RESOLUTIONS OF MONOMIAL IDEALS VIA QUASI-TREES

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Abstract

We examine ways in which simplicial complexes can be used for describing, classifying, and studying multigraded free resolutions of monomial ideals. By using homgenizations of frames and dehomogenizations of resolutions we can, under appropriate circumstances, describe the structure of a resolution of a monomial ideal by a simiplicial complex. We discuss the successes and failures of this approach. We finish by applying the tools we have presented to quasi-trees, providing a new proof to a theorem of Herzog, Hibi, and Zheng which classifies monomial ideals with minimal projective dimension.

List of Abbreviations and Symbols Used

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
k	Arbirtary Field
S	Polynomial ring $k[x_1, ..., x_n]$
$S(-m), S(-\alpha)$	Shifted Polynomial ring
R	S/I for some ideal $I \subseteq S$
$\mathbf{M}, \mathbf{G}, \mathbf{F}$	Chain complex of modules
∂_i	i^{th} differential map of a chain complex
β_i	Betti number of a module
$\beta_{i,j}$	Graded Betti number of a module
$\beta_{i,m}$	Multigraded Betti number of a module
$H_i(\mathbf{M})$	i^{th} homology module of the complex $\mathbf{M}$
$\tilde{H}_i(\mathbf{M})$	i^{th} reduced homology module of The complex $\mathbf{M}$
$\Delta.\Gamma$	Simplicial complex
Δ_W	Induced subcomplex of Δ on the set W
$\Delta^\vee$	Alexander dual complex of the simplicial complex Δ
$\dim(F), \dim(\Delta)$	Dimension of a face, dimension of a simplicial complex
$\mathbf{C}(\Delta), \mathbf{C}(\Delta; k)$	Simplicial chain complex of Δ, with coefficents in k
$\tilde{H}_i(\Delta; k)$	i^{th} reduced homology module of Δ with coefficents in k
L_I	lcm-lattice of the monomial ideal I
$\mathrm{mdeg}(m)$	Multidegree of a monomial
$\mathbf{T}_I$	Taylor resolution of the monomial ideal I
Γ_I	Scarf complex of I
J_Δ	Nearly Scarf Ideal of the simplicial complex Δ
$\mathcal{N}(\Delta)$	Stanley-Reisner ideal of the simplicial complex Δ
$k[\Delta]$	Stanley-Reisner ring of the simplicial complex Δ

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Chapter 1

Introduction

This thesis is focused on introducing some of the theory involved in the study of monomial ideals. Monomial ideals frequently lend themselves to combinatorial descriptions, which makes them particularly interesting. The additional information gained from a combinatorial vantage point enriches the algebra, by allowing us to compute, classify, and analyze in new and unique ways.

The first chapter focuses on the general algebraic landscape in which we are working, namely graded rings, modules, and complexes. We begin with the definition of the standard graded and multigraded polynomial ring and reintroduce many of the basis facts and definitions of ring and module theory within this context. We end the chapter with minimal graded and multigraded free resolutions where we give a discuss uniqueness and existence, describe some invariants that arise from these minimal resolutions, and some choice theorems which will have particular relevance in later sections.

The second chapter is focused on the combinatorics that we need. We discuss simplicial complexes, giving all the necessary preliminary definitions and discussing the algebraic description of a simplicial complex as a complex of abelian groups and boundary maps. We then give definitions and some results for simplicial trees and a quasi-trees.

The third chapter is the heart of the thesis, bringing together the algebraic and combinatorial information of the previous two chapters. We begin by describing homogenization, which takes a simplicial complex and a monomial ideal I and returns a candidate for a resolution of I. We discuss under what conditions is this process effective and the successes and failures of this approach in general. Later in the chapter we also provide ways to generate monomial ideals using simplicial complexes, giving us a method of classifying special types of ideals about which we can make more pointed statements.

The final chapter focuses on restricting theory of Chapter 3 to Quasi-trees. We constructively prove how to describe the minimal resolution of $\mathcal{N}(\Delta^\vee)$ when Δ is a quasi-tree. We also provide a new proof to a result given by Herzog, Hibi, and Zheng which classifies monomial ideals with minimal projective dimension.

Chapter 2

Graded Objects

2.1 Rings and Ideals

Since the aim of this thesis is to characterize resolutions of some class of monomial ideals in the polynomial ring $S = k[x_1, ..., x_n]$ where k is a field, we will develop our theory in this context. What this means going forward is that we will almost immediately restrict our attention to polynomial rings and their quotient rings. After some introductory definitions we will make this restriction more precise.

Definition 2.1. A **graded ring** ($\mathbb{Z}$-**graded ring**) is a ring R with a direct sum decomposition $R = \bigoplus_{d \in \mathbb{Z}} R_d$ as an abelian group, such that $R_i R_j \subset R_{i+j}$ for $i, j \in \mathbb{Z}$

Definition 2.2. A proper ideal, I of a graded ring R is called **graded** (or **homogeneous**) if I has a direct sum decomposition $I = \bigoplus_{d \in \mathbb{Z}} I_d$ as an abelian group such that $I_d = I \cap R_d$ for all $d \in \mathbb{Z}$

With these definitions we could, for any ring R, set $R = R_0$ (and $R_d = 0$ for $d \neq 0$) so that R and all its ideals are graded. Of course, if we are going to get any use out of these definitions, we are going to want to be more restrictive in the rings and the gradings of them that we consider.

If $S = k[x_1, ..., x_n]$, then we say that a **monomial** is an element of the form $m = x_1^{\alpha_1}...x_n^{\alpha_n} = \mathbf{x}^{\overline{\alpha}}$, where $\alpha_i \in \mathbb{N}$, has **degree** $\sum_{i=1}^{n} \alpha_i \in \mathbb{N}$, denoted $\deg(m)$, and **multidegree** $\overline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, denoted $\mathrm{mdeg}(m)$. It is clear that

$$(m_1 \cdot m_2) = \deg(m_1) + \deg(m_2)$$

and

$$\mathrm{mdeg}(m_1 \cdot m_2) = \overline{\alpha}_1 + \overline{\alpha}_2$$

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We also on occasion talk about lcm's and gcd's of multidegrees α_1 and α_2, by which we mean

$$\text{lcm}(\alpha_1, \alpha_2) = \text{mdeg}\big(\text{lcm}(m_1, m_2)\big)$$

$$\gcd(\alpha_1, \alpha_2) = \text{mdeg}\big(\gcd(m_1, m_2)\big)$$

where $\text{mdeg}(m_1) = \alpha_1$ and $\text{mdeg}(m_2) = \alpha_2$. We say that a polynomial $f \in S$ is **homogeneous** if every monomial in f has the same degree. We denote the collection of all homogeneous polynomial of degree i in S by S_i. From the definition we see that each S_i is a k-vector space whose basis is indexed by monomials of degree i with distinct multidegrees. What this mean is that $S = \bigoplus_{d \in \mathbb{Z}} S_i$ as an abelian group, with $S_d = 0$ for $d < 0$ and that $S_i S_j = S_{i+j}$. Therefore, S is a ($\mathbb{N}$-) graded ring and we call this grading the **standard grading** on S.

We can also talk about graded ideals of a graded ring. There are several equivalent definitions we could use to define a graded ideal and we give them here.

Proposition 2.3 ([17], p.2). *Let J be an ideal of the graded ideal R. The following are equivalent:*

1) $J = \bigoplus_{i \in \mathbb{N}} J_i$, where $J_i = J \cap R_i$

2) If $f \in J$, then $f = f_1 + ... + f_j$ where the f_i are homogeneous and in J.

3) If $\widetilde{J}$ is the ideal generated by all homogeneous elements in J then $J = \widetilde{J}$

4) J has a system of homogeneous generators

*If J satisfies any of these four condition, we say that J is a **graded ideal**.*

Note that the definition of a graded ideal depends on the grading of R. If there are more than one possible gradings for a ring R, then we must specify which grading we are referring to when we say that J is a graded ideal of R. Also, if R is a graded ring and J is a graded ideal of R, then the quotient ring R/J is also a graded ring, with graded components $(R/J)_i = R_i/J_i$.

It will at times be useful to use a more refined grading than the standard grading. Instead of considering S as the direct sum of the finite dimensional k-vector spaces, we can consider it as the direct sum over the one dimensional k-vector spaces indexed by the monomials of distinct multidegrees in S. If $m \in S$ is a monomial, then we denote by S_m the vector space to which it belongs. We see that $S = \bigoplus S_m$ and $S_m \cdot S_{m'} = S_{mm'}$. The only way this decomposition differs from the definition of a graded ring is by the way we index our direct sum. Since the monomials of S are defined by their multidegrees, which are in one-to-one correspondence with the elements of $\mathbb{N}^n$, we call this grading a **multigrading** or $\mathbb{N}^n$-**grading** of S (when we talk about modules we will use a $\mathbb{Z}^n$-grading). The definition for multigraded ideals and quotient rings is analogous to that of graded ideals.

It is worth taking note of what condition (4) of Proposition 2.3 tell us about multi-graded ideals of the polynomial ring S. For and ideal $I \subseteq S$ to be multigraded, we must have that I has a system of homogeneous generators. But the homogeneous components of I under the multigrading are $I_m = I \cap S_m$, and each S_m is a one dimensional k-vector space with generator m, i.e. an element in S is homogeneous with respect to the multigrading of S if and only if it is a scalar multiple of a monomial. Therefore, I is a multigraded ideal of S if and only if it has a system of monomial generators, that is, if and only if I is a monomial ideal.

For the rest of the material, we will consistently denote the polynomial ring in n variables over the field k as $S = k[x_1, ..., x_n]$, and use A, B, etc. to refer to the polynomial ring in specific examples where the number of variables has been fixed. In each case, we will use either the standard grading or the multigrading that we have described, and specify which wherever it is unclear.

2.2   Graded Modules and Homomorphisms

In the following material we will use R to denote the quotient ring $R = S/I$ of the polynomial ring by a graded ideal with respect to either the standard grading or the multigrading we have given.

To save ourselves from repetition and tedium, we will develop the theory in terms

of graded objects (modules, homomorphisms, complexes, resolutions etc.) and take for granted that what we present translates to multigraded objects in an obvious way, by decomposing each S_i into a direct sum of one dimensional vector spaces indexed by the monomials of degree i. It is, however, worth reminding ourselves that we are working towards describing monomial ideals. So, even though we stay in the more general setting of standard graded rings and modules, we should keep in the back of our mind that the modules we care about are specifically those of the form $M = I$ or $M = R = S/I$ where I is a monomial ideal, and the grading is with respect to the multigrading on S.

Definition 2.4. For a graded ring R we define a **graded R-module**, M, to be an R-module with a direct sum decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ as an abelian group, such that $R_i M_j \subset M_{i+j}$ for $i, j \in \mathbb{Z}$

The M_i are called the **homogeneous components** of M. Elements of M_i are said to have **degree** i. Since $R_0 = k$ and $R_0 M_i = M_i$, we see that each M_i is a k-vector space. Furthermore, we can say the following about the structure of M,

Proposition 2.5 ([17], p.5)**.** *For a graded R-module M,*

1) *There exist a homogeneous set of generators of M.*

2) *The degrees of the elements in a system of homogeneous generators of M determine the grading on M.*

These facts may seem unsurprising, but they are worth mentioning. Specifically, we will see that we may, in some cases, want to shift the degrees of homogeneous generators of certain modules, so that we can get some desired properties. By the above proposition, we will be altering the grading of the modules we are working with, and the way in which we do this will be important. What we mean by shifting the degrees of the homogeneous generators of a module is given by the following definition.

Definition 2.6. Let M be a graded R-module and let $p \in \mathbb{Z}$. We denote by $M(-p)$ the graded R-module such that $M(-p)_d = M_{d-p}$ for all $d \in \mathbb{Z}$. We say that $M(-p)$ is the module M **shifted p degrees**, and we call p the **shift**. In the multigraded setting we

denote a shift by the multidegree of a monomial m as either $M(-m)$ or $M(-\bar{\alpha})$, where $\bar{\alpha} = \mathrm{mdeg}(m)$ is the multidegree of m, and this shift is such that $M(-m)_{m'} = M_{m'/m}$, where $M_{m'/m} = 0$ if the multidegree of m'/m has any negative exponents.

This is a well defined notion for a graded R-module M, since

$$R_i M(-p)_j = R_i M_{j-p} \subseteq M_{i+j-p} = M(-p)_{i+j}$$

Moreover, we see that $M(-p) \cong M$. Since the elements of a module, and the relations between them, are unchanged by a shift of degrees the map which sends $x \longmapsto x$ for every $x \in M$ is a well defined isomorphism. However, shifting is not a well defined notion on a ring, since changing the degree of $R_0 = k$ will lead to problems with the grading under multiplication. The way in which to approach such a scenario is to treat R as a module over itself. This way, the action of R on $R(-p)$ is still multiplication in R, but the grading of $R(-p)$ will be well defined.

Next we would like to turn our attention graded free modules, and the graded analogy to the fact that every module is the homomorphic image of a free module. In order to to talk about this we first need to know what the graded versions of submodules and homomorphism are.

Definition 2.7. If M is a graded R-module, then a submodule $N \subset M$ is said to be **graded** (or **homogeneous**) if N has a direct sum decomposition $N = \bigoplus_{d \in \mathbb{Z}} N_d$ as an abelian group such that $N_d = N \cap M_d$ for all $d \in \mathbb{Z}$

Recall that, for ideals $J \subseteq S$, Proposition 2.3 gave four equivalent conditions that tell us when J is graded. We can generalize these conditions to the setting of modules and submodules and get the same result. This generalization extends to quotient modules as well. That is, in the same way the quotient ring $R = S/I$ inherits its grading from S by setting $R_i = S_i/I_i$, we get that the quotient module $U = M/N$, of graded modules M and N, inherits it grading from M by setting $U_i = M_i/N_i$.

Definition 2.8. Let M, N be graded R-modules. A module homomorphism $\phi : M \to N$ is said to have **degree** i if $\phi(M_d) \subset N_{d+i}$ for all $d \in \mathbb{Z}$. We call such a homomorphism a

graded homomorphism of modules.

A useful consequence of this definition is that a homomorphism, $\phi : M \longrightarrow N$, is graded if and only if it sends homogeneous elements of M to homogeneous elements of N.

Proposition 2.9 ([17], p.8). *If $\phi : M \longrightarrow N$ is a graded homomorphism of R-modules, then $\ker(\phi)$ is a graded submodule of M, $\mathrm{im}(\phi)$ is a graded submodule of N.*

Remark 2.10. If a graded module homomorphism $\phi : M \longrightarrow N$ has degree i, we can consider how ϕ behaves when we apply it to $M(-i)$, i.e. if $\psi : M(-i) \longrightarrow M$ is the canonical isomorphism between $M(-i)$ and M which sends each element to itself, then let $\phi' = \phi \circ \psi$. What we get is that

$$\phi'(M(-i)_d)) = \phi \circ \psi(M(-i)_d) = \phi(M_{d-i}) \subset N_{d-i+i} = N_d$$

so that ϕ' is a graded homomorphism of degree 0. Since $M(-i) \cong M$, what this means is that we can edit the degree of a graded R module homomorphism using shifted R-modules. Under the right circumstances, we can do more than this, which we will show in the following example.

Example 2.11. Let $A = k[x,y]$, $M = A \oplus A$, and $\phi : A \oplus A \longrightarrow A$ be the homomorphism which sends $(f,g) \mapsto x^2 y f + y^4 g$.

We need to establish a grading on the modules we are using. For A, we use the standard grading. For $A \oplus A$, we define $(A \oplus A)_d = A_d \oplus A_d$ where A_d is the dth graded component of A with respect to the standard grading. So an element $(f,g) \in A \oplus A$ is homogeneous of degree d if and only if f and g are both homogeneous of degree d in A with respect to the standard grading. With this grading we see that ϕ is not a graded homomorphism, since it send the homogeneous element $(1,1) \in M_0$ to the element $x^2 y + y^4$, which is not homogeneous with respect to the standard grading on A. However, $M \cong M' = A(-3) \oplus A(-4)$ under the mapping

$$\psi : M \longrightarrow M'$$
$$(f,g) \longmapsto (f,g)$$

With these shifts the dth graded component of M' is $M'_d = A_{d-3} \oplus A_{d-4}$. Under this grading an element (f, g) is homogeneous of degree d if and only if f is homogeneous of degree $d - 3$, and g is homogeneous of degree $d - 4$, with respect to the standard grading on A. We can define a new map

$$\phi' : M' \longrightarrow A$$
$$(f, g) \longmapsto x^2 y f + y^4 g$$

In fact, $\phi = \phi' \circ \psi$, so ϕ and ϕ' are essentially the same map, with the only difference the way we treat the grading. It is no longer the case that $(1, 1)$ is a homogeneous element in M'. When we apply ϕ' to the homogeneous components of M' we get

$$\phi'(M'_d) = \phi'(A_{d-3} \oplus A_{d-4}) = x^2 y A_{d-3} + y^4 A_{d-4} \subseteq A_3 A_{d-3} + A_4 A_{d-4}$$
$$\subseteq A_d + A_d = A_d$$

So ϕ' is a graded module homomorphism of degree zero.

In the above example we had that the domain of ϕ was a graded free module and the image of each basis element under ϕ was a homogeneous element. Under these conditions we were able to adjust the grading of the domain so that that ϕ became a graded homomorphism of degree zero, even though it was unchanged as a map of sets. The method above can be generalized in a rigorous way to give us the following result:

Theorem 2.12 ([17], p.9). *Let M be a finitely generated graded R-module. Then $M \cong F/U$, where F is a finite direct sum of shifted free R-modules, U is a graded submodule of F, and the isomorphism has degree 0.*

This theorem is just the graded version of the fact that every module is the quotient of a free module ([1], p.21). We will be able to use this fact to show that we can construct a graded free resolution for any module (we will define what a graded free resolution is in the next section).

2.3 Chain Complexes and Free Resolutions

In this section we are going to define, as the title suggests, chain complexes, free resolutions, and their graded versions.

Definition 2.13. A **complex (chain complex)** $\mathbf{M}$ of R-modules is a sequence of R-module homomorphisms:

$$\mathbf{M} : ... \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} M_{-1} \xrightarrow{\partial_{-1}} M_{-2} \xrightarrow{\partial_{-2}} ...$$

such that $\partial_i \circ \partial_{i+1} = 0$ for all $i \in \mathbb{Z}$. The collection of maps $\partial = \{\partial_i\}$ is called the **differential** of $\mathbf{M}$

It is an obvious consequence of the definition that $\operatorname{im}(\partial_{i+1}) \subset \ker(\partial_i)$. Since images and kernels are modules themselves, and one is a submodule of the other, we can take their quotient if we like, and often this is a useful thing to do. We give these quotients a name.

Definition 2.14. The **homology** of a complex $\mathbf{M}$ is defined to be the collection of modules $H_i(\mathbf{M}) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$. The elements of $\ker(\partial_i)$ are called **cycles** and the elements of $\operatorname{im}(\partial_{i+1})$ are called **boundaries**. For each i, we call H_i the ith **homology module of** $\mathbf{M}$. The complex $\mathbf{M}$ is called **exact**, or **acyclic**, if $H_i(\mathbf{M}) = 0$ (which is equivalent to saying $\ker(\partial_i) = im(\partial_{i+1})$ for all $i \in \mathbb{Z}$.

Note that we did not specify that R, or the modules in $\mathbf{M}$, be graded. Of course, we can, and will, refine the definition of chain complexes to suit a graded context.

Definition 2.15. A complex $\mathbf{M}$ is said to be **graded** if each M_i in $\mathbf{M}$ is a graded module and, each ∂_i has degree 0,

The fact that we require each module to be graded is of no surprise, but requiring each map in the differential to have degree 0 might seem unnecessarily restrictive. However, we recall that if the differential does not have degree 0, we can apply shifts to each module so that it does, without making any real change to the map (see Example 2.11). The upshot of

doing this is that we are able to put a grading on the complex:

$$
\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \oplus & & \oplus & & \oplus & & \\
\cdots \xrightarrow{\partial_{i+2}} & M_{i+1,j-1} & \xrightarrow{\partial_{i+1}} & M_{i,j-1} & \xrightarrow{\partial_i} & M_{i-1,j-1} & \xrightarrow{\partial_{i-1}} & \cdots \\
& & \oplus & & \oplus & & \oplus & & \\
\cdots \xrightarrow{\partial_{i+2}} & M_{i+1,j} & \xrightarrow{\partial_{i+1}} & M_{i,j} & \xrightarrow{\partial_i} & M_{i-1,j} & \xrightarrow{\partial_{i-1}} & \cdots \\
& & \oplus & & \oplus & & \oplus & & \\
\cdots \xrightarrow{\partial_{i+2}} & M_{i+1,j-1} & \xrightarrow{\partial_{i+1}} & M_{i,j-1} & \xrightarrow{\partial_i} & M_{i-1,j-1} & \xrightarrow{\partial_{i-1}} & \cdots \\
& & \oplus & & \oplus & & \oplus & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where each row is now a complex of k-vector spaces between the graded components of equal degree in each M_i. If this complex were multigraded then each row would be a complex of one dimensional vector spaces. We will a denote grading of $\mathbf{M}$ as $\bigoplus_{d \in \mathbb{Z}} \mathbf{M}_d$, where each $\mathbf{M}_d$ is the k-vector space complex on the degree d graded components of the M_i (the notation is similar when we use a multigrading, replacing the d's with monomials m). We call each $\mathbf{M}_d$ the dth **graded component** of $\mathbf{M}$. This grading comes with a rather nice property.

Proposition 2.16 ([17], p.16)**.** *A graded complex* $\mathbf{M}$ *is exact if and only if each* $\mathbf{M}_d$ *is exact.*

This is a very beneficial result, since it reduces questions we will have about graded complexes and resolutions (resolutions are complexes with some extra properties) to questions about sequences of vector spaces, which are generally much easier to work with. When we begin to look specifically at monomial ideals and their resolutions, many of the results we get are proven via this reduction.

Definition 2.17. A **free resolution** of an R-module M is an chain complex of the form

$$
\mathbf{F}: \quad \cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0
$$

such that each F_i is a free R-module (isomorphic to a direct sum of copies of R), $H_0(\mathbf{F}) \cong M$, and $H_i(\mathbf{F}) = 0$ for $i \geq 1$. If M and $\mathbf{F}$ are graded, and the isomorphism $H_0(\mathbf{F}) \cong M$

has degree 0, then we say that $\mathbf{F}$ is a **graded free resolution** of M. We define the **length** of $\mathbf{F}$ to be $\max\{i \mid F_i \neq 0\}$.

We should make clear that the length of a free resolution need not be finite. Also, note that since each ∂_i is a homomorphism between free modules, we can describe each ∂_i completely using matricies.

Example 2.18. Let $A = k[x_1, x_2, x_3, x_4]$ and $I = (x_1x_2, x_2x_3, x_4)$, then a graded free resolution of A/I is

$$
0 \longrightarrow A(-4) \xrightarrow{\begin{bmatrix} x_4 \\ -x_3 \\ x_1 \end{bmatrix}} \begin{matrix} A(-3) \\ \oplus \\ A(-3) \\ \oplus \\ A(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x_3 & x_4 & 0 \\ -x_1 & 0 & x_4 \\ 0 & -x_1x_2 & -x_2x_3 \end{bmatrix}} \begin{matrix} A(-2) \\ \oplus \\ A(-2) \\ \oplus \\ A(-1) \end{matrix} \xrightarrow{\begin{bmatrix} x_1x_2 & x_2x_3 & x_4 \end{bmatrix}} A \longrightarrow 0
$$

We can easily show that every differential map is graded, has degree 0, and that $\partial_i \circ \partial_{i+1} = 0$. To illustrate this, we examine the map

$$
\partial_3 : A(-4) \longrightarrow A(-3) \oplus A(-3) \oplus A(-3)
$$
$$
f \longmapsto (x_4 f, -x_3 f, x_1 f)
$$

If f is homogeneous of degree d in $A(-4)$, then it is homogeneous of degree $d-1$ in $A(-3)$, so that $\deg(x_i f) = d$ in $A(-3)$. This means ∂_3 is homogeneous and has degree 0. Moreover, if we apply ∂_2 to $\partial_3(f)$ we get

$$
\begin{aligned}
\partial_2(\partial_3(f)) &= \partial_2((x_4 f, -x_3 f, x_1 f)) \\
&= (x_3(x_4 f) + x_4(-x_3 f), -x_1(x_4 f) + x_4(x_1 f), -x_1 x_2(-x_3 f) - x_2 x_3(x_1 f)) \\
&= (0, 0, 0)
\end{aligned}
$$

Repeating these calculations for the other ∂_i's will verify our claim. What is left to show is that the 0th homology complex is A/I (this is clear from the definition of ∂_1) and that the ith homology complex is 0 when $i \geq 1$ (so far we only know that $\text{im}(\partial_{i+1}) \subseteq \ker(\partial_i)$ but have not shown equality). With our current understanding of free resolutions, this is possible to prove, but would require some computational diligence. We will spare the details, and

return to this example in a later section, where we will have the tools to show that this complex is exact without having to do the calculations.

We know what a graded free resolution of a graded module is but we do not yet know how to find one, or if it is even possible to find one for any given graded module. If M is a finitely generated graded R-module, then we can always find a graded free resolution of M.

Construction 2.19 ([17], p.17)**.** Given a graded finitely generated R-module M, we will construct a graded free resolution of M by induction on homological degree.

Step 0

Choose homogeneous generators $m_1, ..., m_r$ of M. Define a free R-module

$$F_0 = R(-a_1) \oplus ... \oplus R(-a_r)$$

where $a_i = \deg(m_i)$ for $i = 1, ..., r$. For $1 \leq j \leq r$ denote, by f_j, the basis element of $R(-a_j)$. Thus, $\deg(f_j) = a_j$. Define the map

$$d_0 : F_0 \longrightarrow M$$
$$f_j \longmapsto m_j$$

for $1 \leq j \leq r$.

Assume by induction, that F_i and d_i are defined.

Step $i + 1$

Set $M_{i+1} = \ker(d_i) \subseteq F_i$. Choose homogeneous generators $l_1, ..., l_s$ of M_{i+1} (note that since F_i is a finitely generated module over the Noetherian ring $R = S/I$, F_i is a Noetherian R-module, so such a finite generating set for $\ker(d_i)$ exists). Define a free module

$$F_{i+1} = R(-c_1) \oplus ... \oplus R(-c_s)$$

where $c_i = \deg(l_i)$ for $i = 1, ..., s$. For $1 \leq j \leq s$ denote by g_j the basis element of $R(-c_j)$. Thus, $\deg(g_j) = c_j$. Define the map

$$d_{i+1} : F_{i+1} \longrightarrow M_{i+1} \subset F_i$$

$$g_j \longmapsto l_j$$

for $1 \leq j \leq s$. We see that this construction is such that each d_i has degree 0, $\ker(d_i) = \operatorname{im}(d_{i+1})$ when $i \geq 1$, and $M \cong F_0/\operatorname{im}(d_1) = H_0(\mathbf{F})$ (and this map has degree 0). Hence what we have constructed is the graded free resolution of M

$$\mathbf{F}: \quad \cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

It is often useful to follow up with an example to help clarify what it is that we are actually doing. This would require us to compute the kernel of each of the d_i and to do this we would need to make a detour into Gröbner basis theory. This is too far from the main focus of this thesis and is not useful for the constructions of resolutions we will discuss in later sections. So, we will take for granted that computing the kernel of each d_i can be calculated and forgo providing an example (to see how to do this see [5], section 4.4).

Remark 2.20. Instead of considering a free resolution $\mathbf{F}$ of an R-module M as a sequence of maps, we can also consider it as a single R-module

$$F = \bigoplus_{i \geq 0} F_i$$

where the differential is now a module homomorphism $\partial : F \longmapsto F$. Since each F_i is a free module we see that F is a free module, and if we fix a basis for each of the F_i, then the union of these bases becomes a basis for F. In later sections (and in keeping consistency with the terminology used by Peeva and Velasco in [17], [18]), when we refer to the basis of a free resolution, it will be the basis of the R-module F, formed as the union of the bases of the F_i, to which we are referring. Moreover, if $\mathbf{F}$ is (multi)graded and the basis that we

are referring to is (multi)homogeneous with respect to this grading, then we say that $\mathbf{F}$ has a (multi)homogeneous basis.

There is no need for $\mathbf{F}$ to be a resolution for this idea to hold its meaning. If we have a complex, possibly not exact, of free modules then we can define its basis in exactly the same manner.

2.4   Minimal Free Resolutions

Free resolutions provide us with new information about a module. However, we may have more than one free resolution of a given module. A natural question to ask is: Given an R-module M, what information (if any) is consistent across all free resolutions of M? Moreover, is there an obvious candidate for the free resolution of M which best presents this information? In our setting of graded free resolutions of graded R-modules, where $R \cong S/I$ and graded with respect to the standard grading (or multigrading) on the polynomial ring S, the answer to both of these questions is yes. The answer to our second question is given by the following definition.

Definition 2.21. A graded free resolution of a graded finitely generated R-module M is **minimal** if $\partial_{i+1}(F_{i+1}) \subseteq \mathbf{m}F_i$ for all $i \geq 0$ (recall that $\mathbf{m} = (x_1, ..., x_n)$).

We should make a couple of comments about this definition. The first is that it is in no way clear from the definition why such a condition would make a resolution minimal, or what exactly is being minimized. The legitimacy of the definition will be made clear after we give few more results and definitions. The second comment is that, since each ∂_i in a free resolution can be represented by a matrix, minimality amounts to checking that each of these matrices has entries in $\mathbf{m}$.

Example 2.22. Let $R = k[x]/(x^3)$ and $M = k = R/xR$. The graded free resolution of M

$$\cdots \longrightarrow R(-4) \xrightarrow{\ x\ } R(-3) \xrightarrow{\ x^2\ } R(-1) \xrightarrow{\ x\ } R \longrightarrow 0$$

is minimal, since the differential map at each step is either multiplication by x or by x^2, both of which lie in the maximal ideal $(x) \subset R$.

A first step in showing that, for a graded R-module M, a minimal resolution is of M is a useful object to study, is to show that they exist. Not only is this the case, but we may also show that they satisfy uniqueness properties as well.

Theorem 2.23 ([17] p.29). *The graded free resolution we built in Construction 2.19 is minimal if and only if at each step we choose a minimal homogeneous system of generators for the kernel of the differential.*

Theorem 2.24 ([17], p.30). *Let M be a graded finitely generated R-module. Up to an isomorphism, there exists a unique minimal graded free resolution of M*

Remark 2.25. If $\mathbf{F}$ and $\mathbf{G}$ are two minimal graded free resolutions of M, Theorem 2.24 tells us that they are isomorphic chain complexes. This means that we have the commutative diagram

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$
$$\Big\downarrow{\cong} \qquad \Big\downarrow{\cong} \qquad \Big\downarrow{\cong}$$
$$\cdots \xrightarrow{\bar{\partial}_3} G_2 \xrightarrow{\bar{\partial}_2} G_1 \xrightarrow{\bar{\partial}_1} G_0 \longrightarrow 0$$

Since ∂_i and $\bar{\partial}_i$ are all maps of degree 0, we get that the isomorphisms between each F_i and G_i have the same degree. We also have the commutative diagram

$$F_0 \xrightarrow{\phi} H_0(\mathbf{F}) \xrightarrow{\psi} M$$
$$\cong \Big\downarrow \qquad\qquad\qquad \Big\downarrow{\mathrm{id}_M}$$
$$G_0 \xrightarrow{\bar{\phi}} H_0(\mathbf{G}) \xrightarrow{\bar{\psi}} M$$

where $\phi, \psi, \bar{\phi}, \bar{\psi}$, and id_M all have degree 0. Therefore, the isomorphism $F_0 \cong G_0$ has degree 0 and, as a result, so do the isomorphisms $F_i \cong G_i$. Theorem 2.12 tells us that there are degree zero isomorphisms $F_i \cong G_i \cong \bigoplus_{p \in \mathbb{Z}} R(-p)^{\beta_{i,p}}$ for every $i \geq 0$. This, being true for all minimal graded free resolutions, motivates the following definition.

Definition 2.26. The ith **Betti number** of M over R is defined as $\beta_i^R(M) = rank(F_i)$, where the F_i are the free modules which appear in the minimal graded free resolution $\mathbf{F}$ of M. Since $\mathbf{F}$ is graded, each free module F_i is a direct sum of modules of the form $R(-p)$.

We define the **graded Betti numbers** of M by

$$\beta_{i,p}^R(M) = \text{number of summands in } F_i \text{ of the form } R(-p)$$

for an integer p. Similarly, If $\mathbf{F}$ is multigraded, we define the **multigraded Betti numbers** of M to be

$$\beta_{i,m}^R(M) = \text{number of summands in } F_i \text{ of the form } R(-m)$$

for a monomial m. The definition tells us that for a fixed i, $\beta_i^R(M) = \sum_p \beta_{i,p}^R(M) = \sum_m \beta_{i,m}^R(M)$. Furthermore, if our minimal resolution admits a multigrading, we will have that for each i,

$$F_i = \bigoplus_{m \in R} R(-m) = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{m \in R_p} R(-m) = \bigoplus_{p \in \mathbb{Z}} R(-p)$$

and we conclude that $\beta_{i,p}^R(M) = \sum_{m \in R_p} \beta_{i,m}^R(M)$. We now have the information needed to make sense of why minimal resolutions are called minimal.

Theorem 2.27 ([5], p.72)**.** *Let M be a finitely generated graded R-module and $\mathbf{F}$ a (not necessarily minimal) graded free resolution of M, with $F_i = \bigoplus_p R(-p)^{b_{i,p}}$ for each $i \in \mathbb{Z}$. Then*

$$\beta_{i,p}(M) \le b_{i,p}$$

for all $i, p \in \mathbb{Z}$.

The inequality still holds when we sum over p and we get the same result for the $\beta_i^R(M)$. So, the minimality of a graded free resolution is with respect to the ranks of the free modules at each step. If $\mathbf{F}$ is a graded free resolution of a module M with length k then, for all $j \in \mathbb{N}$, $rank(F_{k+j}) = 0 \ge \beta_{k+j}^R(M) \ge 0$. This means that a minimal graded free resolution of M is minimal with respect to length as well and we give this minimal length its own distinction.

Definition 2.28. The **projective dimension** of an R-module M is defined as

$$\mathrm{pd}_R(M) = \max\{i \mid \beta_i^R(M) \neq 0\}$$

where the $\beta_i^R(M)$ are the Betti numbers of M.

Remark 2.29. Let $\mathbf{F}$ be the free resolution of M

$$\mathbf{F} : \cdots \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

Since $\mathbf{F}$ is exact, we get that each of the differential maps ∂_i factors through $\ker(\partial_{i-1})$ for $i \geq 2$, and ∂_1 factors through $\ker(\epsilon)$. This gives us the following commutative diagram.

Where $F_{i+1} \longrightarrow F_i \longrightarrow \ker(\partial_{i-1})$, $0 \longrightarrow \ker \partial_1 \longrightarrow F_i$, and $F_i \longrightarrow \ker(\partial_{i-1}) \longrightarrow 0$ are exact. What this means is that not only do we have a free resolution for M, we have a free resolution for $\ker(\epsilon)$ and each $\ker(\partial_i)$ (up to a shift of indicies). If $\mathbf{F}$ is minimal, so are the resolutions we get for $\ker(\epsilon)$ and $\ker(\partial_i)$.

In particular, if $M = S/I$ then, for a minimal free resolution of M, $F_0 = S$ and $\ker(\epsilon) = I$. From what we have just seen, we can conclude that the minimal free resolution of S/I will give us a minimal free resolution of I as well. Moreover, we have that

$$pd(I) = pd(S/I) - 1 \quad \text{and} \quad \beta_{i,p}(I) = \beta_{i+1,p}(S/I)$$

This is a fact that we will exploit when we study monomial ideals in later sections.

Example 2.30. Recall the minimal graded free resolution from example 2.18

$$0 \longrightarrow A(-4) \xrightarrow{\begin{bmatrix} x_4 \\ -x_3 \\ x_1 \end{bmatrix}} \begin{matrix} A(-3) \\ \oplus \\ A(-3) \\ \oplus \\ A(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x_3 & x_4 & 0 \\ -x_1 & 0 & x_4 \\ 0 & -x_1x_2 & -x_2x_3 \end{bmatrix}} \begin{matrix} A(-2) \\ \oplus \\ A(-2) \\ \oplus \\ A(-1) \end{matrix} \xrightarrow{[\, x_1x_2 \ x_2x_3 \ x_4 \,]} A \longrightarrow 0$$

The graded Betti numbers A/I and I are:

$$\beta_{0,0}(A/I) = 1$$

$$\beta_{0,1}(I) = \beta_{1,1}(A/I) = 1$$

$$\beta_{0,2}(I) = \beta_{1,2}(A/I) = 2$$

$$\beta_{1,3}(I) = \beta_{2,3}(A/I) = 3$$

$$\beta_{2,4}(I) = \beta_{3,4}(A/I) = 1$$

and all others are zero. The total Betti numbers are

$$\beta_0(A/I) = \beta_{0,0}(A/I) = 1$$

$$\beta_0(I) = \beta_1(A/I) = \beta_{1,1}(A/I) + \beta_{1,2}(A/I) = 1 + 2 = 3$$

$$\beta_1(I) = \beta_2(A/I) = \beta_{2,3}(A/I) = 3$$

$$\beta_2(I) = \beta_3(A/I) = \beta_{3,4}(A/I) = 1$$

and all the others are zero. We also see that $pd_R(I) = 2$ and $pd_R(A/I) = 3$.

We now know that for a finitely generated graded R-module M a minimal graded free resolutions always exists, it is unique up to isomorphism, and we know how to go about constructing one, though the construction algorithm may never terminate. With this in mind we often call a minimal graded free resolution of M "the" minimal graded free resolution of M.

While this is relatively satisfying, we may wish to be greedy and ask for more. The more that we ask for is that our minimal graded free resolutions be finite. We already saw in Example 3 that this is not always the case for finitely generated R-modules (R, as always,

is graded and of the form S/I). So, the question is, what further restrictions do we need to make in order to get this result? One possibility is to restrict ourselves to the standard graded polynomial ring S.

Theorem 2.31 (Hilbert's Syzygy Theorem, [5], p.68)**.** *Let* $S = k[x_1, ..., x_n]$. *If* M *is a finitely generated* S*-module, then any minimal graded free resolution of* M *has length at most* n.

Since ideals are also submodules of S, and S is Noetherian (Hilbert's basis theorem, see [5] p.5), we have that this result hold for all ideals of S and in particular all monomial ideals. Moreover, Construction 2.19 provides us with an algorithm which will allow us to compute a minimal free resolution for each ideal in S.

2.5   Regular Elements

Is this section we briefly discuss regular elements and regular sequences (See [17] for a more detailed account). Colloquially, a regular element is a ring element which is a non-zero divisor on a module M. What this means precisely is

Definition 2.32. Let M be an R-module. An element $r \in R$ is said to be M-**regular** if $rm \neq 0$ for all $m \neq 0$, $m \in M$.

A natural extension of this definition is that of a regular sequence.

Definition 2.33. Let M be an R-module. A sequence $\mathbf{f} = f_i, ..., f_q$ of elements of R is called an M-**regular sequence** if the following two conditions are satisfied.

1) f_i is a regular element of $M/(f_1, ..., f_{i-1})M$, for $i = 1, ..., q$

2) $M/\mathbf{f}M \neq 0$

It is important to note that, in general, regular sequences are sensitive to the order of the elements in the sequence.

Example 2.34. Let $M = R = k[x, y, z]$ and consider the elements x, $y(1-x)$, $z(1-x)$. Then the sequence x, $y(1-x)$, $z(1-x)$ is an M-regular sequence. To show this, we begin

with x. Since $k[x, y, z]$ is a domain, x is clearly an M-regular element. To show that the other elements are regular we first observe that

$$\frac{M}{xM} = \frac{R}{(x)} = \frac{k[x, y, z]}{(x)} \cong k[y, z]$$

and under this isomorphism

$$y(1 - x) \longmapsto y, \ z(1 - z) \longmapsto z$$

Since y is regular on $k[y, z]$ and z is regular on $k[y, z]/y \cong k[z]$ we can conclude that $x, \ y(1 - x), \ z(1 - x)$ is a regular sequence.

If we now look at the sequence $y(1 - x), \ z(1 - x), \ x$ then we have that $y(1 - x)$ is regular on M but $z(1 - x)$ is not regular on $M/y(1 - x)M \cong k[x, y, z]/(y(1 - x))$ since $y \cdot z(1 - x) = z \cdot y(1 - x) = 0$ in $k[x, y, z]/(y(1 - x))$ and $y, \ z(1 - x)0 \neq 0$ in $k[x, y, z]/(y(1 - x))$. So, $y(1 - x), \ z(1 - x), \ x$ is not an M-regular sequence.

However, for a graded R-module M, if each $f_i \in \mathbf{m}$ is homogeneous and belongs to the maximal ideal $\mathbf{m}$, then every permutation of $f_1, ..., f_q$ is again an M-regular sequence ([17], p.53).

One feature of regular elements, and the feature which we will focus on, is that we can use them to manipulate free resolutions in a predictable manner. These manipulations will allow us to take a free resolution of an R-module M, and form a resolution of $M/(f)M$, where f is an M-regular element. Before we can explicitly state these results, we need to define what it means to take a tensor product of a chain complex and a module.

Let U be an R-module and let $\mathbf{M}$ be the complex of R-modules

$$\mathbf{M} : ... \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} M_{i-2} \xrightarrow{\partial_{i-2}} M_{i-3} \xrightarrow{\partial_{i-3}} ...$$

We can define a new chain complex $\mathbf{M} \otimes U$ as the complex

$$\mathbf{M} \otimes U : ... \xrightarrow{\partial_{i+1} \otimes 1_U} M_i \otimes U \xrightarrow{\partial_i \otimes 1_U} M_{i-1} \otimes U \xrightarrow{\partial_{i-1} \otimes 1_U} M_{i-2} \otimes U \xrightarrow{\partial_{i-2} \otimes 1_U} \cdots$$

Where 1_U is the identity map on U.

Theorem 2.35 ([17], p.84). *Let M be an R-modules and $f \in R$ be both R-regular and M-regular. If $\mathbf{F}$ is a free resolution of M over R, then $\mathbf{F} \otimes_R R/(f)$ is a free resolution of M/fM over $R/(f)$. In addition, if f is homogeneous and $\mathbf{F}$ is graded, then $\mathbf{F} \otimes_R R/(f)$ is graded. Furthermore, if $\mathbf{F}$ is minimal and $f \in \mathbf{m}$ then $\mathbf{F} \otimes_R R/(f)$ is minimal.*

Remark 2.36. Let R, M, f, and $\mathbf{F}$ be as above. If $f' \in R$ is such that its image in $R/(f)$ is $R/(f)$-regular and M/fM-regular (i.e. f, f' is both an R-regular sequence and an M-regular sequence) then we can apply Theorem 2.35 to get that

$$\left(\mathbf{F} \otimes_R \frac{R}{(f)} \right) \otimes_{R/(f)} \frac{R/(f)}{f'(R/(f))} \cong \mathbf{F} \otimes_R \left(\frac{R}{(f)} \otimes_{R/(f)} \frac{R/(f)}{f'(R/(f))} \right) \cong \mathbf{F} \otimes_R \frac{R/(f)}{f'(R/(f))}$$

Is a free resolution of $\dfrac{M/fM}{f'(M/fM)}$ over $\dfrac{R/(f)}{f'(R/(f))}$. Using the fact that

$$f' \left(\frac{R}{(f)} \right) \cong \frac{(f') + (f)}{(f)}$$

and

$$f' \left(\frac{M}{fM} \right) \cong \frac{f'M + fM}{fM}$$

we can conclude that

$$\frac{R/(f)}{f'(R/(f))} \cong \frac{R}{(f, f')}$$

and

$$\frac{M/fM}{f'(M/fM)} \cong \frac{M}{(f, f')M}$$

Therefore, we can say that if f, f' is both an R-regular and M-Regular sequence, then $\mathbf{F} \otimes_R R/(f, f')$ is a free resolution of $M/(f, f')M$. Also, as Theorem 2.35 says, if f and f' are also homogeneous and in the maximal ideal $\mathbf{m}$, then $\mathbf{F} \otimes_R R/(f, f')$ will be a

minimal resolution. Moreover, by repeating the above arguments we can extend the results of Theorem 2.35 to regular sequences of any length.

Chapter 3

Simplicial Complexes and Simplicial Trees

3.1 Simplicial Complexes, Simplicial Chain Complexes

Definition 3.1. Let $V = \{v_1, ..., v_n\}$ be a finite set. A (finite) **simplicial complex**, Δ, on V is a collection of non-empty subsets of V such that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$ and $\{v_i\} \in \Delta$ for $i = 1, ..., n$. The elements of Δ are called **faces**. Faces containing one element are called **vertices** and maximal faces are called **facets**. For each face $F \in \Delta$, we define $\dim(F) = |F| - 1$ to be the **dimension of the face** F. We define $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$ to be the **dimension of the simplicial complex** Δ. If Δ is a simplicial complex with only 1 facet and r vertices, we call Δ an **r-simplex**.

Definition 3.2. A simplicial complex Δ with vertex set $V = \{v_1, ..., v_n\}$ is **connected** if for every $v_i, v_j \in V$ there is a sequence of faces $F_0, ..., F_k$ such that $v_i \in F_0$, $v_j \in F_k$ and $F_i \cap F_{i+1} \neq \emptyset$ for $i = 0, ..., k - 1$.

It is easy to see from the definition that a simplicial complex can be described completely by its facets, since every face is a subset of a facet and every subset of every facet is in a simplicial complex. So, for a simplicial complex Δ with facets $F_0, ..., F_q$, we use the notation $\langle F_0, ..., F_q \rangle$ to describe Δ.

Also, we can, and often will, present a simplicial complex geometrically when the dimension is small enough. We describe 0-dimensional facets as points, 1-dimensional facets as lines, 2-dimensional facets as solid triangles, and 3-dimensional facets as solid tetrahedrons. Intersecting these shapes at the appropriate subfaces will give us all of the information we need to describe a simplicial complex (see [16],Chapter 1). This is best shown through example.

Example 3.3. If Δ is the simplicial complex whose facets are $\{v_1, v_2\}, \{v_1, v_3\}$, and $\{v_2, v_3, v_4\}$. Then we have the following representation for Δ:
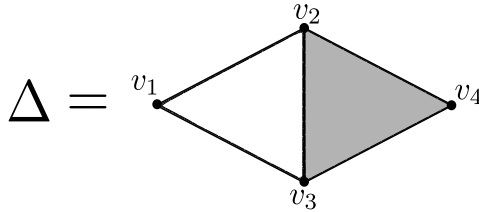

Figure 3.1: Geometric representation of a simplicial complex

Another way we can describe a simplicial complex is by building a chain complex which is specific to the simplicial complex we are working with. It is worth mentioning here that while we will use terminology related to modules to maintain consistency with previous definitions and ideas discussed in this thesis, abelian groups are $\mathbb{Z}$-modules and vice versa. So while we will proceed to discuss complexes of $\mathbb{Z}$-modules, we are in fact giving a description in terms of abelian groups. The objects we will be describing are as follows:

Construction 3.4 ([10], pp.104-106)**.** Let Δ be a simplicial complex on the vertex set $\{v_0, ..., v_n\}$. Let $C_k(\Delta)$ be the free $\mathbb{Z}$-module whose basis is indexed by the k-dimensional faces of Δ. For each $k \in \{1, ..., n\}$ we define a map $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ as follows: If $F \in \Delta$ is the k-dimensional face on the vertices $\{v_{i_0}, ..., v_{i_k}\}$, with corresponding basis element $e_F \in C_k(\Delta)$, then

$$\partial_k(e_F) = \sum_{j=0}^{k} (-1)^j e_{F \setminus \{v_{i_j}\}}$$

If we set $C_{k+1}(\Delta)$ and $C_{(-1)}(\Delta)$ to be the 0 module, with maps $\partial_{k+1} = \partial_0 = 0$ then we get a sequence of module homomorphisms:

$$\mathbf{C}(\Delta) : \ 0 \longrightarrow C_k(\Delta) \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_3} C_2(\Delta) \xrightarrow{\partial_2} C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \longrightarrow 0$$

We call this the **simplicial chain complex** of Δ and we call the homology module

$$H_i(\mathbf{C}(\Delta)) = H_i(\Delta) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$$

the ith **simplicial homology module** of Δ.

Definition 3.5. The f-**vector** of a d-dimensional simplicial complex Δ is the sequence $f(\Delta) = (f_0, ..., f_d) = \big(\mathrm{rank}(C_0(\Delta)), ..., \mathrm{rank}(C_d(\Delta))\big)$, so that each f_i is the number of i-dimensional faces of Δ.

It is not clear from the definition that this is in fact a chain complex. It can be shown, by a calculation that is more tedious than enlightening, that $\partial_{i-1} \circ \partial_i = 0$ for $i = 1, ..., k = 1$, and we conclude that:

Proposition 3.6 ([10], p.105). $\mathbf{C}(\Delta)$ *is a chain complex of $\mathbb{Z}$-modules (abelian groups).*

Remark 3.7. Every simplicial complex Δ gives rise to a chain complex of free $\mathbb{Z}$-modules, but it is not the case that every chain complex of $\mathbb{Z}$-modules gives rise to a simplicial complex. The question my then arise: If we are given a chain complex of free $\mathbb{Z}$-modules, can we determine if this chain complex has the form $\mathbf{C}(\Delta)$ for some simplicial complex Δ?

If a chain complex of free $\mathbb{Z}$-modules were of the form $\mathbf{C}(\Delta)$, the rank of each C_i would determine the number of i-dimensional faces, and the differential maps ∂_{i+1} and ∂_i would indicate how each of these i-dimensional faces attaches to faces of dimension $i + 1$ and dimension i. This gives us good indication as to what the simplicial complex Δ would have to be.

Example 3.8. Let Δ be the 3-simplex and $\mathbf{C}(\Delta)$ be the simplicial chain complex

$$\mathbf{C}(\Delta): \ 0 \longrightarrow \mathbb{Z}^1 \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix}\right]} \mathbb{Z}^3 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{smallmatrix}\right]} \mathbb{Z}^3 \longrightarrow 0$$

We can reconstruct Δ using only information present in $\mathbf{C}(\Delta)$.

Since $C_0(\Delta) = \mathbb{Z}^3$ we know that Δ has three vertices (0-dimensional faces), call them v_0, v_1, and v_2. Also, since $C_1(\Delta) = \mathbb{Z}^3$ we know that Δ has three 1-dimensional faces (i.e. edges), call them F_0, F_1, and F_2 for the moment. Applying the definition of the differential in Construction 3.4 to F_0 we get

$$\partial_1(e_{F_0}) = \sum_{j=0}^{1}(-1)^j e_{F_0 \setminus \{v_{i_j}\}} = e_{F_0 \setminus \{v_{i_0}\}} - e_{F_0 \setminus \{v_{i_1}\}}$$

and applying the matrix given in $\mathbf{C}(\Delta)$, which is also ∂_1, to the basis element e_{F_0} we get

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = e_{\{v_1\}} - e_{\{v_0\}}$$

Upon comparing the two resulting expression we conclude that $v_{i_0} = v_0$ and $v_{i_1} = v_1$ and so $F_0 = \{v_0, v_1\}$ (the edge between v_0 and v_1). Repeating this process for F_1 and F_2 we find that $F_1 = \{v_0, v_2\}$, and $F_2 = \{v_1, v_2\}$.

Lastly, we see that Δ has a single 2-dimensional face, F_3. Again, we will compare our two definitions of ∂_2 to deduce what F_3 is. From the definition, we have

$$\partial_2(e_{F_3}) = \sum_{j=0}^{2}(-1)^j e_{F_3 \setminus \{v_{i_j}\}} = e_{F_3 \setminus \{v_{i_0}\}} - e_{F_3 \setminus \{v_{i_1}\}} + e_{F_3 \setminus \{v_{i_2}\}}$$

and the map from the chain complex of Δ is

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = e_{F_0} - e_{F_1} + e_{F_2}$$

So $F_3 \setminus \{v_{i_0}\} = F_0 = \{v_0, v_1\}$, $F_3 \setminus \{v_{i_1}\} = F_2 = \{v_0, v_1\}$, and $F_3 \setminus \{v_{i_2}\} = F_2 = \{v_1, v_2\}$, and we see that $v_{i_0} = v_2$, $v_{i_1} = v_1$, and $v_{i_2} = v_0$, meaning $F_3 = \{v_0, v_1, v_2\}$. We now know

what all of our faces are, and our simplicial complex is the 3-simplex

$$\Delta = \big\{ \{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0, v_1, v_2\} \big\}$$

Which is exactly what we expect.

It should be mentioned that not every finite chain complex of free $\mathbb{Z}$-modules describes a simplicial complex, so we cannot always apply the methods of Example 3.8 to any chain complex of $\mathbb{Z}$ and recover a simplicial complex.

Remark 3.9. (See [10], pp.109-110) The chain complex of a nonempty simplicial complex is never exact. The best case scenario is that $H_i(\mathbf{C}(\Delta)) = 0$ for $i \geq 1$, and $H_0(\mathbf{C}(\Delta)) = \mathbb{Z}$. To remedy this we can set $C_{(-1)} = \mathbb{Z}$ and define $\partial_0(e_{v_i}) = 1$ for each basis element e_{v_i} in C_0. We can verify that this is still a chain complex, which we call the **augmented simplicial chain complex** of Δ, and we denote it as $\widetilde{\mathbf{C}}(\Delta; k)$. The homology modules of the augmented simplicial chain complex of Δ are called the **reduced homology modules** of Δ and the ith one is denoted $\widetilde{H}_i(\Delta)$ (even though $H_i(\Delta) = \widetilde{H}_i(\Delta)$ when $i \geq 1$).

Definition 3.10. A simplicial complex Δ is called **acyclic** when $\widetilde{H}_i(\Delta) = 0$ for all $i \geq 0$

It may seem that this distinction between the chain complex and the augmented chain complex of a simplicial complex is a waste of time, since it does not tell us anything new about our simplicial complex. However, when we begin to use simplicial complexes as a tool for finding free resolution of monomial ideals, augmented chain complexes and reduced homology will be the more appropriate definitions to work with.

Another consideration we might make is regarding the use of free $\mathbb{Z}$-modules in the simplicial chain complex. We could have, in our definition, defined the chain group as free k-modules, for some field k (or something more exotic than this if we like, see [10], p.153). The choice of $\mathbb{Z}$ is due to topological considerations, and the development of the theory in this regard. If we choose to work over a field as opposed to $\mathbb{Z}$ then the combinatorial description of our simplicial complex that is contained in the chain complex will still be preserved (that is, we will still be able to reconstruct a simplicial complex from its chain complex). This is because the coefficients of our differential are always ± 1, which are

elements of every field. Since we are interested in exploiting combinatorial properties of simplicial complexes, and not topological properties, this generalization to working over a field will be one which we will make.

This does not mean that there will be no consequences in making this generalization. While the differential maps remain relatively unchanged, what can change are the homology modules of the complex. If, for example, $H_n(\Delta) = 2\mathbb{Z}$, for some simplicial complex Δ and some $n \in \mathbb{Z}$, when working over the integers, then we would have that $H_n(\Delta) = 0$ if we are working over the field $\mathbb{Z}/2\mathbb{Z}$. So, homology is sensitive to the choice of the field we are working with. To be clear about our context, we denote by $H_n(\Delta; k)$, the **homology of Δ with coefficients in** k (and we denote the chain groups by $C_n(\Delta; k)$ when we need to be clear about our context. Again, see [10], p.153 for details).

For our treatment of monomial ideals, we have defined the S to be the polynomial ring over some field k, which we do not specify, which means that k could have any possible characteristic. However, being unable to explicitly say what the characteristic of k is, we typically present the information as though k has characteristic zero, noting which results and calculations are dependent on the characteristic of k.

3.2   Simplicial Trees and Quasi-trees

In the main result of our investigation, we focus on two specific types of simplicial complexes, called simplicial trees and quasi-trees. They are the simplicial complex analogy of a graph tree. In fact, we can view graphs as 1-dimensional simplicial complexes, and when we do, the definition we make for simplicial trees will give us graph trees when we restrict to the 1-dimensional case. In order to define what a simplicial tree is and describe some of its properties, we are going to need some definitions.

Definition 3.11. For a simplicial complex Δ with vertex set V and $W \subseteq V$, we define the **induced subcomplex on** W in Δ, denoted Δ_W, to be the set $\Delta_W = \{F \in \Delta | F \subseteq W\}$

Definition 3.12. For a simplicial complex Δ, we define a **subcollection** of Δ to be a simplicial complex whose facets are also facets of Δ. If Δ has facets $F_1, ..., F_q$, then the subcollection which has facets $F_{i_1}, ..., F_{i_k}$ is the simplicial complex $\langle F_{i_1}, ..., F_{i_k} \rangle$

Both of the above definitions describe some type of subcomplex of a simplicial complex. The first definition uses the smallest faces of the complex and works up, the second definition uses the maximal faces of the complex and works down. This allows for the subcomplexes they define to be quite different.

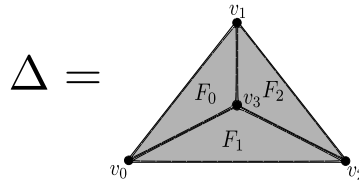**Example 3.13.** If Δ is the following simplicial complex

Figure 3.2: A simplicial complex (example of subcomplexes)

Then we can describe two subcomplexes of Δ as follows: The first complex is the induced subcomplex Δ_W on the vertex set $W = \{v_0, v_1, v_2\}$, and the second is the subcollection $\langle F_1, F_2 \rangle$.
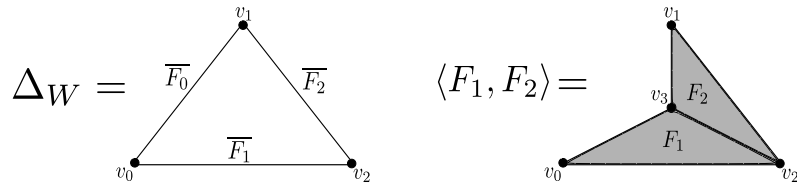
Figure 3.3: Induced subcomplex vs. Subcollection

Clearly these subcomplexes are quite different. Δ_W is a simplicial complex on three vertices. It has 3 facets, none of which are facets of Δ, and $\dim(\Delta_W) = 1 \neq \dim(\Delta)$. On the other hand, $\langle F_1, F_2 \rangle$ is a simplicial complex on four vertices. It has 2 facets, both of which are facets of Δ, and $\dim(\langle F_1, F_2 \rangle) = 2$. Moreover, Δ_W cannot be described as a subcollection of Δ, and $\langle F_1, F_2 \rangle$ cannot be described as the induced subcollection in Δ of any subset $W \subseteq \{v_0, v_1.v_2.v_3\}$.

Definition 3.14. (Faridi, [6]) A facet F of a simplicial complex Δ is called a **leaf** if either F is the only facet of Δ or for some facet $G \in \Delta$ we have that $F \cap (\Delta \setminus \langle F \rangle) \subseteq G$. In the second scenario, the facet G is said to be the **joint** of F.

To be clear, by $\Delta \setminus \langle F \rangle$ we mean the subcollection of Δ that is generated on all the facets of Δ with the exception of F (i.e. if $\Delta = \langle F, F_1, ..., F_q, \rangle$, then $\Delta \setminus \langle F \rangle = \langle F_1, ..., F_q \rangle$). In the above example, neither Δ nor Δ_W contained a leaf. However, in $\langle F_1, F_2 \rangle$ both F_1 and F_2 are leaves.

If a facet F of a simplicial complex Δ is a leaf, then F necessarily has a **free vertex**, which is a vertex of Δ that belongs to exactly one facet. If the leaf F of Δ did not have a free vertex then all vertices of F would belong to $\Delta \setminus F \subset G$, and we would conclude that F is a subface of G, hence not a facet.

We are now ready to define a simplicial tree and describe some of its properties.

Definition 3.15. (Faridi, [6]) A connected simplicial complex Δ is a **simplicial tree** if every nonempty subcollection of Δ has a leaf. If Δ is not necessarily connected, but every subcollection has a leaf, then Δ is called a **forest**.

Example 3.16. Consider the simplicial tree Γ, and simplicial complex Δ (not a simplicial tree)
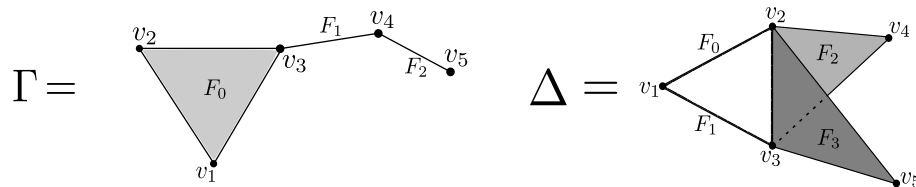
Figure 3.4: Example of when a complex is a simplicial tree

We see that Γ and Δ each have a leaf, and that Γ is a simplicial tree. Upon further inspection of Δ, we find that the subcollection $\langle F_0, F_1, F_2 \rangle$ does not have a leaf, hence Δ is not a simplicial tree.

It is clear from the definition that if Δ is a simplicial tree and Γ is a subcollection of Δ, then Γ is a simplicial forest. We are also able to show the following properties of simplicial trees.

Theorem 3.17 (Faridi, [9])**.** *An induced subcomplex of a simplicial tree is a simplicial forest.*

Proposition 3.18 (Faridi, [9]). *Simplicial trees are acyclic.*

The above two properties will give good justification as to why we wish to study simplicial trees in the context of resolutions of monomial ideals. When we develop the idea of generating resolutions from simplicial complexes we will find that, because of these properties, when we restrict to simplicial trees things simplify nicely and we are able to give some classifications. One of the properties of simplicial trees that we will make particular use of is that whenever Δ is a simplicial tree we can always order the facets $F_1, ..., F_q$ of Δ so that F_i is a leaf of the induced subcollection $\langle F_1, ...F_i \rangle$. Such an ordering on the facets is called a **leaf order** and it is used to make the following definition.

Definition 3.19. (Zheng, [24]) A connected simplicial complex Δ is a **quasi-tree** if Δ has a leaf order. If Δ has a leaf order but is not connected, we say that Δ is a **quasi-forest**.

It follows from the definitions of simplicial trees and quasi-that every simplicial tree is also a quasi-tree. To show that not every quasi-tree is a simplicial tree we provide the following example.

Example 3.20. Let Δ be the following simplicial complex.
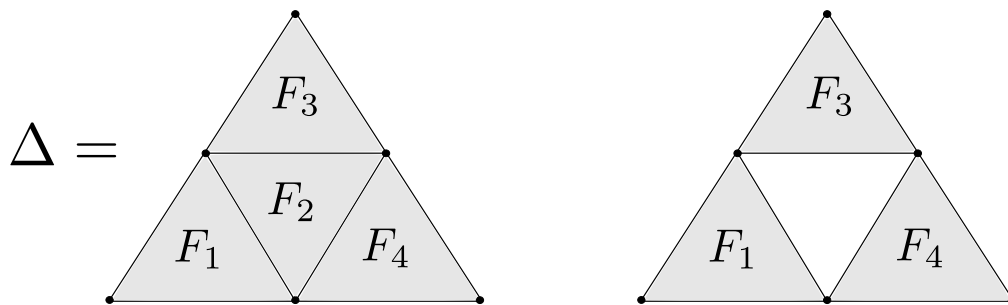
Figure 3.5: quasi-tree that is not a simplicial tree

The ordering given on the facets of Δ satisfies definition 3.19 so Δ is a quasi-tree. However, the subcollection $\langle F_1, F_3, F_4 \rangle$ does not have a leaf, hence Δ is not a simplicial tree.

Equivalently, we could have defined quasi-trees to be simplicial complexes such that every induced subcomplex has a leaf. The equivalence of these definitions is proven below.

Proposition 3.21. *A simplicial complex Δ with vertex set V is a quasi-forest if and only if*
for every subset $W \subset V$, Δ_W has a leaf.

Proof. ($\Rightarrow$) Since Δ has a leaf order, we may label the facets of Δ, $F_0, ..., F_q$, so that F_i is
a leaf of $\Delta_i = \langle F_0, ..., F_i \rangle$. For a subset $W \subset V$, choose the smallest i such that W is a
subset of the vertex set of Δ_i, which we will denote V_i.

We claim that the complex induced on W in Δ_i is Δ_W. It is clear that $(\Delta_i)_W \subseteq \Delta_W$.
To see the converse, let F be a face of Δ_W, then $F \subseteq F_j$ for some facet $F_j \in \Delta$. If $j \leq i$
then $F \in \Delta_i$ and we are done. If $j > i$ then let F_k be the joint of F_j in Δ_j and note that
$k < j$. Since $F \subseteq W \subseteq \Delta_i \subseteq \Delta_j \setminus \langle F_j \rangle$ we have that $F \subseteq F_j \cap (\Delta_j \setminus \langle F_j \rangle) \subset F_k$. If $k \leq i$
then we are done. If not we may iterate this argument as many times as necessary until we
get a facet $F_a \in \Delta_i$ for which $F \subseteq F_a$. Hence $(\Delta_i)_W = \Delta_W$.

We will show that $F_i \cap W$ is a leaf of Δ_W. Since $F_i \in \Delta_i$, $F_i \cap W$ is a face of Δ_W.
Also, $V_i = V_{i-1} \cup \{$free vertices of F_i in $\Delta_i\}$ which means that $W \cap \{$free vertices of F_i in $\Delta_i\} \neq$
$\emptyset$, otherwise W would be contained in the vertex set of Δ_{i-1}. Therefore $F_i \cap W$ is not a
subset of any other face in Δ_W, i.e. $F_i \cap W$ is a facet of Δ_W. If F_j is the joint of F_i in
Δ_i, then for any face $F \in \Delta$, $F \cap F_i \cap W \subset F_j \cap F_i \cap W$. This means that any facet of
Δ_W (except for $F_i \cap W$) that contains $F_j \cap F_i \cap W$ is a joint for $F_i \cap W$ in Δ_W, since the
faces of Δ_W are also faces of Δ. If no such facet exist (except for $F_i \cap W$) then $F_i \cap W$ is
disjoint from the rest of Δ_W. In either scenario, $F_i \cap W$ is a leaf of Δ_W.

($\Leftarrow$) This is done by induction on the size of the vertex set V of Δ. For $|V| = 1$ or 2,
a quick inspection shows that all simplicial complexes with vertex set V have a leaf order
and every induced subcomplex has a leaf. Now assume that every simplicial complex on
$\leq n$ vertices for which every induced subcomplex has a leaf is a quasi-forest.

Suppose Δ is a simplicial complex on $n + 1$ vertices and that every induced subcom-
plex of Δ has a leaf. Since Δ is an induced subcomplex of itself, it also has a leaf, call it
F, with free vertices $v_1, ..., v_k$. The simplicial complex $\Delta \setminus \langle F \rangle$ is given by the induced
subcomplex Δ_W where $W = V \setminus \{v_1, ..., v_k\}$. Every induced subcomplex of Δ_W has a leaf
and Δ_W is a simplicial complex on $\leq n$ vertices, hence Δ_W has a leaf order $G_1, ..., G_j$.
This gives us a leaf order $G_1, ..., G_j, F$ for Δ. $\qquad \square$

Chapter 4

Monomial Ideals

Now that we have some knowledge about monomial resolutions and simplicial complexes, we are ready to begin developing the theory that is of most interest to us; using the combinatorial properties of simplicial complexes to study monomial ideals. We should, however, introduce some basic concepts, notation, and definitions that will be used throughout the rest of the thesis.

Every monomial ideal in S has a unique minimal set of monomial generators. When we say that $I = (m_1, ..., m_r)$ is a monomial ideal, what is meant is that $m_1, ..., m_r$ are monomials and they are the unique minimal set of monomial generators for I. If we consider the set

$$L_I = \left\{ \mathrm{lcm}(m_{i_1}, ..., m_{i_j}) \mid \{i_1, ..., i_j\} \subseteq \{1, ..., r\} \right\}$$

Where $\mathrm{lcm}(\emptyset)$ is defined to be 1, then what we get in this case is a partially ordered set, ordered under divisibility. In fact, this set has even more structure.

Definition 4.1. Let $(P, \leq)$ be a partially ordered set. For $x, y, z \in P$ we say that z is the **join** (least upper bound) of x and y if

1) $x \leq z$ and $y \leq z$

2) If $w \in P$, $x \leq w$ and $y \leq w$, then $z \leq w$.

Similarly, we say that z is the **meet** (greatest lower bound) of x and y if

1) $z \leq x$ and $z \leq y$

2) If $w \in P$, $w \leq x$ and $w \leq y$, then $w \leq z$.

Definition 4.2. A partially ordered set P is called a **lattice** if every pair of element has a meet and a join. If P has a least element $\hat{0}$ then the elements that cover $\hat{0}$ in the Hasse

diagram of P are called the **atoms** of P (for our purposes, these are the elements joined by an edge to $\hat{0}$ in the Hasse diagram).

For a monomial ideal I the set L_I is a lattice, which we call the lcm-**lattice** of I. The element $1 \in I$ takes the role of $\hat{0}$ and the atoms of L_I are the minimal generators of I.

Example 4.3. Let I be the ideal $(x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3 x_4)$ The Hasse diagram of L_I
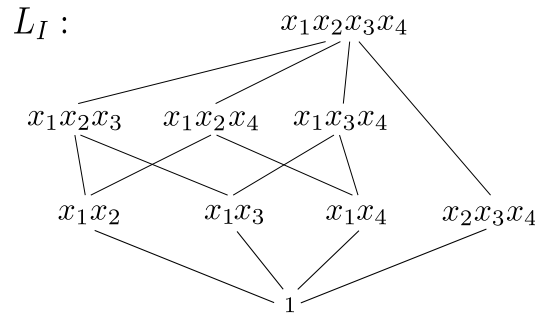
Figure 4.1: The lcm-lattice of $(x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3 x_4)$

4.1 Frames and Homogenization

We will first focus on a technique which will take a simplicial complex on r vertices and a monomial ideal with r generators to a chain complex of free S-modules. This is done using a process called homogenization on the augmented simplicial chain complex with coefficients in k, for a given simplicial complex. Under the right conditions, this construction will yield a resolution for our ideal in question, though it need not be minimal.

Definition 4.4. (Peeva, Velasco, [18]) Let $\mathbf{U}$ be a complex of finite dimensional k-vector spaces with differential ∂ and a fixed basis (see Remark 2.20), such that

1) $U_i = 0$ for $i < 0$ and there is a $j \in \mathbb{N}$ for which $U_i = 0$ when $i > j$

2) $U_0 = k$

3) $U_1 = k^r$

4) $\partial_1(e_j) = 1$ for every basis vector e_j in U_1

We call such a complex a **frame** (or an r-**frame**).

If, for some simplicial complex Δ on vertex set $\{v_1, ..., v_r\}$, we consider the augmented simplicial chain complex with coefficients in k then what we get is a frame, with the caveat that we will need to shift the homological degree of the complex by 1 (i.e. re-index). Conditions (2) and (4) comes from the augmentation of the complex, condition (3) comes from C_0 (now C_1) having a basis indexed by the vertices of Δ. Condition (1) is satisfied when we re-index, because our complexes are assumed to be finite.

It should be noted that not all frames correspond to simplicial chain complexes. It is also not the case that simplicial chain complexes are the only combinatorial/topological object from which we can derive frames. There are merits to considering these other objects but we will not be referring to them in what follows.

Definition 4.5. (Peeva, Velasco, [18]) For a monomial ideal $I = (m_1, ..., m_r)$, with lcm-lattice L_I, let $\mathbf{G}$ be a multigraded complex of finitely generated free multigraded S-modules with differential d and a fixed multihomogeneous basis with multidegrees in L_I, such that

1) $G_i = 0$ for $i < 0$ and there is a $j \in \mathbb{N}$ for which $G_i = 0$ when $i > j$

2) $G_0 = S$

3) $G_1 = S(-m_1) \oplus ... \oplus S(-m_r)$

4) $d_1(e_j) = m_j$ for each basis element e_j of G_1

We call such a complex an I-**complex**.

Remark 4.6. The four conditions of Definition 4.5 guarantee that $H_0(\mathbf{G}) = S/I$, which makes $\mathbf{G}$ a candidate for a free resolution of S/I which, as we saw in Remark 2.29, is equivalent to finding a free resolution of I. The only other property of an I-complex is that it has a multihomogeneous basis with multidegrees in L_I.

We also see that the definition of a frame is similar to the definition of an I-complex, and we can relate the two as follows.

Construction 4.7 (Peeva, Velasco, [18])**.** Let $I = (m_1, ..., m_r)$ be a monomial ideal, and let

$$\mathbf{U}: \quad 0 \longrightarrow k^{b_t} \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_3} k^{b_2} \xrightarrow{\partial_2} k^{b_1=r} \xrightarrow{\partial_1} k \longrightarrow 0$$

be an r-frame, with differential ∂. We will inductively construct an I-complex

$$\mathbf{G}: \quad 0 \longrightarrow \bigoplus_{j=1}^{b_t} S(-\bar{\alpha}_{t,j}) \xrightarrow{d_t} \cdots \xrightarrow{d_3} \bigoplus_{j=1}^{b_2} S(-\bar{\alpha}_{2,j}) \xrightarrow{d_2} \bigoplus_{j=1}^{b_1=r} S(-\bar{\alpha}_{1,j}) \xrightarrow{d_1} S \longrightarrow 0$$

with differential d and multidegrees $\bar{\alpha}_j$, via the following:

1) Set $G_0 = S$ and $G_1 = S(-m_1) \oplus ... \oplus S(-m_r)$ and $d_1(e_j) = m_j$ for each basis element e_j of G_1

2) At the ith step (for $i \geq 2$), Let $\bar{v}_1, ..., \bar{v}_{b_i}$ and $\bar{u}_1, ..., \bar{u}_{b_{i-1}}$ be the given bases of U_i and U_{i-1} respectively, and let $u_1, ..., u_{b_{i-1}}$ be the basis of $G_{i-1} = \bigoplus_{j=1}^{b_{i-1}} S(-\bar{\alpha}_{i-1,j})$ chosen at the previous step of the induction. We define $G_i \cong S^{b_i}$ with basis $v_1, ..., v_{b_i}$. If $\partial_i(\bar{v}_j) = \sum_{s=1}^{b_{i-1}} a_{s,j} \bar{u}_s$ where $a_{s,j} \in k$, then set

   i) $\text{mdeg}(v_j) = \text{lcm}\{\text{mdeg}(u_s) | a_{s,j} \neq 0\}$, and note that $\text{lcm}(\emptyset) = 1$

   ii) $G_i = \bigoplus_{j=1}^{b_i} S(-\text{mdeg}(v_j))$

   iii) $d_i(v_j) = \sum_{s=1}^{b_{i-1}} a_{s,j} \dfrac{\text{mdeg}(v_j)}{\text{mdeg}(u_s)} u_s$

We say that the complex $\mathbf{G}$ is obtained from $\mathbf{U}$ by I-**homogenization** (or that $\mathbf{G}$ is the I-homogenization of $\mathbf{U}$).

    This construction is weighed down by notation, but is not nearly as tedious as it may seem. It is more instructive to consider an example of homogenization before trying to decipher the precise details of the above construction.

Example 4.8. Let $I = (x_1x_4, x_1x_2, x_1x_3, x_2x_3x_4)$. Suppose we would like to I-homogenize the following 4-frame

$$0 \longrightarrow k \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \\ 1 \\ 0 \end{smallmatrix}\right]} k^4 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right]} k^4 \xrightarrow{[1\ 1\ 1\ 1]} k \longrightarrow 0$$

The first thing we do is choose an ordering on the monomial generators of I. We will, for simplicity, use the order in which they appear in our presentation, i.e. $m_1 = x_1x_4, m_2 = x_1x_2$, etc. The first step in our algorithm tells us to define $G_0 = S$ and $G_1 = S(x_1x_4) \oplus S(x_1x_2) \oplus S(x_1x_3) \oplus S(x_2x_3x_4)$. Step 2 tells us how to define the differential which, since G_0 is one copy of S with no shift, will send the homogeneous basis element with degree m_i to m_i. So, our partially homogenized chain complex is:

$$0 \longrightarrow k \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \\ 1 \\ 0 \end{smallmatrix}\right]} k^4 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix}\right]} \begin{matrix} S(-x_1x_4) \\ \oplus \\ S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \\ \oplus \\ S(-x_2x_3x_4) \end{matrix} \xrightarrow{[x_1x_4\ x_1x_2\ x_1x_3\ x_2x_3x_4]} S \longrightarrow 0$$

To determine what happens with G_2 and d_2 we see that (using the notation described in the algorithm) $\partial_2(\bar{v}_1) = \bar{u}_1 - \bar{u}_2$. So we set

$$\mathrm{mdeg}(v_1) = \mathrm{lcm}(\mathrm{mdeg}(u_1), \mathrm{mdeg}(u_2)) = \mathrm{lcm}(x_1x_4, x_1x_2) = x_1x_2x_4$$

$$d_2(v_1) = (x_1x_2x_4/x_1x_4)u_1 - (x_1x_2x_4/x_1x_2)u_2 = x_2u_1 - x_4u_2$$

If we make similar calculations for the other basis elements we get

$$0 \longrightarrow k \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \\ 1 \\ 0 \end{smallmatrix}\right]} \begin{matrix} S(-x_1x_2x_4) \\ \oplus \\ S(-x_1x_3x_4) \\ \oplus \\ S(-x_1x_2x_3) \\ \oplus \\ S(-x_1x_2x_3x_4) \end{matrix} \xrightarrow{\left[\begin{smallmatrix} x_2 & x_3 & 0 & x_2x_3 \\ -x_4 & 0 & x_3 & 0 \\ 0 & -x_4 & -x_2 & 0 \\ 0 & 0 & 0 & -x_1 \end{smallmatrix}\right]} \begin{matrix} S(-x_1x_4) \\ \oplus \\ S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \\ \oplus \\ S(-x_2x_3x_4) \end{matrix} \xrightarrow{[x_1x_4\ x_1x_2\ x_1x_3\ x_2x_3x_4]} S \longrightarrow 0$$

and we can repeat this process for G_3 and d_3 to get the complete I-homogenization of this

frame

$$0 \longrightarrow S(-x_1x_2x_3x_4) \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_4 \\ 0 \end{bmatrix}} \begin{matrix} S(-x_1x_2x_4) \\ \oplus \\ S(-x_1x_3x_4) \\ \oplus \\ S(-x_1x_2x_3) \\ \oplus \\ S(-x_1x_2x_3x_4) \end{matrix} \xrightarrow{\begin{bmatrix} x_2 & x_3 & 0 & x_2x_3 \\ -x_4 & 0 & x_3 & 0 \\ 0 & -x_4 & -x_2 & 0 \\ 0 & 0 & 0 & -x_1 \end{bmatrix}} \begin{matrix} S(-x_1x_4) \\ \oplus \\ S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \\ \oplus \\ S(-x_2x_3x_4) \end{matrix} \xrightarrow{\begin{bmatrix} x_1x_4 & x_1x_2 & x_1x_3 & x_2x_3x_4 \end{bmatrix}} S \longrightarrow 0$$

Remark 4.9. The first remark we should make about homogenization is that the order we put on the monomial generators of the ideal has a direct consequence on the properties of the homogenization. In particular, for a frame U and monomial ideal I, the I-homogenization of U may result in a resolution of I with respect to one ordering of the generators and may fail to be a resolution with respect to another ordering on the generators.

The second remark is that, when our frame is $\widetilde{C}(\Delta; k)$ for some simplicial complex Δ and field k, we can represent the I-homogenization of a frame pictorially. In the simplicial chain complex of Δ the basis for each chain group is indexed by the faces. So when we homogenize, it is like attaching a multidegree to each face of the complex, which is the lcm of the subfaces. Moreover, if homogenizing the frame of a simplicial complex results in a resolution, then we can bound the Betti numbers $\beta_i(I)$ by the number of faces of Δ of dimension i, i.e. by the entries of the f-vector of Δ. If the resolution is minimal, then these values would be equal (note that these are the Betti numbers for I to which we refer and not S/I).

Example 4.10. Let us consider the same ideal I and frame as in the previous example. The frame that we used is actually the $\widetilde{C}(\Delta; k)$ for the following simplicial complex Δ
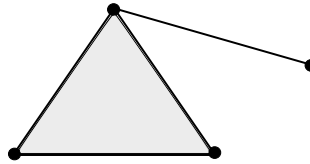

Figure 4.2: A simplicial complex (example of I-homogenization)

Picking an order on the generators of I is equivalent to giving each vertex of Δ a monomial label. The order we had chosen gives us
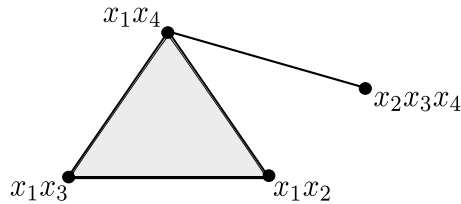
Figure 4.3: Partial I-homogenization of a simplicial complex

As we already mentioned, when we homogenize, what we end up doing is assigning a multidegree to each of the faces of Δ. If a face F had dimension d, then the homogenization algorithm tells us to set

$$\mathrm{mdeg}(F) = \mathrm{lcm}(\mathrm{mdeg}(G) \mid G \text{ is a subface of } F)$$

and working backwards we can describe the multidegree of F, with respect to the generators of I, as

$$\mathrm{mdeg}(F) = \mathrm{lcm}(\mathrm{mdeg}(m_i) \mid m_i \in F)$$

so the complete homogenization of our current example is
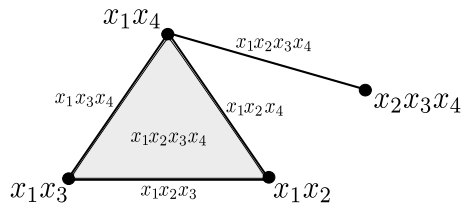
Figure 4.4: Full I-homogenization of a simplicial complex

Since we can deduce the differential maps of the simplicial chain complex for Δ from this pictorial representation, we can similarly deduce the differential of the I-homogenization of Δ (up to a change of sign of the entries in each ∂_i). This means that all the information needed to describe the I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ is available in this presentation. In fact, since the multidegrees of each face is determined by the labels on its vertices, the homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ is completely determined by the ordering of the generators of I (i.e. the labelling of the vertices of Δ). Typically, we will just label the vertices, and not the faces, of a simplicial complex in order to denote the homogenization

with respect to that choice of ordering on the vertices.

We now know how to get from an r-frame (in particular, from a simplicial complex), and monomial ideal I, to a chain complex of free S-modules which may or may not be a resolution of I. Determining whether or not we do indeed get a resolution amounts to examining specific subcomplexes of our r-frame. In order to do this, we need to be able to get from an I-complex back to an r-frame.

Definition 4.11 (Peeva, Velasco. [18])**.** Let $\mathbf{G}$ be an I-complex. We call $\mathbf{U} = \mathbf{G} \otimes_S S/(x_1 - 1, ..., x_n - 1)$ the **frame of G** (or the **dehomogenization** of $\mathbf{G}$).

The generators of $(x_1 - 1, ..., x_n - 1)$ equate x_i and 1 in the above tensor product. This means that each G_i in $\mathbf{G}$ becomes a k-vector space of the same rank as G_i, and each differential map becomes the matrix of the coefficients of its entries. Since the definition of an I-complex and an r-frame are so similar, it is not surprising that the dehomogenization of the an I-complex yields an r-frame, where r is the number of minimal generators of I.

Example 4.12. Using the same I and Δ as the previous two examples we got the I-complex

$$
0 \longrightarrow S(-x_1x_2x_3x_4) \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_4 \\ 0 \end{bmatrix}} \begin{matrix} S(-x_1x_2x_4) \\ \oplus \\ S(-x_1x_3x_4) \\ \oplus \\ S(-x_1x_2x_3) \\ \oplus \\ S(-x_1x_2x_3x_4) \end{matrix} \xrightarrow{\begin{bmatrix} x_2 & x_3 & 0 & x_2x_3 \\ -x_4 & 0 & x_3 & 0 \\ 0 & -x_4 & -x_2 & 0 \\ 0 & 0 & 0 & -x_1 \end{bmatrix}} \begin{matrix} S(-x_1x_4) \\ \oplus \\ S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \\ \oplus \\ S(-x_2x_3x_4) \end{matrix} \xrightarrow{\begin{bmatrix} x_1x_4 & x_1x_2 & x_1x_3 & x_2x_3x_4 \end{bmatrix}} S \longrightarrow 0
$$

The dehomogenization of this complex gives us the frame

$$
0 \longrightarrow k \xrightarrow{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} k^4 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}} k^4 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}} k \longrightarrow 0
$$

which is just the frame that we started with before homogenizing. It may come as no surprise that when we dehomogenized the homogenization of a frame, it returned to us the frame we began with and it can be shown that this is always the case.

Proposition 4.13 (Peeva, Velasco, [18])**.** *Let $I = (m_1, ..., m_r)$ be a monomial ideal. If $\mathbf{G}$ is the I-homogenization of a frame $\mathbf{U}$, then $\mathbf{U}$ is the frame of $\mathbf{G}$*

Remark 4.14. The converse statement of Proposition 4.13 need not be true. In Example 4.8 we gave an I-homogenization $\mathbf{G}$ of a frame. To show that the converse of Proposition 4.13 does not hold it would suffice to present an I-complex $\mathbf{G}'$ with the same frame as $\mathbf{G}$, but which cannot be constructed via I-homogenization. Consider the complex

$$
\mathbf{G}' : 0 \longrightarrow S(-x_1x_2x_3x_4) \xrightarrow{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}
\begin{matrix} S(-x_1x_2x_3x_4) \\ \oplus \\ S(-x_1x_2x_3x_4) \\ \oplus \\ S(-x_1x_2x_3x_4) \\ \oplus \\ S(-x_1x_2x_3x_4) \end{matrix}
\xrightarrow{\begin{bmatrix} x_2x_3 & x_2x_3 & 0 & x_2x_3 \\ -x_3x_4 & 0 & x_3x_4 & 0 \\ 0 & -x_2x_4 & -x_2x_4 & 0 \\ 0 & 0 & 0 & -x_1 \end{bmatrix}}
\begin{matrix} S(-x_1x_4) \\ \oplus \\ S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \\ \oplus \\ S(-x_2x_3x_4) \end{matrix}
$$

$$
\xrightarrow{\begin{bmatrix} x_1x_4 & x_1x_2 & x_1x_3 & x_2x_3x_4 \end{bmatrix}} S \longrightarrow 0
$$

We can easily verify that conditions (1)−(4) of Definition 4.5 are satisfied, and this complex has a multihomogeneous basis with multidegrees in L_I, so $\mathbf{G}'$ is an I-complex. The dehomogenization of this I-complex is

$$
0 \longrightarrow k \xrightarrow{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} k^4
\xrightarrow{\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}} k^4 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}} k \longrightarrow 0
$$

So the frame matches that $\mathbf{G}$ as well. However, if it were possible to construct $\mathbf{G}'$ via I-homogenization, G_0' and G_1' determine that the ordering on the generator of I be the same order that we chose when finding the I-homogenization $\mathbf{G}$. As mentioned in the comments at the end of Example 4.10, this choice of ordering completely determines the I-homogenization. Therefore, since $\mathbf{G}' \neq \mathbf{G}$, we can conclude that $\mathbf{G}'$ cannot be constructed via homogenization.

As mentioned before, we can determine whether or not the I-homogenization of a frame is a resolution by considering certain subcomplexes of the frame we begin with. The subcomplexes that we are interested in are the following.

Definition 4.15. Let $\mathbf{G}$ be an I-complex, where I is a monomial ideal, and let $m \in I$ be a monomial. Denote by $\mathbf{G}(\leq m)$ the subcomplex of $\mathbf{G}$ that is generated by the multihomogeneous basis elements whose multidegrees divide m.

These subcomplexes are worth considering because of our next theorem.

Theorem 4.16 (Peeva, Velasco, [18])**.** *Let* **G** *be an* I-*complex.*

1) *For each monomial* $m \in I$, *the component of* **G** *of multidegree* m *is isomorphic to the frame of the complex* $\mathbf{G}(\leq m)$.

2) *The complex* **G** *is a free multigraded resolution of* S/I *if and only if for all multidegrees* $m \in L_I$ *the frame of the complex* $\mathbf{G}(\leq m)$ *is exact.*

This theorem is quite useful, since it tells us exactly when a homogenized frame yields a resolution. The first statement is proved by interpreting how the frame of the complex $\mathbf{G}(\leq m)$ is determined and comparing it to the definition of the component of **G** of multidegree m. With the first statement in hand we see, by Proposition 2.16, that **G** is a resolution if and only if $\mathbf{G}(\leq m)$ is exact for each multidegree m, and noting that $\mathbf{G}(\leq m) \cong \mathbf{G}(\leq m')$ for some $m' \in L_I$.

Example 4.17. If we recall example 2.18 in section 2.3 we had the complex

$$0 \longrightarrow A(-x_1x_2x_3x_4) \xrightarrow{\begin{bmatrix} x_4 \\ -x_3 \\ x_1 \end{bmatrix}} \begin{matrix} A(-x_1x_2x_3) \\ \oplus \\ A(-x_1x_2x_4) \\ \oplus \\ A(-x_2x_3x_4) \end{matrix} \xrightarrow{\begin{bmatrix} x_3 & x_4 & 0 \\ -x_1 & 0 & x_4 \\ 0 & -x_1x_2 & -x_2x_3 \end{bmatrix}} \begin{matrix} A(-x_1x_2) \\ \oplus \\ A(-x_2x_3) \\ \oplus \\ A(-x_4) \end{matrix} \xrightarrow{\begin{bmatrix} x_1x_2 & x_2x_3 & x_4 \end{bmatrix}} A \longrightarrow 0$$

and we made the claim that this was indeed a graded free resolution of A/I, but we did not show that this complex was exact, since it would require computing the kernels of the differential matricies explicitly. Now, however, if we recognize the fact that this complex is the I-homogenization of the simplex on three vertices
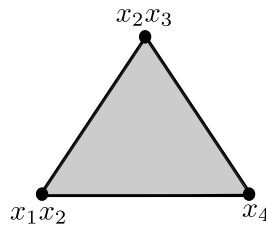

Figure 4.5: The I-homogenization of the simplex on 3 vertices

we may apply the results of our last theorem. Moreover, when we have that our complex is the I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$, for some simplicial complex Δ, the frame of $\mathbf{G}(\leq m)$

is given by the induced subcomplex Δ_W, where $W = \{m_i \mid \mathrm{mdeg}(m_i)$ divides $m\}$ (Peeva, [18]). When the simplicial complex and vertex label are clear, we will denote these induced subcomplexes as Δ_m, when it is not clear we will maintain the $\mathbf{G}(\leq m)$ notation. For our specific example, all induced subcomplexes of Δ fall into one of three possible cases:
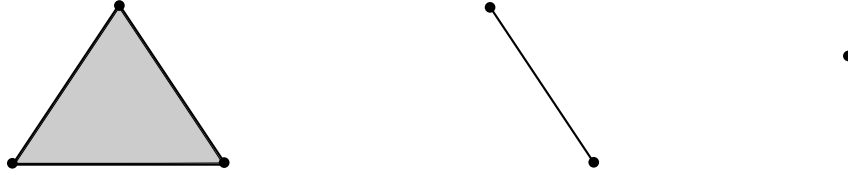

Figure 4.6: All induced subcomplexes of the simplex on three vertices

All of these are contractible, hence acyclic. This means the reduced homology is always zero and the frame of each $\mathbf{G}(\leq m)$ is exact, so that we do indeed have a multigraded free resolution of A/I.

If I is a monomial ideal and Δ is a simplicial complex, then we say that Δ **supports a resolution of** I (or that I has a **resolution supported on** Δ) when the I-homogenization of the augmented chain complex of Δ is a resolution of S/I.

Remark 4.18. If, for some monomial ideal I and simplicial complex Δ, the I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ were a resolution, we would also like to know if it is minimal. We recall that the differential of $\widetilde{\mathbf{C}}(\Delta; k)$ is such that

$$\partial(\bar{e}_F) = \sum_{j=0}^{t} (-1)^j \bar{e}_{F \setminus \{v_{i_j}\}}$$

where $F = \{v_{i_0}, ..., v_{i_t}\}$, and $\bar{e}_F$ is the basis element of C_t indexed by F. The homogenization of ∂ would give

$$d(e_F) = \sum_{j=0}^{t} (-1)^j \frac{\mathrm{mdeg}(e_F)}{\mathrm{mdeg}(e_{F \setminus \{v_{i_j}\}})} e_{F \setminus \{v_{i_j}\}}$$

$$= \sum_{j=0}^{t} (-1)^j \frac{\mathrm{lcm}\{\mathrm{mdeg}(e_{F \setminus \{v_{i_l}\}}) \mid v_{i_l} \in F\}}{\mathrm{mdeg}(e_{F \setminus \{v_{i_j}\}})} e_{F \setminus \{v_{i_j}\}}$$

The minimality condition for a resolution is that $d(e_F) \in \mathbf{m}$ for every multihomogenous basis element e_F. Therefore, we would need to check that

$$\frac{\mathrm{lcm}\{\mathrm{mdeg}(e_{F \setminus \{v_{i_l}\}}) \mid v_{i_l} \in F\}}{\mathrm{mdeg}(e_{F \setminus \{v_{i_j}\}})} \in \mathbf{m}$$

for every $v_{i_j} \in F$. To state it more directly, we give the following proposition

Proposition 4.19 ([3]). *Let I be a monomial ideal and let Δ be a simplicial complex. If the I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ is a resolution, then it is minimal if and only if for every face $F \in \Delta$, $G \subset F$, we have that* $\mathrm{mdeg}(G) \neq \mathrm{mdeg}(F)$.

Note that $\mathrm{mdeg}(e_F)$ and $\mathrm{mdeg}(F)$ are referring to the same thing, but we use the latter for convenience.

We saw in our above example that when we want to determine whether or not the homogenization of a simplicial complex is a resolution we need to check whether or not certain induced subcomplexes are acyclic. If our simplicial complex is a simplicial tree, then we know that it is acyclic. We also know that every induced subcomplex is a forest, and since a connected forest is a tree (hence acyclic) we get the following result.

Theorem 4.20 (Faridi, [9]). *Let Δ be a simplicial tree and $I = (m_1, ..., m_r)$ be a monomial ideal. The I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ is a multigraded free resolution of S/I if and only if Δ_m is connected for every $m \in L_I$.*

Example 4.21. Consider the I and Δ we had in example 4.10 and let $\mathbf{F}$ be the I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ where $m_1 = x_1 x_4$, $m_2 = x_1 x_2$, $m_3 = x_1 x_3$, $m_4 = x_2 x_3 x_4$. Let $\mathbf{G}$ be the I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ where $m_1 = x_2 x_3 x_4$, $m_2 = x_1 x_2$, $m_3 = x_1 x_3$, $m_4 = x_1 x_4$
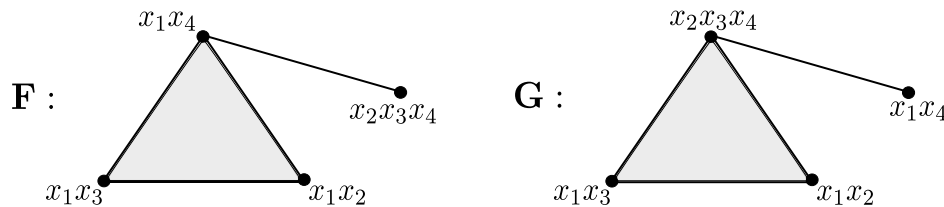
Figure 4.7: Two homogenizations of the same frame

We would like decide whether or not these homogenizations are also resolutions and, if they are, whether or not the resolutions they give are minimal. We notice that Δ is a simplicial tree (in fact, all simplicial complexes with 2 or less facets are simplicial trees), so we can apply the results of theorem 4.20. The lcm-lattice of I is

Figure 4.8: The lcm-lattice of I

and it is easy to see that $\mathbf{F}(\leq m)$ and $\mathbf{G}(\leq m)$ will be a single vertex (hence connected) when m is one of the generators of I. This means that we only need to check that the frames of $\mathbf{F}(\leq m)$ and $\mathbf{G}(\leq m)$ are connected when $m \in \{x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_1x_2x_3x_4\}$. Checking these gives

Figure 4.9: All subcomplexes of $\mathbf{F}$ and $\mathbf{G}$ induced by elements of L_I
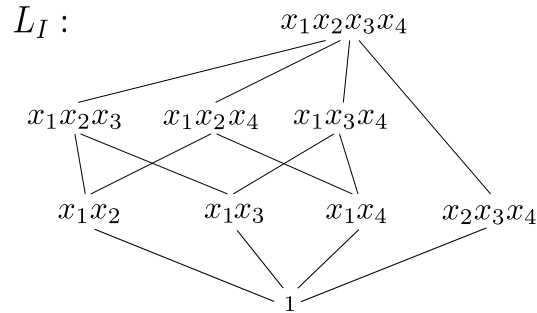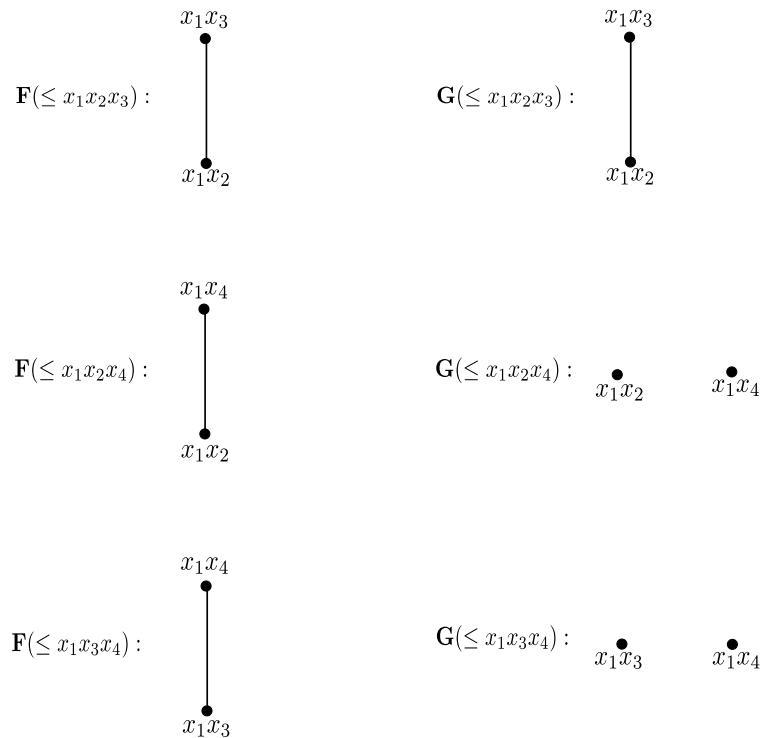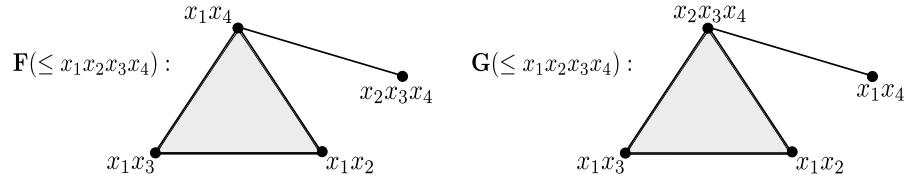
We see that that frames of $\mathbf{G}(\leq x_1 x_2 x_4)$ and $\mathbf{G}(\leq x_1 x_3 x_4)$ are not connected, so $\mathbf{G}$ is not a resolution of S/I. However, the frame for each $\mathbf{F}(\leq m)$ is connected, and we can conclude that $\mathbf{F}$ is a resolution of S/I. Moreover, the only faces of Δ that have the same multidegree are the two facets. Since neither is a subface of the other, we may conclude that $\mathbf{F}$ is minimal as well.

Remark 4.22. In general, suppose Δ is a simplicial tree and we homogenize $\widetilde{\mathbf{C}}(\Delta; k)$ with respect to some labelling of the vertices by monomials. If Δ_m is not connected for some $m \in L_I$ then there are two vertices, with monomial labels m_i and m_j which divide m, that are not connected in Δ_m. Since $m' = \mathrm{lcm}(m_i, m_j)$ divides m, we see that $\Delta_{m'}$ cannot be connected either. This means that, if Δ is a simplicial tree, it is sufficient to check the multidegrees m such that $m = \mathrm{lcm}(m_i, m_j)$ for every pair of vertex labels of Δ.

4.2 Resolutions of Monomial Ideals

In the previous section we developed a way of building I-complexes via the I-homogenization of frames. Moreover, we have given a criterion for when the I-homogenization of a frame is a resolution and gave specific examples where the I-homogenization was successful in finding a free resolution and examples where this failed to be the case. What we would like to do is give more precise statements about the success or failure of the I-homogenization. This amounts to imposing restrictions on the structures that we start with, be it the monomial ideal I or the frame which we choose to homogenize.

For our discussion, we will always restrict the frame of our resolution to be of the form $\widetilde{\mathbf{C}}(\Delta : k)$ for a simplicial complex Δ. What we would like to do is, for a given monomial

ideal $I = (m_1, ..., m_r)$, determine effective ways of choosing a simplicial complex Δ on r vertices for which the I-homogenization of $\widetilde{\mathbf{C}}(\Delta : k)$.

Definition 4.23 (Taylor, [21]). Let $I = (m_1, ..., m_r)$ be a monomial ideal and let Δ be the simplex on r vertices. The I-homogenization of $\widetilde{\mathbf{C}}(\Delta; k)$ is called the **Talyor resolution**, and we denote it by $\mathbf{T}_I$.

The name "Taylor resolution" certainly suggests that this I-homogenization yields a resolution of S/I, and this can be shown quite easily. The simplex Δ on r vertices has only one facet, therefore it is a simplicial tree. Moreover, the underlying graph of Δ is the complete graph on r vertices. If for some monomial m, we have that $m_i, \ m_j \in \Delta_m$, then by definition, m_i, m_j, and $\text{lcm}(m_i, m_j)$ divide m. Since $\text{lcm}(m_i, m_j)$ is the multidegree of the edge between m_i and m_j, we may conclude that Δ_m is connected for every $m \in L_I$. So, by Theorem 4.20, we see that this is indeed a resolution of S/I.

Example 4.24. The complex from Example 2.18 and Example 4.17 is the I-homogenization of the simplex on three vertices. Therefore, it is the Taylor resolution of I.

Not only was this example a resolution, it was minimal as well. More often than not, this is far from the case. While we are more interested in minimal resolutions of ideals, there are still advantages to having the Taylor resolution at our disposal. The most prominent is its simplicity and effectiveness. The simplex has a structure that is very easy to describe and, because of its symmetry, will work regardless of the ordering you put on the generators of I, which is not the case with most simplicial complexes.

There is another simplicial complex whose frame we can always homogenize to get a resolution of a monomial ideal. It has the trade off of being a bit more computationally tedious than the Taylor resolution, but it gives a resolution that provides a better bound (or at the very least, the same bound) on the Betti numbers of an ideal.

Construction 4.25 (Lyubeznik, [14]). Let I be a monomial ideal, and fix a total ordering $\prec$ on the minimal generators of I. Label the minimal generators of I as $m_1, ..., m_r$ so that $m_i \prec m_j$ whenever $i < j$. Let $\mathbf{G}$ be I-homogenization of the r-simplex Δ, i.e. $\mathbf{G}$ is the Taylor resolution of I. For each face F of Δ, define $\min(F) = \min_\prec\{m_i : m_i \ \text{divides}$

$\mathrm{mdeg}(F)\}$ (note that $\min(F)$ need not be a vertex of F). We say that a face F is **rooted** if for every nonempty subface $G \subseteq F$ we have that $\min(G) \in G$. Set $\Lambda_{I,\prec} = \{F \in \Delta : F$ is rooted$\}$. The rooted property gives us that $\Lambda_{I,\prec}$ is a simplicial complex, which we call the **Lyubeznik simplicial complex**, and its corresponding sub-complex in $\mathbf{G}$ (the I-homogenization of $\Lambda_{I,\prec}$) the **Lyubeznik resolution** of I, which we denote $\mathbb{L}_{I,\prec}$

Again, we have defined this I-homogenization to be a resolution, which is always the case. The proof relies on showing that the frame of every induced subcomplex $\mathbb{L}_{I,\prec}(\leq m)$ is a cone, hence acyclic, and the result follows by Theorem 4.16.

We should note that the I-homogenization of is with respect to the same ordering, $\prec$, on the generators of I. Also, we should note that this construction works for any ordering, but the resolutions that we get may differ with different orderings of the generators of I.

Example 4.26. Let $I = (x_1x_5x_6, x_2x_4x_6, x_3x_4x_5, x_4x_5x_6)$. We give three Lyubeznik resolutions of I under three different monomial orderings (recall that these visual presentations determine an I-complex by using the simplicial complex to indicate a frame, and using the labelling of the vertices to indicate the order on the generators of I by which we homogenize, see Example 4.10).

$$x_3x_4x_5 \prec x_4x_5x_6 \prec x_1x_5x_6 \prec x_2x_4x_6$$

$$x_3x_4x_5 \prec x_1x_5x_6 \prec x_2x_4x_6 \prec x_4x_5x_6$$

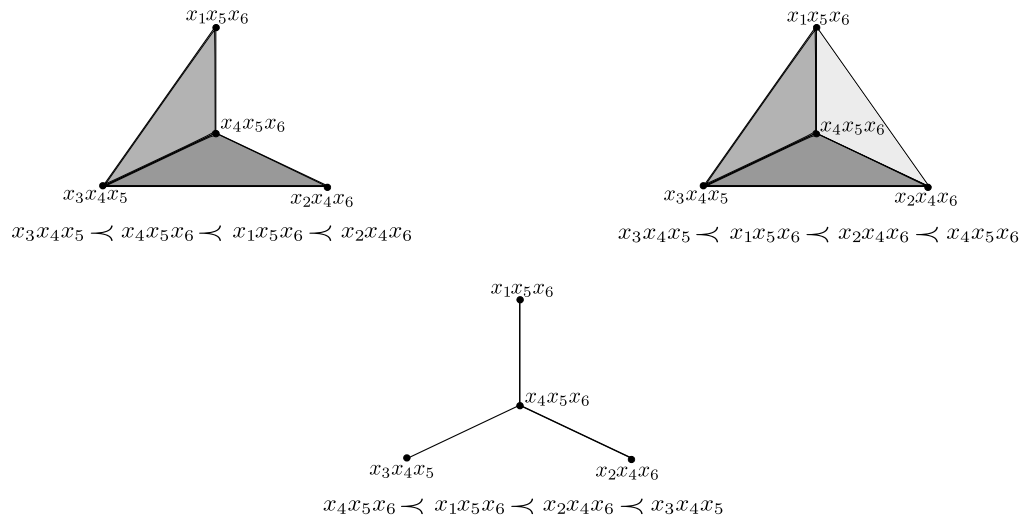$$x_4x_5x_6 \prec x_1x_5x_6 \prec x_2x_4x_6 \prec x_3x_4x_5$$

Figure 4.10: The Lyubeznik resolution of I, under three different monomial orders

Because the generator $x_4x_5x_6$ divides the multidegree of every edge, its position in the ordering determines what edges stay and what edges are left out. Also, note that none

of these resolutions are the Taylor resolution of I, and that the rightmost resolution is the minimal resolution of I.

In this small example, we were able to quickly point out which generators played an important role in what resolution we obtained. Seeing this, we could make a good guess as to what ordering will give us the smallest resolution of S/I. As it stands, this is most that we can hope for, that is, there are currently no methods for determining what ordering of the monomial generator will work best, save for trail and error and some heuristic reasoning ([15], Remark 6.4).

4.3 The Scarf complex

We now have an easy way of generating resolutions of a monomial ideal I via the Taylor complex. The Lyubeznik complex generated a resolution of I, that was closer to being a minimal resolution of I, by removing non-rooted faces with respect to some monomial order. We may wonder if there is a way in which to further remove faces in order to get a minimal resolution for I which is supported on a simplicial complex. The answer is no, not in general, and we will discuss this soon. However, the question is still worth considering and will lead us to some useful theory.

If we recall Proposition 4.19 we see that for a monomial ideal I, if we consider the Taylor resolution of I then it is minimal if no face has the same multidegree as one of its subfaces. So what we would like to do, in parallel to the construction of the Lyubenik resolution, is pick a collection of these faces which will give us another simplicial complex, Γ, such that no face and subface share the same multidegree in the Taylor resolution, then I-homogenize Γ. If the I-homogenization of Γ is a resolution, then it is minimal. The simplicial complex we use, and the I-complex we get, are given by the following definition.

Construction 4.27 (Bayer, Peeva, Sturmfels, [2]). Let $I = (m_1, ..., m_r)$ be a monomial ideal, and $\mathbf{G}$ be the I-homogenization of the r-simplex Δ, i.e. the Taylor resolution of I. Let Γ_I denote the following simplicial complex

$$\Gamma_I = \{F \in \Delta : \mathrm{mdeg}(F) \neq \mathrm{mdeg}(G), \ \forall G \in \Delta\}$$

We call Γ_I the **Scarf simplicial complex** of I, and its I-homogenization the **Scarf complex** of I, which we will also denote Γ_I. The multidegrees of the multihomogeneous basis (see Remark 2.20) of the Scarf complex are called the **Scarf multidegrees**.

Proposition 4.19 tells us that if a simplicial complex Δ supports a resolution of a monomial ideal I, and for every pair of faces $G \subset F$ in Δ, $\mathrm{mdeg}(G) \neq \mathrm{mdeg}(F)$. Since all faces of the Scarf complex have distinct multidegrees, we get the following result.

Theorem 4.28 (Bayer, Peeva, Sturmfels, [2])**.** *Let I be a monomial ideal. If the Scarf complex of I is a resolution of S/I then this resolution is minimal.*

The statement of Theorem 4.28 suggest that the Scarf complex of a monomial ideal is not always a resolution. However, if for a monomial ideal I we get that the Scarf complex of I is a resolution, then we call it a **Scarf resolution**.

Example 4.29. Consider I and $\mathbf{F}$ from Example 4.21. We saw that $\mathbf{F}$ was the multigraded minimal free resolution of I, so let us see how this compares to both the Taylor resolution and the Scarf complex of $I = (x_1x_4, x_1x_2, x_1x_3, x_2x_3x_4)$.
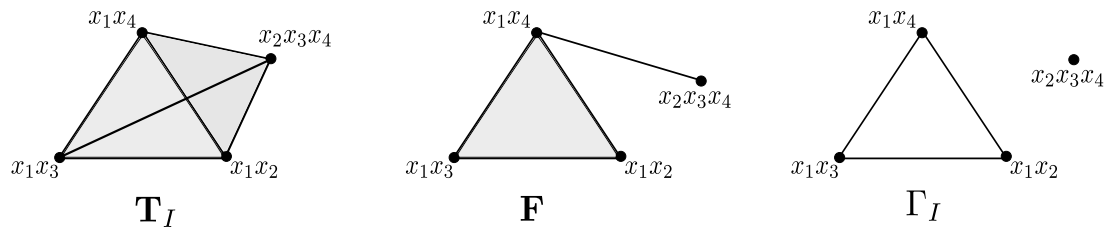
Figure 4.11: The Taylor resolution, minimal free resolution, and Scarf complex of I

Even if we did not have the minimal free resolution of I to compare Γ_I to, we would still be able to see right away that the Scarf simplicial complex of I is not acyclic, hence Γ_I cannot support a resolution of I (this is a consequence of Theorem 4.16, using $m = x_1x_2x_3x_4$). We should also note that Γ_I is a subcomplex of $\mathbf{F}$. It can be shown that this is always the case.

Theorem 4.30 ([17], p.231)**.** *Let $\mathbf{F}$ be a minimal multigraded free resolution of I. Then Γ_I is a subcomplex of $\mathbf{F}$.*

This theorem is proven via the Taylor resolution. Since the Taylor resolution gives an upper bound on the betti numbers of $\mathbf{F}$, we can embed both $\mathbf{F}$ and Γ_I in $\mathbf{T}_I$ and make comparisions as subcomplexes of a common complex. What this theorem tells us is that, for any monomial ideal, we can find simplicial complexes whose I-homogenizations give upper and lower bounds on the betti numbers of I (see [15] for a full proof).

Unlike the Taylor resolution, which is always a simplex, the structure of the Scarf complex is much less predictable. In fact, nearly every simplicial complex can appear as the support of the Scarf complex of some monomial ideal. Exactly which complexes do and do not arise in such a case is given by the following theorem.

Theorem 4.31 (Phan, [19])**.**

1) *A finite simplicial complex with r vertices is the Scarf complex of a monomial ideal if and only if it is not the boundary of the simplex on r vertices.*

2) *A finite simplicial complex Δ supports a Scarf resolution if and only if Δ is acyclic.*

The forward implication of (1) in Theorem 4.31 can be proven by assigning to each simplicial complex Δ which is not the boundary of a simplex, an ideal J_Δ, for which the dehomogenization of the Scarf complex of J_Δ is $\widetilde{\mathbf{C}}(\Delta; k)$. The ideal J_Δ can be described as follows (as given in [17], p.233).

For each face $F \in \Delta$, introduce a variable x_F and consider the polynomial ring $k[x_F | F \in \Delta, F \neq \emptyset]$. For each vertex $v \in \Delta$ we can introduce a monomial

$$m_v = \prod_{v \notin F \in \Delta} x_F$$

and define J_Δ to be the ideal $(m_v \mid v \in \Delta) \subset k[x_F | F \in \Delta, F \neq \emptyset]$. We call J_Δ the **nearly-Scarf ideal** of Δ (Peeva, Velasco, [18]) and the Scarf complex of J_Δ is Δ.

The nearly-Scarf ideal J_Δ is not the only monomial ideal whose Scarf complex is the simplicial complex Δ, however, there are features of nearly Scarf ideals which make them interesting to consider. Particularly, we can show that the lcm-lattice of J_Δ consists of the Scarf multidegrees of J_Δ and the top element $\prod_{F \in \Delta} x_F$. We can use this fact to construct,

from the J_Δ-homogenization of Δ, the minimal free resolution of S/J_Δ.

Theorem 4.32 (Peeva, Velasco, [18]). *Let J be a monomial ideal in S whose* lcm*-lattice consists of the Scarf multidegrees and a top element y. Let Γ be the Scarf complex of J, and*

$$\widetilde{\mathbf{C}}(\Gamma; k) : 0 \longrightarrow C_{\dim(\Gamma)}(\Gamma; k) \longrightarrow \cdots \longrightarrow C_0(\Gamma; k) \longrightarrow C_{-1}(\Gamma; k) \longrightarrow 0$$

be the augmented chain complex of Γ, with coefficients in k and differential ∂. For each i, choose a set $\{q_1, ..., q_p\}$ of cycles in $C_i(\Gamma; k)$ whose classes in $\widetilde{H}_i(\Gamma; k)$ form a basis and set

$$\phi_i : k^{\dim(\widetilde{H}_i(\Gamma; k))} \longrightarrow \ker(\partial_i), \ e_j \longmapsto q_j$$

where the e_j are the standard basis elements of $k^{\dim(\widetilde{H}_i(\Gamma; k))}$. Let $\mathbf{U}$ be the complex

$$\mathbf{U} : 0 \longrightarrow k^{\dim(\widetilde{H}_{\dim(\Gamma)}(\Gamma; k))} \longrightarrow k^{\dim(\widetilde{H}_{\dim(\Gamma)-1}(\Gamma; k))} \oplus C_{\dim(\Gamma)}(\Gamma; k) \longrightarrow \cdots$$

$$\cdots \longrightarrow C_0(\Delta) \oplus k^{\dim(\widetilde{H}_{-1}(\Gamma; k))} \longrightarrow C_{-1}(\Delta) \longrightarrow 0$$

with differential $\partial \oplus \phi$. The J-homogenization of the complex $\mathbf{U}$ is the multigraded minimal free resolution of S/J.

Example 4.33. Consider the simplicial complex Γ:
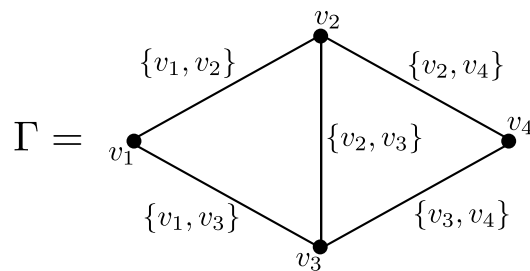

Figure 4.12: Example of a nearly Scarf ideal

The nearly Scarf ideal of Γ is

$$J_\Gamma = (x_2 x_3 x_4 x_{23} x_{24} x_{34}, \ x_1 x_3 x_4 x_{13} x_{34}, \ x_1 x_2 x_4 x_{12} x_{24}, \ x_1 x_2 x_3 x_{12} x_{13} x_{23})$$

and the augmented chain complex $\widetilde{\mathbf{C}}(\Gamma; k)$ is

$$0 \longrightarrow k^5 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{smallmatrix}\right]} k^4 \xrightarrow{[\,1\ 1\ 1\ 1\,]} k \longrightarrow 0$$

This complex is exact at $C_{-1}(\Gamma; k)$ and $C_0(\Gamma; k)$. If $b_1, ..., b_5$ form the standard basis for $C_2(\Gamma; k)$ then a basis for $\widetilde{H}_2(\Gamma; k)$ is generated by the elements $q_1 = b_1 - b_2 + b_3$ and $q_2 = b_3 - b_4 + b_5$. So, following theorem 4.32, we make the exact chain complex

$$0 \longrightarrow k^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{smallmatrix}\right]} k^5 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{smallmatrix}\right]} k^4 \xrightarrow{[\,1\ 1\ 1\ 1\,]} k \longrightarrow 0$$

which is the simplicial chain complex of the simplicial tree
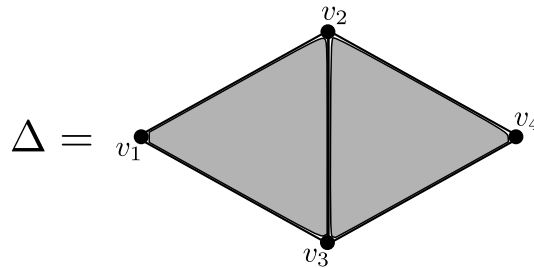
Figure 4.13: Filling in the homology of Γ, as per Theorem 4.32

If we J_Γ-homogenize Δ using the labelling $v_1 = x_2x_3x_4x_{23}x_{24}x_{34}$, $v_2 = x_1x_3x_4x_{13}x_{34}$, $v_3 = x_1x_2x_4x_{12}x_{24}$, and $v_4 = x_1x_2x_3x_{12}x_{13}x_{23}$ of the vertices of Δ, then the resulting J_Γ-complex is the minimal multigraded free resolution of J_Γ.

Remark 4.34. As mentioned before, it is not always the case that a monomial ideal has a minimal resolution supported on a simplicial complex. It was shown in (Velasco, [22]) that if a simplicial complex Δ has certain topological properties, then the nearly Scarf ideal of Δ does not have a resolution supported on a CW-complex, and the class of CW-complexes contains all simplicial complexes.

This result tells us that the structure of resolutions of monomial ideals are not easily described in generality using combinatorics. Another indication of such complexity is

given in (Reiner, Welker, [20]), where it is shown that the structure of a minimal resolution is sensitive to the characteristic of the field k over which the polynomial ring is defined.

We began this section by making restrictions on the frames that we consider, in order to make more precise statements about resolutions of monomial ideals. This allowed us to bound the Betti numbers of a monomial ideal from both above and below. However, we also saw that with our restrictions we would be unable to describe minimal free resolutions of monomial ideals in generality. A natural course of action is to try and determine for what monomail ideals I are simplicial complexes sufficient for describing the free resolution of I. The task would then be to find families of monomial ideals which share similar properties, and for which the minimal resolution is supported on a simplicial complex. In later sections, we will talk about some of the ways in which this can be done.

4.4 Polarization

Up to this point, all of the examples of monomial ideals have been squarefree. Also, while much of the theory is not specific to squarefree monomial ideals, some of it certainly is. This bias towards the squarefree case is deliberate, and with good reason. It turns out that, if what we are interested in is finding minimal free resolutions of monomial ideals, it is enough to study the squarefree ideals. This is because of a construction known as polarization.

Construction 4.35 ([17], pp.89). Let $I = (m_1, ..., m_r)$ be a monomial ideal. For any monomial $m = q_1 \cdots q_n$ where $q_i = x_i^{c_i}$ for $i = 1, ..., n$. We say that

$$
\tilde{q}_i = \begin{cases} 1 & \text{if } c_i = 0, \\ x_i \displaystyle\prod_{j=1}^{c_i-1} t_{i,j} & \text{if } c_i > 0 \end{cases}
$$

is the **polarization of** q_i, $\widetilde{m} = \tilde{q}_1 \cdots \tilde{q}_n$ is the **polarization of** m, and that $I_{\text{pol}} = (\widetilde{m}_1, ..., \widetilde{m}_r)$ is the **polarization of** I. Because of the additional variables, I_{pol} lives in the polynomial ring $S_{\text{pol}} = S[x_1, t_{1,1}, ..., t_{1,p_1}, ..., x_n, t_{n,1}, ..., t_{n,p_n}]$ where $p_i = \max\{c \mid x_i^{c+1}$ divides one of $m_1, ..., m_r\}$.

Example 4.36. Let $I = (x_1^2 x_3^3, \ x_2^3 x_3^2) \subset k[x_1, x_2, x_3]$. Then the polarization of I is

$$I_{\text{pol}} = (x_1 t_{1,1} x_3 t_{3,1} t_{3,2}, \ x_2 t_{2,1} t_{2,2} x_3 t_{3,1}) \subset k[x_1, t_{1,1}, x_2, t_{2,1}, t_{2,2}, x_3, t_{3,1}, t_{3,2}]$$

We see that polarization will take an ideal and present it as a squarefree monomial ideal in some larger polynomial ring. What is more important is knowing how information about I_{pol} translates to information about I. Consider the ideal

$$J = \{x_i - t_{i,j} | 1 \leq i \leq n, 1 \leq j \leq p_i\}$$

in S_{pol}. We see that J contains the relations between the $t_{i,j}$'s and the x_i's they replaced under polarization. The consequence of taking quotients gives

$$\frac{S_{\text{pol}}}{J} \cong S \quad \text{and} \quad \frac{S_{\text{pol}}}{I_{\text{pol}} + J} \cong \frac{S_{\text{pol}}}{I_{\text{pol}}} \otimes \frac{S_{\text{pol}}}{J} \cong \frac{S}{I}$$

Taking this tensor product is referred to as **depolarization**. The generators of the ideal J form an S_{pol}-regular sequence of homogeneous elements (see [18], p.86), so combining the isomorphisms above with Theorem 2.35 and Remark 2.36 gives us the following result.

Theorem 4.37 ([17], p.89). *Let I be a monomial ideal of S. The minimal free resolution of S/I can be attained from the minimal free resolution of $S_{\text{pol}}/I_{\text{pol}}$ by depolarization.*

With this result we can conclude that, when it comes to finding resolutions of monomial ideals, it is enough to focus on ideals with squarefree generators.

4.5 The Stanley-Reisner Ideal and The Alexander Dual

As previously discussed, there is a desire to describe families of monomial ideals. One way to do this is by defining the Stanley-Reisner ideal of a simplicial complex. This allows us to focus on any number of families of ideals, determined by known families of simiplicial complexes.

Definition 4.38 (Hochster, [13])**.** Let Δ be a simplicial complex on the vertex set $\{x_1, ..., x_r\}$. The **Stanley-Reisner ideal** of Δ is defined to be $\mathcal{N}(\Delta) = (x_{i_1} \cdots x_{i_p} | \{x_{i_1}, ..., x_{i_p}\} \notin \Delta)$. The **Stanley-Reisner ring** is defined to be $k[\Delta] = S/\mathcal{N}(\Delta)$.

We see that the Stanley-Reisner ideal is a squarefree monomial ideal generated by the minimal "non-faces" of Δ. The definition focuses on the going up containment of elements of an ideal in contrast to the going down containment of faces in a simplicial complex. As a result, the non-zero squarefree monomials in $k[\Delta]$ are in one-to-one correspondence with the faces of Δ.

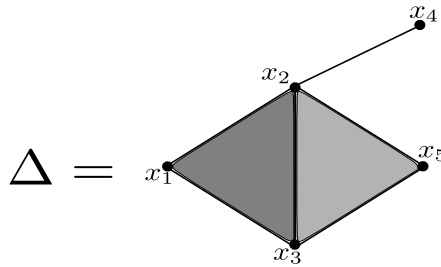**Example 4.39.** Let Δ be the following simplicial complex

Figure 4.14: A simplicial complex (example of the Stanley-Reisner ideal)

The Stanley-Reisner ideal for Δ is

$$\mathcal{N}(\Delta) = (x_1 x_4, x_1 x_5, x_3 x_4, x_4 x_5, x_1 x_2 x_4, x_1 x_2 x_5, x_2 x_3 x_4, x_2 x_4, x_3 x_4 x_5, x_1 x_2 x_3 x_4,$$
$$x_1 x_2 x_3 x_5, x_1 x_2 x_4 x_5, x_1 x_3 x_4 x_5, x_2 x_3 x_4 x_5)$$
$$= (x_1 x_4, x_1 x_5, x_3 x_4, x_4 x_5)$$

Definition 4.40. For a simplicial complex Δ on the vertex set $\{x_1, ..., x_r\}$ we define the **Alexander dual complex** of Δ as $\Delta^\vee = \{\{x_1, ..., x_r\} \setminus \tau \mid \tau \notin \Delta\}$

The dual of a simplicial complex is again a simplicial complex. If $\tau \subset \sigma$ and $\tau \notin \Delta$, then $\sigma \notin \Delta$ either, since Δ is closed under taking subsets. If $F \in \Delta^\vee$, then $F = \{x_1, ..., x_r\} \setminus \tau$, for some $\tau \notin \Delta$ and any subface of F is of the form $\{x_1, ..., x_r\} \setminus \sigma$, where $\tau \subset \sigma$, which is also in $\Delta^\vee$.

We can get another squarefree monomial ideal from Δ by taking the Stanley-Reisner ideal of the Alexander dual of Δ.

$$\mathcal{N}(\Delta^\vee) = (x_{i_1} \cdots x_{i_p} | \{x_{i_1}, ..., x_{i_p}\} \notin \Delta^\vee)$$

If $\{x_{i_1}, ..., x_{i_p}\} \notin \Delta^\vee$ then it is not of the form $\{x_1, ..., x_r\} \setminus \tau$ where $\tau \notin \Delta$. This would mean that it is of the form $\{x_1, ..., x_r\} \setminus \tau$ for $\tau \in \Delta$. So we get that $\mathcal{N}(\Delta^\vee)$ is generated by the monomials which represent the compliments of the faces in Δ. Since $\tau \subset \sigma$ implies that $\{x_1, ..., x_r\} \setminus \sigma \subset \{x_1, ..., x_r\} \setminus \tau$ we get that the monomials which correspond to the complements of the facets in Δ generate $\mathcal{N}(\Delta^\vee)$. We have shown that (see also Faridi [7])

Lemma 4.41. *Let $\Delta = \langle F_1, ..., F_q \rangle$ be a simplicial complex on the vertex set $V = \{x_1, ..., x_n\}$. The minimal generating set of $\mathcal{N}(\Delta^\vee)$ is*

$$\{ \prod_{x_i \notin F_i} x_i, ..., \prod_{x_i \notin F_q} x_i \}$$

Example 4.42. If we use the same Δ as in the previous example, we get that the Alexander dual of Δ is
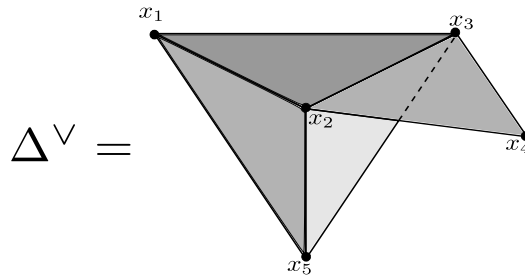
Figure 4.15: The Alexander dual complex

and $\mathcal{N}(\Delta^\vee) = (x_1 x_4, x_4 x_5, x_1 x_3 x_5)$

We now have a one-to-one correspondence between facets of Δ and minimal generators of $\mathcal{N}(\Delta^\vee)$. Moreover, we will see that if we make restrictions on the structure of Δ, it will allow us to prove results about $\mathcal{N}(\Delta^\vee)$ via this correspondence.

Another feature of the Stanley-Reisner ideal of a simplicial complex Δ is Hochster's formula, which allows us to deduce the Betti numbers of I_Δ. It can also be reformulated to

give the Betti numbers of $\mathcal{N}(\Delta^\vee)$ directly from the structure of Δ. In order to do this we need to consider induced subcomplexes of Δ, which are described in Definition 3.11, and the link of a set of vertices which we define now.

Definition 4.43. Let Δ be a simplicial complex, and W a subset of the vertex set of Δ. The **link** of W is the set

$$\text{lk}_\Delta(W) = \{F \mid F \cup W \in \Delta, \ F \cap W = \emptyset\}$$

With this definition in hand we can state our result for calculating betti numbers of Stanley-Reisner ideals.

Theorem 4.44 (Hochster's Formula, [13])**.** *Let Δ be a simplicial complex with vertex set V. Then the following equations hold.*

$$\beta_{i,j}(\mathcal{N}(\Delta)) = \sum_{\substack{A \subset V \\ |A| = j}} \dim_k(\widetilde{H}_{j-i-2}(\Delta_A; k))$$

$$\beta_{i,j}(\mathcal{N}(\Delta^\vee)) = \sum_{\substack{A \subset V \\ |A| = j}} \dim_k(\widetilde{H}_{i-1}(\text{lk}_\Delta(V \setminus A); k))$$

In a paper by Faridi ([8]), these formulas are interpreted for the case where Δ is a simplicial tree. We would like to make specific note of the result for $\mathcal{N}(\Delta^\vee)$.

Theorem 4.45 (Faridi, [8])**.** *Let Δ be a simplicial tree with vertex set V of cardinality n. Then $\mathcal{N}(\Delta^\vee)$ has projective dimension 1, and its Betti numbers are*

$$\beta_{i,j}(\mathcal{N}(\Delta^\vee)) = \begin{cases} \left|\{F \in \Delta \ : \ F \text{ a facet}, \ |F| = n - j\}\right| & i = 0 \\ \displaystyle\sum_{\substack{A \subseteq V \\ |A| = j}} (\{\# \text{ of connected components of } \text{lk}_\Delta(A^c)\} - 1) & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

It should be noted that the statements made about $\mathcal{N}(\Delta^\vee)$ when $i = 0, 1$ are true for all simplicial complexes Δ and the statement for when $i \geq 2$ is specific to the structure of a simplicial tree.

Chapter 5

Quasi-Trees and Resolutions

In the last chapter we introduced Hochster's formula, which provides us with information about the graded Betti numbers of monomial ideals of the for $\mathcal{N}(\Delta)$ and $\mathcal{N}(\Delta^\vee)$. This is certainly useful information, but it does not completely characterize the minimal free resolution of $\mathcal{N}(\Delta^\vee)$. For example, in a paper by Hibi, Kimura, and Murai ([12]) it was shown that the total Betti numbers of a nearly Scarf ideal J_Δ of any simplicial complex Δ will always correspond to the entries of the f-vector of some acyclic simiplicial complex Γ. However, we know that there are nearly Scarf complexes which cannot have a minimal free resolution supported on a simplicial complex ([22]).

We also saw that if we apply Hochster's formula to a simplicial tree Δ we get that $\mathrm{pd}(\mathcal{N}(\Delta^\vee)) = 1$. In this case, where the projective dimension is small, it is possible to avoid some of the subtleties of characterizing the structure of the minimal resolution. Specifically, the minimal resolution of $\mathcal{N}(\Delta^\vee)$ is always supported on a simplicial complex Γ. For Γ to support the minimal resolution of $\mathcal{N}(\Delta^\vee)$ we would need $\dim(\Gamma) = 1$, i.e. Γ would have to be a graph. Moreover, since the frame of a resolution must be acyclic (Theorem 4.16) it must be that G is a (graph) tree. We construct this tree in the proof of Theorem 5.1. The construction we provide for the resolution of $\mathcal{N}(\Delta^\vee)$, when Δ is a simplicial tree, relies on the fact that Δ admits a leaf order, so the result extends to quasi-trees.

Theorem 5.1. *If Δ is a quasi-forest, then $\mathcal{N}(\Delta^\vee)$ has a minimal resolution which is supported on a tree.*

We will prove this by constructing a resolution of $\mathcal{N}(\Delta^\vee)$ which is supported on a graph tree. The minimality of this resolution is guaranteed by the following lemma.

Lemma 5.2. *If I is a monomial ideal which has a resolution supported on a tree T then that resolution is minimal.*

Proof. If $m_1, ..., m_r$ are the minimal generators of I then T would have to have r vertices and $r - 1$ edges. When we regard T as a simplicial complex we get the simplicial chain complex

$$\mathbf{C}(T; k) : 0 \longrightarrow k^{r-1} \xrightarrow{\ \partial_2\ } k^r \xrightarrow{\ (11...1)\ } k \longrightarrow 0$$

where ∂_2 is a matrix in which every column has one entry equal to 1, one entry equal to -1, and the rest equal to zero. Fix a basis $u_{i,j}$ for $\mathbf{C}(T; k)$. The I-homogenization of T would then give a resolution of I of the form

$$\mathbf{G} : \ 0 \longrightarrow \bigoplus_{j=1}^{r-1} S(-\alpha_{2,j}) \xrightarrow{\ d_2\ } \bigoplus_{j=1}^{r} S(-\alpha_{1,j}) \xrightarrow{\ d_1\ } S \longrightarrow 0$$

with multihomogeneous basis $e_{i,j}$ such that $\mathrm{mdeg}(e_{i,j}) = \alpha_{i,j}$. We know that

$$\alpha_{1,j} = \mathrm{mdeg}(e_{1,j}) = \mathrm{mdeg}(m_j)$$

for $j = 1, ..., r$ and the $\alpha_{2,j}$ are given by

$$\alpha_{2,j} = \mathrm{mdeg}\big(\mathrm{lcm}(\mathrm{mdeg}(e_{1,s})| \ a_{s,j} \neq 0)\big)$$

where the $a_{s,j}$ come from the boundary map

$$\partial_2(u_{2,j}) = \sum_{s=1}^{q} a_{s,j} u_{1,s}$$

For each j, exactly 2 of the $a_{s,j} \neq 0$, so the multidegrees of the $e_{2,j}$ are actually of the form $\mathrm{mdeg}(e_{2,j}) = \mathrm{mdeg}(\mathrm{lcm}(m_{i_1}, m_{i_2}))$ where m_{i_1} and m_{i_2} are minimal generators of I. With this in mind we consider the boundary map

$$d_2(e_{2,j}) = \sum_{s=1}^{q} a_{s,j} \frac{\mathrm{mdeg}(e_{2,j})}{\mathrm{mdeg}(e_{1,s})} e_{1,s}$$

which tells us that the matrix representation of d_2 has entries

$$[d_2]_{s,j} = a_{s,j} \frac{\mathrm{mdeg}(e_{2,j})}{\mathrm{mdeg}(e_{1,s})}$$

If $a_{s,j} = 0$ then $[d_2]_{s,j} = 0$. If $a_{s_1,j}, a_{s_2,j} \neq 0$ then we have that $\mathrm{mdeg}(e_{2,j}) = \mathrm{lcm}(m_{s_1}, m_{s_2})$. Since m_{s_1}, m_{s_2} are minimal generators of I we know that m_{s_1} and m_{s_2} strictly divide $\mathrm{mdeg}(e_{2,j}) = \mathrm{lcm}(m_{s_1}, m_{s_2})$, so that $[d_2]_{s,j} \in \mathbf{m}$ for all s, j. By construction, all entries of d_1 are in $\mathbf{m}$ and we can conclude that this resolution is minimal. $\qquad \square$

Proof. (of Proposition 5.1): First we shall construct a tree T whose vertices will be labelled by the monomial generators of $\mathcal{N}(\Delta^\vee)$. Then we will show that the forest induced by the lcm of any two of the vertex labels is connected. Theorem 4.20 and Remark 4.22 show that this is sufficient to conclude that T supports a resolution of $\mathcal{N}(\Delta^\vee)$.

To construct the tree we do the following:

1) Order the facets of Δ as $F_0, ..., F_q$, so that F_i is a leaf of $\Delta_i = \langle F_1, ..., F_i \rangle$.

2) Start with the one vertex tree $T_0 = (V_0, E_0)$ where $V_0 = \{v_0\}$ and $E_0 = \emptyset$

3) For $i = 1, ..., q$ do the following:

   - Pick $u < i$ such that F_u is a joint of the leaf F_i in Δ_i

   - Set $V_i = V_{i-1} \cup \{v_i\}$

   - Set $E_i = E_{i-1} \cup \{(v_i, v_u)\}$

What we get is a graph $T = (V_q, E_q)$ which, by construction, is a tree. To complete our construction we determine a labelling of the vertices of T by which to homogenize. To do this we label the vertex v_i with the monomial

$$m_i = \prod_{x_j \in W \setminus F_i} x_j$$

where $W = \{x_1, ..., x_n\}$ is the vertex set of Δ. These labels are the monomial generators of $\mathcal{N}(\Delta^\vee)$, so we have constructed a tree and specified a labelling. The I-homogenization

of T with respect to this labelling results in the I-complex $\mathbf{F}_T$. We are left with proving that $\mathbf{F}_T$ is a resolution.

Since T is a tree, and hence a simplicial tree, to show that $\mathbf{F}_T$ supports a resolution of $\mathcal{N}(\Delta^\vee)$ it is sufficient to show that T is connected on the subgraphs $T_{i,j}$ which are the induced subgraphs on the vertices m_k such that $m_k|\mathrm{lcm}(m_i, m_j)$, for any minimal generators m_i, m_j in I. We first observe that

$$\mathrm{lcm}(m_i, m_j) = \prod_{x_l \in W \setminus F_i \cap F_j} x_l$$

so that

$$m_k|\mathrm{lcm}(m_i, m_j) \iff F_i \cap F_j \subset F_k$$

Now, to show that every $T_{i,j}$ is connected we first make the set

$$A_{i,j} = \{0 \leq k \leq n : m_k|\mathrm{lcm}(m_i, m_j)\} = \{0 \leq k \leq n : F_i \cap F_j \subset F_k\}$$

and let l be the smallest integer in $A_{i,j}$. We will show that for each $k \in A_{i,j}$, there is a path in $T_{i,j}$ connecting v_k and v_l.

If $k \in A_{i,j}$, $k \neq l$ then we can consider the facet F_k in Δ_k which is a leaf, so it has a joint F_{k_J} for some $k_J < k$. Since $l < k$, F_l is a facet of Δ_k as well. This means that

$$F_i \cap F_j \subset F_k \cap F_l \subset F_{k_J} \implies F_i \cap F_j \subset F_{k_J} \implies k_J \in A_{i,j}$$

Since $k_J \in A_{i,j}$ for any joint of $F_k \in \Delta_k$, it is true for the specific joint we used in Step (3) of our construction of T. We may also conclude that $k_J \geq l$, by the minimality of l. Hence it is the case that the edge $\{v_k, v_{k_J}\} \in T$ which in turn implies that $\{v_k, v_{k_J}\} \in T_{i,j}$. Since $l \leq k_J < k$, we can iterate this argument for k_J and its joint in Δ_{k_J}, and so on, finitely many times to get a path from v_k to v_l in $T_{i,j}$. $\qquad\square$

Remark 5.3. The first remark that we would like to make is that this result tells us that the projective dimension of the ideal $\mathcal{N}(\Delta^\vee)$ is 1 when Δ is a quasi-tree. This fact, along with its converse (that is, if $\mathcal{N}(\Delta^\vee)$ has projective dimension 1, then Δ is a quasi-forest)

is already known, and was proven by Herzog, Hibi, and Zheng in [11] using different methods. We will also show that the converse of Theorem 5.1 holds, again using different methods than those given in [11]. In the proof provided by Herzog, Hibi, and Zheng the authors worked with the Hilbert-Burch Theorem [4], interpreting aspects of this theorem in the context of the Stanley-Reisner ring of the Alexander Dual of a quasi-tree.

The second remark is that in the construction of T, we had some choice as to what joint we chose for a facet F_k in the simplicial complex Δ_k, hence the tree that we constructed is not unique. Furthermore, the proof follows through regardless of our choices, so that any tree that we may have constructed would give us a resolution of $\mathcal{N}(\Delta^\vee)$.

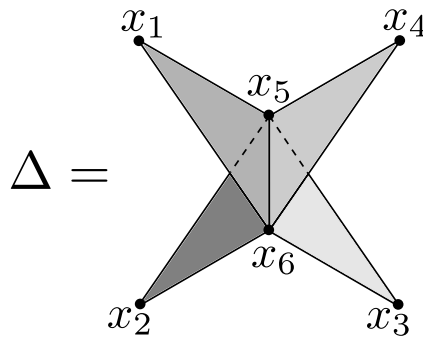**Example 5.4.** Let Δ be the simplicial tree

Figure 5.1: Quasi-tree with many leaf orders

Every order on the facets of Δ is a leaf order, every facet is a leaf, and every facet is the joint of every other facet. This means that if we use the construction given in the proof of Theorem 5.1, we could produce any tree on four vertices. The monomial generators of $\mathcal{N}(\Delta^\vee)$ are $x_1x_2x_3$, $x_1x_2x_4$, $x_1x_3x_4$, $x_2x_3x_4$ and the lcm of any two of these generators is $x_1x_2x_3x_4$, so that each $T_{i,j} = T$ for any tree T we choose to consider. Hence, the $T_{i,j}$ are always connected and we get a minimal free resolution of $\mathcal{N}(\Delta^\vee)$.

It was already mentioned that there is a converse statement to Theorem 5.1. In order to to give a proof of this converse, we are going to need a couple of auxiliary results.

Theorem 5.5. *A monomial ideal I has $pd(I) = 1$ if and only if I has a minimal resolution supported on a (graph) tree*

Proof. ($\Longleftarrow$) Clear.

($\Longrightarrow$) If $pd(I) = 1$ then S/I has a minimal resolution of the form

$$0 \longrightarrow S^t \xrightarrow{\psi} S^r \xrightarrow{\phi} S \longrightarrow 0$$

Where $\phi(e_i) = m_i$ for the basis elements e_i of S^r, and $\psi(g_j) = f_j$ where the g_j form a basis of S^t and the f_j form a minimal generating set of $\ker(\phi)$. It is shown (see [5], Corollary 4.13) that $\ker(\phi)$ can be generated (though not necessarily minimally) by the elements

$$\frac{\operatorname{lcm}(m_i, m_j)}{m_i} e_i - \frac{\operatorname{lcm}(m_i, m_j)}{m_j} e_j$$

Let $f_1, ..., f_t$ be a minimal generating set of $\ker(\phi)$ which have this form. This gives us a complete description of the map ψ as a matrix with exactly two non-zero monomial entries in each column with coefficients corresponding to those appearing in the f_i (i.e one column entry has coefficient 1 and the other has coefficient -1). Dehomogenizing this resolution gives us the sequence of vector spaces

$$0 \longrightarrow k^t \xrightarrow{A} k^r \xrightarrow{(11...1)} k \longrightarrow 0 \tag{5.1}$$

which is exact (Theorem 4.16) and where A is a matrix in which every column has exactly one entry which is 1, one entry which is -1, and the rest equal to zero. If we consider each basis element of k^r as a vertex and each basis element e_i of k^t as an edge between the two verticies determined by the basis elements of k^r to which e_i is sent, we may construct a graph G (as shown in Example 3.8) for which $\tilde{\mathbf{C}}(G; k)$ is the chain complex 5.1. Since this chain complex is exact the graph G is acyclic, hence a tree (this would also imply that $t = r - 1$). $\qquad\square$

Lemma 5.6. *Let Δ be a simplicial complex with vertex set $V = \{x_1, ..., x_n\}$, let $W = \{x_1, ..., x_t\} \subseteq V$, and let Δ_W be the subcomplex of Δ induced on W. If $m_1, ..., m_r$ are the minimal generators of $\mathcal{N}(\Delta^\vee)$ then the generators of $\mathcal{N}\big((\Delta_W)^\vee\big)$ are a subset of $\{\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t)\}$*

Before we begin it is worth noting that restricting to the first t vertices is notationally convenient, but the statement will hold for any subset of V (just make an appropriate relabelling of the vertices).

Proof. Recalling Lemma 4.41, we know that if we present Δ as $\langle F_1, ..., F_r \rangle$ then the generators of $\mathcal{N}(\Delta^\vee)$ have the form $m_i = \prod\limits_{x_j \in V \setminus F_i} x_j$. We also know that the facets of Δ_W are subsets of the facets of Δ, so we can present Δ_W as $\langle \overline{F}_{i_1}, ..., \overline{F}_{i_s} \rangle$, where $\{i_1, ..., i_s\} \subseteq \{1, ..., r\}$ and $\overline{F}_{i_j} \subseteq F_{i_j}$. Since $\overline{F}_{i_j} = F_{i_j} \cap W$ we get that

$$W \setminus \overline{F}_{i_j} = W \setminus (F_{i_j} \cap W) = (V \setminus F_{i_j}) \cap W$$

and the generators of $\mathcal{N}((\Delta_W)^\vee)$ are

$$\overline{m}_{i_j} = \prod_{\substack{x_s \notin \overline{F}_{i_j} \\ x_s \in W}} x_s = \prod_{\substack{x_s \in V \setminus F_{i_j} \\ x_s \in W}} x_s = \gcd(m_{i_j}, x_1 \cdots x_t)$$

so $\overline{m}_{i_j} \in \{\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t)\}$. $\qquad\square$

Remark 5.7. In the above proof we used the fact that there is a correspondence between the facets of Δ_W and a subset of the facets of Δ. If F_q is a facet of Δ where $q \notin \{i_1, ..., i_s\}$ we still have that $F_q \cap W$ is a face of Δ_W. Therefore, $F_q \cap W$ must be a subset of some facet $\overline{F}_{i_j}$ of Δ_W. With this information we can deduce that

$$\gcd(m_q, x_1 \cdots x_t) = \left(\gcd(m_{i_j}, x_1 \cdots x_t) \right) \prod_{\substack{x_s \in F_{i_j} \setminus F_q \\ x_s \in W}} x_s$$

This tells us that $\gcd(m_q, x_1 \cdots x_t) \in \mathcal{N}((\Delta_W)^\vee)$. What this allows us to do is say that

$$\mathcal{N}((\Delta_W)^\vee) = \left(\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t) \right)$$

With this fact we are able to prove the following corollary of Lemma 5.6.

Corollary 5.8. *Let Δ be a simplicial complex with vertex set $V = \{x_1, ..., x_n\}$. Let $W = \{x_1, ..., x_t\}$ for some $t \leq n$ and let $S' = k[x_1, ..., x_t]$. Then*

$$\frac{S'}{\mathcal{N}((\Delta_W)^\vee)} \cong \frac{S}{\mathcal{N}(\Delta^\vee)} \otimes_S \frac{S}{(x_{t+1} - 1, ..., x_n - 1)}$$

Proof. Let $m_1, ..., m_r$ be the minimal generators for $\mathcal{N}(\Delta^\vee)$. Remark 5.7 tells us that

$$\mathcal{N}((\Delta_W)^\vee) = \big(\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t) \big)$$

Which is the same as saying that we can form the generators of $\mathcal{N}((\Delta_W)^\vee)$ by taking the the generators of $\mathcal{N}(\Delta^\vee)$ and setting the variables $x_{t+1}, ..., x_n$ equal to 1. When we are using quotient modules we can do this by adding the desired relations to the ideal by which we are taking the quotient. Specifically, what we mean is

$$\frac{S'}{\mathcal{N}((\Delta_W)^\vee)} \cong \frac{S}{\mathcal{N}(\Delta^\vee) + (x_{t+1} - 1, ..., x_n - 1)}$$

Moreover, we have that

$$\frac{S}{\mathcal{N}(\Delta^\vee) + (x_{t+1} - 1, ..., x_n - 1)} \cong \frac{S}{\mathcal{N}(\Delta^\vee)} \otimes_S \frac{S}{(x_{t+1} - 1, ..., x_n - 1)}$$

and we have our desired result. $\square$

With these additional results we are able to provide a new proof the following theorem.

Theorem 5.9 (Herzog, Hibi, Zheng, [11])**.** *Let Δ be a simplicial complex, then $pd(\mathcal{N}(\Delta^\vee)) = 1$ if and only if Δ is a quasi-forest*

Proof. ($\Leftarrow$) Follows from proposition 5.1.

($\Rightarrow$) Without loss of generality let $W = \{x_1, ..., x_k\}$. Recalling Lemma 3.21, it is enough to show that Δ_W has a leaf to conclude that Δ is a quasi-forest. Let **F** be the

minimal free resolution

$$0 \longrightarrow S^{r-1} \longrightarrow S^r \longrightarrow S \longrightarrow 0$$

of $S/\mathcal{N}(\Delta^\vee)$. The elements $x_{t+1} - 1, ..., x_n - 1$ form an $S/\mathcal{N}(\Delta^\vee)$-sequence (see [17], p.86), so we can use repeated applications of Theorem 2.35 described in Remark 2.36. The result of this is the resolution

$$\mathbf{F} \otimes_S \frac{S}{(x_{t+1} - 1, ..., x_n - 1)}$$

of $S'/\mathcal{N}\big((\Delta_W)^\vee\big)$, where $S' = k[x_1, ..., x_t]$. Since the length of the resulting resolution is no greater than the length of $\mathbf{F}$, we find that $\mathrm{pd}(\mathcal{N}((\Delta_W)^\vee)) \leq \mathrm{pd}(\mathcal{N}(\Delta^\vee)) = 1$.

If $\mathrm{pd}(\mathcal{N}((\Delta_W)^\vee)) = 0$, then it must be the case that $\mathcal{N}((\Delta_W)^\vee) = 0$ which can only happen if Δ_W is a simplex, so it has a leaf.

If $\mathrm{pd}(\mathcal{N}((\Delta_W)^\vee)) = 1$, then Theorem 5.5 tells us that $\mathcal{N}((\Delta_W)^\vee)$ has a minimal resolution supported on a tree T. Choose a labelling of the vertices of T for which the $\mathcal{N}((\Delta_W)^\vee)$-homogenization yields a resolution, let $\overline{m}_l$ be the label of one of the free vertices of T and let $\overline{m}_j$ be the label of the vertex which shares an edge with m_l. For any other minimal generator $\overline{m}_i$ of $\mathcal{N}((\Delta_W)^\vee)$ we must have that $\overline{m}_j | \mathrm{lcm}(\overline{m}_l, \overline{m}_i)$ or else we would contradict the results of Theorem 4.20. In the proof of Theorem 5.1 we saw that

$$\overline{m}_j \big| \mathrm{lcm}(\overline{m}_l, \overline{m}_i) \Longleftrightarrow \overline{F}_l \cap \overline{F}_i \subset \overline{F}_j$$

Which is exactly the condition needed for $\overline{F}_l$ to be a leaf of Δ_W with joint $\overline{F}_j$. Hence, we can conclude that Δ is a quasi-forest. $\qquad\square$

Corollary 5.10. *Let* $I = (m_i, ..., m_r)$ *be a squarefree monomial ideal such that* $gcd(\{m_1, ..., m_k\}) = 1$. *Then* I *has* $pd(I) = 1$ *if and only if* $I = \mathcal{N}(\Delta^\vee)$ *for some quasi-forest* Δ.

Proof. $I = \mathcal{N}(\Delta^\vee)$ for some simplicial complex Δ if and only if $gcd(\{m_1, ..., m_r\}) = 1$ and the rest follows from Theorem 5.9. $\qquad\square$

If $f = gcd(\{m_1, ..., m_r\}) \neq 1$ then we note that the ideal $J = (m_1/f, ..., m_r/f)$ has the same minimal resolution as I (in the sense that they are both homogenizations of the same frame, see [18]), so we can apply the above results. This means that we have essentially characterized the minimal resolutions for all monomial ideals with $pd(I) = 1$ using quasi-trees.

Chapter 6

Conclusion

The process of homogenizing frames is the key concept behind many of the constructions that we have given, as well as others we have not discussed. Historically, many of these constructions were treated individually, modifying an analogous process in each case. The introduction of frames provides a common theoretical foundation on which we can speak of all of these cases simultaneously.

In the fourth chapter we prove our main results by making general observations about the structure of resolutions of monomial ideals I with $\mathrm{pd}(I) = 1$, specifically that they are always minimal and can be supported on a tree. Once we recognize this we can make the correspondence between ideals of minimal projective dimension and ideals of the form $\mathcal{N}(\Delta)$ where Δ is a simplicial tree. The approach of Herzog, Hibi, and Zheng is based on the Hilbert-Birch theorem (see [4], p.502), which classifies modules of $\mathrm{pd}(I) = 1$, and realizing components of this theorem in the setting of the Alexander dual of the Stanley-Reisner ideal of a simplicial complex (see [11] for their proof). The proof presented here has the added benefit of allowing us to explicitly construct the resolution of $\mathcal{N}(\Delta^{\vee})$ if Δ is a quasi-tree.

Bibliography

[1] Michael F. Atiyah and Ian G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[2] Dave Bayer, Irena Peeva, and Bernd Sturmfels. Monomial resolutions. *Math. Res. Lett.*, 5(1-2):31–46, 1998.

[3] Dave Bayer and Bernd Sturmfels. Cellular resolutions of monomial modules. *J. Reine Angew. Math.*, 502:123–140, 1998.

[4] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[5] Viviana Ene and Jürgen Herzog. *Gröbner bases in commutative algebra*, volume 130 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[6] Sara Faridi. The facet ideal of a simplicial complex. *Manuscripta Math.*, 109(2):159–174, 2002.

[7] Sara Faridi. Simplicial trees are sequentially Cohen-Macaulay. *J. Pure Appl. Algebra*, 190(1-3):121–136, 2004.

[8] Sara Faridi. The Betti numbers of Stanley-Reisner ideals of simplicial trees. arXiv:1302.4635, 2013.

[9] Sara Faridi. Monomial resolutions supported by simplicial trees. *J. Commut. Algebra*, 6(3):347–361, 2014.

[10] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[11] Jürgen Herzog, Takayuki Hibi, and Xinxian Zheng. Dirac's theorem on chordal graphs and Alexander duality. *European J. Combin.*, 25(7):949–960, 2004.

[12] Takayuki Hibi, Kyouko Kimura, and Satoshi Murai. Betti numbers of chordal graphs and f-vectors of simplicial complexes. *J. Algebra*, 323(6):1678–1689, 2010.

[13] Melvin Hochster. Cohen-Macaulay rings, combinatorics, and simplicial complexes. In *Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975)*, pages 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26. Dekker, New York, 1977.

[14] Gennady Lyubeznik. A new explicit finite free resolution of ideals generated by monomials in an R-sequence. *J. Pure Appl. Algebra*, 51(1-2):193–195, 1988.

[15] Jeff Mermin. Three simplicial resolutions. In *Progress in commutative algebra 1*, pages 127–141. de Gruyter, Berlin, 2012.

[16] James R. Munkres. *Elements of algebraic topology*. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[17] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.

[18] Irena Peeva and Mauricio Velasco. Frames and degenerations of monomial resolutions. *Trans. Amer. Math. Soc.*, 363(4):2029–2046, 2011.

[19] Jeffry Phan. Minimal monomial ideals and linear resolutions. arXiv:math/0511032, 2005.

[20] Victor Reiner and Volkmar Welker. Linear syzygies of Stanley-Reisner ideals. *Math. Scand.*, 89(1):117–132, 2001.

[21] Diane Taylor. *Ideals generated by monomials in an R-sequence*. PhD thesis, University of Chicago, 1966.

[22] Mauricio Velasco. Minimal free resolutions that are not supported by a CW-complex. *J. Algebra*, 319(1):102–114, 2008.

[23] Xinxian Zheng. *Homological properties of monomail ideals associated to quasi-trees and lattices*. PhD thesis, Universität Duisburg-Essen, 2004.

[24] Xinxian Zheng. Resolutions of facet ideals. *Comm. Algebra*, 32(6):2301–2324, 2004.