INFORMATION TO USERS

EXTENSION THEOREMS AND MINIMAX INEQUALITIES WITH APPLICATIONS TO MATHEMATICAL ECONOMICS

By

Zhou Wu

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
AT
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
August 29, 1997

The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

Canada

DALHOUSIE UNIVERSITY

FACULTY OF GRADUATE STUDIES

The undersigned hereby certify that they have read and recommend to the Faculty of

Graduate Studies for acceptance a thesis entitled "Extension Theorems and Minimax

Inequalities with Applications to Mathematical Economics"

_____

by   Zhou Wu

in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Dated: August 25, 1997

External Examiner

Research Supervisor

Examining Committee

ii
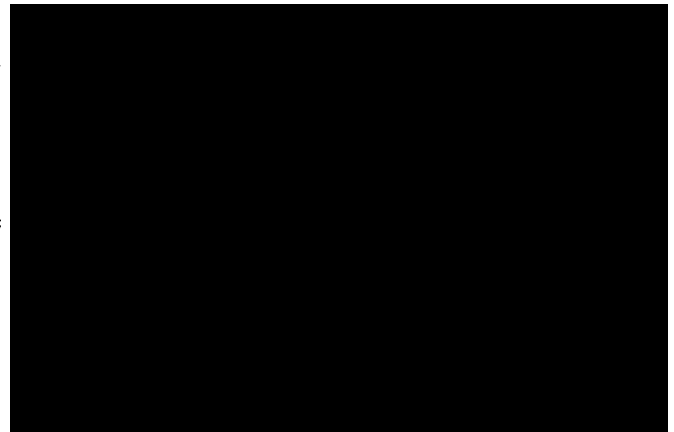
DALHOUSIE UNIVERSITY

Date: **August 29, 1997**

Author: **Zhou Wu**

Title: **Extension Theorems and Minimax Inequalities
            with Applications to Mathematical Economics**

Department: **Mathematics, Statistics and Computing Science**

Degree: **Ph.D.**   Convocation: **Fall**   Year: **1997**

Signature of Author

To my parents

Contents

Abstract

This thesis presents some results in nonlinear analysis and their applications to mathematical economics.

Chapters 2 and 3 are the most important and technical part of this thesis. In Chapter 2, duals of two important results due to Gale and Mas-Colell in 1975 and Shafer and Sonnenschein in 1975 respectively in equilibrium theory are presented. Two examples of how to use these results to resolve equilibrium problems with closed preferences are given. Chapter 3 is devoted to establishing a random version of a Tietze type extension theorem together with its applications. The proof of the theorem is very complicated and technical. We think that the work of Mas-Colell *et al* can be classified as "selection" type in the sense that the proofs almost always invoke Michael's selection theorem or similar ones. Our work can be classified as "extension" type in the sense that we need Tietze type exte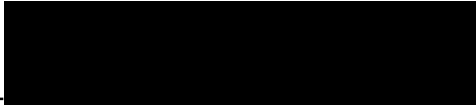nsion theorems to do the proofs, although in a related result, we still have to use the powerful Ky Fan minimax inequality. It is worthwhile to point out that as far as we know, this is the first time that Tietze type extension theorems have been used to study economic problems with closed preferences.

In Chapter 4, we generalize many minimax inequalities. These include Fan's, Sion's, Granas and Liu's and Tan and Yu's results. Applications to fixed point theory, variational inequalities, complementarity problems and abstract economies are given.

Chapter 5 presents a fixed point theorem for correspondences on [0,1]. This is an interesting result for its simplicity and elementary proof. Some examples comparing it to related work and also some simple applications to game theory are included.

Acknowledgments

Chapter 1

Introduction

1.1 Some Basic Concepts and Notation

In this section, we shall introduce some basic concepts and notation that will be used throughout this thesis. More specific concepts and notation will be presented when they are needed.

In this thesis, $\mathbb{R}$ is the set of all real numbers. If A is a set, we shall denote by 2^A the family of all subsets of A and by $\mathcal{F}(A)$ the family of all non-empty finite subsets of A. If A is a subset of a topological space X, we shall denote by $int_X(A)$ the interior of A in X, by $cl_X(A)$ the closure A in X and by $\partial_X(A)$ the boundary of A in X. If A is a subset of a vector space, we shall denote by $co(A)$ the convex hull of A. By a *correspondence* F defined on X with values in Y, denoted by $F : X \to 2^Y$, we mean that to each $x \in X$, F assigns a unique subset $F(x)$ of Y, i.e., an element of 2^Y. A correspondence is also called a *multifunction, set-valued map* or *multi-valued map*, etc.

For a correspondence $F : X \to 2^Y$, we let GrF denote the graph of F, i.e., $GrF = \{(x,y) \in X \times Y : y \in F(x)\}$. Also we define $F^{-1} : Y \to 2^X$ by $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for each $y \in Y$ and $\overline{F} : X \to 2^Y$ by $\overline{F}(x) = \{y \in Y : (x,y) \in cl_{X \times Y}(GrF)\}$ for each $x \in X$.

Let X and Y be topological spaces and $F : X \to 2^Y$ be a correspondence. Then

F is called (1) *upper semicontinuous* if for each open subset U of Y, the set $\{x \in X : F(x) \subset U\}$ is open in X; (2) *lower semicontinuous* if for each closed C of Y, the set $\{x \in X, F(x) \cap C \neq \emptyset\}$ is closed in X; (3) *continuous* if it is both upper semicontinuous and lower semicontinuous.

1.2   An Introduction to Games and Abstract Economies

There are many different kinds of games. In this thesis, we only consider *non-cooperative games*, which are closely related to another concept, *abstract economies*. The latter are also called *non-cooperative generalized games*.

The theory of non-cooperative games studies the behavior of players (also called agents) in any situation where each player's optimal choice may depend on his forecast of the choices of his opponents. Let I be the (finite or infinite) set of players, the set of choices of player i ($i \in I$) is denoted by X_i. Elements of X_i are called *strategies* and X_i is player i's *strategy set*. Let $X = \Pi_{i \in I} X_i$ be the set of strategy vectors. Each strategy vector determines an outcome. Players have preferences over outcomes and this induces preferences over strategy vectors. For convenience we shall work with preferences over strategy vectors. There are two ways to do this. The first is to describe player i's preferences by a binary relation $\tilde{P}_i$ defined on X. Then $\tilde{P}_i(x)$ is the set of all strategy vectors preferred to x. Since the player i only has control over the ith component of x, it is more useful to describe player i's preferences in terms of the good reply set. Given a strategy vector $x \in X$ and a strategy $y_i \in X_i$, let (x^i, y_i) denote the strategy vector obtained from x when player i chooses y_i and the other players keep their choices fixed. We say that y_i is a *good reply* for player i to strategy vector x if $(x^i, y_i) \in \tilde{P}_i$. This defines a correspondence $P_i : X \to 2^{X_i}$, called the good reply correspondence by $P_i(x) = \{y_i \in X_i : (x^i, y_i) \in \tilde{P}_i\}$. It will be convenient to describe preferences in terms of the good reply correspondence P_i rather than preference relation $\tilde{P}_i$. So we will use the former description (i.e., P_i) in

this thesis. Note however that we may lose some information by doing this. Given a good reply correspondence P_i, it will not generally be possible to reconstruct the preference relation $\tilde{P}_i$, unless we know that $\tilde{P}_i$ is transitive. Thus *a game in strategic form* is a tuple $(X_i, P_i)_{i \in I}$ where each $P_i : X \to 2^{X_i}$.

A shortcoming of this model of games is that frequently there are situations in which the choices of players cannot be made independently. To take such possibilities into account, game theorists introduce a correspondence $F_i : X \to 2^{X_i}$ which tells which strategies are actually feasible for player i, given the strategy vector of the others. Note that for the sake of technical convenice, we refer to F_i as a correspondence of the strategies of all players including player i. In modeling most situations, F_i will be independent of player i's choice. The jointly feasible strategy vectors are thus the fixed points of the correspondence $F = \Pi_{i \in I} F_i : X \to 2^X$. A game with the added feasibility or constraint correspondence is called a *generalized game* or *abstract economy*. The reason for the latter name is because this abstract model is found useful in proofs of the general equilibrium theorems in mathematical economics. It is specified by a tuple $(X_i, F_i, P_i)_{i \in I}$ where $F_i, P_i : X \to 2^{X_i}$.

A *Nash Equilibrium* of a strategic form game or abstract economy is a strategy vector x for which no player has a good reply. For a game, an equilibrium is an $x \in X$ such that $P_i(x) = \emptyset$ for each i. For an abstract economy, an equilibrium is an $x \in X$ such that $x \in F(x)$ and $F_i(x) \cap P_i(x) = \emptyset$ for each i.

Nash [59] proved the existence of equilibria for games where preferences are representable by continuous quasi-concave utilities, and the strategy sets are simplexes. Debreu [20] proved the existence of equilibrium for abstract economies. He assumed that strategy sets are contractible polyhedra, that the feasibility correspondences have closed graphs, that the maximized utilities are continuous and that the sets of utility maximizers over each constraint set are contractible. These conditions are jointly assumptions on utility and feasibility. The simplest way to make separate assumptions is to assume that strategy sets are compact and convex, that utilities are continuous

and quasi-concave, and that the constraint correspondences are continuous with compact convex values. Arrow and Debreu [3] used Debreu's result to prove the existence of Walrasian equilibrium for an economy and coined the term abstract economy.

Since Nash's and Debreu's work in the fifties, a large number of papers have been produced generalizing, improving or developing it. Among them, we would like only to mention Gale and Mas-Colell's, and Shafer and Sonnenchein's papers here because the main topic of Chapter 2 is studying duals of the results in these two papers. Gale and Mas-Colell [32] proved a fixed point theorem which allowed them to prove the existence of equilibria for a game without ordered preferences. They assumed that strategy sets are compact convex sets and that the good reply correspondences are convex valued and have **open graphs**. In the work following this paper, it was found that the *open graph* in his original result can be replaced by *open lower section* or *lower semicontinuity*. Shafer and Sonneschein proved the existence of equilibria for abstract economies without ordered preferences. They assumed that the good reply correspondences have **open graphs** and satisfy the convexity/irreflexivity condition $x_i \notin co(P_i(x))$. They also assumed that feasibility correspondences are continuous with compact convex values. The problem is:

Can the good reply correspondences in Gale and Mas-Colell's and Shafer and Sonnenchein's models be **upper semicontinuous with closed convex values** *instead of the conditions such as* **open graph or similar ones** *?*

We give a complete answer to this problem and do more work on its extensions in Chapter 2. We point out here that Gale and Mas-Colell's result can be considered as the generalization of the Fan-Browder fixed point theorem and our corresponding result can be considered as the generalization of the Kakutani-Fan-Glickberg fixed point theorem. Further, while we can write some theorems that can include both situations, we only present one of these in finite dimensional spaces and give two applications of it. The reason for doing this is that we want to illustrate the spirit of studying closed preference games rather than becaming involved in complicated discussions of the structures of infinite dimensional spaces used in this area.

Another direction in general equilibrium theory is to take the structure of the space of the agents into account in the economics models. This was initialized by Aumann and Schmeidler. Aumann [5] and [6] extended Debreu's abstract economy theory that permits the set of agents to be a measure space. Aumman resolved this problem by assuming the set of agents is an atomless measure space which means the influence of each agent is "negligible". The similar argument was applied in game theory by Schmeidler [66]. We explore the equilibrium problems for this kind of economics models in Chapter 3.

The tools for studying these problems are the Tiezte type extension theorems and the Ky Fan inequality. As far as we know, this is the first time that the Tiezte type extension theorems have been used to study equilibrium problems in mathematical economics. At the beginning of this study, we did not know there existed any Tiezte type extension theorems for correspondences. So we developed one and used it to resolve the equilibrium problems. However, after we carefully checked several hundred papers, we found there indeed were a few papers dealing with the set-valued extension problems. We felt a little disappointed at first but this was remedied when we found we could use several construction techniques to develop a random version Tiezte type extension theorem. The proof of this theorem is very complicated and technical. It is one of the author's favorite results in this thesis.

1.3 An Introduction to Some Minimax Inequalities and Counter Examples to a Conjecture of K. K. Tan

In Chapter 4, we shall study the generalizations of the Ky Fan minimax inequality and their applications to fixed point theory, variational inequalities, complementarity problems and abstract economies.

In 1972, Ky Fan proved a celebrated minimax theorem [30] which is equivalent to

the following:

Theorem 1.3.1 *Let E be a topological vector space and X be a non-empty compact convex subset of E. Let f be a real-valued function defined on $X \times X$ such that*

(1) for each $x \in X$, $f(x,x) \leq 0$;

(2) for each fixed $x \in X$, $f(x,y)$ is a lower semicontinuous function of y on X;

(3) for each fixed $y \in X$, $f(x,y)$ is a quasi-concave function of x on X.

Then there exists $\hat{y} \in X$ such that $f(x,\hat{y}) \leq 0$ for all $x \in X$.

In the above theorem, the space E is not required to be Hausdorff as Ky Fan originally stated. This fact was observed by Ding and Tan [23]. This slight improvement sometimes is important since in the study of unifying approach to existence of Nash equilibria by Balder [8], he has to use the above minimax theorem under non-Hausdorff settings.

Our work is to weaken the continuity of the function, the compactness and the convexity conditions in Theorem 1.3.1. This is motivated by Baye *et al* [11] who studied the characterizations of the existence of equilibria in games with discontinuous and non-quasi-concave payoffs. Using one of our results, we also give a generalization of the following minimax theorem due to Sion [74]:

Theorem 1.3.2 *Let X be a convex subset of a liner topological space, Y be a compact convex subset of a linear topological space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that*

(1) for all $y \in Y$ and $\lambda \in \mathbb{R}$, the set $\{x \in X : g(x,y) \geq \lambda\}$ is convex;

(2) for all $x \in X$ and $\lambda \in \mathbb{R}$, the set $\{y \in Y : g(x,y) \leq \lambda\}$ is convex.

Then

$$\min_{y \in Y} \sup_{x \in X} f = \sup_{x \in X} \min_{y \in Y} f.$$

The importance of Sion's weakening of continuity to semicontinuity was that it indicated that many kinds of minimax problems had equivalent formulations in terms of subsets of $X \times Y$, and led to Fan's 1972 work [30] on sets with convex sections and

minimax inequality (Theorem 1.3.1), which have since found many applications in economic theory. So K. K. Tan postulated if another minimax theorem, the Kneser minimax theorem, had a geometric form, or a similar one such as intersection form. For convenience, we state Kneser's minimax theorem [53] as follows:

Theorem 1.3.3 *Let X be a non-empty convex set in a vector space and Y be a non-empty compact convex subset of a topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each $x \in X$, $f(x, y)$ is lower semicontinuous and convex on Y, and for each $y \in Y$, $f(x, y)$ is concave on X. Then*

$$\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y).$$

The following is a conjecture of K. K. Tan, which is an attempt to formulate an intersection form for Kneser's minimax theorem:

Problem 1.3.4 *Let X be a non-empty convex subset of a vector space, Y be a non-empty compact convex subset of a topological vector space, $F : X \to 2^Y \backslash \{\emptyset\}$ be such that*

(i) for all $x \in X$, $F(x)$ is closed and convex;

(ii) for all $y \in Y$, $X \backslash F^{-1}(y)$ is convex.

Then $\cap_{x \in X} F(x) \neq \emptyset$.

Since Kneser's minimax theorem also plays a very important role in game theory, mathematical economics and studying variational inequalities, if the above conjecture were true, we could write a geometric form for it which in turn could be used to obtain some significant results. Unfortunately, it is not true. We have two counter examples as follows:

Example 1.3.5 *Let $I = [0, 1], X = Y = I$. We define $F : X \to 2^Y \backslash \{\emptyset\}$ by*

$$F(x) = \begin{cases} \{0\}, & 0 \leq x \leq \frac{1}{2}; \\ \{1\}, & \frac{1}{2} < x \leq 1. \end{cases}$$

Then

(a) For each $x \in X$, $F(x)$ is closed and convex.

(b) For any $y \in Y$, if $y = 0$, $X \backslash F^{-1}(y) = X \backslash [0, 1/2] = (1/2, 1]$; if $y = 1$, $X \backslash F^{-1}(y) = X \backslash (1/2, 1] = [0, 1/2]$; if $0 < y < 1$, $F^{-1}(y) = \emptyset$. So (ii) is satisfied.

But $\cap_{x \in X} F(x) = \emptyset$. ∎

Example 1.3.6 *Let* $I = [0, 1]$, $X = Y = I$. *We define* $F : X \to 2^Y \backslash \{\emptyset\}$ *by*

$$F(x) = \begin{cases} [0, 1/4], & 0 \le x \le \frac{1}{2}; \\ [3/4, 1], & \frac{1}{2} < x \le 1. \end{cases}$$

Then

(a) For each $x \in X$, $F(x)$ is closed and convex.

(b) For any $y \in Y$, if $y \in [0, 1/4]$, $X \backslash F^{-1}(y) = X \backslash [0, 1/2] = (1/2, 1]$; if $y \in [3/4, 1]$, $X \backslash F^{-1}(y) = X \backslash (1/2, 1] = [0, 1/2]$; if $y \in (1/4, 3/4)$, $F^{-1}(y) = \emptyset$. So (ii) is satisfied.

But $\cap_{x \in X} F(x) = \emptyset$. ∎

We note that in this thesis, the Kneser minimax theorem is mainly used to study the existence of solutions for variational inequalities.

1.4   An Introduction to a Simple Fixed Point Problem

In Chapter 5, we study a fixed point problem on [0, 1].

The problem is motivated by the work of Milgrom and Roberts [58] and Guillerme [36]. The authors of the two papers found an interesting fact about a function on $[0, 1]$: A real function $f : [0, 1] \to [0, 1]$ that is upper semicontinuous on the right and lower semicontinuous on the left has a fixed point. So our problem is:

Does a correspondence that is upper semicontinuous on the right and lower semicontinuous on the left have a fixed point?

We give a yes answer to this problem. Some examples comparing it to related work and also some simple applications to game theory are included.

Chapter 2

Equilibria for Games and Abstract Economies: Duals for Gale-Mas-Colell's and Shafer-Sonnenschein's Theorems

2.1   Introduction

Let us start with the following Fan-Browder fixed point theorem that appears in [29] and [16]:

Theorem 2.1.1 *Let X be a nonempty compact convex subset of a topological vector space E, and $F : X \to 2^X$ a multifunction with nonempty convex values such that for each $x \in X$, $F^{-1}(x)$ is open. Then F has a fixed point.*

In studying game theory and mathematical economics, we need to study a family of correspondences from the product space of several spaces into each of them. In 1975 and 1979, Gale and Mas-Colell in [32] and [33] proved the following theorem:

Theorem 2.1.2 *Let I be a finite index set. For each $i \in I$, let X_i be a nonempty compact convex subset of $\mathbb{R}^{n_i}$, and $F_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ a lower semicontinuous correspondence with convex values. Then there exists $x \in X$ such that for each $i \in I$ either $x_i \in F_i(x)$ or $F_i(x) = \emptyset$.*

Originally, in Gale-Mas-Colell [32] the correspondences in the above theorem were assumed to have open graphs instead of lower semicontinuous. In Gale-Mas-Colell [33], they gave the above form while commenting that the proof of the above theorem was the same as the original one, which used Michael's selection theorem. In [22], Deguire and Lassonde proved the following theorem which partially generalizes Theorem 2.1.2.

Theorem 2.1.3 *Let I be any index set. For each $i \in I$, let X_i be a nonempty compact convex subset of a Hausdorff locally convex space E_i, and $F_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ be a correspondence with convex values such that for any $y_i \in X_i$, $F_i^{-1}(y_i)$ is open in X. Further, for any $x \in X$, there exists $i \in I$ such that $F_i(x) \neq \emptyset$. Then there exist $x \in X$ and $i \in I$ such that $x_i \in F_i(x)$.*

In 1975, Shafer and Sonnenschein [69] proved the following theorem for abstract economies:

Theorem 2.1.4 *Let I be finite and let $(X_i, F_i, P_i)_{i \in I}$ be an abstract economy such that for each i,*
(1) $X_i \subset \mathbb{R}^{n_i}$ is nonempty, compact and convex;
(2) F_i is a continuous correspondence with nonempty compact convex values;
(3) $Gr P_i$ is open in $X \times X_i$;
(4) $x_i \notin co(P_i(x))$ for all $x \in X$.
Then there is an equilibrium.

On the other hand, Theorem 2.1.1 can be thought of as a dual of the following well-known Kakutani-Fan-Glicksberg theorem (Kakutani [43]; Fan [28]; Glicksberg [34]):

Theorem 2.1.5 *Let X be a nonempty compact convex subset of a Hausdorff locally convex space E, $F : X \to 2^X$ be an upper semicontinuous multifunction with nonempty closed convex values. Then F has a fixed point.*

Our problem is whether the conditions in Theorem 2.1.2, Theorem 2.1.3 and Theorem 2.1.4 such as lower semicontinuity or having an open graph, etc. can be replaced by upper semicontinuity (or similar conditions)? i.e, are there corresponding duals for these theorems? We give a positive answer to this problem and expand on the original problem. Also some applications to general equilibrium theory in mathematical economics are included.

2.2 A Star-shaped Extension Theorem and an Existence Theorem of Equilibria for a Game with Star-shaped Good Replies

A set X in a linear space is said to be *star-shaped* if there exists $x_0 \in X$ such that for any $x \in X$, $t x_0 + (1 - t)x \in X$ for all $t \in [0, 1]$. Such an x_0 is called a *center* of the star-shaped set X.

Proposition 2.2.1 *Let X, Y be two star-shaped sets in a linear space, then $X + Y$ is also a star-shaped set.*

Proof. Let x_0 be a center of X and y_0 be a center of Y. Let $z_0 = x_0 + y_0 \in X + Y$. For any $z = x + y \in X + Y$ and $t \in [0, 1]$, $t z_0 + (1 - t)z = [t x_0 + (1 - t)x] + [t y_0 + (1 - t)y] \in X + Y$. So $X + Y$ is also a star-shaped set. ∎

The following lemma is a remark in Aubin and Ekeland [4] on page 108.

Lemma 2.2.2 *If F is a compact-valued correspondence from a metric space X to a metric space Y, then F is upper semicontinuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\text{for all } x \in O(x_0, \delta), \ F(x) \subset O(F(x_0), \varepsilon),$$

where $O(x_0, \delta) := \{x \in X : d(x, x_0) < \delta\}$ and $O(F(x_0), \varepsilon) := \cup_{y \in F(x_0)} O(y, \varepsilon)$.

We frequently refer to the following result, which is Proposition 11 on page II.34 in Bourbaki [15].

Lemma 2.2.3 *Let E be a metrizable locally convex space. The topology of E can be defined by a distance that is invariant under translations, and for which the open balls are convex.*

We now prove a Tietze-Dugundji extension theorem for upper semicontinuous correspondences with nonempty compact star-shaped values.

Theorem 2.2.4 *Let M be a nonempty closed subset of the metrizable space X, E a metrizable locally convex space, and $F : M \to 2^E$ an upper semicontinuous correspondence with nonempty compact star-shaped values. Then there exists an upper semicontinuous correspondence $\tilde{F} : X \to 2^E$ with nonempty compact star-shaped values such that $\tilde{F}|_M = F$ and $\tilde{F}(x) \subset co(F(M))$ for each $x \in X$.*

Proof. (1) Let d be a metric inducing the topology on X. By Lemma 2.2.3, the topology on E can be induced by a metric ρ on E which is invariant under translations, and for which the open balls are convex.

To each $y \in X \backslash M$, we assign an open ball U_y in $X \backslash M$ with center at y and $diam(U_y) < d(U_y, M)$, where $diam(U_y)$ denotes the diameter of the set U_y. This gives us a covering (U_y) of $X \backslash M$. Since X is a metric space, there is a partition of unity $\{f_y\}_{y \in X \backslash M}$ on $X \backslash M$ subordinated to the covering $\{U_y : y \in X \backslash M\}$, that is, $f_y : X \backslash M \to [0, 1]$ is continuous for each $y \in X \backslash M$ and is zero outside of U_y, while each $x \in X \backslash M$ has an open neighborhood $V(x)$ in $X \backslash M$ such that all but a finite number of f_y are identically zero on $V(x)$ and

$$\sum_{y \in X \backslash M} f_y(x) = 1, \text{ for all } x \in X \backslash M.$$

For each $y \in X \backslash M$ we choose an $m_y \in M$ such that $d(m_y, U_y) < 2d(M, U_y)$ and define $\tilde{F} : X \to 2^E$ by

$$\tilde{F}(x) = \begin{cases} F(x), & \text{if } x \in M; \\ \sum_{y \in X \backslash M} f_y(x) F(m_y), & \text{if } x \in X \backslash M. \end{cases}$$

By Proposition 2.2.1, $\tilde{F}(x)$ is a nonempty compact star-shaped subset of $co(F(M))$ for each $x \in X$.

(2) We now show that it is upper semicontinuous at each point in $X \backslash M$.

For $r > 0$ and $p \in E$, let $O(p, r) = \{x \in E : \rho(p, x) < r\}$. First let $x_0 \in X \backslash M$. Then we can find an open neighborhood $V(x_0)$ of x_0 in $X \backslash M$ (since M is closed in X) such that all but only a finite number of f_y, $y \in X \backslash M$ are identically zero. We denote the latter as $f_{y_1}, \ldots, f_{y_n}$. For any $\varepsilon > 0$, since each $F(m_{y_i})$ is compact (hence bounded), we can find $\delta > 0$ such that $\delta y \in O(0, \varepsilon/n)$ for all $y \in \cup_{i=1}^n F(m_{y_i})$.

Since each $f_{y_i}(x)$ is continuous, there exists an open neighborhood $V'(x_0)$ of x_0 in $X \backslash M$ such that for each $i \in \{1, \ldots, n\}$, $|f_{y_i}(x) - f_{y_i}(x_0)| < \delta$ for all $x \in V'(x_0)$. Let $V''(x_0) = V'(x_0) \cap V(x_0)$, then $V''(x_0)$ is an open neighborhood of x_0 in $X \backslash M$. Let $i \in \{1, \ldots, n\}$. For each $x \in V''(x_0)$ and any $y \in F(m_{y_i})$, we have

$$\begin{aligned} \rho(f_{y_i}(x)y, f_{y_i}(x_0)y) &= \rho((f_{y_i}(x) - f_{y_i}(x_0))y, 0) \\ &\leq \varepsilon/n, \end{aligned}$$

It follows that

$$f_{y_i}(x)y \in f_{y_i}(x_0)F(m_{y_i}) + O(0, \varepsilon/n),$$

so that

$$f_{y_i}(x)F(m_{y_i}) \subset f_{y_i}(x_0)F(m_{y_i}) + O(0, \varepsilon/n).$$

Therefore

$$\sum_{i=1}^n f_{y_i}(x)F(m_{y_i}) \subset \sum_{i=1}^n f_{y_i}(x_0)F(m_{y_i}) + O(0, \varepsilon);$$

i.e., $\tilde{F}(x) \subset \tilde{F}(x_0) + O(0, \varepsilon)$ for all $x \in V''(x_0)$, which implies that $\tilde{F}$ is upper semicontinuous at x_0.

(3) We now show that it is upper semicontinuous at each point in $\partial_X(M)$.

(3.1) Let $x_0 \in \partial_X(M)$ be given. Then $\tilde{F}(x_0) = F(x_0)$. Since F is upper semicontinuous, for any $\varepsilon > 0$, by Lemma 2.2.2 there exists $\delta_1 > 0$ such that $F(x) \subset F(x_0) + O(0, \varepsilon)$, i.e., $F(x) \subset \tilde{F}(x_0) + O(0, \varepsilon)$ for all $x \in O(x_0, \delta_1) \cap M$.

(3.2) If $x \in X \backslash M$ and $f_y(x) \neq 0$ for some $y \in X \backslash M$, then by the construction of f_y, we have $x \in U_y$. Applying the triangle inequality yields

$$d(m_y, x) \leq d(m_y, U_y) + diam(U_y) \leq 3d(M, U_y) \leq 3d(x_0, x),$$

and therefore

$$d(m_y, x_0) \leq d(m_y, x) + d(x, x_0) \leq 4d(x_0, x).$$

(3.3) Take $\delta_2 = \delta_1/4$. For any $x \in O(x_0, \delta_2) \cap (X \backslash M)$, if $f_y(x) \neq 0$ for some $y \in X \backslash M$, we have $d(m_y, x_0) \leq 4d(x_0, x) < 4\delta_2 = \delta_1$. Hence $F(m_y) \subset \tilde{F}(x_0) + O(0, \varepsilon)$.

(3.4) For any $x \in O(x_0, \delta_2) \cap (X \backslash M)$, we have $\sum_{y \in X \backslash M} f_y(x) = 1$ and only a finite number of $f_y(x)$'s are not zero.

Note that $O(0, \varepsilon)$ is convex, we have

$$\sum_{y \in X \backslash M} f_y(x) F(m_y) \subset \sum_{y \in X \backslash M} f_y(x) F(x_0) + O(0, \varepsilon),$$

or

$$\tilde{F}(x) \subset \tilde{F}(x_0) + O(0, \varepsilon)$$

for any $x \in O(x_0, \delta_2) \cap (X \backslash M)$.

(3.5) For any $x \in M \cap O(x_0, \delta_2) \subset M \cap O(x_0, \delta_1)$, we have

$$\tilde{F}(x) = F(x) \subset F(x_0) + O(0, \varepsilon) = \tilde{F}(x_0) + O(0, \varepsilon).$$

(3.6) By (3.4) and (3.5), for any $x \in O(x_0, \delta_2)$, we have $\tilde{F}(x) \subset \tilde{F}(x_0) + O(0, \varepsilon)$. Thus F is upper semicontinuous at $x_0 \in \partial_X M$. ∎

We remark that Theorem 2.2.4 is a partial generalization of Theorem 2.1 in [54] in that F may have star-shaped values instead of convex values while the space E is a metrizable locally convex space instead of a Hausdorff locally convex space.

Let us recall the definition of an acyclic space. A compact metrizable space X is said to be *acyclic* if

(1) X is nonempty;

(2) the homology groups $H_q(X)$ vanish for $q > 0$;

(3) the reduced 0-th homology group $\tilde{H}_0(X)$ vanishes.

Obviously nonempty compact convex or star-shaped sets in a metrizable locally convex space are acyclic.

The following lemma is the Eilenberg-Montgomery fixed point theorem [27]:

Lemma 2.2.5 *Let X be an acyclic absolute neighborhood retract and $F: X \to 2^X$ an upper semicontinuous correspondence such that for every $x \in X$ the set $F(x)$ is acyclic. Then F has a fixed point.*

The following fact is easy to prove.

Lemma 2.2.6 *Let F be an upper semicontinuous correspondence from the topological space X to the topological space Y, then the set $\{x \in X : F(x) \neq \emptyset\}$ is a closed subset of X.*

Now we give the following theorem:

Theorem 2.2.7 *Let I be countable. For each $i \in I$, let X_i be a nonempty compact convex subset of the metrizable locally convex space E_i, and $F_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ an upper semicontinuous correspondence with closed star-shaped values. Then there exists $x \in X$ such that for each $i \in I$, either $x_i \in F_i(x)$ or $F_i(x) = \emptyset$.*

Proof. We know that $E := \Pi_{i \in I} E_i$ is also a locally convex space when equipped with the product topology. Since I is countable and each E_i is metrizable, E is metrizable (refer to Corollary 7.3, page 191 in [25]). By the Tychonoff theorem (refer to Theorem 1.4, page 224 in [25]), X is a compact subset of E. Obviously, X is convex. Hence X is acyclic and is an absolute neighborhood retract. For each $i \in I$,

let $C_i = \{x \in X : F_i(x) \neq \emptyset\}$. By Lemma 2.2.6, C_i is a closed subset of X. Define $\tilde{F}_i : X \to 2^{X_i}$ as follows:

(1) If $C_i = \emptyset$, let $\tilde{F}_i(x) = X_i$ for all $x \in X$;

(2) If $C_i = X$, let $\tilde{F}_i(x) = F_i(x)$ for all $x \in X$;

(3) If C_i is a proper nonempty subset of X, by Theorem 2.2.4, there exists an upper semicontinuous correspondence $\tilde{F}_i : X \to 2^{X_i}$ with nonempty closed star-shaped values such that $\tilde{F}_i(x) = F_i(x)$ for all $x \in C_i$.

Define $F : X \to 2^X$ by $F = \Pi_{i \in I} \tilde{F}_i$. Then F is an upper semicontinuous correspondence (by Lemma 3 of Fan [28]) with nonempty closed star-shaped values in X. By the Eilenberg-Montgomery fixed point theorem (Lemma 2.2.5), there exists $x \in X$ such that $x \in F(x)$. Now if $F_i(x) \neq \emptyset$, we have $\tilde{F}_i(x) = F_i(x)$, which in turn implies $x_i \in F_i(x)$. ∎

A direct application of Theorem 2.2.7 yields the following result:

Theorem 2.2.8 *Let $(X_i, P_i)_{i \in I}$ be a game, where I is countable. For each $i \in I$, let X_i be a nonempty compact convex subset of a metrizable locally convex space E_i. Suppose that for each i, $P_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ is an upper semicontinuous correspondence with closed star-shaped values and $x_i \notin P_i(x)$ for all $x \in X$. Then there exists an equilibrium $x \in X$, i.e., $P_i(x) = \emptyset$ for each $i \in I$.*

Remark: Note that in the Eilenberg-Montgomery fixed point theorem, the values of the correspondence are assumed to be acyclic only. We wonder whether the sum of two acyclic sets is acyclic. If it were true, we could replace the star-shaped values with acyclic ones in Theorem 2.2.4 and hence Theorem 2.2.7 and Theorem 2.2.8 could be improved in this way. However, K. Johnson gives the following example which tells us that even the sum of two contractible sets is not necessarily an acyclic set.

Example 2.2.9 *Let two figures in the plane have the shapes of C and I, where the height of I is equal to that of the gap of C. Obviously they are contractible and hence they are acyclic. But the sum of them is a set with the homology group $H_1 \neq 0$.*

Nevertheless, we have the following open problem:

Let $I = \{1, \ldots, n\}$. For each $i \in I$, let X_i be an acyclic absolute neighborhood retract and $F_i : X := \Pi_{j=1}^n X_j \to 2^{X_i}$ an upper semicontinuous correspondence with acyclic or empty-set values. Does there exist $x \in X$ such that for each $i \in I$ either $x_i \in F_i(x)$ or $F_i(x) = \emptyset$?

We next give an existence theorem for equilibria of a game in which the preferences are majorized by upper semicontinuous correspondences. Let us recall some concepts and notation introduced by Tan and Yuan in [76]:

Let X be a topological space, Y a nonempty subset of a vector space E. Let $\theta : X \to E$ be a single-valued map and $\phi : X \to 2^Y$ be a correspondence. Then $\phi : X \to 2^Y$ is said to be of *class $\mathcal{U}_\theta$* if (a) for each $x \in X, \theta(x) \notin \phi(x)$ and (b) ϕ is upper semicontinuous with closed and convex values in Y; (2) ϕ_x is a $\mathcal{U}_\theta$-*majorant* of ϕ at x if there is an open neighborhood $N(x)$ of x in X and $\phi_x : N(x) \to 2^Y$ such that (a) for each $z \in N(x)$, $\theta(z) \subset \phi_x(z)$ and $\theta(z) \notin \phi_x(z)$ and (b) ϕ_x is upper semicontinuous with closed and convex values; (3) ϕ is said to be $\mathcal{U}_\theta$-*majorized* if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a $\mathcal{U}_\theta$-majorant ϕ_x of ϕ at x.

The following result is a particular case of Theorem 2.1 in [76]:

Lemma 2.2.10 *Let X be a metrizable space and Y a subset of a metrizable topological vector space E. Let $\theta : X \to E$ and $P : X \to 2^Y \setminus \{\emptyset\}$ be $\mathcal{U}_\theta$-majorized. Then there exists a correspondence $\Phi : X \to 2^Y$ of class $\mathcal{U}_\theta$ such that $P(x) \subset \Phi(x)$ for each $x \in X$.*

Here we shall deal with $X = \Pi_{i \in I} X_i$ and $\theta = \pi_i : X \to X_i$, the projection from X onto X_i. In the following, we shall write $\mathcal{U}$ instead of $\mathcal{U}_\theta$.

Theorem 2.2.11 *Let $(X_i, P_i)_{i \in I}$ be a game, where I is countable. For each $i \in I$, let X_i be a nonempty compact convex subset of a metrizable locally convex space E_i. Suppose that for each $i \in I$, $P_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ is a $\mathcal{U}$-majorized correspondence*

such that the set $C_i := \{x \in X : P_i(x) \neq \emptyset\}$ is closed in X. Then there exists an equilibrium.

Proof. For each i, since P_i is $\mathcal{U}$-majorized, by Lemma 2.2.10, there exists a correspondence $\Phi_i : C_i \to 2^{X_i}$ of class $\mathcal{U}$ such that $P_i(x) \subset \Phi_i(x)$ for all $x \in C_i$. Since C_i is closed in X, by Theorem 2.2.4 there exists an upper semicontinuous correpondence $\Psi_i : X \to 2^{X_i}$ with nonempty star-shaped values such that $\Psi_i|_{C_i} = \Phi$. Now define $F : X \to 2^X$ by $F = \Pi_{i \in I} \Psi_i$. Then F is an upper semicontinuous correspondence (by Lemma 3 of Fan [28]) with nonempty closed star-shaped values in X. By the Eilenberg-Montgomery fixed point theorem (Lemma 2.2.5), there exists $x \in X$ such that $x \in F(x)$. It is easy to check that x is an equilibrium. ∎

2.3 Equilibria for Games and Abstract Economies in Locally Convex Spaces

In this section, we shall study equilibria for games and abstract economies in locally convex spaces.

Let E be a topological vector space. We shall denote by E' the continuous dual of E, by $\langle w, x \rangle$ the pairing between E' and E for $w \in E'$ and $x \in E$ and by $Re\langle w, x \rangle$ the real part of $\langle w, x \rangle$.

First we have

Theorem 2.3.1 *Suppose that I is a (possibly uncountable) index set. For each $i \in I$, let X_i be a nonempty compact convex subset of the Hausdorff locally convex space E_i, $F_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ be an upper semicontinuous correspondence with closed convex values. Then there exists $x \in X$ such that for each $i \in I$, either $x_i \in F_i(x)$ or $F_i(x) = \emptyset$.*

Proof. Suppose that the conclusion is false, then for any $x \in X$ there must exist $i \in I$ such that $x_i \notin F_i(x) \neq \emptyset$. By the Hahn-Banach theorem, there exists $p_i \in E_i'$

such that

$$Re\langle p_i, x_i \rangle > Re\langle p_i, y_i \rangle \quad \text{for all } y_i \in F_i(x).$$

For each $i \in I$, let $\pi_i : X \to X_i$ be the projection map. Define

$$V_{p_i} = \{x \in X : Re\langle p_i, \pi_i(x) \rangle > Re\langle p_i, y_i \rangle \text{ for all } y_i \in F_i(x)\}.$$

Since F_i is upper semicontinuous, $V_{p_i}(x)$ is open. Now we have

$$X \subset \bigcup_{i \in I} \bigcup_{p_i \in E_i'} V_{p_i}.$$

Since X is compact, it is covered by finitely many $V_{p_i^j}$, where $p_i^j \in E_i', j = 1, \dots, n_i$ for finitely many i in I. Without loss of generality, we suppose $i = 1, \dots, m$. So

$$X = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} V_{p_i^j}.$$

Let $\{f_i^j\}, i = 1, \dots, m$ and $j = 1, \dots, n_i$ be a partition of unity subordinated to this cover and let $\phi : X \times X \to \mathbb{R}$ be defined as follows

$$\phi(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} f_i^j(x) Re\langle p_i^j, \pi_i(x) - y_i \rangle.$$

Since for each fixed $y \in X$, $x \mapsto \phi(x, y)$ is continuous and for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is concave, there exists $\bar{x} \in X$ such that $\phi(\bar{x}, y) \leq 0$ for all $y \in X$ by the Ky Fan inequality in [30] (refer to Theorem 1.3.1).

Now since there is at least one i such that $F_i(\bar{x}) \neq \emptyset$ (see the beginning of this proof), we can take $\bar{y}$ as follows: let $\bar{y}_i$ be any point in $F_i(\bar{x})$ if it is nonempty and be any point in X_i if $F_i(\bar{x})$ is empty. We prove that $\phi(\bar{x}, \bar{y}) > 0$. In fact, if $f_i^j(\bar{x}) > 0$, then $\bar{x} \in V_{p_i^j}$, which in turn implies that $Re\langle p_i, \pi_i(\bar{x}) \rangle - Re\langle p_i, \bar{y}_i \rangle > 0$ since $\bar{y}_i \in F_i(\bar{x})$. Note that there must be some $f_i^j(\bar{x}) > 0$ since $\sum_{i=1}^{m} \sum_{j=1}^{n} f_i^j(\bar{x}) = 1$. This is a contradiction. ∎

It is easy to write an equilibrium existence theorem for a game corresponding to Theorem 2.3.1, which we omit here.

Remark. Theorem 2.3.1 generalizes the well-known Fan-Glicksberg fixed point theorem. However, we know that the Himmelberg fixed point theorem [40] is a little more general than the Fan-Glicksberg fixed point theorem. We formulate the following problem here:

Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of Hausdorff locally convex space E_i and D_i a nonempty compact subset of X_i. If for each $i \in I$, $F_i : X := \Pi_{j \in I} X_j \to 2^{D_i}$ is an upper semicontinuous correspondence with closed convex values, does there exist $x \in D := \Pi_{j \in I} D_j$ such that for each $i \in I$ either $x_i \in F_i(x)$ or $F_i(x) = \emptyset$?

The following lemma was proved in [63]:

Lemma 2.3.2 *Let M be a nonempty closed subset of a metric space X, E a metrizable locally convex space, $F : M \to 2^E$ an upper semicontinuous correspondence with nonempty closed convex values and $\phi : X \to 2^E$ a continuous correspondence with nonempty closed convex values such that $F(y) \subset \phi(y)$ for each $y \in M$. Further suppose that the closure of $\phi(N) := \cup_{x \in N} \phi(x)$ is compact for any bounded subset N of X. Then there exists an upper semicontinuous correspondence $\tilde{F} : X \to 2^E$ with nonempty closed convex values such that $\tilde{F}|_M = F$ and $\tilde{F}(x) \subset \phi(x)$ for each $x \in X$.*

Note that in [63], E is required to be a normed space. The proof remains valid when E is a metrizable locally convex space. This can be done by Lemma 2.2.3 and the following two lemmas.

Lemma 2.3.3 *Let X be a metrizable space and Y a nonempty compact convex subset of the metrizable locally convex space E. Suppose that $F : X \to 2^Y$ is an upper semicontinuous correspondence with compact values, then $T : X \to 2^Y$ defined by $T(x) := cl_Y(co(F(x)))$ for each $x \in X$ is also upper semicontinuous.*

Proof. Since E is a metrizable locally convex space, by Lemma 2.2.3, the topology of E can be defined by a metric d that is invariant under translations, and for which

the open balls are convex by Lemma 2.2.3. We will also denote any metric on X which can induce the topology as d.

Let $\varepsilon > 0$ and $x_0 \in X$. Since F is upper semicontinuous, by Lemma 2.2.2, there exists $\delta > 0$ such that for all $x \in O(x_0, \delta) =: \{x \in X : d(x_0, x) < \delta\}$, $F(x) \subset F(x_0) + O(0, \varepsilon/2)$. Now let any $z \in coF(x)$. Then there exist $z_1, \ldots, z_n$ in $F(x)$ and nonnegative numbers $\lambda_1, \ldots, \lambda_n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $z = \sum_{i=1}^{n} \lambda_i z_i$. We take $y_1, \ldots, y_n$ in $F(x_0)$ with $z_i \in y_i + O(0, \varepsilon/2)$ for all $i = 1, \ldots, n$. Then we have

$$\sum_{i=1}^{n} \lambda_i z_i \in \sum_{i=1}^{n} \lambda_i y_i + \sum_{i=1}^{n} \lambda_i O(0, \varepsilon/2).$$

Since $O(0, \varepsilon)$ is convex, we have

$$\sum_{i=1}^{n} \lambda_i z_i \in \sum_{i=1}^{n} \lambda_i y_i + O(0, \varepsilon/2)$$

or $coF(x) \subset coF(x_0) + O(0, \varepsilon/2)$. So we have $cl_Y(coF(x)) \subset cl_Y(coF(x_0)) + O(0, \varepsilon)$, i.e., $T(x) \subset T(x_0) + O(0, \varepsilon)$. By Lemma 2.2.2 again, T is upper semicontinuous. ∎

The following appears as Proposition 1 in [83].

Lemma 2.3.4 *Let X be a topological space and Y a normal topological space. If $F : X \to 2^Y$ is upper semicontinuous, then the correspondence $clF : X \to 2^Y$ defined by $clF(x) = cl_Y(F(x))$ is also upper semicontinuous.*

Theorem 2.3.5 *Suppose that I is countable. Let $(X_i, F_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$*

(i) X_i is a nonempty compact convex subset of a metrizable locally convex space E_i;

(ii) $F_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ is a continuous correspondence with nonempty compact convex values;

(iii) $P_i : X \to 2^{X_i}$ is upper semicontinuous;

(iv) $x_i \notin cl_{X_i}(co(P_i(x)))$ for all $x \in X := \Pi_{j \in I} X_j$.

Then there is an equilibrium.

Proof. Note that I is countable, X is a subset of the metrizable space $E :=$ $\Pi_{j \in I} E_j$. It is also compact and convex. For each $i \in I$, define $G_i : X \to 2^{X_i}$ by

$$G_i(x) = F_i(x) \cap cl_{X_i}(co(P_i(x))) \text{ for each } x \in X.$$

Since P_i is upper semicontinuous, $x \mapsto cl_{X_i}(P_i(x))$ is upper semicontinuous by Lemma 2.3.4. By Lemma 2.3.3, $cl_{X_i}(co(cl_{X_i}(P_i(x))))$ is upper semicontinuous. Note that $cl_{X_i}(co(cl_{X_i}(P_i(x)))) = cl_{X_i}(co(P_i(x)))$ for any $x \in X$, $cl_{X_i}(co(P_i(x)))$ is upper semicontinuous. Further since F_i is upper semicontinuous, G_i is also upper semicontinuous by Lemma 2.2 in Tan and Yuan [78]. Now let $M_i := \{x \in X : G_i(x) \neq \emptyset\}$, then M_i is closed. Note that for each $x \in M_i$, we have $G_i(x) \subset F_i(x)$. By Lemma 2.3.2, there exists an upper semicontinuous correspondence $\tilde{G}_i(x)$ with nonempty closed convex values such that $\tilde{G}_i(x)|_{M_i} = G_i(x)$ and $\tilde{G}_i(x) \subset F_i(x)$ for each $x \in X$. Now define $G : X \to 2^X$ by $G = \Pi_{i \in I} \tilde{G}_i$, then G is upper semicontinuous correspondence (by Lemma 3 of Fan [28]) with nonempty closed convex values. By the Fan-Glicksberg fixed point theorem, G has a fixed point $x \in X$. Obviously, x is an equilibrium. ∎

Remark. Recently Kim and Lee [47] claimed a "theorem" as follows:

Suppose that I is a (possibly uncountable) index set. Let $(X_i, F_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$

(i) X_i is a nonempty convex subset of a locally convex topological space E_i and D_i is a nonempty compact subset of X_i;

(ii) $clF_i : X \to 2^{D_i}$ is an upper semicontinuous correspondence such that $F_i(x)$ is convex for each $x \in X =: \Pi_{j \in I} X_j$;

(iii) $P_i : X \to 2^{X_i}$ is upper semicontinuous with convex values;

(iv) the set $W_i := \{x \in X : (F_i \cap P_i)(x) \neq \emptyset\}$ is closed;

(v) $x_i \notin cl_{X_i}(P_i(x))$ for all $x \in W_i$.

Then there is an equilibrium $x \in D$, i.e., for each $i \in I$, $x_i \in clF_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$.

If it were true, it would be much more general than Theorem 2.3.5. A very simple example shows that their result is wrong:

Example 2.3.6 *Let $I = \{1\}$, $X_1 = X = [0, 1]$. Suppose*

$$F_1(x) = \begin{cases} [0, 1] & \text{if } x = 1/2; \\ \{1 - x\} & \text{if } x \neq 1/2. \end{cases}$$

$$P_1(x) = \begin{cases} \{1/3\} & \text{if } x = 1/2; \\ \emptyset & \text{if } x \neq 1/2. \end{cases}$$

Then F_1 and P_1 are upper semicontinuous with closed convex values. Further, $W_1 = \{1/2\}$ is closed. The only fixed point of F_1 is $1/2$. However, $F_1(1/2) \cap P_1(1/2) = \{1/3\} \neq \emptyset$. ∎

We think that a reasonable conjecture is as follows:

Suppose that I is a (possibly uncountable) index set. Let $(X_i, F_i, P_i)_{i \in I}$ be an abstract economy such that for each i

(i) X_i is a nonempty convex subset of a locally convex topological space E_i and D_i is a nonempty compact subset of X_i;

(ii) $F_i : X \to 2^{D_i}$ is a continuous correspondence such that $F_i(x)$ is a nonempty convex closed set for each $x \in X := \Pi_{j \in I} X_j$;

(iii) $P_i : X \to 2^{X_i}$ is upper semicontinuous with convex values;

(iv) $x_i \notin cl_{X_i}(P_i(x))$ for all $x \in X$.

Then there is an equilibrium $x \in D$, i.e., for each $i \in I$, $x_i \in F_i(x)$ and $F_i(x) \cap P_i(x) = \emptyset$.

Since most Tietze-Dugundji extension theorems require that the domain of the map to be extended be a metric space, the extension techniques we have used can not be applied to the open problem. There are a very few exceptions (for single-valued maps) in the literature which do not have such a requirement, but they are too crude to be used for our problems. It is not known whether the Tietze-Dugundji extension theorem holds when the domain of the map to be extended is a closed (even compact) subset of a (non-metrizable) locally convex space. This is probably a "classical question". Sticking on this technique may not be a good idea. Possibly

other approaches (think about Theorem 2.2.7 and Theorem 2.3.1!) can resolve this question.

2.4   Equilibrium Existence Theorems with Closed Preferences

In this section, we first present an equilibrium theorem for an abstract economy with mixed preferences (which means that some of the preferences have open graphs, and the others are upper semicontinuous with closed values). Then, as applications, we provide two examples to illustrate how the general result established in this section can be used to give the existence of equilibria of economic models such as the pure exchange model and the Arrow-Debreu model.

The discussions are restricted to $\mathbb{R}^n$ spaces. The reason for this is that we want to illustrate how to resolve general equilibrium problems with closed preferences rather than becoming involved in the explanation of the complex structures of infinite dimensional spaces used in this area. Also some concepts are restated or reexplained with their underlying settings.

2.4.1   Introduction

Following Debreu, $\Gamma = (X_i, F_i, U_i)_{i=1}^N$ is called a *generalized N-person game* (or *an abstract economy*) if for each person (agent) $i = 1, \ldots, N$, X_i is a choice set, $F_i : X := \Pi_{j \in I} X_j \to 2^{X_i}$ is a constraint correspondence and $U_i : X \to \mathbb{R}$ is a utility (payoff) function. The objective of the ith agent is for each $\hat{x} \in X$ to choose an action x_i which maximizes $U_i(\hat{x}_i, \cdots, \hat{x}_{i-1}, \cdot, \hat{x}_{i+1}, \cdots, \hat{x}_N)$ subject to $x_i \in F_i(\hat{x})$. The vector $\hat{x} = (\hat{x}_1, \cdots, \hat{x}_n)$ of actions is an equilibrium for Γ if $\hat{x}_i$ maximizes $U_i(\hat{x}_1, \cdots, \hat{x}_{i-1}, \cdot, \hat{x}_{i+1}, \hat{x}_N)$ subject to $x_i \in F_i(\hat{x})$ for each $i = 1, \ldots, N$. This notion of equilibria is a natural extension of the concept of an equilibrium introduced

by Nash [59] for non-cooperative N-person games. If each X_i is a non-empty compact convex subset of $\mathbb{R}^l$, each F_i is continuous with non-empty convex values and each U_i is continuous on X and quasi-convex in x_i, then Debreu [20] showed that an equilibrium exists. Arrow and Debreu [3] used this result to prove the existence of a competitive equilibrium.

Following Shafer [67], $E = (Y_i, w_i, V_i)_{i=1}^N$ is said to be an economy if N is the number of individuals, and for each $i = 1, \ldots, N$, $Y_i \subset \mathbb{R}^l$ is a consumption set of the ith individual, $w_i \in \mathbb{R}_+^l$ is the ith individual's initial endowment vector, and $V_i : Y_i \to \mathbb{R}$ is a utility function which represents the preferences of the ith individual. Let $Y = \Pi_{i=1}^N Y_i$ and $\Delta = \{p \in \mathbb{R}_+^l : \Sigma_{i=1}^l p_i = 1\}$ denote the set of normalized prices. For two elements $x, y \in \mathbb{R}^l$, xy will be used to represent the scalar product of x and y. A competitive equilibrium for E is a point $(\hat{y}, \hat{p}) \in Y \times \Delta$ such that the following three conditions are satisfied:

(1) $\Sigma_{i=1}^N \hat{y}_i \leq \Sigma_{i=1}^N w_i$;

(2) $\hat{p}\hat{y} = \hat{p}w_i$ for each $i = 1, \ldots, N$;

(3) for $i = 1, \ldots, N$, $\hat{y}_i$ is the solution to 'maximize V_i subject to $\hat{p}y_i \leq \hat{p}w_i$ and $y_i \in Y_i$.'

We associate E with an $(N+1)$-person generalized game $\Gamma = (Y_i, F_i, V_i)_{i=0}^N$ in the following manner: For each $i = 1, \ldots, N$, the ith person has a utility function V_i, a choice set Y_i and a constraint correspondence F_i defined by

$$F_i(y, p) = \{y_i \in Y_i : py_i \leq pw_i\}$$

for each $(y, p) \in Y \times \Delta$. The 0th person, called a *market player*, has a utility (payoff) function V_0 which is defined by

$$V_0(y, p) = p(\Sigma y_i - \Sigma w_i)$$

for each $(y, p) \in Y \times \Delta$, a choice set Δ and a constraint correspondence F_0 defined by

$$F_0(y, p) = \Delta$$

for each $(y, p) \in Y \times \Delta$. Then, if each Y_i is compact and convex, $w_i \in intY_i$ and each V_i is continuous and quasi-concave, this $(N+1)$-person generalized game Γ will satisfy the sufficient conditions mentioned above for the existence of an equilibrium. It is easy to see that (y^*, p^*) is an equilibrium for E if for each $i = 0, \ldots, N$, y_i^* is a maximum of V_i. Note that the above argument remains valid if each agent's utility function V_i is assumed to depend not only on his own consumption y_i, but also on the consumptions of the other agents and on the prices p. Thus, Arrow and Debreu also showed how to prove the existence of a competitive equilibrium with consumption externalities and price dependent preferences.

We now follow Shafer's idea to extend the above result to the existence of an equilibrium in an abstract economy. Given an N-person generalized game $\Gamma = (X_i, F_i, U_i)_{i=1}^N$, for each $i = 1, \ldots, N$, consider the correspondence $P_i : X \to 2^{X_i}$ defined by

$$P_i(x) = \{z_i \in X_i : U_i(x_1, \cdots, x_{i-1}, z_i, x_{i+1}, \cdots, x_n) > U_i(x)\}$$

for each $x \in X$. Note that $\hat{x}$ is an equilibrium of Γ if and only if $P_i(\hat{x}) \cap F_i(\hat{x}) = \emptyset$ and $\hat{x}_i \in F_i(\hat{x})$ for each $i = 1, \ldots, N$. We shall consider abstract economies in which the individual preferences are given by a *preference correspondence* P_i rather than by a utility function. In this formulation, preferences, which depend on the actions of others, are not required to be either transitive or asymmetric.

2.4.2 An Equilibrium Theorem for an Abstract Economy

We have the following equilibrium existence result for an abstract economy.

Theorem 2.4.1 *Let* $\Gamma = (X_i, F_i, P_i)_{i=1}^N$ *be an abstract economy and* $I_0 \cup I_1 = \{1, \ldots, N\}$, *where* I_0 *and* I_1 *are disjoint. Suppose that*

(i) for each $i = 1, \ldots, N$, X_i *is a nonempty, compact and convex subset of* $\mathbb{R}^{l_i}$;

(ii) for each $i = 1, \ldots, N$, F_i *is a continuous correspondence with nonempty compact convex values;*

(iii) (a) for $i \in I_0$, P_i *has an open graph and convex values,*

(b) for $i \in I_1$, P_i is upper semicontinuous with closed convex values;

(iv) for each $i = 1, \ldots, N$, $x_i \notin P_i(x)$ for all $x \in X$.

Then Γ has an equilibrium.

Proof. Let $X = \Pi_{i=1}^N X_i$.

(1) Fix an $i \in I_0$. Let $U_i = \{x \in X : F_i(x) \cap P_i(x) \neq \emptyset\}$. Since F_i is lower semicontinuous and P_i has open graph, $F_i \cap P_i$ is lower semicontinuous by Lemma 4.2 of Yannelis [85]. Hence U_i is open. By Michael's selection theorem [56], there exists a continuous function f_i such that $f_i(x) \in F_i(x) \cap P_i(x)$ for each $x \in U_i$.

Define $G_i : X \to 2^{X_i}$ by

$$G_i(x) = \begin{cases} F_i(x), & \text{if } x \notin U_i, \\ \{f_i(x)\}, & \text{if } x \in U_i. \end{cases}$$

Then $G_i(x)$ is upper semicontinuous with nonempty compact convex values.

(2) Fix an $i \in I_1$. Let $M_i = \{x \in X : F_i(x) \cap P_i(x) \neq \emptyset\}$. Then M_i is closed since $F_i \cap P_i$ is upper semicontinuous by Lemma 2.2 in Tan and Yuan [78]. Note that for each $x \in M_i$, $F_i(x) \cap P_i(x) \subset F_i(x)$, by Lemma 2.3.2, there exists an upper semicontinuous correspondence $G_i : X \to 2^{X_i}$ with nonempty compact convex values such that $G_i|_{M_i} = F_i \cap P_i$ and $G_i(x) \subset F_i(x)$ for each $x \in X$.

(3) Now define $G : X \to 2^X$ by $G(x) = \Pi_{i=1}^N G_i(x)$ for each $x \in X$, then G is upper semicontinuous (by Lemma 3 of Fan [28]) with nonempty compact convex values. By the Kakutani fixed point theorem (Kakutani [43]), G has a fixed point $x \in X$. Clearly, x is an equilibrium for Γ. ∎

Remark 1. Suppose $F_i(x) = X_i$ for each $i = 1, \ldots, N$ and for all $x \in X$. (1) If $I_1 = \emptyset$, then Theorem 2.4.1 reduces to the fixed point theorem (Theorem 2.1.2) in Gale and Mas-Colell [32] (2) If $I_0 = \emptyset$, Theorem 2.4.1 can be regarded as a dual of Gale and Mas-Colell's fixed point theorem.

Remark 2. (1) if $I_1 = \emptyset$, Theorem 2.4.1 reduces to the abstract result (Theorem 2.1.4) in Shafer and Sonnenschein [69]. (2) If $I_0 = \emptyset$, Theorem 2.4.1 can be regarded

as a dual of Shafer and Sonnenschein's abstract result.

Remark 3. By Lemma 2.3 of Tan and Yuan [79], it follows that the condition (iii)(a) in Theorem 2.4.1 is equivalent to the condition that " P_i is lower semicontinuous with convex and open values (which may be empty)".

2.4.3   Application 1

In this section, we shall give an application of our Theorem 2.4.1 to economic models.

Suppose there are n *traders*. For each trader $i \in I = \{1, \ldots, n\}$, the *trading set* X_i is a subset of $\mathbb{R}^l$ and there is a *preference correspondence* P_i from X_i to 2^{X_i}. There is also a subset Y of $\mathbb{R}^l$ which is called the *technology* of the economy (Gale and Mas-Colell [32]).

An *allocation* x is an n-tuple $(x_1, \cdots, x_n)$ where $x_i \in X_i$ for each $i \in I$. Thus $x \in X = \Pi_{i \in I} X_i$. An allocation is said to be *feasible* if $\Sigma_{i=1}^n x_i \in Y$.

For a given allocation $x = (x_1, \cdots, x_n)$, x_0 denotes the sum $\Sigma_{i=1}^n x_i$.

For two points $x = (x^1, \ldots, x^l)$ and $y = (x^1, \ldots, y^l)$ in $\mathbb{R}^l$, we say that $x \geq y$ if $x^i \geq y^i$ for $i = 1, \ldots, l$; and $x > y$ if $x^i > y^i$ for $i = 1, \ldots, l$;

Let Δ be the unit $(l-1)$-simplex. Each element p of Δ will be called a price vector. In formulating equilibrium models, we describe the way in which an allocation x is associated with the price vector p as follows: For each $i \in I$, let the trader i have income $\alpha_i(p)$ at the price p. In the pure exchange model, it is assumed that the trader i has an initial endowment vector w_i and his income is then given by $\alpha_i(p) = pw_i$. The Arrow-Debreu model [3] involves a more complicated set of income functions. It is assumed that the technology set Y is the sum of some *sub-technologies* $Y_1, \cdots, Y_r$ which are to be thought of as firms, and trader i is provided with a *portfolio vector* $\theta_i = (\theta_{i1}, \cdots, \theta_{ir})$, where θ_{ij} represents trader $i's$ share of the firm Y_j. The Arrow-Debreu income functions are then given by

$$\alpha_i(p) = pw_i + \Sigma_{j=1}^r \theta_{ij} \sup pY_j.$$

In the present treatment, by following Gale and Mas-Colell [32], we wish to allow for more general income functions. For any $p \in \Delta$, define the *profit function* $\Pi(p)$ by

$$\Pi(p) = \sup pY.$$

Since Y maybe unbounded, it follows that Π may be infinite. We define $\Delta' \subset \Delta$ by

$$\Delta' = \{p \in \Delta : \Pi(p) < \infty\}.$$

The following lemma is implicitly contained in the proof of Lemma 2 in Gale and Mas-Colell [32]. We shall give a detailed proof here.

Lemma 2.4.2 *Suppose $Y \subset \mathbb{R}^l$ is closed convex, contains the negative orthant and has a bounded intersection other than $\{0\}$ with the positive orthant. Let e be a point in Y and $\hat{Y} = \{y : y \in Y \text{ and } y \geq e\}$. Suppose for some $p \in \mathbb{R}_+^l$ and $z \in \hat{Y}$, $pz = \max_{y \in \hat{Y}} py$. Then there is $q \in \Delta$ such that $qz = \max_{y \in Y} qy$.*

Proof. Since $pz = \max_{y \in \hat{Y}} py$, the set $D := \{y \in \mathbb{R}^l : y > z\}$ is non-empty open convex and disjoint from $\hat{Y}$. D is also disjoint from Y. Otherwise, let $\bar{y} \in D \cap Y$. Then $\bar{y} > z \geq e$ so that $\bar{y} \in \hat{Y}$ which contradicts $pz = \max_{y \in \hat{Y}} py$. Since Y is convex, by the Hahn-Banach theorem, we can find $q \in \mathbb{R}^l$ such that $q\bar{y} > qy$ for all $\bar{y} \in D$ and $y \in Y$. Note that since Y contains the negative orthant, it follows that $q = (q_1, \ldots, q_n) \in \mathbb{R}_+^l$. Take $y = z$ and a sequence $(\bar{y}_n)$ in D converging to z. We have $qz = \max_{y \in Y} qy$. Obviously, $q \geq 0$ and $q \neq 0$. If $\sum_{i=1}^l q_i \neq 1$, replace it with $q/\sum_{i=1}^l q_i \in \Delta$ which is required. ∎

Our hypothesis on Y will guarantee that Δ' is non-empty. One also verifies that Δ' is convex. We also assume the existence of n real-valued functions α_i on Δ' (to be called *income functions*) satisfying the following formula:

$$\Sigma_{i=1}^n \alpha_i(p) = \Pi(p)$$

for all $p \in \Delta'$.

By following Aliprantis *et al*'s book [1], we have the following definition.

A **Walrasian** (or, a **competitive**) **equilibrium** for the model described above consists of a price vector $\hat{p}$ in Δ and an allocation $\hat{x}$ such that

(1) $\hat{p}\hat{x} \leq \alpha_i(\hat{p})$ for all $i \in I$ (budget inequality);

(2) for each $i \in I$, if $x_i \in P_i(\hat{x}_i)$, then $\hat{p}x_i > \hat{p}\hat{x}_i$ (preference condition);

(3) $\hat{x}$ is feasible (balance of supply and demand).

The following lemma is Lemma 3 in Debreu [21].

Lemma 2.4.3 *Suppose X is a non-empty compact convex subset of $\mathbb{R}^l$ and $D = \{(p, w) \in \mathbb{R}^{l+1} : \min_{x \in X} px \leq w\}$. Let $\beta : D \to 2^X$ be defined by $\beta(p, w) = \{x \in X : px \leq w\}$. If $w^0 > \min_{x \in X} p^0 x$, then β is continuous at (p^0, w^0).*

The following lemma is contained in the proof of Theorem 4 in Debreu [21] [page 707-710], but here we shall give an explicit proof based on Debreu's idea.

Lemma 2.4.4 *Suppose X is a non-empty compact convex subset of $\mathbb{R}^l$ and $\alpha(p)$ is a continuous function on a non-empty compact subset Y of $\mathbb{R}^l$. If $\alpha(p) > \min_{x \in X} px$ for all $p \in Y$, then correspondence $A : Y \to 2^X$ defined by $A(p) = \{x \in X : px \leq \alpha(p)\}$ is continuous.*

Proof. Since Y is compact and A is non-empty and closed-valued, A can be shown to have a closed graph. Hence A is upper semicontinuous. It remains to show that A is also lower semicontinuous.

Let p^0 be any point in Y. To show A is lower semicontinuous at p^0, we consider a sequence $(p^i)_{i=1}^{\infty}$ in Y converging to p^0 and a point $z^0 \in A(p^0)$.

Let $w^i = \alpha(p^i)$ for $i = 1, \cdots$, then $(w^i)_{i=1}^{\infty}$ converges to $w^0 = \alpha(p^0)$. For $i = 0, 1, \cdots$, since $w^i = \alpha(p^i) > \min_{x \in X} p^i x$, $(p^i, w^i) \in D := \{(p, w) \in \mathbb{R}^{l+1} : \min_{x \in X} px \leq w\}$. By Lemma 2.4.3, $\beta : D \to 2^X$ defined by $\beta(p, w) = \{x \in X : px \leq w\}$ is continuous at (p^0, w^0). So for $z^0 \in A(p^0) = \beta(p^0, w^0)$, there exists a sequence $(z^i)_{i=1}^{\infty}$, where $z^i \in \beta(p^i, w^i) = A(p^i)$ for each $i = 1, \ldots$, converging to z^0. Thus A is lower semicontinuous.

∎

Now we have the following theorem:

Theorem 2.4.5 *The following conditions are sufficient for the existence of an equilibrium:*

(i) the set Y is closed convex, contains the negative orthant, and has a bounded intersection with the positive orthant;

(ii) for each $i = 1, \ldots, n$, the set X_i is non-empty closed convex and bounded below;

(iii) for each $i = 1, \ldots, n$, the preference P_i is upper semicontinuous with closed convex values and is irreflexive (i.e., $x_i \notin P_i(x)$ for all $x \in X$);

(iv) for each $i = 1, \ldots, n$, the function $\alpha_i(p)$ is continuous and satisfies $\alpha_i(p) > \inf p X_i$ for all $p \in \Delta'$.

Proof. As each X_i is bounded below, it follows there exists a vector e such that for any non-empty subset S of $\{1, \cdots, n\}$, we have $e < y$ for all $y \in \Sigma_{i \in S} X_i$.

Without loss of generality, we may assume that $e < 0$ so that $e \in Y$. Now define $\hat{Y} = \{y \in Y : y \geq e\}$. Note that $\hat{Y}$ contains all feasible $x_0 = \Sigma_{i=1}^n x_i$ and by the condition of Y, $\hat{Y}$ is also bounded above as well as below. Thus there exists a vector f such that $f > \hat{Y}$. By the feasibility, $x_0 = \Sigma_{j=1}^n x_i < f - e$, so that $x_i < f - \Sigma_{j \neq i} x_j < f - e$. Following Gale and Mas-Colell [32], we define the set $\Delta'' \subset \Delta'$ as follows

$$\Delta'' = \{p \in \Delta' : py = \Pi(p) \text{ for some } y \in \hat{Y}\}.$$

Take any $p \in \Delta'$. Since $\hat{Y}$ is compact, there is $z \in \hat{Y}$ such that $p \cdot z = \sup_{y \in \hat{Y}} p \cdot y$. By Lemma 2.4.2, there is $q \in \Delta$ such that $q \cdot z = \sup_{y \in Y} q \cdot y = \Pi(q)$ so that $q \in \Delta''$. Thus Δ'' is non-empty. It is easy to see Δ'' is closed.

Also let Δ^* be the convex hull of Δ''. Then $\Delta^* \subset \Delta'$ since Δ' is convex and Δ^* is also closed as Δ'' is closed.

By the definition of Δ'', we shall show that there is a nonempty finite set $X_i' \subset X_i$ such that $\min p X_i' < \alpha_i(p)$ for all $p \in \Delta^*$. Indeed, for each $p \in \Delta^*$, let $x_p \in X_i$

be such that $px_p < \alpha_i(p)$. By the continuity of α_i, it follows that there is, for each $p \in \Delta^*$, an open neighborhood V_p of p in Δ^* such that $qx_p < \alpha_i(q)$ for each $q \in V_p$. The family of open sets $\{V_p : p \in \Delta^*\}$ covers Δ^* and since Δ^* is compact, there is a finite subcover $\{V_{p_1}, \cdots, V_{p_k}\}$. Then take $X_i' = \{x_{p_1}, \cdots, x_{p_k}\}$. Let r be sufficiently large and we define $\hat{X}_i = \{x_i \in X_i : x_i \leq r\}$. Without loss of generality, we may assume that $X_i' \subset \hat{X}_i$ for all $i = 1, \ldots, n$.

Now for each $i = 1, \ldots, n$, define $F_i : \Delta^* \times \hat{X} \to 2^{\hat{X}_i}$ by

$$F_i(p, x) = \{z_i \in \hat{X}_i : pz_i \leq \alpha_i(p)\}$$

for each $(p, x) \in \Delta^* \times \hat{X}$, where $\hat{X} = \Pi_{i=1}^n \hat{X}_i$. By the construction of $\hat{X}_i$ above and the condition (iv), it follows that $F_i(p, x)$ is nonempty and convex for each $(p, x) \in \Delta^* \times \hat{X}$ and, moreover, by Lemma 2.4.4, F_i is continuous with non-empty closed and convex values.

Now define $F_0 : \Delta^* \times \hat{X} \to 2^{\Delta^*}$ by

$$F_0(p, x) = \Delta^*$$

for each $(p, x) \in \Delta^* \times \hat{X}$. Of course, F_0 is continuous with non-empty closed convex values for all $(p, x) \in \Delta^* \times \hat{X}$. Finally, define $P_0 : \Delta^* \times \hat{X} \to 2^{\Delta^*}$ by

$$P_0(p, x) = \{q \in \Delta^* : q \cdot (\Sigma_{i=1}^n x_i) - \Pi(q) > p \cdot (\Sigma_{i=1}^n x_i) - \Pi(p)\}$$

for each $(p, x) \in \Delta^* \times \hat{X}$. It follows that P_0 has an open graph, each $P_0(p, x)$ is convex (maybe empty), and $p \notin P_0(p, x)$ for all $(p, x) \in \Delta^* \times \hat{X}$.

Identify P_i with $\bar{P}_i : \Delta^* \times \hat{X} \to 2^{\hat{X}_i}$ defined by $\bar{P}_i(p, x) = P_i(x_i)$ for each $(p, x) \in \Delta^* \times \hat{X}$. By the condition (iii), it follows that the abstract economy $(\hat{X}_i, F_i, P_i)_{i=0}^n$ satisfies all hypotheses of Theorem 2.4.1, where $\hat{X}_0 = \Delta^*$. By Theorem 2.4.1 with $I_0 = \{0\}$ and $I_1 = \{1, \ldots, n\}$, there exists $(\hat{p}, \hat{x}) \in \Delta^* \times \hat{X}$ such that $\hat{p} \in \Delta^*$ and

$$F_0(\hat{p}, \hat{x}) \cap P_0(\hat{p}, \hat{x}) = \Delta^* \cap \{q \in \Delta^* : q(\sum_{i=1}^n \hat{x}_i) - \Pi(q) > \hat{p}(\sum_{i=1}^n \hat{x}_i) - \Pi(\hat{p})\} = \emptyset$$

which implies that for each $q \in \Delta^*$, we have

$$q(\sum_{i=1}^{n} \hat{x}_i) - \Pi(q) \leq \hat{p}(\sum_{i=1}^{n} \hat{x}_i) - \Pi(\hat{p}).$$

Moreover, for each $i = 1, 2, \cdots, n$, $\hat{x}_i \in F_i(\hat{p}, \hat{x})$ for $i = 1, 2, \cdots, n$ and $F_i(\hat{p}, \hat{x}) \cap P_i(\hat{p}, \hat{x}) = \emptyset$.

Now we prove that $(\hat{p}, \hat{x})$ is a Walrasian equilibrium. First we claim that $\hat{x}_0 = \sum_{i=1}^{n} \hat{x}_i$ is feasible, i.e., $\hat{x}_0 = \sum_{i=1}^{n} \hat{x}_i \in Y$. Suppose the contrary, i.e. $\hat{x}_0 \notin Y$. We may choose λ such that $y_\lambda = \lambda e + (1 - \lambda)\hat{x}_0 \in Y$ and $y_\alpha = \alpha e + (1 - \alpha)\hat{x}_0 \notin Y$ for all $\alpha < \lambda$.

Since $\hat{x}_0 > e$, by a similar argument as in the proof of Lemma 2.4.2, there exists q in Δ such that $qy_\lambda = \Pi(q)$ and $q\hat{x}_0 > \Pi(q)$. Note that $y_\lambda \in \hat{Y}$, it follows that $p \in \Delta'' \subset \Delta^*$. Therefore, we have $q(\hat{x}_0) - \Pi(q) \leq \hat{p}(\hat{x}_0) - \Pi(\hat{p})$. Thus $\hat{p}(\hat{x}_0) > \Pi(\hat{p})$. However, we do have that $\hat{p}(\hat{x}_0) = \sum_{i=1}^{n} \hat{p}(\hat{x}_i) \leq \Sigma_{i=1}^{n} \alpha_i(\hat{p}) = \Pi(\hat{p})$, which is a contradiction. Thus we must have $\hat{x}_0 = \sum_{i=1}^{n} \hat{x}_i \in Y$.

Finally, we wish to show that $\hat{p}x_i > \hat{p}\hat{x}_i$, for each $x_i \in P_i(\hat{x}_i)$. Since $P_i(\hat{p}, \hat{x}) \cap F_i(\hat{p}, \hat{x}) = \emptyset$, it follows that for each $x_i \in P_i(\hat{p}, \hat{x}_i)$, $x_i \notin F_i(\hat{p}, \hat{x}_i)$, i.e. $\hat{p}x_i > \alpha_i(\hat{p})$. As $\alpha_i(\hat{p}) \geq \hat{p}\hat{x}_i$, it follows that $\hat{p}x_i > \hat{p}\hat{x}_i$ for all $x_i \in P_i(\hat{x}_i)$. Therefore $(\hat{p}, \hat{x})$ is a Walrasian equilibrium and the proof is complete. ∎

2.4.4 Application 2

As another application of Theorem 2.4.1, we shall generalize one of Shafer and Sonnenschein's results in [70].

We now consider the pure exchange economy $\mathcal{E}$ with n consumers and l commodities. For each $i = 1, \ldots, n$, the ith consumer is specified by his consumption set X_i which is a subset of $\mathbb{R}_+^l$, his initial holdings ω_i which is a point in $\mathbb{R}_+^l$, and a preference indicator P_i. A preference indicator may take different forms. In economies without externalities, it may be either a utility function $U_i : X_i \to \mathbb{R}$ or an irreflexive relation $P_i \subset X_i \times X_i$ which was used by Gale and Mas-Colell [32]. Also it is more general

than the utility function formulation since P_i is not required to be asymmetric or transitive. If $U_i(x) > U_i(y)$, or alternatively if $(x, y) \in P_i$, we then say that the ith consumer prefers x to y. In economies with externalities we allow the preference of each individual to depend not only on his own consumption, but also on the consumption of each consumer and price. A price vector p is a point in $\mathbb{R}_+^l$. An allocation $x = (x_1, \cdots, x_n) \in X = \Pi_{i \in I} X_i$ specifies a consumption for each consumer.

A *competitive equilibrium* for the economy $\mathcal{E}$ is defined as an allocation price pair $(\hat{p}, \hat{x}) \in \mathbb{R}_+^l \times X$ such that for each $i = 1, 2, \cdots, n$,

(1) $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot \omega_i$ for all i (budget inequality);

(2) if $x_i \in P_i(\hat{p}, x_i)$, then $\hat{p} \cdot x_i > \hat{p} \cdot \omega_i$ (preference condition).

(3) $\sum_{i=1}^n \hat{x}_i \leq \sum_{i=1}^n \omega_i$ (demand cannot exceed supply).

We shall study the existence of competitive equilibrium for the pure economy $\mathcal{E}$ in which each preference indicator P_i: $\mathbb{R}_+^l \times X \to 2^{X_i}$ is such that $P_i(p, \cdot)$ is an irreflexive relation in $X \times X_i$ for each fixed price vector $p \in \mathbb{R}_+^l$, i.e., $x_i \notin P_i(p, x)$. Also we assume that P_i is upper semicontinuous with closed values instead of being lower semicontinuous or having open graph (which is the usual form used in the literature, e.g., see Gale and Mas-Colell [32], Shafer and Sonnenschein ([69], [70]), Debreu [21] and references therein).

Theorem 2.4.6 *Let $\mathcal{E} = (X_i, \omega_i, P_i)_{i=1}^n$ be an economy satisfying for each $i = 1, \cdots, n$,*

(i) the consumption set X_i is a nonempty compact convex subset of $\mathbb{R}^l$;

(ii) the ith initial endowment $\omega_i \in int X_i$;

(iii) the preference indicator $P_i : \mathbb{R}_+^l \times X \to 2^{X_i}$ is upper semicontinuous with closed convex values such that $x_i \notin P_i(p, x)$ for each $(p, x) \in \mathbb{R}_+^l \times X$, where $X = \Pi_{i=1}^n X_i$.

Then there exists a competitive equilibrium for the economy $\mathcal{E}$.

Proof. Let $\Delta = \{p \in \mathbb{R}_+^l : \sum_{i=1}^l p_i = 1\}$. For each $i = 1, \cdots, n$, we define

$F_i : \Delta \times X \to 2^{X_i}$ by

$$F_i(p, x) = \{z_i \in X_i : pz_i \leq p\omega_i\}$$

for each $(p, x) \in \Delta \times X$. Then F_i has nonempty closed convex values. As $\omega_i \in int X_i$, F_i is both upper and lower semicontinuous by Lemma 2.4.4. Now we define the correspondence $F_0 : \Delta \times X \to 2^{\Delta}$ by

$$F_0(p, x) = \Delta$$

for each $(p, x) \in \Delta \times X$ and define another correspondence $P_0 : \Delta \times X \to 2^{\Delta}$ by

$$P_0(p, x) = \{q \in \Delta : q(\Sigma_{i=1}^n x_i - \Sigma_{i=1}^n \omega_i) > p(\Sigma_{i=1}^n x_i - \Sigma_{i=1}^n \omega_i)\}$$

for each $(p, x) \in \Delta \times X$. Clearly, F_0 is continuous and P_0 has an open graph. Also, $p \notin P_0(p, x)$ for each $(p, x) \in \Delta \times X$. Thus the family $(X_i, F_i, P_i)_{i=0}^n$, where $X_0 = \Delta$, satisfies all hypotheses of Theorem 2.4.1. By Theorem 2.4.1, there exists $(\hat{p}, \hat{x}) \in \Delta \times X$ such that for each $i = 1, \cdots, n$, we have that $\hat{x}_i \in F_i(\hat{p}, \hat{x})$ and $F_i(\hat{p}, \hat{x}) \cap P_i(\hat{p}, \hat{x}) = \emptyset$, and $\hat{p} \in F_0(\hat{p}, \hat{x})$ and $F_0(\hat{p}, \hat{x}) \cap P_0(\hat{p}, \hat{x}) = \emptyset$. That is, we have the following:

(a) $\hat{p} \in \Delta$ and $\hat{p} \cdot \hat{x}_i \leq \hat{p}w_i$ for $i = 1, \ldots, n$;

(b) $\hat{p} \cdot (\sum_{i=1}^n \hat{x}_i - \sum_{i=1}^n w_i) \geq q(\sum_{i=1}^n \hat{x}_i - \sum_{i=1}^n w_i)$ for all $q \in \Delta$; and

(c) $P_i(\hat{p}_i, \hat{x}) \cap \{x_i \in X_i : \hat{p} \cdot x_i \leq \hat{p}w_i\} = \emptyset$ for $i = 1, \ldots, n$.

By (a), we have

$$\hat{p} \sum_{i=1}^n \hat{x}_i \leq \hat{p} \sum_{i=1}^n \omega_i. \tag{2.1}$$

We claim that $\sum_{i=1}^n \hat{x}_i \leq \sum_{i=1}^n \omega_i$. Suppose not, let $z = (z_1, \ldots, z_l) =: \sum_{i=1}^n \hat{x}_i - \sum_{i=1}^n \omega_i$. Then there must be some k such that $z_k > 0$. Take $q = (q_1, \ldots, q_l) \in \Delta$ such that $q_k = 1$ and $q_i = 0$ for $i \neq k$. Then we have $qz = z_k > 0$ so that $\hat{p}(\sum_{i=1}^n \hat{x}_i - \sum_{i=1}^n \omega_i) > 0$ by (b). This contradicts to (2.1). So we have $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. Further, by (a) and (c), $(\hat{p}, \hat{x})$ is an equilibrium for $\mathcal{E}$. ∎

2.5 Comment

In correspondence theory, there are two techniques, "extension" and "selection", that are commonly used. The former is represented by the Dugundji-Ma theorem [54], which is often used to resolve problems involving "upper semicontinuity" ; the latter is represented by Michael's theorem [56], which is often used to resolve problems involving " lower semicontinuity". In Theorem 2.4.1, we used both techniques while remarking that there exists some duality between the existence theorems of fixed points for correspondences that have open graphs and that are upper semicontinuous. In fact, this can be attributed to duality between " extension " and "selection", or more fundamentally, "upper semicontinuity" and "lower semicontinuity". The better the underlying spaces, the better the duality between the two categories of problems.

The following theorem from [17] is also interesting.

Theorem 2.5.1 (Cellina and Solimini) *Let E be a Banach space, X a metric space and D a closed subset of X. Let $F : X \to 2^E$ be a lower semicontinuous correspondence with nonempty closed convex values; let $\phi : D \to 2^E$ be continuous with nonempty closed bounded values. Further, suppose that $\phi(x) \subset F(x)$ for all $x \in D$. Then there exists continuous correspondence $\tilde{\phi} : X \to 2^E$ with closed bounded values such that*

$$\tilde{\phi}(x) \subset F(x) \text{ for } x \in X,$$

and

$$\tilde{\phi}(x) = \phi(x) \text{ for } x \in D.$$

This theorem can be regarded as a dual of Theorem 2.3.2. Note that we can replace "ϕ has nonempty closed bounded values" with "ϕ has nonempty closed bounded convex values" if we impose some common conditions on the image of F, for example, requiring that the image of F be contained in a nonempty compact convex subset of E. This can be done by Lemma 2.3.3 and Proposition 2.6 in [56] (refer to Lemma 4.3.8 in this thesis).

The following table roughly summarizes the duality shown between "upper semi-continuity" and "lower semicontinuity".

Lower semicontinuous	Upper semicontinuous
Michael selection Theorem	Ma-Dugundji extension Theorem
Fan-Browder Theorem	Kakutani-Fan-Browder Theorem
Gale-Mas-Colell Theorem	Theorem 2.2.7 or Theorem 2.4.1
Deguire-Lassonde Theorem	Theorem 2.3.1
Cellina and Solimini Theorem	Pruszko Theorem
Shafer-Sonnenschein Theorem	Theorem 2.3.5 or Theorem 2.4.1

Note: The Michael selection Theorem refers to some results in Section 3 in [56]; the Ma-Dugundji extension Theorem refers to Theorem 2.1 in [54] (also see Theorem 2.2.4 in this thesis); the Fan-Browder Theorem refers to Theorem 2.1.1; the Kakutani-Fan-Browder refers to Theorem 2.1.5; the Gale-Mas-Colell Theorem refers to Theorem 2.1.2; the Deguire-Lassonde theorem refers to Theorem 2.1.3; the Cellina and Solimini Theorem refers to Theorem 2.5.1; the Pruszko Theorem refers to Theorem 2.3.2; the Shafer-Sonnenschein Theorem refers to Theorem 2.1.4.

Chapter 3

Equilibria for Abstract Economies with a Measure Space of Players

3.1 Introduction

As we saw in Chapter 2, the Tietze type extension theorem may be used to study equilibria theory in mathematical economics. This provides a new approach in this area.

The following are examples of how the Tietze extension theorem has been developed: Dugundji proved an extension theorem in [24] for single valued maps, which generalizes the Tietze extension theorem; Ma [54] generalized Dugundji's theorem to compact convex-valued correspondences; and Pruszko [63] proved a completely continuous extension theorem for convex-valued selections. These results have been used in topological degree theory and in studying generic structures of underlying spaces.

On the other hand, Hanš [39] proved a random version of Tietze's extension theorem. Andrus and Brown [2] proved a random version of Dugundji's theorem which extended Hanš' result. Also Bocsan *et al* proved a particular type of continuous random theorem (the same result was presented in Papageorgiou [62]). A slightly more general version of this theorem was also presented in [42].

In this chapter, we shall prove an extension theorem for a correspondence defined

on the product space of a measure space and a metric space. The extension theorem we shall prove (somewhat) corresponds to the Pruszko extension theorem (refer to Lemma 2.3.2). Finally, we apply it to study the equilibria for abstract economies.

3.2   Notation and Definitions

Let X be a topological space. X is said to be *Polish* if X is separable and metrizable by a complete metric, and *Souslin* if X is metrizable and the continuous image of a Polish space.

Let T denote a measurable space with σ-algebra $\mathcal{T}$. If necessary, $(T, \mathcal{T})$ will be used to imply that T is associated with the σ-algebra $\mathcal{T}$. In case there is a σ-finite (respectively, finite) measure μ defined on $\mathcal{T}$ we say that $(T, \mathcal{T}, \mu)$, or simply T, is a *σ-finite* (respectively, *finite*) measurable space; and if there is a complete measure μ defined on $\mathcal{T}$ we call T a *complete* measurable space. If $(T_1, \mathcal{T}_1)$ and $(T_2, \mathcal{T}_2)$ are two measurable spaces, $(T_1 \times T_2, \mathcal{T}_1 \otimes \mathcal{T}_2)$ will denote the measurable space where the product σ-algebra $\mathcal{T}_1 \otimes \mathcal{T}_2$ on $T_1 \times T_2$ is generated by the sets $A \times B$, where $A \in \mathcal{T}_1$ and $B \in \mathcal{T}_2$.

A correspondence $F : T \to 2^X$ is *measurable* (*weakly measurable*, *B-measurable*) iff $F^{-1}(B) := \{t \in T : F(t) \cap B \neq \emptyset\}$ is measurable for each closed (resp., open, Borel) subset B of X. If $F : Y \to 2^X$ where Y is a topological space, then the assertion that F is measurable(weakly measurable, etc.) means that F is measurable (weakly measurable, etc.) when Y is assigned the σ-algebra $\mathcal{B}$ of Borel subsets of Y. Likewise, if $F : T \times Y \to 2^X$, then the various kinds of measurability of F are always defined in terms of the product σ-algebra $\mathcal{T} \otimes \mathcal{B}(Y)$ on $T \times Y$. In addition, if T_1 and T_2 are two measurable spaces, a correspondence $F : T_1 \to 2^{T_2}$ is said to have a *measurable graph* if GrF belongs to the product σ-algebra $\mathcal{T}_1 \otimes \mathcal{T}_2$. Further, F is said to be (1) *random* if it is measurable in the first variable; (2) *upper semicontinuous* (respectively, *continuous*) if it is upper semicontinuous (respectively, continuous) in the second variable; (3) *random upper semicontinuous*(respectively, *random continuous*) if F is

both random and upper semicontinuous (respectively, continuous).

Let $(T, \mathcal{T}, \mu)$ be a complete finite measure space, Y be a Banach space and $L_1(\mu, Y)$ denote the space of equivalence classes of Y-valued Bochner integrable functions $f : T \to Y$ normed by

$$\|f\| = \int_T \|f(t)\| d\mu(t).$$

A correspondence $\phi : T \to 2^Y$ is said to be *integrably bounded* if there exists a map $g \in L_1(\mu)$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \phi(t)\} \leq g(t)$.

Now we recall the notion of a *separable measure space*. Let M be the measure algebra of $(T, \mathcal{T}, \mu)$ which is the factor algebra of $\mathcal{T}$ modulo the μ-null sets. M is a metric space with the distance given by the measure of the symmetric difference. If M is separable, then the measure space $(T, \mathcal{T}, \mu)$ is called separable. A well-known fact is that for a separable Banach space Y, $L_1(\mu, Y)$ is separable if $(T, \mathcal{T}, \mu)$ is.

If X is a metric space, except specified, d will be the metric on X. Now suppose that A, B are two subsets of the metrc space X, we define (1) $d(x, A)$ as ∞ if A is empty; (2) $d(A, B)$ as $\inf_{x \in A} d(x, B)$ if A is nonempty and ∞ otherwise.

3.3 A Random Extension Theorem

The following three lemmas are respectively Proposition 2.1, Proposition 2.2 and Theorem 3.5 (iii) in Himmelberg [41].

Lemma 3.3.1 *For any correspondence $F : T \to 2^X$, $\mathcal{B}$-measurability implies measurability, and if X is perfectly normal, measurability implies weak measurability.*

Lemma 3.3.2 *If $F : T \to 2^X$ is measurable or weakly measurable, the set $\{t \in T : F(t) \neq \emptyset\}$ is measurable.*

Lemma 3.3.3 *Let T be a complete σ-finite measure space, X a separable complete metric space, and $F : T \to 2^X$ a correspondence with closed values. Then the following statements are all equivalent:*

a) F is B-measurable;

b) F is measurable;

c) F is weakly measurable;

d) $t \to d(x, F(t))$ is a measurable function of t for each $x \in X$;

e) GrF is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable.

Note: Since the notation and terminology are quite messy in this area, we have tried as hard as possible to make the results cited here consistent with those defined in this chapter. For instance, the term "complete" measure space in [41] means a σ-finite and complete measure space in our terminology. So when these cited results are really needed to be checked, please take care of the notation and terminology (sometimes they are very tricky).

We shall frequently refer to the following result (Theorem III. 14 in [19]):

Lemma 3.3.4 *Let T be a measurable space, X a separable metrizable space, U a metrizable space and $\phi : T \times X \to U$. If ϕ is measurable in t and continuous in x, then ϕ is (jointly) measurable.*

The following is Lemma 4.6 in [49].

Lemma 3.3.5 *Let T be a measurable space, X an arbitrary topological space and W_n, $n = 1, 2, \ldots$, correspondences from T to X with measurable graphs. Then the correspondences $\cup_n W_n(\cdot)$, $\cap_n W_n(\cdot)$, and $X \backslash W_n(\cdot)$ have measurable graphs.*

The following result is Theorem III.23 in [19].

Lemma 3.3.6 *Let T be a σ-finite measure space, and X a complete separable metric space. If G belongs to $\mathcal{T} \otimes \mathcal{B}(X)$, its projection $proj_T(G)$ belongs to $\mathcal{T}$.*

Lemma 3.3.7 *Let T be a σ-finite measure space, and X a complete separable metric space. If $F : T \to 2^X$ has a measurable graph, the set $\{t \in T : F(t) \neq \emptyset\}$ is measurable.*

Proof. Since the set $\{t \in T : F(t) \neq \emptyset\} = Proj_T(GrF)$, by Lemma 3.3.6, it is measurable. ∎

Lemma 3.3.8 *Let T be a σ-finite measure space, X a complete separable metric space, and $F : T \to 2^X$ a correspondence with a measurable graph. Then for every $x \in X$, $d(x, F(\cdot))$ is a measurable function.*

Proof. First we have $S = \{t \in T : F(t) \neq \emptyset\}$ belongs to $\mathcal{T}$ by Lemma 3.3.7. Next $\{s \in S : d(x, F(s)) < \lambda\} = \{s \in S : F(s) \cap B(x, \lambda) \neq \emptyset\} = Proj_T[G_F \cap (T \times B(x, \lambda))]$. The conclusion follows from Lemma 3.3.6. ∎

Lemma 3.3.9 *Let T be a σ-finite measure space, and X a complete separable metric space. Let $W : T \to 2^X$ be a weakly measurable correspondence or have a measurable graph. Then the correspondence $V : T \to 2^X$ defined by*

$$V(t) = \{x \in X : d(x, W(t)) > \lambda\}, \text{ where } \lambda \text{ is any fixed real number,}$$

has a measurable graph. The conclusion still holds if $>$ is replaced by any of $<$, $\geq$ or $\leq$.

Proof. Part of this lemma was given in Lemma 4.8 in [49], or Lemma 4.8 in [50]. But it seems to us that the same minor error appears in the two papers ($[\lambda, \infty]$ should be replaced by $(\lambda, \infty]$ in the proofs). Consider the function $g : T \times X \to [0, \infty]$ given by $g(t, x) = d(x, W(t))$. If W is a weakly measurable correspondence (respectively, has a measurable graph), by Theorem 3.3 in [41] (respectvely, by Lemma 3.3.8), $g(t, x)$ is measurable in t and continuous in x. Thus g is therefore jointly measurable, i.e., measurable with respect to the product σ-algebra $\mathcal{T} \otimes \mathcal{B}(X)$ by Lemma 3.3.4. For a real number λ, define $U_\lambda(t) = \{x \in X : d(x, W(t)) > \lambda\}$. Then $GrU = g^{-1}((\lambda, \infty])$. Hence U has a measurable graph.

The other conclusions can be proved similarly. ∎

The following lemma is Lemma 4.5 in [49].

Lemma 3.3.10 *Let* $(T_i, \mathcal{T}_i)$ *for* $i = 1, 2, 3$ *be measurable spaces,* $y : T_1 \to T_3$ *a measurable function and* $F : T_1 \times T_2 \to 2^{T_3}$ *a correspondence with a measurable graph, i.e.,* $GrF \in \mathcal{T}_1 \otimes \mathcal{T}_2 \otimes \mathcal{T}_3$. *Let* $W : T_1 \to 2^{T_2}$ *be defined by*

$$W(t) = \{x \in T_2 : y(t) \in F(t, x)\}.$$

Then W *has a measurable graph, i.e.,* $GrW \in \mathcal{T}_1 \otimes \mathcal{T}_2$.

Lemma 3.3.11 *Let* T *be a* σ-*finite measure space,* X *a complete separable metric space, and* $W, V : T \to 2^X$ *two correspondences with measurable graphs. Then the function* $g : T \to [0, \infty]$ *defined by*

$$g(t) = d(W(t), V(t)) \quad \text{for each } t \in T$$

is a measurable function.

Proof. Since $S_1 := \{t \in T : W(t) \neq \emptyset\}$ and $S_2 := \{t \in T : V(t) \neq \emptyset\}$ are measurable by Lemma 3.3.7, $S := S_1 \cap S_2$ is measurable. By Lemma 3.3.9, we know $V_\lambda : T \to 2^X$ defined by

$$V_\lambda(t) = \{x \in T : d(V(t), x) < \lambda\}, \quad \text{where } \lambda \text{ is any fixed real number,}$$

has measurable graph.

Now

$$
\begin{aligned}
\{s \in S : d(W(s), V(s)) < \lambda\} &= \{s \in S : W(s) \cap V_\lambda(s)\} \\
&= Proj_T(GrW \cap Gr(V_\lambda))
\end{aligned}
$$

is measurable by Lemma 3.3.4 (Lemma 4.2 in [49]). ∎

The following lemma is Theorem III.9 in [19].

Lemma 3.3.12 *Let* T *be a measurable space and* X *a separable metric space, and* $F : T \to 2^X$ *a correspondence such that* $F(t)$ *is a nonempty complete subset of* X *for each* $t \in T$. *Then the following properties are equivalent.*

(1) F is weakly measurable,

(2) $d(x, F(\cdot))$ is measurable for every $x \in X$,

(3) F admits a sequence of measurable selections (f_i) such that $F(t) = cl(\cup\{f_i(t)\})$.

Lemma 3.3.13 *Let T be a σ-finite measure space, and X a separable complete metric space. Suppose that $F : T \to 2^X$ is a correspondence with nonempty closed values and has a measurable graph, then F has a measurable selection.*

Proof. By Lemma 3.3.8, for every $x \in X$, $d(x, F(\cdot))$ is a measurable function. By Lemma 3.3.12, F has a measurable selection. ∎

Let K be a nonempty subset of a Hausdorff locally convex space E, and E' the continuous dual of E. The function $\delta^* : E' \to \mathbb{R} \cup \{\infty\}$ defined by

$$\delta^*(x') := \delta^*(x'|K) := \sup_{x \in K} \langle x', x \rangle$$

is called the *support function* of K.

The following result is a particular form of Theorem III.37 in [19].

Lemma 3.3.14 *Let T be a σ-finite measure space, E a Hausdorff locally convex space, X a convex Souslin subset of E and $F : T \to 2^X$ a correspondence with nonempty convex compact values. Then the following properties are equivalent:*

(1) for every $x' \in E'$, $\delta^*(x'|F(\cdot))$ is measurable.

(2) F has a measurable graph.

Lemma 3.3.15 *Let T be a complete σ-finite measure space, X a separable complete metric space, and $f : T \to [0, \infty)$ a measurable function. Then $F : T \to 2^X$ defined by $F(t) := \{x \in X : d(0, x) \le f(t)\}$ for each $t \in T$ is measurable.*

Proof.

Define $g : T \times X \to \mathbb{R}$ by $g(t, x) = d(0, x) - f(t)$. Then g is measurable in t and continuous in x. By Theorem 6.4 in [41], $F(t) = \{x \in X : g(t, x) \le 0\}$ is measurable. ∎

Lemma 3.3.16 *Let T be a σ-finite measure space and X a nonempty compact subset of the Fréchet space E. Let F and $G : T \to 2^X$ be two correspondences with nonempty closed values. If F and G have measurable graphs, then $F + G$ has a measurable graph.*

Proof. For any $x' \in E'$, consider the support functions $\delta^*(x'|F(\cdot))$ and $\delta^*(x'|G(\cdot))$. By Lemma 3.3.14, they are measurable. Now by [(26) on page 31 in [4]], $\delta^*(x'|F(\cdot) + G(\cdot)) = \delta^*(x'|F(\cdot)) + \delta^*(x'|G(\cdot))$ is measurable. By Lemma 3.3.14 again, $F + G$ has measurable graph. ∎

The following theorem is one of the main results in this thesis. It says that under some conditions, a measurable upper semicontinuous correspondence $T \times X \to 2^Y$ can be extended to a correspondence with the same properties but with *nonempty* values. It is very useful to the study of abstract economies with measurable spaces of agents. The main idea of the proof is this: for each fixed $t \in T$, X can be represented by two disjoint subsets: the one (part1) on which $F(t, \cdot)$ is empty and the other one (part2) on which $F(t, \cdot)$ is nonempty. By carefully constructing a partition of unity and using some set approximation techniques, we can define different (nonempty) values for $F(t, \cdot)$ on part1. This new correspondence still keeps the measurablity and continuity (and even some more) as the original one has. But to realize this idea is not easy. The proof of the theorem is very technical (at least I think so) and extremely complicated (you will know this after you have read it).

Theorem 3.3.17 *Let T be a complete σ-finite measure space, X a complete separable metric space and Y a nonempty compact convex subset of the separable Fréchet space E. Let $F, \phi : T \times X \to 2^Y$ be correspondences with compact convex values, and ϕ nonempty valued. Suppose that*

(i) F and ϕ are upper semicontinuous and continuous, respectively;

(ii) F and ϕ are measurable and for each $(t, x) \in T \times X$, $F(t, x) \subset \phi(t, x)$.

*Then there exists $\tilde{F} : T \times X \to 2^Y$ with **nonempty** compact convex values such that*

(a) $\tilde{F}$ is upper semicontinuous;

(b) $\tilde{F}$ is measurable and for each $(t, x) \in T \times X$, $\tilde{F}(t, x) \subset \phi(t, x)$;

(c) $\tilde{F}(t, x) = F(t, x)$ if $F(t, x) \neq \emptyset$.

Proof.

Define $M : T \to 2^X$ by $M(t) = \{x \in X : F(t, x) \neq \emptyset\}$ for each $t \in T$. Then $M(t)$ is closed for each $t \in T$ since $F(t, \cdot)$ is upper semicontinuous. Without loss of generality, we suppose that $M(t) \neq \emptyset$ for each $t \in T$. For if not, we consider $T_1 = Proj_T GrF$. By Lemma 3.3.4, T_1 is measurable. After we get an extension $\hat{F}$ of F on $T_1 \times X$ with nonempty compact convex values such that

(a') $\hat{F}$ *is upper semicontinuous;*

(b') $\hat{F}$ *is measurable and for each* $(t, x) \in T_1 \times X$, $\hat{F}(t, x) \subset \phi(t, x)$;

(c') $\hat{F}(t, x) = F(t, x)$ *if* $F(t, x) \neq \emptyset$,

we simply define $\tilde{F}(t, x) = \phi(t, x) \neq \emptyset$ for all $(t, x) \in (T \backslash T_1) \times X$. Then $\tilde{F}$ is an extension of F as demanded.

In the following, d will be the metric on X and ρ will be the metric on Y which induces the topology on Y such that the metric is invariant under translations, and for which the open balls are convex (by Lemma 2.2.3).

(1) Since $GrM = \{(t, x) \in T \times X : F(t, x) \neq \emptyset\}$ which is measurable by Lemma 3.3.2 (i.e., Proposition 2.2 [40]), M is a correspondence with a measurable graph.

(2) Let $X_0 = \{x_n : n = 1, \ldots\}$ be a dense subset of X. Define $W_n(t) = \{x : 3d(x, x_n) < d(x_n, M(t))\}$ for each $n = 1, 2, \ldots$. We show that W_n has a measurable graph. Obviously $W_n(t)$ may be an empty set.

Consider $g_n : T \times X \to [0, \infty]$ given by $g_n(t, x) = d(x_n, M(t)) - 3d(x, x_n)$. By Lemma 3.3.9, $d(x_n, M(t))$ is a measurable function. So $g_n(t, x)$ is measurable for each x and is continuous for each t. Thus g_n is jointly measurable by Lemma 3.3.4 (i.e., Lemma III.14, [19]). Now $GrW_n = g_n^{-1}[(0, \infty]]$ is measurable.

(3) Define $N : T \to 2^X$ by $N(t) = X \backslash M(t)$ for each $t \in T$. Then N has a measurable graph by Lemma 3.3.5. Further, for each $t \in T$, $\{W_n(t), n = 1, \ldots\}$ covers $N(t)$.

(4) For each $m = 1, 2, \ldots$ define the operator $()_m : 2^X \to 2^X$ by

$$(W)_m = \{w \in W : d(w, N(t) \backslash W) \geq 1/2^m\}.$$

(4.1) We show that $(W)_m$ is closed in $N(t)$ for any set $W \subset N(t)$.

Let $w_n \in (W)_m$ such that $w_n \to w \in N(t)$. If $w \notin (W)_m$, then either $w \notin W$, i.e. $w \in N(t) \backslash W$, which implies that $d(w, N(t) \backslash W) = 0$ or $d(w, N(t) \backslash W) < 1/2^m$. This is impossible since d is continuous.

(4.2) For each $m = 1, \ldots$, we show that $(W(\cdot))_m$ has a measurable graph if $W(\cdot)$ has a measurable graph.

First $N(\cdot) \backslash W(\cdot)$ has a measurable graph by Lemma 3.3.5. Also $(W(\cdot))_m$ has a measurable graph by Lemma 3.3.9.

(4.3) Let $V_1(t) = W_1(t)$ for $t \in T$. For $n = 2, \ldots$ and $t \in T$, let

$$V_n(t) = W_n(t) \backslash \bigcup_{k=1}^{n-1} (W_k(t))_n.$$

Then $V_n(t)$ is open in $N(t)$ by (4.1). Since $W_n(\cdot)$ has a measurable graph, $V_n(t)$ has a measurable graph by (4.2) and Lemma 3.3.5. Further, $\{V_n(t), n = 1, 2, \ldots\}$ is a locally finite open cover of the set $N(t)$.

(5) If $N(t) \neq \emptyset$, let $f_n(t, \cdot)$ be the partition of unity subordinated to the open cover $\{V_n(t) : n = 1, 2, \ldots\}$ defined by

$$f_n(t, x) = \frac{d(x, N(t) \backslash V_n(t))}{\sum_{k=1}^{\infty} d(x, N(t) \backslash V_k(t))} \qquad \text{for each } n = 1, 2, \cdots.$$

Then $f_n(t, x)$ is measurable in t on $\{t \in T, N(t) \neq \emptyset\}$ and continuous in x. Note that since $N(\cdot)$ has a measurable graph by (3), the set $\{t \in T, N(t) \neq \emptyset\}$ is measurable by Lemma 3.3.7.

(6) For each $n = 1, 2, \ldots$, define $M_n' : T \to 2^X$ by

$$M_n'(t) = \{x \in X : d(x, W_n(t)) \leq 2d(M(t), W_n(t))\} \qquad \text{for each } t \in T.$$

Then M_n' has a measurable graph. This can be proved by Lemma 3.3.11 and by the same trick as in by (2).

Now for each $n = 1, 2, \ldots$, define $G_n : T \to 2^X$ by

$$G_n(t) = \{x \in M(t) : d(x, W_n(t)) \leq 2d(M(t), W_n(t))\} \text{ for each } t \in T.$$

This is equivalent to

$$G_n(t) = M(t) \cap M'_n(t) \quad \text{for each } t \in T.$$

Then G_n has a measurable graph by Lemma 3.3.5. Without loss of generality, we may assume for each $n = 1, \ldots, \{t \in T : G_n(t) \neq \emptyset\} \neq \emptyset$. By Lemma 3.3.13, we can choose a measurable function $y_n(t) \in G_n(t)$ such that $y_n(t)$ is measurable on $\{t \in T : G_n(t) \neq \emptyset\}$. By Theorem 8.1 in [40], $y_n(t)$ has a measurable extension on T.

(7) Let $B_r = \{y \in E : \rho(y, 0) \leq r\}$.

(7.1) Let

$$s_n(t, z) = 2\rho(F(t, y_n(t)), \phi(t, z)),$$
$$\psi_n(t, z) = F(t, y_n(t)) + B_{s_n(t,z)},$$
$$R(t, z) = \sum_{n=1}^{\infty} f_n(t, z)(\psi_n(t, z) \cap \phi(t, z)),$$
$$S_i(t, x) = clco(\bigcup_{\{z \in X_0 : d(z, x) < 1/i\}} R(t, z)),$$
$$S(t, x) = \bigcap_{i=1}^{\infty} S_i(t, x).$$

Define $\tilde{F} : T \times X \to 2^Y$ by

$$\tilde{F}(t, x) = \begin{cases} F(t, x), & \text{if } F(t, x) \neq \emptyset \text{ (i.e., } x \in M(t) \text{);} \\ S(t, x), & \text{if } F(t, x) = \emptyset \text{ (i.e., } x \in N(t)). \end{cases}$$

(7.2) For any $x \in N(t)$, $S(t, x)$ is nonempty since $S_i(t, x)$ is a decreasing sequence of nonempty compact sets.

(7.3) We shall show that $\tilde{F}(t, x) \subset \phi(t, x)$ for all $(t, x) \in T \times X$.

By condition (ii) and the definition of $\tilde{F}(t, x)$, we see that $\tilde{F}(t, x) \subset \phi(t, x)$ for each $x \in M(t)$.

Now consider $x \in N(t)$. By the upper semicontinuity of $\phi(t, \cdot)$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\phi(t, y) \subset \phi(t, x) + O(0, \varepsilon)$ provided $y \in O(x, \delta)$. Therefore

$$R(t, z) \subset \phi(t, x) + O(0, \varepsilon) \text{ for each } z \in O(x, \delta) \cap X_0,$$

because the sets $\phi(t,x)$ and $\phi(t,x) + O(0,\varepsilon)$ are convex.

This shows that $\tilde{F}(t,x) \subset \phi(t,x) + O(0,2\varepsilon)$ for any $\varepsilon > 0$. Since $\phi(t,x)$ is compact, we have $\tilde{F}(t,x) \subset \phi(t,x)$.

(8) We claim that for each fixed $z \in X$, $R(t,z)$ is measurable.

(8.1) We shall first prove that $F(t,y_n(t))$ is measurable.

In fact, for each $y \in Y$, $d(y, F(t,x))$ is measurable in (t,x) by condition (ii) and Lemma 3.3.8. By (6), $Gr(y_n)$ is measurable. By Lemma III.39 in [19], $d(y, F(t,y_n(t)))$ is measurable. Hence $F(t,y_n(t))$ is measurable by Lemma 3.3.3.

(8.2) We shall now show that for each fixed $z \in X$, $R(\cdot, z)$ is measurable.

Fix an arbitrary $z \in X$. We know that $\phi(\cdot, z)$ has a measurable graph. Let $g(y,t) = d(y, F(t,y_n(t))) - s_n(t,z)$. Also $s_n(\cdot, z)$ is measurable for the by Lemma 3.3.11. By Lemma 3.3.15, $B_{s_n(t,z)}$ is measurable in t. By Lemma 3.3.16, $\psi_n(\cdot, z)$ is measurable.

Note that for each $f' \in E'$,

$$\delta^*(f'|R(t,z)) = \sum_{n=1}^{\infty} f_n(t,z)\delta^*(f'|(\psi_n(t,z) \cap \phi(t,z)))$$

is measurable, $R(\cdot, z)$ is measurable by Lemma 3.3.14.

(9) We shall show that $\tilde{F}$ is measurable.

Let $U = \{(t,x) : F(t,x) = \emptyset\}$ and B be any nonempty closed subset of Y.

(9.1) For each $i = 1, \ldots$, define $G_i(t,x) = \cup\{R(t,z) : z \in X_0, d(z,x) < 1/i\}$. We claim that G_i is measurable.

Note that

$$\{(t,x) \in T \times X : G_i(t,x) \cap B \neq \emptyset\}$$
$$= \cup_{z \in X_0}(\{t \in T : R(t,z) \cap B \neq \emptyset\} \times \{x \in X : d(x,z) < 1/i\})$$
$$\in \mathcal{T} \otimes \mathcal{B}(X).$$

Therefore, G_i is measurable so that it has a measurable graph.

(9.2) For any $f' \in E'$, by Lemma 3.3.14, $\delta^*(f'|G_i(\cdot))$ is measurable. Since $\delta^*(f'|G_i(\cdot)) = \delta^*(f'|clcoG_i(\cdot))$ (refer to [4], page 27), $\delta^*(f'|clcoG_i(\cdot))$ is measurable.

By Lemma 3.3.14, $clcoG_i(\cdot)$ is measurable. Since $S_i = clcoG_i$, S_i is measurable which in turn implies it has measurable graph.

(9.3) Since $S = \cap_{i=1}^{\infty} S_i$, S has measurable graph by Lemma 3.3.5. By Lemma 3.3.3, S is measurable.

(9.4) Now

$\{(t, x) \in T \times X, \tilde{F}(t, x) \cap B \neq \emptyset\}$

$= \{(t, x) \in U : F(t, x) \cap B \neq \emptyset\} \cup \{(t, x) \notin U : S(t, x) \cap B \neq \emptyset\}$

$= (\{(t, x) \in T \times X : F(t, x) \cap B \neq \emptyset\} \cap U) \cup (\{(t, x) \in T \times X : S(t, x) \cap B \neq \emptyset\} \backslash U)$

$\in \mathcal{T} \otimes \mathcal{B}(Y).$

Thus $\tilde{F}$ is measurable.

(10) Next we shall show that for each $t \in T$, $\tilde{F}(t, x)$ is upper semicontinuous on X.

Fixed an arbitrary $t \in T$. If $N(t) = \emptyset$, $\tilde{F}(t, x) = F(t, x)$ is upper semicontinuous on X. Now suppose $N(t) \neq \emptyset$.

(10.1) It is easy to see that $\tilde{F}(t, \cdot)$ is upper semicontinuous continuous at each point in $int_X(M(t))$.

(10.2) To prove the upper semicontinuity of $\tilde{F}(t, \cdot)$ at any point $x_0 \in N(t)$, choose a ball $O(x_0, r)$ which intersects a finite number of sets of the open cover $\{V_n(t)\}$.

We observe that the set $\tilde{F}(t, x_0) = S(t, x_0)$ is the Hausdorff limit of the sequence of compact sets

$$S_i(t, x) = cl(co \cup_{\{z \in X_0 : d(z, x_0) \leq 1/i\}} R(t, z)) \text{ provided } 1/i < r .$$

Therefore, for every $\varepsilon > 0$ there exists an integer i_0 such that for all $i \geq i_0$,

$$S_i(t, x_0) \subset S(t, x_0) + O(0, \varepsilon).$$

For any $y \in N(t)$ with $d(y, x_0) < \delta := \min\{1/(2i_0), r\}$, we have

$$\{z \in X_0 : d(z, y) < 1/(2i_0)\} \subset \{z \in X_0 : d(z, x_0) < 1/i_0\}.$$

Hence, $S_{2i_0}(t, y) \subset S_{i_0}(t, x_0)$ and

$$S_{2i_0}(t, y) \subset S_{i_0}(t, x_0) \subset S(t, x_0) + O(0, \varepsilon).$$

i.e., $\tilde{F}(t, y) \subset \tilde{F}(t, x_0) + O(0, \varepsilon)$ for any $y \in O(x_0, \delta)$.

(10.3) Now we shall show that $\tilde{F}(t, \cdot)$ is upper semicontinuous at each point in $\partial_X M(t)$.

Let $x_0 \in \partial_X M(t)$. Then $\tilde{F}(t, x_0) = F(t, x_0)$.

If $x \in N(t)$, we have $x \in V_n(t)$ for some n. Applying the triangle inequality yields

$$\begin{aligned}
d(x_n, M(t)) &\leq d(x_0, x_n) \\
&\leq d(x_0, x) + d(x, x_n) \\
&\leq d(x_0, x) + d(x_n, M(t))/3.
\end{aligned}$$

Hence

$$d(x_n, M(t)) \leq 3d(x_0, x)/2.$$

Now take (v_i) in $W_n(t)$ such that $d(y_n(t), v_i) = \lim_i d(y_n(t), v_i)$. Note that

$$d(y_n(t), x_n) \leq d(y_n(t), v_i) + d(v_i, x_n),$$

and by (2), we have

$$\begin{aligned}
d(y_n(t), x_n) &\leq \lim_i d(y_n(t), v_i) + d(x_n, M(t))/3 \\
&= d(y_n(t), W_n(t)) + d(x_n, M(t))/3 \\
&\leq 2d(M(t), W_n(t)) + d(x_n, M(t))/3.
\end{aligned}$$

Now

$$\begin{aligned}
d(y_n(t), x_0) &\leq d(y_n(t), x_n) + d(x_n, x) + d(x, x_0) \\
&\leq 2d(M(t), W_n(t)) + d(x_n, M(t))/3 + d(x_n, M(t))/3 + d(x, x_0)
\end{aligned}$$

$$\leq \quad (2+2/3)d(x_n, M(t)) + d(x_0, x)$$

$$\leq \quad 8/3 \cdot 3d(x_0, x)/2 + d(x_0, x)$$

$$\leq \quad 5d(x_0, x);$$

i.e,

$$d(y_n(t), x_0) \leq 5d(x_0, x). \tag{$*$}$$

For any $\varepsilon > 0$, by condition (i), there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$F(t, x) \subset O(F(t, x_0), \varepsilon/8) \text{ for each } x \in M(t) \cap O(x_0, \delta_1)$$

and

$$H(\phi(t, y), \phi(t, x_0)) < \varepsilon/8 \text{ for each } O(x_0, \delta_2)$$

where H is the Hausdorff distance. Since by condition (ii), $F(t, x_0) \subset \phi(t, x_0)$ for $x_0 \in M(t)$, so

$$F(t, x_0) \subset \phi(t, x_0) \subset O(\phi(t, y), 2\varepsilon/8).$$

Now

$$F(t, x) \quad \subset \quad F(t, x_0) + O(0, \varepsilon/8)$$

$$\subset \quad O(\phi(t, y), 2\varepsilon/8) + O(0, \varepsilon/8)$$

$$\subset \quad O(\phi(t, y), 3\varepsilon/8).$$

Hence, $d(\phi(t, y), F(t, x)) < 3\varepsilon/8$ for each $x \in M(t) \cap O(x_0, \delta_1)$ and $y \in O(x_0, \delta_2)$.

Let $\delta = \min\{\delta_1/5, \delta_2\}$ and $x \in O(x_0, \delta)$, then $y_n(t) \in O(x_0, \delta_1) \cap M(t)$ for all $n \in \mathbb{N}$ such that $x \in V_n(t)$ by $(*)$. So we have

$$s_n(t, x) = 2\rho(F(t, y_n(t)), \phi(t, x)) < 2 \cdot 3\varepsilon/8 = 6\varepsilon/8 \text{ for any } x \in O(x_0, \delta).$$

Therefore,

$$\psi_n(t, x) \quad = \quad F(t, y_n(t)) + B_{s_n(t,x)}$$

$$\subset \quad F(t, y_n(t)) + O(0, 6\varepsilon/8)$$

$$\subset \quad F(t, x_0) + O(0, \varepsilon/8) + O(0, 6\varepsilon/8)$$

$$\subset \quad F(t, x_0) + O(0, 7\varepsilon/8).$$

Since $\{V_i(t), i \in \mathbb{N}\}$ is a locally finite covering of $N(t)$ by (4), there exists an open neighborhood $O_N(x_0)$ in $N(t)$ such that $\{i : O_N(x_0) \cap V_i(t) \neq \emptyset\}$ is finite. Hence for $x \in O_N(x_0)$,

$$
\begin{aligned}
\sum_{i=1}^{\infty} f_i(t,x)(\psi_i(t,x) \cap \phi(t,x)) &\subset \sum_{i=1}^{\infty} f_i(t,x)(F(t,x_0) + O(0, 7\varepsilon/8)) \\
&\subset F(t,x_0) + O(0, 7\varepsilon/8).
\end{aligned}
$$

For any $i \in N$ with $1/i < \delta - d(x_0, x)$,

$$
\begin{aligned}
S_i(t,x) &= clco(\cap_{z \in X_0 : d(z,x) < 1/i} R(t,z)) \\
&\subset cl(F(t,x_0) + O(0, 7\varepsilon/8)) \\
&\subset F(t,x_0) + O(0,\varepsilon).
\end{aligned}
$$

Hence for any $x \in O_N(x_0)$,

$$
\begin{aligned}
\tilde{F}(t,x) &= S(t,x) \\
&= \cap_{i=1}^{\infty} S_i(t,x) \\
&\subset F(t,x_0) + O(0,\varepsilon) \\
&= \tilde{F}(t,x_0) + O(0,\varepsilon).
\end{aligned}
$$

Also for any $x \in O(x_0, \delta) \cap M(t)$,

$$
\begin{aligned}
\tilde{F}(t,x) &= F(t,x) \\
&\subset O(F(t,x_0), \varepsilon/8) \\
&\subset O(\tilde{F}(t,x_0), \varepsilon).
\end{aligned}
$$

Therefore $\tilde{F}$ is upper semicontinuous at every point in $\partial_X M(t)$. ∎

Remark. Suppose that X, Y and T are as in Theorem 3.3.17. Lemma 3.3.4 (i.e., Lemma III.14 in [19]) claims that a single-valued map $f : T \times X \to Y$ is (jointly) measurable if it is measurable in the first variable and continuous in the second variable. One might ask if a nonempty closed valued correspondence $F : T \times X \to$

2^Y that is measurable in the first variable and continuous in the second variable is (jointly) measurable. Indeed, this is true. Consider the function $g : Y \times T \times X \to \mathbb{R} \cup \{\infty\}$ defined by $g(y, t, x) = d(y, F(t, x))$ for each $(y, t, x) \in Y \times T \times X$. For each fixed $(t, x) \in T \times X$, $g(\cdot, t, x)$ is continuous. For each fixed $y \in Y$, note that $g(y, t, x)$ is measurable in t and continuous in x, hence $g(y, t, x)$ is measurable in (t, x). By Lemma 3.3.12, $F(t, x)$ is measurable. The next question is whether a (nonempty) closed valued correspondence $F : T \times X \to 2^Y$ that is measurable in the first variable and upper semicontinuous in the second variable is measurable. The answer for this problem is no. This can be seen by the following example from [83].

Example 3.3.18 *Let $T := [0, 1]$ be equipped with the σ-algebra $\mathcal{T}$ of Lebesgue measurable sets, $X := [0, 1]$ be equipped with the Borel σ-algebra $\mathcal{B}(X)$ and $Y := [0, 1]$. Let M be a subset of $[0, 1]$ which is not Lebesgue measurable. Define $F : T \times X \to 2^Y$ by $F(\omega, x) := [0, \mathcal{X}_W(\omega, x)]$ for each $(\omega, x) \in T \times X$, where $\mathcal{X}_W$ denotes the characteristic function with respect to W and $W := \{(\omega, \omega) : \omega \in M\}$.*

First we note that $W = \{(\omega, \omega) : \omega \in M\} \notin \mathcal{T} \otimes \mathcal{B}([0, 1])$. If this were not true, $M = Proj_T(W)$ would be measurable by Lemma 3.3.6. It is easy to check that F is random upper semicontinuous with nonempty closed and convex values. However the mapping F is not jointly measurable as the set

$$F^{-1}(\{1\}) = \{(\omega, x) \in T \times X : 1 \in F(\omega, x)\} = \{(\omega, \omega) : \omega \in M\},$$

which is not a Lebesgue measurable set. ∎

Nevertheless, we have the following conjecture. Roughly speaking, it states that a random upper semicontinuous correspondence has a nonempty valued extension. Note that in this case, we only require F to be measurable in the first variable rather than measurable in both variables as in Theorem 3.3.17. If it is true, it can be used to resolve some different economics problems (see the next section).

Let T be a complete σ-finite measure space, X a complete separable metric space and Y a nonempty compact convex subset of a Fréchet space. Let $F, \phi : T \times X \to 2^Y$

be correspondences with compact convex values, and ϕ nonempty-valued. Suppose

(i) F and ϕ are upper semicontinuous and continuous, respectively;

(ii) F and ϕ are random (i.e., measurable in the **first variable**) and for each $(t, x) \in T \times X$, $F(t, x) \subset \phi(t, x)$.

Then there exists a correspondence $\tilde{F} : T \times X \to 2^Y$ with **nonempty** compact convex values such that

(a) $\tilde{F}$ is upper semicontinuous;

(b) $\tilde{F}$ is random and for each $(t, x) \in T \times X$, $\tilde{F}(t, x) \subset \phi(t, x)$;

(c) $\tilde{F}(t, x) = F(t, x)$ if $F(t, x) \neq \emptyset$.

Let $(T, \mathcal{T})$ be a measurable space, X be a topological space, and $F : T \times X \to 2^X$ be a correspondence. Then the (single-valued) map $f : T \to X$ is said to be (1) a *deterministic fixed point* of F if $f(t) \in F(t, f(t))$ for all $t \in T$ and (2) a *random fixed point* if f is measurable and $f(t) \in F(t, f(t))$ for all $t \in T$. It should be noted here some authors define a random fixed point of F to be a measurable map f such that $f(t) \in F(t, f(t))$ for almost every $t \in T$, for example, see [60] and [64].

The following lemma is Theorem 2.3 in [76].

Lemma 3.3.19 *Let $(T, \mathcal{T})$ be a measurable space, $\mathcal{T}$ a Souslin family, X a topological space, and X_0 a Souslin subset of X. Suppose that $F : T \times X_0 \to 2^X$ is such that $GrF \in \mathcal{T} \otimes \mathcal{B}(X_0 \times X)$. Then F has a random fixed point if and only if F has a deterministic fixed point, i.e., for each $t \in T$, $F(t, \cdot)$ has fixed point.*

Theorem 3.3.20 *Let T be a complete σ-finite measurable metric space, and X a nonempty compact convex subset of a separable Fréchet space Y. Let $F : T \times X \to 2^X$ be a correspondence with compact convex values, and further,*

(i) *for each t, $F(t, \cdot) : X \to 2^X$ is upper semicontinuous,*

(ii) *$F : T \times X \to 2^X$ is measurable.*

Then there exists a measurable function $f : T \to X$ such that for each $t \in T$ either $f(t) \in F(t, f(t))$ or $F(t, f(t)) = \emptyset$.

Proof.   Since X is compact, it is a separable and complete metric space. Let $\phi : X \to 2^X$ be defined by $\phi(x) = X$ for all $x \in X$. By Theorem 3.3.17, there exists a measurable correspondence $\tilde{F} : T \times X \to 2^X$ with nonempty compact convex values such that

(a) $\tilde{F}(t,x) = F(t,x)$ if $F(t,x) \neq \emptyset$;

(b) for each $t \in T$, $\tilde{F}(t,\cdot) : X \to 2^X$ is upper semicontinuous.

By the Fan-Glicksberg fixed point theorem, for each $t \in T$, $\tilde{F}(t,\cdot)$ has a fixed point. Since X is a compact metric space, $\mathcal{B}(X \times X) = \mathcal{B}(X) \otimes \mathcal{B}(X)$ (see [65], page 113). Note that $Gr\tilde{F}$ belongs to $T \otimes \mathcal{B}(X) \otimes \mathcal{B}(X) = T \otimes \mathcal{B}(X \times X)$. Thus $\tilde{F}$ has a random fixed point $f : T \to X$ by Lemma 3.3.19. Now for each $t \in T$ such that $F(t,f(t)) \neq \emptyset$, we have $f(t) \in \tilde{F}(t,f(t)) = F(t,f(t))$. ∎

Theorem 3.3.21 *Let T be a complete σ-finite measurable metric space, X a nonempty compact convex subset of a separable Fréchet space Y, $F : T \times X \to 2^X$ a correspondence with compact convex values and $\phi : T \times X \to 2^Y$ a correspondence with nonempty compact convex values and further,*

(i) for each $t \in T$, $F(t,\cdot) : X \to 2^X$ is upper semicontinuous and $\phi(t,\cdot) : X \to 2^Y$ is continuous.

(ii) $F, \phi : T \times X \to 2^X$ are measurable.

(iii) for each $t \in T$, $x \notin F(t,x)$ for all $x \in X$.

Then there exists a measurable function $f : T \to X$ such that for each $t \in T$ $f(t) \in \phi(t,f(t))$ and $F(t,f(t)) \cap \phi(t,f(t)) = \emptyset$.

Proof.   Define $G : T \times X \to 2^X$ by $G(t,x) = F(t,x) \cap \phi(t,x)$ for each $(t,x) \in T \times X$. Then $G(t,x) \in \phi(t,x)$ for each $(t,x) \in T \times X$. Let $U = \{(t,x) \in T \times X : G(t,x) \neq \emptyset\}$. If U is empty, the conclusion follows trivially. We now consider that U is not empty.

Note that G is measurable by Lemma 3.3.3 and for each fixed $t \in T$, $G(t,\cdot)$ is upper semicontinuous by Lemma 2.2 in Tan and Yuan [78]. By Theorem 3.3.17, there exists an upper semicontinuous correspondence $\tilde{G}$ with nonempty compact convex values

such that $\tilde{G}|_U = G$ and $\tilde{G}(t,x) \subset \phi(t,x)$ for each $(t,x) \in T \times X$. So by Theorem 3.3.20, there exists a measurable function $f : T \to X$ such that $f(t) \in \tilde{G}(t,f(t))$ for all $t \in T$. Obviously, $f(t) \in \phi(t,f(t))$ for all $t \in T$. Now for each $t \in T$, $G(t,f(t))$ must be an empty set. Otherwise, we have $f(t) \in \tilde{G}(t,f(t)) = G(t,f(t))$ which contradicts condition (iii). Thus the conclusion holds. ∎

3.4 An Equilibrium Existence Theorem for an Abstract Economy

Throughout this section, $(T,\mathcal{T},\mu)$ will be a finite, positive, complete, and separable measure space of agents. Let Y be a separable Banach space. For any correspondence $H : T \to 2^Y$, $L_1(\mu, H)$ denotes the subset of $L_1(\mu, Y)$ consisting of those $x \in L_1(\mu, Y)$ for which $x(t) \in H(t)$ for almost all t in T. Following Kim *et al* [48], we define the notion of an abstract economy with a measure space of players as follows:

An *abstract economy* Γ is a tuple $[(T,\mathcal{T},\mu), H, P, F]$, where

(1) $(T,\mathcal{T},\mu)$ is a measure space of agents;

(2) $H : T \to 2^Y$ is a strategy correspondence;

(3) $P : T \times L_1(\mu, H) \to 2^Y$ is a preference correspondence such that $P(t,x) \subset H(t)$ for all $(t,x) \in T \times L_1(\mu, H)$;

(4) $F : T \times L_1(\mu, H) \to 2^Y$ is a constraint correspondence such that $F(t,x) \subset H(t)$ for all $(t,x) \in T \times L_1(\mu, H)$.

An *equilibrium* for Γ is a point $\hat{x}(t) \in L_1(\mu, H)$ such that for almost all $t \in T$ the following conditions are satisfied:

(i) $\hat{x}(t) \in F(t,\hat{x})$;

(ii) $P(t,\hat{x}) \cap F(t,\hat{x}) = \emptyset$.

The following lemma has been proved in many ways. It seems that the simplest proof is the one in Khan and Papageorgiou [45] which was reproduced in Yannelis [85].

Lemma 3.4.1 *Let Y be a separable Banach space, and $H : T \to 2^Y$ an integrably bounded, nonempty convex valued correspondence such that for all $t \in T$, $H(t)$ is a weakly compact subset of Y. Let $\theta : T \times L_1(\mu, H) \to 2^Y$ be a nonempty closed convex valued correspondence such that*

(a) for each $(t, x) \in T \times L_1(\mu, H)$, $\theta(t, x) \subset H(t)$;

(b) for each fixed $x \in L_1(\mu, H)$, $\theta(\cdot, x) : T \to 2^Y$ has a measurable graph;

(c) for each fixed $t \in T$, $\theta(t, \cdot) : L_1(\mu, H) \to 2^Y$ is upper semicontinuous in the sense that the set $\{x \in L_1(\mu, H) : \theta(t, x) \subset V\}$ is weakly open in $L_1(\mu, H)$ for every norm open subset V of Y.

Then the correspondence $F : L_1(\mu, H) \to 2^{L_1(\mu, H)}$ defined by $F(x) = \{y \in L_1(\mu, H) : y(t) \subset \theta(t, x) \text{ for almost all } t \in T\}$ is upper semicontinuous in the sense that the set $\{x \in L_1(\mu, H) : F(x) \subset V\}$ is relatively weakly open in $L_1(\mu, H)$ for every relatively weakly open subset V of $L_1(\mu, H)$.

We now state the assumptions in order to obtain an equilibrium for the generalized abstract economy.

(i) $H : T \to 2^Y$ is a correspondence such that it is integrably bounded and for all $t \in T$, $H(t)$ is a nonempty weakly compact convex subset of Y.

(ii) $F : T \times L_1(\mu, H) \to 2^Y$ is a correspondence such that

(ii.1) $\{(t, x, y) \in T \times L_1(\mu, H) \times Y : y \in F(t, x)\} \in \mathcal{T} \times \mathcal{B}_w(L_1(\mu, H)) \times \mathcal{B}(Y)$ where $\mathcal{B}_w(L_1(\mu, H))$ is the Borel σ-algebra for the weak topology on $L_1(\mu, H)$ and $\mathcal{B}(Y)$ is the Borel σ-algebra for the norm topology on Y.

(ii.2) for all $(t, x) \in T \times L_1(\mu, H)$, $F(t, x)$ is a nonempty closed convex set in $H(t)$;

(ii.3) for each $t \in T$, $F(t, \cdot)$ is continuous from the weak topology of $L_1(\mu, H)$ to the strong (norm) topology of Y.

(iii) $P : T \times L_1(\mu, H) \to 2^Y$ is a correspondence such that:

(iii.1) $\{(t, x, y) \in T \times L_1(\mu, H) \times Y : y \in P(t, x)\} \in \mathcal{T} \otimes$ $\mathcal{B}_w(L_1(\mu, H)) \otimes \mathcal{B}(Y)$;

(iii.2) for all $(t, x) \in T \times L_1(\mu, H)$, $P(t, x)$ is a closed convex subset in $H(t)$;

(iii.3) $x(t) \notin P(t, x)$ for all $x \in L_1(\mu, H)$ and for almost all $t \in T$.

(iii.4) for each $t \in T$, $P(t, \cdot)$ is upper semicontinuous from the weak topology of $L_1(\mu, H)$ to the strong (norm) topology of Y.

Theorem 3.4.2 *Let* $\Gamma = [(T, \mathcal{T}, \mu), H, P, F]$ *be an abstract economy satisfying (i)-(iii). Then* Γ *has an equilibrium.*

Proof. First note that $L_1(\mu, H)$, endowed with the weak topology, is compact by Theorem 4.2 in Papageorgiou [61]. Since the relatively weak topology of a weakly compact subset of a separable Banach space is metrizable, $L_1(\mu, H)$ is a compact metrizable space (Theorem V.6.3.3, page 434 in [26]).

Define $\phi : T \times L_1(\mu, H) \to 2^Y$ by $\phi(t, x) = F(t, x) \cap P(t, x)$. Then for each $(t, x) \in T \times H$, $\phi(t, x)$ is closed convex and for each fixed $t \in T$, $\phi(t, \cdot)$ is upper semicontinuous by Lemma 2.2 in [79]. By Lemma 3.3.5, ϕ has a measurable graph.

Let $U = \{(t, x) \in T \times L_1(\mu, H) : \phi(t, x) \neq \emptyset\}$. By Theorem 3.3.17, there exists $G : T \times L_1(\mu, H) \to 2^Y$ such that

(1) G has a measurable graph;

(2) For each $t \in T$, $G(t, \cdot) : L_1(\mu, H) \to 2^Y$ is upper semicontinuous with nonempty closed convex values;

(3) (a) $G(t, x) = \phi(t, x)$ if $(t, x) \in U$ and

    (b) $G(t, x) \subset F(t, x)$ for all $(t, x) \in T \times L_1(\mu, H)$.

Define $\psi : L_1(\mu, H) \to 2^{L_1(\mu, H)}$ by

$$\psi(x) = \{y \in L_1(\mu, H) : \text{ for almost all } t \in T, \ y(t) \in G(t, x) \ \}.$$

Since $H(t)$ is integrably bounded and has a measurable graph, $L_1(\mu, H)$ is nonempty by the Aumann measurable selection theorem [7], and obviously it is convex since $H(t)$ is so. Since for each $x \in L_1(\mu, H)$, $G(\cdot, x)$ has a measurable graph, ψ is nonempty valued as a consequence of the Aumann measurable selection theorem [7]. Furthermore, since G is convex valued, so is ψ. By Lemma 3.4.1, ψ is weakly upper semicontinuous. Therefore, by the Fan-Glicksberg fixed point theorem (Theorem 2.1.5), there exists $\hat{x} \in L_1(\mu, H)$ such that $\hat{x} \in \psi(\hat{x})$. It can now be easily checked that the fixed point is by construction an equilibrium for Γ. ∎

Chapter 4

Generalizations of the Ky Fan Minimax Inequality with Applications

4.1 Introduction

Motivated by the work of Baye *et al* [11] and Tian [84], we first establish several generalizations of the Ky Fan type minimax inequalities in Section 4.2. In Section 4.3, we study the fixed point versions and maximal element versions of the minimax inequalities obtained in Section 4.2. These results improve the corresponding results of Ding and Tan [23] and include a fixed point theorem of Tarafdar [81] as a corollary and an existence theorem for an equilibrium point of an abstract economy. In Section 4.4, we give some von Neumann type minimax inequalities which are in turn applied to obtain some generalizations of Tan and Yu's recent work. Section 4.5 is devoted to studying variational inequalities. Related topics such as complementarity problems and fixed point theorems are also included. We point out here that our results do not require the topological spaces to be Hausdorff which were assumed in many papers such as in Baye *et al* [11], Tian [84] and Tarafdar [81].

4.2 Minimax Inequalities of the Ky Fan Type

We begin with the following lemma from [23] which is a slightly modified version of Lemma 1 of Fan in [29].

Lemma 4.2.1 *Let X and Y be nonempty sets in a topological vector space and $F : X \to 2^Y$ be such that*

(i) *for each $x \in X, F(x)$ is a nonempty closed subset of Y;*

(ii) *for each $A \in \mathcal{F}(X), co(A) \subset \cup_{x \in A} F(x)$;*

(iii) *there exists an $x_0 \in X$ such that $F(x_0)$ is compact.*

Then $\cap_{x \in X} F(x) \neq \emptyset$.

Now we are ready to establish

Theorem 4.2.2 *Let E be a topological vector space, X a nonempty convex subset of E and $f, g : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ satisfy the following conditions*

(i) *for each $x, y \in X$ with $f(x, y) > 0$, we have*

(i.1) *$g(x, y) > 0$;*

(i.2) *there exist some point $s \in X$ and some open neighborhood $N_X(y)$ of y in X such that $f(s, z) > 0$ for all $z \in N_X(y)$;*

(ii) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A), \min_{x \in A} g(x, y) \leq 0$;*

(iii) *there exist a nonempty closed and compact subset K of X and $x_0 \in X$ such that $g(x_0, y) > 0$ for all $y \in X \backslash K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Proof.

Define $F, G : X \to 2^X$ by

$$F(x) = \{y \in X : f(x, y) \leq 0\} \text{ for all } x \in X,$$

$$G(x) = \{y \in X : g(x, y) \leq 0\} \text{ for all } x \in X.$$

We first show that $\cap_{x\in X}F(x) = \cap_{x\in X}cl_X(F(x))$ and for this it is clear that it is sufficient to show $\cap_{x\in X}F(x) \supset \cap_{x\in X}cl_X(F(x))$. Suppose, on the contrary, there is some $y \in \cap_{x\in X}cl_X(F(x))$ such that $y \notin \cap_{x\in X}F(x)$. Then $y \notin F(x)$ for some $x \in X$ and thus $f(x,y) > 0$. By (i.2), there is some $s \in X$ and some open neighborhood $N_X(y)$ of y in X such that $f(s,z) > 0$ for all $z \in N_X(y)$. Thus $y \notin cl_X(F(s))$ which is a contradiction.

Next we prove that $\cap_{x\in X}cl_X(G(x)) \neq \emptyset$. We observe that

(1) If A is any finite subset of X, then $co(A) \subset \cup_{x\in A}G(x)$; for if this were false, then there would be a finite subset $\{z_1,\ldots,z_m\}$ of X and $z \in co(\{z_1,\ldots,z_m\})$ with $z \notin \cup_{j=1}^m G(z_j)$ so that $g(z_j,z) > 0$ for all $j = 1,\ldots,m$ which contradicts (ii). Hence $co(A) \subset \cup_{x\in A}cl_X(G(x))$.

(2) Note that x_0 must be in K, otherwise, we have $g(x_0,x_0) > 0$ which contradicts (ii). By (iii), $G(x_0) \subset K$. Since K closed, $cl_X(G(x_0)) \subset K$. Since K is compact, $cl_X(G(x_0))$ is compact.

(3) For each $x \in X$, $G(x)$ is nonempty by (ii); thus $cl_X(G(x))$ is nonempty and closed.

It follows that the map $clG : X \to 2^X$, defined by $(clG)(x) = cl_X(G(x))$ for all $x \in X$, satisfies all the hypotheses of Lemma 4.2.1. Hence by Lemma 4.2.1, $\cap_{x\in X}cl_X(G(x)) \neq \emptyset$.

Now by (i.1), we have $G(x) \subset F(x)$ so that $cl_X(G(x)) \subset cl_X(F(x))$ for all $x \in X$. Hence

$$\emptyset \neq \cap_{x\in X}cl_X(G(x)) \subset \cap_{x\in X}cl_X(F(x)) = \cap_{x\in X}F(x).$$

Take any $\hat{y} \in \cap_{x\in X}cl_X(G(x))$, then $\hat{y} \in K$ and $\hat{y} \in \cap_{x\in X}F(x)$. Hence $F(x,\hat{y}) \leq 0$, for all $x \in X$. ∎

We point out here that when X is compact and $f = g$, Theorem 4.2.2 was implicitly contained in the proof of Theorem 1 of Baye et al in [11]. Also it closely relates to the results in Tian [84]. As we noted in the introduction, the topological vector space E need not be Hausdorff. In addition, Theorem 4.2.2 improves Theorem 1 of Shih and Tan in [72] which in turn generalizes Ky Fan's minimax inequality in [30]

in several ways.

Theorem 4.2.3 *Let E be a topological vector space, X a nonempty convex subset of E and $f, g : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ satisfy the following conditions:*

(i) *for each $x, y \in X$ with $f(x, y) > 0$, we have $g(x, y) > 0$;*

(ii) *for each nonempty compact subset C of X and for each $x, y \in C$ with $f(x, y) > 0$, there exist $s \in C$ and an open neighborhood N of y in C such that $f(s, z) > 0$ for all $z \in N$;*

(iii) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A), \min_{x \in A} g(x, y) \leq 0$;*

(iv) *there exist a nonempty closed and compact subset K of X and $x_0 \in X$ such that $g(x_0, y) > 0$ for all $y \in X \backslash K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Proof.

(1) First we observe that if C is any nonempty compact subset of X,

$$\cap_{x \in C} \{y \in C : f(x, y) \leq 0\} = \cap_{x \in C} cl_C(\{y \in C : f(x, y) \leq 0\}).$$

Suppose not, then there is $y \in \cap_{x \in C} cl_C(\{y \in C : f(x, y) \leq 0\})$ such that $f(x_0, y) > 0$, for some $x_0 \in C$. By (ii), there exist $s \in C$ and an open neighborhood N of y in C such that $f(s, z) > 0$ for all $z \in N$. Thus $N \cap \{z \in C : f(s, z) \leq 0\} = \emptyset$ so that $y \notin cl_C(\{z \in C : f(s, z) \leq 0\})$ which is a contradiction.

(2) Define $F : X \to 2^K$ by

$$F(x) = \{y \in K : f(x, y) \leq 0\} \text{ for all } x \in X.$$

We shall prove that $\cap_{x \in X} cl_K F(x) \neq \emptyset$.

Let $\{x_1, x_2, \ldots, x_n\}$ be any finite subset of X. Set $C = co(\{x_0, x_1, \ldots, x_n\})$, then C is nonempty compact convex. Note that by (iii), $g(x, x) \leq 0$ for all $x \in X$. Define $G : C \to 2^C$ by $G(x) = \{y \in C : g(x, y) \leq 0\}$ for all $x \in C$. Note that

(a) if A is any finite subset of C, then $co(A) \subset \cup_{x \in A} G(x)$; for if this were false, then there would be a finite subset $\{z_1, \ldots, z_m\}$ of C and $z \in co(\{z_1, \ldots, z_m\})$ with $z \notin \cup_{j=1}^m G(z_j)$ so that $g(z_j, z) > 0$ for all $j = 1, \ldots, m$ which contradicts (iii);

(b) for each $x \in C$, $cl_C(G(x))$ is closed in C and is therefore also compact. By Lemma 4.2.1, $\cap_{x \in C} cl_C(G(x)) \neq \emptyset$.

Since for each $x \in C$, $G(x) \subset \{y \in C : f(x,y) \leq 0\}$, hence $\cap_{x \in C} cl_C G(x) \subset \cap_{x \in C} cl_C \{y \in C : f(x,y) \leq 0\} = \cap_{x \in C} \{y \in C : f(x,y) \leq 0\}$ by (1).

Take any $\bar{y} \in \cap_{x \in C} cl_C(G(x))$. Since $x_0 \in C$ and $G(x_0) \subset K$ by (iv), $\bar{y} \in cl_C(G(x_0)) \subset cl_X(G(x_0)) = cl_K(G(x_0)) \subset K$. Moreover, since $f(x_i, \bar{y}) \leq 0$ for each $i = 1, \ldots, n$, $\cap_{i=1}^{n} F(x_i) \neq \emptyset$. Thus $\{F(x) : x \in X\}$ has the finite intersection property. By the compactness of K, $\cap_{x \in X} cl_K(F(x)) \neq \emptyset$.

(3) We next shall show that

$$\cap_{x \in X} cl_K(F(x)) = \cap_{x \in X} F(x).$$

Suppose not, then there is $y_0 \in \cap_{x \in X} cl_K(F(x))$ with $y_0 \notin \cap_{x \in X} F(x)$ so that $y_0 \notin F(\bar{x}_0)$ for some $\bar{x}_0 \in X$. Let $K_0 = K \cup \{\bar{x}_0\}$, then K_0 is compact, $\bar{x}_0, y_0 \in K_0$ and $f(\bar{x}_0, y_0) > 0$. By (ii), there exist $s_0 \in K_0$ and an open neighborhood N_0 of y_0 in K_0 such that $f(s_0, z) > 0$ for all $z \in N_0$. Thus $N_0 \cap F(s_0) = \emptyset$. Note that $\bar{x}_0 \neq y_0$; it follows that $N_0' = N_0 \setminus \{\bar{x}_0\}$ is an open neighborhood of y_0 in K such that $N_0' \cap F(s_0) = \emptyset$. Thus $y_0 \notin cl_K(F(s_0))$ which is impossible. Hence $\cap_{x \in X} F(x) = \cap_{x \in X} cl_K(F(x)) \neq \emptyset$.

By (2) and (3), $\cap_{x \in X} F(x) \neq \emptyset$. Choose any $\hat{y} \in \cap_{x \in X} F(x)$, then $f(x, \hat{y}) \leq 0$ for all $x \in X$. ∎

If for each fixed $x \in X$, $y \mapsto f(x,y)$ is lower semicontinuous on each nonempty compact subset of X, condition (ii) in Theorem 4.2.3 is satisfied. Hence Theorem 4.2.3 improves Theorem 2.2 in [77].

The following result is a slightly modified Lemma 2 in [51] (page 229):

Lemma 4.2.4 *(Minty's lemma) Let E be a topological vector space and X be a nonempty convex subset of E. Suppose $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:*

(i) *for each $y \in X$, $x \longmapsto f(x,y)$ is concave;*

(ii) *for any* $x, y \in X$, *$f(x, y) \leq 0$ implies $f(y, x) \geq 0$ and for each $x \in X$,*
$f(x, x) \leq 0$;

(iii) *for each $x \in X$, $y \longmapsto f(x, y)$ is lower semicontinuous along line segments in*
X.

Define $F, F^* : X \to 2^X$ by

$$F(x) = \{y \in X : f(x, y) \leq 0\} \text{ for each } x \in X;$$

$$F^*(x) = \{y \in X : f(y, x) \geq 0\} \text{ for each } x \in X.$$

Then $\cap_{x \in X} F(x) = \cap_{x \in X} F^*(x)$.

Proof.

We first note that $F(x) \subset F^*(x)$ for all $x \in X$ by (ii). Therefore $\cap_{x \in X} F(x) \subset \cap_{x \in X} F^*(x)$.

Conversely, let $y \in \cap_{x \in X} F^*(x)$, then $y \in F^*(x)$ for all $x \in X$. Fix any $\bar{x} \in X$. For each $t \in (0, 1)$, define $w(t) = t\bar{x} + (1 - t)y$, then $0 \leq f(y, t\bar{x} + (1 - t)y)$ since X is convex and $y \in F^*(w(t))$. For each $x \in X$, $f(x, x) \leq 0$ by (ii) and $x \longmapsto f(x, y)$ is concave by (i), so we have for each $t \in (0, 1)$,

$$
\begin{aligned}
0 &\leq f(y, t\bar{x} + (1 - t)y) - f(t\bar{x} + (1 - t)y, t\bar{x} + (1 - t)y)/(1 - t) \\
&\leq f(y, t\bar{x} + (1 - t)y) - (tf(\bar{x}, t\bar{x} + (1 - t)y) + (1 - t)f(y, t\bar{x} + (1 - t)y))/(1 - t) \\
&= -tf(\bar{x}, t\bar{x} + (1 - t)y)/(1 - t);
\end{aligned}
$$

it follows that $f(\bar{x}, t\bar{x} + (1 - t)y) \leq 0$ for all $t \in (0, 1)$; by taking $t \to 0^+$, (iii) implies $f(\bar{x}, y) \leq 0$. Hence $y \in F(\bar{x})$. Since $\bar{x}$ is arbitrary in X, $y \in \cap_{x \in X} F(x)$. Therefore we also have $\cap_{x \in X} F^*(x) \subset \cap_{x \in X} F(x)$. ∎

Theorem 4.2.5 *Let E be a topological vector space, X a nonempty convex subset of*
E and $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ satisfy the following conditions:

(i) *for each fixed $y \in X$, $x \longmapsto f(x, y)$ is concave;*

(ii) *for any $x, y \in X$, $f(x, y) \leq 0$ implies $f(y, x) \geq 0$ and for each $x \in X$,*
$f(x, x) \leq 0$;

(iii) *for each* $x \in X, y \longmapsto f(x,y)$ *is lower semicontinuous along line segments in* X;

(iv) *for each* $x, y \in X$ *with* $f(y,x) < 0$, *there exist some point* $s \in X$ *and some open neighborhood* $N_X(y)$ *of* y *in* X *such that* $f(z,s) < 0$ *for all* $z \in N_X(y)$;

(v) *there exist a nonempty closed and compact subset* K *of* X *and* $x_0 \in X$ *such that* $f(x_0, y) > 0$ *for all* $y \in X \backslash K$.

Then there exists $\hat{y} \in K$ *such that* $f(x, \hat{y}) \leq 0$ *for all* $x \in X$.

Proof. Define $F, F^* : X \to 2^X$ by

$$F(x) = \{y \in X : f(x,y) \leq 0\},$$

$$F^*(x) = \{y \in X : f(y,x) \geq 0\}$$

for each $x \in X$. We have:

(1) $\cap_{x \in X} cl_X(F(x)) \neq \emptyset$. Indeed, we observe that:

(a) By (ii), for each $x \in X$, $x \in F(x)$, so that $F(x)$ is nonempty. Hence $cl_X(F(x))$ is nonempty and closed.

(b) For any $A \in \mathcal{F}(X)$, we show that $co(A) \subset \cup_{x \in A} F(x) \subset \cup_{x \in A} cl_X(F(x))$. Suppose not, then there is a finite subset $\{x_1, \dots, x_n\}$ of X such that $co\{x_1, \dots, x_n\} \not\subset \cup_{i=1}^n F(x_i)$. Let $x = \sum_{i=1}^n \lambda_i x_i \notin F(x_i)$ for each $i = 1, \dots, n$, where $\lambda_i \geq 0$ for each $i = 1, \dots, n$ with $\sum_{i=1}^n \lambda_i = 1$. Then for each i, $f(x_i, x) > 0$. By (i),

$$f(x,x) = f(\sum_{i=1}^n \lambda_i x_i, x) \geq \sum_{i=1}^n \lambda_i f(x_i, x) > 0$$

which contradicts (ii).

(c) By (v), we have $F(x_0) \subset K$ so that $cl_X(F(x_0))$ is compact.

By Lemma 4.2.1, $\cap_{x \in X} cl_X(F(x)) \neq \emptyset$.

(2) By Lemma 4.2.4, $\cap_{x \in X} F(x) = \cap_{x \in X} F^*(x)$.

(3) By (iv), we have $\cap_{x \in X} cl_X(F^*(x)) = \cap_{x \in X}(F^*(x))$. Hence

$$\cap_{x \in X} F(x) \subset \cap_{x \in X} cl_X(F(x)) \subset \cap_{x \in X} cl_X(F^*(x)) = \cap_{x \in X}(F^*(x)).$$

By (1) and (2), $\cap_{x\in X} F(x) = \cap_{x\in X} cl_X(F(x)) \neq \emptyset$. Take any $\hat{y} \in \cap_{x\in X} F(x)$, then $\hat{y} \in K$ and $f(x,\hat{y}) \leq 0$ for all $x \in X$. ∎

Another Fan's type minimax inequality is the following one which generalizes Theorem 1 in [23].

Theorem 4.2.6 *Let E be a topological vector space, X be a nonempty convex subset of E and $f \to \mathbb{R} \cup \{+\infty, -\infty\}$ be such that*

(i) *for each nonempty compact subset C of X and for each $x, y \in C$ with $f(x,y) > 0$, there exist $s \in C$ and an open neighborhood $N_C(y)$ of y in C such that $f(s,z) > 0$ for all $z \in N_C(y)$;*

(ii) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x\in A} f(x,y) \leq 0$;*

(iii) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X\backslash K$, there is $x \in co(X_0 \cup \{y\})$ with $f(x,y) > 0$.*

Then there is $\hat{y} \in K$ such that $f(x,\hat{y}) \leq 0$ for all $x \in X$.

Proof.

(1) First we observe that if C is any nonempty compact subset of X,

$$\cap_{x\in C}\{y \in C : f(x,y) \leq 0\} = \cap_{x\in C} cl_C(\{y \in C : f(x,y) \leq 0\}).$$

Suppose not, there is $y \in \cap_{x\in C} cl_C(\{y \in C : f(x,y) \leq 0\})$ so that for some $x_0 \in C, f(x_0, y) > 0$. By (i), there exist $s \in C$ and an open neighborhood $N_C(y)$ of y in C such that $f(s,z) > 0$ for all $z \in N_C(y)$. Thus $N_C(y) \cap \{z \in C : f(s,z) \leq 0\} = \emptyset$ so that $y \notin cl_C(\{z \in C : f(s,z) \leq 0\})$ which is a contradiction.

(2) Define $F : X \to 2^K$ by

$$F(x) = \{y \in K : f(x,y) \leq 0\} \text{ for all } x \in X.$$

We shall now show that the family $\{cl_K(F(x)) : x \in X\}$ has the finite intersection property. Indeed, let $\{x_1, \ldots, x_n\}$ be any finite subset of X and let $C = co(X_0 \cup \{x_1, \ldots, x_n\})$, then C is nonempty and compact. Note that $f(x,x) \leq 0$ for all $x \in X$

by (ii). Define $G : C \to 2^C$ by $G(x) = \{y \in C : f(x,y) \leq 0\}$ for each $x \in C$. We observe that:

(a) If A is any finite subset of C, then $co(A) \subset \cup_{x \in A} G(x)$; for if this were false, then there would exist a finite subset $\{z_1, \ldots, z_m\}$ of C with $co(\{z_1, \ldots, z_n\}) \not\subset \cup_{j=1}^m G(z_j)$. Choose any $z \in co(\{z_1, \ldots, z_m\})$ with $z \notin \cup_{j=1}^m G(z_j)$ so that $f(z_j, z) > 0$ for all $j = 1, \ldots, m$ and hence $\min_{1 \leq j \leq m} f(z_j, z) > 0$ which contradicts (ii).

(b) For each $x \in C$, $cl_C(G(x))$ is closed in C and is therefore compact.

By Lemma 4.2.1, we must have $\cap_{x \in C} cl_C G(x) \neq \emptyset$. By (1), $\cap_{x \in C} G(x) = \cap_{x \in C} cl_C G(x)$. Take any $\bar{y} \in \cap_{x \in C} G(x)$, then $\bar{y} \in C$ and $f(x, \bar{y}) \leq 0$ for all $x \in C$. In particular, $f(x_j, \bar{y}) \leq 0$ for all $i = 1, \ldots, n$. If $\bar{y} \in X \backslash K$, then by (iii), there is $x \in co(X_0 \cup \{y_0\}) \subset C$ with $f(x, \bar{y}) > 0$ so that $\bar{y} \notin G(x)$ which is a contradiction. Thus $\bar{y} \in K$. It follows that $\bar{y} \in \cap_{j=1}^n F(x_j)$ and hence $\{F(x) : x \in X\}$ has the finite intersection property.

By the compactness of K, $\cap_{x \in X} cl_K(F(x)) \neq \emptyset$.

(3) Next we shall show that

$$\cap_{x \in X} cl_K(F(x)) = \cap_{x \in X} F(x).$$

Suppose not, then there is $y_0 \in \cap_{x \in X} cl_K(F(x))$ with $y_0 \notin \cap_{x \in X} F(x)$ so that $y_0 \notin F(x_0)$ for some $x_0 \in X$. Let $K_0 = K \cup \{x_0\}$, then K_0 is compact, $x_0, y_0 \in K_0$ and $f(x_0, y) > 0$. By (i), there exist $s_0 \in K_0$ and an open neighborhood N_0 of y_0 in K_0 such that $f(s_0, z) > 0$ for all $z \in N_0$. Thus $N_0 \cap F(s_0) = \emptyset$, so that $y_0 \notin cl_K(F(s_0))$ which is impossible. Hence $\cap_{x \in X} F(x) = \cap_{x \in X} cl_K(F(x))$.

By (2) and (3), $\cap_{x \in X} F(x) \neq \emptyset$ Choose any $\hat{y} \in \cap_{x \in X} F(x)$, then $\hat{y} \in K$ and $f(x, \hat{y}) \leq 0$ for all $x \in X$. ∎

4.3   Fixed Point Theorems and Equilibria of One-agent Economy

We shall first prove that Theorem 4.2.2 implies the following fixed point theorem:

Theorem 4.3.1 *Let E be a topological vector space, X be a nonempty convex subset of E and $F, G : X \to 2^X$ be such that*

(i) *for each $x \in X, F(x) \subset G(x)$;*

(ii) *for each $y \in X, F(y) \neq \emptyset$ implies that there exists $s \in X$ such that $y \in int_X(F^{-1}(s))$;*

(iii) *there exists a nonempty closed and compact subset K of X such that $\cap_{y \in X \setminus K} G(y) \neq \emptyset$;*

(iv) *for each $y \in K, F(y) \neq \emptyset$;*

(v) *for each $x \in X, G(x)$ is convex.*

Then there exists $\bar{y} \in X$ such that $\bar{y} \in G(\bar{y})$.

Proof.

Define $f, g : X \times X \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1, & \text{if } x \in F(y), \\ 0, & \text{if } x \notin F(y), \end{cases}$$

$$g(x, y) = \begin{cases} 1, & \text{if } x \in G(y), \\ 0, & \text{if } x \notin G(y) \end{cases}$$

for all $x, y \in X$. It is obvious that condition (i.1) of Theorem 4.2.2 is satisfied. If there are $x, y \in X$ such that $f(x, y) > 0$, then $F(y) \neq \emptyset$. By (ii), there exist $s \in X$ and an open neighborhood $N_X(y)$ of y in X such that for each $z \in N_X(y), s \in F(z)$; that is, $f(s, z) > 0$ for all $z \in N_X(y)$. Thus condition (i.2) of Theorem 4.2.2 is also satisfied. By (iii), take any $x_0 \in \cap_{y \in X \setminus K} G(y)$, we have $g(x_0, y) = 1 > 0$ for all $y \in X \setminus K$. Therefore condition (iii) of Theorem 4.2.2 is also satisfied.

If hypothesis (ii) of Theorem 4.2.2 were also satisfied, then by Theorem 4.2.2, there would exist $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$. It follows that $F(\hat{y}) = \emptyset$ which contradicts (iv). Therefore condition (ii) of Theorem 4.2.2 does not hold. Hence there exist $A \in \mathcal{F}(X)$ and $\bar{y} \in co(A)$ such that $\min_{x \in A} g(x, \bar{y}) > 0$; i.e., $x \in G(\bar{y})$ for all $x \in A$. Therefore $\bar{y} \in co(A) \subset G(\bar{y})$ by (v). ∎

Theorem 4.3.1 implies the following result which is Theorem 1 in [35] due to Granas *et al.*

Theorem 4.3.2 *Let X be a nonempty compact convex subset of a topological vector space E and $A : X \to 2^X$ a mapping with nonempty values satisfying the following conditions:*

(i) $A^{-1}(y)$ *is convex for every $y \in X$;*

(ii) *there exists a selection $B : X \to 2^X$ of A with nonempty values ,i.e. $B(x) \subset A(x)$ for every $x \in X$ such that*

(ii.1) $B^{-1}(y) \neq \emptyset$ *for every $y \in X$;*

(ii.2) $B(x)$ *is open for every $x \in X$.*

Then there exists $w \in X$ such that $w \in A(w)$.

Proof.

For each $x \in X$, let $F(x) = B^{-1}(x)$, $G(x) = A^{-1}(x)$ and $K = X$. By (a), for each $x \in X$, $F(x) \neq \emptyset$. Let $s \in F(x) = B^{-1}(x)$, then $x \in B(s)$. By (b), $N = B(s)$ is an open neighborhood of x in X contained in $F^{-1}(s)$ so that $x \in int_X(F^{-1}(s))$. Thus condition (ii) of Theorem 4.3.2 is satisfied. All other conditions of Theorem 4.3.2 are easily verified, hence there exists $w \in X$ such that $w \in G(w)$, i.e., $w \in A(w)$. ∎

Now we shall show that Theorem 4.3.1 implies the following existence theorem of a maximal element:

Theorem 4.3.3 *Let E be a topological vector space, X a nonempty convex subset of E and $M, N : X \to 2^X$ such that*

(i) *for each $x \in X$, $M(x) \subset N(x)$;*

(ii) *for $x \in X, M(x) \neq \emptyset$ implies there exists some $s \in X$ such that $y \in int_X(M^{-1}(s))$;*

(iii) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, there is an $x \in A$ such that $x \notin N(y)$;*

(iv) *there exists a nonempty closed and compact subset K of X such that $\cap_{x \in X \backslash K} N(x) \neq \emptyset$.*

Then there exists $\hat{y} \in K, M(\hat{y}) = \emptyset$.

Proof. Suppose on the contrary that $M(x) \neq \emptyset$ for all $x \in K$. Let $F, G : X \to 2^X$ be defined by $F(x) = M(x)$ and $G(x) = co(N(x))$ for each $x \in X$. Then by Theorem 4.3.1, there is $y \in X$ such that $y \in G(y) = co(N(y))$. Now let $y = \sum_{i=1}^{n} \lambda_i x_i$ where $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i > 0$, and $x_i \in N(y)$ for each $i = 1, \ldots, n$. Let $A = \{x_i : i = 1, \ldots, n\}$, then $y \in co(A)$ and for each $x \in A, x \in N(y)$, which contradicts (iii). ∎

The following corollary is Theorem 4 of Metha and Tarafdar in [55]:

Corollary 4.3.4 *Let K be a nonempty compact convex subset of a topological vector space E and $T : K \to 2^K$ such that*

(i) $T(x)$ *is convex for $x \in X$;*

(ii) $x \notin T(x)$;

(iii) *if $T(x) \neq \emptyset$, there exists $y \in K$ such that $x \in int_X(T^{-1}(y))$.*

Then there exists $\bar{x} \in K$, such that $T(\bar{x}) = \emptyset$.

Proof. Let $M = N = T$ in Theorem 4.3.3, we only need to prove that for any $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, there is $x \in A$ such that $x \notin T(y)$. Suppose not, there exist $A \in \mathcal{F}(K)$ and $y \in co(A)$ such that for all $x \in A, x \in T(y)$. By (i), we have $y \in co(A) \subset T(y)$ which contradicts (ii). ∎

In Theorem 4.3.2 and Corollary 4.3.4, we note that E is not required to be Hausdorff which was assumed in [35] and [55], respectively.

Remark. We see that Theorem 4.2.2 $\Rightarrow$ Theorem 4.3.1 $\Rightarrow$ Theorem 4.3.3. We shall now show that Theorem 4.3.3 $\Rightarrow$ Theorem 4.2.2.

Let

$$M(y) = \{x \in X : f(x, y) > 0\},$$

$$N(y) = \{x \in X : g(x, y) > 0\}$$

for each $y \in X$. Then the conditions of Theorem 4.3.3 can be easily verified. By Theorem 4.3.3, there exists $\hat{y} \in K$ such that $M(\hat{y}) = \emptyset$; i.e., $f(x, \hat{y}) \leq 0$ for all $x \in X$.

73

Thus Theorem 4.2.2, Theorem 4.3.1 and Theorem 4.3.3 are all equivalent.

Other fixed point versions equivalent to Theorem 4.3.1 can be similarly stated and verified as those in [23] and [77]. Similarly we can obtain fixed point theorems and existence theorem of maximal elements which are equivalent to Theorem 4.2.6. However, we shall only state the following two equivalent versions of Theorem 4.2.6.

Theorem 4.3.5 *Let E be a topological vector space, X a nonempty convex subset of E and $G : X \to 2^X$ a nonepmty valued correpondence such that*

(i) *for each nonempty compact subset C of X and each $y \in C$ with $G(y) \cap C \neq \emptyset$, there is $s \in C$ such that $y \in int_C(G^{-1}(s) \cap C)$;*

(ii) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$ there is $co(X_0 \cup \{y\}) \cap G(y) \neq \emptyset$.*

Then there exists $\hat{y} \in X$ such that $\hat{y} \in co(G(\hat{y}))$.

Theorem 4.3.6 *Let X be a nonempty convex subset of a topological vector space and $G : X \to 2^X$. Suppose that*

(i) *for each nonempty compact subset C of X and each $y \in C$ with $C \cap G(y) \neq \emptyset$, there is $s \in C$ such that $y \in int_C(G^{-1}(s) \cap C)$;*

(ii) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, there is $x \in A$ such that $x \notin G(y)$;*

(iii) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$, $co(X_0 \cup \{y\}) \cap G(y) \neq \emptyset$.*

Then there exists $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$.

Proof of Theorem 4.2.6 $\Rightarrow$ Theorem 4.3.5:

Let $f : X \times X \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x \in G(y), \\ 0, & \text{if } x \notin G(y) \end{cases}$$

for all $x, y \in X$. Since for each $x \in X$, $G(x) \neq \emptyset$, by Theorem 4.2.6 there must be some $A \in \mathcal{F}(X)$ and $y \in co(A)$ such that $\min_{x \in A} f(x,y) > 0$ i.e. for each $x \in A$, $x \in G(y)$. Hence $y \in co(A) \subset co(G(y))$. ∎

Proof of Theorem 4.3.5 ⇒ Theorem 4.3.6:

Suppose that for each $x \in X$, $G(x) \neq \emptyset$, then all conditions of Theorem 4.3.5 are satisfied. By Theorem 4.3.5, there exists $y \in X$ such that $y \in co(G(y))$. Now let $y = \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i > 0$, $x_i \in G(y)$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \lambda_i = 1$. Let $A = \{x_i : i = 1, \ldots, n\}$, then $y \in co(A)$ and for each $x \in A, x \in G(y)$, which contradicts (ii). So there must be $\hat{y} \in X$ such that $G(\hat{y}) = \emptyset$. By (iii), $\hat{y}$ must be in K. ■

Proof of Theorem 4.3.6 ⇒ Theorem 4.2.6:

Define $G : X \to 2^X$ by $G(y) = \{x \in X : f(x,y) > 0\}$ for each $y \in X$. It is easy to verify that all conditions of Theorem 4.3.6 are satisfied. So by Theorem 4.3.6, there is $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$; i.e., $f(x, \hat{y}) \leq 0$ for all $x \in X$. ■

A quadruple $(X; A, B; P)$ is a *one-agent abstract economy* or a *one-person generalized game* if X is a nonempty convex subset of a topological vector space, $A, B : X \to 2^X$ are constraint correspondences with nonempty values and $P : X \to 2^X$ is a preference correspondence. An *equilibrium point* for $(X; A, B; P)$ is a point $\hat{x} \in X$ such that $\hat{x} \in cl_X(B(\hat{x}))$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Theorem 4.3.7 *Let $(X; A, B; P)$ be a one-agent abstract economy such that*

(i) *for each $x \in X$, $x \notin co(P(x) \cap A(x))$ and $co(A(x)) \subset B(x)$;*

(ii) *for each nonempty compact subset C of X and each $x \in C$ such that $C \cap A(x) \neq \emptyset$, there is $s \in C$ such that $x \in int_C(A^{-1}(s) \cap C)$;*

(iii) *for each nonempty compact subset C and each $x \in C$ such that $C \cap A(x) \cap P(x) \neq \emptyset$, there is $s \in C$ with $x \in int_C(A^{-1}(s) \cap (P^{-1}(s) \cup M) \cap C)$, where $M = \{x \in X : x \notin cl_X(B(x))\}$;*

(iv) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$,*

$$co(X_0 \cup \{y\}) \cap A(y) \cap P(y) \neq \emptyset.$$

Then $(X; A, B; P)$ has an equilibrium point $\hat{x} \in K$.

Proof.

Suppose that for each $x \in X$, we have either $x \notin cl_X(B(x))$ or $A(x) \cap P(x) \neq \emptyset$. Define $G : X \to 2^X$ by

$$G(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \in cl_X(B(x)), \\ A(x), & \text{if } x \notin cl_X(B(x)). \end{cases}$$

We shall prove that for any compact subset C of X and for any $x \in C$ such that $C \cap G(x) \neq \emptyset$, there is $s \in C$ such that $x \in int_C(G^{-1}(s) \cap C)$. Indeed, suppose $C \cap G(x) \neq \emptyset$. If $x \notin cl_X(B(x))$, then $G(x) = A(x)$. Hence $A(x) \cap C = G(x) \cap C \neq \emptyset$. By (ii), there is $s \in C$ such that $x \in int_C(A^{-1}(s) \cap C)$. Since $x \notin cl_X(B(x))$, there is an open neighborhood $N(x)$ of x in X such that $N(x) \subset X \backslash cl_X(B(x))$. Take $N_1(x) = N(x) \cap int_C(A^{-1}(s) \cap C)$, then $N_1(x)$ is an open neighborhood of x in C such that for any $z \in N_1(x)$, $s \in A(z) = G(z)$. Therefore $x \in int_C(G^{-1}(s) \cap C)$. If $x \in cl_X(B(x))$, then $C \cap A(x) \cap P(x) \neq \emptyset$. By (iii), there is $s \in C$ such that $x \in int_C(A^{-1}(s) \cap (P^{-1}(s) \cup M) \cap C)$. Note that

$$\begin{aligned} G^{-1}(s) &= [(P^{-1}(s) \cap A^{-1}(s)) \cap (X \backslash M)] \cup [A^{-1}(s) \cap M] \\ &= A^{-1}(s) \cap [(P^{-1}(s) \cap (X \backslash M)) \cup M] \\ &= A^{-1}(s) \cap (P^{-1}(s) \cup M). \end{aligned}$$

Hence $x \in int_C(G^{-1}(s) \cap C)$. Thus all the conditions of Theorem 4.3.5 are satisfied. By Theorem 4.3.5, there exists $\hat{x} \in X$ such that $\hat{x} \in co(G(\hat{x}))$. If $\hat{x} \in cl_X(B(\hat{x}))$, then $\hat{x} \in co(A(\hat{x}) \cap P(\hat{x}))$ which contradicts (i). If $\hat{x} \notin cl_X(B(\hat{x}))$, then $\hat{x} \in co(A(\hat{x})) \subset B(\hat{x})$ which is impossible. Therefore there must exist $\hat{x} \in X$ such that $\hat{x} \in cl_X(B(\hat{x}))$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$, that is, $\hat{x}$ is an equilibrium point for $(X; A, B; P)$. By (iv), $\hat{x}$ is necessarily in K. ∎

We need the following simple result which is Proposition 2.6 of Michael in [56]:

Lemma 4.3.8 *Let X be a topological space and Y be a nonempty convex subset of a topological vector space. If $F : X \to 2^Y$ is lower semicontinuous, then $coF : X \to 2^Y$ defined by $coF(x) = co(F(x))$ for each $x \in X$ is lower semicontinuous.*

Now we give an existence theorem for approximate equilibria.

Theorem 4.3.9 *Let X be a nonempty convex subset of a topological vector space E. Let $A, B : X \to 2^X$ be correspondences with nonempty values and $P : X \to 2^X$. Suppose that*

(i) *A is lower semicontinuous such that for each $x \in X$, $x \notin co(A(x) \cap P(x))$ and $co(A(x)) \subset B(x)$;*

(ii) *for each nonempty compact subset C of X and each $x \in C$ such that $C \cap A(x) \cap P(x) \neq \emptyset$, there is $s \in C$ with $x \in int_C(A^{-1}(s) \cap P^{-1}(s) \cap C)$.*

(iii) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$,*

$$co(X_0 \cup \{y\}) \cap A(y) \cap P(y) \neq \emptyset.$$

Then for each open convex neighborhood V of zero in E, the one-agent abstract economy $(X; A, \overline{B_V}; P)$ has an equilibrium point $\hat{x} \in K$; that is, there exists a point $x_V \in K$ such that $x_V \in \overline{B_V}(x_V)$ and $A(x_V) \cap P(x_V) = \emptyset$, where $B_V(x) = (B(x) + V) \cap X$ for each $x \in X$.

Proof. Suppose the conclusion does not hold, then there exists an open convex neighborhood V of zero in E such that for each $x \in X$, we have either $x \notin cl_X(B_V(x))$ or $A(x) \cap P(x) \neq \emptyset$. Define $A_V : X \to 2^X$ by $A_V(x) = (co(A(x)) + V) \cap X$ for each $x \in X$. Then A_V has an open graph in $X \times X$ by Lemma 4.3.8 and by Lemma 4.1 of Chang in [14] such that for each $x \in X$, $A_V(x) \subset B_V(x)$. Let $M_V = \{x \in X : x \notin \overline{B_V}(x)\}$. Define $G_V : X \to 2^X$ by

$$G_V(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \notin M_V, \\ A_V(x), & \text{if } x \in M_V. \end{cases}$$

Suppose C is a nonempty compact subset of X and $x \in C$ such that $C \cap G_V(x) \neq \emptyset$.

Case 1. If $x \in M_V$, then $G_V(x) = A_V(x)$. Hence $A_V(x) \cap C = G_V(x) \cap C \neq \emptyset$. Take any $s \in A_V(x) \cap C$. Since A_V has an open graph, $A_V^{-1}(s) \cap C$ is open in C. Since $x \in M_V$, there is an open neighborhood $N(x)$ of x in X such that $N(x) \subset M_V$. Take

$N_1(x) = N(x) \cap A_V^{-1}(s) \cap C$, then $N_1(x)$ is an open neighborhood of x in C such that for any $z \in N_1(x), s \in A_V(z) = G_V(z)$. Therefore $x \in int_C(G^{-1}(s) \cap C)$.

Case 2. If $x \notin M_V$, then $A(x) \cap P(x) \cap C \neq \emptyset$. By (ii), there is $s \in C$ such that $x \in int_C(A^{-1}(s) \cap P^{-1}(s) \cap C)$. Note that

$$
\begin{aligned}
G_V^{-1}(s) &= [(P^{-1}(s) \cap A^{-1}(s)) \cap (X \backslash M_V)] \cup [A_V^{-1}(s) \cap M_V] \\
&\supset A^{-1}(s) \cap [(P^{-1}(s) \cap (X \backslash M_V)) \cup M_V] \\
&= A^{-1}(s) \cap (P^{-1}(s) \cup M_V) \\
&\supset A^{-1}(s) \cap P^{-1}(s).
\end{aligned}
$$

Hence $x \in int_C(G_V^{-1}(s) \cap C)$.

Finally, by (iii), there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \backslash K$, $co(X_0 \cup \{y\}) \cap G_V(y) \neq \emptyset$.

Thus all the conditions of Theorem 4.3.5 are satisfied. By Theorem 4.3.5, there exists $\hat{x} \in X$ such that $\hat{x} \in co(G_V(\hat{x}))$. If $\hat{x} \in \overline{B_V}(\hat{x})$, then $\hat{x} \in co(A(\hat{x}) \cap P(\hat{x}))$ which contradicts (i). If $\hat{x} \notin \overline{B_V}(\hat{x})$, then $\hat{x} \in co(A_V(\hat{x})) = co(A(x)) + V \subset B_V(\hat{x})$ which is impossible. Therefore for any open convex neighborhood V of zero in E, there must exist $x_V \in X$ such that $x_V \in \overline{B_V}(x_V)$ and $A(x_V) \cap P(x_V) = \emptyset$; that is, x_V is an equilibrium point for $(X; A, \overline{B_V}; P)$. By (iii), x_V is necessarily in K. ∎

The following result is Lemma 5.3 in Tan and Yuan [77]:

Lemma 4.3.10 *Let X be a topological space, Y a nonempty subset of a topological vector space E, $\mathcal{B}$ a fundamental system of open neighborhoods of zero in E and $B : X \to 2^Y$ a nonempty valued correspondence . For each $V \in \mathcal{B}$, let $B_V : X \to 2^Y$ be defined by $B_V(x) = (B(x) + V) \cap Y$ for each $x \in X$. If $\hat{x} \in X$ and $\hat{y} \in Y$ satisfy $\hat{y} \in \cap_{V \in \mathcal{B}} \overline{B_V}(\hat{x})$, then $\hat{y} \in \overline{B}(\hat{x})$.*

Theorem 4.3.11 *In Theorem 4.3.9, if, in addition, E is a locally convex topological vector space and the set $\{x \in X : A(x) \cap P(x) \neq \emptyset\}$ is open in X, then the one-agent abstract economy $(X; A, \overline{B}; P)$ has an equilibrium point in K.*

Proof. Let $\mathcal{B}$ be a fundamental system of open convex neighborhoods of zero in E. For each $V \in \mathcal{B}$, by Theorem 4.3.9 there exists $\hat{x}_V \in K$ such that $\hat{x}_V \in \overline{B_V}(\hat{x}_V)$ and $A(\hat{x}_V) \cap P(\hat{x}_V) = \emptyset$. It follows that the set $Q_V := \{x \in K : x \in \overline{B_V}(x) \text{ and } A(x) \cap P(x) = \emptyset\}$ is a nonempty closed subset of K.

Now we shall prove $\{Q_V\}_{V \in \mathcal{B}}$ has the finite intersection property. Let $\{V_1, \ldots, V_n\}$ be any finite subset of $\mathcal{B}$. Let $V = \cap_{i=1}^n V_i$. Then V is also an open convex neighborhood of zero in E such that $\cap_{i=1}^n Q_{V_i} \supset Q_V \neq \emptyset$. Therefore, the family $\{Q_V : V \in \mathcal{B}\}$ has the finite intersection property. Since K is compact, $\cap_{V \in \mathcal{B}} Q_V \neq \emptyset$. Now take any $\hat{x} \in \cap_{V \in \mathcal{B}} Q_V$. We have $\hat{x} \in \overline{B_V}(\hat{x})$ for each $V \in \mathcal{B}$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. By Lemma 4.3.10, $\hat{x} \in \overline{B}(\hat{x})$. Thus $\hat{x} \in K$ is an equilibrium point of $(X; A, \overline{B}; P)$. ∎

We point out that our results in this section are closely related to those in [23] and [77] and the references therein. Further applications to abstract economies with more than one agent (finite or infinite agents) can be done without difficulty.

4.4 Minimax Inequalities of the von Neumann Type

In this section, we shall develop some von Neumann type minimax inequalities. We begin with the following theorem:

Theorem 4.4.1 *Let X and Y be nonempty convex sets, each in a topological vector space and let f, u, v, g be four real-valued functions on $X \times Y$ and $\rho \in \mathbb{R}$. Assume*

(i) *for each $x, y \in X$ with $f(x, y) > \rho$, we have*

(i.1) $u(x, y) > \rho$;

(i.2) *there exist some point $s \in X$ and some open neighborhood $N_Y(y)$ of y in Y such that $f(s, z) > \rho$ for all $z \in N_X(y)$;*

(ii) *for each $x, y \in X$ with $g(x, y) < \rho$, we have*

(ii.1) $v(x, y) < \rho$;

(ii.2) *there exist some point $c \in Y$ and some open neighborhood $N_X(x)$ of x in X such that $g(z,c) < \rho$ for all $z \in N_X(x)$;*

(iii) *there exist a nonempty compact convex subset K of $X \times Y$ and $(\bar{x}_0, \bar{y}_0) \in X \times Y$ such that $u(\bar{x}_0, y) > \rho$ and $v(x, \bar{y}_0) < \rho$ for all $(x,y) \in X \times Y \backslash K$.*

(iv) *for each $A \in \mathcal{F}(X \times Y)$ and for each $(x,y) \in co(A)$, there is $(\bar{x}, \bar{y}) \in A$ such that either $u(\bar{x}, y) \leq \rho$ or $v(x, \bar{y}) \geq \rho$.*

Then there exists a point $(x_0, y_0) \in K$ such that $f(x, y_0) \leq \rho$ for all $x \in X$ or $g(x_0, y) \geq \rho$ for all $y \in Y$.

Proof.

Define $F, G : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$F((\bar{x}, \bar{y}), (x,y)) = \begin{cases} 1, & \text{if } f(\bar{x}, y) > \rho \text{ and } g(x, \bar{y}) < \rho, \\ 0, & \text{otherwise;} \end{cases}$$

$$G((\bar{x}, \bar{y}), (x,y)) = \begin{cases} 1, & \text{if } u(\bar{x}, y) > \rho \text{ and } v(x, \bar{y}) < \rho, \\ 0, & \text{otherwise} \end{cases}$$

for each $((\bar{x}, \bar{y}), (x,y)) \in (X \times Y) \times (X \times Y)$.

(1) Note that by (i.1) and (ii.1), for each $((\bar{x}, \bar{y}), (x,y)) \in (X \times Y) \times (X \times Y)$ with $F((\bar{x}, \bar{y}), (x,y)) > 0$, we have $G((\bar{x}, \bar{y}), (x,y)) > 0$.

(2) If $F((\bar{x}, \bar{y}), (x,y)) > 0$, then by (i.2), there exist some $s \in X$ and an open neighborhood $N_Y(y)$ of y in Y such that $f(s, z) > \rho$ for all $z \in N_Y(y)$; by (ii.2), there exist some $c \in Y$ and some open neighborhood $N_X(x)$ of x in X such that $g(t, c) < \rho$ for all $t \in N_X(x)$. Then we have $F((s, c), (t, z)) > 0$ for all $(t, z) \in N_X(x) \times N_Y(y)$.

(3) By (iii), we have $G((\bar{x}_0, \bar{y}_0), (x,y)) > 0$ for all $(x,y) \in X \times Y \backslash K$.

Suppose that the assertion of the theorem were false. Then for each point $(\bar{x}, \bar{y}) \in K$, there exists $(x,y) \in X \times Y$ such that $f(x, \bar{y}) > \rho$ and $g(\bar{x}, y) < \rho$ so that $F((x,y), (\bar{x}, \bar{y})) > 0$. Hence by Theorem 4.2.2, there must be some $A \in \mathcal{F}(X \times Y)$ and some $(x,y) \in co(A)$ such that $\min_{(\bar{x}, \bar{y}) \in A} G((\bar{x}, \bar{y}), (x,y)) > 0$; but this would violate (iv). ■

Corollary 4.4.2 *Let X and Y be nonempty compact convex sets, each in a topological vector space and let f, u, v, g be four real-valued functions on $X \times Y$. Assume*

(i) *for each $x, y \in X$ and $\lambda \in \mathbb{R}$ with $f(x, y) > \lambda$, we have*

(i.1) *$u(x, y) > \lambda$;*

(i.2) *there exist some point $s \in X$ and some open neighborhood $N_Y(y)$ of y in Y such that $f(s, z) > \lambda$ for all $z \in N_Y(y)$;*

(ii) *for each $x, y \in X$ and $\lambda \in \mathbb{R}$ with $g(x, y) < \lambda$, we have*

(ii.1) *$v(x, y) < \lambda$;*

(ii.2) *there exist some point $c \in Y$ and some open neighborhood $N_X(x)$ of x in X such that $g(t, c) < \lambda$ for all $t \in N_X(x)$;*

(iii) *$u \leq v$ on $X \times Y$;*

(iv) *for each fixed $y \in Y$, $x \longmapsto u(x, y)$ is quasi-concave on X and for each fixed $x \in X$, $y \longmapsto v(x, y)$ is quasi-convex on Y.*

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Proof. We shall prove by contradiction. Suppose that there is $\rho \in \mathbb{R}$ such that

$$\sup_{x \in X} \inf_{y \in Y} g(x, y) < \rho < \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Then clearly condition (iii) of Theorem 4.4.1 is satisfied for $K = X \times Y$. We shall show that condition (v) of Theorem 4.4.1 is also satisfied. Suppose not, there is $A = \{(x_i, y_i) : i = 1, \dots, n\} \in \mathcal{F}(X \times Y)$ and $(\bar{x}, \bar{y}) = \sum_{i=1}^n \lambda_i (x_i, y_i) \in co(A)$, where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$ for each $i = 1, \dots, n$ such that for each $i = 1, \dots, n$, $u(x_i, \bar{y}) > \rho$ and $v(\bar{x}, y_i) < \rho$. Then by (v) we have $v(\bar{x}, \bar{y}) < \rho < u(\bar{x}, \bar{y})$, which contradicts (iii).

Now by Theorem 4.4.1, there exists a point $(x_0, y_0) \in X \times Y$ such that either $f(x, y_0) \leq \rho$ for all $x \in X$ or $g(x_0, y) \geq \rho$ for all $y \in Y$. This is a contradiction. ∎

Suppose that X is a nonempty convex subset of a vector space. Recall that a real-valued function f on X is *quasi-convex* if for each real number λ, $\{x \in X : f(x) < \lambda\}$ is convex. f is *quasi-concave* if $-f$ is quasi-convex. By taking $f = u = v = g$ in Corollary 4.4.2, we have the following generalization of Theorem 3.1 of Sion in [74]:

Corollary 4.4.3 *Let X and Y be nonempty compact convex sets, each in a topological vector space and let f be a real-valued function on $X \times Y$. Assume*

(i) *for each $x, y \in X$ and $\lambda \in \mathbb{R}$ with $f(x, y) > \lambda$, there exist some point $s \in X$ and some open neighborhood $N_Y(y)$ of y in Y such that $f(s, z) > \lambda$ for all $z \in N_Y(y)$;*

(ii) *for each $x, y \in X$ and $\lambda \in \mathbb{R}$ with $f(x, y) < \lambda$, there exist some point $c \in Y$ and some open neighborhood $N_X(x)$ of x in X such that $f(t, c) < \lambda$ for all $t \in N_X(x)$;*

(iii) *for each fixed $y \in Y$, $x \longmapsto f(x, y)$ is quasi-concave on X and for each fixed $x \in X$, $y \longmapsto f(x, y)$ quasi-convex on Y.*

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

Corollary 4.4.4 *If the compactness of Y in Corollary 4.4.2 is dropped, the conclusion of Corollary 4.4.2 still holds.*

Proof. Suppose that there is $\lambda \in \mathbb{R}$ such that

$$\sup_{x \in X} \inf_{y \in Y} g(x, y) < \lambda < \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Then for each $x \in X$ there is $y \in Y$ with $g(x, y) < \lambda$. Thus by (ii.2) of Corollary 4.4.2, there exist some $c \in Y$ and some open neighborhood $N_X^c(x)$ of x in X such that $g(t, c) < \lambda$ for all $t \in N_X^c(x)$. Because X is compact, we can find a finite subset $\{c_1, \ldots, c_n\}$ of Y such that $\cup_{i=1}^n N_X^{c_i} = X$. Take $N = \{c_1, \ldots, c_n\}$, then for any $x \in X$ there is $y \in N$ with $g(t, y) < \lambda$, for all $t \in N_X^y(x)$ so that $\sup_{x \in X} \inf_{y \in N} g(x, y) \leq \lambda$.

Take $g' = g|_{X \times co(N)}$ and $f' = f|_{X \times co(N)}$. Note that for any $(x, y) \in X \times co(N)$ with $f(x, y) > \lambda$, there exist some $s \in X$ and an open neighborhood $N_Y(y)$ of y in Y such that $f(s, z) > \lambda$ for all $z \in N_Y(y)$; thus $f'(s, z) > \lambda$ for all $z \in N_{co(N)}(y) = N_Y(y) \cap co(N)$ which is an open neighborhood of y in $co(N)$. By Corollary 4.4.2, $\sup_{x \in X} \inf_{y \in co(N)} g'(x, y) \geq \inf_{y \in co(N)} \sup_{x \in X} f'(x, y)$.

But

$$\begin{aligned}
\sup_{x \in X} \inf_{y \in co(N)} g'(x, y) &\leq \sup_{x \in X} \inf_{y \in N} g(x, y) \leq \lambda \\
&< \inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in co(N)} \sup_{x \in X} f'(x, y).
\end{aligned}$$

This is a contradiction. ■

The following is a generalization of Theorem 1 of Tan and Yu [80].

Theorem 4.4.5 *Let X be a nonempty compact convex subset of a topological vector space E and Y a nonempty convex subset of a topological vector space F. Suppose that the real-valued function $f : X \times Y \to \mathbb{R}$ and the set-valued map $T : X \to 2^Y$ satisfy the following conditions:*

(i) *for each $(x, y) \in X \times Y$ and $\lambda \in \mathbb{R}$ with $f(x, y) > \lambda$, there exist some point $s \in X$ and some open neighborhood $N_Y(y)$ of y in Y such that $f(s, z) > \lambda$ for all $z \in N_Y(y)$;*

(ii) *for each $(x, y) \in X \times Y$ and $\lambda \in \mathbb{R}$ with $f(x, y) < \lambda$, there exist some point $c \in Y$ and some open neighborhood $N_X(x)$ of x in X such that $f(t, c) < \lambda$ for all $t \in N_X(x)$;*

(iii) *for each fixed $y \in Y$, $x \longmapsto f(x, y)$ is quasi-concave on X and for each fixed $x \in X$, $y \longmapsto f(x, y)$ quasi-convex on Y.*

(iv) *for each $x \in X, T(x)$ is compact convex and there is $y \in T(x)$ such that $f(x, y) \leq 0$.*

(v) *for each $x \in X$ with $\{u \in X : f(u, y) > 0 \text{ for all } y \in T(x)\} \neq \emptyset$, there is $\bar{x} \in X$ such that $x \in int_X\{v \in X : f(\bar{x}, y) > 0 \text{ for all } y \in T(v)\}$.*

Then there exists $(\hat{x}, \hat{y}) \in X \times Y$ with $\hat{y} \in T(\hat{x})$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

Proof.

Define $F : X \to 2^Y$ by

$$F(x) = \{u \in X : f(u, y) > 0 \text{ for all } y \in T(x)\}$$

for each $x \in X$. Fix an $x \in X$. Let $u_1, u_2 \in F(x)$ be given, then $f(u_1, y) > 0$ and $f(u_2, y) > 0$ for all $y \in T(x)$. By (iii), for each $\lambda \in [0, 1]$ and each $y \in T(x)$,

$$f(\lambda u_1 + (1 - \lambda)u_2, y) \geq \min\{f(u_1, y), f(u_2, y)\} > 0$$

so that $\lambda u_1 + (1 - \lambda)u_2 \in F(x)$. Thus $F(x)$ is convex for each $x \in X$.

By (iv), $x \notin F(x)$ for all $x \in X$. By (v), if $F(x) \neq \emptyset$, there is $\bar{x} \in X$ such that $x \in int_X F^{-1}(\bar{x})$.

By Corollary 4.3.4, there exists $\hat{x} \in X$ such that $F(\hat{x}) = \emptyset$. Thus, for each $u \in X$, there is $y_u \in T(\hat{x})$ such that $f(u, y_u) \leq 0$. It follows that $\inf_{y \in T(\hat{x})} f(u, y) \leq 0$ for all $u \in X$.

Since all the conditions of Corollary 4.4.3 hold, we have

$$\inf_{y \in T(\hat{x})} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in T(\hat{x})} f(x, y) \leq 0.$$

Fix an arbitrary $\varepsilon > 0$. Let $A_\varepsilon = \{y \in T(\hat{x}) : \sup_{x \in X} f(x, y) \leq \varepsilon\}$, then clearly A_ε is nonempty. We shall prove that A_ε is also closed in $T(\hat{x})$. Indeed, if $y_0 \in T(\hat{x})$ and $y_0 \notin A_\varepsilon$, we have $\sup_{x \in X} f(x, y_0) > \varepsilon$. Then by (i) , there exist $s \in X$ and an open neighborhood $N(y_0)$ of y_0 in Y such that $f(s, z) > \varepsilon$ for all $z \in N(y_0)$. Let $N_0(y_0) = N(y_0) \cap T(\hat{x})$, then $N_0(y_0)$ is an open neighborhood of y_0 in $T(\hat{x})$ such that $f(s, z) > \varepsilon$ for all $z \in N_0(y_0)$, i.e. $N_0(y_0) \cap A_\varepsilon = \emptyset$. Hence A_ε is closed in $T(\hat{x})$.

Since the family $\{A_\varepsilon : \varepsilon > 0\}$ has the finite intersection property, $\cap_{\varepsilon > 0} A_\varepsilon \neq \emptyset$ by the compactness of $T(\hat{x})$. Choose any $\hat{y} \in \cap_{\varepsilon > 0} A_\varepsilon$, then $\hat{y} \in T(\hat{x})$ and $\sup_{x \in X} f(x, \hat{y}) \leq \varepsilon$ for all $\varepsilon > 0$; i.e., $f(x, \hat{y}) \leq 0$ for all $x \in X$. ∎

Let X be a topological space such that $X = \cup_{n=1}^{\infty} C_n$ where C_n is an increasing sequence of nonempty compact sets. Then a sequence $\{x_n\}_{n=1}^{\infty}$ is said to be escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ if for each $n \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $y_k \notin C_n$ for all $k \geq M$. The concept of *escaping sequences* was introduced by Border in [13], page 34.

Theorem 4.4.6 *Let X be a nonempty subset of a topological vector space E such that $X = \cup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of X and Y be a nonempty convex subset of a topological vector space F. Suppose that the real-valued function $f : X \times Y \to \mathbb{R}$ and the set-valued map $T : X \to 2^Y$ satisfy the following conditions:*

(i) *for each $n \in \mathbb{N}$, each $(x, y) \in C_n \times Y$ and $\lambda \in \mathbb{R}$ with $f(x, y) > \lambda$, there exist some point $s \in C_n$ and some open neighborhood $N_Y(y)$ of y in Y such that $f(s, z) > \lambda$ for all $z \in N_Y(y)$;*

(ii) *for each $(x, y) \in X \times Y$ and $\lambda \in \mathbb{R}$ with $f(x, y) < \lambda$, there exist some point $c \in Y$ and some open neighborhood $N_X(x)$ of x in X such that $f(t, c) < \lambda$ for all $t \in N_X(x)$;*

(iii) *for each fixed $y \in Y$, $x \longmapsto f(x, y)$ is quasi-concave on X and for each fixed $x \in X$, $y \longmapsto f(x, y)$ quasi-convex on Y.*

(iv) *for each $x \in X$, $T(x)$ is compact convex and there is $y \in T(x)$ such that $f(x, y) \le 0$;*

(v) *for each $n \in \mathbb{N}$ and each $x \in C_n$ with $\{u \in C_n : f(u, y) > 0$ for all $y \in T(x)\} \neq \emptyset$, there is $\bar{x} \in C_n$ such that $x \in int_{C_n} \{v \in C_n : f(\bar{x}, y) > 0$ for all $y \in T(v)\}$;*

(vi) *for each sequence $\{x_n\}_{n=1}^{\infty}$ in X, where $x_n \in C_n$ for each $n = 1, 2, \ldots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, and each sequence $\{y_n\}_{n=1}^{\infty}$, where $y_n \in T(x_n)$ for each $n = 1, 2, \ldots$, there exist $n_0 \in \mathbb{N}$ and $x'_{n_0} \in C_{n_0}$ with $f(x'_{n_0}, y_{n_0}) > 0$.*

Then there exists $(x^*, y^*) \in X \times Y$ with $y^* \in T(x^*)$ such that $f(x, y^*) \le 0$ for all $x \in X$.

Proof.

For each $n \in \mathbb{N}$, by Theorem 4.4.5, there is $(x_n, y_n) \in C_n \times Y$ with $y_n \in T(x_n)$ such that $f(x, y_n) \le 0$ for all $x \in C_n$.

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ were escaping from X relative to $\{C_n\}_{n=1}^{\infty}$. By (vi), there exist $n_0 \in N$ and $x'_{n_0} \in C_{n_0}$ with $f(x'_{n_0}, y_{n_0}) > 0$ which is a contradiction. Therefore the sequence $\{x_n\}_{n=1}^{\infty}$ is not escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, so that some subsequence of $\{x_n\}_{n=1}^{\infty}$ must lie entirely in some $C_{n'_1}$. Since $C_{n'_1}$ is compact, there exist a subnet $\{z_\alpha\}_{\alpha \in \Gamma}$ of $\{x_n\}_{n=1}^{\infty}$ in $C_{n'_1}$ and $x^* \in C_{n'_1}$ such that $z_\alpha \to x^*$. Denote $z_\alpha = x_{n(\alpha)}$, where $n(\alpha) \to \infty$.

If $\{u \in X : f(u, y) > 0$ for all $y \in T(x^*)\} \neq \emptyset$, there exists $n'_2 \ge n'_1$ such that $\{u \in C_{n'_2} : f(u, y) > 0$ for all $y \in T(x^*)\} \neq \emptyset$. By (v), there is $\bar{x} \in C_{n_2}$ such that $x^* \in int_{C_{n'_2}} \{v \in C_{n'_2} : f(\bar{x}, y) > 0$ for all $y \in T(v)\}$. Since $z_\alpha \to x^*$, there is α_0 such that

$n(\alpha_0) \geq n'_2$ and $f(\bar{x}, y) > 0$ for all $y \in T(z_{\alpha_0})$. Since $y_{n(\alpha_0)} \in T(x_{n(\alpha_0)}) = T(z_{\alpha_0})$, we have $f(\bar{x}, y_{n(\alpha_0)}) > 0$ which contradicts the fact that $\bar{x} \in C_{n(\alpha_0)}$ and $f(\bar{x}, y_{n(\alpha_0)}) \leq 0$. Therefore $\{u \in X : f(u, y) > 0 \text{ for all } y \in T(x^*)\} = \emptyset$; i.e. $\inf_{y \in T(x^*)} f(u, y) \leq 0$ for all $u \in X$.

By Corollary 4.4.4, we have

$$\inf_{y \in T(x^*)} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in T(x^*)} f(x, y) \leq 0.$$

Similar to the last step in the proof of Theorem 4.4.5, we can prove that there exist $y^* \in T(x^*)$ such that $\sup_{x \in X} f(x, y^*) \leq 0$; i.e., $f(x, y^*) \leq 0$ for all $x \in X$. ∎

4.5 Applications to Variational Inequalities and Related Problems

Let E be a topological vector space, X a nonempty subset of E and $T : X \to 2^{E'}$. Then T is said to be (1) *almost monotone* if for each $x \in X$, $\sup_{u \in T(x)} Re\langle u, y - x \rangle \geq 0$ implies $\inf_{w \in T(y)} Re\langle w, y - x \rangle \geq 0$ and (2) *monotone* if for each $x, y \in X$, $\sup_{u \in T(x)} Re\langle u, y - x \rangle \leq \inf_{w \in T(y)} Re\langle w, y - x \rangle$. Clearly if T is monotone, then T is almost monotone. The converse is not true in general as the following example illustrates:

Example 4.5.1 *Let $A = [1, \infty)$. Define $T : A \to 2^{\mathbb{R}}$ by $T(x) = [1, x+1]$ for all $x \in A$. Suppose $x, y \in A$ such that $x < y$. Choose $u = x + 1 \in T(x)$ and $w = 1 \in T(y)$. Then $\langle u - w, x - y \rangle = x(x - y) < 0$, which shows that T is not monotone.*

On the other hand, let $x, y \in A$. Clearly, if $\sup_{w \in T(y)} \langle w, x - y \rangle \geq 0$, then $x \geq y$ so that $\inf_{u \in T(x)} \langle u, x - y \rangle \geq 0$. Hence T is almost monotone. ∎

Recall that if $y \in E$, the set

$$I_X(y) = \{x \in E : x = y + r(u - y) \text{ for some } u \in X \text{ and } r > 0\}$$

is called the *inward set* of y with respect to X [38].

As an application of Theorem 4.2.5, we have the following generalized variational inequality for almost monotone operators.

Theorem 4.5.2 *Let E be a Hausdorff topological vector space and X a nonempty convex subset of E. Let $T : X \to 2^{E'}$ almost monotone and upper semicontinuous from line segments in X to the weak* topology of E' such that each $T(x)$ is weak* compact. Further suppose that there exist a nonempty weakly closed and weakly compact subset K of X and $x_0 \in X$ such that $\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle > 0$ for all $y \in X \backslash K$. Then there exists $\hat{y} \in K$ such that $\inf_{w \in T(\hat{y})} \langle w, \hat{y} - x \rangle \leq 0$ for all $x \in I_X(\hat{y})$. If, in addition, $T(\hat{y})$ is convex, then there exists a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in cl_E(I_X(\hat{y}))$.*

Proof. Define $f : X \times X \to \mathbb{R}$ by $f(x, y) = \inf_{w \in T(y)} Re\langle w, y - x \rangle$ for all $x, y \in X$. Then:

(i) For each fixed $y \in X$, $x \longmapsto f(x, y)$ is concave.

(ii) For any $x, y \in X$ with $f(x, y) \leq 0$, that is, $\inf_{w \in T(y)} Re\langle w, y - x \rangle \leq 0$, we have $\sup_{w \in T(y)} Re\langle w, x - y \rangle \geq 0$. Since T is almost monotone, $\inf_{u \in T(x)} Re\langle u, x - y \rangle \geq 0$, i.e., $f(y, x) \geq 0$.

(iii) Fix any $x \in X$ and $\lambda \in \mathbb{R}$, let $A_\lambda = \{z \in L : f(x, z) \leq \lambda\}$ where $L = \{tu + (1 - t)v : 0 \leq t \leq 1\}$ and $u, v \in X$. We shall show that A_λ is closed in L. Let $(z_{t_\alpha})_{\alpha \in \Gamma}$ be a net in A_λ converging to $z_{t_0} \in L$ where $z_t = tu + (1 - t)v, t \in [0, 1]$. Fix an arbitrary $\varepsilon > 0$, then for each $\alpha \in \Gamma$, there is $w_{t_\alpha} \in T(z_{t_\alpha})$ such that

$$\lambda + \epsilon > Re\langle w_{t_\alpha}, z_{t_\alpha} - x \rangle$$
$$= t_\alpha Re\langle w_{t_\alpha}, u - x \rangle + (1 - t_\alpha) Re\langle w_{t_\alpha}, v - x \rangle.$$

Since T is upper semicontinuous from line segments in X to the weak* topology on E' and each $T(z_t)$ is nonempty weak* compact, $\cup_{t \in [0,1]} T(z_t)$ is weak* compact in E'. Thus there is a subnet $(w_{t_{\alpha'}})_{\alpha' \in \Gamma'}$ of $(w_{t_\alpha})_{\alpha \in \Gamma}$ such that $w_{t_{\alpha'}}$ converges to $w_0 \in E'$ in weak* topology. Since for each $z \in L$, $T(z)$ is weak* closed and T is upper semicontinuous, $w_0 \in T(z_{t_0})$. Thus we have

$$\lambda + \varepsilon \geq t_0 Re\langle w_0, u - x \rangle + (1 - t_0) Re\langle w_0, v - x \rangle$$

$$= Re\langle w_0, z_0 - x \rangle$$

$$\geq \inf_{w \in T(z_{t_0})} Re\langle w, z_{t_0} - x \rangle.$$

Since ε is arbitrary, we have $\inf_{w \in T(z_{t_0})} Re\langle w, z_{t_0} - x \rangle \leq \lambda$. Therefore for any fixed $x \in X$, $y \longmapsto f(x, y)$ is lower semicontinuous from line segments in X to the weak* topology on E'.

Clearly, for each $x \in X$, $f(x, x) = 0$.

(iv) If $f(y, x) = \inf_{u \in T(x)} Re\langle u, x - y \rangle < 0$, then there exists $u_0 \in T(x)$ such that $Re\langle u_0, x - y \rangle < 0$. Thus there is an weakly open neighborhood $N_X(y)$ of y in X such that $Re\langle u_0, x - z \rangle < 0$ for all $z \in N_X(y)$. Hence $f(z, x) = \inf_{u \in T(x)} Re\langle u, x - z \rangle < 0$ for all $z \in N_X(y)$.

(v) By assumption, there exist a nonempty weakly closed and weakly compact subset K in X and $x_0 \in X$ such that $f(x_0, y) = \inf_{w \in T(y)} Re\langle w, y - x_0 \rangle > 0$ for all $y \in X \backslash K$.

Equip E with the weak topology, then all conditions of Theorem 4.2.5 are satisfied. Hence there is $\hat{y} \in K$ such that $f(x, \hat{y}) = \inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x \rangle \leq 0$ for all $x \in X$. Let $x \in I_X(\hat{y})$, then $x = \hat{y} + r(u - \hat{y})$ for some $u \in X$ and $r > 0$. Thus $\hat{y} - x = r(u - \hat{y})$ so that

$$\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x \rangle = r \inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - u \rangle \leq 0.$$

If $T(\hat{y})$ is convex, then by Kneser's minimax theorem [53],

$$\min_{w \in T(\hat{y})} \sup_{x \in I_X(\hat{y})} Re\langle w, \hat{y} - x \rangle = \sup_{x \in I_X(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x \rangle \leq 0.$$

Therefore there exists a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in I_X(\hat{y})$. Since $\hat{w}$ is continuous, we conclude that $Re\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in cl_E(I_X(\hat{y}))$. ∎

If X is a cone in a topological vector space E, we shall denote by $\hat{X}$ the set $\{w \in E' : Re\langle w, x \rangle \geq 0 \text{ for all } x \in X\}$, then $\hat{X}$ is also a cone in E', called the dual cone of X in E'. A result of S. C. Fang (e.g. see [18]) can be easily modified to give the following:

Lemma 4.5.3 *Let X be a cone in a vector space E and $T : E \rightarrow 2^{E'}$. Then following statements are equivalent:*

(a) *There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in X$.*

(b) *There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \hat{X}$.*

By using Lemma 4.5.3, we have the following existence theorem of a generalized complementarity problem:

Theorem 4.5.4 *Let E be a Hausdorff topological vector space and X a cone in E. Let $T : X \rightarrow 2^{E'}$ be almost monotone and upper semicontinuous from line segments in X to the weak* topology on E' such that for each $x \in X$, $T(x)$ is weak* compact convex. Suppose there exist a nonempty weakly closed and weakly compact subset K of X and $x_0 \in X$ such that $\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle > 0$ for all $y \in X \backslash K$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} \rangle = 0$ and $\hat{w} \in \hat{X}$.*

Theorem 4.5.5 *Let $(E, \|\cdot\|)$ be a reflexive Banach space and X be a nonempty closed convex subset of E. Let $T : X \rightarrow 2^{E'}$ be upper semicontinuous along line segments in X to the weak topology on E' such that each $T(x)$ is weakly compact convex and T is almost monotone. Suppose there is $x_0 \in X$ such that*

$$\lim_{\substack{\|y\| \to \infty \\ y \in X}} \inf_{w \in T(y)} Re\langle w, y - x_0 \rangle > 0. \tag{5.1}$$

Then there exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in cl_E(I_X(\hat{y}))$.

Proof. Let $\alpha = \lim_{\substack{\|y\| \to \infty \\ y \in X}} \inf_{w \in T(y)} Re\langle w, y - x_0 \rangle$. Then by (5.1), $\alpha > 0$. Let $M > 0$ be such that $\|x_0\| \leq M$ and $\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle > \frac{\alpha}{2}$ for all $y \in X$ with $\|y\| > M$. Let $K = \{x \in X : \|x\| \leq M\}$; then K is a nonempty weakly compact subset of X. Note that for any $y \in X \setminus K$, $\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle > \frac{\alpha}{2} > 0$. The conclusion now follows from Theorem 4.5.2. ∎

If H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the corresponding induced norm $\| \cdot \|$ and X is a nonempty subset of H, we shall denote by $bcc(X)$ the family of all nonempty bounded closed convex subsets of X.

Now we shall give an existence theorem for a variational inequality in Hilbert space. It can be reduced to a fixed point theorem under suitable conditions.

Theorem 4.5.6 *Let H be a Hilbert space and X a nonempty convex subset of H. Let $T : X \to bcc(H)$ upper semicontinuous from line segments in X to the weak topology on H such that $I - T$ is almost monotone. Suppose that there exist a nonempty weakly compact subset K of X and an $x_0 \in X$ such that $\inf_{w \in T(y)} Re\langle y - w, y - x_0 \rangle > 0$ for all $y \in X \backslash K$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that*

$$Re\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0 \text{ for all } x \in cl_E(I_X(\hat{y})).$$

Moreover, if either $\hat{y}$ is an interior point of X in H or $p(\hat{y}) \in cl_E(I_X(\hat{y}))$, where $p(\hat{y})$ is the projection of $\hat{y}$ on $T(\hat{y})$, then $\hat{y}$ is a fixed point of T, i.e., $\hat{y} \in T(\hat{y})$.

Proof. Equip H with the weak topology. Since T is upper semicontinuous from line segments in X, $I - T : X \to bcc(H)$ is also upper semicontinuous along line segments in X. Now $I - T$ satisfies all the hypotheses of Theorem 4.5.2. Thus by Theorem 4.5.2, there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{y} - \hat{w}, \hat{y} - x \rangle \leq 0 \text{ for all } x \in cl_E(I_X(\hat{y})).$$

Now if $\hat{y}$ is an interior point of X in H, then above inequality implies that $\hat{y} = \hat{w} \in T(\hat{y})$. Next suppose that $p(\hat{y}) \in cl_E(I_X(\hat{y}))$. Note that the projection $p(\hat{y})$ of $\hat{y}$ on $T(\hat{y})$ has the property, and in fact is characterized by (e.g., see Theorem I.2.3 in [51], Page 9.

$$p(\hat{y}) \in T(\hat{y}) \text{ and } Re\langle p(\hat{y}) - \hat{y}, w - p(\hat{y}) \rangle \geq 0 \text{ for all } w \in T(\hat{y}).$$

Since $\hat{w} \in T(\hat{y})$, we have

$$0 \leq Re\langle p(\hat{y}) - \hat{y}, \hat{w} - p(\hat{y}) \rangle$$

$$= Re\langle p(\hat{y}) - \hat{y}, \hat{w} - \hat{y} + \hat{y} - p(\hat{y})\rangle$$

$$= Re\langle p(\hat{y}) - \hat{y}, \hat{w} - \hat{y}\rangle - \|\hat{y} - p(\hat{y})\|^2.$$

Therefore, $\|\hat{y} - p(\hat{y})\|^2 \leq Re\langle p(\hat{y}) - \hat{y}, \hat{w} - \hat{y}\rangle \leq 0$; thus $\hat{y} = p(\hat{y}) \in T(\hat{y})$. ∎

The following fixed point theorem, closely relates to Theorem 7 in [9], is an immediate consequence of Theorem 4.5.6.

Theorem 4.5.7 *Let H be a Hilbert space and X a nonempty convex subset of H. Let $T : X \rightarrow bcc(H)$ be upper semicontinuous from line segments in X to the weak topology on H such that $I - T$ is almost monotone. Suppose that there exist a nonempty weakly compact subset K of X and $x_0 \in X$ such that*

(i) *for each $y \in K \cap \partial_H(X)$, $p(y) \in cl_E(I_X(y))$, where $p(y)$ is the projection of y on $T(y)$ and*

(ii) *for each $y \in X \backslash K$, $\inf_{w \in T(y)} Re\langle y - w, y - x_0\rangle > 0$.*

Then T has a fixed point in K.

Corollary 4.5.8 *Let H be a Hilbert space and X a nonempty bounded convex subset of H. Let $T : X \rightarrow bcc(H)$ be upper semicontinuous from line segments in X to the weak topology of H such that $I - T$ is almost monotone. If for each $y \in \partial_H(X)$, $p(y) \in cl_E(I_X(y))$ where $p(y)$ is the projection of y on Ty, then T has a fixed point in X.*

Corollary 4.5.9 *Let H be a Hilbert space and X a nonempty bounded closed convex subset of H. Let $T : X \rightarrow bcc(X)$ be upper semicontinuous from line segments in X to the weak topology of H such that $I - T$ is almost monotone. Then T has a fixed point in X.*

Chapter 5

A Note on Fixed Point Theorems for Semi-continuous Correspondences on [0,1]

5.1   Introduction

A function either *continuous* or *nondecreasing* from $[0,1]$ to itself has a fixed point. What are common points between these two results? This problem has been investigated independently by Milgrom and Roberts [58] and Guillerme [36]. Their findings can be summarized as:

Theorem 5.1.1 *Suppose that the real function* $f : [0,1] \to [0,1]$ *is upper semicontinuous on the right and lower semicontinuous on the left, then it has a fixed point.*

On the other hand, Strother [75] asserted that a continuous multivalued function on $[0,1]$ has a fixed point. This is a surprising result in the sense that a fixed point theorem for a multivalued function usually requires the function to have convex values. For example, the Kakutani theorem, or some theorems involving a continuous selection (for which Michael's [56] result is often needed). Do we have an analogy to Theorem 5.1.1 for multivalued functions? The answer is positive. Strother's result

was proved also by constructing a continuous selection. But our result is not. Moreover, we shall give an example to show that correspondences satisfying the hypothesis of our theorem may not allow a continuous selection. Also some simple applications to game theory are included.

5.2 Fixed Point Theorems

First we give some definitions. Let $T : \mathbb{R} \to 2^Y$ be a correspondence, where Y is a metric space. T is said to be *upper semicontinuous on the right (RUS)* if for any $\bar{x} \in \mathbb{R}$ and any sequence (x_n) such that $x_n \downarrow \bar{x}$, and every sequence y_n converging to $\bar{y}$ such that $y_n \in T(x_n)$ for every n, then $\bar{y} \in T(\bar{x})$; T is said to be *lower semicontinuous on the left (LLS)* if for any sequence $x_n \uparrow \bar{x}$ and every $\bar{y} \in T(\bar{x})$, there is a sequence (y_n) converging to $\bar{y}$ such that $y_n \in T(x_n)$ for every n.

Now we are ready to prove an intermediate value theorem.

Proposition 5.2.1 *Let $T : [0,1] \to 2^{\mathbb{R}}$ be an RUS and LLS correspondence. Suppose that $T([0,1])$ is contained in some bounded set in $\mathbb{R}$ and for any $y \in T(0)$ (if there is such y), $y \geq 0$; for any $y \in T(1) \neq \emptyset, y \leq 0$. Then there exists $x \in [0,1]$ such that $0 \in T(x)$.*

Proof.

Part (I). We first prove that for any $x \in [0,1)$, $T(x) \neq \emptyset$. Suppose not, let $x_0 = \sup\{x : T(x) = \emptyset\}$.

(1) We claim that $T(x_0) \neq \emptyset$. If $x_0 = 1$, $T(x_0) \neq \emptyset$ by hypothesis. If $x_0 < 1$, take any $x_n \downarrow x_0$. Then $T(x_n)$ is not empty. Take any $y_n \in T(x_n)$, we can assume that y_n converges to some y_0. Then $y_0 \in T(x_0) \neq \emptyset$.

(2) If $x_0 = 0$, by (1), the conclusion of Part (1) holds. If $x_0 > 0$, we can find $x_n \uparrow x_0$ and $T(x_n) = \emptyset$. This is not possible for $T(x_0) \neq \emptyset$.

Part(II). Define $x_L = \inf\{x : \text{there exists some } y \in T(x) \text{ such that } y \leq 0\}$. Obviously x_L exists. If $x_L = 1$, we have $0 \in T(1)$, for otherwise, take any $y_L \in T(1)$.

Then $y_L < 0$. Now take any sequence (x_n) in $[0,1]$ such that $x_n \uparrow x_L$. Since T is LLS , we can find (y_n) such that $0 \le y_n \in T(x_n)$ for each n and $y_n \to y_L$. This is a contradiction. In fact, $T(1) = \{0\}$ in this case.

Now suppose that $0 \le x_L < 1$. Then there is a sequence (x_n) in $[0,1]$ such that $x_n \downarrow x_L$ and there is a sequence (y_n) such that for each n, $y_n \in T(x_n)$ and $y_n \le 0$. Since the set $T([0,1])$ is bounded, we can have a subsequence (y_{n_i}) of (y_n) converging to some $y_L = \lim y_{n_i} \le 0$. Since T is (RUS), we have $y_L \in T(x_L)$. If $x_L = 0$, then $y_L = 0$. If $x_L > 0$, we can find a sequence (x_n) in $[0,1]$ such that $x_n \uparrow x_L$. Since T is LLS, there exists (y_n) such that for each n, $y_n \in T(x_n)$ and $y_n \to y_L$. Since for each n, $y_n > 0$, we have $y_L \ge 0$. So $y_L = 0$, i.e., $0 \in T(x_L)$. ∎

Theorem 5.2.2 *Let $T : [0,1] \to 2^{[0,1]}$ be an RUS and LLS correspondence with $T(1)$ nonempty. Then T has a fixed point.*

Proof. Define $G : [0,1] \to 2^{\mathbb{R}}$ by $G(x) = \{y - x : \text{ for } y \in T(x)\}$, then G satisfies all conditions of Theorem 1. So there is an $\bar{x}$ such that $0 \in G(\bar{x})$ which means that $\bar{x} \in T(\bar{x})$. ∎

Example 5.2.3 *Let $F : [0,1] \to 2^{[0,1]}$ be defined by*

$$F(x) = \begin{cases} [1/3 - (x+1)/8, 2/3 + (x+1)/8] & x < 1/2; \\ [1/3, 2/3] \cup \{3/4\} & x = 1/2; \\ [2/5, 3/5] \cup \{3/4\} & x > 1/2. \end{cases}$$

It is easy to check that F is RUS and LLS, but it is neither upper semicontinuous nor lower semicontinuous. Moreover, it has non-convex values. Therefore the usual Kakutani fixed point theorem cannot be applied. ∎

In [75], Strother proved the result we mentioned in Section 5.1 by constructing a continuous selection of the continuous correspondence. We give an example to show that our conditions may not allow a continuous selection.

Example 5.2.4 *Let $F : [0,1] \to 2^{[0,1]}$ be defined by*

$$F(x) = \begin{cases} [0,1] & x \leq 1/2; \\ \{|\sin(1/(x-1/2))|, |\cos(1/(x-1/2))|\} & x > 1/2. \end{cases}$$

Then F is RUS and LLS, but it is not possible that F has a continuous selection. For if it had, we denote it by f and suppose $f(1/2) = a \in [0,1]$. If $a = \sin(\pi/4) = \cos(\pi/4)$, we take $x_n = 1/2 + 1/(n\pi)$, then $F(x_n) = \{0,1\}$, $d(a, F(x_n)) = (1 - \sin(\pi/4))$. If $a \neq \sin(\pi/4)$, take $x_n = 1/2 + 1/(n\pi + \pi/4)$, then $d(a, F(x_n)) = |a - \sin(\pi/4)|$. So $F(x)$ has no continuous selection. This also demonstrates that F is not continuous for otherwise F would have a continuous selection by Strother's Theorem. ∎

Again, since F is not convex-valued, the Kakutani fixed point theorem cannot be applied.

Theorem 5.2.2 does not contain Theorem 5.1.1, that is, for a monotone multivalued function in the usual sense, it is easy to show that it has a fixed point. We give another simple result in which the meaning of monotonicity is defined as follows.

A multivalued function $F : [0,1] \to 2^{[0,1]}$ is said to be *monotone* if for any $x < y$, and any $u \in F(x), v \in F(y)$ we have $\min\{u,v\} \in F(x)$ and $\max\{u,v\} \in F(y)$.

Theorem 5.2.5 *Let $F : [0,1] \to 2^{[0,1]}$ be monotone with nonempty closed values, then it has a fixed point.*

Proof.

Let $f : [0,1] \to [0,1]$ be defined by $f(x) = \min F(x)$. Then f is non-decreasing. For any $x < y$,

$$f(x) = \min F(x) \leq \min F(y) = f(y).$$

Otherwise we have $\min\{f(x), f(y)\} = f(y) \notin F(x)$. Thus f, and hence F, has a fixed point by Theorem 5.1.1. ∎

5.3   Applications to Game Theory

Consider a *two-person game* in which the two players are identified as Player 1 and Player 2. The strategy spaces for them are sets X_1 and X_2, respectively. The best responses of Player 1 and Player 2 are defined by correspondences $T_1 : X_2 \to 2^{X_1}$ and $T_2 : X_1 \to 2^{X_2}$, respectively. Let $T : X_1 \times X_2 \to 2^{X_1} \times 2^{X_2}$ be defined by $T(x) = T_1(x_2) \times T_2(x_1)$ for $x = (x_1, x_2) \in X_1 \times X_2$. The set of *Nash equilibrium points* of this game is defined as the set of fixed points of T. Note that such a game has a non-empty set of equilibria if and only if the composite correspondence $T_1 T_2 : X_1 \to 2^{X_1}$ or $T_2 T_1 : X_2 \to 2^{X_2}$ has a fixed point.

Theorem 5.3.1 *For a two-person game on $[0,1] \times Y$ where Y is a compact metric space, let $T_1(y)$ be the best response of Player 1 and $T_2(x)$ the best response of Player 2. We suppose that T_1 is continuous with nonempty closed values and T_2 is RUS and LLS with nonempty closed values, then there is a Nash equilibrium for this game.*

Proof.

We define $T : [0,1] \to 2^{[0,1]}$ by $T = T_1 T_2$. If T has a fixed point $\hat{x}$, then there is $\hat{y} \in T_2(\hat{x})$ such that $\hat{x} \in T_1(\hat{y})$. So $(\hat{x}, \hat{y})$ will be a Nash equilibrium of the game.

Now we prove that T is RUS. Suppose that $x_n \downarrow x$ and $z_n \in T(x_n) \to z$. Then there is $y_n \in T_2(x_n)$ such that $z_n \in T_1(y_n)$. Since Y is compact, without loss generality, we suppose that (y_n) converges to y. Then $z \in T_1(y)$ since T_1 is continuous with nonempty closed values (hence it is closed). But $y \in T_2(x)$ since T_2 is RUS. Therefore $z \in T_1 T_2(x) = T(x)$. That is, T is RUS.

Next we shall prove that T is LLS. Suppose that $x_n \uparrow x$ and $z \in T(x)$. First there is $y \in T_2(x)$ such that $z \in T_1(y)$. Since T_2 is LLS, there is $y_n \in T_2(x_n)$ such that $y_n \to y$. Since $y_n \to y$ and $z \in T_1(y)$, there is $z_n \in T_1(y_n)$ such that $z_n \to z$ since T_1 is lower semicontinuous. Now we have $z_n \in T_1 T_2(x_n)$ such that $z_n \to z \in T(x)$. Thus T is LLS.

So by Theorem 5.2.2, there is a fixed point $\hat{x} \in T(\hat{x})$, i.e. there is a Nash equilibrium for the two person game. ■

Corollary 5.3.2 *For a two-person game on $[0,1] \times Y$ where Y is a Banach space, let $T_1(y)$ be the best response of Player 1 and $T_2(x)$ the best response of Player 2. We suppose that T_1 is lower semicontinuous with nonempty closed convex values and T_2 is RUS and LLS with nonempty closed values, then there is a Nash equilibrium for this game.*

Proof. In fact, by Theorem 3.2″ in Michael [56] there is a continuous selection of T_1. So the result follows. ∎

Bibliography

[1] C. D. Aliprantis, D. J. Brown and O. Burkinshaw, *Existence and Optimality of Competitive Equilibria*, Springer-Verlag, New York, 1989.

[2] G. F. Andrus and L. Brown, Measurable extension theorems, *J. Math. Anal. Appl.* **96** (1983), 454-462.

[3] K. J. Arrow and G. Debreu, Existence of an equilibrium for a competitive economy, *Econometrica* **22** (1954), 265-290.

[4] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, New York, 1984.

[5] R. J. Aumann, Markets with a continuum of agents, *Econometrica* **32** (1964), 39-50.

[6] R. J. Aumann, Existence of competitive equilibria in markets with a continuum of agents, *Econometrica* **34** (1966), 1-17.

[7] R. J. Aumann, Measurable utility and the measurable choice theorem, *in* "La Desision," pp. 15-26, CNRS, Aix-en-Provence, 1967.

[8] E. J. Balder, A unifying approach to existence of Nash equilibria, *International J. of Game Theo.* **24** (1995), 79-94.

[9] J. S. Bae, W. K. Kim & K. K. Tan, Another generalization of Ky Fan's minimax inequality and its application, *Bull. Inst. Math. Acad. Sinica* **21** (1993), 229-244.

[10] C. Baiocchi and A. Capelo, *Variational and Quasivaritional Inequalities Applications to Free-boundary Problems*, John Wiley & Sons, New York, 1984.

[11] M. Baye, G. Tian and J. X. Zhou, Characterizations of the existence of equilibria in games with discontinuous and nonquasiconcave payoffs, *Rev. Econom. Stud.* **60** (1993), 935-948.

[12] H. Ben-El-Mechaiekh, P. Deguire et A. Granas, Points fixes et coincidences pour les fonctions multivoques II(Applications de Type ϕ et ϕ^*), *C. R. Acad. Sci. Paris* **295** (1982), 381-384.

[13] K. C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, London, 1985.

[14] S. Y. Chang, On the Nash equilibrium, *Soochow J. Math.* **16**, 241-248 (1990).

[15] N. Bourbaki, *Topological Vector Spaces*, Springer-Verlag, New York, 1987.

[16] F. E. Browder, The fixed-point theory of multi-valued mappings in topological spaces, *Math. Ann.* **177** (1968), 283-301.

[17] A. Cellina and S. Solomini, Continuous extensions of selections, *Bull. Polish. Acad. Sci.* **35** (1987), 573-581.

[18] D. Chan, and J. S. Pang, The generalized quasi-variational inequality problem, *Math. Oper. Res.* **7** (1982), 211-222.

[19] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, Vol.580, Springer-Verlag, New York, 1977.

[20] G. Debreu, A social equilibrium existence theorem, *Proc. Nat. Acad. Sci.* **38** (1952), 886-893.

[21] G. Debreu, Existence of competitive equilibrium, in Handbook of Mathematical Economics, Vol. 2, Chapter 7, pp.697-743, North-Holland, 1982.

[22] P. Deguire and M. Lassonde, Familles sélectantes, *Topol. Methods in Nonlinear Anal.* **5** (1995), 261-269.

[23] X. P. Ding and K. K. Tan, A minimax inequality with applications to existence of equilibrium point and fixed point theorems, *Colloq. Math.* **63** (1992), 233-247.

[24] J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* **1** (1951), 353-367.

[25] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[26] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Wiley, New York, 1966.

[27] S. Eilenberg and D. Montgomery, Fixed point theorems for multi-valued transformations, *Amer. J. Math.* **68** (1946), 214-222.

[28] K. Fan, Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci.* **38** (1952), 121-126.

[29] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* **142** (1961), 305-310.

[30] K. Fan, A minimax inequality and applications, in *Inequalities* III, Ed. O. Shisha, pp. 103-113, Academic Press, New York, 1972.

[31] K. Fan, Some properties of convex sets related to fixed point theorems, *Math. Ann.* **266** (1984), 519-537.

[32] D. Gale and A. Mas-Colell, An equilibrium existence theorem for a general model without ordered preferences, *J. Math. Econo.* **2** (1975), 9-15.

[33] D. Gale and A. Mas-Colell, Corrections to an equilibrium existence theorem for a general model without ordered preferences, *J. Math. Econo.* **6** (1979), 297-298.

[34] I. Glicksberg, A further generalization of the Kakutani fixed-point theorem with applications to Nash equlibrium points, *Proc. Amer. Math. Soc.* **3** (1952), 170-174.

[35] A. Granas, H. Ben-El-Mechaikh and P. Deguire, A non-linear alternative in convex analysis: Some consequences, *C. R. Acad. Sci. Paris* **295** (1982), 257-259.

[36] J. Guillerme, Intermediate value theorems and fixed point theorems for semi-continuous functions in product spaces, *Proc. Amer. Math. Soc.* **123** (1995), 2119-2122.

[37] P. T. Harker and J. S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problem: A Survey of theory, algorithms, and applications, *Math. Programming* **48** (1990), 161-220.

[38] B. Halpern and G. Bergman, A fixed point theorem for inward and outward maps, *Trans. Amer. Math. Soc.* **130** (1968), 353-358.

[39] O. Hanš, Measurability of extensions of continuous random transforms, *Ann. Math. Stat.* **30** (1959), 1152-1157.

[40] C. J. Himmelberg, Fixed points of compact multifunctions, *J. Math. Anal. App.* **38** (1972),205-207.

[41] C. J. Himmelberg, Measurable relations, *Fund. Math.* **87** (1975), 53-72.

[42] Y. Hou, The extensions of continuous random operators, *Math. Phys. Acta* **11** (1991), 420-429.

[43] S. Kakutani, A generalization of Brouwer's fixed point theorem, *Duke Math. J.* **8** (1941), 457-459.

[44] D. Kandilakis and N. S. Papageorgiou, Properties of measurable multifunctions with stochastic domain and their applications, *Math. Japonica* **35** (1990), 629-643.

[45] M. Ali Khan and N. S. Papageorgiou, On Cournot-Nash equilibria in generalized quantitative games with a continuum of players, University of Illinois, Urbana, 1985.

[46] W. K. Kim, Remarks on a generalized quasi-variational inequality, *Proc. Amer. Math. Soc.* **103** (1988), 667-668.

[47] W. K. Kim and K. H. Lee, Existence of equilibrium and separation in generalized games, *J. Math. Anal. Appl.* **207** (1997), 316-325.

[48] T. Kim, K. Prikry, and N. C. Yannelis, Equilibria in abstract economies with a measure space of agents and with an infinite dimensional strategy space, *J. Approx. Theory* **56** (1989), 256-266.
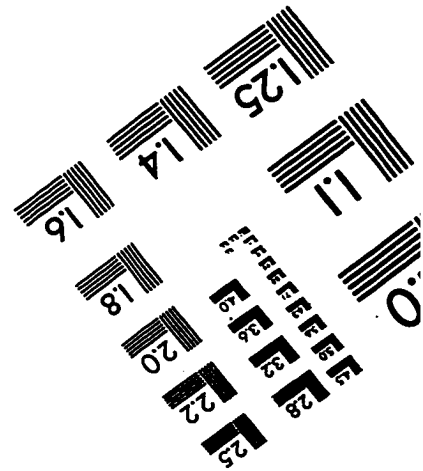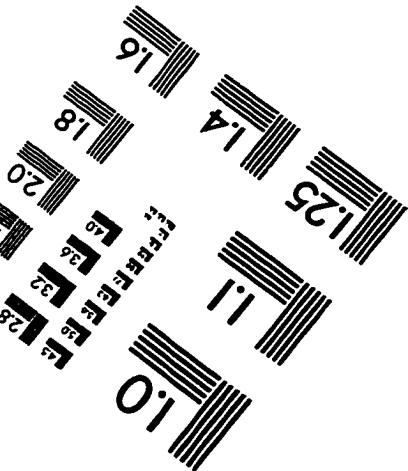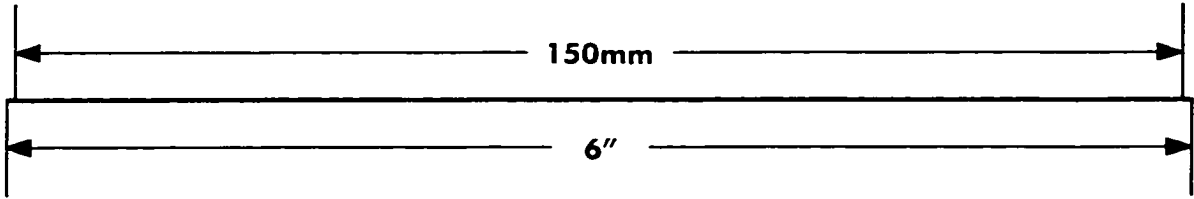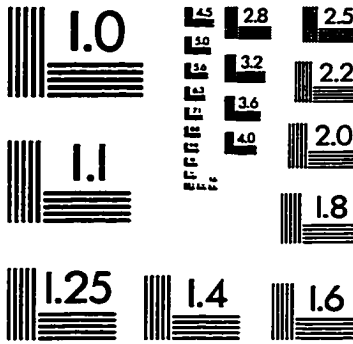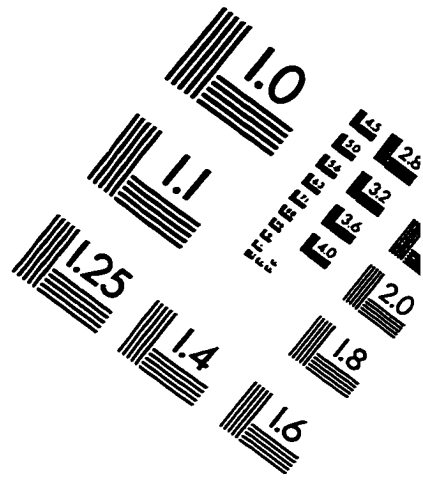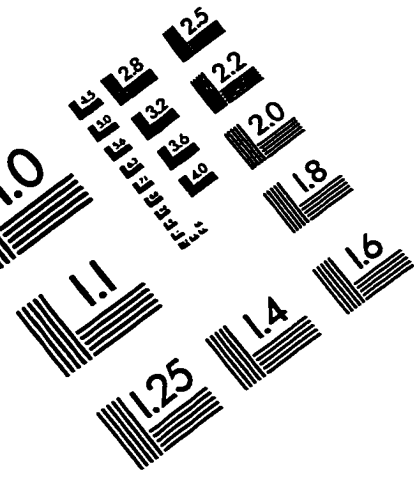
[49] T. Kim, K. Prikry, and N. C. Yannelis, On a Carathéodory-type selection theorem, *J. Math. Anal. Appl.* **135** (1988), 664-670.

[50] T. Kim, K. Prikry, and N. C. Yannelis, Caratheodory-type selections and random fixed point theorems, *J. Math. Anal. Appl.* **122** (1987), 393-407.

[51] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[52] E. Klein and A. C. Thompson, *Theory of Correspondences: including applications to mathematical economics*, John Wiley & Sons, 1984.

[53] H. Kneser, Sur un théorème fondamental de la théorie des jeux, *C. R. Acad. Sci. Paris* **234** (1952), 2418-2420.

[54] T-W. Ma, Topological degree of set-valued compact fields in locally convex spaces, *Dissertationes. Math.* **92** (1972), 1-47.

[55] G. Metha and E. Tarafdar, Infinite-dimensional Gale-Nikaido-Debreu theorem and a fixed-point theorem of Tarafdar, *J. Economic Theo.* **41** (1987), 333-339.

[56] E. Michael, Continuous selections I, *Ann. Math.* **63** (1956), 361-382.

[57] U. Mosco, Implicit variational problems and quasivaritional inequalities, in *Nonlinear Operators and the Calculus of Variations* (Eds J. P. Gossez, E. L. Dozo, J. Mawhin and L. Waelbroeck), 83-156, Proc. Summer School (Bruxelles, 1975), Lecture Notes in Mathematics, **534**, Springer, Berlin, 1976.

[58] P. Milgrom and J. Roberts, Comparing equilibria, *Amer. Econ. Rev.* **84** (1994), 441-459.

[59] J. Nash, Equlibrium points in N-person Games, *Proc. Nat. Acad. Sci.* **36** (1950), 48-49.

[60] A. Nowak, Applications of random fixed point theorems in the theory of generalized random differential equations, *Bull. Polish. Acad. Sci. Math.* **34** (1986), 487-494.

[61] N. Papageorgiou, Representation of set valued operators, *Tran. Amer. Math. Soc.* **292** (1985), 557-572.

[62] N. Papageorgiou, On measurable multifunctions with stochastic domain, *J. Austral. Math. Soc. (Series A)* **45** (1988), 204-216.

[63] T. Pruszko, Completely continuous extensions of convex-valued selectors, *Nonlinear Analysis* **27** (1996), 781-784.

[64] L. E. Rybinski, Random fixed points and viable random solutions of functional differential inclusions, *J. Math. Anal. Appl.* **142**(1989), 53-61.

[65] M. F. Saint-Beuve, On the existence of von Neuamnn-Aumann's theorem, *J. Funct. Anal.* **17** (1974), 112-129.

[66] D. Schmeidler, Equilibrium points of nonatomic games, *J. Stat. Phys.* **7** (1973), 295-300.

[67] W. Shafer, Equilibrium in abstract economies without ordered preferences, *J. Math. Econo.* **2** (1975), 345-348.

[68] W. Shafer, Equilibrium in economies without ordered preferences or free disposal, *J. Math. Econo.* **3** (1976), 135-137.

[69] W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, *J. Math. Econo.* **2** (1975), 345-348.

[70] W. Shafer and H. Sonnenschein, Some theorems on the existence of competitive equilibrium, *J. Economic Theory* **11** (1975), 83-93.

[71] M. H. Shih and K. K. Tan, Generalized quasi-variational inequalities in locally convex topological spaces, *J. Math. Anal. Appl.* **108** (1985), 333-343.

[72] M. H. Shih and K. K. Tan, A minimax inequality and Browder-Hartman-Stampacchia variational inequalities for multi-valued monotone operators, *Proceedings of the Fourth FRANCO-SEAMS Joint Conference* (Chiang Mai, Thailand, 1988), DAR TR-88-2.

[73] M. H. Shih and K. K. Tan, Generalized bi-variational inequalities in locally convex topological spaces, *J. Math. Anal. Appl.* **143** (1989), 66-85.

[74] M. Sion, On general minimax theorems, *Pacific J. Math.* **8** (1958), 171-176.

[75] W. L. Strother, On an open question concerning fixed points, *Proc. Amer. Math. Soc.* **4** (1953), 988-993.

[76] K. K. Tan and X. Z. Yuan, Equilibria of generalized games with U-majorized preference correspondences, *Econo. Letter* **41** (1993), 379-383.

[77] K. K. Tan and X. Z. Yuan, A minimax inequality with applications to existence of equilibrium points, *Bull. Austral. Math. Soc.* **47** (1993), 483-503.

[78] K. K. Tan and X. Z. Yuan, Maximal elements and equilibria for U-majorized preferences, *Bull. Austral. Math. Soc.* **49** (1994), 47-54.

[79] K. K. Tan and X. Z. Yuan, Lower semicontinuity of multivalued mappings and equilibrium points, Proceedings of the First World Congress of Nonlinear Analysts, Tampa, Florida, Augest 19-26, 1992, (Ed. V. Lakshmikantham), Walter de Gruyter, Berlin, pp. 1849-1860, 1996.

[80] K. K. Tan and J. Yu, Minimax inequalities and generalizations of Gale-Nikaido-Debreu Lemma, *J. Optim. Theo. Appl.* **82** (1994), 105-120.

[81] E. Tarafdar, On nonlinear variational inequalities, *Proc. Amer. Math. Soc.* **67** (1977), 95-98.

[82] E. Tarafdar and X. Z. Yuan, Non-compact generalized quasi-variational inequality in locally convex topological vector spaces, *Nonlinear World* **1** (1994), 373-383.

[83] E. Tarafdar, P. Watson and X. Z. Yuan, Jointly measurable selections of condensing Carathéodory set-valued mappings and its applications to random fixed points, *Nonlinear Analysis* **28** (1997), 39-48.

[84] G. Tian, Necessary and sufficient conditions for maximization of a class of preference relations, *Rev. Econom. Stud.* **60** (1993), 949-958.

[85] N. C. Yannelis, Equilibria in noncooperative models of competition, *J. Economic Theory* **41** (1987), 96-111.

[86] X. Z. Yuan and E. Tarafdar, Existence of equilibria of generalized games without compactness and paracompactness, *Nonlinear Analysis.* **26** (1996), 893-902.

IMAGE EVALUATION
TEST TARGET (QA—3)

150mm

6"