ROOTS OF CHROMATIC AND INDEPENDENCE POLYNOMIALS

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In loving memory of Allan Hickman.

Contents

Abstract

Two polynomials arise naturally from the notion of an independent set of vertices in a graph G: (i) the *chromatic polynomial*, $\pi(G,x) = \sum_{k\geq 1} r_k\, x^{(k)}$, where r_k is the number of partitions of $V(G)$ into k independent sets (and $x^{(k)}$ a falling factorial); and (ii) the *independence polynomial*, $i_G(x) = \sum_{k\geq 0} i_k\, x^k$, where i_k is the number of independent sets in V of cardinality k. We study here the roots of these polynomials, each respect to a specific graph operation: chromatic roots with respect to edge subdivision, independence roots with respect to graph composition.

For a connected graph G of co-rank $k = |E| - |V| + 1$, it is known that any chromatic root z satisfies $|z-1| \leq k$. We prove that large subdivisions of G, while not changing its co-rank, draw the chromatic roots close to the disk $|z-1| \leq 1$. For 'uniform' subdivisions, we describe the limit points of the roots, and in turn characterize graphs having a subdivision with a chromatic root with negative real part. In fact, infinitely many such roots are achievable from graphs of corank 2, the 3–ary *theta graphs*. And for each $3 \leq k \leq 8$, the k–ary theta graph with path lengths 2 has a chromatic root z which maximizes $|z-1|$.

Independence polynomials are (essentially) closed under graph composition. We prove this, and apply it to families of well covered and comparability graphs, finding the topological closures of both real and complex independence roots. For higher composites of a graph G with itself, we prove that their independence roots converge (in the Hausdorff topology) to the *Julia set* of $i_G(x) - 1$, thereby associating a fractal with G. For graphs with independence number 2, we determine when these fractals are connected. Further, the join of all sufficiently many copies of *any* graph

has a disconnected fractal, proving the existence of many connected graphs with a disconnected *independence fractal.*

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Chapter 1

Introduction and Background

1.1 Introduction

One method of investigating a combinatorial sequence is to associate a generating function with the sequence, and study the function itself, including its roots (or 'zeros'). In this thesis, we will study the roots of two generating functions – chromatic and independence polynomials – which arise naturally from the notion of an *independent set* of vertices in a graph. In section 1.3, we will lay out the plan for the thesis. For now, we simply introduce the two polynomials.

The *chromatic polynomial* of a graph $G(V, E)$ is the function $\pi(G,x) = \sum_{k\geq 1} r_k x^{(k)}$, where r_k is the number of ways of partitioning V into k independent sets, and $x^{(k)}$ is the falling factorial $x(x-1)\cdots(x-k+1)$. For x a positive integer, $\pi(G,x)$ counts the number of proper vertex colourings of G using at most x given colours. Expanding, then collecting terms of like degree, leads to an expansion of the form $\pi(G,x) = \sum_{k\geq 0}(-1)^k b_k x^{n-k}$, where $n = |V|$. As it happens, the sequence $(b_0, b_1, \ldots, b_{n-1})$ is also combinatorial: b_k counts the number of broken circuits (cf. [10]) in G.

Initial interest ([12, 13, 14]) in the roots of chromatic polynomials stemmed from the former four colour conjecture, which states that 4 cannot be a chromatic root of any planar graph. Roots of chromatic polynomials have since emerged as a rather fascinating topic in its own right, having attracted considerable attention (cf. [9, 11, 18, 21, 22, 45, 57, 60]). Chromatic polynomials also arise in statistical physics as zero-temperature limits of the partition function of the Potts model antiferromagnet on G (cf. [55]). Its roots are closely linked to phase transitions [61], and as such have received interest from physicists [52, 53, 54] in addition to mathematicians.

The *independence polynomial* of $G(V, E)$ is the function $i_G(x) = \sum_{k\geq 0} i_k x^k$, where i_k is the number of independent sets in V having cardinality k. It is the generating function for the 'independence vector' $(i_0, i_1, \ldots, i_\beta)$, and generalizes the more familiar *matching polynomial* (cf. [39]), $M(G,x) = \sum_{k\geq 0}(-1)^k m_k x^{n-2k}$, where m_k is the number of matchings in G on k edges. Indeed, a matching in G is an independent set in $L(G)$ (the line graph of G), and mere inspection reveals that $M(G,x) = x^n \cdot i_{L(G)}(-x^{-2})$.

Matching polynomials also arise in statistical physics, as monomer-dimer partition functions (cf. [42]). Again, the roots are associated with phase transitions. The roots of matching polynomials are all real (cf. [42]). While the same is *not* true of independence polynomials, it is known ([35]) that at least one their roots *is* real; in fact, a root of smallest modulus is necessarily real. Further results on independence polynomials and their roots can be found in [23, 35, 40, 42].

1.2 Background

Some basic terminology and results will be needed for the chapters which follow.

1.2.1 Graphs

A graph $G(V, E)$ consists of a finite, nonempty set V (the *vertices* of G) and a multiset E of one- and two-element subsets of V (the *edges* of G). If not all of the elements of E are distinct, then G is said to have *parallel edges*, since there are pairs of vertices in G with at least two edges between them. A graph with no parallel edges is a *simple* graph. Those edges in E which are one-element subsets of V are called *loops*.

Two vertices u and v in G are *adjacent* if they are joined by an edge, that is, if

$\{u, v\} \in E$. We often write uv instead of $\{u, v\}$. The vertices u and v are the *ends* of $e = uv$, and e is *incident* with both u and v; both u and v are incident with e. The *degree* of a vertex $v \in V$, written $\deg v$, is the number of edges in G which are incident with v. The maximum degree of a vertex in G is denoted by $\Delta(G)$. If every vertex in G has degree k, then G is *k-regular.*

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G such that for all pairs of vertices u and v in $V(H)$ it is true that $uv \in E(H)$ if and only if $uv \in E(G)$, then H is an *induced subgraph* of G.

A *cycle* of length $k \geq 2$ in G is a finite sequence $v_0e_1v_1e_2 \ldots e_kv_0$ whose terms are alternately vertices and edges, with $e_i = v_{i-1}v_i$ for $i = 1, \ldots, k$, and such that no vertex is repeated. The *girth* of G is the length of the shortest cycle in G (or ∞ if G has no cycles). If G has n vertices and consists of only a cycle, then G is an *n-cycle*, written $G = C_n$. Removing any single edge from an n-cycle leaves a *path* P_n on n vertices; P_n has $n - 1$ edges (the *length* of the path).

If G and H are simple graphs such that $V(G) = V(H) = V$, and, for all distinct u and v in V, $uv \notin E(G)$ if and only if $uv \in E(H)$, then H (resp., G) is the *complement* of G (resp., H). We write $H = \overline{G}$ (or $G = \overline{H}$). The *union* $G = G_1 \cup G_2$ of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is the graph with vertices $V(G) = V(G_1) \cup V(G_2)$ and edges $E(G) = E(G_1) \cup E(G_2)$. If, in addition, $V_1 \cap V_2 = \emptyset$, then G is the *disjoint union* of G_1 and G_2, written $G = G_1 \dot{\cup} G_2$.

The simple graph G on n vertices having an edge between every pair vertices is the *complete graph* K_n. Its complement is the *empty graph* of order n, written $\overline{K_n}$.

For a graph $G(V, E)$, a subset U of V is *independent* if, for all u and v in U, edge uv is not in E. The largest size of an independent set in G is the *independence number* of G. A *k-partite* graph is a graph whose vertex set can be partitioned into k independent subsets. The graph is called *bipartite* if $k = 2$. If G is a k-partite graph, and $U_1, \ldots, U_k$ is a partition of $V(G)$ into k independent subsets of cardinality $n_1, \ldots, n_k$, respectively, and $E(G)$ consists exactly of all possible edges between the sets $U_1, \ldots, U_k$, then G is *complete k-partite*, and we write $G = K_{n_1,\ldots,n_k}$. It is not hard to see that $K_{n_1,\dots,n_k} = \overline{K}_{n_1} + \cdots + \overline{K}_{n_k}$. If $k = 2$, then G is *complete bipartite.*

A graph is *connected* if there exists a path between any two of its vertices, and is *disconnected* otherwise. The *components* of a graph are its maximal connected subgraphs.

The *connectivity* of G, written $\kappa(G)$, is the minimum number of vertices whose removal disconnects G or reduces it to K_1. G is said to be *k-connected* if $\kappa(G) \geq k$. A *cut set* in G is a subset of $V(G)$ whose removal increases the number of components in G. If $v \in V(G)$ and $\{v\}$ is a cutset in G, then v is called a *cut vertex.* So a graph with cut vertices is not 2-connected. A *block* in G is a subgraph that is maximal subject to being either K_2 or 2-connected. An edge e of G whose removal increases the number of componenets of G is called a *bridge.*

A *colouring* of a graph G is an assignment of colours to its vertices in such a way that no two adjacent vertices get the same colour. If (at most) k colours are used, we have a *k-colouring* of G, and G is *k-colourable.* More precisely , a k-colouring of G is a function $f : V(G) \to \{1, \dots, k\}$ such that, for all u and v in $V(G)$, we have that $uv \in E(G)$ implies $f(u) \neq f(v)$. The *chromatic number*, $\chi(G)$, is the minimum number of colours required to colour G. Any colouring of G partitions $V(G)$ into independent subsets, which we call *colour classes*, one class for each colour used.

1.2.2 Roots of Polynomials

Throughout the thesis, we will make use of fundamental results on the roots (or zeros) of (univariate) polynomials with real and complex coefficients. We collect these results here, beginning with a necessary condition for a rational number to be a root of an integer polynomial. The notation $k[x]$ denotes the ring of polynomials in x with coefficients from k.

Theorem 1.2.1 (Rational Root Theorem) *Let* $f(x) = \sum_{i=1}^{n} a_i x^i \in \mathbb{Z}[x]$ *with* $a_n \neq 0$. *If* $a/b \in \mathbb{Q}$ *is a root of* $f(x)$, *with* $\gcd(a, b) = 1$, *then* $a \mid a_0$ *and* $b \mid a_n$.

A polynomial f with real coefficients is *Hurwitz quasi-stable* [5, 58] if every root of f has nonpositive real part. The statement of the Hermite-Biehler Theorem proved in Gantmacher [37] is actually a criterion for deciding whether every root of a real polynomial has strictly negative real part. Wagner [58] deduced from this an analogous criterion for Hurwitz quasi-stability. It is the latter which we shall call the Hermite-Biehler Theorem. As in [58], a polynomial is *standard* if it is either identically zero or has positive leading coefficient, and is said to have *only nonpositive zeros* if it is either identically zero or has all of its roots real and nonpositive.

Theorem 1.2.2 (Hermite-Biehler) *Let $P(x) \in \mathbb{R}[x]$ be standard, and write $P(x) = P_e(x^2) + xP_o(x^2)$. Set $t = x^2$. Then $P(x)$ is Hurwitz quasi-stable if and only if both $P_e(t)$ and $P_o(t)$ are standard, have only nonpositive zeros, and $P_o(t) \prec P_e(t)$.*

Roughly speaking (and made precise in [58]), the notation $P_o(t) \prec P_e(t)$ says that the roots of $P_o(t)$ 'interlace' the roots of $P_e(t)$, but we need not concern ourselves with that here. In fact, we shall use only the following.

Corollary 1.2.3 *If either $P_e(t)$ or $P_o(t)$ has a nonreal root (and is not identically zero), then $P(x)$ is not Hurwitz quasi-stable.*

Now Sturm's Theorem (cf. [46]) gives rise to a useful test for deciding whether a real polynomial has a nonreal root. We say that two consecutive terms of a sequence $s = (a_0, a_1, \ldots, a_k)$ of nonzero real numbers have a *sign variation* if they have opposite signs, and denote by $\operatorname{Var} s$ the number of sign variations of s. If s contains zero entries, then $\operatorname{Var} s$ is defined to be the number of sign variations of the subsequence of nonzero terms of s.

The *Sturm sequence* of a real polynomial $f(t)$ of positive degree is $f_0, f_1, f_2, \cdots$, where $f_0 = f$, $f_1 = f'$, and, for $i \geq 2$, $f_i = -\operatorname{rem}(f_{i-1}, f_{i-2})$, where $\operatorname{rem}(g, h)$ denotes the remainder upon dividing g by h. The sequence is terminated at the last nonzero f_i, which is easily seen to be a constant times the greatest common divisor of f and f' (just compare the process to the Euclidean Algorithm).

Theorem 1.2.4 (Sturm's Theorem) *Let $f(t) \in \mathbb{R}[t]$ have positive degree, and suppose $(f_0, f_1, \ldots, f_k)$ is its Sturm sequence. Let $a < b$ be reals that are not roots of f. Then the number of distinct roots of f in (a, b) is $V(a) - V(b)$, where $V(c) \equiv Var(f_0(c), f_1(c), \ldots, f_k(c))$.*

A proof can be found in [46]. Now let us say that the Sturm sequence $(f_0, f_1, \ldots, f_k)$ of $f(t)$ *has gaps in degree* if there is a $j \leq k$ such that $\deg f_j < \deg f_{j-1} - 1$. If there is a $j \leq k$ such that f_j has negative leading coefficient, then we say the Sturm sequence *contains a negative leading coefficient.*

We shall make important use of the following corollary to Sturm's Theorem which is not explicitly found in the literature (a similar statement, though stated incorrectly, is found in [4, p.176]).

Corollary 1.2.5 *Let $f(t)$ be a real polynomial whose degree and leading coefficient are positive. Then $f(t)$ has all real roots if and only if its Sturm sequence has no gaps in degree and no negative leading coefficients.*

Proof Let us begin with a few observations. We write $f = g \cdot h$, where $g = \gcd(f, f')$. Then the number of distinct roots of f is exactly $\deg h$, as the roots of g are the multiple roots of f. Consider the Sturm sequence $S_f(t) = (f_0, f_1, \ldots, f_k)$ of f. Recall that $f_0 = f$ and f_k is a (nonzero) constant times g. Then since $\deg f = \deg g + \deg h$, we have that the number of terms in $S_f(t)$ is $k + 1 \leq \deg h + 1$, with equality exactly when it has no gaps in degree.

We define $V(-\infty)$ and $V(\infty)$ to be, respectively, $V(-M)$ and $V(M)$, where $M > 0$ is any number large enough that all real roots of each f_i $(i = 0, 1, \ldots, k)$ lie in $(-M, M)$. It is clear that

$$V(-\infty) = \mathrm{Var}\left((-1)^{\deg f_0}\mathrm{lcoeff}\, f_0, (-1)^{\deg f_1}\mathrm{lcoeff}\, f_1, \ldots, (-1)^{\deg f_k}\mathrm{lcoeff}\, f_k\right)$$

and

$$V(\infty) = \mathrm{Var}\left(\mathrm{lcoeff}\, f_0, \mathrm{lcoeff}\, f_1, \ldots, \mathrm{lcoeff}\, f_k\right),$$

where $\mathrm{lcoeff}\,\psi$ denotes the leading coefficient of ψ.

All real roots of f lie in $(-M, M)$ as $f = f_0$. Then Sturm's Theorem says that the number of distinct real roots of f is $V(-\infty) - V(\infty)$.

With these observations, we prove the result. If $S_f(t)$ has gaps in degree, then it has $k+1 < \deg h + 1$ terms, so in particular $V(-\infty) \leq k < \deg h$, and so

$$\begin{aligned}
\#\text{ distinct real roots of } f &= V(-\infty) - V(\infty) \\
&\leq V(-\infty) \\
&< \deg h \\
&= \#\text{ distinct roots of } f,
\end{aligned}$$

which implies that f has a nonreal root. If $S_f(t)$ has no gaps in degree but has a negative leading coefficient, then let j be the first i such that $\operatorname{lcoeff} f_i < 0$. Then $\operatorname{lcoeff} f_{j-1} > 0$ (as $\operatorname{lcoeff} f_0$ is), and since $\deg f_j = \deg f_{j-1} - 1$, we have that $(-1)^{\deg f_{j-1}} \operatorname{lcoeff} f_{j-1}$ and $(-1)^{\deg f_j} \operatorname{lcoeff} f_j$ have the same sign, so $V(-\infty) < k = \deg h$, and again f has a nonreal root.

Conversely, if $S_f(t)$ has no gaps in degree and no negative leading coefficients, then $k = \deg h$, $V(-\infty) = k$, and $V(\infty) = 0$, so

$$\begin{aligned}
\#\text{ distinct real roots of } f &= V(-\infty) - V(\infty) \\
&= \deg h \\
&= \#\text{ distinct roots of } f,
\end{aligned}$$

which says f has all real roots. □

For c any *positive* real number, it is easy to see that if, on obtaining the term f_j $(0 \leq j \leq k)$ in the construction of the Sturm sequence $(f_0, f_1, \ldots, f_k)$ of $f(t)$, we were to change f_j to cf_j before continuing, then the resulting sequence would differ from $(f_0, f_1, \ldots, f_k)$ only in that some f_i's would now become cf_i, and so clearly that sequence could be used in place of $(f_0, f_1, \ldots, f_k)$ when applying Theorem 1.2.4 or Corollary 1.2.5. In fact, we may perform multiplications like this at any number of steps (by repeatedly applying the above argument), and we consider *any* to be a Sturm sequence of $f(t)$. We shall make use of this observation in Section 2.1.2.

1.2.3 The Riemann Sphere

In Chapter 3, we will undertake a study of fractals which arise in a natural way from independence polynomials of graphs. In the process, we will make nontrivial use of results from the theory of iterating rational functions, an understanding of which requires knowledge of a few specific facts about metric spaces – the Riemann sphere, in particular. This material, which we now review, can be found in the works of Barnsley [6] and Beardon [7].

We assume the reader is familiar with the definition of a metric space $(\mathbb{X}, d)$, a *complete* metric space, and a *compact* subset of a metric space. The collection $\mathcal{H}(\mathbb{X})$ of compact subsets of a metric space $(\mathbb{X}, d)$ forms a metric space in itself: if A and B are two compact subsets of $(\mathbb{X}, d)$, then the *distance* $d(A, B)$ from A to B is given by

$$d(A, B) = \max_{a \in A} \min_{b \in B} d(a, b).$$

The definition is not symmetric. However, if we define

$$h_d(A, B) = \max\left(d(A, B), d(B, A)\right),$$

then h_d is a metric on $\mathcal{H}(\mathbb{X})$; in fact, if $(\mathbb{X}, d)$ is a complete metric space, then $(\mathcal{H}(\mathbb{X}), h_d)$ is a complete metric space.

On the space $\mathbb{X} = \mathbb{C}$ of complex numbers, the *Euclidean* metric $|\cdot|$ measures the distance between points z and w as $|z - w|$; it is well known that $(\mathbb{C}, |\cdot|)$ is a complete metric space. As is the case with $(\mathbb{R}, |\cdot|)$, however, the notion of a sequence 'converging to infinity' requires a separate definition. While the fractals we will consider in chapter 3 are bounded in $(\mathbb{C}, |\cdot|)$, their study involves iteration theory of polynomials, where it is important to eliminate this special significance of infinity.

First, an abstract point, denoted by ∞, is adjoined to $\mathbb{C}$, forming the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, called the *extended* complex plane, or *Riemann sphere*. To obtain a metric on $\mathbb{C}_\infty$, we construct a model of $\mathbb{C}_\infty$ as a sphere, as follows. Identify $\mathbb{C}$ with the horizontal plane in $\mathbb{R}^3$ containing the origin, and let S be the sphere in $\mathbb{R}^3$ with unit radius and centered at the origin. The *stereographic projection* of $\mathbb{C}$ into S is the map π which takes a complex number z and projects it linearly towards the top point $(0,0,1)$ of S until it meets S at a point $\pi(z)$; also, we set $\pi(\infty) = (0,0,1)$. This leads to two natural metrics on $\mathbb{C}_\infty$: the *chordal* and *spherical* metrics σ and σ_0, respectively, are defined as follows. If z and w are two points in $\mathbb{C}_\infty$, then

$$\begin{aligned}\sigma(z,w) &= \text{the Euclidean distance (in } \mathbb{R}^3\text{) between } \pi(z) \text{ and } \pi(w);\\ \sigma_0(z,w) &= \text{the great circle distance (on } S\text{) between } \pi(z) \text{ and } \pi(w).\end{aligned}$$

Recall that two metrics d_1 and d_2 on $\mathbb{X}$ are *equivalent* if there exist positive real constants c_1 and c_2 such that

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y)$$

for all $x, y \in \mathbb{X}$. The chordal and spherical metrics on $\mathbb{C}_\infty$ turn out to be equivalent. Further, on a bounded subset of $(\mathbb{C}, |\cdot|)$, each is equivalent to the Euclidean metric, $|\cdot|$:

Theorem 1.2.6 *Let $\mathbb{X}$ be a bounded subset of $(\mathbb{C}, |\cdot|)$. Then, on $\mathbb{X}$, the Euclidean, spherical and chordal metrics are equivalent.*

Proof It suffices to show that $|\cdot|$ is equivalent to σ on $\mathbf{X}$. As $\mathbf{X}$ is bounded with respect to $|\cdot|$, there exists a number $M > 0$ such that $\mathbf{X}$ is contained in an $M-$ radius of 0. For any two points z and w in $\mathbf{X}$, then, $\sigma(z,w) = \dfrac{2|z-w|}{(1+|z|)^{1/2}(1+|w|)^{1/2}}$ (a well known explicit formula (cf. [7]) for $\sigma(z,w)$) satisfies

$$\frac{2}{1+M^2} \cdot |z-w| \leq \sigma(z,w) \leq 2|z-w|,$$

and $|\cdot|$ is thus equivalent to σ on $\mathbf{X}$, completing the proof. □

From this, we may conclude:

Theorem 1.2.7 *Let $\mathbb{X}$ be a compact subset of $(\mathbb{C}, |\cdot|)$, and $\{A_n\}$ a sequence in $\mathcal{H}(\mathbb{X})$. Let $A \in \mathcal{H}(\mathbb{X})$. Then $A_n \to A$ with respect to h_{σ_0} (or h_σ) if and only if $A_n \to A$ with respect to $h_{|\cdot|}$.*

In chaper 3, we consider sequences $\{A_n\}$ which arise from root sets of an iterated graph-theoretic polynomial, and which are always bounded in $(\mathbb{C}, |\cdot|)$. We are naturally led to consider their limit in that space. Theorem 1.2.7 tells us that their limit in $(\mathbb{C}_\infty, \sigma_0)$ is identical, which then enables us to make important use of iteration theory in describing that limit.

1.3 An Overview of the Thesis

The roots of a graph polynomial are studied from various standpoints, from bounding the roots, to determining accumulation points for the roots, to finding regions in the complex plane or real line containing no root at all. The effects that familiar graph-theoretic operations have on the roots are of interest as well, and this will be a common thread in the thesis.

Our study of roots of chromatic polynomials ('chromatic roots') in chapter 2 is directly related to the operation of *subdividing* an edge of G, that is, replacing an edge with a path. If G consists of nothing more than two vertices joined by one or more edges, then subdividing edges in G gives rise to what are known as *generalized theta graphs.* Numerical calculations suggest that among all generalized theta graphs on k paths, the one with a chromatic root z that maximizes $|z-1|$ is the graph with all path lengths equal to 2; we will prove that for $k \leq 8$, this is indeed the case. Roots of chromatic polynomials lying in the left-half plane have also received attention, and we will prove that infinitely many such are achievable among theta graphs on 3 paths, thereby providing a family of minimal *co-rank* (cf. chapter 2) having chromatic roots with negative real part. Moreover, a graph G will possess a subdivision having a chromatic root with negative real part if and only if G has a theta-subgraph. At the same time, however, large subdivisions of all the edges in G draw the roots close to the disk $|z-1| \leq 1$.

In chapter 3, we turn our attention to roots of independence polynomials ('independence roots'). The work is based largely on the *lexicographic product* (or graph *composition*) operation, which involves 'replacing' each vertex of G with a graph H. We begin the chapter with the observation that independence polynomials are essentially closed under graph composition, a result which we then exploit to show that the real roots of independence polynomials have closure $(-\infty, 0]$, while the complex roots are dense in $\mathbb{C}$, even for some restricted families of graphs. We then consider, for an arbitrary graph G, what happens to the independence roots of the graphs $G[G]$, $(G[G])[G]$, and so on. The independence polynomials are iterates (with respect to function composition) of $i(G,x)$ (the independence polynomial of G), and their roots approach (in a very strong sense) the *Julia set* of $i(G,x)-1$, thereby associating with G a fractal, $\mathcal{I}(G)$, which we call the *independence attractor* of G. We are led to ask: when is $\mathcal{I}(G)$ connected? For graphs with independence number 2, as well as a few infinite families of graphs with arbitrarily high independence numbers, we provide a complete answer to the question.

Chapter 2

Roots of Chromatic Polynomials

The *chromatic polynomial*, $\pi(G,x)$, of a graph G is the polynomial whose value at each positive integer x is the number of functions $f : V \rightarrow \{1,\ldots,x\}$ such that $uv \in E$ implies $f(u) \neq f(v)$, where V and E are the sets of vertices and edges of G, respectively. The roots of $\pi(G,x)$ are often called the *chromatic roots* of G, and in general a chromatic root is any (complex) number which is a root of some chromatic polynomial. Initial interest in chromatic roots stemmed from the former Four Colour Conjecture, which states that no planar graph has 4 as a chromatic root. The study of chromatic roots has since emerged as a rather fascinating topic in its own right, having attracted considerable attention (c.f. [9, 11, 18, 20, 21, 22, 45, 50, 57, 60]). However, much remains unknown.

We shall need a few basic facts about chromatic polynomials (all of which are discussed in [10]), which we review now.

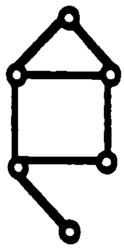
Let G be a graph, possibly containing parallel edges or loops. Parallel edges have no effect on the chromatic polynomial, and it follows directly from the definition that if G has a loop then indeed $\pi(G,x) \equiv 0$. If G has no loops, $\pi(G,x)$ is a monic polynomial in x of degree $|V|$, whose coefficients are integers that alternate in sign.

The well known *deletion-contraction* reduction states that for e any edge of G, $\pi(G,x) = \pi(G-e,x) - \pi(G \bullet e,x)$, where $G \bullet e$ is the contraction of e in G, and is obtained by removing e and identifying its end vertices. It is not hard to verify that the formula holds even if e is a parallel edge or loop. We sometimes rewrite the deletion-contraction $\pi(G - e, x) = \pi(G, x) + \pi(G \bullet e, x)$, which we will refer to as *addition-contraction.*

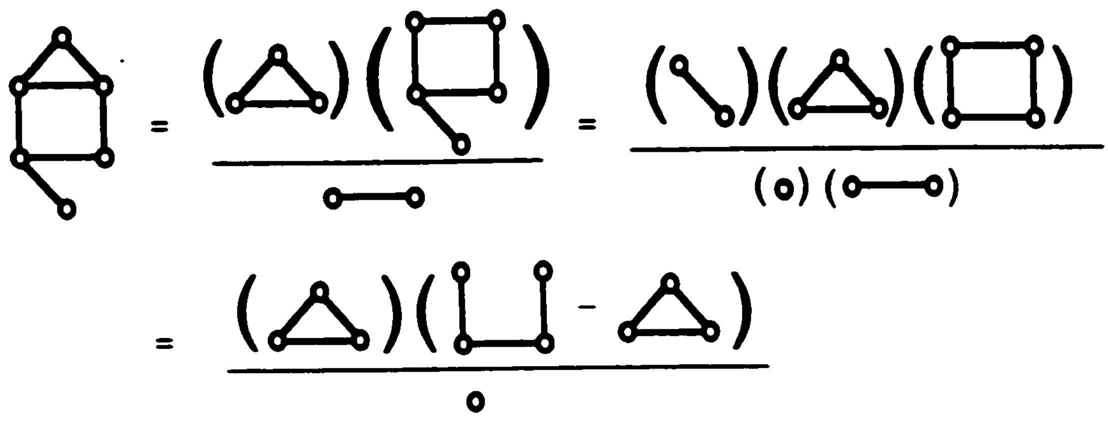
Also, if two graphs G and H intersect exactly on a complete graph, K_p, on p vertices, then $\pi(G \cup H, x) = \pi(G, x)\pi(H, x)/\pi(K_p, x)$; this is sometimes referred to as the *Complete Cutset Theorem.*

Finally, the chromatic polynomials of K_n, T_n (any tree of order n), and C_n (the cycle of order n) are given by $x(x-1)\cdots(x-n+1)$, $x(x-1)^{n-1}$, and $(x-1)^n + (-1)^n(x-1)$, respectively.

Here is an example of a chromatic polynomial calculation; let G be the graph shown below:



We can use deletion-contraction and the complete cutset theorem to calculate $\pi(G, x)$ as follows:



$$\begin{aligned}\pi(G,x) &= \frac{x(x-1)(x-2)\left(x(x-1)^3 - x(x-1)(x-2)\right)}{x} \\ &= x(x-1)^2(x-2)(x^2-3x+3) \\ &= x^6 - 7x^5 + 20x^4 - 29x^3 + 21x^2 - 6x\end{aligned}$$

One avenue of investigation into chromatic roots has been determining bounds on their moduli in terms of various graph parameters. Very recently, Sokal proved the following, which confirms a 1972 conjecture of Biggs, Damerell, and Sands [11]:

Theorem 2.0.1 **([54])** *If G is a graph of maximum degree k, then every chromatic root z of G lies in the disc $|z| < 7.963907\,k$.*

While the constant 7.963907 found in [54] can likely be improved, the linearity of the bound is best possible, since the complete graph K_{k+1} (which has maximum degree k) has a chromatic root at $z = k$.

A complementary approach is to bound the chromatic roots in terms of the *co-rank* (or *cycle rank*) of the graph. Recall that a connected graph with n vertices and m edges has corank $m - n + 1$. Brown has has recently proved:

Theorem 2.0.2 **([21])** *If G is a graph of corank $k \geq 1$, then every chromatic root z of G lies in the disc $|z - 1| \leq k$.*

Unlike the bound in terms of maximum degree, however, it is not known whether linear growth with corank is best possible.

It is natural to ask for suitably restricted *subclasses* of graphs which satisfy a sublinear bound in either maximum degree or co-rank, and indeed in [29] a sublinear bound on the chromatic roots of *both* corank and maximum degree was proven for graphs called *generalized theta graphs*. Our numerical explorations with theta graphs suggest an interesting conjecture which extends that result, and in section 2.1.1 we prove our conjecture for 'small' theta graphs.

Roots of chromatic polynomials lying in the left-half plane have also attracted interest, and very little is known about them. In section 2.1.2 we prove that indeed infinitely many chromatic roots with negative real part are achievable among theta graphs of co-rank 2, thereby providing a family of minimal corank having chromatic roots with negative real part.

While theta graphs may be a very restricted subclass of graphs*, we prove in section 2.2.4 that having a theta subgraph is both necessary and sufficient for a graph to possess a *subdivision* whose chromatic polynomial has a root with negative real part. In section 2.2.3 we are able to describe the limit points of the chromatic roots of 'uniform' subdivisions of a graph, and finally, in section 2.2.5 we prove that large subdivisions of *any* graph draw the chromatic roots close to the disk $|z-1| \leq 1$.

2.1 Generalized Theta Graphs

For a positive integer k, the k-ary *generalized theta graph* $\Theta_{s_1,\ldots,s_k}$, is formed by taking a pair of vertices u, v (called the *endvertices*) and joining them by k internally disjoint paths of lengths $s_1, \ldots, s_k \geq 1$. The adjective, 'generalized', is more philological than mathematical, and we will often drop it in what follows. For brevity, we denote by $\Theta_{(k;s)}$ the k-ary theta graph whose path lengths are all equal to s.

Let us derive an expression for the chromatic polynomials of generalized theta graphs. Observe that if one adjoins to a generalized theta graph $\Theta_{s_1,\ldots,s_k}$ a new edge e between the two endvertices, the resulting graph is a collection of cycles (of lengths s_i+1) that overlap in a complete graph K_2 (namely the edge e), so that

$$\pi(\Theta_{s_1,\ldots,s_k}+e,x) = \frac{\prod_{i=1}^{k}\left[(x-1)^{s_i+1}+(-1)^{s_i+1}(x-1)\right]}{[x(x-1)]^{k-1}}. \tag{2.1}$$

Likewise, contraction of the endvertices of $\Theta_{s_1,\ldots,s_k}$ yields a collection of cycles (of

*Sokal has recently proved that chromatic roots of theta graphs are dense in all of $\mathbf{C}$, except possibly the disk $|z-1| \leq 1$. Thus, theta graphs are not such a "restricted" family after all.

lengths s_i) that overlap in a complete graph of order 1, so that

$$\pi(\Theta_{s_1,\ldots,s_k} \bullet e, x) = \frac{\prod_{i=1}^{k} [(x-1)^{s_i} + (-1)^{s_i}(x-1)]}{x^{k-1}} . \tag{2.2}$$

It follows that the chromatic polynomial of $\Theta_{s_1,\ldots,s_k}$ is given by

$$\begin{aligned} \pi(\Theta_{s_1,\ldots,s_k}, x) &= \frac{\prod_{i=1}^{k} [(x-1)^{s_i+1} + (-1)^{s_i+1}(x-1)]}{[x(x-1)]^{k-1}} \\ &\quad + \frac{\prod_{i=1}^{k} [(x-1)^{s_i} + (-1)^{s_i}(x-1)]}{x^{k-1}} \\ &= \frac{(-1)^{\sum_{i=1}^{k} s_i - 1}(1-x)}{x^{k-1}} \left[\prod_{i=1}^{k} [(1-x)^{s_i} - 1] \right. \\ &\quad \left. - (1-x)^{k-1} \prod_{i=1}^{k} [(1-x)^{s_i - 1} - 1] \right] . \end{aligned} \tag{2.3}$$

Therefore, we need only concern ourselves with the roots of

$$f_{s_1,\ldots,s_k}(y) = \prod_{i=1}^{k} (y^{s_i} - 1) - y^{-1} \prod_{i=1}^{k} (y^{s_i} - y) , \tag{2.4}$$

where $y = 1 - x$. All of our calculations in section 2.1.1 will be expressed in terms of the variable y.

Let us now dispose of some trivial cases. If $k = 1$, the theta graph Θ_{s_1} is isomorphic to the path P_{s_1}, so that its chromatic roots are 0 and 1. If $k = 2$, the theta graph Θ_{s_1,s_2} is isomorphic to the cycle $C_{s_1+s_2}$, so that its chromatic roots (other than $z = 1$) all lie on the circle $|z - 1| = 1$. We shall therefore assume henceforth that $k \geq 3$.

If one or more of the path lengths s_i equals 1, then the second product in (2.4) vanishes, and all the chromatic roots (other than $z = 1$) again lie on the circle $|y| = 1$, i.e. $|z - 1| = 1$. We shall therefore assume henceforth that all path lengths s_i are at least 2.

The above method for computing $\pi(\Theta_{s_1,\ldots,s_k}, z)$ was employed previously by Read and Tutte [51, pp. 29–30] for the case $k = 3$.

2.1.1 Theta Graphs with ≤ 8 Paths

A k-ary theta graph clearly has maximum degree k and corank $k-1$ (except for the trivial case $k = 1$ with $s_1 \geq 2$, which has maximum degree 2). The main result of [29], stated below, thus proves that the chromatic roots of the family of all k-ary theta graphs are bounded sublinearly in both co-rank and maximum degree.

Theorem 2.1.1 ([29]) *The chromatic roots of any k-ary generalized theta graph lie in the disc $|z-1| \leq [1+o(1)]\, k/\log k$, where $o(1)$ denotes a constant $C(k)$ that tends to zero as $k \to \infty$.*

This bound is asymptotically saturated by the graph $\Theta_{(k;2)}$ with all path lengths equal to 2 (which is isomorphic to the complete bipartite graph $K_{2,k}$).

Now, numerical computations suggest that among all k-ary theta graphs $\Theta_{s_1,\ldots,s_k}$, the one with a chromatic root that maximizes $|z-1|$ is the graph with all path lengths s_i equal to 2, i.e. the graph $\Theta_{(k;2)} \simeq K_{2,k}$. We conjecture that this is indeed the case. The general bound of Theorem 2.1.1 is not strong enough to prove this conjecture. Nevertheless, by different techniques we shall show the validity of this conjecture for all $k \leq 8$. As before, it suffices to consider $k \geq 3$ and $s_1, \ldots, s_k \geq 2$. Denote by $\rho(s_1, \ldots, s_k)$ denotes the maximum modulus of a root of $f_{s_1,\ldots,s_k}(y)$.

Our method is based on the following trivial bound (see e.g. [47, Theorem 27.1]):

Proposition 2.1.2 *Let $P(y) = \sum_{j=0}^{n} a_j y^j$ be a polynomial of degree n (so that $a_n \neq 0$), and let R be the unique nonnegative real solution of*

$$|a_n| R^n - \sum_{j=0}^{n-1} |a_j| R^j = 0 . \tag{2.5}$$

Then all the roots of P lie in the disc $|y| \leq R$. Moreover, in (2.5) the numbers $|a_j|$ for $0 \leq j \leq n-1$ can be replaced by any numbers $b_j \geq |a_j|$; this only makes the bound weaker.

At first sight it is surprising that such a crude estimation method — which throws away all the sign or phase information in the coefficients of P — could yield reasonably sharp results. And indeed, we do not entirely understand why it works so well in our application — but it does! Here is the key trick: since the polynomial $f_{s_1,\ldots,s_k}(y)$ defined in (2.4) is divisible by $(y-1)^k$, we are free to pull out a factor $(y-1)^l$ with any $0 \le l \le k$ before applying Proposition 2.1.2. It turns out that the right choice is to take $l = 1$. That is, we define the polynomial $\phi_{s_1,\ldots,s_k}(y)$ by

$$\phi_{s_1,\ldots,s_k}(y) \;=\; \frac{f_{s_1,\ldots,s_k}(y)}{y-1}\,. \tag{2.6}$$

Let $[k]$ denote the set $\{1,\ldots,k\}$, and let $\binom{[k]}{l}$ denote the set of all subsets of $[k]$ of cardinality l. For a subset $X = \{i_1, i_2, \ldots, i_l\}$ of $[k]$, let us define

$$s_X = \sum_{j=1}^{l} s_{i_j}.$$

We then have:

$$\begin{aligned}
f_{s_1,\ldots,s_k}(y) &= \prod_{i=1}^{k}(y^{s_i}-1) - y^{-1}\prod_{i=1}^{k}(y^{s_i}-y) & (2.7)\\
&= \sum_{m=0}^{k}(-1)^m \sum_{X\subseteq\binom{[k]}{k-m}} y^{s_X} - \sum_{m=0}^{k}(-1)^m \sum_{X\subseteq\binom{[k]}{k-m}} y^{s_X+m-1} & (2.8)\\
&= \sum_{m=0}^{k}(-1)^m \sum_{X\subseteq\binom{[k]}{k-m}} y^{s_X}(1-y^{m-1}) & (2.9)\\
&= y^{(\sum s_i)-1}(y-1) - \sum_{m=2}^{k}(-1)^m \sum_{X\subseteq\binom{[k]}{k-m}} y^{s_X}(y^{m-1}-1) & (2.10)
\end{aligned}$$

and hence

$$\phi_{s_1,\ldots,s_k}(y) \;=\; y^{(\sum s_i)-1} - \sum_{m=2}^{k}(-1)^m \sum_{X\subseteq\binom{[k]}{k-m}} y^{s_X}(1+y+\cdots+y^{m-2})\,. \tag{2.11}$$

We can now implement Proposition 2.1.2 by defining $h_{s_1,\ldots,s_k}(y)$ to be the polynomial obtained from $\phi_{s_1,\ldots,s_k}(y)$ by changing all subleading signs to $-$ as in (2.5), and letting $r(s_1,\ldots,s_k)$ be the unique positive root of $h_{s_1,\ldots,s_k}(y)$. We then have $\rho(s_1,\ldots,s_k) \leq r(s_1,\ldots,s_k)$, where (recall) $\rho(s_1,\ldots,s_k)$ is the maximum modulus of a root of $f_{s_1,\ldots,s_k}(y)$. Unfortunately, this bound is unsuitable for our present purposes, as it turns out that $r(s_1,\ldots,s_k)$ is not a monotonically decreasing function of the path lengths $s_1,\ldots,s_k$. We therefore throw away a bit more, by disregarding all sign cancellations among the subleading terms of (2.11), and define

$$\widetilde{h}_{s_1,\ldots,s_k}(y) \;=\; y^{(\sum s_i)-1} - \sum_{m=2}^{k} \sum_{X\subseteq\binom{[k]}{k-m}} y^{s_X}(1+y+\cdots+y^{m-2})\,. \tag{2.12}$$

[Thus, the coefficient of y^j in (2.12) is in general *larger* in magnitude than in (2.11).] Let $\widetilde{r}(s_1,\ldots,s_k)$ be the unique positive root of $\widetilde{h}_{s_1,\ldots,s_k}(y)$. Then it follows immediately from Proposition 2.1.2 that

$$\rho(s_1,\ldots,s_k) \;\leq\; r(s_1,\ldots,s_k) \;\leq\; \widetilde{r}(s_1,\ldots,s_k)\,, \tag{2.13}$$

or in other words:

Proposition 2.1.3 *Every chromatic root z of $\Theta_{s_1,\ldots,s_k}$ lies in the disc $|z-1| \leq \widetilde{r}(s_1,\ldots,s_k)$.*

We now analyze the behaviour of the upper bound $\widetilde{r}(s_1,\ldots,s_k)$:

Proposition 2.1.4 *$\widetilde{r}(s_1,\ldots,s_k)$ is symmetric in $s_1,\ldots,s_k$ and strictly decreasing in each s_i.*

Proof The symmetry is obvious. To prove the decreasing property, fix $s_1,\ldots,s_k$ and set $r=\widetilde{r}(s_1,s_2,\ldots,s_k)$; by symmetry, it suffices to show that $\widetilde{r}(s_1+1,s_2,\ldots,s_k) < r$. Now, equation (2.12) implies that $r>1$. From equation (2.12) we can rewrite $\widetilde{h}_{s_1,s_2,\ldots,s_k}(y)$ in the form

$$\widetilde{h}_{s_1,s_2,\ldots,s_k}(y) \;=\; y^{(\sum_{i=1}^{k} s_i)-1} - y^{s_1}A(y) - B(y) \tag{2.14}$$

where A and B are polynomials in y with nonnegative integer coefficients that do not depend on s_1. Moreover, the degrees of $y^{s_1}A(y)$ and $B(y)$ are less than $(\sum_{i=1}^{p} s_i) - 1$, and $B(0) = 1$. It follows that $r^{(\sum_{i=1}^{k} s_i)} - r^{s_1}A(r) > 0$. Now from (2.14) we have

$$\begin{aligned} \tilde{h}_{s_1+1,s_2,\ldots,s_k}(r) &= r^{\sum_{i=1}^{k} s_i} - r^{s_1+1}A(r) - B(r) \\ &= r \cdot \left(r^{(\sum_{i=1}^{k} s_i)-1} - r^{s_1}A(r)\right) - B(r) \\ &> r^{(\sum_{i=1}^{k} s_i)-1} - r^{s_1}A(r) - B(r) \\ &= \tilde{h}_{s_1,s_2,\ldots,s_k}(r) \\ &= 0\,. \end{aligned} \tag{2.15}$$

It follows that $\tilde{r}(s_1 + 1, s_2, \ldots, s_k) < r = \tilde{r}(s_1, s_2, \ldots, s_k)$, completing the proof. □

What may be surprising is how well the roots of h and $\tilde{h}$ bound the roots of f for small k (see Table 2.1). In particular, they are good enough to prove the following. By $\rho_{(k;2)}$ we mean $\rho(\underbrace{2, 2, \ldots, 2}_{k})$.

Theorem 2.1.5 *For $3 \leq k \leq 8$, we have $\rho(s_1, \ldots, s_k) \leq \rho_{(k;2)}$, with equality only when $s_1 = \cdots = s_k = 2$. In other words, among all k-ary theta graphs, the graph with a chromatic root that maximizes $|z - 1|$ is the one with all path lengths equal to 2.*

Proof By direct calculation (see Table 2.1) we have $\tilde{r}(2, 2, 3) < \rho(2, 2, 2)$, $\tilde{r}(2, 2, 2, 3) < \rho(2, 2, 2, 2)$ and $\tilde{r}(2, 2, 2, 2, 3) < \rho(2, 2, 2, 2, 2)$. The result for $3 \leq k \leq 5$ then follows immediately from Propositions 2.1.3 and 2.1.4. For $k = 6, 7$, a bit more work is needed: the calculations show that $\rho(2, \ldots, 2, 2, 3)$, $\tilde{r}(2, \ldots, 2, 2, 4)$ and $\tilde{r}(2, \ldots, 2, 3, 3)$ are all bounded above by $\rho(2, \ldots, 2, 2, 2)$, so that the result again follows from Propositions 2.1.3 and 2.1.4. Finally, for $k = 8$, the calculations show that $\rho(2, \ldots, 2, 2, 3)$, $\rho(2, \ldots, 2, 2, 4)$, $\tilde{r}(2, \ldots, 2, 2, 5)$ and $\tilde{r}(2, \ldots, 2, 3, 3)$ are all bounded above by $\rho(2, \ldots, 2, 2, 2)$, which is again sufficient. □

This method of proof relies in an essential way on the fact that, for $3 \leq k \leq 8$, only finitely many of the upper bounds $\tilde{r}(s_1, \ldots, s_k)$ are larger than the true

Path length sequence $(s_1, \ldots, s_k)$	Actual value $\rho(s_1, \ldots, s_k)$	Upper bound $r(s_1, \ldots, s_k)$	$\tilde{r}(s_1, \ldots, s_k)$
(2, 2, 2)	1.5247025799	1.5905667405	1.5905667405
(2, 2, 3)	1.3247179572	1.4655712319	1.4655712319
(2, 2, 2, 2)	1.9635530390	2.0652388409	2.0959187459
(2, 2, 2, 3)	1.6180339887	1.8003794650	1.9038165409
(2, 2, 2, 2, 2)	2.3602010481	2.4788311017	2.5569445891
(2, 2, 2, 2, 3)	1.9596554046	2.0481965587	2.3283569921
(2, 2, 2, 2, 4)	1.9125157044	2.0726410424	2.2158195963
(2, 2, 2, 2, 5)	2.0227195761	2.1137657905	2.1572723181
(2, 2, 2, 2, 6)	1.9492237868	2.0928219450	2.1267590770
(2, 2, 2, 2, 2, 2)	2.7305222731	2.8521866737	2.9891971006
(2, 2, 2, 2, 2, 3)	2.3291754791	2.4702504048	2.7400794700
(2, 2, 2, 2, 2, 4)	2.3208606055	2.4487347678	2.6342641478
(2, 2, 2, 2, 3, 3)	2.0524815723	2.2641426827	2.5176585462
(2, 2, 2, 2, 2, 2, 2)	3.0823336669	3.1959268744	3.4006086206
(2, 2, 2, 2, 2, 2, 3)	2.6933092033	2.8543267466	3.1395749040
(2, 2, 2, 2, 2, 2, 4)	2.7030241913	2.8316875864	3.0429807861
(2, 2, 2, 2, 2, 3, 3)	2.3573224846	2.4527687226	2.8983449779
(2, 2, 2, 2, 2, 2, 2, 2)	3.4201564280	3.5685068590	3.7959050193
(2, 2, 2, 2, 2, 2, 2, 3)	3.0446178232	3.2040479885	3.5278440533
(2, 2, 2, 2, 2, 2, 2, 4)	3.0625912820	3.2129169213	3.4402140830
(2, 2, 2, 2, 2, 2, 2, 5)	3.0953618332	3.1953189320	3.4125677445
(2, 2, 2, 2, 2, 2, 3, 3)	2.6885399588	2.8486049323	3.2745245420
(2, 2, 2, 2, 2, 2, 2, 2, 2)	3.7468849281	3.9272779941	4.1781887719
(2, 2, 2, 2, 2, 2, 2, 2, 3)	3.3836067543	3.5282506474	3.9060114610
(2, 2, 2, 2, 2, 2, 2, 2, 4)	3.4054981704	3.5867024115	3.8263498519
(2, 2, 2, 2, 2, 2, 2, 2, 5)	3.4292505541	3.5677746122	3.8040844502
(2, 2, 2, 2, 2, 2, 2, 2, 6)	3.4182415134	3.5704257784	3.7980747620
(2, 2, 2, 2, 2, 2, 2, 2, 7)	3.4200422197	3.5685857538	3.7964779130
(2, 2, 2, 2, 2, 2, 2, 2, 8)	3.4203605983	3.5684058522	3.7960560504
(2, 2, 2, 2, 2, 2, 2, 2, 9)	3.4200947731	3.5685249008	3.7959448158
(2, 2, 2, 2, 2, 2, 2, 2, 10)	3.4201605551	3.5685220773	3.7959155041
(2, 2, 2, 2, 2, 2, 2, 2, 11)	3.4201602535	3.5685079914	3.7959077815
(2, 2, 2, 2, 2, 2, 2, 2, 12)	3.4201547358	3.5685071412	3.7959057470
(2, 2, 2, 2, 2, 2, 2, 2, 13)	3.4201566935	3.5685072051	3.7959052110
(2, 2, 2, 2, 2, 2, 2, 3, 3)	3.0254986086	3.2079141314	3.6449248003

Table 2.1: Values of $\rho(s_1, \ldots, s_k)$ and its upper bounds $r(s_1, \ldots, s_k) \leq \tilde{r}(s_1, \ldots, s_k)$.

value $\rho(2,\ldots,2) \equiv \rho_{(k;2)}$. Unfortunately, this fails for $k = 9$: indeed, we have $\lim_{s_k\to\infty} \tilde{r}(s_1,\ldots,s_{k-1},s_k) = \tilde{r}(s_1,\ldots,s_{k-1})$ and in particular $\lim_{s_9\to\infty} \tilde{r}(2,\ldots,2,s_9) = \tilde{r}_{(8;2)} \approx 3.7959050193 > 3.7468849281 \approx \rho_{(9;2)}$. A genuinely new method, therefore, seems to be required to prove Theorem 2.1.5 for $k \geq 9$.

2.1.2 Theta Graphs with 3 Paths: An Explicit Family having Roots with Negative Real Part

Since the coefficients of any chromatic polynomial alternate in sign, no real root of a chromatic polynomial is negative. But can a chromatic root have negative real part? Based on the chromatic roots of all graphs on at most 8 vertices, the following was conjectured in 1980.

Conjecture 2.1.6 [33] *There are no chromatic roots with negative real part.*

In 1991 Read and Royle computed the chromatic polynomials of all 3–regular graphs on at most 16 vertices, noting that graphs with high girth appear to be contributing the roots with smallest real part. They proceeded to plot the chromatic roots of all 3–regular graphs of girth at least 5 on 18 vertices, and observed the following, thus providing the smallest known counterexample to the above conjecture.

Proposition 2.1.7 [50] *There are graphs of order* 18 *having a chromatic root with negative real part.*

They noted the same for the 3–regular graphs of girth at least 6 on 20 vertices, and girth at least 7 on 26 vertices. More recently, Shrock and Tsai [53] have shown how badly the conjecture fails by showing that as $k \to \infty$, the k–ary theta graphs (i.e. graphs formed from two vertices joined by k internally disjoint paths) had chromatic roots whose real parts tended to $-\infty$. By different techniques (namely the Hermite-Biehler and Sturm theorems on the roots of real polynomials), we show here that indeed infinitely many chromatic roots with negative real part are achievable among 3–ary theta graphs themselves. These provide examples of the smallest corank.

To this end, we restrict our attention now to the subfamily $\{\Theta_{a,a,a} : a \geq 2\}$ of generalized theta graphs whose u–v paths all have the same length. The smallest such graph having a chromatic root with negative real part is found (by direct calculation) to be $\Theta_{8,8,8}$. We can say much more.

Theorem 2.1.8 *For $a \geq 8$ the graph $\Theta_{a,a,a}$ has a chromatic root with negative real part.*

Proof Setting $k = 3$ and $s_1 = s_2 = s_3 = a$ in equation (2.3), we find (after a little simplification) that

$$\pi(\Theta_{a,a,a}, x) = \frac{(-1)^{3a-1}(1-x)}{x}\left[(1-x)^{3a-1} - 3(1-x)^a + 2 - x\right]. \tag{2.16}$$

For $a \geq 8$, we need to show that $\pi(\Theta_{a,a,a}, -x)$ has a root with *positive* real part, i.e., that

$$\psi_a(x) \equiv (1+x)^{3a-1} - 3(1+x)^a + 2 + x \tag{2.17}$$

is *not* Hurwitz quasi-stable. So let $a \geq 8$, and expand $\psi_a(x)$ into its even and odd parts:

$$\psi_a(x) = P_e^a(x^2) + xP_o^a(x^2). \tag{2.18}$$

Now set $t = x^2$ (as in Theorem 1.2.2). Several calculations suggested that $P_e^a(t)$ always appears to have a nonreal root (for $a \geq 8$), and by Corollary 1.2.3 of the Hermite-Biehler Theorem (Theorem 1.2.2), it is enough to show that this is indeed the case. To that end, it would suffice (by Theorem 1.2.4) to show that its Sturm sequence contains either a negative leading coefficient or gaps in degree. Our computations, however, suggest that this does not occur until close to the end of its Sturm sequence. So let us instead consider the polynomial

$$\phi_a(t) \equiv t^{\deg P_e^a(t)} P_e^a(1/t), \tag{2.19}$$

which clearly has a nonreal root if and only if $P_e^a(t)$ does. Moreover, we can establish the following.

Lemma 2.1.9 *For $a \geq 14$, the Sturm sequence of $\phi_a(t)$ has as its fifth term a polynomial with negative leading coefficient.*

We shall see, also, that *before* the fifth term in the sequence, there are neither gaps in degree nor any negative leading coefficients. However, since Lemma 2.1.9 tells us the leading coefficient of the fifth term *is* negative, we conclude that $\phi_a(t)$ has a nonreal root for $a \geq 14$, and in fact for $a \geq 8$ upon verifying the cases $8 \leq a \leq 13$ directly. Hence, to prove Theorem 2.1.8, it remains only to prove Lemma 2.1.9. We assume that $a \geq 14$ is *even* (the odd case is handled similarly). Then from (2.17), (2.18), and (2.19) we find that

$$\begin{aligned}\phi_a(t) &= \left[\binom{3a-1}{2} - 3\binom{a}{2}\right] t^{\frac{3a-4}{2}} + \left[\binom{3a-1}{4} - 3\binom{a}{4}\right] t^{\frac{3a-6}{2}} \\ &\quad + \cdots + \left[\binom{3a-1}{a} - 3\binom{a}{a}\right] t^{\frac{2a-2}{2}} + \binom{3a-1}{a+2} t^{\frac{2a-4}{2}} \\ &\quad + \binom{3a-1}{a+4} t^{\frac{2a-6}{2}} + \cdots + \binom{3a-1}{3a-2}.\end{aligned}$$

Let us denote the first five terms of the Sturm sequence of ϕ_a by $\phi_a^0 (= \phi_a)$, $\phi_a^1 (= \phi_a')$, ϕ_a^2, ϕ_a^3, and ϕ_a^4. Then it is clear, from the form of ϕ_a and the division process, that lcoeff(ϕ_a^4) is a real valued function of a, which we shall show is always negative (for $a \geq 14$ even). We shall see that only the first 7 terms of ϕ_a are needed. To begin, we have

$$\phi_a = bt^n + ct^{n-1} + dt^{n-2} + et^{n-3} + ft^{n-4} + gt^{n-5} + ht^{n-6} + \cdots,$$

where $b = \binom{3a-1}{2} - 3\binom{a}{2}$, $c = \binom{3a-1}{4} - 3\binom{a}{4}$, $\ldots$, $h = \binom{3a-1}{14} - 3\binom{a}{14}$, and $n = \frac{3a-4}{2}$. Note that $b, c, \ldots, h$ and n are polynomials in a with rational coefficients. Now

$$\begin{aligned}\phi_a' &= bnt^{n-1} + c(n-1)t^{n-2} + d(n-2)t^{n-3} + e(n-3)t^{n-4} \\ &\quad + f(n-4)t^{n-5} + g(n-5)t^{n-6} + h(n-6)t^{n-7} + \cdots.\end{aligned}$$

Dividing ϕ_a by ϕ_a', we find

$$\begin{aligned}-\text{rem}(\phi_a, \phi_a') &= \frac{c^2(n-1) - 2bdn}{bn^2} t^{n-2} + \frac{cd(n-2) - 3ben}{bn^2} t^{n-3} \\ &\quad + \frac{ce(n-3) - 4bfn}{bn^2} t^{n-4} + \frac{cf(n-4) - 5bgn}{bn^2} t^{n-5} \\ &\quad + \frac{cg(n-5) - 6bhn}{bn^2} t^{n-6} + \cdots.\end{aligned}$$

Since $bn^2 > 0$, we can (by an observation made in Section 1.2.2) clear the denominators by choosing $\phi_a^2 = bn^2 \cdot (-\mathrm{rem}(\phi_a, \phi_a'))$. Now it turns out that $c^2(n-1) - 2bdn$ is always positive (for $a \geq 14$ even). Indeed, it is a polynomial in b, c, d, and n with integer coefficients, each of which (recall) are themselves polynomials in a with rational coefficients. Carrying out the substitutions in Maple, we obtain an exact expression for $c^2(n-1) - 2bdn \in \mathbb{Q}[a]$, and find that it has positive leading coefficient, and so is positive beyond its largest real root, a bound for which is obtained by applying a standard result (c.f. [30, p.197]) to the polynomial. It is then verified directly that the polynomial $c^2(n-1) - 2bdn$ is also positive for those (even) values of a between 14 and that bound.

Moving on to the next term in the Sturm sequence, we divide ϕ_a' by ϕ_a^2, and find

$$\begin{aligned} -\mathrm{rem}(\phi_a', \phi_a^2) &= \frac{u}{(c^2(n-1) - 2bdn)^2} t^{n-3} + \frac{v}{(c^2(n-1) - 2bdn)^2} t^{n-4} \\ &\quad + \frac{w}{(c^2(n-1) - 2bdn)^2} t^{n-5} + \cdots, \end{aligned}$$

where

$$\begin{aligned} u &= -\left(c^2(n-1) - 2bdn\right)\left\{d(n-2)\left(c^2(n-1) - 2bdn\right)\right. \\ &\qquad\qquad \left. -bn\left(ce(n-3) - 4bfn\right)\right\} \\ &\quad + \left(cd(n-2) - 3ben\right)\left\{c(n-1)\left(c^2(n-1) - 2bdn\right)\right. \\ &\qquad\qquad \left. -bn\left(cd(n-2) - 3ben\right)\right\}, \end{aligned}$$

$$\begin{aligned} v &= -\left(c^2(n-1) - 2bdn\right)\left\{e(n-3)\left(c^2(n-1) - 2bdn\right)\right. \\ &\qquad\qquad \left. -bn\left(cf(n-4) - 5bgn\right)\right\} \\ &\quad + \left(ce(n-3) - 4bfn\right)\left\{c(n-1)\left(c^2(n-1) - 2bdn\right)\right. \\ &\qquad\qquad \left. -bn\left(cd(n-2) - 3ben\right)\right\}, \end{aligned}$$

and

$$w = -\left(c^2(n-1) - 2bdn\right)\left\{f(n-4)\left(c^2(n-1) - 2bdn\right)\right.$$

$$-bn\,(cg(n-5)-6bhn)\}$$
$$+\,(cf(n-4)-5bgn)\left\{c(n-1)\left(c^2(n-1)-2bdn\right)\right.$$
$$\left.-bn\,(cd(n-2)-3ben)\right\}.$$

Multiplying by the positive number $\left(c^2(n-1)-2bdn\right)^2$, we choose

$$\phi_a^3 = ut^{n-3}+vt^{n-4}+wt^{n-5}+\cdots.$$

Again, we have verified (with the aid of Maple) that indeed u is always positive (for $a \geq 14$ even). Let us move on to the fifth term in the Sturm sequence. Dividing ϕ_a^2 by ϕ_a^3, we find

$$\begin{aligned}-\mathrm{rem}(\phi_a^2,\phi_a^3) &= \frac{1}{u^2}\left(v\left\{(cd(n-2)-3ben)\,u-\left(c^2(n-1)-2bdn\right)v\right\}\right. \\ &\qquad \left.-u\left\{(ce(n-3)-4bfn)\,u-\left(c^2(n-1)-2bdn\right)w\right\}\right)t^{n-4} \\ &\quad +\cdots.\end{aligned}$$

Multiplying by the positive number u^2, we choose

$$\begin{aligned}\phi_a^4 &= \left(v\left\{(cd(n-2)-3ben)\,u-\left(c^2(n-1)-2bdn\right)v\right\}\right. \\ &\qquad \left.-u\left\{(ce(n-3)-4bfn)\,u-\left(c^2(n-1)-2bdn\right)w\right\}\right)t^{n-4}+\cdots.\end{aligned}$$

We denote by $\mathrm{lcoeff}\,(\phi_a^4)$ the coefficient of t^{n-4} in ϕ_a^4 (no confusion arises in doing so, as we are about to see that this coefficient is never zero, and so really *is* the leading coefficient of ϕ_a^4). So $\mathrm{lcoeff}\,(\phi_a^4)$ is a polynomial in $b, c, \ldots, h$, and n with integer coefficients. Substituting (once again) our expressions for $b, c, \ldots, h$ as polynomials in a with rational coefficients, we obtain an exact expression for $\mathrm{lcoeff}\,(\phi_a^4) \in \mathbb{Q}[a]$, the first few terms of which are approximately

$$-342.7311661a^{70}+22877.24435a^{69}-718377.3180a^{68}+\cdots.$$

In particular, the first term is negative, and so $\mathrm{lcoeff}\,(\phi_a^4)$ is negative for a sufficiently large, which is what we want. Applying a standard result (c.f. [30, p.197]) to the polynomial $\mathrm{lcoeff}\,(\phi_a^4)$, we obtain 134 as a bound on its largest real root, and so the proof of Lemma 2.1.9 for a even is completed by verifying directly that $\text{lcoeff}(\phi_a^4)$ is also negative for $a = 14, 16, 18, \ldots, 134$.

We mentioned that the proof for a odd is similar. There we find

$$\text{lcoeff}(\phi_a^4) \approx -342.7311661a^{70} + 21277.83225a^{69} - 617481.3554a^{68} + \cdots,$$

in which case an analysis like the previous paragraph is carried out.

This completes the proof of Lemma 2.1.9, and so of Theorem 2.1.8. □

We remark that the smallest generalized theta graphs (in terms of number of vertices) having a chromatic root with negative real part are $\Theta_{4,5,5,5,5}$ and $\Theta_{5,6,6,6}$, each of order 21.

2.2 Large Subdivisions of Graphs

We constructed theta graphs by taking a K_2- bond and replacing its edges by paths. In general, if e is an edge of a graph G, then by subdividing e we mean replacing e by a path. A *subdivision* of G is any graph formed by subdividing (one or more) edges in G. It is natural to ask what effect this operations has on the roots of the chromatic polynomial. In Section 2.2.2 we shall derive an expression for the chromatic polynomials of subdivisions of G. This expression simplifies considerably in the case of *uniform* subdivisions of G (c.f. Section 2.2.3). Their chromatic polynomials form what is known as a recursive family of polynomials, and we will apply a theorem of Beraha, Kahane, and Weiss [8] to describe the limits of their roots. We will see that the circle $|z-1| = 1$ plays a key role in describing the limits of chromatic roots of uniform subdivisions of a graph, and is itself among those limits (Figure 2.1 shows the roots of a subdivision (a theta graph with paths of length 10, 10 and 11) of $K_4 - e$ along with the circle $|z-1| = 1$).

We will derive two interesting consequences of our work here. Firstly, recall that in the previous section we determined the family of graphs with minimal co-rank

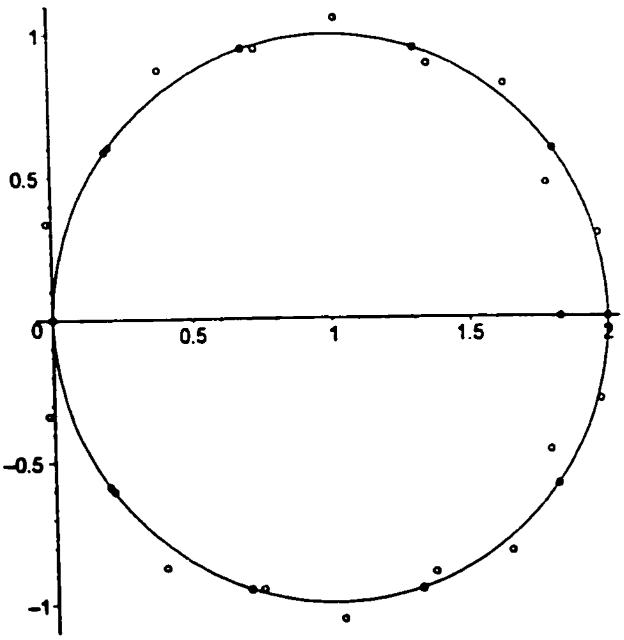


Figure 2.1: The chromatic roots of a subdivision of $K_4 - e$ with $|z-1| = 1$.

having chromatic roots with negative real part (some further graphs having chromatic roots with negative real part are provided in [50, 52, 53, 55]). A corollary to our expansion of chromatic polynomials of uniform subdivisions of graphs leads to a complete characterization of those graphs which have a subdivision having a chromatic root with negative real part.

Secondly, experimental evidence of chromatic roots of subdivisions leads to the observation that the roots tend to be drawn towards the unit circle centered at $z = 1$. We show that in fact for any $\varepsilon > 0$, the chromatic roots of *all* large subdivisions of a graph have their roots in $|z-1| < 1+\varepsilon$ (improving a recent result [20] which only proves *some* subdivision has its roots in this region).

2.2.1 Recursive Families of Polynomials

Before we proceed onto a discussion of the roots of chromatic polynomials of subdivisions of graphs, we need to state (in detail) an analytic result on particular families of polynomials (namely, *recursive familes*). We begin with the following definition.

Definition 2.2.1 *If $\{f_n(x)\}$ is a family of (complex) polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $\{f_n(x)\}$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathcal{R}(\{f_n(x)\})$, where $\mathcal{R}(\{f_n(x)\})$ is the union of the roots of the $f_n(x)$.*

Now (as in [8]) a family $\{f_n(x)\}$ of polynomials is a *recursive family of polynomials* if the $f_n(x)$ satisfy a homogenous linear recurrence

$$f_n(x) = \sum_{i=1}^{k} a_i(x) f_{n-i}(x), \tag{2.20}$$

where the $a_i(x)$ are fixed polynomials, with $a_k(x) \not\equiv 0$. The number k is the *order* of the recurrence.

We can form from (2.20) its associated *characteristic equation*

$$\lambda^k - a_1(x)\lambda^{k-1} - a_2(x)\lambda^{k-2} - \cdots - a_k(x) = 0, \tag{2.21}$$

whose roots $\lambda = \lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity (c.f. [1, 43]).

If these roots, say $\lambda_1(x), \lambda_2(x), \ldots, \lambda_k(x)$, are distinct, then the general solution to (2.20) is known [8] to be

$$f_n(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^n, \tag{2.22}$$

with the 'usual' variant (cf. [8]) if some of the $\lambda_i(x)$ were repeated. The functions $\alpha_1(x), \alpha_2(x), \ldots, \alpha_k(x)$ are determined from the initial conditions, that is, the k linear equations in the $\alpha_i(x)$ obtained by letting $n = 0, 1, \ldots k-1$ in (2.22) or its variant. The details are found in [8].

Beraha et al. [8] proved the result below on recursive families of polynomials and their roots.

Theorem 2.2.2 ([8]) *If $\{f_n(x)\}$ is a recursive family of polynomials, then a complex number z is a limit of roots of $\{f_n(x)\}$ if and only if there is a sequence $\{z_n\}$ in $\mathbb{C}$ such that $f_n(z_n) = 0$ for all n and $z_n \to z$ as $n \to \infty$.*

The main result of their paper characterizes precisely the limits of roots of a recursive family of polynomials.

Theorem 2.2.3 ([8]) *Under the non-degeneracy requirements that in (2.22) no $\alpha_i(x)$ is identically zero and that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus, then $z \in \mathbb{C}$ is a limit of roots of $\{f_n(x)\}$ if and only if either*

(i) *two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or*

(ii) *for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$ have, and $\alpha_j(z) = 0$.*

This result has found application to the chromatic roots of *recursive families of graphs* (cf. [11]), that is, families of graphs whose Tutte (and therefore chromatic) polynomials satisfy a homogeneous linear recurrence; see [9, 50] for some examples. It is also proved in [8] that the first non-degeneracy requirement in the statement of the theorem is equivalent to $f_n(x)$ satisfying no lower order (homogeneous, linear) recurrence. What we shall need here is the following.

Corollary 2.2.4 *Suppose $\{f_n(x)\}$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \cdots + \alpha_k(x)\lambda_k(x)^n \tag{2.23}$$

where the $\alpha_i(x)$ and $\lambda_i(x)$ are fixed nonzero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then the limits of roots of $\{f_n(x)\}$ are exactly those z satisfying (i) *or* (ii) *of Theorem 2.2.3.*

Proof It is enough to show that $f_n(x)$ satisfies a k-th order homogeneous linear recurrence, for then Theorem 2.2.3 applies as the non-degeneracy requirements are satisfied here. Such a recurrence is:

$$f_n(x) = a_1(x)f_{n-1}(x) + a_2(x)f_{n-2}(x) + \cdots + a_k(x)f_{n-k}(x) \quad (n \geq k)$$

together with the initial polynomials

$$f_j(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^j \quad (j = 0, \ldots, k-1),$$

where the $a_i(x)$ are such that

$$(\lambda - \lambda_1(x)) \cdots (\lambda - \lambda_k(x)) \equiv \lambda^k - a_1(x)\lambda^{k-1} - a_2(x)\lambda^{k-2} - \cdots - a_k(x).$$

This completes the proof. □

In the next section, we derive an expression for the chromatic polynomials of subdivisions of a graph, that when restricted to the uniform case, provides a recursive family of polynomials. The Beraha-Kahane-Weiss Theorem will then allow us to derive some precise information on the limit points of these chromatic roots.

2.2.2 An Expression for the Chromatic Polynomials of Subdivisions

Let us assume, for the remainder of the chapter, that G is a graph, *without loops*, which may indeed have parallel edges; its vertex and edge sets V and E have cardinalities n and m, respectively. Also, any parallel edges and/or loops resulting from the contraction of an edge at any time are not to be thrown away.

We derive now a rather technical expression of the chromatic polynomial of a general subdivision of a graph. What is crucial is the expansion of this polynomial in terms of powers of $1 - x$ and coefficients that depend only on the underlying graph (and *not* the exact subdivision we have taken).

Theorem 2.2.5 *Let $E' = \{e_1, \ldots e_k\} \subseteq E$, and $G^{e_1,\ldots,e_k}_{l_1,\ldots,l_k}$ the graph obtained from G by subdividing edge e_i into a path of length l_i ($i = 1, \ldots, k$). Then*

$$\begin{aligned}
\pi(G^{e_1,\ldots,e_k}_{l_1,\ldots,l_k}, x) &= \frac{(-1)^{\sum_{i=1}^k l_i}}{x^k}\Big\{\pi(G - E', x)(1-x)^{\sum_{i=1}^k l_i} \\
&- \sum_{1 \le i_1 < \cdots < i_{k-1} \le k} f_{i_1,\ldots,i_{k-1}}(x)(1-x)^{\sum_{j=1}^{k-1} l_{i_j}} \\
&+ \sum_{1 \le i_1 < \cdots < i_{k-2} \le k} f_{i_1,\ldots,i_{k-2}}(x)(1-x)^{\sum_{j=1}^{k-2} l_{i_j}} - \cdots \\
&+ (-1)^{k-1} \sum_{1 \le i_1 \le k} f_{i_1}(x)(1-x)^{l_{i_1}} + (-1)^k g_{E'}(x)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
g_{E'}(x) &= \pi(G, x) + (1-x)\sum_{i=1}^{k} \pi(G \bullet e_i, x) \\
&+ (1-x)^2 \sum_{1 \le i_1 < i_2 \le k} \pi(G \bullet e_{i_1} \bullet e_{i_2}, x) + \cdots \qquad (2.24)\\
&+ (1-x)^{k-1} \sum_{1 \le i_1 < \cdots < i_{k-1} \le k} \pi(G \bullet e_1 \bullet \cdots \bullet e_{k-1}, x) \\
&+ (1-x)^k \, \pi(G \bullet e_1 \bullet \cdots \bullet e_k, x)
\end{aligned}$$

and the f's (and clearly $g_{E'}$) are polynomials that depend on G and $E' = \{e_1, \ldots, e_k\}$, but <u>not</u> on $l_1, \ldots, l_k$.

This can be proved by induction on k. Let us examine the cases $k = 1$ and $k = 2$, the latter being sufficiently descriptive of the general argument which is tedious but no more difficult. Because of the degree of symbolism involved, it will be convenient, for the remainder of this section only, to denote the chromatic polynomial $\pi(H, x)$ of a graph H by the symbol H itself, and it will be clear from the context whether we are actually referring to the graph or its chromatic polynomial.

For $k = 1$, we are subdividing a single edge, e, of G into a path of length l, say. Tossing e into the graph G^e_l, and contracting, we have

$$G_l^e = (G_l^e + e) + (G_l^e + e) \bullet e.$$

Now $G_l^e + e$ creates a cycle of length $l+1$, intersecting G exactly on e, while $(G_l^e + e) \bullet e$ produces a cycle of length l which intersects $G \bullet e$ on a single vertex. Hence

$$\begin{aligned} G_l^e &= \frac{C_{l+1} \cdot G}{x(x-1)} + \frac{C_l \cdot G \bullet e}{x} \\ &= \frac{(-1)^{l+1}(1-x)((1-x)^l - 1)G}{x(x-1)} + \frac{(-1)^l(1-x)((1-x)^{l-1} - 1)G \bullet e}{x} \\ &= \frac{(-1)^l}{x}\left\{(G + G \bullet e)(1-x)^l - (G + (1-x)G \bullet e)\right\}, \end{aligned} \tag{2.25}$$

which establishes the result for $k = 1$.

Now for $k = 2$, we want an expression for the chromatic polynomial of $G_{l,s}^{e,f}$, the graph resulting from subdividing edges e and f of G into paths of length l and s, respectively. This we derive from the case $k = 1$ (more specifically, from (2.25)):

$$G_{l,s}^{e,f} = (G_l^e)_s^f = \frac{(-1)^s}{x}\left\{(G_l^e + G_l^e \bullet f)(1-x)^s - (G_l^e + (1-x)G_l^e \bullet f)\right\}.$$

It is clear that $G_l^e \bullet f = (G \bullet f)_l^e$. Thus, from (2.25),

$$\begin{aligned} G_l^e + G_l^e \bullet f &= \frac{(-1)^l}{x}\{(G + G \bullet e + G \bullet f + G \bullet f \bullet e)(1-x)^l \\ &\quad -(G + G \bullet f + (1-x)(G \bullet e + G \bullet f \bullet e))\}, \end{aligned}$$

and

$$\begin{aligned} G_l^e + (1-x)G_l^e \bullet f &= \frac{(-1)^l}{x}\{(G + G \bullet e + (1-x)(G \bullet f + G \bullet e))(1-x)^l \\ &\quad -(G + (1-x)(G \bullet e + G \bullet f) + (1-x)^2 G \bullet f \bullet e)\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} G^{e,f}_{l,s} &= \frac{(-1)^{s+l}}{x^2}\{(G+G\bullet e+G\bullet f+G\bullet f\bullet e)(1-x)^{s+l} \\ &\quad -(G+G\bullet f+(1-x)(G\bullet e+G\bullet f\bullet e))(1-x)^s \\ &\quad -(G+G\bullet e+(1-x)(G\bullet f+G\bullet e))(1-x)^l \\ &\quad +(G+(1-x)(G\bullet e+G\bullet f)+(1-x)^2G\bullet f\bullet e)\}\,. \end{aligned}$$

The 'coefficient' of $(1-x)^{s+l}$ above is exactly $G-e-f$, as $G+G\bullet e = G-e$ and $G\bullet f+G\bullet f\bullet e = (G\bullet f)-e$, and clearly $(G\bullet f)-e = (G-e)\bullet f$, giving

$$\begin{aligned} G+G\bullet e+G\bullet f+G\bullet f\bullet e &= (G-e)+(G-e)\bullet f \\ &= (G-e)-f. \end{aligned}$$

This establishes the case $k=2$ from the previous case ($k=1$). □

We will see that Theorem 2.2.5 has some deep consequences in terms of the location of chromatic roots, especially when restricted to the natural subfamily of uniform subdivisions.

2.2.3 Uniform Subdivisions and the Limits of their Chromatic Roots

When subdividing each edge of E' the same number of times, Theorem 2.2.5 specializes to the following.

Theorem 2.2.6 *Let $E' = \{e_1,\ldots e_k\} \subseteq E$, and $G^{E'}_l$ the graph obtained from G by subdividing each edge of E' into a path of length l. Then*

$$\begin{aligned} \pi(G^{E'}_l,x) &= \frac{(-1)^{kl}}{x^k}\{\pi(G-E',x)(1-x)^{kl}-f_{k-1}(x)(1-x)^{(k-1)l} \\ &\quad +f_{k-2}(x)(1-x)^{(k-2)l}-\cdots+(-1)^{k-1}f_1(x)(1-x)^l+(-1)^k g_{E'}(x)\}\,, \end{aligned} \tag{2.26}$$

where $g_{E'}$ is given by (2.24), and the f's (again) are polynomials that depend on G (and E') but not on l.

The key point is that expansion (2.26) expresses $\pi(G_l^{E'}, x)$ as a recursive family, and hence we can employ the power of the Beraha-Kahane-Weiss Theorem. In doing so, we find *all* of the limit points of the uniform subdivisions $\{G_l^{E'} : l \geq 1\}$ of G (without explicitly finding the chromatic roots of each of these graphs!).

We need yet one bit of technical notation; $\tilde{E}'$ will denote the edges of E' that are not bridges.

Theorem 2.2.7 *If E' is a subset of E containing at least one edge that is not a bridge of G, then the limits of the chromatic roots of the family $\{G_l^{E'}\}$ are exactly*

(i) *the circle $|z-1| = 1$,*

(ii) *the roots of $\pi(G - \tilde{E}', x)$ outside $|z-1| = 1$, and*

(iii) *the roots of $g_{\tilde{E}'}(x)$ inside $|z-1| = 1$.*

Proof Let us first examine the case where E' contains *no* bridges, in which case $\tilde{E}' = E'$. Clearly $\pi(G-E', x)$ is not identically zero. And neither is $g_{E'}(x)$, for suppose $g_{E'}(x) \equiv 0$. Then, from (2.26), we would have that $(1-x)^l$ divides $G_l^{E'}$. However, it is well known (c.f. [60]) that the multiplicity of 1 as a chromatic root of a graph is the number of blocks in the graph. Since E' contains no bridges, for each l the graph $G_l^{E'}$ has the same number of blocks as G, so that the multiplicity of 1 as a chromatic root of $G_l^{E'}$ cannot possibly go to infinity with l, a contradiction.

Now ignoring the factor $\frac{(-1)^{kl}}{x^k}$ in (2.26) and rewriting $(-1)^k g_{E'}(x)$ as $(-1)^k g_{E'}(x) 1^l$, we can apply Corollary 2.2.4, and therefore (i) and (ii) of Theorem 2.2.3, to get the limits of the chromatic roots of $\{G_l^{E'}\}$.

Applying (i) of Theorem 2.2.3, we immediately get the circle $|z-1| = 1$ as limits, by setting all of the $|\lambda_i(z)|$ equal, i.e.,

$$|1-z|^k = |1-z|^{k-1} = \cdots = |1-z| = 1.$$

This is the only situation where (i) of Theorem 2.2.3 gives any limits, for setting any fewer than all of the $|\lambda_i(z)|$ equal here will, upon applying (i), amount to finding values z such that $|z-1| = 1$ and $|z-1| > 1$, which is impossible.

Moving on to (ii) of Theorem 2.2.3, one application gives the roots z of $\pi(G-E', x)$ such that

$$|1-z|^k > |1-z|^i \text{ for all } i = 0, 1, \ldots, k-1,$$

that is, the roots z of $\pi(G-E', x)$ such that $|1-z| > 1$.

Another application of (ii) gives the roots z of $g_{E'}$ such that

$$1 > |1-z|^i \text{ for all } i = 1, 2, \ldots, k,$$

that is, the roots z of $g_{E'}$ such that $|1-z| < 1$.

Finally, applying (ii) to any j such that $0 < j < k$ would amount to finding roots z of f_j such that $|1-z| < 1$ and $|1-z| > 1$, which is clearly impossible.

Now in the case where E' does contain some bridges, subdividing a bridge $e \in E'$ is merely the replacement of a bridge by a path, from which it follows (quite easily, using the complete cutset theorem) that $\pi(G_l^{E'}, x) = (x-1)^{l-1}\pi(G_l^{E'-e}, x)$. Repeating this argument recursively over all the bridges in E', we have that the limits of the chromatic roots of $\{G_l^{E'}\}$ are exactly those of $\{G_l^{\widetilde{E'}}\}$, from which the result follows from the previous case. This completes the proof. □

When $E' = E$, it is clear that $G-\widetilde{E'}$ is a forest, therefore having no chromatic roots outside $|z-1| = 1$. Also, we write $\widetilde{g}(x)$ instead of $g_{\widetilde{E}}(x)$.

Corollary 2.2.8 *Let G_l be the graph obtained from G by subdividing each edge into a path of length exactly l. Then, if G is not a forest, the limits of the chromatic roots of the family G_l are exactly:*

(i) *the circle $|z-1| = 1$;*

(ii) *the roots of $\widetilde{g}(x)$ inside $|z-1| = 1$.*

Based on direct calculation for various small graphs, it appears that (ii) of Corollary 2.2.8 simply never happens, except of course for the point $z = 1$ which is clearly a root of $\widetilde{g}\,(x)$. In fact, we conjecture that if z is a root of $\widetilde{g}\,(x)$, then $|z-1|$ is either 0 or 1.

2.2.4 Application 1: Chromatic Roots with Negative Real Part

Our first application of our investigation of subdivisions concerns chromatic roots with negative real part.

Very little is known about the chromatic roots lying in the left-half plane. Earlier, we pointed out that it was conjectured [33] in 1980 that in fact there are none at all, and Read and Royle [50] showed recently by direct calculation with cubic graphs that they do exist. In section 2.1.2, we proved the existence of *infinitely* many chromatic roots with negative real part, and that there are graphs of arbitrarily high girth with them. More precisely, we showed the graph $\Theta_{a,a,a}$ has a chromatic root with negative real part for each $a \geq 8$, and that the moduli of these roots get arbitrarily small. (The existence of infinitely many chromatic roots with negative real part was demonstrated independently in [52, 55] by entirely different methods.)

Of course, generalized theta graphs are very specific graphs. But it turns out they are the key ingredient in characterizing the graphs in general which have a *subdivision* having a chromatic root with negative real part. It is from Theorem 2.2.7 that we are able to make this connection, and it is perhaps a bit surprising that most graphs have a subdivision having a chromatic root with negative real part. We say that G has a Θ–subgraph if G has a subgraph isomorphic to some generalized theta graph.

Theorem 2.2.9 *The following are equivalent.*

(i) *G has a subdivision having a chromatic root with negative real part;*

(ii) *G has a block of co-rank at least 2;*

(iii) *G has a Θ-subgraph.*

Proof We prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). It is easy to see that blocks of co-rank 0 are trees, and of co-rank 1 are cycles. Thus, if no block of G has co-rank larger than 1, then the blocks of G are simply bridges and cycles. Hence the blocks of any subdivision of G are K_2's and cycles as well. And since neither K_2 nor any cycle has a chromatic root with negative real part, neither do the subdivisions of G.

(ii) $\Rightarrow$ (iii). Without loss of generality, assume that G is 2-connected and has co-rank at least 2. Let e be an edge of G. Clearly e is not a bridge, and therefore lies on a cycle, say C_1, of G. Now G must have an edge, say e', other than those of C_1 (or else $G = C_1$ and G has co-rank 1). Since G is 2-connected, e and e' both belong to a cycle $C_2(\neq C_1)$ and $e \in C_1 \cap C_2$. Now, C_1 together with a component of $C_1 \oplus C_2$ (the symmetric difference) is a θ-subgraph of G, as required.

(iii) $\Rightarrow$ (i). From the complete cutset theorem, it is enough to show the result holds for 2-connected graphs. So we assume G is 2-connected. Start with a Θ-subgraph of G, and subdivide it into $\Theta_{a,a,a}$ for some fixed $a \geq 8$, obtaining a subdivision H of G containing as a subgraph $H_0 = \Theta_{a,a,a}$. Now let E' be the edges of H that do not lie on H_0. Then $H - E'$ consists exactly of H_0 and possibly some isolated vertices, a graph which we know (cf. section 2.1.2) has a chromatic root with negative real part. Thus if E' is empty, then we are done. And if not, then consider the family $\{H_l^{E'}\}$. Since E' has no bridges and $H - E'$ has a chromatic root z with negative real part, then by (ii) of Theorem 2.2.7, z will be a limit of the chromatic roots of $\{H_l^{E'}\}$. The $H_l^{E'}$'s are indeed subdivisions of G, and it follows that for l sufficiently large, $H_l^{E'}$ has a chromatic root with negative real part. □

From Theorem , we immediately deduce the following.

Corollary 2.2.10 *Every non-empty graph has a series–parallel extension having a chromatic root with negative real part.*

We can sometimes make use of subdivisions to generate, from a single chromatic root with negative real part, an infinite cluster of such chromatic roots. For suppose e is an edge of G which is not a bridge, and that $G-e$ has a chromatic root z with negative real part. Then, from (ii) of Theorem 2.2.7, z is a limit of the chromatic roots of the family $\{G_l^e\}$. If, in addition, z is *not* a chromatic root of $G \bullet e$, then, from (2.25), z will not be a chromatic root of any G_l^e, and so in fact must be a *limit point* of the chromatic roots of the family $\{G_l^e\}$. Together with Theorem 2.2.2, we find a sequence $\{z_l\}$ in $\mathbb{C} \setminus \{z\}$ such that $\pi(G_l^e, z_l) = 0$ and $z_l \to z$.

Fix any $a \geq 8$, for instance, and let $F = \Theta_{a,a,a}+uv$, where u and v are the terminals of $\Theta_{a,a,a}$. Then $F - uv = \Theta_{a,a,a}$, which we know (from Section 2.1.2) has a chromatic root z with negative real part, while $F \bullet uv$ is just three cycles intersecting on a single vertex, a graph having no chromatic roots with negative real part whatsoever, as the chromatic roots of cycles lie on the disk $|z-1| = 1$. Thus $\{F_l^{uv}\}$ is a family of graphs producing infinitely many chromatic roots with negative real part.

2.2.5 Application 2: Bounding the Chromatic Roots of Large Subdivisions

Our second application of our investigation of chromatic roots of subdivisions centers on the location of the roots of *large* subdivisions of a graph. A few plots (see, e.g., Figure 2.1) of the chromatic roots of subdivisions of several small graphs will indicate that subdividing edges tends to draw the chromatic roots closer to the disk $|z-1| \leq 1$. It was shown in [21] that *co-rank* is an upper bound for $|z-1|$ where z is any chromatic root of the graph. The roots of any subdivision of G, therefore, are bounded by 1 plus the co-rank of G. However, more is true: in [20] it was proven that for any $\varepsilon > 0$, there is *some* subdivision of G having all its chromatic roots in $|z-1| < 1+\varepsilon$. This belies the empirical evidence that *all* large subdivisions have their chromatic roots in the salient disc. Our results here are indeed strong enough to prove this fact.

Theorem 2.2.11 *For any $\varepsilon > 0$, there is an $L = L(G, \varepsilon)$ such that, if we subdivide each edge of G into a path of length at least L, then all chromatic roots of the resulting graph G' lie in $|z-1| < 1+\varepsilon$.*

Proof Let $\varepsilon > 0$ be given. Suppose $E = \{e_1, \ldots, e_m\}$ are the edges of G, and that we subdivide edge e_i into a path of length l_i, $i = 1, \ldots, m$. We obtain a graph $G^{e_1,\ldots,e_m}_{l_1,\ldots,l_m}$, whose chromatic polynomial, by Theorem 2.2.5, is $\pi(G^{e_1,\ldots,e_m}_{l_1,\ldots,l_m}, x) = \frac{(-1)^{\sum_{i=1}^{k} l_i}}{x^m}\mathcal{F}_{l_1,\ldots,l_m}(x)$, where

$$\begin{aligned}
\mathcal{F}_{l_1,\ldots,l_m}(x) &= \pi(G-E,x)(1-x)^{\sum_{i=1}^{m} l_i} \\
&- \sum_{1\le i_1<\cdots<i_{m-1}\le k} f_{i_1,\ldots,i_{m-1}}(x)(1-x)^{\sum_{j=1}^{m-1} l_{i_j}} \\
&+ \sum_{1\le i_1<\cdots<i_{m-2}\le k} f_{i_1,\ldots,i_{m-2}}(x)(1-x)^{\sum_{j=1}^{m-2} l_{i_j}} - \cdots \\
&+ (-1)^{m-1}\sum_{1\le i_1\le m} f_{i_1}(x)(1-x)^{l_{i_1}} + (-1)^m g_E(x)
\end{aligned}$$

is the expression (in braces) in Theorem 2.2.5 (with k replaced by m).

As we remarked earlier, the roots of $\pi(G^{e_1,\ldots,e_m}_{l_1,\ldots,l_m}, x)$, and therefore of $\mathcal{F}_{l_1,\ldots,l_m}$, are bounded by the co-rank μ of G. Let $C = C(G, \varepsilon) > 0$ be a bound for the maximum modulus of the f's and g_E on $1+\varepsilon \le |1-z| \le \mu$. Choose $L > 0$ large enough that $\frac{\varepsilon^n(1+\varepsilon)^L}{C} > 2^m - 1$. Suppose that $l_1, \ldots, l_m$ are all larger than L, and, without loss of generality, $l_1 \le l_2 \le \cdots \le l_m$. Let z be such that $1+\varepsilon \le |1-z| \le \mu$; we will show that $|\mathcal{F}_{l_1,\ldots,l_m}(z)| > 0$.

To that end, note that $\pi(G-E, z) = z^n$, whose modulus is at least ε^n. Set $y = |1-z|$. Then, by the triangle inequality,

$$|\mathcal{F}_{l_1,\ldots,l_m}(z)| \ \ge \ |z|^n y^{\sum_{i=1}^{m} l_i} - \sum_{1\le i_1<\cdots<i_{m-1}\le m} |f_{i_1,\ldots,i_{k-1}}(z)|\, y^{\sum_{j=1}^{m-1} l_{i_j}}$$

$$
\begin{aligned}
& - \sum_{1\le i_1<\cdots<i_{m-2}\le m} |f_{i_1,\ldots,i_{k-2}}(z)|\, y^{\sum_{j=1}^{m-2} l_{i_j}} - \cdots \\
& - \sum_{1\le i_1<i_2\le m} |f_{i_1,i_2}(z)|\, y^{l_{i_1}+l_{i_2}} - \sum_{1\le i_1<m} |f_{i_1}(z)|\, y^{l_{i_1}} - |g_E(z)| \\
\ge\ & \varepsilon^n y^{\sum_{i=1}^m l_i} - \sum_{1\le i_1<\cdots<i_{m-1}\le m} C\, y^{\sum_{j=1}^{m-1} l_{i_j}} \\
& - \sum_{1\le i_1<\cdots<i_{m-2}\le m} C\, y^{\sum_{j=1}^{m-2} l_{i_j}} - \cdots - \sum_{1\le i_1<i_2\le m} C\, y^{l_{i_1}+l_{i_2}} \\
& - \sum_{1\le i_1<m} C\, y^{l_{i_1}} - C.
\end{aligned}
$$

On the right hand side of the above there are exactly $2^m - 1$ terms in y (without combining any terms of like degree). We rewrite the expression as

$$
C \cdot \left(\frac{\varepsilon^n}{C} y^{n_1} - y^{n_2} - \cdots - y^{n_p} - 1 \right),
$$

where $p = 2^m - 1$ and $n_1 \ge n_2 \ge \cdots \ge n_p$. In particular, $n_1 = \sum_{i=1}^m l_i$ and $n_2 = \sum_{i=2}^m l_i$. Then

$$
\begin{aligned}
\frac{|\mathcal{F}_{l_1,\ldots,l_m}(z)|}{C} &\ge \frac{\varepsilon^n}{C} y^{n_1} - y^{n_2} - \cdots - y^{n_p} - 1 \\
&= y^{n_p} \left(\frac{\varepsilon^n}{C} y^{n_1-n_p} - y^{n_2-n_p} - \cdots - y^{n_{p-1}-n_p} - 1 \right) - 1 \\
&= y^{n_p}(y^{n_{p-1}-n_p}(\frac{\varepsilon^n}{C} y^{n_1-n_{p-1}} - y^{n_2-n_{p-1}} - \cdots \\
&\quad - y^{n_{p-2}-n_{p-1}} - 1) - 1) - 1 \\
&\ \vdots \\
&= y^{n_p}(y^{n_{p-1}-n_p}(y^{n_{p-2}-n_{p-1}} \cdots y^{n_3-n_4}(y^{n_2-n_3}(\frac{\varepsilon^n}{C} y^{n_1-n_2} - 1) \\
&\quad -1) \cdots - 1) - 1) - 1.
\end{aligned}
$$

Now $n_1 - n_2 = l_1 \ge L$ and $y \ge 1 + \varepsilon > 1$. Hence

$$
\frac{|\mathcal{F}_{l_1,\ldots,l_m}(z)|}{C} > \frac{\varepsilon^n}{C}(1+\varepsilon)^L \underbrace{-1-1\cdots-1-1-1}_{p \text{ such}}
$$

$$= \underbrace{\frac{\varepsilon^n}{C}(1+\varepsilon)^L}_{>p} - p$$
$$> 0,$$

which completes the proof. □

Direct calculations with generalized theta graphs suggests that this may be only half of the story. More specifically, we conjecture that the region $|z-1| < 1+\varepsilon$ in Theorem 2.2.11 can be replaced by $\{z \in \mathbb{C} : 1-\varepsilon < |z-1| < 1+\varepsilon\} \cup \{1\}$.

Returning to the proof of the theorem, all we really needed of $G - E$ was the fact that it has no roots on $1 < |z-1| < \mu$, for then we knew it is bounded away from zero on $1+\varepsilon \leq |z-1| \leq \mu$ for any fixed $\varepsilon > 0$. So in fact any subset E' of E for which every root of $G - E'$ lies in $|z-1| \leq 1$ will do. Conversely, if E' is a subset of edges such that $G - E'$ has a root on $|z-1| > 1$, then by (ii) of Theorem 2.2.7 and the remark immediately following the theorem, there will certainly be an $\varepsilon > 0$ and an l such that $G_l^{E'}$ has a root on $|z-1| \geq 1+\varepsilon$. Hence, in the statement of Theorem 2.2.11, we can replace G by E' if and only if the graph $G - E'$ has all its chromatic roots in $|z-1| \leq 1$.

Chapter 3

Roots of Independence Polynomials

For a graph G with independence number β, let i_k denote the number of independent sets of vertices of cardinality k in G ($k = 0, 1, \ldots, \beta$). Several papers exist (cf. [2, 23, 34, 36, 41]) on the independence sequence $(i_1, i_2, \ldots i_\beta)$ of a graph (or its complement), exploring various such problems. The *independence polynomial* of G,

$$i_G(x) = \sum_{k=0}^{\beta} i_k x^k,$$

is the generating polynomial for the sequence. The path P_4 on 4 vertices, for example, has one independent set of cardinality 0 (the empty set), four independent sets of cardinality 1, and three independent sets of cardinality 2; its independence polynomial is then $i_{P_4}(x) = 1 + 4x + 3x^2$.

As is the case with other graph polynomials, such as chromatic polynomials (cf. [25, 51]), matching polynomials ([38, 39]), and others, it is natural to consider the nature and location of the roots. Interesting in their own right, they can shed some light on the underlying combinatorics as well. Newton (cf. [31]), for example, showed that if a polynomial $f(x) = \sum_{k=0}^{\beta} a_k x^k$ with positive coefficients has all real roots, then the sequence $a_0, a_1, \ldots, a_\beta$ is log-concave (i.e., $a_k^2 \leq a_{k-1}a_{k+1}$ for all k), and hence unimodal (for some k, $a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$). Unimodality conjectures permeate combinatorics (see, e.g., [56]), and it was conjectured in [23] that the independence vector $(i_0, i_1, \ldots, i_\beta)$ of any *well covered* graph (cf. Section 3.1.1 of this thesis for the definition) is unimodal, and some partial results in that regard have been obtained via the roots of independence polynomials (cf. [23]). Further results on independence polynomials and their roots can be found in [23, 35, 40, 42].

One line of research in *chromatic* roots has been determining the topological closures of both the real and complex roots of the set of all chromatic polynomials. It was shown between the works of Jackson [45] and Thomassen [57] that the closure of the set of real roots of chromatic polynomials is $\{0\} \cup \{1\} \cup [32/27, \infty)$. Recent work of Sokal (cf. [55]) shows that chromatic roots in general are dense in the entire complex plane; even when restricted to planar graphs, the complex roots are dense in $\mathbb{C}$, except possibly in the disk $|z-1| \leq 1$. Our first order of business in this chapter will be to answer these same questions for roots of *independence* polynomials.

Much of our work stems from the following key result. For two graphs G and H, let $G[H]$ be the graph with vertex set $V(G) \times V(H)$ and such that vertex (a, x) is adjacent to vertex (b, y) if and only if a is adjacent to b (in G) or $a = b$ and x is adjacent to y (in H). The graph $G[H]$ is the *lexicographic product* (or *composition*) of G and H, and can be thought of as the graph arising from G and H by substituting a copy of H for every vertex of G.

Theorem 3.0.12 *Let G and H be graphs. Then the independence polynomial of $G[H]$ is*

$$i_{G[H]}(x) = i_G(i_H(x) - 1). \tag{3.1}$$

<u>Proof</u> By definition, the polynomial $i_G(i_H(x) - 1)$ is given by

$$\sum_{k=0}^{\beta_G} i_k^G \left(\sum_{j=1}^{\beta_H} i_j^H x^j \right)^k, \tag{3.2}$$

where i_k^G is the number of independent sets of cardinality k in G (similarly for i_k^H).

Now, an independent set in $G[H]$ of cardinality l arises by choosing an independent set in G of cardinality k, for some $k \in \{0, 1, \ldots l\}$, and then, within each associated copy of H in $G[H]$, choosing a nonempty independent set in H, in such a way that the total number of vertices chosen is l. But the number of ways of actually doing this is exactly the coefficient of x^l in (3.2), which completes the proof. □

By applying (3.1) to the right families of graphs, we will be able to determine the closures of real and complex independence roots. We will then move on to consider the special case of higher compositions of a graph with itself, asking for where the independence roots are approaching. Since the polynomials involved are essentially higher compositions of $i_G(x) - 1$ with itself, it may not be too surprising to learn that the generic case is convergence to a fractal.

We shall have occasion to make use of an easy recursive formula for calculating independence polynomials.

Proposition 3.0.13 (**[23, 44]**) *For any vertex v of a graph G,*

$$i_G(x) = i_{G-v}(x) + x \cdot i_{G-[v]}(x).$$

where $[v]$, the closed neighbourhood of v, consists of v, together with all vertices incident with v.

Proof For $k \geq 1$, an independent set of k vertices in G either contains v or does not. There are $i_{k-1}^{G-[v]}$ that do, and i_k^{G-v} that do not. Thus, for each $k \geq 1$, the coefficient of x^k is the same in both sides of the above equation; and both sides clearly have constant term 1. The two polynomials are therefore equal. □

Another useful tool is:

Proposition 3.0.14 *For graphs G and H, the independence polynomial of their disjoint union $G \uplus H$ is*

$$i_{G \uplus H}(x) = i_G(x) \cdot i_H(x).$$

Proof For $k \geq 0$, an independent set of k vertices in $G \uplus H$ arises by choosing an independent set of j vertices in G (for some $j \in \{0, 1, \ldots, k\}$) and then an independent set of $k - j$ vertices in H. The number of ways of doing this over all $j = 0, 1, \ldots k$ is exactly the coefficient of x^k in $i_G(x) \cdot i_H(x)$. Hence, since both sides of the above equation have the same coefficients, they are identical polynomials. □

3.1 Location of Independence Roots of some Families of Graphs

As advertised, we shall now find the topological closures of real and complex independence roots. As the coefficients of any independence polynomial are positive all the way down to the constant term, it is clear that no real independence root is nonnegative. Nevertheless, we have:

Theorem 3.1.1 *Complex roots of independence polynomials are dense in all of* $\mathbb{C}$, *while real independence roots are dense in* $(-\infty, 0]$.

We will prove the theorem by considering very specific families of graphs, taking their lexicographic product, and examining the roots of the independence polynomials which arise. The upshot will be the truth of the Theorem 3.1.1 even for some very restricted families of graphs, namely well covered and comparability graphs.

3.1.1 Well Covered Graphs

A graph is *well covered* if every maximal set of independent vertices has the same cardinality. The graph C_4, for instance, is well covered with independence number 2, while C_6, a graph with independence number 3, is not well covered, since it contains maximal independent subsets of cardinality 2. Well covered graphs have attracted considerable attention; see [49] for an extensive survey.

Well covered graphs are closed under lexicographic product.

Proposition 3.1.2 *If G and H are well covered, then $G[H]$ is also well covered.*

Proof Any independent set in $G[H]$ arises by choosing an independent set W in G, and for each vertex in W, an independent set of vertices X in H. Every maximal independent set in $G[H]$ will therefore have cardinality $\beta_G\beta_H$, and so $G[H]$ is well covered. □

Denote by $[1,\beta]$ the set $\{1,2,\ldots,\beta\}$, and (as in [23]) L_β^k (where k is a positive integer) the graph with vertex set $[1,\beta]^k$, in which two k-tuples are joined by an edge if and only if they agree in a coordinate.

Proposition 3.1.3 *For $\beta, k \geq 2$, the graph L_β^k is well covered with independence number β.*

Proof Let $\{v_1, v_2, \ldots, v_s\}$ be an independent set of $s < \beta$ vertices. Then, for each $i = 1,2,\ldots,k$ there is a number $x^{(i)} \in [1,\beta]$ which is not equal to the i-th coordinate of any v_j $(j = 1,2,\ldots,s)$. Thus, the vertex $v_{j+1} = (x^{(1)}, x^{(2)}, \ldots, x^{(k)})$ is not adjacent to any v_j $(j = 1,2,\ldots,s)$, and so $\{v_1, v_2, \ldots v_{s+1}\}$ is not a maximal independent set. Hence, any maximal independent set has cardinality at least β.

On the other hand, let $\{v_1, v_2, \ldots v_s\}$ be a set of $s > \beta$ vertices. Then two of them must agree in the first coordinate, by the Pigeonhole Principle. Therefore, no independent set has cardinality greater than β. This completes the proof. □

The graphs L_β^k were considered in [23], where the following was established.

Theorem 3.1.4 ([23]) *If $2^{k-1} \geq \beta \geq 1$ then the smallest zero $y_\beta^{(k)}$ of $i_{L_\beta^k}(x)$ lies in the interval*

$$-\beta < y_\beta^{(k)} < -\beta(1-2^{-k}).$$

By taking the lexicographic product of the L_β^k with complete graphs, we find below that among the independence roots which arise are real roots that are dense in

$(-\infty, 0]$. Complete graphs are obviously well-covered (with independence number 1), and $i_{K_n}(x) = 1 + nx$. Proposition 3.1.2 then implies that $L^k_\beta[K_n]$ is well-covered, and, by equation (3.1), $i_{L^k_\beta[K_n]}(x) = i_{L^k_\beta}(nx)$.

Theorem 3.1.5 *The real independence roots of the family* $\{L^k_\beta[K_n]\}$ *are dense in* $(-\infty, 0]$.

Proof Note that $i_{L^k_\beta[K_n]}(x) = iL^k_\beta(1 + nx - 1) = i_{L^k_\beta}(nx)$. Let $s \in (-\infty, 0]$ and $\varepsilon > 0$ be given. Begin by choosing a positive integer n large enough that the interval $n \cdot (s - \varepsilon, s + \varepsilon) = (ns - n\varepsilon, ns + n\varepsilon)$ contains some integer $\beta \leq -2$. Next, from Theorem 3.1.4, we can choose a k such that $i_{L^k_\beta}(x)$ has a root r in that interval. Then $r/n \in (s - \varepsilon, s + \varepsilon)$, and

$$i_{L^k_\beta[K_n]}(\frac{r}{n}) = i_{L^k_\beta}(n \cdot \frac{r}{n}) = i_{L^k_\beta}(r) = 0.$$

completing the proof. □

It turns out that complex independence roots of well covered graphs are dense in all of $\mathbb{C}$. To prove this, we will compose the graphs $L^k_\beta[K_{n_1}]$ with empty graphs $\overline{K_{n_2}}$. The empty graph $\overline{K_n}$ is obviously well covered (with independence number n), and (from Proposition 3.0.14) $i_{\overline{K_n}}(x) = (1 + x)^n$.

Theorem 3.1.6 *The independence roots of the family* $L^k_\beta[K_{n_1}][\overline{K_{n_2}}]$ *are dense in* $\mathbb{C}$.

Proof Set R = {real roots of the family $L^k_\beta[K_{n_1}][\overline{K_{n_2}}]$}; from Theorem 3.1.5 we know that $\overline{R} = (-\infty, 0]$. Let $z \in \mathbb{C}$ and $\varepsilon > 0$. We will show that there exist positive integers β, k, n_1, n_2 such that $i_{L^k_\beta[K_{n_1}][\overline{K_{n_2}}]}(\tilde{z}) = 0$ for some $\tilde{z}$ within an ε-radius of z. From Proposition 3.1, we have

$$i_{L^k_\beta[K_{n_1}][\overline{K_{n_2}}]}(x) = i_{L^k_\beta[K_{n_1}]}(i_{\overline{K_{n_2}}}(x) - 1) = iL^k_\beta[K_{n_1}]((1 + x)^{n_2} - 1).$$

We may assume $z \neq -1$; thus $|z + 1| > 0$. Choose n_2 large enough that some n_2-th root of $-|z + 1|^{n_2}$, say $w = |z + 1|e^{\frac{i(2k+1)\pi}{n_2}}$, is within an $\frac{\varepsilon}{2}$-radius of $z + 1$.

Choose $\delta > 0$ such that $\delta < \frac{\varepsilon}{2}$ and $r = -(|z+1|+\delta)^{n_2} - 1 \in R$ (by continuity, such a δ exists). Then the corresponding n_2-th root of $r + 1 = -(|z+1|+\delta)^{n_2}$, namely $\tilde{w} = (|z+1|+\delta)e^{\frac{i(2k+1)\pi}{n_2}}$, is within an ε-radius of $z+1$, as

$$\begin{aligned} |\tilde{w} - (z+1)| &= |(\tilde{w} - w) + (w - (z+1))| \\ &\leq |\tilde{w} - w| + |w - (z+1)| \\ &< \delta + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Finally, since $r \in R$, there are numbers β, k, n_1 such that $i_{L_\beta^k[K_{n_1}]}(r) = 0$. Set $\tilde{z} = \tilde{w} - 1$. Then

$$|\tilde{z} - z| = |(\tilde{w} - 1) - z| = |\tilde{w} - (z+1)| < \varepsilon,$$

and

$$\begin{aligned} i_{L_\beta^k[K_{n_1}][\overline{K_{n_2}}]}(\tilde{z}) &= i_{L_\beta^k[K_{n_1}]}((1+\tilde{z})^{n_2} - 1) \\ &= i_{L_\beta^k[K_{n_1}]}(\tilde{w}^{n_2} - 1) \\ &= i_{L_\beta^k[K_{n_1}]}((r+1) - 1) \\ &= i_{L_\beta^k[K_{n_1}]}(r) \\ &= 0, \end{aligned}$$

completing the proof. □

Theorems 3.1.5 and 3.1.6 imply that Theorem 3.1.1 is true even when restricted to well covered graphs.

3.1.2 Comparability Graphs

A simple graph G is a *comparability graph* if it has a *transitive orientation*, that is, an orientation of its edges such that if $x \to y$ and $y \to z$ then $x \to z$. Comparability graphs are also closed under graph composition.

Proposition 3.1.7 *If G and H are comparability graphs, then $G[H]$ is also a comparability graph.*

Proof Orient the graph $G[H]$ by $(a,x) < (b,y)$ if and only if $a < b$ (in G) or $a = b$ and $x < y$ (in H). This is a transitive orientation of $G[H]$. For suppose $(a,x) < (b,y)$ and $(b,y) < (c,z)$. If $a = b = c$, then $x < y$ and $y < z$, and so $x < z$ by transitivity of H, whence $(a,x) < (c,z)$. If instead $a = b < c$ or $a < b = c$ or $a < b < c$, then $a < c$ by transitivity of G, and so $(a,x) < (c,z)$. This completes the proof, as there are no other possibilities. □

Contained in the collection of comparability graphs are paths, complete graphs, and empty graphs.

Proposition 3.1.8 *Paths, complete graphs, and empty graphs are all comparability graphs.*

Proof Going from left to right through P_n, the path on n vertices, orient the first edge forward, the second backward, third forward, and so on. This gives a transitive orientation (trivially), and therefore P_n is a comparability graph.

To see that K_n is a comparability graph, simply label its vertices $1, 2, \ldots, n$, and orient edge ij as $i \rightarrow j$ if and only if $i < j$. The transitivity of this orientation follows that of $<$ on $\mathbb{R}$.

Finally, $\overline{K_n}$ is trivially a comparability graph. □

Together with Proposition 3.1.7, this implies:

Corollary 3.1.9 *The graphs $P_{n_1}[K_{n_2}]$ and $P_{n_1}[K_{n_2}][\overline{K_{n_3}}]$ are comparability graphs.*

Analogous to what we did for well covered graphs, we will show that the family $\{P_{n_1}[K_{n_2}]\}$ has real independence roots which are dense in $(-\infty, 0]$, while the complex independence roots of the family $\{P_{n_1}[K_{n_2}][\overline{K_{n_3}}]\}$ are dense in all of $\mathbb{C}$.

We start with paths, themselves. The graph P_5, for example, has independence polynomial $i_{P_5}(x) = 1 + 5x + 6x^2 + x^3$. Its roots are shown in Figure 3.1; they are all real. We can say much more:

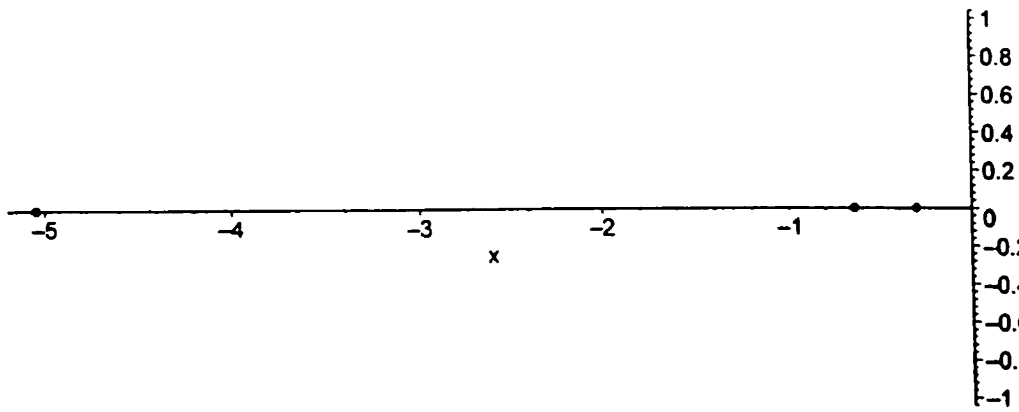


Figure 3.1: The independence roots of P_5.

Theorem 3.1.10 *The independence roots of the family* $\{P_n\}$ *are real and dense in* $(-\infty, -\frac{1}{4}]$.

Proof Since P_n is the line graph of P_{n+1}, $M(P_{n+1}, x) = x^n i_{P_n}(-1/x^2)$, the former being the *matching polynomial* of P_{n+1}, and matching polynomials are known [42] to have only real roots. It follows that $i_{P_n}(x)$ has only real roots as well. The reduction in Proposition 3.0.13 for calculating independence polynomials gives

$$i_{P_n}(x) = i_{P_{n-1}}(x) + x \cdot i_{P_{n-2}}(x) \quad (l \geq 3), \tag{3.3}$$

and so the family $\{i_{P_n}(x)\}$ is recursive; the initial conditions are $i_{P_1}(x) = 1 + x$ and $i_{P_2}(x) = 1 + 2x$. Solving, we find

$$i_{P_n}(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n,$$

where

$$\lambda_1(x), \lambda_2(x) = \frac{1 \pm \sqrt{1+4x}}{2}$$

and

$$\alpha_1(x), \alpha_2(x) = \frac{\sqrt{1+4x} \pm (1+2x)}{2\sqrt{1+4x}}.$$

The non-degeneracy conditions of the Beraha-Kahane-Weiss theorem (Theorem 2.2.3) are therefore satisfied, and part (i) of that theorem implies that among the limits of roots are those z for which

$$|\lambda_1(z)| = |\lambda_2(z)|,$$

which simplifies to

$$|1 + \sqrt{1+4z}| = |1 - \sqrt{1+4z}|,$$

implying that $\sqrt{1+4z}$ is purely imaginary. Thus $1 + 4z \leq 0$, i.e., $z \leq -1/4$, which is what we wanted to show. □

By composing with complete graphs, we can fill up the rest of the negative real axis.

Theorem 3.1.11 *The independence roots of the family $P_{n_1}[K_{n_2}]$ are real and dense in $(-\infty, 0]$.*

Proof From the previous theorem, independence roots of the graphs $P_{n_1}[K_1] = P_{n_1}$ are real and dense in $(-\infty, -1/4]$. Now let $s \in (-1/4, 0)$ and $\varepsilon > 0$ be given. Then there are positive integers n_1 and n_2 for which $i_{P_{n_1}[K_{n_2}]}(x) = i_{P_{n_1}}(n_2 x)$ has a root in $(s - \varepsilon, s + \varepsilon)$. For choose n_2 large enough that $n_2 s \leq -1/4$. Then, from the previous theorem, we can choose a number n_1 such that $i_{P_{n_1}}(x)$ has a root $r \in n_2 \cdot (s - \varepsilon, s + \varepsilon)$. But then $\frac{1}{n_2} \cdot r \in (s - \varepsilon, s + \varepsilon)$ and $i_{P_{n_1}[K_{n_2}]}(\frac{1}{n_2} \cdot r) = i_{P_{n_1}}(n_2 \cdot \frac{r}{n_2}) = i_{P_{n_1}}(r) = 0$, completing the proof. □

Compositions with empty graphs will then fill up the complex plane.

Theorem 3.1.12 *The independence roots of the graphs $P_{n_1}[K_{n_2}][\overline{K_{n_3}}]$ are dense in $\mathbb{C}$.*

Proof The proof is essentially the same as that of Theorem 3.1.6: starting with *any* collection of graphs whose independence roots are dense in $(-\infty, 0]$, compositions with empty graphs yield independence roots which are dense in $\mathbb{C}$. □

Hence, Theorem 3.1.1 remains true when we restrict to comparability graphs.

3.1.3 On Further Classes of Graphs

It may be of interest to study the independence roots of yet further classes of graphs. Some common ones include chordal, interval, claw-free, and line graphs:

- A simple graph G is a *chordal graph* if every cycle in G of length at least 4 has a chord.
- Given a family of intervals, we can define a graph whose vertices are the intervals, with vertices adjacent when the intervals intersect. A graph formed in this way is an *interval graph*, and the family of intervals is an *interval representation* of the graph.
- A simple graph G is *claw-free* if does not contain $K_{1,3}$ as an induced subgraph.
- The *line graph* of a graph H, is the graph $G = L(H)$ whose vertices are the edges of G, with $ef \in E(L(G))$ when e and j share a vertex.

It can be shown (cf. [59]) that interval graphs are chordal, and line graphs are claw-free. The reader can verify that the graphs $P_{n_1}[K_{n_2}]$ are chordal, interval, and claw-free, while graphs $P_{n_1}[K_{n_2}][K_{n_3}]$ $(n_3 \geq 2)$ are neither; paths P_n are line graphs, while graphs $P_{n_1}[K_{n_2}]$ are not. We summarize our results in the table below:

Class	Real Roots		Complex Roots	
	Limits	Generating Family	Limits	Gen. Family
Well Covered	$(-\infty, 0]$	$L^k_\beta[K_n]$	$\mathbb{C}$	$L^k_\beta[K_{n_1}][\overline{K}_{n_2}]$
Comparability	$(-\infty, 0]$	$P_{n_1}[K_{n_2}]$	$\mathbb{C}$	$P_{n_1}[K_{n_2}][\overline{K}_{n_3}]$
Chordal	$(-\infty, 0]$	$P_{n_1}[K_{n_2}]$	*	
Interval	$(-\infty, 0]$	$P_{n_1}[K_{n_2}]$	*	
Claw-Free	$(-\infty, 0]$	$P_{n_1}[K_{n_2}]$	*	
Line	$(-\infty, -\frac{1}{4}]$*	P_n	$(-\infty, -\frac{1}{4}]$*	P_n

* further investigation necessary

3.2 The Independence Attractor of a Graph

Since lexicographic product is associative, we may speak of *powers* $G^k = \underbrace{G[G[G \cdots]]}_{k}$ of a graph G without ambiguity ($G^1 = G$). For $G = P_3$ (a path on three vertices) the independence roots of G^7 are shown in Figure 3.2. It appears that the independence roots of G^k are approaching a fractal-like object as $k \to \infty$. We are lead to ask:

Question 3.2.1 *For a graph G, what happens to the roots of the independence polynomials $i_{G^k}(x)$ as $k \to \infty$?*

In section 3.2.2 we are able to describe precisely where the roots are approaching, and in what sense they do so. We then ask the question of when these limiting sets are connected. In section 3.2.3, we provide a complete answer for graphs with independence number 2. We then answer the question for some infinite families of graphs having arbitrarily high independence numbers, in section 3.2.4.

Some familiarity with iteration theory will precede any reasonable understanding of this material. With that in mind, we collect in the next section the relevant notation, terminology and results from the field, along with some references. Incidentally, while Theorem 3.2.10 will have most direct application for us, it cannot (as far as the author is aware) be found explicitly in the literature.

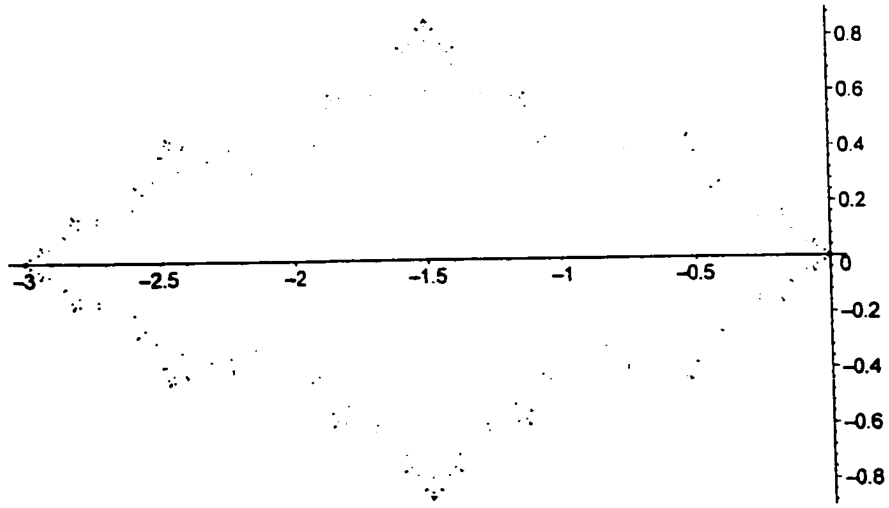


Figure 3.2: The independence roots of G^7, where $G = P_3$.

3.2.1 Julia Sets and the Iteration of Rational Functions

Except where otherwise stated, any definition or assertion made in this section can be found in Beardon's book [7]. Much of the information can also be found among the works of Blanchard [15] and Brolin [19].

In section 1.2.3, we introduced the space $(\mathbb{C}_\infty, \sigma_0)$ – the extended complex plane $\mathbb{C}_\infty$, with the spherical metric σ_0. A rational map on $\mathbb{C}_\infty$ is a function $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials; its *degree* is $\max\{\deg p(z), \deg q(z)\}$. It is well known that each rational map is analytic throughout $\mathbb{C}_\infty$ (neither ∞ nor any poles of f are singularities).

Rational maps of the form

$$\phi(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

are called *Möbius maps*; the condition $ad - bc \neq 0$ ensures that ϕ is one-to-one and thus invertible. Two rational maps f and g are *conjugate* if there is some Möbius map ϕ such that

$$g = \phi \circ f \circ \phi^{\circ(-1)}$$

where $\phi^{\circ(-1)}$ is the inverse of ϕ. It is then easily verified that, for any k,

$$g^{\circ k} = \phi \circ f^{\circ k} \circ \phi^{\circ(-1)},$$

an important property of conjugacy. By $f^{\circ k}$, where $k \geq 1$, we mean $\underbrace{f \circ f \circ \cdots \circ f}_{k}$ if $k \geq 1$. Also, define $f^{\circ(0)}$ to be the identity map.

Associated with each rational map is a set (most often, a fractal) known as its Julia set.

Definition 3.2.2 *A family $\mathcal{F}$ of maps of $\mathbb{C}_\infty$ into itself is equicontinuous at z_0 if and only if for every ε there exists a positive δ such that for all $z \in \mathbb{C}_\infty$, and for all f in $\mathcal{F}$,*

$$\sigma(z_0, z) < \delta \quad \textit{implies} \quad \sigma(f(z_0), f(z)) < \varepsilon.$$

The family $\mathcal{F}$ is equicontinuous on a subset X of $\mathbb{C}_\infty$ if it is equicontinuous at each point z_0 of X.

Roughly speaking, equicontinuity means preservation of proximity. Unless otherwise stated, f is a rational map from $\mathbb{C}_\infty$ into itself.

Definition 3.2.3 *The Fatou set $F(f)$ is the maximal open subset of $\mathbb{C}_\infty$ on which the family $\{f^{\circ k}\}_{k=0}^{\infty}$ is equicontinuous, and the Julia set $J(f)$ is its complement in $\mathbb{C}_\infty$.*

The Julia set of $f(x) = 3x^3 + 9x^2 + 7x$ is shown (in black) in Figure 3.3. A method for computing such pictures is suggested by Theorem 3.2.7 below. The actual Maple procedure used to generate the pictures of Julia sets found in this thesis is provided in the Appendix.

It is not hard to prove the following.

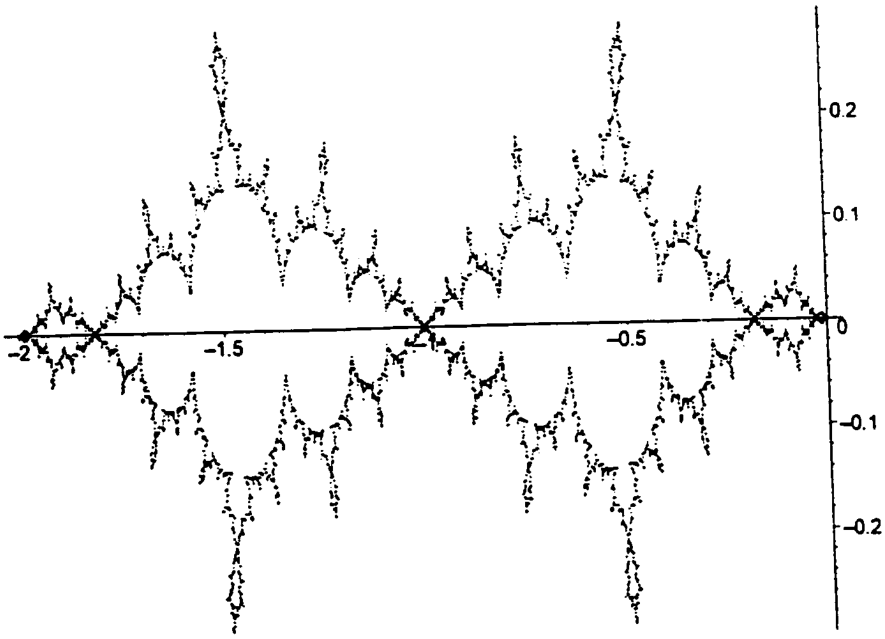


Figure 3.3: The Julia set, $J(3x^3 + 9x^2 + 7x)$

Theorem 3.2.4 *If g is a rational map which is conjugate to f - say $g = \phi \circ f \circ \phi^{\circ(-1)}$ for some Möbius map ϕ - then $F(g) = \phi(F(f))$ and $J(g) = \phi(J(f))$. The sets $J(g)$ and $J(f)$ are then said to be analytically conjugate, as are $F(g)$ and $F(f)$. Furthermore, $F(f^{\circ k}) = F(f)$ and $J(f^{\circ k}) = J(f)$ for any positive integer k.*

By definition, $F(f)$ is open, while $J(f)$ is compact. If $\deg(f) \geq 2$, then $J(f)$ is infinite; in fact, $J(f)$ is a *perfect* set (and hence uncountably infinite by Baire's Category Theorem), that is, $J = Ac(J)$. The sets F and J are each *completely invariant* under f, in that $f(F) = F = f^{\circ(-1)}(F)$ and $f(J) = J = f^{\circ(-1)}(J)$.

Periodic points of rational maps play an important role in iteration theory. A point z_0 is a *periodic point* of f if, for some positive integer k, $f^{\circ k}(z_0) = z_0$. The smallest positive integer k for which $f^{\circ k}(z_0) = z_0$ is the *period* of z_0. The forward orbit $O^+(z_0) = \{f^{\circ k}(z_0)\}$ of a periodic point z_0 is called a *periodic cycle*. Periodic points of period 1 are the fixed points of f.

For a periodic point $z_0 \in \mathbb{C}$ of period k, the number $\lambda = \left(f^{\circ k}\right)'(z_0)$ is the *multiplier* of its periodic cycle, and is independent of the choice of z_0 from the cycle (a simple exercise involving the Chain Rule). The cycle is

(i) *attracting* if $0 < |\lambda| < 1$ (and *super-attracting* if $\lambda = 0$);

(ii) *repelling* if $|\lambda| > 1$;

(iii) *rationally indifferent* if λ is a root of unity; and

(iv) *irrationally indifferent* if $|\lambda| = 1$, but λ is not a root of unity.

Theorem 3.2.5 *Repelling periodic cycles all lie on $J(f)$, and in fact are dense on $J(f)$. Rationally indifferent cycles also lie on $J(f)$. Attracting cycles, on the other hand, lie in $F(f)$. Meanwhile, an irrationally indifferent periodic cycle may lie in $F(f)$ or it may lie on $J(f)$.*

If z_0 is an irrationally indifferent periodic point of period k, lying in $F(f)$, then the component (i.e., maximal open connected subset) F_0 of $F(f)$ containing z_0 is forward invariant under $f^{\circ k}$, and is called a *Siegel disk*. For any point $z \neq z_0$ in F_0, the sequence $\{f^{\circ k}(z), f^{\circ(2k)}(z), \ldots\}$ is dense on a curve – called an *invariant circle* – lying inside F_0.

In fact, a complete classification of the possibilities for periodic components of a rational map is known; and *every* component C of a Fatou set $F(f)$ is eventually periodic under f, that is, for some $j > k \geq 0$, $f^{\circ j}(C) = f^{\circ k}(C)$. These very deep and fascinating results were proved by Sullivan (cf. [7] for references and details). An immediate consequence of his work is:

Theorem 3.2.6 *If f is a polynomial, and $z_0 \in F(f)$, then the forward orbit $O^+(z_0) = \{f^{\circ k}(z_0)\}$ either*

(i) *converges to a periodic cycle, or*

(ii) *settles into a 'periodic cycle' of Siegel disks, becoming dense on an invariant circle in each.*

Now, the *backward orbit* of a point $z \in \mathbb{C}_\infty$ is the set

$$O^-(z) = \cup_{k=0}^{\infty} f^{\circ(-k)}(z).$$

We are using the notation $f^{\circ(-1)}$ for the *set-valued* inverse of f (a function on subsets of $\mathbb{C}_\infty$), and $f^{\circ(-k)} = \underbrace{f^{\circ(-1)} \circ f^{\circ(-1)} \circ \cdots \circ f^{\circ(-1)}}_{k}$. If $O^-(z)$ happens to be finite, then z is called *exceptional.* For example, if $f(x) = x^n$, then both 0 and ∞ are exceptional points of f. A rational map f has at most two exceptional points, and they necessarily belong to $F(f)$. The point ∞ is an exceptional point for every *polynomial* p; thus, $\infty \in F(p)$ and, since $J(p)$ is closed, $J(p)$ will be *bounded* as a subset of $(\mathbb{C}, |\cdot|)$, where $|\cdot|$ is the Euclidean metric on $\mathbb{C}$, discussed in section 1.2.3. Backward orbits are closely linked to the Julia set:

Theorem 3.2.7 *Let f be a rational map with* $\deg(f) \geq 2$.

- *If z is not exceptional, then $J(f) \subseteq Cl\ (O^-(z))$.*
- *If $z \in J(f)$, then $J(f) = Cl\ (O^-(z))$.*

Since the inverse images $f^{\circ(-k)}(z_0)$ are finite, they are necessarily compact. Instead of looking at the entire inverse orbit $O^{-1}(z_0)$, we could ask whether the sets $f^{\circ(-k)}(z_0)$ will actually converge to $J(f)$ under the Hausdorff metric h_{σ_0} on $\mathcal{H}(\mathbb{C}_\infty)$ (cf. section 1.2.3). As it turns out, the generic case *is* convergence to $J(f)$. While a result of this nature cannot be found explicitly in the literature, however, there are enough tools available to piece one together. Our efforts in this regard will pay off immensely in the sections that follow. One tool we will need is:

Theorem 3.2.8 (cf. [7], p. 71) *Let f be a rational map of degree at least two, and E a compact subset of $\mathbb{C}_\infty$ such that for all $z \in F(f)$, the sequence $\{f^{\circ k}(z)\}$ does not accumulate at any point of E. Then for any open set U containing $J(f)$, $f^{\circ(-k)}(E) \subseteq U$ for all sufficiently large k.*

Another is:

Theorem 3.2.9 (cf. [7], p. 149) *Let f be a rational map of degree at least two, W a domain that meets J, and K any compact set containing no exceptional points of f. Then for all sufficiently large k, $f^{\circ k}(W) \supset K$.*

(A domain is an open connected set). Together with Theorem 3.2.6, these are strong enough to prove:

Theorem 3.2.10 *Let f be a polynomial, and z_0 a point which does not lie in any attracting cycle or Siegel disk of f. Then*

$$\lim_{k\to\infty} f^{\circ(-k)}(z_0) = J(f).$$

Proof* Let $\varepsilon > 0$ be given. Establishing the above limit is equivalent (cf. [6], p. 35) to proving that, for all sufficiently large k,

$$\text{(i)} \quad f^{\circ(-k)}(z_0) \subseteq J(f) + \varepsilon$$

and

$$\text{(ii)} \quad J(f) \subseteq f^{\circ(-k)}(z_0) + \varepsilon,$$

where $A + \varepsilon = \{z : \sigma_0(z,a) < \varepsilon \text{ for some } a \in A\}$, the "dilation of A by a ball of radius ε" ([6], p. 35).

To prove (i), note first that if $z_0 \in J(f)$, then $f^{\circ(-k)}(z_0) \subseteq J(f) \subseteq J(f) + \varepsilon$ for all k. If instead $z_0 \in F(f)$, then, since we are assuming that z_0 lies in neither an attracting cycle nor a Siegel disk, Theorem 3.2.6 implies that no point z in $F(f)$ will have a forward orbit which accumulates at z_0 (recall from Theorem 3.2.5 and the remarks immediately following the theorem imply that any non-attractive (and thus irrationally indifferent) periodic cycle in $F(f)$ necessarily lies in a Siegel disk). Hence, the set $E = \{z_0\}$ satisfies the hypothesis of Theorem 3.2.8, and therefore $f^{\circ(-k)}(z_0) \in J(f) + \varepsilon$ for all sufficiently large k.

*The author is indebted to Professors John Milnor and Robert Devaney for sharing with him their own thoughts on the possible correctness of this result, at a time when the author had not yet completed his proof.

To prove (ii), we begin by choosing a positive number $\delta < \varepsilon/2$, and covering $J(f)$ with finitely many open balls of radius δ (such a covering exists since $J(f)$ is compact). The point z_0 is not exceptional, since exceptional points are necessarily periodic points in $F(f)$. For each ball W in the covering $\mathcal{W}$, Theorem 3.2.9 implies that, for all sufficiently large k, $f^{\circ k}(W) \supset \{z_0\}$, and hence $f^{\circ(-k)}(z_0) \cap W \neq \emptyset$. Since there are only finitely many such balls, we then have that, for all sufficiently large k, $f^{\circ(-k)}(z_0) \cap W \neq \emptyset$ for each ball W. Finally, since $\varepsilon > 2\delta$, it follows that, for all such k, $f^{\circ(-k)}(z_0) + \varepsilon \supset \mathcal{W} \supset J(f)$. □

3.2.2 Independence Attractors of Graphs: a General Theory

We set out now to describe just where the independence roots of powers G^k of a graph G are 'approaching' as $k \to \infty$. The upshot will be an association of a fractal with a graph. For each $k \geq 1$ the set, Roots (i_{G^k}), of roots of i_{G^k} is a finite – and therefore, compact – subset of $(\mathbb{C}, |\cdot|)$. We ask whether the *limit* of the sequence $\{\text{Roots } (i_{G^k})\}$ with respect to the Hausdorff metric $h_{|\cdot|}$ on $\mathcal{H}(\mathbb{C})$ (cf. section 1.2.3) exists in general. In fact, it does.

Definition 3.2.11 *The independence attractor of a graph G is the set*

$$\mathcal{I}(G) = \lim_{k\to\infty} \text{Roots } (i_{G^k}), \tag{3.4}$$

where the limit is taken in $(\mathcal{H}(\mathbb{C}), h_{|\cdot|})$.

That $\mathcal{I}(G)$ actually exists for every graph G is part of Theorem 3.2.14 below, the main result of this section. Let us begin with a simple but important characterization of the right hand side of equation (3.4). For each $k \geq 2$, associativity of graph composition allows us to write $G^k = G^{k-1}[G]$, and Proposition 3.1 then implies that

$$i_{G^k} = i_{G^{k-1}} \circ (i_G - 1),$$

which in turn leads to the relation

$$\text{Roots } (i_{G^k}) = (i_G - 1)^{\circ(-1)} \left(\text{Roots } (i_{G^{k-1}})\right).$$

Also,

$$\text{Roots }(i_G) = (i_G - 1)^{\circ(-1)}(-1).$$

Hence,

Proposition 3.2.12 *Setting $f_G(x) \equiv i_G(x) - 1$, we have*

$$\text{Roots }(i_{G^k}) = f_G^{\circ(-k)}(-1)$$

for each $k \geq 1$. Therefore,

$$\mathcal{I}(G) = \lim_{k\to\infty} f_G^{\circ(-k)}(-1). \tag{3.5}$$

With this observation, we are almost in a position to apply the results of the previous section to the problem of describing $\mathcal{I}(G)$, but one point remains to be addressed. We are attempting to take the limit (3.5) in $(\mathcal{H}(\mathbb{C}), h_{|\cdot|})$, as opposed to $(\mathcal{H}(\mathbb{C}_\infty), h_{\sigma_0})$. However, note the following:

Lemma 3.2.13 *The sequence $\{f_G^{\circ(-k)}(-1)\}$ is bounded in $(\mathbb{C}, |\cdot|)$.*

<u>Proof</u> The result will be obvious if we can prove the existence of a number $R > 0$ such that $|z| > R$ implies $|f_G(z)| > R$. To that end, for $f_G(x) = \sum_{k=1}^{\beta} i_k x^k$, let $C = \max_{1\leq k\leq \beta-1} i_k$, and set $R = \max\left(\sqrt[n-1]{\frac{2}{|i_\beta|}}, \frac{C}{|i_\beta|} + 1\right)$. By the triangle inequality,

$$\begin{aligned}
|z| > R \;\Rightarrow\; |f_G(z)| &\geq i_\beta|z|^\beta - \left(i_{\beta-1}|z|^{\beta-1} + i_{\beta-2}|z|^{\beta-2} + \cdots + i_1|z|\right) \\
&\geq i_\beta|z|^\beta - C\left(|z|^{\beta-1} + |z|^{\beta-2} + \cdots + |z|\right) \\
&= i_\beta|z||z|^{\beta-1} - C|z|\frac{|z|^{\beta-1} - 1}{|z| - 1} \\
&= |z|\left(i_\beta|z|^{\beta-1} - C\frac{|z|^{\beta-1} - 1}{|z| - 1}\right) \\
&> R\left(i_\beta|z|^{\beta-1} - C\frac{|z|^{\beta-1} - 1}{C/i_\beta}\right) \\
&= R \cdot i_\beta \\
&\geq R,
\end{aligned}$$

which is what we wanted to show. □

Applying Theorem 1.2.7, we may instead consider the limit (3.5) as being taken in $(\mathcal{H}(\mathbb{C}_\infty), h_{\sigma_0})$, which means that we can apply the techniques and results of the previous section on iteration theory.

Theorem 3.2.14 answers completely our question in general, and is the foundation for everything that follows. Empty graphs will not be of interest, for if $G = \overline{K_n}$, then $G^k = \overline{K_{n^k}}$ and $i_{G^k}(x) = (1+x)^{n^k}$, whence $\mathcal{I}(G) = \{-1\}$.

Theorem 3.2.14 *Let G be a non-empty graph, and denote by $\eta(G)$ the multiplicity of -1 as a root of i_G. Set $f_G = i_G - 1$.*

(i) *If $\eta(G) \leq 1$, then*

$$\mathcal{I}(G) = J(f_G), \text{ the Julia set of } f_G.$$

(ii) *If $\eta(G) \geq 1$, then*

$$\mathcal{I}(G) = Cl\ (\cup_{k\geq 1} \text{Roots } (i_{G^k})) = Cl\ \left(\cup_{k\geq 1} f_G^{\circ(-k)}(-1)\right).$$

In case (ii), *i_{G^k} is divisible by $(i_{G^{k-1}})^{\eta(G)}$ for each $k \geq 2$, and*

$$\lim_{k\to\infty} (\text{Roots } (i_{G^k}) \setminus \text{Roots } (i_{G^{k-1}})) = \lim_{k\to\infty} \text{Roots } \left(\frac{i_{G^k}}{(i_{G^{k-1}})^{\eta(G)}}\right) = J(f_G). \qquad (3.6)$$

Further, for $\eta(G) > 1$, $\mathcal{I}(G)$ is partitioned by the set, $\cup_{k\geq 1}$ Roots (i_{G^k}), and its accumulation points, $J(f_G)$.

Proof If G has independence number 1, then $G = K_n$ for some $n \geq 2$, and $i_G(x) = 1 + nx$. It follows easily that for each $k \geq 1$, $i_{G^k}(x) = 1 + n^k x$, whose only root is $-1/n^k$, which tends to 0 as k tends to infinity. Thus, $\mathcal{I}(G) = \{0\}$ for graphs G with independence number 1. As $i_G(x) = 1 + nx$ where $n \geq 1$, we have $\eta(G) = 0$;

also, $J(nx) = \{0\}$ as the forward orbit of any non-zero point will go off to infinity. This proves the result for graphs with independence number 1.

Let G be a non-empty graph whose independence number is at least 2. For the cases $\eta(G) = 0$ and $\eta(G) = 1$, we will show that the hypothesis of Theorem 3.2.10 is satisfied for $f = f_G$ and $z_0 = -1$. To that end, note first that since f_G has integer coefficients, the forward orbit $\{f_G^{\circ k}(-1)\}$ of -1 lies entirely in $\mathbb{Z}$, and therefore could not possibly be dense on any invariant circle in a Siegel disk.

Consider first the case $\boldsymbol{\eta(G) = 0}$: here, $f_G(-1) \neq -1$ as $f_G(-1)+1 = i_G(-1) \neq 0$. We will show that in fact -1 cannot be a periodic point of f_G of *any* order. Together with our observation above, this will show that the hypothesis of Theorem 3.2.10 is satisfied here. We argue by contradiction. Suppose $f_G^{\circ k}(-1) = -1$ for some $k \geq 2$. Then $f_G(\underbrace{f_G^{\circ(k-1)}(-1)}_{\in \mathbb{Z}}) + 1 = 0$, and so $f_G^{\circ(k-1)}(-1)$ is an integer root of $f_G + 1$, a polynomial having positive integer coefficients and constant term 1. Applying the Rational Root Theorem, it follows easily that $f_G^{\circ(k-1)}(-1) = -1$. Repeating this argument, we conclude that $f_G(-1) = -1$, a contradiction. Therefore, -1 is not a periodic point of f_G, and the hypothesis of Theorem 3.2.10 is satisfied.

Next, consider the case $\boldsymbol{\eta(G) = 1}$: here, $f_G(-1) = -1$ and $f'_G(-1) = (f_G + 1)'(-1) = i'_G(-1) \neq 0$. Since f'_G has integer coefficients, this implies that $|f'_G(-1)| \geq 1$. If $|f'_G(-1)| = 1$, then $f'_G(-1)$ is either $+1$ or -1, implying that -1 is a *rationally* indifferent fixed point of f_G. If instead $|f'_G(-1)| > 1$, then -1 is a *repelling* fixed point of $J(f_G)$. In either case, the point $z_0 = -1$ belongs to $J(f_G)$, and once again the hypothesis of Theorem 3.2.10 is satisfied. In addition, as $-1 \in J(f_G)$, it follows from Theorem 3.2.7 that $J(f_G) = Cl\ (\cup_{k\geq 1} f_G^{-k}(-1))$.

When $\boldsymbol{\eta(G) > 1}$, $f_G(-1) = -1$ and $f'_G(-1) = (f_G+1)'(-1) = i'_G(-1) = 0$; thus, $z_0 = -1$ is an *attracting* fixed point of f_G, and so Theorem 3.2.10 does not apply. Let $\varepsilon > 0$ be given. We need to show that

(a) $f_G^{\circ(-k)}(-1) \subseteq Cl\left(\cup_{k\geq 1} f_G^{\circ(-k)}(-1)\right) + \varepsilon$; and

(b) $Cl\left(\cup_{k\geq 1} f_G^{\circ(-k)}(-1)\right) \subseteq f_G^{\circ(-k)}(-1) + \varepsilon$

for all sufficiently large k.

Part (a) is obvious, as $f_G^{\circ(-k)}(-1) \subseteq \cup_{k\geq 1} f_G^{\circ(-k)}(-1)$ for all k. Part (b) is not difficult either: Since $f(-1) = -1$, we have $-1 \in f_G^{\circ(-1)}(-1)$. Applying $f_G^{\circ(-1)}$ to both sides, we find that $f_G^{\circ(-1)}(-1) \subseteq f_G^{\circ(-2)}(-1)$. By induction, then, $f_G^{\circ(-(k-1))}(-1) \subseteq f_G^{\circ(-k)}(-1)$ for each $k \geq 2$. Consider the family of open sets $\{f_G^{\circ(-k)}(-1) + \varepsilon\}$. From what we have just shown, this forms an *increasing* open cover of the compact set $Cl\left(\cup_{k\geq 1} f_G^{\circ(-k)}(-1)\right)$, and hence the latter will be contained in $\{f_G^{\circ(-k)}(-1) + \varepsilon\}$ for all sufficiently large k, which is what we wanted to show.

Now we prove (3.6). Let G be a graph such that $\eta(G) \geq 1$. Then -1 is a root of i_G. Let R denote the set of roots of i_G; note that $R = f_G^{\circ(-1)}(-1)$. As G is non-empty, $i_G(x)$ is *not* of the form $(1+x)^\beta$, and so $\widetilde{R} = R \setminus \{-1\}$ is non-empty. Consider any element r of $\widetilde{R}$. As $f_G(r) = -1$, $f_G^{\circ k}(r) \in \mathbb{Z}$ for all $k \geq 1$. Thus, r cannot be a periodic point of f_G, for if it were, then $r \in \mathbb{Z}$. However, since $i_G(x)$ has positive coefficients and constant term 1, its only integer root, by the Rational Root Theorem, is -1. And so $r = -1$, contradicting the fact that $r \in \widetilde{R}$. Now, as $f_G^{\circ k}(r) \in \mathbb{Z}$ for all $k \geq 1$, the sequence $\{f_G^{\circ k}(r)\}$ will not be dense on any curve, and so r can lie in no Siegel disk either. Hence, the hypothesis of Theorem 3.2.10 is satisfied for each member r of the finite set $\widetilde{R}$, and so

$$\lim_{k\to\infty} f_G^{\circ(-k)}(\widetilde{R}) = J(f_G).$$

Write $i_G(x) = (1+x)^{\eta(G)}\psi_G(x)$, where $\psi_G(x)$ has roots $\widetilde{R}$. Then $i_{G^2}(x) = i_G(f_G(x)) = (i_G(x))^{\eta(G)}\, \psi_G(f_G(x))$. By induction, we find

$$i_{G^k}(x) = (i_{G^{k-1}}(x))^{\eta(G)}\, \psi_G\left(f_G^{\circ(k-1)}(x)\right),$$

and therefore

$$\frac{i_{G^k}(x)}{(i_{G^{k-1}}(x))^{\eta(G)}} = \psi_G\left(f_G^{\circ(k-1)}(x)\right)$$

which has roots $f_G^{\circ(-(k-1))}(\widetilde{R})$. Also,

$$\begin{aligned}
&\text{Roots }(i_{G^k}) \setminus \text{Roots }(i_{G^{k-1}}) \\
&= f_G^{\circ(-(k-1))}(R) \setminus f_G^{\circ(-(k-2))}(R) \\
&= \left(\cup_{j=0}^{k-1} f_G^{\circ(-j)}(\widetilde{R}) \cup \{-1\}\right) \setminus \left(\cup_{j=0}^{k-2} f_G^{\circ(-j)}(\widetilde{R}) \cup \{-1\}\right) \\
&= f_G^{\circ(-(k-1))}(\widetilde{R}).
\end{aligned}$$

The claim $f_G^{\circ(-(k-1))}(R) = \cup_{j=0}^{k-1} f_G^{\circ(-j)}(\widetilde{R}) \cup \{-1\}$ made in the second last equality is seen by a simple inductive argument, noting that $R = \widetilde{R} \cup \{-1\}$ and $f_G^{\circ(-1)}(-1) = \widetilde{R} \cup \{-1\}$.

To see why the last equality (above) holds, suppose that $r, s \in \widetilde{R}$ were such that $f_G^{\circ(-(k-1))}(r) \cap f_G^{\circ(-j)}(s) \neq \emptyset$ for some $j < k-1$. Then $r = f^{\circ(k-1-j)}(s) = -1$, contradicting the assumption that $r \in \widetilde{R}$. This completes the proof of (3.6).

Finally, since $\lim_{k \to \infty} f_G^{\circ(-k)}(\widetilde{R}) = J(f_G)$, and $J(f_G)$ is a perfect set (i.e., equal to its accumulation points), the union $\cup_{k \geq 0} f_G^{\circ(-k)}(\widetilde{R})$ will accumulate to no less than $J(f_G)$. It can accumulate to no more than $J(f_G)$ either, since any accumulation point necessarily lies in the limiting set. Hence,

$$J(f_G) = Ac\left(\cup_{k \geq 0} f_G^{\circ(-k)}(\widetilde{R})\right) = Ac\left(\cup_{k \geq 0} f_G^{\circ(-k)}(-1)\right).$$

And for $\eta(G) > 1$, $J(f_G)$ is disjoint from $\cup_{k \geq 0} f_G^{\circ(-k)}(-1)$, as -1, being an *attractive* fixed point of f_G, does not belong to $J(f_G)$. This concludes the proof of the theorem. □

Early into the proof, we observed that the independence attractor of any graph with independence number 1 is just $\{-1\}$. Things get more interesting as we consider graphs with higher independence number, and we will sometimes find it convenient to work with a polynomial g_G to which f_G is conjugate.

Theorem 3.2.15 *If, for a graph G, a Möbius transformation ϕ and polynomial g_G are such that $g_G = \phi \circ f_G \circ \phi^{\circ(-1)}$, then*

(i) f_G *fixes* -1 *if and only if* g_G *fixes* $\phi(-1)$,

(ii) *the multiplicity of* -1 *as a root of* $f_G(z)+1$ *equals the multiplicity of* $\phi(-1)$ *as a root of* $g_G(z)-\phi(-1)$, *and*

(iii) $\mathcal{I}(G) = J(f_G)$ *if and only if* $\mathcal{I}(G) = \phi^{\circ(-1)}(J(g_G))$.

In short, $\phi(-1)$ is to g_G what -1 is to f_G.

Proof To see (i), observe that $g_G(\phi(-1)) = \phi(f_G(\phi^{\circ(-1)}(-1))) = \phi(f_G(-1))$, which (since ϕ is injective) is equal to $\phi(-1)$ if and only if $f_G(-1) = -1$.

To see why (ii) holds, just note that $g_G(z) = \phi(-1) \Leftrightarrow \phi(f_G(\phi^{\circ(-1)}(z))) = \phi(-1) \Leftrightarrow f_G(\phi^{\circ(-1)}(z)) = -1$.

Finally, property (iii) follows immediately from the conjugacy property of Julia sets, mentioned in Theorem 3.2.4. Specifically, since $g_G = \phi \circ f_G \circ \phi^{\circ(-1)}$, $J(g_G) = \phi(J(f_G))$. Applying $\phi^{\circ(-1)}$ gives $J(f_G) = \phi^{\circ(-1)}(J(g_G))$. □

As Julia sets are typically fractals, we are in essence associating a fractal $\mathcal{I}(G)$ with a graph G. The question arises as to the possible connections between the two objects. How are graph-theoretic properties encoded in the fractals? What does $\mathcal{I}(G)$ say about G itself? Figure 3.4 shows the independence attractor of $G = 5K_2$, the disjoint union of five K_2's. Since $i_G(x) = (1+2x)^5$, $\mathcal{I}(G) = J((1+2x)^5 - 1)$. The attractor appears disconnected, in contrast to $\mathcal{I}(P_3)$ (cf. Figure 3.2). We ask here:

Question 3.2.16 *For which graphs G is $\mathcal{I}(G)$ connected?*

We can give a complete answer for graphs with independence number 2.

3.2.3 Graphs With Independence Number 2

Let G be a non-empty graph with independence number 2, having n vertices and m non-edges (i.e., $\overline{G}$ has exactly m edges). (Recall that empty graphs have independence

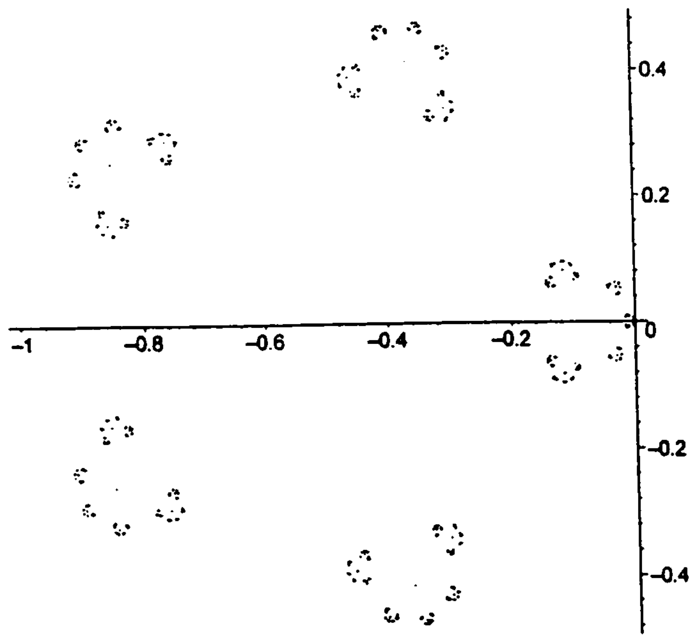


Figure 3.4: The independence attractor, $\mathcal{I}(5K_2) = J\left((1+2x)^5 - 1\right)$.

attractor $\{-1\}$, and are therefore not of interest). Then

$$f_G(x) = i_G(x) - 1 = mx^2 + nx. \tag{3.7}$$

As G is non-empty, $i_G(x) \neq (1+x)^2$ and so -1 will have multiplicity no greater than 1 as a root of $i_G(x)$. Applying Theorem 3.2.14, we have:

Theorem 3.2.17 *If G is a non-empty graph with independence number 2, then* $\mathcal{I}(G) = J(f_G)$.

The *Mandelbröt set* $\mathcal{M}$ is the set of all complex numbers c for which the Julia set of the polynomial $x^2 + c$ is connected. For any other value of c, $J(x^2+c)$ is not only disconnected, but *totally disconnected* (cf. [32], p. 246). Julia sets of this type are often called *fractal dust*. A plot of the Mandelbröt set (a subset of the complex c–plane) is shown in Figure 3.5. A well known fact (cf. [32]) is that $\mathcal{M}$ is contained in the disk $|c| \leq 2$, .

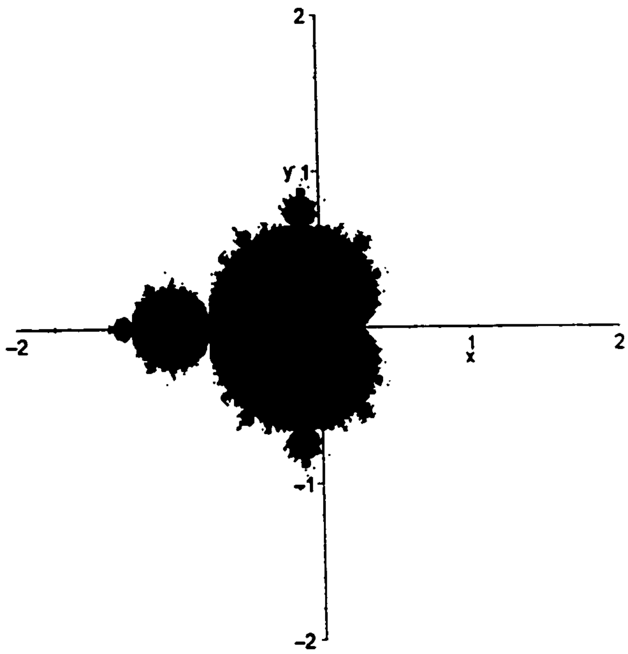


Figure 3.5: The Mandelbröt set.

Let us then work with a polynomial of the form x^2+c to which $f_G(x)$ is conjugate. It is straightforward to check that

$$g_G = \phi \circ f_G \circ \phi^{\circ(-1)},$$

where

$$g_G(x) = x^2 + \frac{n}{2} - \left(\frac{n}{2}\right)^2 \tag{3.8}$$

and

$$\phi(x) = mx + \frac{n}{2}. \tag{3.9}$$

For future reference,

$$\phi^{\circ(-1)}(x) = \frac{1}{m}x - \frac{n}{2m}. \tag{3.10}$$

Notice that the constant term $c = \frac{n}{2} - \left(\frac{n}{2}\right)^2$ in $g_G(x)$ is independent of m. This means that the connectivity of $\mathcal{I}(G)$ will depend only on how many *vertices* G has; the fact that G is non-empty implies that $n \geq 3$. The location of $\mathcal{I}(G)$, though, depends on both the numbers of vertices and edges in G, as Theorems 3.2.19, 3.2.20, and 3.2.21 below imply the following:

Theorem 3.2.18 *If G is a non-empty graph with independence number 2 having n vertices and m non-edges, and $z \in \mathcal{I}(G)$, then*

(i) $-\frac{n}{m} \leq \mathcal{R}e\,(z) \leq 0$, *and*

(ii) $\mathcal{I}m\,(z) = 0$, *unless $n = 3$, in which case* $-\frac{\sqrt{3}}{2m} \leq \mathcal{I}m\,(z) \leq \frac{\sqrt{3}}{2m}$.

Graphs for which $\beta = 2$, $n = 3$

There are exactly two graphs with independence number 2 on $n = 3$ vertices, namely $K_1 \uplus K_2$, the disjoint union of a point and an edge, and P_3, the path on three vertices. Their independence polynomials are $i_{K_1 \uplus K_2}(x) = 2x^2 + 3x + 1$ and $i_{P_3}(x) = x^2 + 3x + 1$; thus, $\mathcal{I}(K_1 \uplus K_2) = J(2x^2 + 3x)$ and $\mathcal{I}(P_3) = J(x^2 + 3x)$.

For either graph G, equation (3.8) says that $f_G(x)$ is conjugate to the polynomial $g_G(x) = x^2 - \frac{3}{4}$. For $G = K_1 \uplus K_2$, equation (3.9) tells us that $\phi(x) = 2x + \frac{3}{2}$ and so $\phi^{\circ(-1)}(x) = \frac{1}{2}x - \frac{3}{4}$, while for $G = P_3$, $\phi(x) = x + \frac{3}{2}$ and $\phi^{\circ(-1)}(x) = \frac{1}{2}x - \frac{3}{2}$. Since $-\frac{3}{4}$ lies in the Mandelbröt set, $J(x^2 - \frac{3}{4})$ is connected. By Theorem 3.2.15, $\mathcal{I}(G) = \phi^{\circ(-1)}J(x^2 - \frac{3}{4})$, and since, for either graph G, $\phi^{\circ(-1)}$ is a mere scaling and shifting, $\mathcal{I}(G)$ must also be connected.

With a little work, one can determine a box containing $J(x^2 - \frac{3}{4})$. The author can prove that for $g(x) = x^2 - \frac{3}{4}$ and $z \in \mathbb{C}$, if either $|\mathcal{R}e\,(z)| > \frac{3}{2}$ or $|\mathcal{I}m\,(z)| > \frac{\sqrt{3}}{2}$, then $|g^k(z)| \to \infty$ as $z \to \infty$. Surely this is known, so we will not clutter this section with the details. The box $[-\frac{3}{2}, \frac{3}{2}] \times [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$ containing $J(g(x))$ is in fact best possible, as the points $\pm\frac{3}{2}$ and $\pm\frac{\sqrt{3}}{2}i$ all lie in $J(g(x))$: the point $z = \frac{3}{2}$ is a repelling fixed point of g, and $g^{\circ 2}(\pm\frac{\sqrt{3}}{2}i) = g(-\frac{3}{2}) = \frac{3}{2} \in J(g(x))$. Applying $\phi^{\circ(-1)}$ to the box will give a tight box containing $\mathcal{I}(G)$. We have proved:

Theorem 3.2.19 *If G is a graph with independence number 2 on $n = 3$ vertices, then*

$$\mathcal{I}(G) = \phi^{\circ(-1)}\left(J\left(x^2 - \frac{3}{4}\right)\right),$$

where either

(i) $G = K_1 \uplus K_2$ *and* $\phi^{\circ(-1)}(x) = \frac{1}{2}x - \frac{3}{4}$, *or*

(ii) $G = P_3$ *and* $\phi^{\circ(-1)}(x) = \frac{1}{2}x - \frac{3}{2}$.

The attractor, $\mathcal{I}(G)$, is connected, therefore, and

(i) $\mathcal{I}(G = K_1 \uplus K_2) = J(2x^2 + 3x) \subseteq [-\frac{3}{2}, 0] \times [-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}]$, *while*

(ii) $\mathcal{I}(G = P_3) = J(x^2 + 3x) \subseteq [-3, 0] \times [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$.

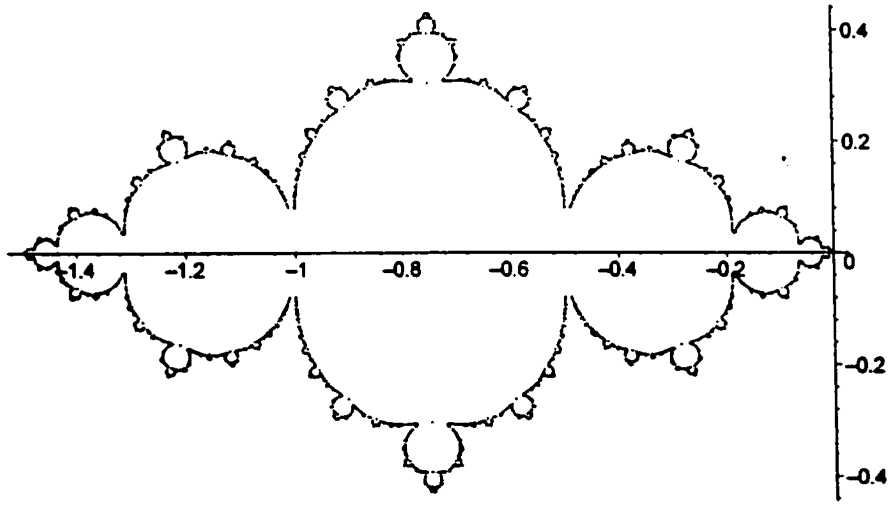


Figure 3.6: The independence attractor, $\mathcal{I}(K_1 \uplus K_2)$.

Plots of $\mathcal{I}(K_1 \uplus K_2)$ and $\mathcal{I}(P_3)$ are shown in Figures 3.6 and 3.7, respectively. That they appear to have the same 'shape' agrees with the fact that each is just a shifting and scaling of $J\left(x^2 - \frac{3}{4}\right)$.

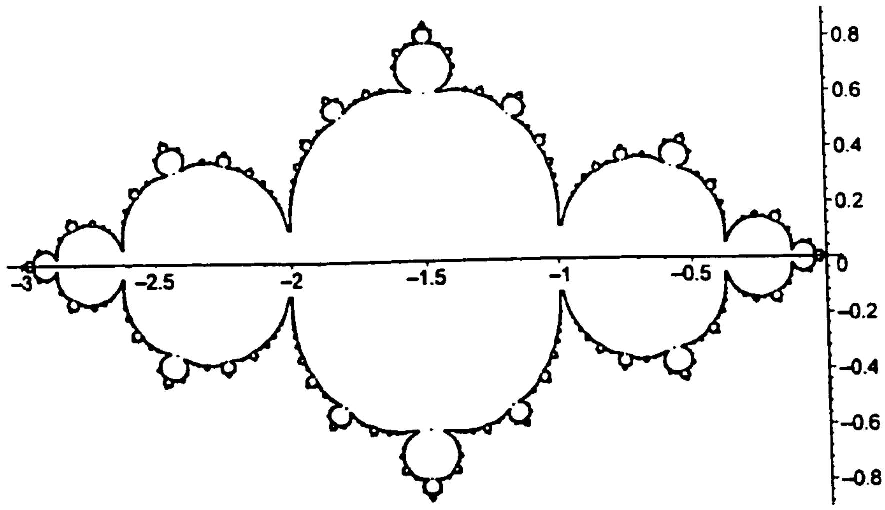


Figure 3.7: The independence attractor, $\mathcal{I}(P_3)$.

Graphs for which $\beta = 2$, $n = 4$

For a graph G with independence number 2 on $n = 4$ vertices (and m non-edges), equations (3.8) and (3.9) tell us that $f_G(x)$ is conjugate to $g_G(x) = x^2 - 2$ via $\phi(x) = mx + 2$, that is, $g_G = \phi \circ f_G \circ \phi^{\circ(-1)}$. Now $J(x^2 - 2)$ is well-known (cf. [32], p. 226) to be the interval $[-2, 2]$; applying the map $\phi^{\circ(-1)}(x) = \frac{1}{m}x - \frac{2}{m}$ to this interval gives

Theorem 3.2.20 *If G is a graph with independence number 2 having $n = 4$ vertices and m non-edges, then*

$$\mathcal{I}(G) = \left[\frac{-4}{m}, 0\right].$$

The graph $G = K_4 - e$ has independence number $\beta = 2$, $n = 4$ vertices, and $m = 1$ non-edge. Then $f_G(x) = x^2 + 4x$ and, from Theorem 3.2.20, $\mathcal{I}(G) = [-4, 0]$, as shown in Figure 3.8.

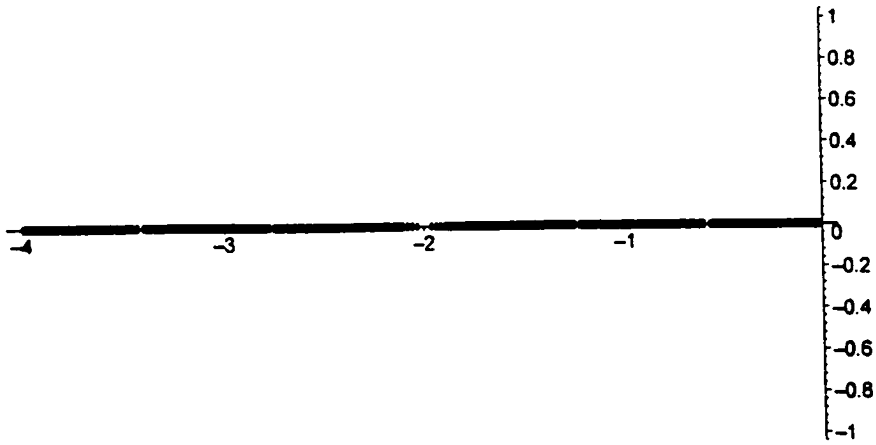


Figure 3.8: The independence attractor, $\mathcal{I}(K_4 - e) = [-4, 0]$.

Graphs for which $\beta = 2$, $n \geq 5$

If G is a graph with independence number 2 on $n \geq 5$ vertices, then $c = n/2 - (n/2)^2 < -2$, which lies outside the Mandelbröt set. This implies that $J(x^2 + c)$, and hence $\mathcal{I}(G) = \phi^{\circ(-1)}(J(x^2 + c)))$, a mere scaling and shifting (by equation (3.10)) of $J(x^2 + c)$, is fractal dust. In fact, we have:

Theorem 3.2.21 *If G is a graph with independence number 2 having $n \geq 5$ vertices and m non-edges, then $\mathcal{I}(G)$ is a dusty subset of the interval $\left[-\frac{n}{m}, 0\right]$.*

<u>Proof</u> It remains to show that $\mathcal{I}(G)$ is real and contained in $\left[-\frac{n}{m}, 0\right]$. We will do this by proving that the independence roots of each power G^k all belong to $\left[-\frac{n}{m}, 0\right]$. First, we prove:

Lemma 3.2.22 *The independence roots, R_k, of G^k satisfy:*

$$\begin{aligned} R_1 &= -\frac{n}{2m} \pm \frac{1}{2m}\sqrt{n^2 - 4m}; \\ R_k &= -\frac{n}{2m} \pm \frac{1}{2m}\sqrt{n^2 + 4mR_{k-1}} \ (k > 1). \end{aligned}$$

Proof of Lemma 3.2.22 The expression for R_1 follows directly from equation (3.7) and the quadratic formula. For each $k > 1$, we know that we can find R_k by solving the equation

$$mx^2 + nx = R_{k-1},$$

for x, which produces the above expression for R_k. □

Next, we can show:

Theorem 3.2.23 *For each $k \geq 1$, we have $-\frac{n}{m} \leq R_k \leq 0$.*

Proof of Theorem 3.2.23 By induction on k. For $k = 1$, we note that the discriminant $n^2 - 4m$ of (3.7) is ≥ 0, by applying Turan's theorem[†] to $\overline{G}$. So R_1 is real. Using Lemma 3.2.22 and the simple observation that $4m \geq 0$, we have:

$$\begin{aligned} -\frac{n}{2m} - \frac{1}{2m}\sqrt{n^2 - 4m} \ &\leq R_1 \leq \ -\frac{n}{2m} + \frac{1}{2m}\sqrt{n^2 - 4m} \\ -\frac{n}{2m} - \frac{1}{2m}\sqrt{n^2} \ &\leq R_1 \leq \ -\frac{n}{2m} + \frac{1}{2m}\sqrt{n^2} \\ -\frac{n}{m} \ &\leq R_1 \leq \ 0. \end{aligned}$$

Now suppose the result is true up to before $k > 1$. Then $n^2 + 4mR_{k-1} \geq n^2 - 4m \cdot \frac{n}{m} = n^2 - 4n = n(n-4) \geq 0$. Thus, from Lemma 3.2.22 we have that R_k is real and

$$\begin{aligned} -\frac{n}{2m} - \frac{1}{2m}\sqrt{n^2 + 4mR_{k-1}} \ &\leq R_k \leq \ -\frac{n}{2m} + \frac{1}{2m}\sqrt{n^2 + 4mR_{k-1}} \\ -\frac{n}{2m} - \frac{1}{2m}\sqrt{n^2} \ &\leq R_k \leq \ -\frac{n}{2m} + \frac{1}{2m}\sqrt{n^2} \\ -\frac{n}{m} \ &\leq R_1 \leq \ 0, \end{aligned}$$

and so the result holds for k as well, completing the proof. □

Hence, as the roots at each step are all real and contained in $\left[-\frac{n}{m}, 0\right]$, the limiting set, $\mathcal{I}(G)$, will also be real and contained in $\left[-\frac{n}{m}, 0\right]$. This completes the proof of Theorem 3.2.21. □

[†]Turan's theorem (cf. [59], p.33) states that for a graph having no triangles, $m \leq \frac{n^2}{4}$, where n and m are its numbers of vertices and edges, respectively.

The graph $K_2 \uplus K_3$ has independence number $\beta = 2$, $n = 5$ vertices, and $m = 6$ non-edges. Then $f_G(x) = 6x^2 + 5x$ and, by Theorem 3.2.21, $\mathcal{I}(G)$ is a totally disconnected subset of $[-\frac{5}{6}, 0]$, as shown in Figure 3.9.

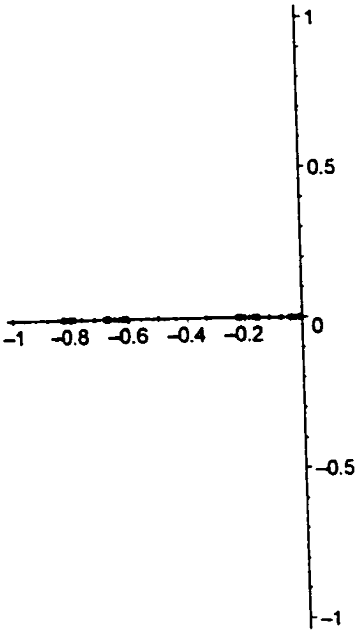


Figure 3.9: The independence attractor, $\mathcal{I}(K_2 \uplus K_3) = J(6x^2 + 5x)$.

3.2.4 Beyond Independence Number 2

If G is a nonempty graph, and -1 is a root of $i_G(x) = f_G(x) + 1$ of multiplicity $\eta(G) > 1$, then Theorem 3.2.14 implies that $\mathcal{I}(G)$ is partitioned by $\cup_{k \geq 1} \text{Roots}\,(i_{G^k})$ and its accumulation points, $J(f_G)$. While this implies that $\mathcal{I}(G)$ is disconnected, it is perhaps more interesting to know whether its collection of accumulation points, $J(f_G)$, is connected.

Perhaps, then, we should reformulate our question. Equation (3.6) tells us that the 'new' roots at each step will converge to $J(f_G)$, even for $\eta(G) = 1$. In fact, the equation is also valid when $\eta(G) = 0$, since, in that case, $\text{Roots}\, i_{G^k} \cap \text{Roots}\, i_{G^{k-1}} = \emptyset$. To see this, suppose the intersection were non-empty. Then, for some z, $f_G^{\circ k}(z) = f_G^{\circ(k-1)}(z) = -1$, that is, $f_G(f_G^{\circ(k-1)}(z)) = f_G^{\circ(k-1)}(z) = -1$, which says that $f_G(-1) =$

-1, contradicting the fact that $\eta(G) = 0$. We are led to make the following definition.

Definition 3.2.24 *For a nonempty graph G, the* <u>*independence fractal*</u> $\tilde{\mathcal{I}}(G)$ *of G is the set*

$$\tilde{\mathcal{I}}(G) = \lim_{k\to\infty} (\text{Roots } (i_{G^k}) \setminus \text{Roots } (i_{G^{k-1}})) = \lim_{k\to\infty} \text{Roots} \left(\frac{i_{G^k}}{(i_{G^{k-1}})^{\eta(G)}}\right) = J(f_G),$$

where (as before) $\eta(G)$ is the multiplicity of -1 as a root of i_G, and $f_G = i_G - 1$.

Thus, $\tilde{\mathcal{I}}(G)$ is always $J(f_G)$, whereas $\mathcal{I}(G)$ consists of $J(f_G)$ and possibly more, depending on whether $\eta(G) > 1$. And we ask instead:

Question 3.2.25 *When is $\tilde{\mathcal{I}}(G)$ connected?*

(The name, *independence fractal*, is not entirely accurate, since not every Julia set is a fractal. Recall, for instance, that $\tilde{\mathcal{I}}(K_4 - e) = J(x^2 + 4x) = [-1, 0]$. A fractal (cf. [6]) is generally considered to be a compact subset of $(\mathbb{C}_\infty, \sigma_0)$, whose Hausdorff dimension (cf. [7]) strictly exceeds its topological dimension.)

If G is a graph with independence number $\beta = 3$, then

$$f_G(x) = tx^3 + mx^2 + nx, \tag{3.11}$$

where n, m and t are the numbers of vertices, edges and triangles in $\overline{G}$, respectively. With a little effort, we can find a conjugate polynomial $g_G(x)$ with no x^2– term:

$$g_G(x) = x^3 + ax + b = \phi \circ f_G \circ \phi^{\circ(-1)}(x), \tag{3.12}$$

where

$$a = n - \frac{m^2}{3t}\ ,\ b = \frac{m}{3\sqrt{t}}\left(\frac{2m^2}{9t} - n + 1\right)$$

and $\phi(x) = \sqrt{t}x + \dfrac{m}{3\sqrt{t}}$.

Since there are now two parameters a and b, the Mandelbröt set (those values of a and b for which $J(x^3+ax+b)$ is connected) is a subset of $\mathbb{C}\times\mathbb{C}$, and is much more complicated than the Mandelbröt set for quadratics. The first extensive study of this set was carried out jointly by Branner and Hubbard in [16] and [17]. For quartics, quintics and beyond, very little is known about the Mandelbröt sets.

The graph $G = C_6$ has independence number 3, and $f_G(x) = 2x^3 + 9x^2 + 6x$. Its independence attractor is shown in Figure 3.10, and appears to be disconnected. On the other hand, the graph H whose complement consists of three triangles intersecting on a cut vertex, appears to have a connected indpendence fractal, shown in Figure 3.11.

Independence fractals of graphs of independence number 3 are likely sufficiently intricate to occupy a thesis in their own right, and we shall not endeavour to investigate them here. Instead, we end the chapter with an analysis of two infinite families of graphs of arbitrarily high independence numbers.

The following result from iteration theory will be useful. *Critical points* of a polynomial play a key role in the connectivity of its Julia set:

Theorem 3.2.26 (cf. [7]) *Let f be a polynomial of degree at least two.*

- *Its Julia set $J(f)$ is connected if and only if the forward orbit of each of its critical points is bounded in* $(\mathbb{C}, |\cdot|)$.
- *Its Julia set $J(f)$ is totally disconnected if (but not only if) the forward orbit of each of its critical points is unbounded in* $(\mathbb{C}, |\cdot|)$.

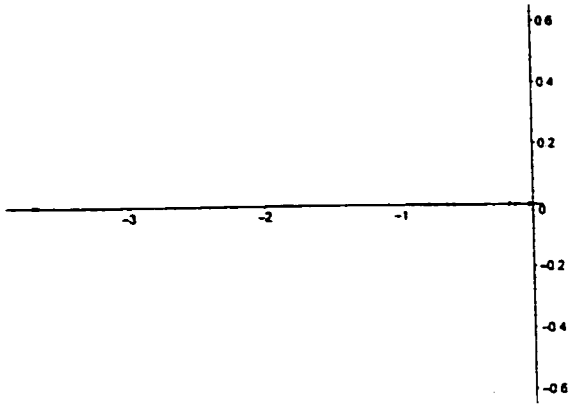


Figure 3.10: The independence attractor, $\mathcal{I}(C_6) = \bar{\mathcal{I}}(C_6) = J(2x^3 + 9x^2 + 6x)$.

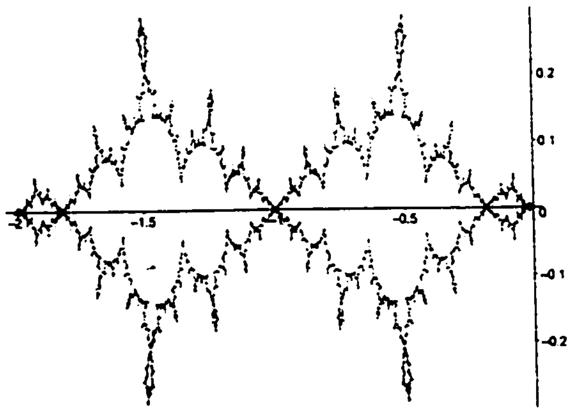


Figure 3.11: The independence attractor, $\mathcal{I}(H) = \bar{\mathcal{I}}(H) = J(3x^3 + 9x^2 + 7x)$, where H is the complement of three K_3's intersecting on a cut vertex.

The families aK_b and $K_{\underbrace{a,a,\ldots,a}_{b}}$

Denote by aK_b be the graph $\underbrace{K_b \uplus K_b \uplus \cdots \uplus K_b}_{a}$, the disjoint union of b copies of the complete graph K_a. The independence attractors of $3K_2$ and $4K_2$ are shown in Figures 3.12 and 3.13, respectively.

We have already described the independence attractors of empty graphs and complete graphs, and so we may assume that both a and b are greater than 1. Now

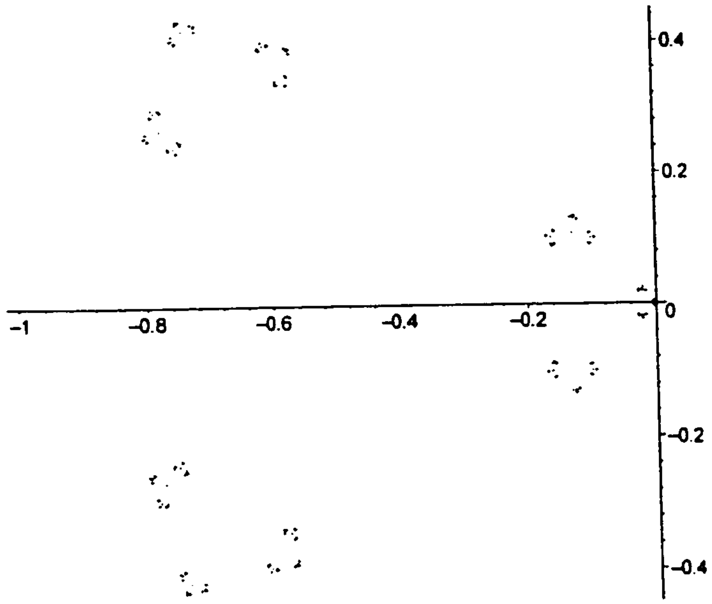


Figure 3.12: The independence attractor, $\mathcal{I}(3K_2)$.

$aK_b = \overline{K_a}[K_b]$, and since $f_{\overline{K_a}}(x) = (1+x)^a - 1$ and $f_{K_b}(x) = bx$, we have

$$f_{aK_b}(x) = f_{\overline{K_a}}(bx) = (1+bx)^a - 1,$$

and

$$f'_{aK_b}(x) = ab(1+bx)^{a-1},$$

whose only critical point is $z = -1/b$. By Theorem 3.2.26, then, $\bar{\mathcal{I}}(G)$ will either be connected or totally disconnected, depending on whether the forward orbit of $z = -1/b$ is bounded or unbounded, respectively, in $(\mathbb{C}, |\cdot|)$.

Now, $f_{aK_b}(-1/b) = 0^a - 1 = -1$.

<u>Case 1</u>: $b = 2$, a even. Then $f_{aK_b}(x) = f_{aK_2}(x) = (1+2x)^a - 1$. Now $f_{aK_2}(-1/b) = -1$, $f_{aK_2}(-1) = (1-2)^a - 1 = 0$, and $f_{aK_2}(0) = 0$. Hence, the forward orbit of $-1/b$ converges to 0, and is therefore bounded in $(\mathbb{C}, |\cdot|)$. Thus, $\bar{\mathcal{I}}(aK_2)$ is connected.

<u>Case 2</u>: $b \geq 3$, a even. Then $f_{aK_b}(-1/b) = -1$, and $f_{aK_b}(-1) = (1-b)^a - 1 \geq 2^a - 1 > 1$. And $z > 1 \Rightarrow f_{aK_b}(z) > (1+2z)^1 - 1 = 2z > z + 1$. Hence, the forward orbit of $-1/b$ is unbounded in $(\mathbb{C}, |\cdot|)$, and $\bar{\mathcal{I}}(aK_b)$ is totally disconnected.

<u>Case 3</u>: $b \geq 2$, a odd. Then Then $f_{aK_b}(-1/b) = -1$, and $f_{aK_b}(-1) = (1-b)^a - 1 \leq (1-2)^3 - 1 = -2 < -1$. And $z < -1 \Rightarrow f(z) < (1+2z)^1 - 1 = 2z = z + z < z - 1$.

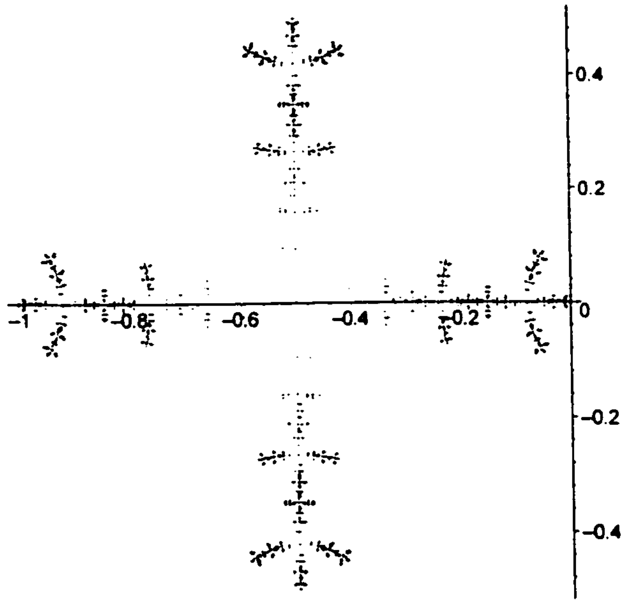


Figure 3.13: The independence attractor, $\mathcal{I}(4K_2)$.

Hence, the forward orbit of $-1/b$ is unbounded in $(\mathbb{C}, |\cdot|)$, and $\tilde{\mathcal{I}}(aK_b)$ is totally disconnected.

In all cases, since $b > 1$, -1 is not a fixed point of f_{aK_b}, and thus $\mathcal{I}(G) = \overline{\mathcal{I}}(G)$. We have now proved:

Theorem 3.2.27 *The independence attractor of aK_b is connected if $b = 2$ and a is even; totally disconnected otherwise.*

As we did for graphs with independence number 2, we can find a region inside which $\mathcal{I}(aK_b)$ lies. It lies in the disk

$$\left|z + \frac{1}{b}\right| \leq \frac{1}{b}.$$

First, we prove:

Lemma 3.2.28 *For each $k \geq 1$, each independence root, F_k, of the graph $(aK_b)^k$ satisfies:*

$$\begin{aligned} F_1 &= -\frac{1}{b}, \text{ and} \\ F_k &= -\frac{1}{b} + \frac{1}{b}\omega \quad (k > 1), \end{aligned}$$

where $\omega^a = F_{k-1}$ *for some independence root* F_{k-1} *of* $(aK_b)^{k-1}$.

Proof The expression for F_1 comes from the fact that $i_{aK_b}(x) = (1+bx)^a$. For each $k > 1$, we can find the points F_k by solving the equations

$$(1+bx)^a - 1 = F_{k-1}$$

for x, thereby producing the above expression for F_k. □

From this, we can prove:

Theorem 3.2.29 *For each* $k \geq 1$, $|F_k + \frac{1}{b}| \leq \frac{1}{b}$.

Proof By induction on k. It is true for $k = 1$, since $F_1 = -\frac{1}{b}$ implies $|F_1 + \frac{1}{b}| = 0 < \frac{1}{b}$. Now suppose the result is true for a number $k-1$ ($k > 1$). Then, by Lemma 3.2.28, we have:

$$\begin{aligned}\left|F_k + \frac{1}{b}\right| &= \frac{1}{b}\left|1 + F_{k-1}\right|^{1/a} \\ &= \frac{1}{b}\left|(F_{k-1} + \frac{1}{b}) + (1 - \frac{1}{b})\right|^{1/a} \\ &\leq \frac{1}{b}\left(\left|F_{k-1} + \frac{1}{b}\right| + \left|1 - \frac{1}{b}\right|\right)^{1/a} \\ &\leq \frac{1}{b}\left(\frac{1}{b} + \left(1 - \frac{1}{b}\right)\right)^{1/a} \\ &= \frac{1}{b},\end{aligned}$$

and so the result holds for k as well, completing the proof. □

Finally, since roots at each step are at most a distance of $\frac{1}{b}$ from $-\frac{1}{b}$, the same must be true of the limiting set.

Theorem 3.2.30 *The independence attractor of the graph* aK_b *lies in the disk*

$$\left|z + \frac{1}{b}\right| \leq \frac{1}{b}.$$

Further, the bound is best possible. To see this, let $z = \frac{1}{b}(\omega - 1)$ be any root ω of unity shifted 1 unit to the left, and scaled by $\frac{1}{b}$. Then $\left|z + \frac{1}{b}\right| = \left|\frac{1}{b}\omega\right| = \frac{1}{b}$, and $f_{aKb}(z) = (1+bz)^a - 1 = (\omega)^a - 1 = 1 - 1 = 0$. And $0 \in J(f_G)$ for <u>any</u> nonempty graph G, since $f_G(0) = 0$, while $f_G'(0) \neq 0$, so that 0 is either a repulsive or rationally indifferent fixed point of f_G, thereby lying on $J(f_G)$. Thus, since $f_{aK_b}(z) = 0 \in J(f_G)$, we must have $z \in J(f_G)$.

Our next family is complete multipartite graphs $K_{\underbrace{a, a, \ldots, a}_{b}} = K_b[\overline{K_a}]$. Then

$$f_{K_{\underbrace{a, a, \ldots, a}_{b}}}(x) = f_{K_b[\overline{K_a}]}(x) = b \cdot f_{\overline{K_a}}(x) = b(1+x)^a - b.$$

Again, we assume that both a and b are greater than 1. Since then -1 is not a fixed point of $f_{K_b[\overline{K_a}]}$, we have $\mathcal{I}(K_b[\overline{K_a}]) = \tilde{\mathcal{I}}(K_b[\overline{K_a}]) = J\left(b(1+x)^a - b\right)$.

The independence attractors of the graphs $K_{3,3} = K_2[\overline{K_3}]$ and $K_{4,4} = K_2[\overline{K_4}]$ are shown in Figures 3.14 and 3.15, respectively. They appear to have the same 'shape' as the attractors of their counterparts, $\overline{K_3}[K_2]$ and $\overline{K_4}[K_2]$ (Figures 3.12 and 3.13), respectively.

We can explain this phenomenon. Set $\phi(x) = bx$. Then ϕ is a Möbius map, and

$$\begin{aligned} f_{K_b[\overline{K_a}]}(\phi(x)) &= f_{K_b[\overline{K_a}]}(bx) \\ &= b(1+bx)^a - b \\ &= b \cdot f_{\overline{K_a}[K_b]}(x) \\ &= \phi(f_{\overline{K_a}[K_b]}(x)). \end{aligned}$$

Thus, $f_{K_b}[\overline{K_a}]$ and $f_{K_b}[\overline{K_a}]$ are analytically conjugate (cf. Theorem 3.2.4), and

$$\begin{aligned} \mathcal{I}\left(K_b[\overline{K_a}]\right) &= J\left(f_{K_b[\overline{K_a}]}\right) \\ &= \phi\left(J\left(f_{\overline{K_a}[K_b]}\right)\right), \text{ from Theorem 3.2.4} \\ &= b \cdot J\left(f_{\overline{K_a}[K_b]}\right) \\ &= b \cdot \mathcal{I}\left(\overline{K_a}[K_b]\right). \end{aligned}$$

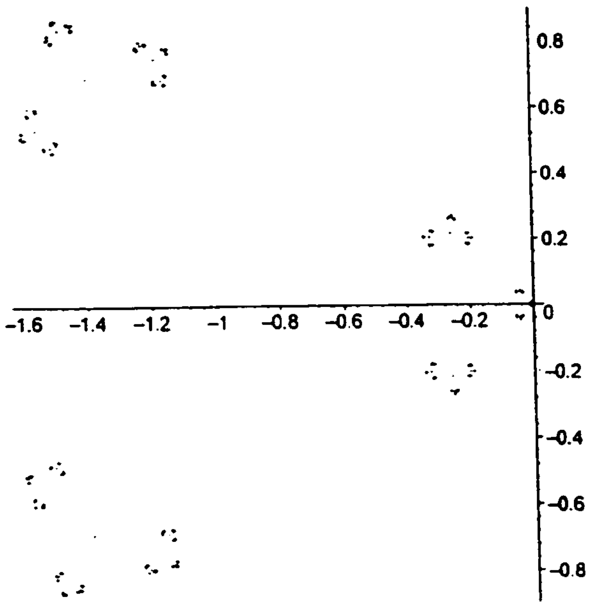


Figure 3.14: The independence attractor, $\mathcal{I}(K_{3,3})$.

This observation, together with our analysis of $\mathcal{I}(\overline{K_a}[K_b])$ above, enables us to conclude:

Theorem 3.2.31 *The independence attractor of $K_{\underbrace{a,a,\ldots,a}_{b}}$ is connected if $b = 2$ and a is even; totally disconnected otherwise. The attractor lies in the disk $|z+1| \leq 1$, and this bounding disk is best possible.*

The situation in general is this:

Theorem 3.2.32 *For a graph G,*

$$f_{K_n[G]}(nx) = n \cdot f_G(nx) = n \cdot f_{G[K_n]}(x).$$

That is,

$$f_{K_n[G]} \circ \phi = \phi \circ f_{G[K_n]},$$

where ϕ is the Mobius map $x \mapsto nx$. Hence,

$$\bar{\mathcal{I}}(K_n[G]) = n \cdot \bar{\mathcal{I}}(G[K_n]). \qquad \square$$

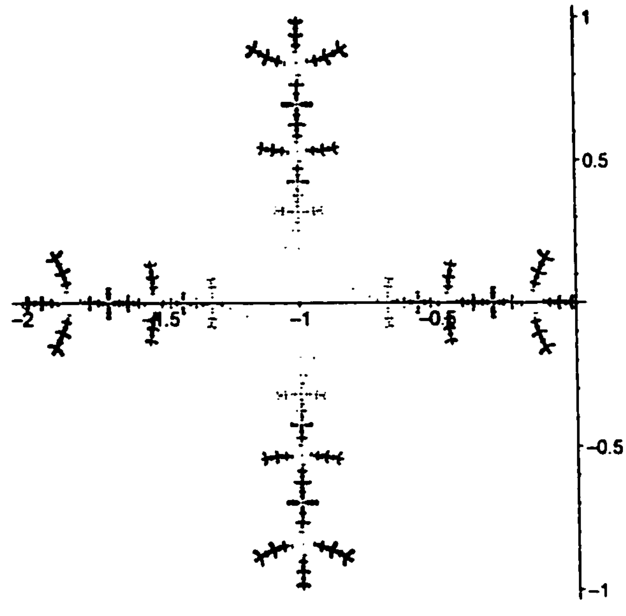


Figure 3.15: The independence attractor, $\mathcal{I}(K_{4,4})$.

We can also deduce the following.

Theorem 3.2.33 *Every nonempty graph G with independence number at least two is an induced subgraph of a graph H with the same independence number, whose independence fractal is disconnected.*

Proof Since $f_G(x)$ has degree at least 2, an argument similar to that of Lemma 3.2.13 shows that there exists a real number $R > 1$ such that $|z| > R \Rightarrow |f_G(z)| > 2|z|$, which implies that the forward orbit of z is unbounded in $(\mathbb{C}, |\cdot|)$.

Now, not every critical point of f_G is a root of f_G. Indeed, for a root r of both f_G' and f_G, its multiplicity as a root of f_G is one greater than its multiplicity as a root of f_G'. But $\deg f_G = \deg f_G' + 1$, and so, if every critical point of f_G were a root of f_G, then in fact f_G must have only one critical point c, and $f_G(x) = a(x+c)^{\beta}$. But we know that $x | f_G(x)$, and so $c = 0$ and $f_G(x) = ax^{\beta}$. But this could only be the case if $\beta = 1$, which it is not.

Let c then be a critical point of f_G for which $f_G(c) = w \neq 0$, and choose a positive integer n large enough that $|n \cdot w| > R$. For the graph $G[K_n]$, we have $f_{G[K_n]}(x) =$

$f_G(nx)$, a critical point of which is c/n. But then $f_{G[K_n]}(c/n) = f_G(c) = w$, and $|f_{G[K_n]}^{\circ k}(w)| = |f_G^{\circ k}(nw)| \to \infty$ as $k \to \infty$. Hence, by Theorem 3.2.26, the graph $G[K_n]$, which has independence number β, and of which G is an induced subgraph, has a disconnected independence fractal. □

The argument we used, together with the fact (Theorem 3.2.32) that $G[K_n]$ and $K_n[G]$ have analytically conjugate independence fractals, enables us to deduce the following as well, with which we conclude the chapter.

Theorem 3.2.34 *If G is a nonempty graph with independence number at least 2, then for all sufficiently large n, the join of n copies of G has a disconnected independence fractal.*

It would seem, then, that graph connectedness and independence fractal connectedness are unrelated.

Chapter 4

Open Problems and Future Directions

There are many avenues that one could explore.

4.1 Chromatic Roots

Several conjectures and open problems came up in our work on roots of chromatic polynomials in Chapter 2. While we were interested mainly in *large* subdivisions of a graph, computational explorations with *real* roots of small subdivisions suggests the following.

Conjecture 4.1.1 *If each edge of G is subdivided at least once, then no real chromatic root of the resulting graph G' is greater than 2.*

Here is a partial result in this regard:

Theorem 4.1.2 *If we subdivide each edge of G into a path of even length, then no real chromatic root of the resulting graph is* 2 *or more.*

Proof For convenience, we will denote the chromatic polynomial $\pi(H, x)$ of a graph H by the symbol H itself. We argue by induction on the number m of edges in a graph. For $m = 1$, the result is clear. Now let $m \geq 2$ and suppose the result holds for all graphs of size at most $m-1$. Let G be a graph with m edges $e_1, \ldots, e_m$. Subdividing e_i into a path of even length l_i ($i = 1, \ldots, m$), we obtain a graph $G^{e_1,\ldots,e_m}_{l_1,\ldots,l_m}$, whose chromatic polynomial, by equation (2.25), is given by

$$\begin{aligned} G^{e_1,\ldots,e_m}_{l_1,\ldots,l_m} &= (G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}})^{e_m}_{l_m} \\ &= \frac{(-1)^{l_m}}{x}\left(\underbrace{(G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} + G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} \bullet e_m)}_{G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} - e_m}(1-x)^{l_m}\right. \\ &\qquad \left. - \underbrace{(G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} + (1-x)\, G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} \bullet e_m)}_{(G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} - e_m) - x G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} \bullet e_m}\right) \\ &= \frac{(-1)^{l_m}}{x}\left((G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} - e_m)\left((1-x)^{l_m} - 1\right) + x\,(G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} \bullet e_m)\right), \end{aligned}$$

and for $x \geq 2$, this is indeed positive, as $G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} - e_m = (G - e_m)^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}}$ and $G^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}} \bullet e_m = (G \bullet e_m)^{e_1,\ldots,e_{m-1}}_{l_1,\ldots,l_{m-1}}$ are both positive, by assumption, and $(1-x)^{l_m} - 1$ is positive since l_m is even. □

The number 2 here is best possible, for consider the graph $\Theta_{1,1,1}$. Among its even subdivisions (in the sense of Theorem 4.1.2) are the graphs $\Theta_{2l,2l,2l}$ ($l \geq 1$), whose chromatic polynomials are given (cf. equation (2.16) by

$$\pi(\Theta_{2l,2l,2l}, x) = \frac{1-x}{x}\left((1-x)^{6l-1} - 3(x-1)^{2l} + 2 - x\right).$$

With this expression, we can verify that, for any given $\varepsilon > 0$, the graph $\Theta_{2l,2l,2l}$ will have a real chromatic root between $2 - \varepsilon$ and 2 for l sufficiently large.

4.1.1 Bounding Chromatic Roots in terms of Corank

As in [29], let $\rho(G) = \max\{|z-1| \colon \pi(G,z) = 0\}$, and, for each positive integer $k \geq 1$,

$$\rho_k = \max\{\rho(G) \colon \text{graphs } G \text{ of corank } k\}. \tag{4.1}$$

Certainly, $\rho_0 = \rho_1 = 1$. And $\rho_{k+1} \geq \rho_k$, for if G has corank k, then the disjoint union of G and a cycle C_n is a graph of corank $k+1$, whose chromatic roots are those of G and the cycle. Further, we may restrict our attention to 2-connected graphs; indeed, if we separate G into its 2-connected components and then glue these components back together along a common edge, the resulting graph G' has the same corank as G, and, by the complete intersection theorem mentioned in Chapter 2, has the same (distinct) chromatic roots as G'.

Theorem 2.0.2 tells us that $\rho_k \leq k$ for $k \geq 1$, while it follows from [29] that $\rho_k \geq \rho_{(k+1;2)} = [1+o(1)]\,k/\log k$, where

$$\rho_{(k+1;2)} = \max\{|z-1| \;:\; z \text{ is a chromatic root of } \Theta_{(k;2)}\}.$$

Whether ρ_k grows linearly or sublinearly is simply not known.

The only 2-connected graphs of corank 2 are 3-ary theta graphs, and so it follows from Theorem 2.1.5 that $\rho_2 = \rho(2,2,2) \approx 1.5247$. Could the same kind of result be true for all k? Not quite; in fact, it fails for $k=3$. Indeed, while $\rho(2,2,2,2) \approx 1.9636$ (cf. Table 2.1), we have $\rho(K_4) = 2$, since $z=3$ is a chromatic root of K_4. Nevertheless, we have found no other counterexample, and so we pose the following.

Conjecture 4.1.3 *For all $k \geq 4$, $\rho_k = \rho_{(k+1;2)}$.*

Since $\rho_{(k+1;2)}$ grows asymptotically as $\frac{k}{\log k}$ (cf. [29]), the truth of this conjecture would imply that indeed ρ_k exhibits sublinear growth.

4.2 Independence Attractors

The relationships between a graph and its independence fractal remains a tantalizing question. Even the restricted question of when an independence fractal is connected seems elusive. Certainly, it does not depend on the connectivity of the graph. We have seen, for instance, that $4K_2$, a disconnected graph, has a connected independence fractal, while Theorem 3.2.34 guarantees the existence of many connected graphs with disconnected independence fractals. We were able to provide a complete answer for graphs with independence number 2 in Section 3.2.3; it may be possible to complete the $\beta = 3$ picture as well.

Just how much about a graph can its independence attractor tell us? Theorem 3.2.32 tells us that $G[K_n]$ and $K_n[G]$ have analytically conjugate independence fractals. Further, since (cf. Section 3.2.1) for any polynomial f, its Julia set is equal to that of $f^{\circ k}$ for any $k \geq 1$, it follows that G^k and G have identical independence fractals. These obversations give a partial answer to the following

Question 4.2.1 *When do two graphs G and H have analytically conjugate independence fractals?*

4.2.1 Independence Attractors of Complexes

The independent sets of vertices in a graph form a *complex*. An (abstract, simplicial) complex on a set X is a collection $\mathcal{C}$ of subsets of X that is closed under containment, i.e., if $A \in \mathcal{C}$ and $B \subseteq A$, then $B \in \mathcal{C}$. The sets in $\mathcal{C}$ are the *faces* of the complex, and faces of cardinality one (the elements of X) are the *vertices* of $\mathcal{C}$. Denoting by f_i the number of faces in $\mathcal{C}$ of cardinality i, define the f-polynomial of $\mathcal{C}$ as

$$f_{\mathcal{C}}(x) = \sum_{i \geq 1} f_i.$$

For a graph G, the collection $\mathcal{C}$ of independent sets of vertices in G is indeed a complex on $V(G)$, as any subset of an independent set is again an independent set; and

$$f_{\mathcal{C}}(x) = f_G(x) = i_G(x) - 1.$$

For two complexes $\mathcal{C}$ and $\mathcal{D}$ on sets X and Y, respectively, we can define their composition $\mathcal{C}[\mathcal{D}]$ as the complex on $X \times Y$ whose faces are all sets $C \times D$, where $C \in \mathcal{C}$ and $D \in \mathcal{D}$. The argument we gave for Theorem 3.2 also proves the following:

$$f_{\mathcal{C} \times \mathcal{D}}(x) = f_{\mathcal{C}}\left(f_{\mathcal{D}}(x)\right). \tag{4.2}$$

Thus, by leaving out the constant term $f_0 = 1$, the polynomials are closed under composition. Define the *attractor* of a complex $\mathcal{C}$ as

$$\text{Frac}\,(\mathcal{C}) = \lim_{k\to\infty} \text{Roots}\,(f_{\mathcal{C}^k})\,.$$

This time, the attractor is the inverse orbit of 0, instead of -1. But 0 is a fixed point of $f_{\mathcal{C}}$, and is not a root of $f'_{\mathcal{C}}$, which implies that $0 \in J(f_{\mathcal{C}})$, and so

$$\text{Frac}\,(\mathcal{C}) = J\,(f_{\mathcal{C}}) = Cl\,\left(\cup_{k\geq 1} \text{Roots}\,(f_{\mathcal{C}^k})\right),$$

illustrating a second advantage to leaving out the constant term: the attractor never contains anything but the Julia set. This does come with a small price: in terms of f- polynomials, Theorems 3.0.13 and 3.0.14 now have the slightly more awkward form

$$f_{\mathcal{C}}(x) = f_{\mathcal{C}-v}(x) + x \cdot \left(f_{\mathcal{C}-[v]}(x) + 1\right)$$

and

$$f_{\mathcal{C}\uplus\mathcal{D}}(x) = (f_{\mathcal{C}}(x) + 1) \cdot (f_{\mathcal{D}}(x) + 1) - 1,$$

where, for v a vertex of $\mathcal{C}$, $\mathcal{C} - v$ consists of all faces in $\mathcal{C}$ that do not contain v, and $\mathcal{C} - [v]$ those faces in $\mathcal{C}$ containing neither v nor any vertex which did not lie in a face (in $\mathcal{C}$) with v.

Probably the simplest example of a complex is an n–simplex, $\mathcal{C}_n$, which consists of an n element set, together with all of its subsets. Since, for each i, f_i is then the binomial coefficient C (n,i), the f-polynomial has the form $f_{\mathcal{C}}(x) = (1+x)^n - 1$. And then, for each $k \geq 1$, $f_{\mathcal{C}}^{\circ k}(x) = (1+x)^{kn} - 1$, whose roots are the kn–th roots of unity, shifted to the left one unit. As $k \to \infty$, the roots become dense on the circle $|z+1| = 1$, and hence

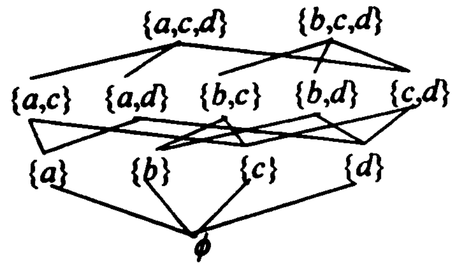
Theorem 4.2.2 *For an n–simplex $\mathcal{C}_n$,*

$$\text{Frac}\,(\mathcal{C}_n) = \{z\,:\,|z+1| = 1\}.$$

Now, if we take an n–simplex C_n and remove one of its faces F of cardinality $m < n$ (along with all faces in C_n containing F), we obtain a complex $C_n^{(m)}$, say, whose f–polynomial can be seen as follows. For each i, the number of faces in C_n of cardinality i which contain F is the binomial coefficient $\mathrm{C}\,(n-m, i-m)$. Hence,

$$\begin{aligned} f_{C_n^{(m)}}(x) &= (1+x)^n - 1 - \sum_{i=1}^{n} \mathrm{C}\,(n-m, i-m)x^i \\ &= (1+x)^n - \sum_{i=m}^{n} \mathrm{C}\,(n-m, i-m)x^i - 1 \\ &= (1+x)^n - x^m(1+x)^{n-m} - 1. \end{aligned}$$

If, for instance, C_4 is the 4– simplex on $\{a, b, c, d\}$, and $C_4^{(2)}$ the complex obtained by removing face $\{a, b\}$ (and those containing it), then $C_4^{(2)}$ is the following complex:



And $f_{C_4^{(2)}}(x) = (1+x)^4 - x^2(1+x)^2 - 1 = 2x^3 + 5x^2 + 4x$. Its attractor, Frac $\left(C_4^{(2)}\right)$, is shown in Figure 4.1. In addition, plots of the attractors Frac $\left(C_4^{(3)}\right)$ and Frac $\left(C_4^{(4)}\right)$ are found in Figures 4.2 and 4.3, respectively.

Let us conclude with a proof of the following.

Theorem 4.2.3 *The complexes C_n, $C_n^{(1)}$ and $C_n^{(2)}$ all have connected attractors.*

Proof Theorem 4.2.2 tells us that Frac (C_n) is connected. Since $C_n^{(1)}$ is nothing but C_{n-1}, then its attractor is also connected. Now consider $C_n^{(2)}$, where $n \geq 3$. Then

$$\begin{aligned} f_{C_n^{(2)}}(x) &= (1+x)^n - x^2(1+x)^{n-2} - 1 \\ &= (1+x)^{n-2}(2x+1) - 1, \end{aligned}$$

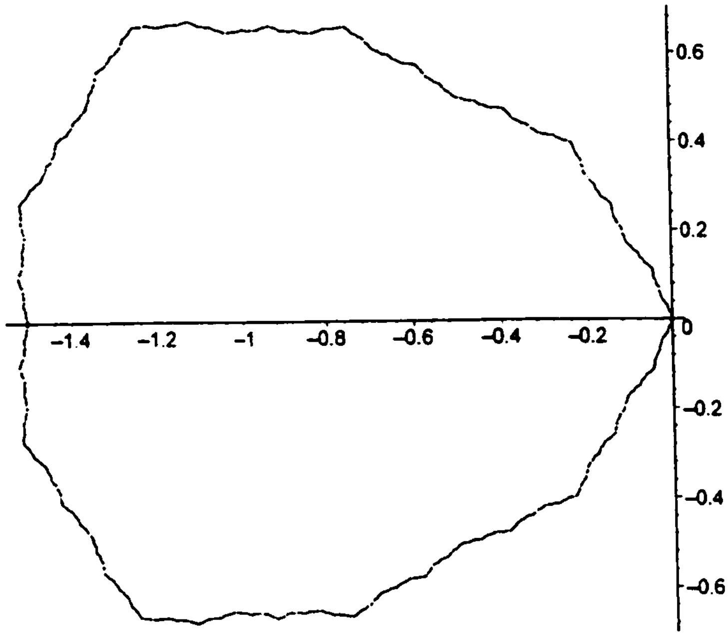


Figure 4.1: The attractor, Frac $\left(\mathcal{C}_4^{(2)}\right)$.

and (by a simple calculation)

$$\left(f_{\mathcal{C}_n^{(2)}}\right)'(x) = (1+x)^{n-3} \cdot (2(n-1)x + n),$$

whose critical points are -1 and $\frac{-n}{2(n-1)}$. If $n = 3$, then

$$f_{\mathcal{C}_3^{(2)}}(x) = (1+x)^3 - x^2(1+x)^{3-2} - 1 = 2x^2 + 3x,$$

whose Julia set, as we saw in Section 3.2.3, is connected.

We may assume, then, that $n \geq 4$. Now $f_{\mathcal{C}_n^{(2)}}(-1) = -1$, and (the reader can verify this)

$$f_{\mathcal{C}_n^{(2)}}\left(\frac{-n}{2(n-1)}\right) = -\underbrace{\left(\frac{n-2}{2(n-1)}\right)^{n-2} \cdot \frac{1}{n-1}}_{\in (0,\frac{1}{4})} - 1,$$

so

$$\left| f_{\mathcal{C}_n^{(2)}}\left(\frac{-n}{2(n-1)}\right) + 1 \right| < \frac{1}{4}.$$

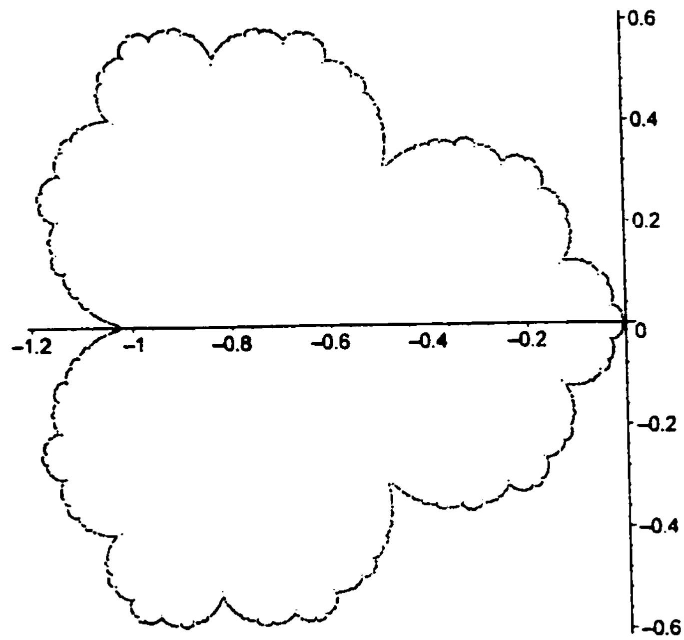


Figure 4.2: The attractor, $\operatorname{Frac}\left(\mathcal{C}_4^{(3)}\right)$.

And $|x+1| < \varepsilon < \frac{1}{4}$ implies

$$\begin{aligned} \left|f_{\mathcal{C}_n^{(2)}}(x) + 1\right| &= |x+1|^{n-2} \cdot |2x+1| \\ &= |x+1|^{n-2} \cdot |2(x+1) - 1| \\ &\leq |x+1|^{n-2}(2|x+1| + 1) \\ &< \varepsilon^{n-2}(2\varepsilon + 1), \end{aligned}$$

which is $< \varepsilon^{n-2} \cdot 2 \leq \varepsilon^2 \cdot 2 < \varepsilon \cdot \frac{1}{4} \cdot 2 = \frac{1}{2}\varepsilon$. Hence, the forward orbit of $\frac{-n}{2(n-1)}$ will converge to -1, and is therefore bounded in $(\mathbb{C}, |\cdot|)$. Thus, since all critical points of $f_{\mathcal{C}_n^{(2)}}$ have bounded forward orbits, its Julia set, $\operatorname{Frac}\left(\mathcal{C}_n^{(2)}\right)$, is connected. □

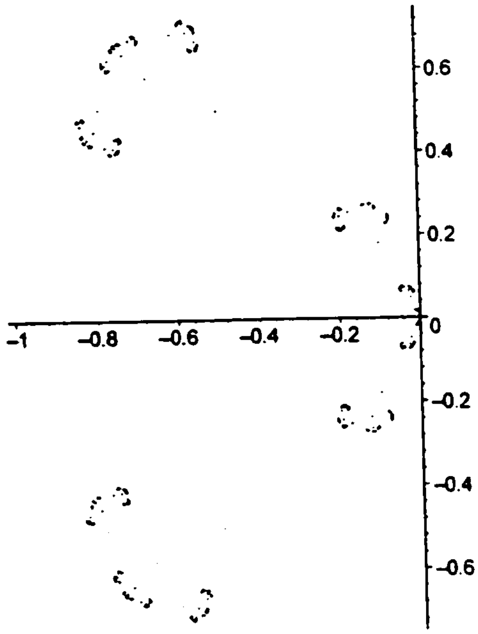


Figure 4.3: The attractor, Frac $\left(\mathcal{C}_4^{(4)}\right)$.

Figures 4.2 and 4.3 show that $\mathcal{C}_4^{(3)}$ is connected, while $\mathcal{C}_4^{(4)}$ is not. Perhaps it is true in general that for some k (possibly dependeing on n), $\mathcal{C}_n^{(j)}$ is disconnected for each $j \geq k$. Computational explorations do suggest that $\mathcal{C}_n^{(n)}$ is always disconnected.

Appendix A

A Maple Procedure for Independence Attractors

Below is the procedure the author used to construct his plots of independence attractors. It takes as input a polynomial $f \in \mathbb{Z}[x]$ (the f– polynomial of a graph or complex), a starting point z (where $z = -1$ for graphs, and $z = 0$ for complexes) and a 'mesh number' N, to be defined momentarily. It then constructs (the beginnings of) the backward orbit $\mathcal{O}^-(z)$ as follows.

First, an imaginary grid is placed over the complex plane: its squares have side $1/N$. Whenever an inverse image X is computed, the grid squares in which the points of X reside are determined. Those points which land in a square previously visited are then discarded, so that their inverse images are not taken.

This is a standard trick (cf. [48]) for computing Julia sets in practice. It allows one to go deeper into the backward orbit, whose sets are actually growing exponentially in size. The result is a more complete, more uniform plot.

The algorithm terminates when all points at given a given step are discarded. That this happens is an immediate consequence of the fact (cf. Section 3.2.1) that Julia sets of polynomials are bounded in $(\mathbb{C}, |\cdot|)$.

```
Attract := proc(f,z,N)
local A,w,r,xx,yy,OldRts,NewRts,pts,i,j,n,symb:
n := degree(f):
A := table():
pts := NULL:
OldRts := [op({fsolve(f,x,complex)} minus {z})]:
pts := pts,[Re(z),Im(z)],op(map([Re,Im],OldRts)):
while nops(OldRts) > 0 do
NewRts := NULL:
for i from 1 to nops(OldRts) do
w := [fsolve(f-OldRts[i],x,complex)]:
for j from 1 to n do
r := w[j]:
xx := ceil(N*(Re(r))):
yy := ceil(N*(Im(r))):
if A[xx,yy]<>1 then
A[xx,yy] := 1:
NewRts := NewRts,r:
pts := pts,[Re(r),Im(r)]:
fi:
od:
od:
OldRts := [NewRts]:
od:
pts := [pts]:
symb := cross:
for i from 1 to min(nops(pts),20) do
if pts[i][2] <> 0 then
symb := point:
```

```
i := min(nops(pts),20):
fi:
od:
if symb=point then
print('Points calculated.  Plotting...'):
else
print('Attractor is Real.  Plotting...'):
fi:
plot(pts,style=point,symbol=symb,colour=black,scaling=constrained);
end;
```

The latter part of the algorithm simply looks at the first 20 points in the list obtained, to "determine" whether the attractor is real. An appropriate plotting symbol is then chosen.

The plot in figure 3.7, for instance, namely $\mathcal{I}(P_3) = J(x^2 + 3x)$ was obtained with the command > `Attract(x2+3*x,-1,100);`

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