subset \mathcal{E}, H_q\}$  is the configuration entropy manifold and  $\mathcal{E}_p = \{\mathbf{p} \mid \mathcal{E}_p \subset \mathcal{E}, H_p\}$  is the impulse entropy manifold.

### 5.3 Solenoid Manifold

We mentioned the divergence  $\text{div} A = \nabla \cdot A = 0$  in the last chapter, thus, with the divergence we obtain a new manifold as following:

$$M = \left\{ \frac{\partial \mathbf{q}}{\partial t}, \frac{\partial \mathbf{p}}{\partial t} \mid M \subset \mathcal{E}, \sigma = \text{div} A, A \in T\mathcal{E} \right\},$$

where  $T\mathcal{E}$  is the tangent bundle of the entropy manifold.

Hence, Panchenkov called the new one a solenoid manifold and stated that the  $\sigma = 0$  as we have proved in the last chapter by the continuity equation.

Then he introduced a physical quantity commonly used in the fluid dynamics: the acceleration potential  $\Theta = \Theta(\mathbf{q}, \mathbf{p})$ . Now we can narrow the entropy manifold to the acceleration potential manifold, namely,

$$\Pi = \left\{ \frac{\partial \mathbf{q}}{\partial t}, \frac{\partial \mathbf{p}}{\partial t} \mid \Pi \subset M, \Theta, \omega_0 \right\},$$

where  $\omega_0$  is a canonical symplectic form, that is  $i(X_\Theta)\omega_0 = d\Theta$ , we can formally identify  $\Theta = -H$ , where  $H$  is an Hamiltonian function.

**Remark 5.3.1.**  $\Theta$  and  $H$  are derived from different variational principles. We may find the projection of  $\Theta$  satisfying a Hamiltonian variational principle in some manifolds. But it is necessary to note that  $\Theta$  and  $H$  are totally different quantities.

Upon the identification  $A = X_\Theta$  we arrive at the following canonical equations:

$$\frac{\partial \mathbf{q}}{\partial t} = -\frac{\partial \Theta}{\partial \mathbf{p}}; \quad (5.1)$$

$$\frac{\partial \mathbf{p}}{\partial t} = \frac{\partial \Theta}{\partial \mathbf{q}}, \quad (5.2)$$

which is shown via the identification, where  $(\frac{\partial \mathbf{q}}{\partial t}, \frac{\partial \mathbf{p}}{\partial t}) \in \Pi$ . In conclusion, it can also be represented as  $A = \nabla \Theta$ , where  $\nabla \Theta = -\frac{\partial \Theta}{\partial \mathbf{p}} + \frac{\partial \Theta}{\partial \mathbf{q}} = \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{p}}{\partial t}$ , which justifies the assumption  $\Theta = -H$  above.

## 5.4 Hilbert Manifold

Finally, Panchenkov introduced another new quantity: the scalar impulse potential  $\Psi = \Psi(\mathbf{q}, t)$ . Introducing  $\Psi$  into the acceleration potential manifold we can finally construct the Hilbert manifold:

$$\Gamma = \{\mathbf{q}, \mathbf{p} \mid \Gamma \subset \Pi, \Psi\}.$$

In the Hilbert manifold it can be obtained immediately from its physical meaning that

$$\frac{\partial \Psi}{\partial t} = \Theta. \quad (5.3)$$

Then, if taking the equation (5.3) partial derivative with respect to  $\mathbf{q}$ , we get

$$\frac{\partial}{\partial \mathbf{q}} \frac{\partial \Psi}{\partial t} = \frac{\partial \Theta}{\partial \mathbf{q}}.$$

Also it follows from (5.2), the last equality

$$\frac{\partial}{\partial \mathbf{q}} \frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \Psi}{\partial \mathbf{q}} = \frac{\partial \mathbf{p}}{\partial t}.$$

Integrating with respect to  $t$ , it yields

$$\mathbf{p} = \frac{\partial \Psi}{\partial \mathbf{q}}. \quad (5.4)$$

Note that  $\text{grad}\Psi = \frac{\partial \Psi}{\partial \mathbf{q}}$ . Hence, the equation (5.4) can be rewritten as

$$\mathbf{p} = \text{grad}\Psi.$$

Also, the Hilbert manifold can be constructed from the point of view of the Hamiltonian function as follows:

Now take the total derivative of  $\Psi$ , we get

$$d\Psi = \frac{\partial \Psi}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial \Psi}{\partial t} dt.$$

By the equations (5.3) and (5.4), it can be written as

$$d\Psi = \mathbf{p}d\mathbf{q} + \Theta dt.$$

Since  $\Theta = -H$ , it is also

$$d\Psi = \mathbf{p}d\mathbf{q} - H dt.$$

Thus, Panchenkov defined a new variable  $\varepsilon = d\Psi$  in the Hilbert manifold as follows:

$$\hat{\Gamma} = \left\{ \mathbf{q}, \mathbf{p} \mid \hat{\Gamma} \subset \Pi, \varepsilon, H, \Psi \right\}.$$

It is the same Hilbert manifold constructed from the point of view of Hamiltonian function, finally, a Hamilton-Jacobi equation can be derived in the Hilbert space, namely:

$$\frac{\partial \Psi}{\partial t} = -H,$$

which describes an evolution equation of a complex system.

Thus we have a model that can be used, for example, to study monitoring problems.

## Chapter 6

# Theory of Monitoring and P-entropy

In the previous chapter we constructed certain geometric structures for the VCM, which is a model for complex systems. It is necessary to mention that the VCM also has subsystems evolving inside of it. The VCM can be seen as a general model for the universe, in which biological, technological, social and natural systems exist and evolve. Such enormous and complex systems are abstractly described as different subsystems evolving in the VCM.

For a given system, one can consider an evolutionary problem. With the Hilbert manifold constructed, it is possible now to consider evolution type problems in this framework. Note that since the Hilbert manifold is a submanifold of the VCM, it “inherits” any invariants defined in the VCM, which we will describe in what follows. Since the total entropy is conserved within the Hilbert manifold, an evolutionary process is described in Panchenkov’s theory as a conversion between the structural entropy  $H_q$  and the impulse entropy  $H_p$ , since  $H_q + H_p = \text{const}$ .

Having introduced certain structures in the Hilbert manifold allowed Panchenkov to geometrically describe the motion, which is composed of the evolution and the events, of complex systems. This approach can be reduced to the study of differential equations with initial or boundary conditions to characterize a trajectory of a system evolving in the Hilbert manifold.

A solution to such a differential equation (or a system of differential equations) describes the evolution of the system in question. If a solution has singularities, they will correspond to a singularity in the Hilbert manifold; such a singularity together with a small neighbourhood forms the kernel of the Hilbert manifold, which is the object used in describing events of the evolution studied. The singularity is also the location where a system collapses and a new system emerges.

## 6.1 Entropy in the Hilbert Manifold

The Hilbert manifold inherits the volume, entropy and mass from the stochastic environment as invariants via the following identification:  $f \rightarrow \rho$ .

### 6.1.1 Phase volume

The phase volume in the Hilbert manifold is an oriented volume defined as follows

$$V = \text{sign}V|V|,$$

$$\text{where } \text{sign}V = \begin{cases} -1, & V < 0, \\ 0, & V = 0, \\ 1. & V > 0. \end{cases}$$

The entropy equation (4.9) is rewritten as:

$$H_f = \text{sign}V \ln |V|.$$

### 6.1.2 Duality on the Hilbert space

Recall that  $\rho$  denotes the density of the virtual continuous medium. The decomposition  $\Omega = \Omega_q \times \Omega_p$  implies

$$\rho = \rho_q \cdot \rho_p,$$

where  $\rho_q$  is the density of the configuration space and  $\rho_p$  is the density of the momentum space.

Then the structural entropy  $H_q$  and the impulse entropy  $H_p$  can be represented as follows:

$$H_q = \text{sign}V_q \ln |V_q| \quad \text{and} \quad H_p = \text{sign}V_p \ln |V_p|.$$

Their sum is the total entropy:  $H_f = H_q + H_p$ .

### 6.1.3 Continuity on the Hilbert space

Since the mass of the Hilbert space is constant, it follows that both the mass of the configuration space and the mass of the momentum space are also constant.

This arrangement leads to the following decomposition of the continuity equation, which assumes the following respective forms on the two spaces:

$$\frac{d\rho_q}{dt} + \rho_q \text{div}U = 0, \quad U = \frac{\partial \mathbf{q}}{\partial t}$$

$$\frac{d\rho_p}{dt} + \rho_p \operatorname{div} V = 0, \quad V = \frac{\partial \mathbf{p}}{\partial t}$$

Hence,

$$\frac{1}{\rho_q} \frac{d\rho_q}{dt} = -\operatorname{div} U,$$

$$\frac{1}{\rho_p} \frac{d\rho_p}{dt} = -\operatorname{div} V,$$

Finally, we have

$$-\frac{d \ln \rho_q}{dt} = \operatorname{div} U,$$

$$-\frac{d \ln \rho_p}{dt} = \operatorname{div} V.$$

Next, Panchenkov defined the following new quantities:  $\sigma_1 := \operatorname{div} U$  and  $\sigma_2 := \operatorname{div} V$ .

## 6.2 The Inner Time

To describe the evolution within the VCM one can employ two types of time: the astronomical time  $t$  and the entropy time  $s$ .

Now look back on the continuity of  $P$ -entropy:

$$\mathbf{A} = \dot{\mathbf{q}} + \dot{\mathbf{p}}, \quad \dot{\mathbf{q}} = \mathbf{A}_{\mathbf{q}}, \quad \sigma_1 = \operatorname{div} \dot{\mathbf{q}},$$

where  $\mathbf{A} = A_q \times A_p$ , which is the flow velocity in the VCM and  $A_q$  is the flow velocity in the configuration space of VCM.

Then by Euler's equations:

$$\mathbf{A}_{\mathbf{q}} = \dot{\mathbf{q}} = \frac{\sigma_1}{n} \mathbf{q}. \quad (6.1)$$

Note  $\mathbf{A}_{\mathbf{q}}$  is measured in  $\frac{[L]}{[T]}$ , while  $\sigma_1 - \frac{1}{[T]}$ ,  $\mathbf{q} - [L]$  and  $n$  is dimensionless.

Panchenkov defined the inner time  $s$  through a generalization of the equation (6.1), namely:

$$\frac{ds}{dt} \triangleq \operatorname{sign} V \frac{\sigma_1}{l}, \quad (6.2)$$

where the dimension of  $s$  is  $[L]$ ,  $t - [T]$  and  $l - \frac{1}{[L]}$ .

We also have the equation:

$$\frac{dH_q}{dt} = \operatorname{sign} V \sigma_1. \quad (6.3)$$

Then it follows from the equations (6.2) and (6.3) that the structural entropy is the inner time, namely,  $H_q = s$ .



### 6.3 The Boundary Conditions

The initial and final conditions are defined as follows:

$$\{H_q = 0, H_p = H_f\} \quad \forall t = t_0;$$

$$\{H_p = 0, H_q = H_f\} \quad \forall t = T,$$

where  $t_0$  is the initial time and  $T$  is the final time.

Also, along with structural conditions:  $\mathbf{q}$  is the wear of the system

$$\mathbf{q} = 0, \quad \forall t = t_0, \quad \mathbf{q} = 0, \quad \forall t = T;$$

$$\mathbf{q} = I, \quad \forall t = t_0, \quad \mathbf{q} = 0, \quad \forall t = T.$$

### 6.4 Evolution Equations

The system theory provides a natural framework for the study of objects on the Hilbert manifold.

A system can be defined by the following three axioms[13]:

- A system is determined by a space of states  $E$  and a space of parameters  $M$ . The latter defines the behaviour of a system;
- The space of states  $E$  contains a subspace (i.e., the number of elements of  $E$  is at least two);
- A system possesses the emergent property.

To describe the evolution of a system on the Hilbert manifold, we need to derive and solve the corresponding evolution equations, solving of which by exact methods in many cases is nearly impossible due to their nonlinearity and complexity in general. Therefore Panchenkov employed the asymptotic method to deal with such equations.

However, not all systems can be studied by the asymptotic method. Thus, we introduce three more axioms and describe a class of systems that can be investigated by asymptotic methods. These axioms are listed below [13].

- There exists a region  $\Omega$  in the Euclidean space  $\mathbb{R}^n$  ( $\Omega \subset \mathbb{R}^n$ ) such that the space of states  $E$  is the dual space on  $\Omega$ :  $E = E(\Omega)$ ;
- The space  $E(\Omega)$  is a metric space;
- The space of parameters  $M(\Omega)$  is a region in the Euclidean space  $\mathbb{R}^k$  ( $M \subset \mathbb{R}^k$ ).

In this view we will denote a system by  $S(E, M)$  or, simply,  $S$ .

Here we attain a generalized equation to describe the evolution process, which can be written as:  $\Gamma\phi = \mathfrak{X}$ , where  $\Gamma$  is a generalized operator that may include boundary or initial conditions or even both of them,  $\phi \in E(\Omega)$  and  $\mathfrak{X}$  is an inhomogeneous part.

We further can obtain  $\phi = L(\epsilon)\phi_{\epsilon 0}$ , where  $L(\epsilon)$  is an evolution operator depending on  $\epsilon$  and an element in the space of the evolution operators  $\epsilon(M_0)$ .

Naturally, it is necessary to talk about the scale of comparison  $\epsilon(M_0)$ :  $\epsilon(M_0)$  is an ordered set of elements  $\delta_\alpha(\epsilon)$ , where  $\epsilon \in M_0$ ,  $\alpha$  is in an index set  $A_\alpha$ , which define the corresponding structure in  $\epsilon(M_0)$ , namely,  $\delta_\alpha(\epsilon) < \delta_\beta(\epsilon)$  if  $\alpha < \beta$ .

Let us construct the corresponding space of operator  $E_\epsilon(M_0)$  such that  $L(\epsilon) \in E_\epsilon$ .

As a result,  $\phi \in E(\Omega)$  is an asymptotic expansion in  $E_0(M_0)$  up to  $\delta_0(\epsilon)$ , in which one has to confirm that the ordered sets must be convergent, thus Panchenkov has assumed [13] there exists a set of elements  $\alpha_\lambda \in E(\Omega)$ ,  $\lambda < \alpha$  for which

$$\| \phi - \sum_{\lambda < \alpha} \alpha_\lambda \delta_\lambda(\epsilon) \|_{E_\epsilon} \sim O[\delta_\alpha(\epsilon)].$$

Depending on the particular space  $E_\epsilon(M_0)$  one can always formalize an asymptotic expansion for function  $\phi$  in terms of different types of  $L(\epsilon)$ , for which Panchenkov has given the asymptotic expansion in the following way:

$$L(\epsilon) = 1 + \delta_1(\epsilon)L_1 + \delta_2(\epsilon)L_2 + \dots + \delta_{\alpha-1}(\epsilon)L_{\alpha-1} + O(\delta_\alpha(\epsilon)).$$

This method is called the *limiting correctness* by Panchenkov and is used to simulate the behaviour of a system as precise as possible. Roughly speaking, the limiting correctness is a much generalized asymptotic method, which is also more accurate. Occasionally, when applying the regular asymptotic method, we have some solutions of a differential equation that fail to cover some boundary or initial conditions or ones at infinity, for which in order to modify the inaccuracy, Panchenkov introduces the limiting incorrectness analysis. With such method one can have a result that can feasibly include the whole behaviour of the original equation.

Thus, let the state of a system  $S$  at  $\epsilon = \epsilon_0$  be described by the following mathematical model:

$$\Gamma_0\phi_0 = \mathfrak{X}_0; \quad \phi_0 \in E(\Omega_0); \quad \epsilon = \epsilon_0, \quad \mathfrak{X}_0 \in X.$$

which can be analysed by the limiting correctness method, that is,

- 1) choose  $\phi \in E_\epsilon(M_0)$ ;
- 2) use the asymptotic expansion in order to simulate the system,  $\lim_{\epsilon \rightarrow 0}(\Gamma\phi - \Gamma_0\phi_0) = 0$  and  $\lim_{\epsilon \rightarrow 0}(\Omega - \Omega_0) = 0$ .

Now the evolution of the systems on the Hilbert manifold can be formed as follows:

$$\Gamma\phi = \mathfrak{X}; \mathbf{q} \in \Omega_q; \phi \in E(\Omega_q); \Omega_q \subset R^n; \mathfrak{X} \in X; \epsilon \in M; M \subset R^+,$$

where  $\phi$  is a dual vector in state space  $E$ .

Note the small parameter  $\epsilon$  here. In order to construct it, Panchenkov has introduced two variables  $t_0$  and  $T$ , where  $t_0$  and  $T$  are accordingly the small character time and the great character time in the evolution. Consequently, the small  $\epsilon$  is defined as  $\epsilon = \frac{t_0}{T}$ , which can be called a *gauge parameter*.

Eventually, we can present the state system  $S$  of a continuum on the Hilbert manifold in what follows:

$$S = \{\phi, t, \mathbf{q}, \epsilon \mid \mathbf{q} \in \Omega_q; \Gamma\phi = \mathfrak{X}; \phi \in E(\Omega_q); \epsilon \in M; M \subset R^+; \mathfrak{X} \in X; t \in J\},$$

where  $J$  is a time interval.

Setting  $\epsilon = 0$ , we reduce the above set to

$$S_0 = \{\phi_0, t, \mathbf{q}, \epsilon \mid \mathbf{q} \in \Omega_{0q}; \Gamma_0\phi_0 = \mathfrak{X}_0; t \in J\}.$$

Provided there exists an isomorphism  $S \rightarrow S_0$ , we can identify the operator  $L(\epsilon) \in E(M_0)$  such that  $\phi = L(\epsilon)\phi_0$  and therefore obtain the corresponding asymptotic expansion  $\Gamma\phi = \mathfrak{X}$ .

However, the asymptotic expansion can not cover all of the original evolution, i.e. its solution may fail to be regular at some initial or boundary points. Such a local incorrectness is called *limiting incorrectness* by Panchenkov. Thus, we need to employ an extra analysis in the area of the local incorrectness, the analysis of which will be used to describe the events on the Hilbert manifold within the framework of Panchenkov's theory.

One can ask why such a limiting incorrectness exists. Suppose an object on the Hilbert manifold evolves in a time interval  $J = [T^-, T^+]$ , the limiting in-correctness can be seen as some sufficiently small time interval  $\delta J = \left\{t \mid \frac{|t|}{T} \leq \epsilon\right\}$  where the evolution of the object is discontinuous. Such a small neighbourhood is called *the kernel of Hilbert manifold* and the system may acquire some new properties through this. Panchenkov called such a process *an event*.

So far we can split the state system  $S$  into two models:

1. The outer interval where the evolution is regular:

$$S_p = \{\phi, t, \mathbf{q}, \epsilon \mid S_p \subset S; t \in J_0, J_0 = J - \delta J\}$$

2. The kernel where the events occur:

$$S_\delta = \{\phi, t, \mathbf{q}, \epsilon | S_\delta \subset S; t \in \delta J\}$$

Again, as  $\epsilon \rightarrow 0$ ,

$$S \rightarrow \begin{cases} S_0 & \text{if } t \in J_0, \\ S_\delta & \text{if } t \in \delta J. \end{cases}$$

We can subsequently use the asymptotic limiting correctness method to characterize both the evolution as well as any events that the object may go through on the Hilbert manifold.

Recall the generalized evolution process  $\Gamma\phi = \mathfrak{X}$  on the Hilbert manifold. Firstly, the operator  $\Gamma$  can be decomposed into three parts:

$$\Gamma = \Gamma_p + \Gamma_c^- + \Gamma_c^+,$$

where  $\Gamma_p$  is the regular operator as  $t \in J_0$ ,  $\Gamma_c^-$  is the left operator as  $t \in \delta J$  and  $\Gamma_c^+$  is the right operator as  $t \in \delta J$ .

**Remark 6.4.1.** The kernel is mathematically a discontinuity and thus regular operator can not characterize it. To resolve this difficulty, Panchenkov utilized the left and right ones in the vicinity of the kernel as it is sufficiently small.

Next, let us choose a set of scale of comparison  $E(\epsilon) = \{\epsilon^\gamma\}$ , where  $\gamma = -1, 0, 1, 2, 3, \dots, n$  and take an asymptotic expansion of  $\phi$  in the kernel  $t \in \delta J$  provided it exists:

$$\phi = \frac{\phi_{-1}}{\epsilon} + \phi_0 + \phi_1\epsilon + \phi_2\epsilon^2 + \dots + O(\epsilon^n); t \in \sigma J; \|\phi_n\| \sim O(1).$$

Also for  $\Gamma_c^-$  and  $\Gamma_c^+$ :

$$\Gamma_c^\pm = \frac{\Gamma_{c-1}^\pm}{\epsilon} + \Gamma_{c0}^\pm + \Gamma_{c1}^\pm\epsilon + \dots + O(\epsilon^n); t \in \delta J.$$

Since the kernel interval is sufficiently small, we can see  $\|\mathfrak{X}\| \sim O(1)$ , accordingly, the asymptotic expansion of  $\Gamma\phi = \mathfrak{X}$  in the kernel  $t \in \delta J$  is:

$$\left( \frac{\Gamma_{c-1}^\pm}{\epsilon} + \Gamma_{c0}^\pm + \Gamma_{c1}^\pm\epsilon + \dots + O(\epsilon^n) \right) \left( \frac{\phi_{-1}}{\epsilon} + \phi_0 + \phi_1\epsilon + \phi_2\epsilon^2 + \dots + O(\epsilon^n) \right) = 0$$

Now open the brackets and collect the equation in terms of  $\epsilon$ :

$$O\left(\frac{1}{\epsilon^2}\right) \text{ terms: } \quad \Gamma_{c-1}^- \phi_{-1} + \Gamma_{c-1}^+ \phi_{-1} = 0; t \in \delta J;$$

$$O\left(\frac{1}{\epsilon}\right) \text{ terms: } \quad (\Gamma_{c-1}^- + \Gamma_{c-1}^+) \phi_0 + (\Gamma_{c0}^- + \Gamma_{c0}^+) \phi_{-1} = 0; t \in \delta J;$$

...

where the ... are the terms of the order higher than  $-1$ .

We are only interested in the  $O(\frac{1}{\epsilon^2})$  terms. Substitute a new expansion  $\phi_{-1} \cong \epsilon\Psi + O(\epsilon)$  in terms of a new parameter  $\|\Psi\| \sim \frac{1}{O(\epsilon)}$  in the  $O(\frac{1}{\epsilon^2})$  terms:

$$\Gamma_{c-1}^-(\epsilon\Psi + O(\epsilon)) + \Gamma_{c-1}^+(\epsilon\Psi + O(\epsilon)) = 0,$$

It follows that

$$\epsilon(\Gamma_{c-1}^-\Psi + \Gamma_{c-1}^+\Psi) + \Gamma_{c-1}^-O(\epsilon) + \Gamma_{c-1}^+O(\epsilon) = 0,$$

Since

$$\Gamma_{c-1}^\pm O(\epsilon) \cong O(\epsilon^n),$$

from which we obtain

$$\epsilon(\Gamma_{c-1}^-\Psi + \Gamma_{c-1}^+\Psi) = O(\epsilon^n);$$

Dividing by the small parameter  $\epsilon$  gives

$$\Gamma_{c-1}^-\Psi + \Gamma_{c-1}^+\Psi \cong O(1); \quad t \in \delta J.$$

Thus,  $S_\delta$  can describe the kernel quite precisely in the vanishing of  $\epsilon$ :

$$S_\delta = \{\phi, \mathbf{q}, t \mid \Gamma_c \phi = 0; \Gamma_c = \Gamma_{c-1}^- + \Gamma_{c-1}^+; t \in \delta J\}.$$

Now let us consider again:

$$\frac{\partial \Psi}{\partial t} = \Theta, \quad \text{the acceleration potential equation;} \quad (6.4)$$

$$\frac{\partial \mathbf{p}}{\partial t} = \frac{\partial \Theta}{\partial \mathbf{q}}, \quad \text{the impulse equation.} \quad (6.5)$$

The acceleration potential on the Hilbert manifold can be written:

$$\Theta = -K - \Pi, \quad (6.6)$$

where  $K = K(\mathbf{p}, t)$  is the ratio of the pressure to the fluid density and  $\Pi = \Pi(\mathbf{q}, t)$  is the potential of the force field acting on a fluid .

Without loss of generality we can assume the ratio takes on the following form:

$$K = \frac{1}{2}\Lambda(\mathbf{p}, \mathbf{p}), \quad \mathbf{p} = \text{grad}\Psi.$$

Panchenkov assumed  $\Pi(\mathbf{q}, t) \in \mathcal{C}^2(\Omega_q)$  and  $\Pi(\mathbf{q}, 0) = 0$ . Define  $\sigma J$  to be a small interval in time. Hence, take  $t \in \sigma J$

$$\bar{\mathbf{p}}(\mathbf{q}, t) = t\mathbf{p}; \quad \bar{\mathbf{p}}(\mathbf{q}, t) \in \mathcal{C}^2(\Omega_q \times \sigma J);$$

$$\bar{\Psi}(\mathbf{q}, t) = t\Psi; \quad \bar{\Psi}(\mathbf{q}, t) \in \mathcal{C}^2(\Omega_q \times \sigma J).$$

Linearising around  $t = 0$ , we get:

$$\bar{\mathbf{p}}(\mathbf{q}, t) = \bar{\mathbf{p}}(\mathbf{q}) + \tilde{\mathbf{p}}(\mathbf{q}, t); \quad \bar{\mathbf{p}} = \bar{\mathbf{p}}(\mathbf{q}, 0) \sim O(t).$$

$$\bar{\Psi}(\mathbf{q}, t) = \bar{\Psi}(\mathbf{q}) + \tilde{\Psi}(\mathbf{q}, t); \quad \bar{\Psi}(\mathbf{q}) = \bar{\Psi}(\mathbf{q}, 0) \sim O(t).$$

Next, via  $t \rightarrow \tau$ ;  $\tau = \frac{t}{\epsilon}$ , such that

$$\mathbf{p} \cong \frac{\bar{\mathbf{p}}(\mathbf{q})}{\epsilon\tau}; \quad \forall \bar{\mathbf{p}}(\mathbf{q}) \sim O(1); \quad \tau \in \sigma\bar{J}; \quad \sigma J = \epsilon\sigma\bar{J};$$

$$\Psi \cong \frac{\bar{\Psi}(\mathbf{q})}{\epsilon\tau} + O(1); \quad \forall \bar{\Psi}(\mathbf{q}) \sim O(1).$$

Correspondingly,

$$K = \frac{1}{2\epsilon^2\tau^2} \|\bar{\mathbf{p}}\|^2; \quad \bar{\mathbf{p}}(\mathbf{q}) = \frac{\partial \bar{\Psi}}{\partial \mathbf{q}};$$

$$\frac{\partial \Psi}{\partial t} = -\frac{\bar{\Psi}}{\epsilon^2\tau^2}; \quad \frac{\partial \mathbf{p}}{\partial t} = -\frac{\bar{\mathbf{p}}}{\epsilon^2\tau^2}. \quad (6.7)$$

**Note:** The negative sign of the equation (6.7) is inherited from the sign of the acceleration potential in the equation (6.6).

Now substitute the above equations into the equations (6.4) and (6.5):

$$-\frac{\bar{\Psi}}{\epsilon^2\tau^2} + \frac{1}{2} \frac{1}{\epsilon^2\tau^2} \|\bar{\mathbf{p}}\|^2 = 0;$$

$$-\frac{\bar{\mathbf{p}}}{\epsilon^2\tau^2} + \frac{1}{2\epsilon^2\tau^2} \frac{\partial}{\partial \mathbf{q}} \|\bar{\mathbf{p}}\|^2 = 0,$$

Also,

$$\Gamma_{c-1}^- + \Gamma_{c+1}^+ = O,$$

from which we get

$$\bar{\mathbf{p}} - \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} \|\bar{\mathbf{p}}\|^2 = 0;$$

$$\bar{\Psi} - \frac{1}{2} \left\| \frac{\partial \bar{\Psi}}{\partial \mathbf{q}} \right\|^2 = 0; \quad \tau \in \sigma\bar{J}.$$

Note that in this case  $\phi = \mathbf{p}$ .

## 6.5 Monitoring Problem

We are now in a position to formulate and summarize the idea of Panchenkov, which was discussed through the previous chapters. In view of the results of the considerations we are led to the conviction that, according to the principle of the maximum entropy, the total entropy is conserved among different systems, such as biological, engineering, social and political ones, and with which we can forecast and even possibly control the behaviour of complex structures, both regular evolutions and events.

### 6.5.1 Virtual continuous medium

Referring to the axioms defining the virtual continuous medium (See Section 4.3.2), which can be expressed as:

$$S = \{\mathbf{q}, \mathbf{p}, t, \rho | \rho = \rho(\mathbf{q}, \mathbf{p}, t), \mathbf{q} \in \Omega_q, \mathbf{p} \in \Omega_p, \Omega = \Omega_q \times \Omega_p; \Omega_q \subset \mathbb{R}^n; \\ \Omega_p \subset \mathbf{T}^*\mathbb{R}^n, \Omega \subset \mathbb{R}^n \oplus \mathbb{R}_n; m; H_f\}.$$

Another three postulates can be drawn from the above[14]:

- 1) Entropy is the main object of the virtual continuous medium;
- 2) The phase volume is oriented.
- 3) The entropy of the virtual continuous medium is determined by the Boltzmann formula for an inertial continuum with respect to  $\rho$ .

More precisely, the oriented phase volume is given by

$$V = \text{sign}V|V|,$$

$$\text{where } \text{sign}V = \begin{cases} -1, & V < 0, \\ 0, & V = 0, \\ 1, & V > 0. \end{cases}$$

$$V = \int_{\Omega} d\Omega.$$

Let us investigate conservation of mass  $m$ . Suppose  $m = \text{constant}$ ,

$$\text{sign}V \int_{\Omega} \rho d\Omega = 1 \Rightarrow \int_{\Omega} \rho d\Omega = \text{sign}V.$$

We introduce the virtual entropy through both:

$$H_f = - \int_{\Omega} \rho \ln \rho d\Omega \Rightarrow H_f = \text{sign}V \ln |V|.$$

### 6.5.2 Fundamental properties of the entropy model

**Theorem 6.1.** *(The preservation of the entropy) In the virtual continuous medium (VCM), the phase volume and the entropy remain constant according to the principle of maximum of entropy:*

$$H_f = \text{constant}, V = \text{constant}$$

and are related in the following equation:

$$H_f = \text{sign}V \ln |V|.$$

Now we have a dual representation of entropy

$$H_f = \begin{cases} - \int_{\Omega} \rho \ln \rho d\Omega, \\ \text{sign}V \ln |V|. \end{cases}$$

The density of the virtual continuous medium is  $\rho$ , the duality of which naturally inherits from the bi-fold representation of the phase space  $\Omega = \Omega_q \times \Omega_p$ ,

$$\rho = \rho_q \cdot \rho_p,$$

where  $\rho_q = \rho_q(\mathbf{q}, t)$  and  $\rho_p = \rho_p(\mathbf{p}, t)$  are the density of the configuration space  $\Omega_q$  and the momentum space  $\Omega_p$  respectively.

Now integrating the two densities correspondingly,

$$\int_{\Omega_q} \rho_q d\mathbf{q} = \text{sign}V_q \quad \text{and} \quad \int_{\Omega_p} \rho_p d\mathbf{p} = \text{sign}V_p.$$

As we know  $H_f = H_q + H_p$ ,

from which we attain the formulae for  $H_q$  and  $H_p$ ,

$$H_q = -\text{sign}V_p \int_{\Omega_q} \rho_q \ln \rho_q d\mathbf{q};$$

$$H_p = -\text{sign}V_q \int_{\Omega_p} \rho_p \ln \rho_p d\mathbf{p}.$$



Then we substitute  $V = V_q \cdot V_p$ , namely,

$$H_q = \text{sign}V \ln |V_q|,$$

$$H_p = \text{sign}V \ln |V_p|.$$

### 6.5.3 Monitoring problem

In order to have a complete description of the movement in the VCM, we must specify how systems evolve in time. The information must be supplemented by such a definition of time that, in virtue of this definition, these time-values can be regarded as magnitudes capable of observation. We introduce time scales for the monitoring problem: the inner time  $H_q$  and the regular time  $t$ .

Panchenkov defined such time variables [14]:

- 1) Astronomical time  $t$ ;
- 2) Time of the existence of the system  $H_q$ .

and the boundary points in a time interval:

- 1) Initial time  $t_0$ ;
- 2) Final time  $T$ .

To illustrate the monitoring problem, let us consider a general biological system.

The system evolves in the VCM:

$$S = \{\mathbf{q}, \mathbf{p}, t, \rho | \rho = \rho(\mathbf{q}, \mathbf{p}, t), \mathbf{q} \in \Omega_q, \mathbf{p} \in \Omega_p, \Omega = \Omega_q \times \Omega_p; \Omega_q \subset \mathbb{R}^n; \Omega_p \subset \mathbb{R}_n; \Omega \subset \mathbb{R}^n \oplus \mathbb{R}_n; m; H_f\}.$$

In order to give a more exact description, we must obtain more properties and geometrical structures, namely, the Hilbert manifold:

$$\Gamma = \{\mathbf{q}, \mathbf{p} | \Gamma \subset \Omega; \Theta; \Psi; \omega_0\}.$$

with the boundary conditions

$$\{H_q = 0; H_p = H_f\} \quad \forall t = t_0,$$

$$\{H_q = 0; H_p = H_f\} \quad \forall t = T.$$

It satisfies the global symmetry:

$$H_f = H_q + H_p.$$

More specifically, the biological evolution can be characterized as follows:

$$H_p \rightarrow H_q \rightarrow H_p,$$

$$\{H_q = 0; H_p = H_f\} \quad \forall t = t_0;$$

$$\{H_q = 0; H_p = H_f\} \quad \forall t = T.$$

**Remark 6.5.1.** Possibly, in many circumstances the following additional conditions are required:

$$\mathbf{q} = 0, \quad \forall t = t_0, \quad \mathbf{q} = 0, \quad \forall t = T;$$

$$\mathbf{q} = I, \quad \forall t = t_0, \quad \mathbf{q} = 0, \quad \forall t = T.$$

Let us investigate the two periods of the evolution of a biological system.

1. Development:  $H_p \rightarrow H_q$ ;

$$H_q = 0, \quad \forall t = t_0; \quad \frac{dH_q}{dt} > 0, \quad t \in (t_0, T_1);$$

$$H_q = H_f, \quad \frac{dH_q}{dt} = 0, \quad \forall t = T_1,$$

where the  $T_1$  is the end time of the development.

2. Senescence:  $H_q \rightarrow H_p$ ;

$$H_q = H_f, \quad \frac{dH_q}{dt} = 0, \quad \forall t = T_1; \quad \frac{dH_q}{dt} < 0, \quad t \in (T_1, T);$$

$$H_q = 0, \quad \forall t = T.$$

As a result of the analysis presented above of the biological system, it is evident that in reality varieties of systems can possibly be characterized with respect to the new conserved quantity, namely, the entropy subjected to certain conditions and the evolution of a system can be described by solving for the differential equation of the structural entropy.

#### 6.5.4 Acceleration potential and potential impulse

Now we revisit the properties and geometrical properties on the Hilbert manifold

$$\Gamma = \{\mathbf{q}, \mathbf{p} \mid \Gamma \subset \Omega; \Theta; \Psi; \omega_0\},$$

in which  $\omega_0$  is a canonical symplectic form, i.e.,  $\omega_0 = \sum_{i=1}^n dq_i \wedge dp_i$ .

As we know,

$$\Theta \rightarrow \mathbf{A} = \nabla\Theta \rightarrow \operatorname{div}\mathbf{A} = 0,$$

where  $\mathbf{A} = \frac{\partial\mathbf{q}}{\partial t} + \frac{\partial\mathbf{p}}{\partial t}$ ;  $\nabla\Theta = (-\frac{\partial\Theta}{\partial\mathbf{p}}, \frac{\partial\Theta}{\partial\mathbf{q}})$ .

Upon the identification  $A = X_\Theta$ , we arrive at

$$\frac{\partial\mathbf{q}}{\partial t} = -\frac{\partial\Theta}{\partial\mathbf{p}};$$

$$\frac{\partial\mathbf{p}}{\partial t} = \frac{\partial\Theta}{\partial\mathbf{q}}.$$

We take another observation: on the Hilbert manifold we have

$$\frac{\partial\Psi}{\partial t} = \Theta.$$

Then it follows that,

$$\frac{\partial\nabla^2\Psi}{\partial t} = \nabla^2\Theta.$$

At this point we have the following equations involving important geometrical structures on the Hilbert manifold:

$$\frac{\partial\mathbf{p}}{\partial t} = \frac{\partial\Theta}{\partial\mathbf{q}}, \quad \mathbf{q} \in \Omega_q, \quad t \in J;$$

$$\frac{\partial\Psi}{\partial t} = \Theta;$$

$$\frac{\partial\nabla^2\Psi}{\partial t} = \nabla^2\Theta,$$

with which we can study the evolution of a system defined in the VCM.

### 6.5.5 Momentum tensor and structural entropy

Since the  $\Omega_q$  is a linear subspace of  $\mathbf{R}^n$ , let us define  $\Psi$  as a quadratic function,

$$\Psi = \frac{1}{2}\Lambda(\mathbf{q}, \mathbf{q}),$$

where  $\Lambda$  is momentum tensor defined by a symmetric  $n \times n$  matrix.

Note,

$$\mathbf{p} = \Lambda\mathbf{q}, \tag{6.8}$$

Furthermore,

$$\nabla^2\Psi = \operatorname{Tr}\Lambda,$$

where  $\operatorname{Tr}\Lambda$  is the trace of the  $n \times n$  matrix  $\Lambda$ .

Observe that  $\mathbf{A} = U + W$ , where  $U = \frac{\partial\mathbf{q}}{\partial t}$  and  $W = \frac{\partial\mathbf{p}}{\partial t}$ .

Hence,

$$\operatorname{div} \mathbf{A} = \operatorname{div} U + \operatorname{div} W.$$

If we set  $\sigma_1 = \operatorname{div} \dot{\mathbf{q}}$ , then

$$\mathbf{p} = \operatorname{grad} \Psi \Rightarrow \sigma_1 = \nabla^2 \Psi = \operatorname{Tr} \Lambda.$$

Note that the  $\sigma_1$  is invariant when  $\Psi$  is a homogeneous function of order 2.

Now we can obtain

$$\frac{dH_q}{dt} = \operatorname{sign} V \sigma_1. \quad (6.9)$$

**Remark 6.5.2.** This is the first fundamental equation of the monitoring theory, characterizing the evolving process of a system in time.

### 6.5.6 The boundary layer

In order to solve for  $H_q$  we need to have corresponding boundary conditions, which can be determined by analysing the kernel of the Hilbert manifold.

We now have the following equations in the neighbourhood of singular points (say corresponding to  $t_0 = 0$ ),

$$\bar{\mathbf{p}} - \frac{1}{2} \frac{\partial}{\partial \mathbf{q}} \|\bar{\mathbf{p}}\|^2 = 0, \quad \bar{\mathbf{p}} = \frac{\mathbf{p}}{t}; \quad (6.10)$$

$$\bar{\Psi} - \frac{1}{2} \left\| \frac{\partial \bar{\Psi}}{\partial \mathbf{q}} \right\|^2 = 0, \quad \bar{\Psi} = \frac{\Psi}{t}; \quad t \in \sigma J. \quad (6.11)$$

The density matrix has the following asymptotic representation

$$\Lambda \simeq \frac{\bar{\Lambda}}{t} + O(1),$$

where one of the values of  $\bar{\Lambda}$  is

$$\bar{\Lambda} = \operatorname{diag} \{ \bar{\Lambda}_{11}, \bar{\Lambda}_{22}, \dots, \bar{\Lambda}_{nn} \},$$

in which  $\bar{\Lambda}_{ii} = 0 \vee 1$ ,  $i = 1, \dots, n$ .

Then

$$\operatorname{Tr} \Lambda \simeq \frac{n}{t}, \quad \forall t \in \sigma J, \quad (6.12)$$

where  $n$  is the dimension of the monitoring system.

### 6.5.7 Conservative environment

Consider  $\Theta$  to be of the following forms:

$$\Theta = -\frac{1}{2}\|\mathbf{p}\|^2 - \frac{1}{2}A(\mathbf{q}, \mathbf{q}),$$

which is common in the problem of fluid dynamics and in which  $A$  is an  $n \times n$  symmetric matrix.

It follows from the Hamilton's canonical equations that

$$\dot{\mathbf{q}} = \mathbf{p},$$

from which we obtain

$$\frac{\partial \mathbf{p}}{\partial t} = -\frac{1}{2} \frac{\partial}{\partial \mathbf{q}} \|\mathbf{p}\|^2 - A\mathbf{q}.$$

From this, we can obtain that the momentum density satisfies the Riccati equation by using  $\mathbf{p} = Tr\Lambda$ ,

$$\dot{\Lambda} = -\Lambda^2 - A.$$

Also we know

$$\frac{dH_q}{dt} = \text{sign}V Tr\Lambda.$$

Without loss of generality we can take  $\text{sign}V = 1$ , then arriving at two problems for a biological system:

1.The event:

$$\dot{\Lambda} = -\Lambda^2 - A, \quad t \in (t_0, T);$$

$$H_q = 0; \quad t = t_0.$$

2.The regular evolution:

$$\frac{dH_q}{dt} = Tr\Lambda,$$

$$H_q = 0; \quad t = t_0.$$

The trajectory of the system on the VCM is accordingly determined by

$$\dot{\mathbf{q}} = -\frac{\partial \Theta}{\partial \mathbf{p}} \rightarrow \dot{\mathbf{q}} = \mathbf{p} \rightarrow \dot{\mathbf{q}} = \Lambda \mathbf{q}. \quad (6.13)$$

**Remark 6.5.3.** This is the second fundamental equation of the monitoring theory, describing the movement of a system in the VCM and, moreover, through the solution of  $\mathbf{q}$  revealing us exactly where the system is.

## 6.6 A Harmonic Model

Let us investigate a 2-dimensional harmonic motion from fluid dynamics:

$$\Theta = -\frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{2}A_1q_1^2 - \frac{1}{2}A_2q_2^2,$$

where  $A_i > 0$ ,  $i = 1, 2$ , with  $\mathbf{q} = 0$  when  $t = 0$  and  $\mathbf{q} = 0$  when  $t = 1$ .

Now introduce a complex coordinate system  $(x, y)$  such that  $z = x + iy$ , in the manner of which we can express the entropy as follows:

$$\bar{H}_q = \bar{H}_q(x, y) \text{ where } H_q = \bar{H}_q|_{y=0},$$

$$\bar{H}_p = \bar{H}_p(x, y) \text{ where } H_p = \bar{H}_p|_{y=0}.$$

Naturally the complex velocity potential [1] is

$$f(z) = \phi(x, y) + i\hat{\psi}(x, y),$$

where  $\phi$  is the velocity potential and  $\hat{\psi}$  is the stream function.

It follows that the complex velocity is

$$W = \frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial x} + i \frac{\partial \hat{\psi}}{\partial x} = u + iv.$$

Since the fluid flow in the VCM is ideal, therefore,  $u = \frac{\partial \phi}{\partial x}$  is the horizontal velocity and  $v = -\frac{\partial \hat{\psi}}{\partial x}$  is the vertical one.

Now we can identify the structural entropy with the stream function  $\hat{\psi}$ , namely,  $\bar{H}_q = \hat{\psi}$ , which leads to  $v = -\frac{\partial \bar{H}_q}{\partial x}$ .

Then consider a rigid body moving forward with a identical concentration along the real axis  $x$ . According to Fick's law of diffusion [16], by assuming that the diffusion region is sufficiently small we can obtain

$$\frac{\partial H_q}{\partial x} = \frac{D}{x}, \tag{6.14}$$

where  $D$  is a diffusion coefficient and will be determined in the following.

Integrating the above equation (6.14) we can have

$$H_q = D \ln |x|. \tag{6.15}$$

It follows from the equations (6.9) and (6.12) that

$$D = \text{sign} Vn.$$

By the recognition  $\Theta = -H$ , the canonical Hamilton's equations are in the following way:

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2.$$

Also recall the equation (6.8), from which we have

$$p_1 = \lambda_1 q_1, \quad p_2 = \lambda_2 q_2.$$

The above dual representations yield the following expressions:

$$\lambda_1 = \frac{\dot{q}_1}{q_1}, \tag{6.16}$$

$$\lambda_2 = \frac{\dot{q}_2}{q_2}. \tag{6.17}$$

Also,

$$\dot{\lambda}_1 = -\lambda_1^2 - A_1,$$

$$\dot{\lambda}_2 = -\lambda_2^2 - A_2.$$

Therefore,

$$\ddot{q}_1 + A_1 q_1 = 0, \tag{6.18}$$

$$\ddot{q}_2 + A_2 q_2 = 0. \tag{6.19}$$

Solving the equation (6.17) subject to  $\mathbf{q}(0) = 0$  and  $\mathbf{q}(1) = 0$  we arrive at

$$q_1 = a_1 \sin(\pi t),$$

where  $a_1$  is a constant.

Similarly,

$$q_2 = a_2 \sin(\pi t),$$

where  $a_2$  is a constant.

Substituting the above  $q_1$  and  $q_2$  into the equations (6.16) and (6.17) correspondingly we get

$$\lambda_1 = \lambda_2 = \pi \cot(\pi t).$$

Asymptotically,

$$\lambda_1 = \lambda_2 \simeq \frac{1}{t} + O(t),$$

which implies that

$$\text{Tr}\Lambda \simeq \frac{2}{t} + O(t).$$

Notice that  $x$  in the equation (6.15) can be generalized to other coordinates, that is, we can say

$$H_q = \text{sign}Vn \ln |\mathbf{q}|,$$

where  $\mathbf{q} = q_1 + q_2$  here.

We can take, without loss of generality,  $\text{sign}V = -1$ , hence,

$$\begin{aligned} H_q &= -2 \ln |q_1 + q_2| \\ &= -2 \ln |a_1 \sin(\pi t) + a_2 \sin(\pi t)|. \end{aligned} \tag{6.20}$$

Let us investigate its kernel, in which we can obtain the above initial conditions. From the equation (6.20), immediately, we can see there are singularities at  $t \in \mathbb{Z}$ . To illustrate we take  $t = 0, 1$ , which can also explain the above boundary conditions.

Now choose a neighbourhood  $\delta J_0 = (0, \epsilon)$  of the singular point 0, where  $\epsilon$  is sufficiently small.

In the  $\delta J_0$  we can asymptotically take

$$\bar{\Psi} = \frac{\Psi}{t} \simeq \frac{\partial \Psi}{\partial t}, \quad \bar{\mathbf{p}} = \frac{\mathbf{p}}{t} \simeq \frac{\partial \mathbf{p}}{\partial t}.$$

It follows the equations (6.4) and (6.5) that

$$\bar{\Psi} = \Theta, \quad \bar{\mathbf{p}} = \frac{\partial \Theta}{\partial \mathbf{q}}.$$

That is,

$$\bar{\Psi}_1 = -\frac{1}{2}\bar{p}_1^2 - \frac{1}{2}A_1q_1^2; \quad \bar{\Psi}_2 = -\frac{1}{2}\bar{p}_2^2 - \frac{1}{2}A_2q_2^2, \tag{6.21}$$

$$\bar{p}_1 = -A_1q_1; \quad \bar{p}_2 = -A_2q_2. \tag{6.22}$$

**Note:** We do not assume  $\Theta = -H$  for the kernel and thus new equations (6.22) do not contradict with the above dual presentations of  $\mathbf{q}$ .

Substituting into (6.10) and (6.11) we can obtain

$$\bar{\Psi}_1 - \frac{1}{2} \left\| \frac{\partial \bar{\Psi}_1}{\partial q_1} \right\|^2 = 0; \quad \bar{\Psi}_2 - \frac{1}{2} \left\| \frac{\partial \bar{\Psi}_2}{\partial q_2} \right\|^2 = 0,$$

$$\bar{p}_1 - \frac{1}{2} \frac{\partial}{\partial q_1} \|\bar{p}_1\|^2 = 0; \quad \bar{p}_2 - \frac{1}{2} \frac{\partial}{\partial q_2} \|\bar{p}_2\|^2 = 0,$$



from which by substituting the above equations (6.21) and (6.22) we arrive at

$$q_1 = q_2 = 0 \Rightarrow \mathbf{q} = 0 \text{ when } t \in \delta J_0.$$

Taking  $\epsilon \rightarrow 0$  we can conclude that  $\mathbf{q} = 0$  when  $t = 0$ .

Analogously,  $\mathbf{q} = 0$  when  $t = 1$  via another kernel analysis in the vicinity of the singular point 1.

## Chapter 7

# Conclusions

In this thesis we have studied the mathematical models based on a generalized notion of entropy introduced by Panchenkov and his principle of maximum of entropy. This setting is quite appropriate for the mathematical treatment of monitoring type problems. We have shown how differential equations can be derived from the aforementioned variational principle and illustrated the procedure with an example. The method based on the Panchenkov's approach can be applied to a broader class of problems in which one can describe mathematically both the evolution and events of a phenomenon in question. At the same time it must be mentioned that the implementation of the method is a very difficult undertaking due to the highly non-trivial mathematical techniques based on advanced asymptotic methods involved.

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