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DECOMPOSABILITY AND TRIANGULARIZABILITY OF POSITIVE OPERATORS ON BANACH LATTICES

By
Mohammad Taghi Jahandideh

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To Mehrdokht, Hossein, and Ehssan

Contents

Abstract

The main results in this thesis are about invariant subspaces of multiplicative semigroups of quasinilpotent positive operators on Banach lattices.

There are some known results that guarantee the existence of a non-trivial closed invariant ideal for a quasinilpotent positive operator on certain spaces, for example on $C_0(\Omega)$ with Ω a locally compact Hausdorff space or on a Banach lattice with atoms. Some recent results also guarantee the existence of non-trivial closed invariant ideals for a compact quasinilpotent positive operator on an arbitrary Banach lattice. In fact it is known that given such an operator T, on a real or complex Banach lattice, there is a nontrivial closed ideal which is invariant under all positive operators that commute with T.

This thesis deals with invariant ideals for families of positive operators on Banach lattices. In particular it studies ideal-decomposable and ideal-triangularizable semigroups of positive operators. We show that in certain Banach lattices compactness is not required for the existence of hyperinvariant closed ideals for a quasinilpotent positive operator. We also show that in those Banach lattices a semigroup of quasinilpotent positive operators might be decomposable without imposing any compactness condition. We generalize the fact that *the only irreducible* C_p*-closed subalgebra of* C_p *is* C_p *itself* to extend some recent reducibility results and apply them to derive some decomposability theorems concerning a collection of quasinilpotent positive operators on reflexive Banach lattices.

We use these results for "ideal-triangularization", i.e., we construct a maximal closed ideal chain, each of whose members is invariant under a certain collection of operators that are related to compact positive operators, or to quasinilpotent positive operators.

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Introduction

There are two notions which gave rise to the questions this thesis attempts to address.

First is the consideration of the existence of invariant subspaces for certain collections of quasinilpotent operators, especially non-compact quasinilpotent operators. Several mathematicians have made progress in this direction and have obtained a number of results concerning certain semigroups of quasinilpotent operators on a Hilbert space. An example for the compact case can be found in [33]. For the non-compact case an application of the Andô-Krieger theorem [48, Theorem 136.9] implies that every quasinilpotent integral operator with non-negative kernel on $\mathfrak{L}_2(\mathcal{X}, \Sigma, \mu)$, where $\mathcal{X}$ is a topological space, μ is a positive Borel measure on $\mathcal{X}$, and

$$\dim(L_2(\mathcal{X}, \Sigma, \mu)) \geq 2,$$

has a nontrivial invariant subspace.

Since 1970, there have been a number of works on invariant order ideals, motivated by interest in the spectral radius of positive operators on Banach lattices. (If there exists a nontrivial closed ideal for a positive operator T on a Banach lattice, then T is called reducible. The term "reducibility" has been used in [1],[2],[3],[4],[5],[6],[34], and [46]. In this thesis, for technical reasons, the term "decomposability" will be used instead of "reducibility" in the application of the results of those papers. For precise definitions and terminology see Chapter 0.) It has been proven [45] that an indecomposable positive operator on a closed ideal of an AM-space with unit must have a positive spectral radius. Other works are based on several attempts at understanding the Andô-Krieger theorem – which states that a band-irreducible positive kernel operator has a strictly positive spectral radius. One of the most important works on invariant ideals is the elegant theorem of de Pagter [34] - which states that every ideal-irreducible compact positive operator on a Banach lattice has a strictly positive spectral radius. In 1992, by a close study of de Pagter's proof, Abramovich, Aliprantis, and Burkinshaw proved that every compact quasinilpotent positive operator T on a Banach lattice E has a nontrivial closed ideal that is invariant under every positive operator S on E that commutes with T (see [1]). Using this theorem, they further established several results that guarantee a positive operator with certain properties has a strictly positive spectral radius. Also in [2],[3],[4],[5] these authors established a number of invariant subspace theorems for positive operators on Banach lattices under weaker conditions. Since in these theorems the invariant subspaces for positive operators are quite often order ideals, one may consider them as invariant ideal theorems.

The second notion is the investigation of a maximal chain of closed ideals of a Banach lattice E each of whose members is invariant under all the operators in a collection Γ of positive operators on E. This concept is the Banach lattice version of triangularizability of operators on a Banach space. The concept of triangularizability of operators on a Banach space has been studied by many authors. Kaplansky [24] gives triangularizability results for semigroups of operators on finite-dimensional spaces. In [27], by considering infinite-dimensional analogues of the known results, the results of McCoy [31] were generalized to get necessary and sufficient conditions that collections of compact operators on an infinite-dimensional Banach space be triangularizable. In [36], one can find the generalization of Kaplansky's result in the case where the spectrum of every operator in a semigroup of trace-class operators is contained in $\{0, 1\}$. Other triangularizability results are given in [10], [25],[33],[36],[37], and [38].

In this thesis, we consider several questions concerning the existence of a nontrivial closed ideal of a Banach lattice E that is invariant under every member of a collection Γ of positive (quasinilpotent) operators on E. After introducing the concept of ideal-triangularizability of operators on a Banach lattice we investigate a number of ideal-triangularizability problems.

In Chapter 0 we cover basic definitions and results for future use. We also introduce some notations and terminology. Most of the material in this chapter comes from [7], [26], [35], and [46].

Chapter 1 is about a special class of operators on a Banach space, called *trace ideals*, whose properties enable us to extend some reducibility and triangularizability results of operators on a Hilbert space. In this chapter we first introduce the concept of operator and trace ideals and use basic properties of these classes of operators to extend Theorem 6.1 of [39]. Then we apply this theorem to extend the results of [33] and [38].

In Chapter 2 we study the decomposability of a collection of positive operators on a Banach lattice E. We improve some well-known decomposability results and establish new decomposability results for a commutative collection of positive operators on AM-spaces and in Banach lattices with atoms. As an example, we prove that a positive quasinilpotent operator T in $C_0(\Omega)$, where Ω is a locally compact Hausdorff space, has a nontrivial closed ideal that is invariant under every positive operator S in $C_0(\Omega)$ that commutes with T. We also derive the decomposability of certain semigroups of integral operators on $C(\mathcal{K})$, where $\mathcal{K}$ is a compact Hausdorff space.

Chapter 3 is devoted to the study of the Banach-lattice version of triangularizabilty of operators. In this chapter we use the same procedure as in [41, Chapter 4] to introduce the notion of ideal-triangularizability of operators on Banach lattices. The results of Chapter 2 enable us to establish some ideal-triangularizabilty results. For example, we show that each quasinilpotent positive operator on $C(\mathcal{K})$, where $\mathcal{K}$ is a compact Hausdorff space, is ideal-triangularizable. We also discuss discrete Banach lattices with order continuous norms. We shall also prove that each quasinilpotent positive operator in such Banach lattices is ideal-triangularizable.

In Chapter 4 we introduce Banach lattices in which one can define an indecomposable quasinilpotent positive operator. The existence of such operators was first established in [45]. In fact, it was shown that if Γ is the circle group and if Θ is the Haar measure on Γ, then, for each $p \in [1, \infty)$, one can construct a quasinilpotent positive operator on $L_p(\Gamma, \Theta)$. We shall use the examples found in [45] to show that, if $(\mathcal{X}, \Sigma, \mu)$ is a σ-finite measure space with no atoms and if $L_p(\mathcal{X}, \Sigma, \mu)$, $1 \leq p < \infty$, is separable, then there exists an indecomposable quasinilpotent positive operator on $L_p(\mathcal{X}, \Sigma, \mu)$. In Chapter 4 we also present some examples and remarks to further illustrate our results in the earlier chapters.

Chapter 0

Basic Concepts

0.0 Prerequisites and Notations

The principal prerequisite for this thesis is a knowledge of the theory of operators on Banach spaces. The terminology of the books [7] and [46] will be employed throughout with the exception that, for technical reasons, the term "decomposability" will be used instead of "reducibility" in the application of the results of [1],[2],[3],[4],[5],[6],[34], [45], and [46]. The following conventions and notations will also be adhered to.

The symbols X, Y, and Z always denote real or complex Banach spaces, and whenever Banach lattices are under discussion we use E, F and G instead of X, Y and Z, respectively. Throughout this thesis we assume that all the Banach spaces have dimension greater than one. The symbols $\mathcal{X}$, Ω, and $\mathcal{K}$ are used to denote a general topological space, a locally compact Hausdorff space, and a compact Hausdorff space, respectively. The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ are used to denote the set of positive integers, the set of real numbers, and the set of complex numbers, respectively.

By a *measure* (without adjectives) we shall always mean a non-negative and countably additive set function μ defined on a σ-algebra Σ of subsets of $\mathcal{X}$. Almost all the measures we shall encounter will be *strictly positive* σ-finite measures, i.e. such that $\mu(U) > 0$ for all nonempty open subsets $U \in \Sigma$ and there exists a sequence $\{\mathcal{X}_n\}_{n=1}^{\infty}$

in Σ such that $\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n$ and $\mu(\mathcal{X}_n) < \infty$ for all $n \in \mathbb{N}$. The symbol $(\mathcal{X}, \Sigma, \mu)$ is used to denote a *measure space.*

The closure of a subset A of $\mathcal{X}$ is denoted by $\overline{A}$. In those cases that $\mathcal{X}$ has several topologies we use the symbol $\overline{A}^{\tau}$ to identify the topology under which closure is taken. The convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ of points in a metric space to a point x is denoted by $x_n \longrightarrow x$.

The space of continuous functions on Ω, the space of bounded continuous functions on Ω, the space of continuous functions on Ω whose support is compact, and the space of continuous functions on Ω which vanish at infinity, are denoted by $C(\Omega)$, $C_b(\Omega)$, $C_c(\Omega)$, and $C_0(\Omega)$, respectively. It is known that $C_0(\Omega)$ is the completion of $C_c(\Omega)$, relative to the metric defined by the supremum norm. Of course,

$$C_c(\Omega) \subseteq C_0(\Omega) \subseteq C(\Omega)$$

and they are equal whenever Ω is compact.

If $1 \leq p < \infty$ the Banach space of all μ-measurable functions f on $\mathcal{X}$ such that $\int_{\mathcal{X}} |f|^p \, d\mu < \infty$ is denoted by $L_p(\mathcal{X}, \Sigma, \mu)$ or simply by $L_p(\mu)$ or $L_p(\mathcal{X})$ if there is no ambiguity. If $\mathcal{X}$ is countable and μ is the counting measure, we denote the corresponding L_p-space by l_p. The Banach space of all functions f on the set of positive integers such that $f(n) \longrightarrow 0$ as $n \longrightarrow \infty$ with sup-norm is denoted by c_0.

By an *operator* $T : X \longrightarrow Y$ between two Banach spaces, we shall mean a continuous linear mapping. The class of all operators between two Banach spaces X and Y is denoted by $\mathfrak{B}(X, Y)$ and the symbol $\mathfrak{B}(X)$ is used whenever $X = Y$. The class of all linear mappings $T : V \longrightarrow W$ between two vector spaces is denoted by $\mathfrak{L}(V, W)$ and the symbol $\mathfrak{L}(V)$ is used whenever $V = W$. Unless otherwise stated, the topology on $\mathfrak{B}(X, Y)$ is the norm topology.

The symbols $\sigma(T)$ and $r(T)$ denote the *spectrum* and the *spectral radius* of an operator $T \in \mathfrak{B}(X)$, respectively. If $r(T) = 0$, T is called a *quasinilpotent* operator.

By the term (*multiplicative*) *semigroup* of operators on X, we shall mean a subset $\mathcal{S}$ of $\mathfrak{B}(X)$ such that $ST \in \mathcal{S}$ whenever $S, T \in \mathcal{S}$.

There are other notations that appear in the appropriate sections.

0.1 Vector Lattices

The function spaces which appear in real analysis are usually ordered in a natural way. This order is related to the norm and is important in the study of the space as a Banach space. In the sequel we study partially ordered Banach spaces whose order and norm are related according to the following Definitions.

Definitions 0.1.1 A partially ordered vector space V over the real numbers is called an *ordered vector space* if

(i) $x \leq y$ implies $x + z \leq y + z$, for every $x, y, z \in$ V, and

(ii) $ax \geq 0$. for every $x \geq 0$ in V and every nonnegative real a.

It is called a *vector lattice* (or a *Riesz space*) if it also satisfies

(iii) for all $x, y \in$ V there exists a *least upper bound* $x \vee y$ and a *greatest lower bound* $x \wedge y$ in V.

Let V be a vector lattice. For all $x \in$ V, we define $x^+ = x \vee 0$. $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. x^+. x^- and $|x|$ are called the positive part. the negative part. and the modulus (or absolute value) of x. respectively.

Definition 0.1.2 A vector lattice V. endowed with a norm. is called a *normed vector lattice* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in$ V.

If a normed vector lattice is a complete space, then it is called a *Banach lattice.*

By Proposition [46, II.5.2], the lattice operations are norm continuous. The continuity of the lattice operations implies, in particular, that the set

$$\mathsf{V}_+ = \{x \in \mathsf{V} : x \geq 0\}$$

is norm closed. The set V_+ is a convex cone and is called the *positive cone* of V.

Definitions 0.1.3 An *ideal* in a vector lattice V is a linear subspace J for which $y \in$ J whenever $y \in$ V and $|y| \leq |x|$ for some $x \in$ J. An ideal B of V is called a *band* if $A \subseteq$ B and $\sup A \in$ V together imply $\sup A \in$ B.

Let $E = L_p(\mu)$, $1 \leq p < \infty$. Then the subspace

$$J = \{ f \in L_p(\mu) \, : \, f = 0 \;\; a.e. \; [\mu] \;\; on \, S \},$$

where S is a measurable subset of $\mathcal{X}$, is an ideal of E.

The following well-known proposition characterizes all closed ideals of $E = C(\mathcal{K})$. For completeness, we present its proof here.

Proposition 0.1.4 *Let $J \subseteq C(\mathcal{K})$. Then J is a closed ideal of $C(\mathcal{K})$ if and only if there exists a closed subset S_J of $\mathcal{K}$ such that*

$$J = \{ f \in C(\mathcal{K}) \, : \, f(t) = 0 \;\; \forall t \in S_J \}.$$

Furthermore, J is isomorphic to $C_0(U)$, where $U = \mathcal{K} \setminus S_J$ is a locally compact Hausdorff space.

Proof : Suppose J is a closed ideal of $C(\mathcal{K})$. Let $g \in J$ and $f \in C(\mathcal{K})$. Since $|g| \in J$, $\|f\|(|g|) \in J$. Thus the relation

$$|fg| = |f| \cdot |g| \leq \|f\| \cdot |g|$$

implies that $fg \in J$. Therefore; J is in fact an algebraic ideal of $C(\mathcal{K})$. Now the result follows from [23, Theorem 3.4.1]. The converse and the last part of the Proposition are obvious.

■

Definition 0.1.5 Let A be a nonempty subset of a vector lattice V. Then *the ideal generated* by A is the smallest (with respect to inclusion) ideal that contains A. The ideal generated by an element $x \in V$ will be denoted by V_x and will be referred to as a *principal ideal.*

The above definition reveals that an ideal generated by a nonempty subset A of a vector lattice V is of the form

$$\left\{ x \in V \, : \, \exists x_1, \cdots, x_n \in A \;\; and \;\; \lambda_1, \cdots, \lambda_n \in \mathbb{R}^+ \;\; with \;\; |x| \leq \sum_{i=1}^{n} \lambda_i \, |x_i| \right\}.$$

In particular, for each element x of a vector lattice V

$$V_x = \{ y \in V : \exists \lambda > 0 \ with \ |y| \le \lambda |x| \}.$$

Let V be a vector lattice and let J be an ideal of V. Denote by π the canonical map of V onto the quotient vector space V/J. The relation "$\pi(x) \le \pi(y)$ if and only if there exist elements $x_1 \in x + J$ and $x_2 \in y + J$ satisfying $x_1 \le x_2$ in V" obviously gives an ordering under which V/J is a vector lattice.(cf.[46, Proposition II.2.6]). If V is a normed lattice and J is a closed ideal of V then the following Proposition, which will be used freely in the sequel, holds.(cf. [46, Proposition II.5.4]).

Proposition 0.1.6 *For any closed ideal* J *of the normed vector lattice* V*, the quotient* V/J *is a normed vector lattice under its canonical order and norm;* V/J *is a Banach lattice whenever* V *is.*

Definition 0.1.7 A topological vector space V, which is also a vector lattice, is said to have a *quasi-interior* point, if there exists an element $u \in V_+$ such that V_u is dense in V.

It is known, [46, Section II.6], that if μ is σ-finite then each of the Banach lattices $L_p(\mu)$ $(1 \le p \le \infty)$ possesses quasi-interior points. If μ is not σ-finite then none of the spaces $L_p(\mu)$ $(1 \le p < \infty)$ contains quasi-interior elements. If $\mathcal{X}$ is a completely regular topological space, then for the Banach lattice $C_b(\mathcal{X})$ the functions f for which $\inf_{t \in \mathcal{X}} f(t) > 0$ are quasi-interior elements.

Definitions 0.1.8 A subset A of a vector lattice V is called *order bounded* if it is bounded both from above and from below, (i.e. there exist $x, y \in V$ such that $x \le z \le y$ for all $z \in A$. A vector lattice V is called *Dedekind complete* or, briefly, *complete* whenever every nonempty subset that is bounded from above has a supremum (or, equivalently, whenever every nonempty subset bounded from below has an infimum).

It is known, [30, Chapter 4], that the simplest examples of concrete complete

Banach lattices are the $L_p(\mu)$ spaces with $1 \leq p \leq \infty$ (though $L_1(0,1)$ is not a conjugate space). It is also known that, [30, Section 43], the space $C(\mathcal{K})$ is a complete Banach lattice if and only if the closure of every open set in $\mathcal{K}$ is open. Banach lattices generated by unconditional bases are also complete: the supremum can be taken coordinate wise.

0.2 Positive Operators

Definitions 0.2.1 Let V and W be two ordered vector spaces. A linear mapping $T \in \mathfrak{L}(\mathsf{V},\mathsf{W})$ is said to be a *positive linear mapping* (in symbols: $T \geq 0$) if $T(x) \geq 0$ holds for all $x \geq 0$. It is called *strictly positive* (in symbols: $T \gg 0$) if $Tx > 0$ for all $x > 0$. The linear mappings lying in the vector subspace generated by the positive linear mappings are referred to as *regular linear mappings*. We say that $T \leq S$ whenever $S - T$ is positive.

For a linear mapping $T \in \mathfrak{L}(\mathsf{V},\mathsf{W})$ between two vector lattices we say that its *modulus* $|T|$ exists whenever $T \vee (-T)$ exists (in the sense that $T \vee (-T)$ is the supremum of the set $\{T, -T\}$ in $\mathfrak{L}(\mathsf{V},\mathsf{W})$). In this case we write $|T| = T \vee (-T)$.

For a positive linear mapping $T \in \mathfrak{L}(\mathsf{V},\mathsf{W})$, between two vector lattices, its *null ideal* $\mathcal{N}_T$ is defined by

$$\mathcal{N}_T = \{x \in \mathsf{V} \,:\, T(|x|) = 0\,\}\,.$$

Clearly, T is strictly positive if and only if $\mathcal{N}_T = \{0\}$. Also, $\mathcal{N}_T$ is a closed ideal in V whenever V is a normed vector lattice.

Positive linear mappings between Banach lattices are necessarily continuous. In fact:

Proposition 0.2.2 [46, Theorem II.5.3] *Every positive linear mapping from a Banach lattice to a normed vector lattice is continuous.*

The linear mapping $T : C[0,1] \longrightarrow C[0,1]$, defined by

$$Tf(x) = \int_0^x f(t)\,dt,$$

carries positive functions of $C[0,1]$ into positive functions, and is thus an example of a positive linear mapping. It is well known that many linear mappings between Banach spaces arising in classical functional analysis are in fact positive linear mappings.

Suppose V, W are vector lattices. Then $T \in \mathfrak{L}(\mathsf{V},\mathsf{W})$ is called a *lattice homomorphism* if $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in \mathsf{V}$. Since for such T we have

$$Tx = T(x^+) = T(x \vee 0) = Tx \vee 0 = (Tx)^+ \geq 0 \quad \forall x \in \mathsf{V}_+,$$

every lattice homomorphism is positive. Now if $T \in \mathfrak{L}(\mathsf{V},\mathsf{W})$ is positive, then $|Tx| \leq T|x|$ for all $x \in \mathsf{V}$ as $\pm x \leq |x|$ implies $\pm Tx \leq T|x|$.

If V is a vector lattice and if J is an ideal of V; then (cf. [46, Proposition II.2.6]), the canonical map π of V onto V/J is a lattice homomorphism, and hence the image of each ideal of V under π is an ideal of V/J.(cf [46, Proposition II.2.5]). It can also be easily checked that for any ideal $\hat{\mathsf{J}}$ of V/J; $\pi^{-1}(\hat{\mathsf{J}})$ is an ideal of V. Now if V is a vector lattice as well as a topological vector space with a compatible topology, then π is continuous and open and hence for each closed ideal J_0 of V; $\pi(\mathsf{J}_0)$ is a closed ideal of V/J and for each closed ideal $\hat{\mathsf{J}}$ of V/J; $\pi^{-1}(\hat{\mathsf{J}})$ is a closed ideal of V. These facts are also used freely in the sequel.

Definition 0.2.3 Suppose V and W are vector lattices and $T, S \in \mathfrak{L}(\mathsf{V},\mathsf{W})$. If S is positive and if

$$|T(x)| \leq S(|x|) \quad \forall x \in \mathsf{V},$$

then we say that S *dominates* T. A subset $\mathfrak{C}$ of $\mathfrak{L}(\mathsf{V},\mathsf{W})$ is called *majorized* by S if $S \in \mathfrak{L}(\mathsf{V},\mathsf{W})$ and $T \leq S$ for all $T \in \mathfrak{C}$.

It is obvious that every linear mapping on a Banach lattice which is dominated by a positive linear mapping is continuous, and that a positive linear mapping T is dominated by another positive linear mapping S if and only if $0 \leq T \leq S$.

It is known (cf. [7, Theorem 1.10]) that if $T \in \mathfrak{L}(\mathsf{V}, \mathsf{W})$ is a linear mapping between two vector lattices such that $\sup\{|T(y)| \,:\, |y| \leq x\}$ exists in W for each $x \in \mathsf{V}_+$, then $|T|$ exists and

$$|T|(x) = \sup\{|T(y)| \,:\, |y| \leq x\}$$

holds for all $x \in \mathsf{V}_+$. Therefore, when V is an order complete vector lattice, then a linear mapping $T \in \mathfrak{L}(\mathsf{V})$ is dominated by a positive linear mapping S if and only if T is regular, —T— exists and $|T| \leq S$.

Linear mappings dominated by a compact (weakly compact) positive operator have many remarkable properties. The next two propositions introduce some examples.

Proposition 0.2.4 [7, Theorem 16.14] *If in the diagram of positive operators*

$$\mathsf{E} \xrightarrow{M_1} \mathsf{F} \xrightarrow{M_2} \mathsf{G} \xrightarrow{M_3} \mathsf{H}$$

between Banach lattices each M_i is dominated by a compact positive operator, then $M_3M_2M_1$ is a compact operator.

In the above theorem the number of operators cannot be reduced to two (cf. [7, Example 16.18]).

Proposition 0.2.5 [7, Theorem 17.12] *If in the scheme of positive operators*

$$\mathsf{E} \xrightarrow{S_1} \mathsf{F} \xrightarrow{S_2} \mathsf{G}$$

between Banach lattices each S_i is dominated by a weakly compact positive operator, then S_2S_1 is a weakly compact operator.

The following simple fact about positive operators will be used later. Its proof can be established by using the fact that for each positive operator T on E the relation

$$\|T\| = \sup\{\, \|Tx\| \,:\, x \geq 0,\ \|x\| \leq 1\,\}$$

holds.

Lemma 0.2.6 *Let* E *be any Banach lattice and let* $T, S \in \mathfrak{B}(\mathsf{E})$ *be two positive operators such that* $T \leq S$. *Then,* $\|T\| \leq \|S\|$.

0.3 Examples of Banach Lattices

Most classical Banach spaces over $\mathbb{R}$ are, in fact, Banach lattices on which positive linear mappings appear naturally. Clearly every $L_p(\mu)$ space, $1 \leq p \leq \infty$, and every $C(\mathcal{K})$, is a Banach lattice with the pointwise order. Banach lattices of type $C(\mathcal{K})$, $L_p(\mu)$, and l_p where $1 \leq p \leq \infty$ are among the most important examples of Banach lattices; so it is natural that they are also of special interest in our context.

Definitions 0.3.1 A Banach lattice E is said to be

1. an *abstract L-space* (briefly, *AL-space*) whenever

$$\|x + y\| = \|x\| + \|y\| \quad \forall x, y \in \mathsf{E}_+.$$

2. an *abstract M-space* (briefly, *AM-space*), whenever

$$\|x \vee y\| = \|x\| \vee \|y\| \quad \forall x, y \in \mathsf{E}_+ .$$

If the unit ball of an AM-space contains a largest element e, e is called the *unit* of E.

The vector space $C_b(\mathcal{X})$ of all bounded real valued continuous functions on $\mathcal{X}$, endowed with its canonical order is an AM-space with unit e, where $e(s) = 1$ for all $s \in \mathcal{X}$.

The closed vector sublattice c of all convergent real sequences is a separable AM-space with unit $e = (1, 1, \ldots)$, and c_0, the space of all real sequences that vanish at infinity, is an AM-space without unit.

Another important example of an AM-space is the space $D_\phi(\mathcal{X})$ (cf. [46, Example II.7.2]). Let $\phi : \mathcal{X} \longrightarrow \mathbb{R}$ be a strictly positive function. A continuous real function f on $\mathcal{X}$ is called ϕ-dominated if for each $\epsilon > 0$, there exists a compact subset $\mathcal{K}$ of $\mathcal{X}$ such that $|f(s)| \leq \epsilon\phi(s)$ whenever $s \in \mathcal{X} \setminus \mathcal{K}$. Under the norm

$$\|f\|_\phi = \inf\{\lambda : |f| \leq \lambda\phi\} ,$$

the vector lattice of ϕ-dominated functions is a normed vector lattice denoted by $D_\phi(\mathcal{X})$. If ϕ is an upper semi-continuous function, then $D_\phi(\mathcal{X})$ is an AM-space.

The prime example of an AM-space with unit is $C(\mathcal{K})$; in fact:

Proposition 0.3.2 (cf. [46, Section II.7]) *The Banach lattice* E *is an AM-space with unit if and only if there exists a compact Hausdorff space* $\mathcal{K}$ *such that* E *is isomorphic to* $C(\mathcal{K})$.

Note: In conformity with the usage of the term "isomorphism" (without qualifier) to indicate preservation of all structures implied by a given concept, we understand by an *isomorphism* of an ordered vector space V_1 onto an ordered vector space V_2, a linear bijection $T : \mathsf{V}_1 \longrightarrow \mathsf{V}_2$ such that $x \leq y$ in V_1 if and only if $Tx \leq Ty$ in V_2.

The spaces $L_1(\mu)$ and l_1 are AL-spaces and in fact, as the following Proposition shows, the spaces $L_1(\mu)$ are the most general AL-spaces.

Proposition 0.3.3 (cf. [46, Section II.8]) *For every AL-space* E, *there exists a locally compact Hausdorff space* Ω *and a strictly positive Radon measure* μ *on* Ω *such that* E *is isomorphic with* $L_1(\mu)$.

After an equivalent renorming if necessary, every Banach space with an unconditional basis $\{x_n\}_{n=1}^{\infty}$ is a Banach lattice when the order is defined by

$$\sum_{n=1}^{\infty} a_n x_n \geq 0 \quad \Longleftrightarrow \quad a_n \geq 0, \ \forall n\,.$$

This order is called the order induced by the unconditional basis. Such Banach lattices are called discrete Banach lattices with order continuous norm. Recall that:

Definition 0.3.4 A normed vector lattice V is said to have *order continuous norm* if every order convergent filter in V norm converges, (cf. [46, Section II.1]).

Examples of Banach lattices whose order is not induced by an unconditional basis are $L_p(\mu)$, where $1 \leq p \leq \infty$, μ is not purely atomic (and σ-finite), and $C(\mathcal{K})$, where $\mathcal{K}$ is infinite. Every space $L_p[0,1]$, $1 < p < \infty$, has an unconditional basis, namely

Haar basis, but the natural order in these Banach spaces is completely different from the order induced by the basis.

Some examples of Banach lattices whose positive cones are generated by their unconditional bases are c_0 and l_p, $(1 \leq p < \infty)$. In fact, $\{e_n\}_{n=1}^{\infty}$, where $e_n = (\delta_{nm})$, is an unconditional basis for these Banach lattices and, as one can easily check, the positive cone of c_0 and l_p, $(1 \leq p < \infty)$, are generated by this basis.

If we fix a basis $\{x_n\}_{n=1}^{\infty}$ for a Banach space X, then every linear transformation $T \in \mathfrak{L}(X)$ can be identified in the usual manner with an infinite matrix $[T_{ij}]$. Note that a linear transformation $T \in \mathfrak{L}(X)$ with matrix $[T_{ij}]$ is positive if and only if $T_{ij} \geq 0$ holds for each pair (i, j). If the basis $\{x_n\}_{n=1}^{\infty}$ is also unconditional, then every positive linear transformation is automatically continuous, by Proposition 0.2.2.

Some elementary permanence properties of AL- and AM-spaces are simple to describe. We omit their easy verifications.

Lemma 0.3.5 *Each closed vector sublattice of an AM-space (AL-space) is an AM-space (AL-space).*

Lemma 0.3.6 *The quotient of an AM-space (with unit) over a closed ideal is an AM-space (with unit).*

Lemma 0.3.7 *The quotient of an AL-space over a closed ideal is an AL-space.*

Recall that a Banach space X has the *Dunford-Pettis property* if $x_n \xrightarrow{w} 0$ in X and $x'_n \xrightarrow{w} 0$ in X' imply $\lim x'_n(x_n) = 0$. A non-trivial property of AL- and AM-spaces is the Dunford-Pettis property. A. Grothendieck [15] has shown that AL- and AM-spaces have the Dunford-Pettis property.

Proposition 0.3.8 (cf. [7, Corollary 5.19.9]) *If T is a weakly compact operator on an AL- or an AM-space, then T^2 is a compact operator.*

Definition 0.3.9 Suppose E is an ordered vector space. A nonzero element x_0 of E_+ is called an *atom* if for every $y \in \mathsf{E}$ the relation $0 \leq y \leq x_0$ implies $y = \lambda x_0$ for some $\lambda \geq 0$.

As we shall see, in Section 3.3, every closed ideal J and quotient E/J of a discrete Banach lattice with order continuous norm E contains an atom.

Remark 0.3.10 Suppose V is any l_1-relatively complete normed lattice. (cf. [46, Definition II.1.8]). Then the basic concepts about E can be extended to the complexification of E, (cf. [46, Section II.11]).

Remark 0.3.11 Suppose E_C is the complexification of a Banach lattice E. A linear map $T \in \mathfrak{L}(E_C)$ is positive if T is real (i.e. if the range of T is a subset of $\mathbb{R}$) and its restriction $T_0 = T|_E$ is positive. Now it is not difficult to check that the norm of a positive operator T on a real Banach lattice E does not change whenever we consider T as an operator on the complexification of E. Therefore in cases that we deal with the spectral radius of positive operators we shall assume without loss of any generality that our Banach lattices are all real Banach lattices.

0.4 Invariant Subspaces and Invariant Ideals

A closed subspace M of a Banach space X is said to be *invariant* for $\mathfrak{C} \subseteq \mathfrak{L}(X)$ if $T(M) \subseteq M$ for all $T \in \mathfrak{C}$. The collection of all closed invariant subspaces of $\mathfrak{C}$ will be denoted by $Lat(\mathfrak{C})$. If E is a Banach lattice and if $\mathfrak{C} \subseteq \mathfrak{L}(E)$, then the collection of all closed invariant ideals of $\mathfrak{C}$ will be denoted by $Ilat(\mathfrak{C})$. If $\mathfrak{C}$ consists of a single linear mapping T, then we simply use T instead of $\mathfrak{C}$.

A closed ideal J of a Banach lattice E is said to be *p-hyperinvariant* for a positive operator $T \in \mathfrak{B}(E)$, if J is invariant for all $S \in \{T\}'_+$, where

$$\{T\}'_+ = \{S \in \mathfrak{B}(X) \ : \ S \geq 0 \ and \ ST = TS\}.$$

The concept of a p-hyperinvariant ideal was introduced in [1] under the name of "hyperinvariant ideals".

Definitions 0.4.1 Suppose X is a Banach space and $\mathfrak{C} \subseteq \mathfrak{B}(X)$. We say that $\mathfrak{C}$ is *reducible* if $Lat(\mathfrak{C})$ contains an element other than $\{0\}$ and X; otherwise $\mathfrak{C}$ is called *irreducible.* (If $\mathfrak{C}$ is an irreducible algebra it is also called *transitive.*)

Definitions 0.4.2 Suppose E is a Banach lattice and $\mathfrak{C} \subseteq \mathfrak{B}(\mathsf{E})$. We say that $\mathfrak{C}$ is *decomposable* if $Ilat(\mathfrak{C})$ contains an element other than $\{0\}$ and E; otherwise $\mathfrak{C}$ is called *indecomposable.*

It is well known that each compact operator on a Banach space is reducible. Examples of decomposability results can be found in [1],[2],[3],[4],[5],[6] and [34], where it has been shown that each compact quasinilpotent positive operator on an arbitrary Banach lattice of dimension at least 2 has a nontrivial invariant ideal. More examples of reducibility and decomposability will be discussed in the sequel.

Suppose X is a Banach space, $T \in \mathfrak{L}(X)$, and $M \in Lat(T)$. If we define a map $\hat{T} : X/M \longrightarrow X/M$ by $\hat{T}(x+M) = Tx+M$, then $\hat{T}$ is well defined and $\hat{T} \in \mathfrak{L}(X/M)$. The linear mapping $\hat{T}$ will be called the *compression* of T to X/M.

The following simple but useful facts, concerning the compression of linear linear mappings, can be easily verified.

Lemma 0.4.3 *If* E *is a Banach lattice,* $T \in \mathcal{L}(\mathsf{E})$ *is positive, and* $\mathsf{J} \in Ilat\ (T)$, *then the compression* $\hat{T}$ *of* T *to* E/J *is also a positive linear mapping on* E/J.

Lemma 0.4.4 *Suppose* X *is a Banach space,* $T \in \mathfrak{B}(X)$ *and* $M \in Lat\ (T)$. *Let* $\hat{T}$ *be the compression of* T *to* X/M.

(a) *If* T *is quasinilpotent, then* $\hat{T}$ *is a quasinilpotent linear mapping on* X/M.
(b) *If* T *is compact, then* $\hat{T}$ *is a compact operator on* X/M.
(c) *If* T *is weakly compact, then* $\hat{T}$ *is a weakly compact operator on* X/M.

Lemma 0.4.5 *Suppose* E *is a Banach lattice,* $S, T \in \mathfrak{L}(\mathsf{E})$ *and* $\mathsf{J} \in Ilat(\{S, T\})$. *Let* $\hat{S}$ *and* $\hat{T}$ *be the compression of* S *and* T *to* E/J, *respectively.*

(a) *If* T *is a positive linear mapping that dominates* S, *then* $\hat{T}$ *dominates* $\hat{S}$.
(b) *If* $S \leq T$, *then* $\hat{S} \leq \hat{T}$.

Suppose E is a Banach lattice, $\dim E \geq 2$, $T \in \mathfrak{B}(\mathsf{E})$, and T is a compact quasinilpotent positive operator. As mentioned above, T is decomposable. Now suppose that $J_1, J_2 \in Ilat(T)$ with $J_1 \subseteq J_2$ and $\dim(J_2/J_1) \geq 2$. By Proposition 0.1.6, J_2/J_1 is a Banach lattice and by Lemma 0.4.3 and 0.4.4, the compression $\hat{T}$ of T to J_2/J_1 is a compact quasinilpotent positive operator and hence is decomposable. This property of compact operators enables us to find maximal invariant ideal chains for compact operators. We introduce special terminology for operators that possess maximal invariant ideal chains.

Definition 0.4.6 The collection $\mathfrak{C}$ of operators on a Banach lattice E is *ideal-triangularizable* if there is a maximal ideal chain each of whose members is invariant under all the operators in $\mathfrak{C}$; such an ideal chain will be said to be *triangularizing* for $\mathfrak{C}$. If $\mathfrak{C} = \{T\}$, we simply say that T is ideal-triangularizable.

Remark The above terminology is another version of the well-known triangularizability concept. A collection $\mathfrak{C}$ of operators in a Banach space is said to be *triangularizable* if there exists a maximal subspace chain each of whose members is invariant under all the operators in $\mathfrak{C}$.

The concept of *quasinilpotency at a point*, which was first introduced in [2], enables us to derive many decomposability results under weaker conditions. This has been observed by Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in [2] and [3]. We also use this concept in Chapter 2 to extend Theorem 4.3 of [1]. Recall that:

Definition 0.4.7 An operator $T \in \mathfrak{B}(X)$ is quasinilpotent at a point x_0 whenever

$$\lim_{n \to \infty} \|T^n x_0\|^{1/n} = 0 .$$

There might be many operators T on a given Banach lattice E that leaves all bands of E invariant, i.e., $T(\mathsf{B}) \subseteq \mathsf{B}$ holds for each band B of E. Such operators are called *band preserving*. If a band preserving operator T on E is also an *order bounded* operator, i.e., $T(A)$ is an order bounded set for each order bounded subset A of E, then T is called an *orthomorphism*. The set of all orthomorphisms on E is denoted by $Orth(\mathsf{E})$.

Definition 0.4.8 The order ideal $\mathfrak{Z}(\mathsf{E})$ of $Orth(\mathsf{E})$ generated by the identity operator I is called the *center* of E, i.e. $\pi \in Orth(\mathsf{E}) \cap \mathfrak{Z}(\mathsf{E})$ if and only if $|\pi| \leq nI$ for some $n \in \mathbb{N}$.

For examples and properties of band preserving operators and orthomorphisms we refer the reader to [7, Sections 1.1 and 2.8]. For properties of the center of a Banach lattice we refer the reader to [48, Section 20.140].

0.5 Operator Ideals

We start with the fundamental concept of an operator ideal that was introduced by A. Pietsch [35, Chapter 1]. We then introduce some special operator ideals which play an important role in deriving reducibility and decomposability results for collections of operators on Banach spaces or Banach lattices.In what follows the class of all operators between arbitrary Banach spaces will be denoted by $\mathfrak{B}$.

Definition 0.5.1 An *operator ideal* $\mathcal{I}$ is a subclass of $\mathfrak{B}$ satisfying the following conditions:

(i) $I_X \in \mathcal{I}$ for every 1-dimensional Banach space X.

For every pair of Banach spaces X, Y the set

$$\mathcal{I}(X,Y) = \mathcal{I} \cap \mathfrak{B}(X,Y),$$

the so-called (X,Y)-*component* of $\mathcal{I}$, has the following properties:

(ii) If $T_1, T_2 \in \mathcal{I}(X,Y)$ then $T_1 + T_2 \in \mathcal{I}(X,Y)$.

(iii) If $T \in \mathfrak{B}(X_0, X)$, $S \in \mathcal{I}(X,Y)$, and $R \in \mathfrak{B}(Y, Y_0)$, then $RST \in \mathcal{I}(X_0, Y_0)$.

The class of all finite-rank operators $\mathfrak{F}$, the class of all compact operators $\mathfrak{K}$, and the class of all weakly compact operators $\mathfrak{W}$ are three examples of operator ideals. Note that the components $\mathfrak{F}(X) = \mathfrak{F} \cap \mathfrak{B}(X)$, $\mathfrak{K}(X) = \mathfrak{K} \cap \mathfrak{B}(X)$, and $\mathfrak{W}(X) = \mathfrak{W} \cap \mathfrak{B}(X)$, are the familiar *algebraic ideals* of $\mathfrak{B}(X)$. In general, for any operator ideal $\mathcal{I}$, the component $\mathcal{I}(X) = \mathcal{I} \cap \mathfrak{B}(X)$ is an algebraic two-sided ideal of $\mathfrak{B}(X)$.

Definition 0.5.2 A map $\mathfrak{q}$ from an operator ideal $\mathcal{I}$ into $\mathbb{R}^+$ is called a *quasi-norm* if the following conditions are satisfied:

(i) $\mathfrak{q}(I_X) = 1$ for all X.

(ii) There exists a constant $\kappa \geq 1$ such that

$$\mathfrak{q}(T_1 + T_2) \leq \kappa[\,\mathfrak{q}(T_1) + \mathfrak{q}(T_2)\,] \quad \textit{for} \quad T_1, T_2 \in \mathcal{I}(X, Y) \textit{ and for all } X, Y .$$

(iii) If $T \in \mathfrak{B}(X_0, X)$, $S \in \mathcal{I}(X, Y)$, and $R \in \mathfrak{B}(Y, Y_0)$, then

$$\mathfrak{q}(RST) \leq \|R\|\mathfrak{q}(S)\|T\| .$$

A *quasi-normed operator ideal* $[\mathcal{I}, \mathfrak{q}]$ is an operator ideal $\mathcal{I}$ with a quasi-norm $\mathfrak{q}$ such that all linear topological Hausdorff spaces $[\mathcal{I}(X, Y), \mathfrak{q}]$ are complete. If $\kappa = 1$, then $[\mathcal{I}, \mathfrak{q}]$ is called a *normed operator ideal.*

Examples of quasi-normed operator ideals are introduced in the sequel. The next propositions reveal some properties of quasi-normed operator ideals that we need later.

Proposition 0.5.3 ([35, proposition 6.1.4]) *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal. Then $\|T\| \leq \mathfrak{q}(T)$ for all $T \in \mathcal{I}$.*

Remarks Suppose $[\mathcal{I}, \mathfrak{q}]$ is a quasi-normed operator ideal. Definition 0.5.1 shows that for any Banach space X, the component $\mathcal{I}(X)$ is closed under the addition and composition of operators in $\mathfrak{B}(X)$, and hence it is a subalgebra of $\mathfrak{B}(X)$. Now by definition 0.5.2(iii)

$$\mathfrak{q}(ST) = \mathfrak{q}(STI_X) \leq \|S\|\mathfrak{q}(T)\|I_X\| = \|S\|\mathfrak{q}(T)$$

for all $S \in \mathfrak{B}(X)$ and $T \in \mathcal{I}(X)$. Thus

$$\mathfrak{q}(ST) \leq \mathfrak{q}(S)\mathfrak{q}(T)$$

for all $S, T \in \mathcal{I}(X)$ by proposition 0.5.3. Therefore; $\mathcal{I}(X)$ is a *quasi-Banach algebra* under the quasi-norm $\mathfrak{q}$.

We can also conclude from Theorem 1.2.2 of [35] that $\mathfrak{F}(X) \subseteq \mathcal{I}(X)$ for any Banach space X.

The following notation is convenient in the sequel. If X^* is the dual of a Banach space X, $x \in X$, and $\phi \in X^*$, we write $\langle x, \phi \rangle$ for $\phi(x)$. The operator A of rank one on X defined by $Ax = \langle x, \phi \rangle u$, where u and ϕ are fixed members of X and X^* respectively, is denoted by $u \otimes \phi$. Every operator of rank one on X is of this form and $(u \otimes \phi)^* = \phi \otimes \hat{u}$, where $\hat{u}$ denotes the image of u under the natural injection of X into X^{**}. Also $\|u \otimes \phi\| = \|u\| \cdot \|\phi\|$.

Proposition 0.5.4 ([35, Proposition 6.1.5.]) *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal. Then $\mathfrak{q}(x^* \otimes y) = \|x^*\| \cdot \|y\|$ for all $x^* \in X^*$ and $y \in Y$*

Since every operator $F \in \mathfrak{B}(X, Y)$, of rank one, is of the form $F = x^* \otimes y$, where y and x^* are fixed members of Y and X^* respectively, and since $\|F\| = \|x^*\| \cdot \|y\|$, we can state:

Corollary 0.5.5 *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed ideal. Then for each rank-one operator $F \in \mathcal{I}(X, Y)$ we have $\mathfrak{q}(F) = \|F\|$.*

0.6 Integral Operators

Consider a measure space $(\mathcal{X}, \Sigma, \mu)$ and let $M(\mathcal{X}, \mu)$ be the corresponding vector lattice of all real measurable functions. The product measure of μ and μ in the Cartesian product $\mathcal{X} \times \mathcal{X}$ is denoted by $\mu \times \mu$. The product measure is first defined on the semiring Δ of all sets $A \times B$ with $A, B \in \Sigma$ by

$$(\mu \times \mu)(A \times B) = \mu(A)\, \mu(B),$$

(with the convention that $0 \cdot \infty = \infty \cdot 0 = 0$), and the resulting measure on the semiring Δ is then extended by means of the Carathéodory extension procedure.

Let $K(x, y)$ be a real valued $(\mu \times \mu)$-measurable function on $\mathcal{X} \times \mathcal{X}$. For any $f \in M(\mathcal{X}, \mu)$ the function $K(x, y) f(y)$ is $(\mu \times \mu)$-measurable, which implies by Fubini's theorem that for μ-almost every $x \in \mathcal{X}$ the function $K(x, y) f(y)$ is μ-measurable as a function of y. It follows that

$$h_f(x) = \int_{\mathcal{X}} |K(x,y)f(y)|\, d\mu(y)$$

is well defined for these values of x (the value $+\infty$ for $h_f(x)$ is permitted) and by Fubini's theorem the function h_f is μ-measurable on $\mathcal{X}$. The set of all $f \in M(\mathcal{X}, \mu)$ for which the corresponding h_f is finite almost everywhere is called the *domain* of K. and will be denoted by *dom* (K). It is clear that *dom* (K) is an ideal of $M(\mathcal{X}, \mu)$. For $f \in$ *dom* (K) the corresponding function h_f is finite μ-almost everywhere on $\mathcal{X}$, and so

$$g_f(x) = \int_{\mathcal{X}} K(x,y)f(y)\, d\mu(y)$$

is also finite μ-almost everywhere on $\mathcal{X}$. The function g_f is μ-measurable as

$$g_f(x) = \int_{\mathcal{X}} (K(x,y)f(y))^{+}\, d\mu(y) - \int_{\mathcal{X}} (K(x,y)f(y))^{-}\, d\mu(y),$$

and the terms on the right are μ-measurable by Fubini's theorem. Hence the map $f \longrightarrow g_f$ defines a linear mapping T with *dom* (K) as its domain and with range in $M(\mathcal{X}, \mu)$. The linear mapping T is called an *integral operator*. The function K is called the *kernel* of T and is denoted by K_T.

If T is an integral operator with kernel K_T and if J is an ideal of $M(\mathcal{X}, \mu)$, then T is said to be an integral operator on J if J is included in *dom* (K) and T maps J into J.

Suppose ν is a σ-finite positive regular Borel measure on Ω. Let $C_b(\Omega)$ be the Banach lattice of all bounded real valued continuous functions on Ω and suppose K is a ν-measurable function on $\Omega \times \Omega$ such that for each $f \in C_b(\Omega)$ the function Tf defined by

$$Tf(x) = \int_{\Omega} K(x,y)f(y)\, d\nu(y) \qquad x \in \Omega$$

belongs to $C_b(\Omega)$. Then T is an operator on $C_b(\Omega)$ that is also called an integral operator. As an example, let $\Omega = [0,1]$, ν be the Lebesgue measure on $[0,1]$, and

$$K(x,y) = \begin{cases} 0 & if \quad y > x \\ 1 & if \quad y \leq x \end{cases}.$$

Then, $C_b[0,1] = C[0,1]$,

$$Tf(x) = \int_\Omega K(x,y)f(y)\,d\nu(y) = \int_0^x f(y)\,d\nu(y) \quad \forall x \in [0,1],$$

and $Tf \in C_b(\Omega)$. This operator T is known as the *Volterra operator* on $C[0,1]$.

Remark 0.6.1 According to [22, Section 12] there are conditions under which certain class of operators on $C(\Omega)$ can be represented as integral operators. As an example, it is known that each locally compact and locally continuous operator on $C(\Omega)$ can be represented as an integral operator by way of a regular measure (cf. [22, Theorem 12.2]). However, it is not known whether we can find a single regular measure by way of which a semigroup of such operators can be represented simultaneously as integrals.

Chapter 1

Reducibility of Operators on Banach Spaces

There are several known theorems giving sufficient conditions under which a semigroup of compact or quasinilpotent operators, on a Hilbert space, is reducible. As an example, [38, Theorem 2] states that *if every member of a semigroup* $\mathcal{S}$, *of operators on a Hilbert space, is a nonnegative scalar multiple of a compact idempotent and* $r(AB) \leq r(A)r(B)$ *for every pair of* A *and* B *in* $\mathcal{S}$, *then* $\mathcal{S}$ *is reducible.* Another example is [33, Theorem 1], which states that *if* $\mathcal{S}$ *is a semigroup of quasinilpotent operators, on a Hilbert space, and if* $\mathcal{S}$ *contains an operator other than* 0 *in some* C_p, $1 \leq p < \infty$, *class, then* $\mathcal{S}$ *is reducible.*

Our main results in this chapter concern semigroups of compact or quasinilpotent operators on a more general Banach space. We first derive an extension of [39, Theorem 6.1] for a reflexive Banach space X. Using this, we further extend some results of [33] and [38]. Finally we use these facts to establish new reducibility and triangularizability results on Banach Spaces.

1.1 Trace Ideals

Definition 1.1.1 Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal. For a Banach space X we call $\mathcal{I}(X)$ a *trace ideal*, provided

(i) the set of finite-rank operators $\mathfrak{F}(X)$ is q-dense in $\mathcal{I}(X)$, and

(ii) whenever $\{\lambda_n\}$ is the sequence of nonzero eigenvalues of $T \in \mathfrak{F}(X)$, counted according to their multiplicities, the trace

$$tr : \mathfrak{F}(X) \longrightarrow \mathbb{C}$$

defined by $tr(T) = \Sigma_n \lambda_n(T)$, is a q-continuous linear functional on $\mathfrak{F}(X)$.

We call $\mathcal{I}(X)$ a *C-trace ideal* if it is a trace ideal and the functional tr on $\mathcal{I}(X)$ is well-defined as the q-continuous linear extension of the trace on $\mathfrak{F}(X)$.

We call $[\mathcal{I}, \mathrm{q}]$ a *C-trace quasi-normed operator ideal* if $\mathcal{I}(X)$ is a C-trace ideal for every Banach space X.

To present examples of quasi-normed operator ideals, C-trace quasi-normed operator ideals, and C-trace ideals, we should introduce certain definitions and remarks. In this paper we confine ourselves to those examples which are used in the sequel and which collapse to the trace class C_1 when X is a Hilbert space. Recall that whenever H is a Hilbert space then the *trace class* C_1 is the set of all operators T in $\mathfrak{B}(H)$ which satisfy the following condition: *for each orthonormal system* $\{ v_k : k \in \mathbb{N}\}$ *in* H,

$$\sum_{k\in\mathbb{N}} \mid \langle Tv_k, v_k\rangle \mid < \infty .$$

Recall also that for each $T \in C_1$ the trace of T, denoted by $tr(T)$, is defined by the equation

$$tr(T) = \sum_n \lambda_n ,$$

where $\{\lambda_n\}$ is the sequence of nonzero eigenvalues of T, counted according to their multiplicities. It is known that:

(a) C_1 is a Banach space with the norm $|\cdot|_1$ defined by

$$\mid T \mid_1 = \sum_n \mid \lambda_n \mid \qquad T \in C_1 .$$

(b) the class $\mathfrak{F}(H)$ of all finite-rank operators on H is $|\cdot|_1$-dense in C_1,

(c) the functional tr on C_1 is well defined as the $|\cdot|_1$-continuous linear extension of the trace on $\mathfrak{F}(H)$, and

(d) the only irreducible $|\cdot|_1$-closed subalgebra $\mathfrak{A}$ of C_1 is C_1 itself.

The facts (a), (b), and (c) can be found in [41], and the fact (d) is a special case of Theorem 6.1 of [39].

Definition 1.1.2 An operator $T \in \mathfrak{B}(X, Y)$ is called *nuclear* if there is a sequence $\{x_n^*\}_n^\infty$ in X^* and a sequence $\{y_n\}_n^\infty$ in Y such that

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n \quad with \quad \sum_{n=1}^{\infty} \|x_n^*\| \cdot \|y_n\| < \infty . \qquad (1)$$

The set of all nuclear operators is denoted by $\mathfrak{N}$. For each $T \in \mathfrak{N}$ we define

$$\nu(T) = \inf \sum_{n=1}^{\infty} \|x_n^*\| \cdot \|y_n\| ,$$

where the infimum is taken over all so-called *nuclear representations* described above.

Proposition 1.1.3 ([35, Proposition 6.3.2]) *The function ν defines a norm on $\mathfrak{N}(X, Y)$ and with this norm $[\mathfrak{N}, \nu]$ is a normed operator ideal.*

The next Proposition is an easy consequence of Definition 1.1.2.

Proposition 1.1.4 *For any Banach space X, $\overline{\mathfrak{F}(X)}^{\nu} = \mathfrak{N}(X)$, where $\mathfrak{N}(X) = \mathfrak{N} \cap \mathfrak{B}(X)$.*

If $T \in \mathfrak{F}(X)$, with representation

$$T = \sum_{i=1}^{n} x_i^* \otimes x_i : \quad x_i^* \in X^* , \quad x_i \in X ,$$

the expression $\Sigma_{i=1}^n x_i^*(x_i)$ turns out to be independent of the particular representation of T and thus the trace of T can be defined as

$$tr(T) = \sum_{i=1}^{n} x_i^*(x_i) .$$

As is well known, this "matrix trace" is equal to the "spectral trace",

$$tr(T) = \sum_n \lambda_n(T).$$

For (infinite-rank) nuclear operators T with representation (1), the infinite sum $\Sigma_{i=1}^{\infty} x_i^*(x_i)$ converges absolutely. Unfortunately, this sum will not define a notion of trace, since its value may depend on the particular (infinite) representation of T if the space X does not satisfy the "*approximation property*".

Definition 1.1.5 A Banach space X has the *approximation property* (abbreviated *A.P.*), if for every compact subset Y of X and every $\epsilon > 0$ there is a finite-rank operator $F \in \mathfrak{F}(X)$ such that $\|Fx - x\| < \epsilon$ for all $x \in Y$.

A result which goes back to the beginnings of functional analysis asserts that the compact operators on a Hilbert space are exactly those operators which are limits in norm of operators of finite rank. This assertion is also true for Banach spaces with *A.P.*. In fact:

Proposition 1.1.6 *A Banach space X has A.P. if and only if $\overline{\mathfrak{F}(X)} = \mathfrak{K}(X)$, where the closure is taken in the norm topology.*

For a proof of Proposition 1.1.6 refer to, *e.g.*, [28, Theorem 1.e.4]. Another theorem which we apply in the sequel and which requires *A.P.* is the following classical result of Grothendieck [16, I.5.1].

Proposition 1.1.7 *Let X be a Banach space. Then the following are equivalent:*

(a) *X has* A.P.

(b) $|\, tr(T)\, | \leq \nu(T)$ *for all $T \in \mathfrak{F}(X)$, i.e.*

$$tr : \mathfrak{F}(X) \longrightarrow \mathbb{C}$$

is a ν-continuous linear functional.

Corollary 1.1.8 *If the Banach space X has* A.P. *then $\mathfrak{N}(X)$ is a C-trace ideal.*

By Enflo's famous counterexample [13] there are spaces without $A.P.$ In fact, the construction of Davie given in [28] can be used to define a nuclear operator T on c_0 for which trace is well defined, $tr(T) = 1$, and $T^2 = 0$. Thus all the eigenvalues of T are zero and hence

$$1 = tr(T) \neq \sum_{n=1}^{\infty} \lambda_n(T) = 0\,,$$

where $\{\lambda_n(T)\}_{n=1}^{\infty}$ is the sequence of eigenvalues of T. However, there are suitable ideals of operators on a general Banach space, satisfying a compactness criterion, for which the linear functional $\Sigma_{n=1}^{\infty}\lambda_n(\cdot)$ is continuous. Some examples of these ideals are introduced in this section.

Definition 1.1.9 Let $T \in \mathfrak{B}(X,Y)$. The *approximation numbers* $a_n(T)$, $n = 1, 2, \cdots$, of T are defined by

$$a_n(T) = \inf\{\, \|T - S\| \;:\; S \in \mathfrak{B}(X,Y) \quad and \quad rank(S) < n\,\}\,.$$

Let S_1^a denote the collection of all operators T between arbitrary Banach spaces for which $(a_n(T)) \in l_1$ and for such a T define

$$s_1(T) = \|(a_n(T))\|_1\,.$$

Then the following holds (cf. [26. Section 1.d])

Proposition 1.1.10 *The function s_1 defines a quasi-norm on S_1^a and with this quasi-norm $[S_1^a, s_1]$ is a quasi-normed operator ideal.*

Definition 1.1.11 Let $T \in \mathfrak{B}(X,Y)$. The operator T is 2-*summing* if there is a constant $c > 0$ such that for any finite sequence $s = (x_i)_{i=1}^n$ in X

$$\Big(\sum_{i=1}^{n} \|T(x_i)\|^2\Big)^{1/2} \leq c\,\varepsilon_2(s)\,,$$

where

$$\varepsilon_2(s) = \sup\Big\{\Big(\sum_{i=1}^{n} |\, x^*(x_i)\,|^2\Big)^{1/2} \;:\; x^* \in X^* \quad and \quad \|x^*\| = 1\Big\}\,.$$

The smallest constant c is denoted by $\pi_2(T)$ and the collection of all 2-summing operators is denoted by Π_2.

Proposition 1.1.12 (a) *The function π_2 defines a norm on Π_2 and with this norm $[\Pi_2, \pi_2]$ is a normed operator ideal.*

(b) *$[\Pi_2^2, \pi_2^2]$ is a quasi-normed operator ideal, where for each X, Y, $\Pi_2^2(X, Y)$ is the collection of all $T \in \mathfrak{B}(X, Y)$ for which there exist Z, $R \in \Pi_2(X, Z)$, and $S \in \Pi_2(Z, Y)$ such that $T = SR$ and for each $T \in \Pi_2^2$*

$$\pi_2^2(T) = \inf\{\, \pi_2(S)\pi_2(R) \,:\, T = SR \,\}$$

The proof of Proposition 1.1.12 can be found in [26, Section 1.d]. In the following theorem it is shown that the functional trace on $S_1^a(X)$ and $\Pi_2^2(X)$ is well defined and well behaved.

Proposition 1.1.13 ([26, Section 4.a) *The ideals S_1^a and Π_2^2 are trace ideals.*

As we shall see, the existence of a $\mathfrak{q}$-continuous linear functional on $\mathcal{I}(X)$, where $[\mathcal{I}, \mathfrak{q}]$ is a quasi-normed operator ideal, enables us to derive some reducibility results for a semigroup of operators on a Banach space. We have already seen that the functional trace is ν-continuous on $\mathfrak{N}(X)$, whenever X is a Banach space with $A.P.$ (see Corollary 1.1.8). This means that $\mathfrak{N}(X)$ is a C-trace ideal whenever X has $A.P.$. Another application of Proposition 1.1.7, shows that $[\mathfrak{N}, \nu]$ is not a C-trace quasi-normed operator ideal as there are Banach spaces without $A.P.$. However, there are, at least, two examples of such operator ideals as shown in the next Theorem.

Theorem 1.1.14 ([26, Theorem 4.a.6]) *Let X be any Banach space and let $T \in S_1^a(X)$ or $T \in \Pi_2^2(X)$. Then the trace of T is well-defined as the s_1-continuous or π_2^2-continuous (linear) extension of the trace on the operators of finite rank and the trace formula*

$$tr(T) = \sum_{i=1}^{\infty} \lambda_i(T)$$

holds.

The following Lemma is useful in deriving some triangularizability results concerning semigroups of operators in $\mathcal{I}(X)$, where $\mathcal{I}$ is an operator ideal.

Lemma 1.1.15 *Suppose $\mathcal{I}$ is an operator ideal, $T \in \mathcal{I}(X)$, and $M \in Lat(T)$. Then*

(i) $T|_M \in \mathcal{I}(M)$.

(ii) *If X is a reflexive Banach space, then $\hat{T} \in \mathcal{I}(X/M)$, where $\hat{T}$ is the compression of T to X/M.*

Proof: (i) Consider the operator $S = I_X \circ T \circ j$, where $j : M \longrightarrow X$ is the inclusion map. We have $S \in \mathcal{I}(M, X)$ by the condition (iii) of definition 0.5.1, and hence $S \in \mathcal{I}$. On the other hand, $S = T|_M$ and $T|_M \in \mathfrak{B}(M)$. Thus

$$T|_M \in \mathcal{I} \cap \mathfrak{B}(M) = \mathcal{I}(M).$$

(ii) We know that

$$(X/M)^* = M^{\perp} \subseteq X^*,$$

(see, *e.g.*, [11, Theorem III.10.2]). We also know that X/M is a reflexive Banach space as X is reflexive. These facts and an application of the Hahn-Banach Theorem imply that

$$X/M = (X/M)^{**} = (M^{\perp})^* \subseteq X^{**} = X.$$

Let j be the induced inclusion map from X/M to X and let Q be the canonical map from X to X/M. Then $QTj = \hat{T}$, and $QTj \in \mathcal{I}(X/M)$ by the condition (iii) of Definition 0.5.1.

■

1.2 Irreducible Algebras Containing Operators of Finite Rank

Barnes [9] has proved that *a uniformly closed, irreducible algebra of operators on* *a reflexive Banach space that contains a non-zero operator of finite rank necessarily contains all operators of finite rank.* In fact he proved the following:

Proposition 1.2.1 *Suppose X is a reflexive Banach space and $\mathfrak{A}$ is an irreducible subalgebra of $\mathfrak{B}(X)$ such that:*

(i) *$\mathfrak{A}$ contains a finite-rank operator other than 0, and*

(ii) *$\mathfrak{A}$ is a Banach algebra under some norm $|\cdot|$ with the property that $\|A\| \leq |A|$ for all $A \in \mathfrak{A}$ and $\|F\| = |F|$ for operators in $\mathfrak{A}$ of rank one.*

Then $\mathfrak{A}$ contains all finite-rank operators

This case of Barnes's theorem was obtained in [32] in a fairly straightforward manner avoiding any algebraic machinery. We use the method used in [32] to establish a new version of Proposition 1.2.1 concerning quasinormed operator ideals $\mathcal{I}(X)$, where X is a reflexive Banach space.

Lemma 1.2.2 *Suppose X is a Banach space and $\mathcal{I}$ is a quasi-normed operator ideal under the quasi-norm $\mathfrak{q}$. Let $\mathfrak{A}$ be an irreducible $\mathfrak{q}$-closed subalgebra of $\mathcal{I}(X)$. If $\mathfrak{A}$ contains an operator of the form $x_0 \odot \phi_0$, $0 \neq x_0 \in X$, $\phi_0 \in X^*$, then $\mathfrak{A}$ contains $x \odot \phi_0$ for all $x \in X$.*

Proof: Fix $x \in X$. Since $\mathfrak{A}$ is irreducible, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathfrak{A}$ with $\lim_{n\to\infty} \|A_n x_0 - x\| = 0$.

Since $(A_n x_0) \odot \phi_0 = A_n(x_0 \odot \phi_0) \in \mathfrak{A}$ and since, by Corollary 0.5.5, $\mathfrak{q}(F) = \|F\|$ for all rank-one operators in $\mathfrak{A}$, we have:

$$\mathfrak{q}\left[(A_n x_0) \odot \phi_0 - (A_m x_0) \odot \phi_0\right] = \mathfrak{q}[(A_n x_0 - A_m x_0) \odot \phi_0]$$

$$= \|(A_n x_0 - A_m x_0) \odot \phi_0\| = \|A_n x_0 - A_m x_0\| \cdot \|\phi_0\| .$$

Since $\{A_n x_0\}_{n=1}^{\infty}$ is a Cauchy sequence in X, it follows that $\{(A_n x_0) \odot \phi_0\}_{n=1}^{\infty}$ is a $\mathfrak{q}$-Cauchy sequence in $\mathfrak{A}$; hence there exists $A \in \mathfrak{A}$ such that

$$\lim_{n\to\infty} \mathfrak{q}[(A_n x_0) \odot \phi_0 - A] = 0 ,$$

as $\mathfrak{A}$ is q-closed and $\mathcal{I}(X)$ is q-complete. By Proposition 0.5.3 we have

$$\lim_{n\to\infty} \|(A_n x_0)\otimes\phi_0 - A\| = 0\,.$$

which implies $A = x \otimes \phi_0$.

■

Lemma 1.2.3 *Let X be a Banach space and $\mathcal{I}$ be a quasi-normed operator ideal under the quasi-norm $\mathfrak{q}$. Assume that $\mathfrak{A}$ is a $\mathfrak{q}$-closed subalgebra of $\mathcal{I}(X)$ which contains a nonzero operator of finite rank. If $\mathfrak{A}^*$ is irreducible, then $\mathfrak{A}$ contains every operator of finite rank.*

Proof: First observe that the irreducibility of $\mathfrak{A}^*$ implies the irreducibility of $\mathfrak{A}$. Now suppose F is a nonzero finite-rank operator in $\mathfrak{A}$. Then $F\mathfrak{A} = \{FA : A \in \mathfrak{A}\}$ is a subalgebra of $\mathfrak{A}$ whose members leave $M = \{Fx : x \in X\}$ invariant. The restriction $F\mathfrak{A}\,|_M$ is obviously an irreducible algebra of operators on the finite-dimensional space M; and hence, by Burnside's theorem (cf. [21])

$$F\mathfrak{A}\,|_M = \mathfrak{B}(M)\,.$$

Thus there exists $A \in \mathfrak{A}$ such that $FA\,|_M$ is of rank one. Then FAF is in $\mathfrak{A}$ and has rank equal to one.

Let $x_0 \in X$ and $\phi_0 \in X^*$ be such that $FAF = x_0 \otimes \phi_0$. Then, $x \otimes \phi_0 \in \mathfrak{A}$ for all $x \in X$, by Lemma 1.2.2. We show that $x \otimes \phi \in \mathfrak{A}$ for all $\phi \in X^*$.

Define $\Psi(A^*) = \mathfrak{q}(A)$ for all $A \in \mathfrak{A}$. It can be easily seen that Ψ is a quasi-norm on $\mathfrak{A}^*$ and with this quasi-norm $\mathfrak{A}^*$ is a quasi-Banach algebra. Also since $\|A^*\| = \|A\|$,

$$\Psi(F^*) = \|F^*\| \qquad \text{for each rank-one operator} \quad F^* \in \mathfrak{A}^*$$

and

$$\|A^*\| \le \Psi(A^*) \qquad \text{for each} \quad A^* \in \mathfrak{A}^*\,.$$

by Corollary 0.5.5 and Proposition 0.5.3., respectively. Thus by the same argument as in Lemma 1.2.2 we deduce that, since $\phi_0 \otimes \hat{x} \in \mathfrak{A}^*$, $\phi \otimes \hat{x}$ is in $\mathfrak{A}^*$ for all $\phi \in X^*$.

Hence $x \odot \phi \in \mathfrak{A}$ for all $x \in X$ and $\phi \in X^*$. This shows that all rank-one operators are in $\mathfrak{A}$ and consequently $\mathfrak{A}$ contains all operators of finite rank, as every operator of finite rank is a finite sum of operators of rank one.

■

Theorem 1.2.4 *Let X be a reflexive Banach space and $\mathcal{I}$ be a quasi-normed operator ideal under the quasi-norm $\mathfrak{q}$. Assume that $\mathfrak{A}$ is a $\mathfrak{q}$-closed irreducible subalgebra of $\mathcal{I}(X)$ which contains a nonzero operator of finite rank. Then $\mathfrak{A}$ contains all operators of finite rank.*

Proof: It is known that $\mathfrak{A}^*$ is irreducible whenever X is a reflexive Banach space and $\mathfrak{A}$ is an irreducible algebra. Now use Lemma 1.2.3.

■

To prove the main result of this section we recall two well-known results that are also crucial in the sequel. We state them without proof and give a reference.

Proposition 1.2.5 **(Lomonosov's Lemma** [29]**)** *If $\mathfrak{A}$ is a (non-zero) irreducible subalgebra of $\mathfrak{B}(X)$ and K_0 is any compact operator other than 0, then there exists an $A \in \mathfrak{A}$ such that 1 is in the point spectrum of $K = AK_0$.*

Proposition 1.2.6 [20] *Suppose $(X, \|\cdot\|)$ is a Banach space and $T \in \mathfrak{B}(X)$. If $r(T) < 1$, then there is a norm $|||\cdot|||$ on X which is equivalent to $\|\cdot\|$ such that $|||T||| < 1$.*

Note For the case of Hilbert spaces we have the following version of Proposition 1.2.6 which was proved by Rota [42]. *Every bounded linear operator on a Hilbert space whose spectrum lies in the interior of the unit disc is similar to an operator of norm less than one (that is, to a proper contraction).*

The following theorem is the main result of this chapter and it extends Theorem 6.1 of [39].

Theorem 1.2.7 *Suppose $[\mathcal{I}, \mathfrak{q}]$ is a quasi-normed operator ideal, X is a reflexive*

Banach space, and the set of all finite-rank operators $\mathfrak{F}(X)$ is q*-dense in $\mathcal{I}(X)$. Then the only irreducible* q*-closed subalgebra $\mathfrak{A}$ of $\mathcal{I}(X)$ is $\mathcal{I}(X)$ itself.*

Proof: First observe that all of the elements of $\mathcal{I}(X)$ are compact by Proposition 0.5.3. Therefore there is a compact operator $K \in \mathfrak{A}$ with 1 in its point spectrum by Proposition 1.2.5. With no loss of generality assume $r(K) = 1$. Let $\sigma(K) = \sigma_1 \cup \sigma_2$, where

$$\sigma_1 = \{ \lambda \in \sigma(K) : | \lambda | = 1 \}$$

and $\sigma_2 = \sigma(K) \setminus \sigma_1$. Since K is compact σ_1 is finite and σ_2 is closed. We also know that both σ_1 and σ_2 are nonempty, as $1 \in \sigma_1$ and $0 \in \sigma_2$. Thus, by the Riesz Decomposition Theorem (see, e.g., [40, Theorem 2.10]) there are two projections P_1 and P_2 with the following property:

If M_1 and M_2 are the ranges of P_1 and P_2 respectively, then:

(i) M_1 is finite dimensional.

(ii) M_1 and M_2 are complementary and they are invariant under K.

(iii) if $K_1 = K|_{M_1}$ and $K_2 = K|_{M_2}$, then $\sigma(K_1) = \sigma_1$, and $\sigma(K_2) = \sigma_2$.

(iv) after applying a similarity $K_1 = U + N$, with U an $m \times m$ unitary matrix and N an $m \times m$ nilpotent matrix commuting with U.

Since X is a Banach space, if we define a norm on $M_1 \dotplus M_2$ by

$$\|(m_1 \dotplus m_2)\| = \|m_1\| + \|m_2\| \qquad m_1 \dotplus m_2 \in M_1 \dotplus M_2 .$$

then the Inverse Mapping Theorem implies that the linear mapping

$$L : M_1 \dotplus M_2 \longrightarrow X$$

defined by $L(m_1 \dotplus m_2) = m_1 + m_2$ is a homeomorphism, and hence X and $M_1 \dotplus M_2$ are isomorphic. Thus, we can work with $M_1 \dotplus M_2$ instead of X and $K_1 \dotplus K_2$ instead of K.

Since $K \in \mathcal{I}(X)$, K_1 and K_2 are in $\mathcal{I}(M_1)$ and $\mathcal{I}(M_2)$, respectively, by Lemma 1.1.15 (i). We can also verify that $\mathfrak{q}(A_1 \dotplus 0) = \mathfrak{q}(A_1)$ and $\mathfrak{q}(0 \dotplus A_2) = \mathfrak{q}(A_2)$ for all $A_1 \in \mathcal{I}(M_1)$ and $A_2 \in \mathcal{I}(M_2)$.

Consider the operator K_2. Since $\sigma(K_2) = \sigma_2$, $r(K_2) < 1$, and hence, by Proposition 1.1.6, there is an equivalent norm $|||\cdot|||$ on M_2 such that $|||K_2||| < 1$. Hence

$$\lim_{n\to 0} |||K_2|||^n = 0\,.$$

We now distinguish two cases:

1) $N = 0$. In this case $K = U \oplus K_2$. Hence, for each $n \in \mathbb{N}$,

$$K^n = (U^n \oplus K_2^n)\,.$$

Since U is unitary, some subsequence of $\{U^n\}$, say $\{U^{n_i}\}$, approaches the $m \times m$ identity I_m. Since $K^{n_i+1} = U^{n_i+1} \oplus 0 + 0 \oplus K_2^{n_i+1}$, we conclude from

$$K^{n_i+1} - (U^{n_i+1} \oplus 0) = 0 \oplus K_2^{n_i+1}$$

that

$$\begin{aligned} \mathfrak{q}[K^{n_i+1} - (U^{n_i+1} \oplus 0)] &= \mathfrak{q}(0 \oplus K_2^{n_i+1}) \;=\; \mathfrak{q}(K_2^{n_i+1}) \\ &\le \mathfrak{q}(K_2)\,|||K_2^{n_i}||| \;\le\; \mathfrak{q}(K_2)|||K_2|||^{n_i}\,, \end{aligned}$$

which means $\lim_{n_i\to\infty} \mathfrak{q}[K^{n_i+1} - (U^{n_i+1} \oplus 0)] = 0$. Since

$$\begin{aligned} \mathfrak{q}[(U^{n_i+1} \oplus 0) - (U \oplus 0)] &= \mathfrak{q}\{(U \oplus 0)[(U^{n_i} - I_m) \oplus 0]\} \\ &\le \mathfrak{q}(U \oplus 0)\,\|[(U_i^n - I_m) \oplus 0]\| \;\le\; \mathfrak{q}(U)\,\|U^{n_i} - I_m\|\,, \end{aligned}$$

we also have $\lim_{n_i\to\infty} \mathfrak{q}[(U^{n_i+1} \oplus 0) - (U \oplus 0)] = 0$. Therefore

$$\lim_{n\to 0} \mathfrak{q}[K^{n_i+1} - (U \oplus 0)] = 0\,,$$

and hence $U \oplus 0$, which is a finite-rank operator, is in $\mathfrak{A}$. This is because $\{K^{n_i+1}\}$ is a sequence in $\mathfrak{A}$ and $\mathfrak{A}$ is $\mathfrak{q}$-closed.

2) $N \neq 0$. In this case let $k \in \mathbb{N}$ be such that $N^k \neq 0$ and $N^{k+1} = 0$. Then

$$(U + N)^n \;=\; U^n + nU^{n-1}N + \cdots + \frac{n!}{k!(n-k)!}U^{n-k}N^k$$

for every n. Once again, since U is unitary, some subsequence of U^n, say $\{U^{n_i-k}\}$ approaches the identity I_m. Thus if we put $m_i = \frac{n_i!}{k!(n_i-k)!}$, we get

$$\lim_{n_i \to 0} (m_i^{-1})(U+N)^{n_i} = N^k .$$

Since $(m_i^{-1})K^{n_i+1} = (m_i^{-1})(U+N)^{n_i+1} \oplus (m_i^{-1})K_2^{n_i+1}$, and since $\lim_{n_i \to \infty} m_i^{-1} = 0$, we conclude from

$$(m_i^{-1})K^{n_i+1} - ((m_i^{-1})(U+N)^{n_i+1} \oplus 0) = 0 \oplus (m_i^{-1})K_2^{n_i+1}$$

that

$$\lim_{n_i \to \infty} \mathfrak{q}[(m_i^{-1})K^{n_i+1} - ((m_i^{-1})(U+N)^{n_i+1} \oplus 0)] = 0 .$$

by the same method as in case 1. Also since

$$\begin{aligned} &\mathfrak{q}\{[(m_i^{-1})(U+N)^{n_i+1} \oplus 0] - [(U+N)N^k \oplus 0]\} \\ &= \mathfrak{q}\{[(U+N) \oplus 0][((m_i^{-1})(U+N)^{n_i} \oplus 0) - (N^k \oplus 0)]\} \\ &\le \mathfrak{q}[(U+N) \oplus 0] \, \| [(m_i^{-1})(U+N)^{n_i} - N^k] \oplus 0 \| \\ &\le \mathfrak{q}(U+N) \, \| (m_i^{-1})(U+N)^{n_i} - N^k \| . \end{aligned}$$

$\lim_{n_i \to \infty} \mathfrak{q}[(m_i^{-1})(U+N)^{n_i+1} \oplus 0) - ((U+N)N^k \oplus 0)] = 0$ holds. Therefore,

$$\lim_{n_i \to 0} \mathfrak{q}[(m_i^{-1})K^{n_i+1} - ((U+N)N^k \oplus 0)] = 0 .$$

and hence the finite-rank operator $(U+N)N^k \oplus 0$ is in $\mathfrak{A}$, as $\mathfrak{A}$ is $\mathfrak{q}$-closed.

So far we have proved that $\mathfrak{A}$ contains a finite-rank operator. Thus all the conditions of Theorem 1.2.4 are satisfied, and hence $\mathfrak{F}(X) \subseteq \mathfrak{A}$. Therefore,

$$\mathcal{I}(X) = \overline{\mathfrak{F}(X)}^{\mathfrak{q}} \subseteq \mathfrak{A} ,$$

and hence $\mathcal{I}(X) = \mathfrak{A}$, as desired.

■

Corollary 1.2.8 *If X is a reflexive Banach space, then the only irreducible ν-closed subalgebra $\mathfrak{A}$ of $\mathfrak{N}(X)$ is $\mathfrak{N}(X)$ itself.*

Proof: Apply Proposition 1.1.3 , Proposition 1.1.4, and Theorem 1.2.7 with $\mathcal{I} = \mathfrak{N}$ and $\mathsf{q} = \nu$.

■

The following Corollary is another version of Lomonosov's Lemma (Proposition 1.2.5).

Corollary 1.2.9 *Let X be a reflexive Banach space with A.P. Then the only norm-closed irreducible subalgebra $\mathfrak{A}$ of $\mathfrak{K}(X)$ is $\mathfrak{K}(X)$ itself.*

Proof: Apply Theorem 1.2.7 with $\mathcal{I} = \mathfrak{K}$ and $\mathsf{q} = \|\cdot\|$ and use Proposition 1.1.6.

■

Corollary 1.2.10 *Let X be a reflexive Banach space. Then, the only irreducible s_1-closed (π_2^2-closed) subalgebra $\mathfrak{A}$ of $S_1^a(X)$ ($\Pi_2^2(X)$) is $S_1^a(X)$ ($\Pi_2^2(X)$) itself.*

Proof: Apply Theorem 1.2.7 with $\mathcal{I} = S_1^a$ ($\mathcal{I} = \Pi_2^2$) and $\mathsf{q} = s_1$ ($\mathsf{q} = \pi_2^2$) and use Propositionl 1.1.10, 1.1.12, and 1.1.13.

■

1.3 Reducibility of Semigroups of Operators on Banach Spaces

In [38] there are several reducibility results for certain semigroups of compact operators on a Hilbert space, that satisfy a submultiplicativity condition. In this section we apply Theorem 1.2.7 to extend some results of [38] to the cases where a reflexive Banach space is under consideration. The proofs of many of the results that we present here, are similar to those of [38], with some necessary changes, and we include them here for completeness.

The first result is a well known lemma for algebras. We include its short proof to emphasize that it is also valid for semigroups of operators on Banach spaces. Recall that a subset $\mathcal{J}$ of a semigroup $\mathcal{S}$ is a *semigroup ideal*, if JS and SJ belong to $\mathcal{J}$ for all $J \in \mathcal{J}$ and $S \in \mathcal{S}$.

Lemma 1.3.1 *If a semigroup $\mathcal{S}$ of operators on a Banach space is irreducible, then so is every nonzero semigroup ideal $\mathcal{J}$ in $\mathcal{S}$.*

Proof: Let M be a subspace of X and consider the following two subspaces:

(i) M_1 = the closed linear span of $\{ JM : J \in \mathcal{J} \}$, and

(ii) M_2 = the intersection of kernels of all J in $\mathcal{J}$.

If M is an invariant subspace for $\mathcal{J}$ then M_1 and M_2 are invariant subspace for $\mathcal{S}$ and it is easy to check that at least one of them is nontrivial whenever M is.

■

The next result is an easy consequence of our results in Section 1.2. It plays an important role in the sequel.

Lemma 1.3.2 *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal and let X be a reflexive Banach space for which $\overline{\mathfrak{F}(X)}^{\mathfrak{q}} = \mathcal{I}(X)$. Suppose $\mathcal{S}$ is a semigroup in $\mathcal{I}(X)$ and f is a nonzero $\mathfrak{q}$-continuous linear functional on $\mathcal{I}(X)$ whose restriction to $\mathcal{S}$ is zero. Then $\mathcal{S}$ is reducible.*

Proof: Let $\mathfrak{A}$ be the algebra generated by $\mathcal{S}$. Observe that $f|_{\mathcal{S}} = 0$ implies $f|_{\mathfrak{A}} = 0$ and that $\mathfrak{A}$ is irreducible whenever $\mathcal{S}$ is. Now the irreducibility of $\mathfrak{A}$ shows that $\mathfrak{A}$ is $\mathfrak{q}$-dense in $\mathcal{I}(X)$ by Theorem 1.2.7 . which means f is a zero functional on $\mathcal{I}(X)$. a contradiction.

■

In the following we get a stronger result about functionals.

Proposition 1.3.3 *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal and let X be a reflexive Banach space for which $\overline{\mathfrak{F}(X)}^{\mathfrak{q}} = \mathcal{I}(X)$. Suppose $\mathfrak{E}$ is any subset of $\mathcal{I}(X)$ and f is a nonzero $\mathfrak{q}$-continuous linear functional on $\mathcal{I}(X)$ such that*

$$f(A_1 A_2 \cdots A_n) = f(A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)}),$$

for any $n \in N$, any members $A_1, \cdots, A_n$ of $\mathfrak{E}$, and any permutation σ. Then $\mathfrak{E}$ is *reducible.*

Proof: First observe that the permutability condition also holds on $\mathfrak{A}$, where $\mathfrak{A}$ is the algebra generated by $\mathfrak{E}$. If $\mathfrak{A}$ is commutative, then it is reducible, by Lomonosov's theorem. If there are members A and B in $\mathfrak{A}$ with $AB-BA \neq 0$, generate a semigroup ideal $\mathcal{J}$ of $\mathfrak{A}$ by $AB - BA$. The condition on f implies that $f \mid_{\mathcal{J}} = 0$, so that $\mathcal{J}$ is reducible, by Lemma 1.3.2. Thus $\mathfrak{A}$ is reducible by Lemma 1.3.1.

■

Corollary 1.3.4 *If in the hypotheses of Proposition* 1.3.3, *we replace* $\mathfrak{E}$ *by a semigroup* $\mathcal{S}$ *and* f *by a nonzero multiplicative functional on* $\mathcal{I}(X)$, *then* $\mathcal{S}$ *is reducible.*

Corollary 1.3.5 *If in the hypotheses of Proposition* 1.3.3 *we replace* $\mathfrak{E}$ *by a semigroup* $\mathcal{S}$ *and* f *by a nonzero functional on* $\mathcal{I}(X)$ *which is constant on* $\mathcal{S}$, *then* $\mathcal{S}$ *is reducible.*

The next result is crucial in the sequel. We state it without proof and just give a reference.

Theorem 1.3.6 (Ringrose's Theorem [41]**)** (i) *A nest* $\mathcal{N}$ *in a Banach space* X *is maximal if, and only if, for all* $N \in \mathcal{N}$, $\dim(N/N_-) \leq 1$, *where* N_- *denotes the smallest member of* $\mathcal{N}$ *containing all* $M \in \mathcal{N}$ *such that* $M \subset N$ *and* $M \neq N$.

(ii) *If* $T \in \mathfrak{K}(X)$ *leaves each member of a maximal nest* $\mathcal{N}$ *invariant, then its eigenvalues are, with the possible exception of zero, precisely its* diagonal coefficients, *namely the numbers* $\lambda_N(T)$, $N \in \mathcal{N}$, *where* $\lambda_N(T)$ *is the (scalar) operator induced by* T *on* N/N_-.

We also need the following simple fact concerning the direct sum of Banach spaces. This proof can be derived from [11, Proposition III.4.4.].

Proposition 1.3.7 *Suppose* $(X_1, \|\cdot\|_1)$ *and* $(X_2, \|\cdot\|_2)$ *are Banach spaces. Then the space*

$$X_1 \oplus X_2 = \{ (x_1, x_2) : x_1 \in X_1, x_2 \in X_2 \}$$

is a Banach space under the norm

$$\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2 \quad \forall (x_1, x_2) \in X_1 \oplus X_2 .$$

If we define a new norm on $X_1 \oplus X_2$ *by*

$$|||\,(x_1, x_2)\,||| = \max\{\|x_1\|_1, \|x_2\|_2\} \quad \forall (x_1, x_2) \in X_1 \oplus X_2 ,$$

then $|||(\cdot,\cdot)|||$ *is equivalent to* $\|(\cdot,\cdot)\|$ *and*

$$|||A||| \leq \max\{\|A_1\|, \|A_2\|\} ,$$

whenever $A_1 \in \mathfrak{B}(X_1)$, $A_2 \in \mathfrak{B}(X_2)$, *and* $A = A_1 \oplus A_2$.

We are now ready to extend some reducibility results of [38]. We start with a result about a semigroup of compact operators for which the spectral radius of each member is bounded by 1.

Theorem 1.3.8 *Let* $[\mathcal{I}, \mathfrak{q}]$ *be a quasi-normed operator ideal and let* X *be a reflexive Banach space for which* $\mathcal{I}(X)$ *is a trace ideal. Suppose* $\mathcal{S} \subseteq \mathcal{I}(X)$ *is a semigroup with* $r(S) \leq 1$ *for all* $S \in \mathcal{S}$. *Suppose also that* $\mathcal{S}$ *contains a non-zero operator* A *that is not a contraction under any renorming of* X. *Then* $\mathcal{S}$ *is reducible.*

Proof: By Proposition 1.2.6 there would be a new norm $|||\cdot|||$ on X under which $|||A||| < 1$ if $r(A) < 1$.

Therefore $r(A) = 1$ and $\sigma(A) = \sigma_1 \cup \sigma_2$, where

$$\sigma_1 = \{\lambda \in \sigma(A) : |\lambda| = 1\} \quad \textit{and} \quad \sigma_2 = \{\lambda \in \sigma(A) : |\lambda| < 1\} .$$

both σ_1 and σ_2 are nonempty and closed, and σ_1 is finite. Let M_1, M_2, U, and N be as in the proof of Theorem 1.2.7, where K, K_1, and K_2 are replaced by A, A_1, and A_2 respectively. Since $r(A_2) < 1$, as $\sigma(A_2) = \sigma_2$, by another application of Proposition 1.2.6, there is a renorm $|||\cdot|||$ on M_2 such that $|||A_2||| < 1$.

Consider $M_1 \oplus M_2$ in which M_1 is equipped with its original norm and M_2 with $|||\cdot|||$. This induces a new norm

$$|||(m_1, m_2)||| = \|m_1\| + |||m_2||| ,$$

whenever $m_1 \in M_1$ and $m_2 \in M_2$, on $M_1 \oplus M_2$ which is equivalent to $\|(\cdot,\cdot)\|$, and hence to the original norm of X. By Proposition 1.3.7, we can define an equivalent norm $|||\cdot|||'$ on $M_1 \oplus M_2$, and hence on X, such that

$$|||A|||' \leq \max\{\|U+N\|, |||A_2|||\}.$$

This shows that under this new norm, $|||A|||' \leq 1$ whenever $N = 0$, which is impossible by hypothesis. Hence $N \neq 0$ and we conclude that $m \geq 2$ and there exists $k \in \mathbb{N}$ such that $N^k \neq 0$ and $N^{k+1} = 0$. Now a procedure similar to that in the proof of Theorem 1 of [38] and an application of the continuity of spectral radius on $\mathfrak{K}(X)$ (cf [8, Corollary 1.2.7]), show that there exists a finite-rank operator B such that

$$tr(BS) = 0 \qquad \textit{for every } S \in \mathcal{S}.$$

Define $f : \mathcal{I}(X) \longrightarrow \mathbb{C}$ by $f(T) = tr(BT)$. Since the functional tr is $\mathfrak{q}$-continuous on $\mathfrak{F}(X)$, the functional f is $\mathfrak{q}$-continuous on $\mathcal{I}(X)$. Therefore: all the conditions of Lemma 1.3.2 are satisfied with the functional f, and hence $\mathcal{S}$ is reducible.

■

The following is an extension of a result about compact idempotents on a Banach space X (cf. [38, Theorem 3.1]. The strategy of the proof is the same, making use of the results of Section 1.1.

Theorem 1.3.9 *Suppose X is any reflexive Banach space and let $\mathcal{S}$ be a semigroup of operators on X. If every member of $\mathcal{S}$ is a nonnegative scalar multiple of a compact idempotent and*

$$r(AB) \leq r(A)r(B)$$

for every pair A and B in $\mathcal{S}$, then $\mathcal{S}$ is reducible.

Proof: It is easy to verify that the semigroup

$$\mathbb{R}^+\mathcal{S} = \{\, pS \;:\; S \in \mathcal{S},\ p \textit{ a positive number} \,\}$$

satisfies the hypotheses of the theorem and hence we may assume, without any loss of generality, that $\mathcal{S} = \mathbb{R}^+\mathcal{S}$. It is well known that each compact idempotent has finite-rank, hence all members of $\mathcal{S}$ are finite-rank operators and we can assume $\mathcal{S} \subseteq \mathcal{I}(X)$, where $[\mathcal{I}, \mathfrak{q}]$ is a quasi-normed operator ideal, $\mathcal{I}(X)$ is a trace ideal, and the functional tr is $\mathfrak{q}$-continuous on $\mathcal{I}(X)$. (Either $[S_1^a, s_1]$ or $[\Pi_2^2, \pi_2^2]$ would do, by Proposition 1.1.14.) If $\mathcal{S} = \{0\}$ there is nothing to prove; otherwise follow the proof of Theorem 2 of [38] to find an idempotent P of minimal positive rank m with the following property:

$$tr(PJ) = tr(J) \qquad for\ every\ J \in \mathcal{J},$$

where

$$\mathcal{J} = \{\, SPT \,:\, S, T \in \mathcal{S}\,\}.$$

This shows $tr[(I-P)J] = 0$ on $\mathcal{J}$, *i.e.* the linear functional $f : \mathcal{I}(X) \longrightarrow \mathbb{C}$, defined by $f(T) = tr[(I-P)T]$, is zero on $\mathcal{J}$. Since the functional tr is $\mathfrak{q}$-continuous on $\mathcal{I}(X)$, $\mathcal{J}$ is reducible by Lemma 1.3.2. The reducibility of $\mathcal{S}$ now follows from Lemma 1.3.1.

■

We now use the results of Section 0.4, about the compression of an operator, to prove a Corollary of Theorem 1.3.9 that also strengthens it.

Corollary 1.3.10 *Assume all the conditions of Theorem* 1.3.9. *Then* $\mathcal{S}$ *is triangularizable.*

Proof: Let $\mathcal{F}$ be a maximal chain of invariant subspaces of $\mathcal{S}$. We must show that $\mathcal{F}$ is a maximal subspace chain. Suppose not. Then there are members M and N of $\mathcal{F}$ with $M \subset N$ and $\dim(N/M) \geq 2$. Since X is reflexive, N is reflexive and hence N/M is also a reflexive Banach space. It is also easy to verify that the compression $\hat{\mathcal{S}}$ of $\mathcal{S}$ is a semigroup consisting of nonnegative multiples of idempotents. It remains to show that

$$r(\hat{S}\hat{T}) \leq r(\hat{S})r(\hat{T}) \quad for\ every\ \hat{S}, \hat{T} \in \hat{\mathcal{S}}. \qquad (1)$$

Now it can be easily checked that, $\|(\widehat{ST})^n\| \leq \|(ST)^n\|$ for all $n \in \mathbb{N}$, whenever

$S, T \in \mathcal{S}$. Thus,

$$r(\hat{S}\hat{T}) = r(\widehat{ST}) \le r(ST) \le r(S)r(T) \quad \textit{for every } S, T \in \mathcal{S}. \qquad (2)$$

If either $\hat{S} = 0$ or $\hat{T} = 0$ then $\hat{S}\hat{T} = 0$ and hence

$$0 = r(\hat{S}\hat{T}) = r(\hat{S})r(\hat{T}).$$

If $\hat{S}$ and $\hat{T}$ are both nonzero, then S and T are both nonzero and since the spectral radius of each nonzero idempotent is 1, we have $r(\hat{S}) = r(S)$ and $r(\hat{T}) = r(T)$. In this case it follows, from (2), that $r(\hat{S}\hat{T}) \le r(\hat{S})r(\hat{T})$. Thus (1) is satisfied and it follows, from Theorem 1.3.9, that $\hat{\mathcal{S}}$ is reducible. This means there exists a nontrivial invariant subspace $\hat{W}$ of N/M for $\hat{\mathcal{S}}$. An easy verification now show that the set K, defined by

$$W = \{ x \in N : x + M \in \hat{W} \}.$$

is a proper subspace of N that contains M as a proper subspace, and is invariant for $\mathcal{S}$. This contradicts the maximality of $\mathcal{F}$ among invariant chains. Therefore, $\mathcal{F}$ is also a maximal subspace chain, as desired.

■

An application of Corollary 1.3.10 shows that a semigroup satisfying the hypotheses of Theorem 1.3.9 is essentially a semigroup of idempotents.

Corollary 1.3.11 *Assume all the conditions of Theorem 1.3.9. Let*

$$\mathcal{T} = \left\{ \frac{S}{r(S)} : S \in \mathcal{S} \right\},$$

in which we define $0/0 = 0$ whenever $\mathcal{S}$ contains the zero operator. Then $\mathcal{T}$ is a semigroup of idempotents.

Proof: As in the proof of Corollary 4 of [38], we just need to verify that if S_1 and S_2 are in $\mathcal{S}$ and if $S_1S_2 \neq 0$, then $r(S_1S_2) = r(S_1)r(S_2)$. By Corollary 1.3.10, there is a triangularizing chain $\mathcal{F}$ for $\mathcal{S}$. By Theorem 1.3.6, $\mathcal{F}$ contains a pair of subspaces M and N, with $\dim(N/M) = 1$, such that the compression $\widehat{S_1S_2}$ of S_1S_2 to N/M is the scalar $r(S_1S_2)$. Thus, if we consider the corresponding compressions $\hat{S}_1$ and $\hat{S}_2$ of S_1 and S_2, respectively, then $\hat{S}_1$ and $\hat{S}_2$ are nonzero and equal to $r(S_1)$ and $r(S_2)$, respectively. Hence $r(S_1S_2) = r(S_1)r(S_2)$ holds.

■

Definition 1.3.12 Suppose $T \in \mathfrak{B}(X)$. We say that $r(T)$ is the *dominant* member of $\sigma(T)$ if $r(T) \in \sigma(T)$ and $|\lambda| < r(T)$ for all $\lambda \in \sigma(T) \setminus \{r(T)\}$.

The next theorem shows that, under suitable hypotheses, the dominance condition also leads to the reducibility of a semigroup of operators on a Banach space.

Theorem 1.3.13 *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal and let X be a reflexive Banach space for which $\mathcal{I}(X)$ is a trace ideal. Suppose $\mathcal{S} \subseteq \mathcal{I}(X)$, and $\mathcal{S}$ contains a non-quasinilpotent operator. Suppose also that*

$$r(AB) \leq r(A)r(B) \quad \forall A, B \in \mathcal{S}.$$

and $r(A)$ is the dominant member of $\sigma(A)$ for every A in $\mathcal{S}$. Then $\mathcal{S}$ is reducible.

Proof: If for each $S \in \mathcal{S}$ we cannot define an equivalent norm $|||\cdot|||$ on X such that $|||S||| \leq 1$, then, by Theorem 1.3.8, $\mathcal{S}$ is reducible. Otherwise, follow the proof of Theorem 3 of [38] to find: (i) a nonquasinilpotent $S = r(S)I_m \oplus C$, where m is minimal, I_m is the $m \times m$ identity matrix, and $r(C) < r(S)$, and (ii) an idempotent P of rank m in the (uniform) closure $\overline{\mathcal{S}}$ of $\mathcal{S}$, and consider $\mathcal{J} = \overline{\mathcal{S}}P\overline{\mathcal{S}}$. Observe that every members of $\mathcal{J}$ has rank at most m.

(i) Assume that $\mathcal{J}$ contains a nonzero nilpotent operator. Then the inequality of the hypothesis shows that the set $\mathcal{J}_0$ of all nilpotent members of $\mathcal{J}$ is an ideal of $\mathcal{J}$. Since $\mathcal{J}_0 \subseteq \mathcal{I}(X)$, $\mathcal{J}_0$ is reducible by Lemma 1.3.2, and Theorem 1.1.15 (with tr serving as the functional f). Hence Lemma 1.3.1 implies the reducibility of $\mathcal{J}$ and $\overline{\mathcal{S}}$.

(ii) As in case (ii), in the proof of Theorem 3 of [38], every member A of $\mathcal{J}$ would be $r(A)$ times an idempotent of rank m if there is no nonzero nilpotent operator in $\mathcal{J}$. Therefore; by Theorem 1.3.9, $\mathcal{J}$ is reducible, and so is $\mathcal{S}$.

■

The following Corollary is an extension of Theorem 1 of [36] to our broader context.

Corollary 1.3.14 *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal and let X be a reflexive Banach space for which $\mathcal{I}(X)$ is a trace ideal. Suppose $\mathcal{S} \subseteq \mathcal{I}(X)$ is a semigroup and 1 is the dominant member of $\sigma(S)$ for every S in $\mathcal{S}$. Then $\mathcal{S}$ is reducible.*

With exactly the same proof as the one given for Theorem 4 of [38], one can obtain an extension of this theorem simply by using our previous extensions and replacing $N \ominus M$ by M/N, where M and N are the appropriate subspaces of X with $M \subseteq N$.

Remark: Using Lemma 1.3.1, we are able to weaken the hypotheses in many of the previous results. In the following we state one of them as a sample and give no proof. It is obvious that one may state and prove similar versions of the other results.

Theorem 1.3.8* *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal and let X be a reflexive Banach space for which $\mathcal{I}(X)$ is a trace ideal. Suppose $\mathcal{S} \subseteq \mathfrak{B}(X)$ is a semigroup such that $\mathcal{S} \cap \mathcal{I}(X)$ is nonempty and contains an operator A that is not a contraction under any renorming of X. If $r(S) \leq 1$ for every $S \in \mathcal{S}$, then $\mathcal{S}$ is reducible.*

It should be noted that the hypothesis of the dominance is essential in our previous results. It should also be noted that the condition on $\mathcal{I}(X)$, that it be a trace ideal, cannot be removed from the hypotheses of these results (cf [38, Theorem 6 and Theorem 7]).

1.4 Triangularizability of Semigroups of Operators on Banach Spaces.

There are several triangularizability results for certain semigroups of compact operators both in Hilbert and Banach spaces. Examples of such results can be found in [25],[27],[33], and [37].

Our main results in this section concern semigroups of quasinilpotent operators on a reflexive Banach space. We then use these results in Chapter 2 and 3 to establish certain decomposability and ideal-triangularizability (see Definition 0.4.6 below) results. The results of this section are the extensions of those in [33] that can be obtained from the theorems of Section 1.1. The proof of each new version is similar to the proof given in [33] with some necessary changes.

Theorem 1.4.1 *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal and let X be a reflexive Banach space for which $\mathcal{I}(X)$ is a C-trace ideal. If $\mathcal{S}$ is a semigroup of quasinilpotent operators on X and if $\mathcal{S}$ contains an operator other than 0 in $\mathcal{I}(X)$, then Lat $(\mathcal{S})$ contains a nontrivial element.*

Proof: Suppose *Lat* $(\mathcal{S}) = \{\{0\}, X\}$, then the algebra $\mathfrak{A}$ generated by $\mathcal{S}$ has the same trivial lattice of invariant subspaces. Now let $\mathcal{J}$ denote the set of all linear combinations of operators in $\mathcal{S} \cap \mathcal{I}(X)$. Since $\mathcal{S}$ is a semigroup, $\mathfrak{A}$ is an algebra, and since $\mathcal{I}(X)$ is an ideal in $\mathfrak{B}(X)$, $\mathcal{J}$ is a two-sided ideal in $\mathfrak{A}$. Since $\mathfrak{A}$ has no nontrivial invariant subspace. $\mathcal{J}$ has no nontrivial invariant subspace by Lemma 1.2.1. Therefore an application of Theorem 1.2.7 shows that $\mathcal{J}$ is $\mathfrak{q}$-dense in $\mathcal{I}(X)$. Since

$$tr(T) = 0 \qquad \forall T \in \mathcal{J}.$$

and since tr is $\mathfrak{q}$-continuous on $\mathcal{I}(X)$, we have

$$tr(T) = 0 \qquad \forall T \in \mathcal{I}(X).$$

a contradiction.

■

We now derive several corollaries about triangularizing semigroups of operators on a reflexive Banach space.

Corollary 1.4.2 *Suppose $[\mathcal{I}, \mathfrak{q}]$ is a C-trace quasi-normed operator ideal. Suppose also that X is a reflexive Banach space and $\mathcal{S}$ is a semigroup of quasinilpotent operators on X that is generated by a commutative set of polynomially compact operators and a subset of $\mathcal{I}(X)$. Then $\mathcal{S}$ is triangularizable.*

Proof: Choose a chain $\mathcal{F}$ of subspaces that is maximal as a chain of common invariant subspaces of the operators in $\mathcal{S}$. If $\mathcal{F}$ is not maximal as a subspace chain, there must exist a subspace M in $\mathcal{F}$, such that $\dim(M/M_-) \geq 2$ (see [41]). Now consider the compression $\hat{\mathcal{S}}$ of $\mathcal{S}$ to M/M_-. We distinguish two cases:

1) The compression $\hat{T}$ of every $T \in \mathcal{I}(X) \cap \mathcal{S}$ to M/M_- is zero. Thus $\hat{\mathcal{S}}$ is a commutative family of polynomially compact operators, and hence $\hat{\mathcal{S}}$ is reducible by Lomonosov's Theorem.

2) There exists $T \in \mathcal{I}(X) \cap \mathcal{S}$ such that the compression $\hat{T}$ of T to M/M_- is not zero. Since M/M_- is a reflexive Banach space, since $\hat{T} \in \mathcal{I}(M/M_-)$ by Lemma 1.1.15, and since $\hat{\mathcal{S}}$ is a semigroup of quasinilpotent operators by the results of Section 0.4, it follows from Theorem 1.4.1 that $\hat{\mathcal{S}}$ is reducible.

In either case we can find a nontrivial subspace K_0 of M/M_- which is invariant under $\hat{\mathcal{S}}$. As before one can easily check that the subspace

$$K = \{ x \in M_- : x + M_- \in K_0 \}$$

is invariant for $\mathcal{S}$, properly contains M_-, and is properly contained in M. But this contradicts the maximality of $\mathcal{F}$, and hence $\mathcal{F}$ is also maximal as a subspace chain, as desired.

■

Corollary 1.4.3 *Suppose* $[\mathcal{I}, \mathfrak{q}]$ *is a C-trace quasi-normed operator ideal. Suppose also that* X *is a reflexive Banach space. Let* $\mathcal{S}$ *be a semigroup of quasinilpotent operators that is generated by the union of a finite subset* Γ *of* $\mathcal{I}(X)$ *and an operator* $B \in \mathfrak{B}(X)$. *If some power of* B, *say* B^n, *is compact, then*

$$\sum_{A \in \Gamma} A + B$$

is quasinilpotent.

Proof: The semigroup $\mathcal{S}$ is triangularizable by Corollary 1.4.2. Now proceed as in the proof of Corollary 3 of [33] and use Theorem 1.3.6, to prove that $(\sum_{A \in \Gamma} A + B)^n$, and hence $\sum_{A\in\Gamma} A + B$, is quasinilpotent.

■

As a consequence of Theorem 1.4.1, we know that under the conditions of that theorem, a semigroup $\mathcal{S}$ in $\mathfrak{B}(X)$ is reducible. In the following we observe that under a weaker condition we can still prove the reducibility of a semigroup in $\mathfrak{B}(X)$.

Corollary 1.4.4 *Let $[\mathcal{I}, \mathfrak{q}]$ be a quasi-normed operator ideal and let X be a reflexive Banach space for which $\mathcal{I}(X)$ is a C-trace ideal. Suppose $\mathcal{S} \subseteq \mathfrak{B}(X)$ is a semigroup of quasinilpotent operators, $k \in \mathbb{N}$, and*

$$\mathcal{S}^k = \{ S_1 S_2 \cdots S_k : S_i \in \mathcal{S} \; i = 1, \cdots k \} .$$

(a) If there exists an element A in $\mathcal{S}$ such that $A^k \neq 0$ and $A^k \in \mathcal{I}(X)$, then $\mathcal{S}$ is reducible.

(b) If $\mathcal{S}^k \subseteq \mathcal{I}(X)$, then $\mathcal{S}$ is reducible.

Proof: First observe that $\mathcal{S}^k$ is a semigroup ideal.

(a) Since $A^k \in \mathcal{S}^k$, $\mathcal{S}^k$ is reducible by Theorem 1.4.1, and hence $\mathcal{S}$ is reducible by Lemma 1.3.1.

(b) If $\mathcal{S}^k \neq \{0\}$ we use (a). So assume that $\mathcal{S}^k = \{0\}$. Let m be the smallest positive integer such that $\mathcal{S}^m = \{0\}$. If $m = 1$ there is nothing to prove, otherwise there exist $S_1, S_2, \cdots, S_{m-1}$ in $\mathcal{S}$ such that

$$S_0 = S_1 S_2 \cdots S_{m-1} \neq 0 .$$

Therefore

$$S S_0 = 0 \qquad \forall S \in \mathcal{S},$$

and hence $\mathcal{S}$ has a nonzero kernel which is also not the whole space X. Thus $\mathcal{S}$ is reducible.

■

Corollary 1.4.5 *If, in Corollary 1.4.4, $[\mathcal{I}, \mathfrak{q}]$ is a C-trace quasi-normed operator ideal, then, in case (b), $\mathcal{S}$ is triangularizable.*

Proof: First use Lemma 1.1.15 to observe that:

$$(a) \quad (S_1S_2\cdots S_k)|_M = (S_1|_M)(S_2|_M)\cdots(S_k|_M) \in \mathcal{I}(M).$$

whenever $M \in Lat(\mathcal{S})$ and $S_1, \cdots, S_k \in \mathcal{S}$, and

$$(b) \quad \widehat{S_1\cdots S_k} = \widehat{S_1}\cdots\widehat{S_k} \in \mathcal{I}(M_2/M_1).$$

whenever $M_1, M_2 \in Lat(\mathcal{S})$ with $M_1 \subseteq M_2$ and $\widehat{S_1}, \cdots, \widehat{S_k}, \widehat{S_1\cdots S_k}$ are the compressions of $S_1, \cdots, S_k, S_1\cdots S_k$ to M_2/M_1, respectively.

Now use the same technique as in the proof of Corollary 1.4.2 and apply Corollary 1.4.4(b).

■

Chapter 2

Decomposability of Positive Operators on Banach Lattices

2.1 Decomposability Theorems

There are many decomposability theorems concerning compact and non-compact quasinilpotent positive operators. Several such theorems can be found in [1], [2],[3],[4], [5],[6] [34], [45], and [46]. In this section we recall a number of those theorems which are well-known and crucial in our investigation of decomposability problems concerning a collection of quasinilpotent positive operators. Note that by a "positive operator" we mean "a nonzero positive operator".

Declaration: Since in the remaining chapters we deal with quasinilpotent positive operators on Banach lattices, from now on we can — and we shall — assume with no loss of any generality that our Banach lattices are all real Banach lattices (see Remark 0.3.11).

Theorem 2.1.1 [45, Lemma 1 and Theorem 2] *Suppose* E *is either* $C_0(\Omega)$ *or a Banach lattice with atoms. If* $T \in \mathfrak{B}(\mathsf{E})$ *is a quasinilpotent positive operator, then* T *is decomposable.*

B. de Pagter [34] has shown that every positive, indecomposable, compact linear operator $T \in \mathfrak{B}(\mathsf{E})$ has strictly positive spectral radius. In other words:

Theorem 2.1.2 [34, Proposition 2] *Every compact, quasinilpotent, positive operator on a Banach lattice is decomposable.*

In [1], Y.A. Abramovich , C.D. Aliprantis, and O. Burkinshaw obtained the next result which is much stronger than the result stated in Theorem 2.1.2.

Theorem 2.1.3 [1, Theorem 4.3] *Every compact, quasinilpotent, positive operator on a Banach lattice has a nontrivial p-hyperinvariant closed ideal.*

2.2 Decomposability of a Semigroup of Positive Operators in a Trace Ideal

In this section we use Corollary 1.4.3 to establish the decomposability of certain semigroups which are subsets of C-trace ideals.

Theorem 2.2.1 *Suppose $[\mathcal{I}, \mathfrak{q}]$ is a C-trace quasi-normed operator ideal. Suppose also that E is a reflexive Banach lattice. Let $\mathcal{S}$ be a semigroup of quasinilpotent positive operators that is generated by the union of a finite subset $\mathfrak{C}$ of $\mathcal{I}(\mathsf{E})$ and a positive operator, $B \in \mathfrak{B}(\mathsf{E})$. If some power of B is compact, then $\mathcal{S}$ is decomposable.*

Proof: By Corollary 1.4.3, $T = \sum_{A \in \mathfrak{C}} A + B$ is quasinilpotent. Since T is positive, since $S = \sum_{A \in \mathfrak{C}} A$ is compact, and since T dominates S, as $S \leq T$ and both are positive operators, T is decomposable by Corollary 4.3 of [3]. Since we also have $A \leq T$ for all $A \in \mathfrak{C}$, and since $B \leq T$, it follows that $\mathfrak{C} \cup \{B\}$, and hence also $\mathcal{S}$, is decomposable.

■

Theorem 2.2.2 *Suppose $[\mathcal{I}, \mathfrak{q}]$ is a C-trace quasi-normed operator ideal. Suppose also that E is a reflexive Banach lattice. Let $\mathcal{S}$ be a semigroup of quasinilpotent positive operators that is generated by a countable subset $\mathfrak{C}$ of $\mathcal{I}(\mathsf{E})$. Then $\mathcal{S}$ is decomposable.*

Proof: Suppose $\mathfrak{C} = \{ A_n : n \in \mathbb{N} \}$. Without loss of generality we can assume that

$$A_n = (\frac{1}{\kappa^n n^2})B_n \qquad \forall\ A_n \in \mathfrak{C},$$

where $B_n \in \mathcal{I}(\mathsf{E})$ with $\mathfrak{q}(B_n) = 1$ and $\kappa \geq 1$ is the coefficient that appears in the quasi-triangle inequality related to $\mathfrak{q}$. By a simple verification one can show that the sequence $\{S_n\}_{n=1}^{\infty}$ with

$$S_n = \sum_{i=1}^{n} A_i$$

is a $\mathfrak{q}$-Cauchy sequence in $\mathcal{I}(\mathsf{E})$ and hence converges, in $\mathfrak{q}$, to an element A of $\mathcal{I}(\mathsf{E})$. It is clear that A is a positive operator and since

$$\|S_n - A\| \leq \mathfrak{q}(S_n - A) \qquad n \in \mathbb{N},$$

S_n also converges, in norm, to A. We also know that S_n is a compact operator for each n and hence A is compact and

$$\lim_{n \to \infty} r(S_n) = r(A)$$

by the continuity of spectral radius on compact operators [6, Corollary 1.2.7]. Since each S_n is a quasinilpotent operator, by Corollary 1.4.3, $r(A) = 0$. Thus A is a quasinilpotent positive operator in $\mathcal{I}(\mathsf{E})$ and it is decomposable by Theorem 2.1.2. Now since

$$A_n \leq A \qquad n \in \mathbb{N},$$

$\mathfrak{C}$, and hence also $\mathcal{S}$, is decomposable.

■

Corollary 2.2.3 *Suppose $[\mathcal{I}, \mathfrak{q}]$ is a C-trace quasi-normed operator ideal. Suppose also that E is a reflexive Banach lattice. Let $\mathcal{S}$ be a semigroup of quasinilpotent positive operators in $\mathcal{I}(\mathsf{E})$ that is separable in the strong operator topology of $\mathfrak{B}(\mathsf{E})$. Then $\mathcal{S}$ is decomposable.*

Proof: Let $\mathfrak{C} = \{T_n : n \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{S}$ in the strong operator topology. Use the same technique as in the proof of Theorem 2.2.2 to prove that $\mathfrak{C}$ is decomposable. Now let J be a nontrivial closed ideal in $Ilat(\mathfrak{C})$. If $S \in \mathcal{S}$, then there exists a subsequence $\{T_{n_i}\}_{i=1}^{\infty}$ of $\mathfrak{C}$ that converges strongly to S. This means

$$\lim_{i \to \infty} \|T_{n_i}x - Sx\| = 0 \qquad \forall x \in X .$$

Therefore; $Sx \in$ J whenever $x \in$ J, and hence J $\in Ilat(\mathcal{S})$, as desired.

■

The following is a Banach lattice version of Lemma 1.3.1. It enables us to derive some decomposability results under weaker conditions. It certainly plays an important role in the further investigation of decomposability problems.

Lemma 2.2.4 *If a semigroup $\mathcal{S}$ of positive operators on a Banach lattice* E *is indecomposable, then so is every nonzero semigroup ideal $\mathcal{J}$ of $\mathcal{S}$.*

Proof: Suppose I $\in Ilat(\mathcal{J})$ is a nontrivial closed ideal. Let I_1 be the closure of

$$\mathsf{V} = \left\{ x \in \mathsf{E} \,:\, \exists x_1, \cdots x_n \in \mathsf{W} \text{ and } \lambda_1, \cdots, \lambda_n \in \mathbb{R}^+ \text{ with } |x| \le \sum_{i=1}^{n} \lambda_i |x_i| \right\},$$

where

$$\mathsf{W} = \bigcup_{J \in \mathcal{J}} \{ Jx \,:\, x \in \mathsf{I} \} .$$

Then I_1 is a closed ideal of E and $\mathsf{I}_1 \subseteq \mathsf{I}$ as $\mathsf{W} \subseteq \mathsf{I}$. Now for each $S \in \mathcal{S}$ and $x \in \mathsf{V}$

$$|Sx| \le S|x| \le S(\sum_{i=1}^{n} \lambda_i |x_i|) = \sum_{i=1}^{n} \lambda_i S(|J_i y_i|) ,$$

where $J_1, \cdots J_n \in \mathcal{J}$ and $y_1, \cdots y_n \in \mathsf{I}$. Thus

$$|Sx| \le \sum_{i=1}^{n} \lambda_i S J_i |y_i| .$$

Since $\mathcal{J}$ is a semigroup ideal, $SJ_i \in \mathcal{J}$ for each i, and since I is an ideal of E, $|y_i| \in \mathsf{I}$ for each i. Thus $Sx \in \mathsf{V}$ and hence $\mathsf{I}_1 \in Ilat(\mathcal{S})$. We distinguish two cases:

Case 1) $\mathsf{I}_1 \ne \{0\}$. In this case, since $\mathsf{I}_1 \subseteq \mathsf{I}$ and since I is nontrivial, I_1 is a nontrivial invariant ideal for $\mathcal{S}$, a contradiction.

Case 2) $\mathsf{I}_1 = \{0\}$. This means $\mathsf{W} = \{0\}$ and hence $Jx = 0$ for all $J \in \mathcal{J}$ and all $x \in \mathsf{I}$. Thus $\mathsf{I} \subseteq \mathsf{I}_2$, where

$$\mathsf{I}_2 = \bigcap_{J \in \mathcal{J}} \{ x \in \mathsf{E} : J(|x|) = 0 \} .$$

We know that I_2 is a closed ideal of E and it is easy to verify that I_2 is invariant under $\mathcal{S}$. Since $\mathsf{I} \neq \{0\}$ and since $\mathsf{I}_2 \neq \mathsf{E}$, as $\mathcal{J}$ is a nonzero semigroup ideal of $\mathcal{S}$, I_2 is a nontrivial invariant ideal for $\mathcal{S}$, another contradiction.

■

Proposition 2.2.5 *Let $[\mathcal{I}, \mathfrak{q}]$ be a C-trace quasi-normed operator ideal and let E be a reflexive Banach lattice. Suppose $\mathcal{S}$ is a semigroup of quasinilpotent positive operators which is a countably generated subset of $\mathfrak{B}(\mathsf{E})$. If there exists an integer $k \in \mathbb{N}$ such that $\mathcal{S}^k \subseteq \mathcal{I}(\mathsf{E})$, then $\mathcal{S}$ is decomposable.*

Proof: First observe that $\mathcal{S}^k$ is generated by the countable set

$$\mathfrak{C}_k = \{ A_1 A_2 \cdots A_k : A_i \in \mathfrak{C},\ i = 1, \ldots, k \} ,$$

where $\mathfrak{C}$ is the generator of $\mathcal{S}$. Here we also distinguish two cases.

Case 1) $\mathcal{S}^k \neq \{0\}$. Since $\mathcal{S}^k$ is decomposable by Theorem 2.2.2, $\mathcal{S}$ is decomposable by Lemma 2.2.4.

Case 2) $\mathcal{S}^k = \{0\}$. Let m be the smallest positive integer such that $\mathcal{S}^m = \{0\}$. If $m = 1$ there is nothing to prove, otherwise there exists $S_1, S_2, \cdots, S_{m-1}$ in $\mathcal{S}$ such that

$$S_0 = S_1 S_2 \cdots S_{m-1} \neq 0 .$$

Therefore;

$$S S_0 = 0 \qquad \forall S \in \mathcal{S} ,$$

and hence the *null ideal* $\mathsf{N}_{\mathcal{S}}$, defined by

$$\mathsf{N}_{\mathcal{S}} = \bigcap_{S \in \mathcal{S}} \{ x \in \mathsf{E} : S(|x|) = 0 \} ,$$

is a nontrivial invariant ideal for $\mathcal{S}$.

■

2.3 Decomposability of a Commutative Collection of Positive Operators

Our reading of [1] and [34] revealed that there is an extension of Theorem 2.1.3, whose proof is a slight modification of the proofs given in [34] and [1] for Theorems 2.1.2 and 2.1.3, respectively. To do this we need to state an important lemma that has been established in [34].

Lemma 2.3.1 *Let* E *be a Banach lattice whose positive cone contains a quasi-interior point. If* $0 \leq u \leq |x|$ *where* $x, u \in \mathsf{E}$, *then there exists a sequence* $\{\pi_n : n \in \mathbb{N}\}$ *in* $\mathfrak{Z}(\mathsf{E})$ *such that* $\pi_n x \to u$ *(in norm) as* $n \to \infty$ *and* $|\pi_n| \leq I_{\mathsf{E}}$ *for all* n.

The previous lemma was used in the proof of part two of [34, Proposition 2]. In the following we shall indicate that part and give its proof for completeness.

Lemma 2.3.2 *Let* E *be a Banach lattice whose positive cone contains a quasi-interior point. Let* $\mathfrak{A}$ *be a linear manifold of* $\mathfrak{B}(\mathsf{E})$ *with the property that* $\pi A \in \mathfrak{A}$ *whenever* $A \in \mathfrak{A}$ *and* $\pi \in \mathfrak{Z}(\mathsf{E})$. *Then for each* $x \in \mathsf{E}$ *the closure of the set*

$$\mathfrak{A}x = \{\, Ax \,:\, A \in \mathfrak{A} \,\}$$

is a closed ideal of E.

Proof: Suppose that $0 \leq u \leq |Ax|$ in E for some $A \in \mathfrak{A}$. By Lemma 2.3.1 there exists a sequence $\{\pi_n : n \in \mathbb{N}\}$ in $\mathfrak{Z}(\mathsf{E})$ such that $|\pi_n| \leq I_{\mathsf{E}}$ and $\pi_n(Ax) \to u$ (in norm) as $n \to \infty$. Since $\pi_n A \in \mathfrak{A}$ for all n, $u \in \overline{\mathfrak{A}x}$.

Now let I denote the ideal generated by $\mathfrak{A}x$ and take $v \in \mathsf{I}_+$. Then there exist $A_1, A_2, \cdots, A_m \in \mathfrak{A}$ such that $v \leq \sum_{k=1}^{m} |A_k x|$. Thus we can write $v = \sum_{k=1}^{m} v_k$ with $0 \leq v_k \leq |A_k x|$. By the first paragraph $v_k \in \overline{\mathfrak{A}x}$ for each $k \in \{1, 2, \cdots, m\}$, and hence

$v \in \overline{\mathfrak{A}x}$. Therefore

$$\mathfrak{A}x \subseteq \mathsf{I} \subseteq \overline{\mathfrak{A}x},$$

and this implies that $\bar{\mathsf{I}} = \overline{\mathfrak{A}x}$. Since the closure of an ideal is an ideal, $\overline{\mathfrak{A}x}$ is an ideal.

■

We now use the techniques used in [1] and [34] to establish new versions of Proposition 2 of [34] and Theorem 4.3 of [1].

Theorem 2.3.3 *Let $T \in \mathfrak{B}(\mathsf{E})$ be a nonzero compact positive operator. If T is quasinilpotent at some $x_0 > 0$ in E, then T has a nontrivial p-hyperinvariant closed ideal.*

Proof : Since the *null ideal* $\mathsf{N}_T = \{ x \in \mathsf{E} : T(|\, x \,|) = 0 \}$ is a p-hyperinvariant closed ideal for T, we are done if $Tx_0 = 0$. If $Tx_0 \neq 0$ we proceed as follows:

As in the proof of [1, Theorem 4.3], the set

$$\mathsf{F} = \{ x \in \mathsf{E} : \exists y \geq 0 \ \ such\ that\ \ |x| \leq Ty \}^{-}$$

contains a nonzero element and is a p-hyperinvariant closed ideal of T. Furthermore, $\overline{\mathsf{E}_u} = \mathsf{F}$, where $u = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n \|x_n\|} > 0$, and $\{x_n\}$ is a norm-dense sequence in $T(\mathsf{E})$. If $\overline{\mathsf{E}_u} = \mathsf{F} \neq \mathsf{E}$, then $\overline{\mathsf{E}_u}$ is a nontrivial p-hyperinvariant closed ideal for T, otherwise E contains a quasi-interior point and this, by Lemma 2.3.2, shows that:

For each $x \neq 0$ the closure $\overline{\mathfrak{A}x}$ is a nonzero closed ideal which is invariant under T, where

$$\mathfrak{A} = \{ S_1 - S_2 \ : \ S_1, S_2 \in \mathfrak{A}^{+} \},$$

and

$$\mathfrak{A}^{+} = \{ S \in \mathfrak{B}(\mathsf{E}) : 0 \leq S \leq R \ \ for\ \ some\ \ R \in \{T\}'_{+} \}.$$

By the last part of [1, Theorem 4.3], $\overline{\mathfrak{A}x}$ is a p-hyperinvariant closed ideal for T; therefore we are done if we show that there exists $x \neq 0$ in E such that $\overline{\mathfrak{A}x} \neq \mathsf{E}$. The proof is the same as the final step in the proof of [34, Proposition 2] with some minor changes. Suppose, on the contrary, that $\overline{\mathfrak{A}x} = \mathsf{E}$ for all $x \neq 0$, in E. Since x_0

and Tx_0 are not equal to zero, we can choose an open ball $\mathcal{U}$, with center x_0, such that $0 \notin \overline{T(\mathcal{U})}$ and $0 \notin \overline{\mathcal{U}}$. For any $x \in \overline{T(\mathcal{U})}$ we have $x \neq 0$ and so, by hypothesis, $\overline{\mathfrak{A}x} = \mathsf{E}$. In particular $x_0 \in \overline{\mathfrak{A}x}$, hence there exists $S_x \in \mathfrak{A}$ such that $S_x x \in \mathcal{U}$. Let $\mathcal{U}_x$ be an open neighborhood of x such that $S_x(\mathcal{U}_x) \subseteq \mathcal{U}$. Now $\{\mathcal{U}_x : x \in \overline{T(\mathcal{U})}\}$ is an open covering of the compact set $\overline{T(\mathcal{U})}$, hence there exist $x_1, \cdots, x_n$ in $\overline{T(\mathcal{U})}$, with corresponding $\mathcal{U}_j = \mathcal{U}_{x_j}$, $S_j = S_{x_j}$ $(j = 1, \cdots, n)$ such that

$$\overline{T(\mathcal{U})} \subseteq \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n .$$

Since $Tx_0 \in \overline{T(\mathcal{U})}$, there exists $j_1 \in \{1, \cdots, n\}$ such that $Tx_0 \in \mathcal{U}_{j_1}$, so $S_{j_1}Tx_0 \in \mathcal{U}$ and hence $TS_{j_1}Tx_0 \in \overline{T(\mathcal{U})}$. Repeating the argument we obtain a sequence $\{j_m : m \in \mathbb{N}\}$ in $\{1, 2, \cdots, n\}$ such that:

$$g_m = S_{j_m}TS_{j_{m-1}}T \cdots S_{j_1}Tx_0 \in \mathcal{U} \qquad \forall m \in \mathbb{N}.$$

Now write $S_j = S_j^{(1)} - S_j^{(2)}$ with $0 \leq S_j^{(i)} \leq R_j^{(i)} \in \{T\}'_+$ $(j = 1, \cdots, n; i = 1, 2)$ and let $C = \max\{\, \|R_j^{(i)}\| : j \in \mathbb{N}, n;\ i = 1, 2\}$. Then

$$| g_m | = | S_{j_m}T \cdots S_{j_1}Tx_0 | \leq (R_{j_m}^{(1)} + R_{j_m}^{(2)})T \cdots (R_{j_1}^{(1)} + R_{j_1}^{(2)})Tx_0$$
$$= (R_{j_m}^{(1)} + R_{j_m}^{(2)}) \cdots (R_{j_1}^{(1)} + R_{j_1}^{(2)})T^m x_0 .$$

and hence $\|g_m\| \leq (2C)^m \|T^m x_0\|$, *i.e.*

$$\|g_m\|^{1/m} \leq 2C\|T^m x_0\|^{1/m} \quad \forall m \in \mathbb{N}.$$

Since T is quasinilpotent at x_0, $\|g_m\| \longrightarrow 0$ and hence $0 \in \overline{\mathcal{U}}$ contradicting the choice of $\mathcal{U}$.

∎

Definition 2.3.4 Let E be a Banach lattice. A semigroup $\mathcal{S}$ of positive operators in $\mathfrak{B}(\mathsf{E})$ is said to be *convex* if it is closed under addition and positive scalar multiplication.

A trivial example of a convex semigroup is

$$\{\, p(T) : p \text{ is a polynomial with positive coefficients}\}$$

in which T is any positive operator on a given Banach lattice. For nontrivial examples of convex semigroups see Section 4.2. In the following we prove several decomposability theorems for convex semigroups of positive operators.

Theorem 2.3.5 *Let* E *be a Dedekind complete Banach lattice whose positive cone contains a quasi-interior point. Suppose* $\mathcal{S} \subseteq \mathfrak{B}(\mathsf{E})$ *is a convex semigroup of quasinilpotent positive operators that contains a non-zero compact operator* K. *Then* $\mathcal{S}$ *is decomposable.*

Proof: Suppose $\mathcal{S}$ is not decomposable. Let

$$\mathcal{J} = \{\, 0 \leq R \in \mathfrak{B}(\mathsf{E}) \,:\, RS, SR \in \mathcal{S} \;\; \forall S \in \mathcal{S} \},$$

$$\mathfrak{A}^+ = \{\, 0 \leq T \in \mathfrak{B}(\mathsf{E}) \,:\, \exists R \in \mathcal{J} \text{ such that } T \leq R \,\},$$

and

$$\mathfrak{A} = \{\, T_1 - T_2 \,:\, T_1, T_2 \in \mathfrak{A}^+ \,\}.$$

It can be easily checked that $\mathfrak{A}^+$ is a semigroup that is closed under addition and positive scalar multiplication and hence $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}(\mathsf{E})$.

Since each element of $\mathfrak{A}$ is regular, $|T|$ exists for each $T \in \mathfrak{A}$ as E is a complete Banach lattice. Let T be in $\mathfrak{A}$. Then $T = T_1 - T_2$, where $T_1, T_2 \in \mathfrak{A}^+$, and there are $R_1, R_2 \in \mathcal{J}$ such that $T_1 \leq R_1$ and $T_2 \leq R_2$. Since

$$|TK| \leq |T|K \leq T_1K + T_2K \leq R_1K + R_2K$$

and since $R_1K + R_2K \in \mathcal{S}$, $|TK|$ is quasinilpotent by Lemma 0.2.6. Therefore TK is quasinilpotent as $\|(TK)^n\| \leq \|\,|(TK)^n|\,\| \leq \|\,|TK|^n\|$ for each positive integer n.

Finally, as in the proof of Proposition 2 of [34], it is easy to verify that $\pi T \in \mathfrak{A}$ whenever $T \in \mathfrak{A}$ and $\pi \in \mathfrak{Z}(\mathsf{E})$.

By the above observations we can deduce that for each $x \in \mathsf{E}$ the set $\mathfrak{A}x$ is a subspace of E which is invariant under $\mathcal{S}$. Since $I \in \mathfrak{A}$, $\mathfrak{A}x \neq \{0\}$ for each non-zero element x of E.

By Lemma 2.3.2 for each $x \in \mathsf{E}$, $\mathsf{J}_x = \overline{\mathfrak{A}x}$ is a closed ideal of E. Since J_x is invariant under $\mathcal{S}$, we have $\mathsf{J}_x = \mathsf{E}$ for each nonzero element $x \in \mathsf{E}$ as $\mathcal{S}$ is indecomposable.

We now apply Lomonosov's Technique in the proof of Lemma 8.22 of [40] to find a finite number of operators $A_i \in \mathfrak{A}$, $1 \le i \le n$, and a finite number of real numbers $b_i \in [0,1]$, $1 \le i \le n$, such that

$$\sum_{i=1}^{n} b_i A_i K x = x \qquad \text{for some } x \in V,$$

where $V = \{x \in \mathsf{E} : \|x - x_0\| < 1\}$ and $x_0 \in \mathsf{E}$ with $\|x_0\| > 1$. This shows that if $T = \sum_{i=1}^{n} b_i A_i$, then $T \in \mathfrak{A}$ and 1 is in the point spectrum of TK. But, as we observed above, TK is quasinilpotent, a contradiction.

■

Remark: As we mentioned in Section 0.2, many classical Banach lattices are Dedekind complete Banach lattices. It is also known [46, Theorem II.5.11] that all reflexive Banach lattices are Dedekind complete Banach lattices. Therefore the proof of Theorem 2.3.5 is valid for a large number of Banach lattices.

We now investigate the decomposability of a commutative semigroup of positive operators, on certain Banach lattices, without imposing a compactness condition.

Lemma 2.3.6 *Suppose* E *is a Banach lattice whose positive cone* E_+ *contains an atom and suppose* $\mathcal{S}$ *is a convex semigroup of quasinilpotent positive operators on* E*. Then* $\mathcal{S}$ *is decomposable.*

Proof: Suppose $\mathcal{S}$ is indecomposable. Let $x_0 \in \mathsf{E}_+$ be an atom and let J_0 be the one dimensional ideal generated by x_0. Consider the ideal V which is generated by the set

$$A = \{ Sx_0 : S \in \mathcal{S} \}.$$

Since $\mathcal{S}$ is a semigroup of positive operators, V is invariant for $\mathcal{S}$ and hence $\overline{\mathsf{V}} = \mathsf{E}$ as $\mathcal{S}$ is indecomposable. Thus, by [46, Proposition III.1.1],

$$\overline{\mathsf{J}_0 \cap \mathsf{V}} = \mathsf{J}_0 \cap \overline{\mathsf{V}} = \mathsf{J}_0,$$

which shows $\mathsf{J}_0 \cap \mathsf{V} \neq \{0\}$ and hence $\mathsf{J}_0 \subseteq \mathsf{V}$. Since $\mathcal{S}$ is convex, there exists $S \in \mathcal{S}$ such that $x_0 \le Sx_0$. For this S we have $r(S) > 0$ which is a contradiction as $S \in \mathcal{S}$

and S should be quasinilpotent.

∎

Lemma 2.3.7 *Suppose* $\mathsf{E} = C_0(\Omega)$ *and let* $\mathcal{S}$ *be a convex semigroup of quasinilpotent positive operators on* E. *Then* $\mathcal{S}$ *is decomposable.*

Proof: By using Urysohn's lemma (cf. [43, Section 2.12]) we can find a nonzero function $f \in C_0(\Omega)$ with the following properties:

(i) $0 \leq f \leq 1$ on Ω,

(ii) $\mathcal{K}_0 = supp(f)$ is compact.

Suppose $\mathcal{S}$ is indecomposable. If $Sf = 0$ for all $S \in \mathcal{S}$ then the null ideal

$$\mathsf{N}_{\mathcal{S}} = \{ f \in C_0(\Omega) \ : \ Sf = 0 \ \text{for all} \ S \in \mathcal{S} \}$$

is a nontrivial closed ideal of E which is invariant for $\mathcal{S}$, a contradiction. Thus, there exists $S \in \mathcal{S}$ such that $Sf \neq 0$. Now we distinguish two cases:

Case 1) There exists $x_0 \in \Omega$ such that $(Sf)(x_0) = 0$ for all $S \in \mathcal{S}$.
In this case the closed ideal

$$\mathsf{J} = \overline{\{g \in C_0(\Omega) \ : \ |g| \leq Sf \ \textit{for some} \ S \in \mathcal{S}\}}$$

is a nontrivial closed ideal of E which is invariant for $\mathcal{S}$, contradicting the indecomposability of $\mathcal{S}$.

Case 2) Given $x \in \Omega$ there exists $S_x \in \mathcal{S}$ such that $(S_x f)(x) \neq 0$. Hence for each $x \in \mathcal{K}_0$ there exists an $S_x \in \mathcal{S}$ and a neighborhood $\mathcal{V}_x$ of x such that

$$(S_x f)(t) > 0 \quad \forall t \in \mathcal{V}_x.$$

Since $\mathcal{K}_0$ is compact there exists a finite number of elements $S_1, S_2, \cdots, S_n$ of $\mathcal{S}$ and a finite number of open subsets $\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n$ of Ω such that:

$$\mathcal{K}_0 \subset \bigcup_{i=1}^{n} \mathcal{V}_i,$$

and for each $i = 1, 2, \cdots n$

$$(S_i f)(t) > 0 \quad \forall t \in \mathcal{V}_i.$$

Now if we put $S = \sum_{i=1}^{n} S_i$, then $S \in \mathcal{S}$ and

$$(Sf)(t) > 0 \quad \forall t \in \mathcal{K}_0 .$$

Therefore, there exists $\epsilon > 0$ such that $\epsilon f \leq Sf$, and hence $r(S) \geq \epsilon > 0$, another contradiction.

■

Lemma 2.3.8 *Let* E *be any Banach lattice and let* T *be any nonzero quasinilpotent positive operator on* E. *Then the subset* $\mathcal{J}$ *of* $\{T\}'_+$ *defined by*

$$\mathcal{J} = \{\, S \in \{T\}'_+ : \exists\, R \in \{T\}'_+ \text{ such that } S \leq RT \,\},$$

is a nonzero semigroup ideal of $\{T\}'_+$ *all of whose elements are quasinilpotent. Furthermore* $\mathcal{J}$ *is a convex semigroup.*

Proof: Since $I, T \in \{T\}'_+$ and $T = IT$, $T \in \mathcal{J}$ and hence $\mathcal{J}$ is nonzero. Since for each $R \in \{T\}'_+$ we have $r(RT) \leq r(R)r(T)$, $r(RT) = 0$. Thus all of the elements of $\mathcal{J}$ are quasinilpotent as $S \leq RT$ implies $\|S^n\| \leq \|(RT)^n\|$, $n \in \mathbb{N}$, by Lemma 0.2.6. and hence $r(S) \leq r(RT)$.

Now consider $S \in \mathcal{J}$ and $R \in \{T\}'_+$. Then, there exists $R_0 \in \{T\}'_+$ such that $S \leq R_0 T$. Thus $RS \leq RR_0T$ and $SR \leq R_0TR$ and hence $RS, SR \in \mathcal{J}$ as $RR_0, R_0R \in \{T\}'_+$ and $R_0TR = R_0RT$. Therefore $\mathcal{J}$ is an ideal of $\{T\}'_+$. The last part is obvious as $\{T\}'_+$ is itself a convex semigroup.

■

The following two theorems reveal that we may drop the compactness hypothesis in certain situations and find nontrivial p-hyperinvariant closed ideals. even in nonreflexive Banach lattices.

Theorem 2.3.9 *Suppose* E *is a Banach lattice with atoms. Then, every quasinilpotent positive operator* T *on* E *has a nontrivial p-hyperinvariant closed ideal. In particular, every quasinilpotent positive operator* T *on* l_p, $1 \leq p < \infty$ *has a nontrivial p-hyperinvariant closed ideal.*

Proof: Let $\mathcal{J}$ be defined as in the proof of Lemma 2.3.8. Then $\mathcal{J}$ is decomposable by Lemma 2.3.8 and Lemma 2.3.6, and hence $\{T\}'_+$ is decomposable by Lemma 2.2.4.

■

Theorem 2.3.10 *Every quasinilpotent positive operator on $C_0(\Omega)$ has a p-hyperinvariant closed ideal. More generally, if* E *is a closed ideal of an AM-space* F *with unit, then every quasinilpotent positive operator T on* E *has a nontrivial p-hyperinvariant closed ideal.*

Proof: Let $\mathcal{J}$ be defined as in the proof of Lemma 2.3.8. Then $\mathcal{J}$ is decomposable by Lemma 2.3.8 and Lemma 2.3.7, and hence $\{T\}'_+$ is decomposable by Lemma 2.2.4.

For the general case we proceed as follows. By Proposition 0.3.2, F is isomorphic to $C(\mathcal{K})$ for a suitable compact Hausdorff space $\mathcal{K}$, and, by Proposition 0.1.4, E is isomorphic to $C_0(\Omega)$, where Ω is a locally compact open subset of $\mathcal{K}$. The general case now follows from the first part of the theorem.

■

Using the results of this section we may derive some simple decomposability facts concerning a commutative collection of positive operators on certain Banach lattices. The interesting fact is that in some cases such collections need not contain a compact operator.

Proposition 2.3.11 *Suppose $\mathfrak{C}$ is a commutative collection of positive operators on* E *and $T \in \mathfrak{B}(\mathsf{E})$ is a nonzero positive operator that is quasinilpotent at a nonzero element of* E_+.

(a) *If T is compact and $\mathfrak{C}$ contains T, then $\mathfrak{C}$ is decomposable.*

(b) *If $T^3 \neq 0$, if $T^3 \in \mathfrak{C}$, and if T is dominated by a compact operator, then $\mathfrak{C}$ is decomposable.*

(c) *If* E *an AL-space, if $T^2 \neq 0$, if $T^2 \in \mathfrak{C}$, and if T is weakly compact, then $\mathfrak{C}$ is decomposable.*

(d) *If* E *is an AL-space, if $T^4 \neq 0$, if $T^4 \in \mathfrak{C}$, and if T is dominated by a weakly compact operator, then $\mathfrak{C}$ is decomposable.*

Proof: (a) Since $\mathfrak{C} \subseteq \{T\}'_+$, Theorem 2.1.3 gives the desired result.

(b) By Proposition 0.2.4. T^3 is compact and hence $\mathfrak{C}$ is decomposable by (a).

(c) As in the proof of Corollary 2.1.4, T^2 is compact. Now apply (a).

(d) By Proposition 0.2.5, T^2 is weakly compact and hence $\mathfrak{C}$ is decomposable by (c).

■

Proposition 2.3.12 *Suppose* E *is either a closed ideal of an AM-space with unit or a Banach lattice with atoms. Suppose also that* $\mathfrak{C}$ *is a commutative collection of positive operators in* $\mathfrak{B}(\mathsf{E})$. *If* $\mathfrak{C}$ *contains a quasinilpotent operator* T, *then* $\mathfrak{C}$ *is decomposable.*

Proof: Since $\mathfrak{C} \subseteq \{T\}'_+$, an application of Theorems 2.3.9 and 2.3.10 gives the desired result.

■

Other examples of decomposable collections of operators on E, that may not contain a compact operator. can be obtained by using the following simple fact. The proof is omitted.

Lemma 2.3.13 *Suppose* $\mathfrak{C} \subseteq \mathfrak{B}(\mathsf{E})$, $T \in \mathfrak{B}(\mathsf{E})$. T *is a positive operator, and* $\mathsf{J} \in$ *Ilat* (T).

(a) *If* T *dominates all of the operators in* $\mathfrak{C}$, *then* $\mathsf{J} \in$ *Ilat* $(\mathfrak{C})$.

(b) *If all elements of* $\mathfrak{C}$ *are positive and if* $S \leq T$ *for all* $S \in \mathfrak{C}$, *then* $\mathsf{J} \in$ *Ilat* $(\mathfrak{C})$.

Proposition 2.3.14 *Suppose* $\mathfrak{C} \subseteq \mathfrak{B}(\mathsf{E})$, T *is a quasinilpotent positive operator on* E *and either (a)* T *dominates all members of* $\mathfrak{C}$ *or (b) all elements of* $\mathfrak{C}$ *are positive and* T *majorizes* $\mathfrak{C}$. *Then in each of the following cases* $\mathfrak{C}$ *is decomposable:*

(i) E *is a closed ideal of an AM-space,*

(ii) E *is a Banach lattice with atoms,*

(iii) E *is an AL-space and* T *is weakly compact,*

(iv) E *is any Banach lattice and T is compact.*

Another example of a decomposable collection of operators on E that does not contain a compact operator can be obtained by using [3, Theorem 7.1].

Proposition 2.3.15 *Suppose* $\mathcal{S}$ *is a semigroup of positive operators on* E *that is generated by the set* $\mathfrak{C} = \{T, S\}$. *If* $ST = TS$, S *is quasinilpotent at a nonzero positive element of* E, *and* T *dominates a nonzero compact operator, then* $\mathcal{S}$ *is decomposable.*

2.4 Decomposability of a Semigroup of Integral Operators on $C(\mathcal{K})$

It is known that every positive integral operator on a Banach lattice E that is quasinilpotent at some $x_0 > 0$ is decomposable (cf. [3, Corollary 6.4]). Therefore, it is natural to ask if the same result holds for a semigroup of such operators. In general, we do not know the answer. In some special cases, when the quasinilpotency is under discussion, the answer is affirmative as has been observed in [10]. In this section we follow the strategy given in [10] to show that under some conditions a semigroup of positive quasinilpotent integral operators on $C(\mathcal{K})$ is decomposable.

In what follows μ is a non-negative σ-finite regular Borel measure on Ω such that every non-empty open subset of Ω has positive measure. .

Lemma 2.4.1 *Let* U *be a non-empty open subset of* Ω. *Suppose for each* $k \in \mathbb{R}^+$ *the operator* T_k *is an integral operators on* $C_0(\Omega)$ *with a non-negative kernel* K_{T_k} *such that* $K_{T_k}(x, y) \geq k > 0$ *on* $F \times U$, *where* F *is some measurable subset of* Ω *with nonzero measure. Then there exists a measurable subset* G *of* U *of nonzero finite measure such that*

$$\|T_k\| \geq k\mu(G) \qquad \text{for all } k \in \mathbb{R}^+.$$

Proof : Since μ is σ-finite and $\mu(U) > 0$ we can choose a measurable subset A

of U with $0 < \mu(A) < \infty$. Let $f = \chi_A$ and apply the techniques used in the proof of Lusin's Theorem, (cf. [43, Theorem 2.23]), to find a function g in $C_c(\Omega)$ with the following properties:

(i) $g(x) \geq 0 \ \forall x \in \Omega$,

(ii) $\mu(B) < \mu(A)/2$, where $B = \{x \in \Omega : g(x) \neq f(x)\}$, and

(iii) $\|g\|_\infty \leq \|f\|_\infty = 1$.

Fix $k \in \mathbb{R}^+$. Since $C_c(\Omega) \subseteq C_0(\Omega)$ we have

$$T_k g(x) = \int_\Omega K_{T_k}(x,y)g(y)d\mu(y) \geq \int_A K_{T_k}(x,y)g(y)d\mu(y).$$

So for $x \in F$,

$$T_k g(x) \geq k\Big\{\int_{A_1} g(y)d\mu(y) + \int_{A_2} g(y)d\mu(y)\Big\},$$

where $A_1 = \{y \in A : g(y) = f(y) = 1\}$ and $A_2 = \{y \in A : g(y) \neq f(y)\}$. Since $A_2 \subseteq B$, $\mu(A_2) \leq \mu(B) < \mu(A)/2$. Hence $\mu(A_1) = \mu(A) - \mu(A_2) > \mu(A)/2 > 0$, and

$$T_k g(x) \geq k\Big\{\mu(A_1) + \int_{A_2} g(y)d\mu(y)\Big\} \geq k\mu(A_1) \quad \forall x \in F.$$

as $\int_{A_2} g(y)d\mu(y) \geq 0$, and hence $T_k g(x) \geq k\mu(A_1)$ for all $x \in F$. Therefore:

$$\|T_k g\|_\infty = \sup\{T_k g(x) : x \in \Omega\} \geq \sup\{T_k g(x) : x \in F\} \geq k\mu(A_1).$$

So with $G = A_1$ we obtain

$$\|T_k\| = \sup\{\|T_k h\|_\infty : \|h\|_\infty \leq 1\} \geq k\mu(G).$$

■

Lemma 2.4.2 *Suppose T is an integral operator on $C_0(\Omega)$ with a non-negative kernel K_T. If $K_T(x,y) \geq k > 0$ on some rectangle $U \times U$, where U is a non-empty open subset of Ω , then there exists a measurable subset G of U of nonzero finite measure such that $r(T) \geq k\mu(G)$.*

Proof: Use Lemma 2.4.1 to find a measurable subset G with the stated properties given in that Lemma.

Let $K_T^{(n)}$ denote the kernel of T^n. Then for $x, y \in U$.

$$K_T^{(n)}(x,y) = \int_\Omega K_T(x,t_1)K_T(t_1,t_2)\cdots K_T(t_{n-1},y)dt_1\cdots dt_{n-1}$$

$$\geq \int_{U\times U\times\cdots\times U} k^n dt_1 dt_2\cdots dt_{n-1} = k^n\mu(U)^{n-1}.$$

Therefore

$$\|T^n\| \geq k^n\mu(U)^{n-1}\mu(G) \geq k^n\mu(G)^n.$$

which means $\|T^n\|^{1/n} \geq k\mu(G)$ for all n, and hence $r(T) \geq k\mu(G)$.

■

Lemma 2.4.3 *If T is a quasinilpotent integral operator on $C_0(\Omega)$ with non-negative, lower semicontinuous kernel K_T, then $K_T(x,x) = 0$ for all $x \in \Omega$.*

Proof: Suppose not and choose any x_0 with $K_T(x_0, x_0) = 2k > 0$. Lower semicontinuity implies there is an open set U such that $K_T(x,y) \geq k$ for all $(x,y) \in U \times U$. Now apply Lemma 2.4.2 to obtain a subset G of U of nonzero finite positive measure such that $r(T) \geq k\mu(G)$, which contradicts the fact that T is quasinilpotent.

■

Suppose $\mathcal{S}$ is a semigroup of quasinilpotent integral operators on $C_0(\Omega)$ such that every operator on $\mathcal{S}$ has a non-negative, lower semicontinuous kernel. By using Lemma 2.4.3 and an argument similar to the proof of [10, Theorem 3.4] we can show that there exists an open set V of finite measure such that the subspace

$$J = \{ f \in C_0(\Omega) : f = 0 \text{ on } \Omega \setminus V \}$$

is invariant under $\mathcal{S}$. Since J is a closed ideal of $C_0(\Omega)$, we conclude that $\mathcal{S}$ is decomposable. We summarize this observation in the following theorem and use the procedure given in the proof of [10, Theorem 3.4] to give a sketch of its proof.

Theorem 2.4.4 *Let $\mathcal{S}$ be a semigroup of quasinilpotent integral operators on $C_0(\Omega)$ by way of the measure μ, such that every operator in $\mathcal{S}$ has a non-negative, lower semicontinuous kernel. Then $\mathcal{S}$ is decomposable.*

Sketch of Proof : If $\mathcal{S} = \{0\}$, with any open subset V of Ω, the closed ideal $J = \{f \in C_0(\Omega) \ : \ f(t) = 0 \ \textit{for all} \ t \in \Omega \setminus V\}$ is invariant under $\mathcal{S}$. Otherwise choose $T \in \mathcal{S}$, with $T(x_0, y_0) > 0$ for some $(x_0, y_0) \in \Omega \times \Omega$, and use the lower semicontinuity of its kernel and Lemma 2.4.3 to find two open subsets U_0 and V_0 of Ω with the following properties:

(i) $U_0 \cap V_0 = \emptyset$,

(ii) $K_S(y, x) = 0$ whenever $S \in \mathcal{S}$ and $(x, y) \in U_0 \times V_0$.

Now for each $x \in U_0$ define

$$W_x = \{t \in \Omega \ : \ K_S(t, x) = 0 \ \textit{for all} \ S \in \mathcal{S}\}$$

and observe that it is a closed subset of Ω that includes V_0. We distinguish two cases:

(1) $\mu(\Omega \setminus W_x) = 0$ for every $x \in U_0$. In this case put $V = U_0$ and observe that

$$K_S(x, y) = 0 \quad \forall (x, y) \in (\Omega \setminus V) \times V.$$

whenever $S \in \mathcal{S}$.

(2) $\mu(\Omega \setminus W_x) \neq 0$ for some $x \in U_0$. In this case cut U_0 down and relabel if necessary, to assume that x is x_0. Put $V = \Omega \setminus W_{x_0}$ and show that

$$K_S(x, y) = 0 \quad \forall (x, y) \in W_{x_0} \times (\Omega \setminus W_{x_0}).$$

whenever $S \in \mathcal{S}$.

In each case verify that the closed ideal

$$J = \{f \in C_0(\Omega) \ : \ f(t) = 0 \ \textit{for all} \ t \in \Omega \setminus V\}$$

is invariant under $\mathcal{S}$.

■

2.5 Decomposability of Positive Operators on a Discrete Banach lattice

There are several invariant subspace theorems for positive operators acting on a Banach space with basis. These results that obtained by Abramovich, Aliprantis, and Burkinshaw in [2] and [4] for locally quasinilpotent operators are much stronger than the corresponding analogues for quasinilpotent operators. If a discrete Banach lattice with order continuous norm is under discussion, then these results can be considered as invariant ideal theorems. In this section we follow the strategy given in [10] to obtain one of those results for quasinilpotent positive operators. Then we use it to show that a semigroup of quasinilpotent positive operators on a discrete Banach lattice with order continuous norm is decomposable.

Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X. The sequence of linear functionals $\{f_n\}_{n=1}^{\infty}$ defined by

$$f_j(x) = \alpha_j \qquad \left(x = \sum_{i=1}^{\infty} \alpha_i x_i \in X\,,\ j \in \mathbb{N}\right).$$

is called *the sequence of coefficient functionals associated to the basis* $\{x_n\}_{n=1}^{\infty}$ (we shall write: a.f.).

Thus, if $\{x_n\}_{n=1}^{\infty}$ is a basis of the Banach space X and $\{f_n\}_{n=1}^{\infty}$ the a.f., then every $x \in X$ has a unique expansion of the form

$$x = \sum_{i=1}^{\infty} f_i(x)\, x_i\,.$$

The following is the fundamental fact about the coefficient functionals associated to a basis of a Banach space X. (Cf. [47, Theorem I.3.1]).

Lemma 2.5.1 *Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X. Then the coefficient functionals f_n, $n \in \mathbb{N}$, associated to the basis $\{x_n\}_{n=1}^{\infty}$, are continuous on X and there exists a constant M such that*

$$1 \le \|x_n\|\,\|f_n\| \le M \qquad (n \in \mathbb{N}).$$

Now consider the closed cone generated by the basis $\{x_n\}_{n=1}^{\infty}$. It is easy to check that the coefficient functionals f_n, $n \in \mathbb{N}$, associated to the basis $\{x_n\}_{n=1}^{n}$ are automatically positive with respect to this cone.

Recently in [4] several invariant subspace theorems were established for operators acting on a Banach space with basis. In this section we find invariant closed ideals for semigroups of positive operators on a discrete Banach lattice with order continuous norm.

Lemma 2.5.2 *Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X and let $\{f_n\}_{n=1}^{\infty}$ be the a.f. Suppose T is a quasinilpotent operator on X which is positive with respect to the positive cone generated by $\{x_n\}_{n=1}^{\infty}$. Then*

$$T_{jj} = f_j(Tx_j) = 0 \qquad \textit{for each } j \in \mathbb{N},$$

where $[T_{ij}]$ is the matrix representation of T.

Proof: Observe that for each $n \in \mathbb{N}$ we have

$$(T_{jj})^n \leq f_j(T^n x_j) \qquad j \in \mathbb{N}.$$

Since for each $j \in \mathbb{N}$, f_j is a positive linear functional and since, by Lemma 2.5.1 each f_j is continuous, we also have

$$(T_{jj}))^n \leq \|f_j\| \, \|T^n x_j\| \leq \|f_j\| \, \|x_j\| \, \|T^n\| .$$

Another application of Lemma 2.5.1 reveal that there exists a constant $M \geq 1$ such that

$$T_{jj} \leq M^{1/n} \|T^n\|^{1/n} \qquad j \in \mathbb{N}.$$

Therefore, for each $j \in \mathbb{N}$, $T_{jj} = 0$ as T is quasinilpotent.

■

It is easy to verify that if $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for a discrete Banach lattice with order continuous norm E, then, for each $n \in \mathbb{N}$, x_n is an atom for E. Therefore each discrete Banach lattice with order continuous norm is a Banach lattice with atoms, and hence each quasinilpotent positive operator on such spaces is decomposable by Theorem 2.1.1. This fact can also be obtained under weaker conditions (cf. [4]). In the following we prove this fact by another method and use this method to establish the decomposability of a semigroup of quasinilpotent positive operators on a discrete Banach lattice with order continuous norm.

Theorem 2.5.3 *Let* $\{x_n\}_{n=1}^{\infty}$ *be an unconditional basis for a discrete Banach lattice with order continuous norm* E. *Suppose* T *is a nonzero quasinilpotent positive operator on* E. *Then* T *is decomposable.*

Proof: Fix $j \in \mathbb{N}$ and let $\{f_n\}_{n=1}^{\infty}$ be the sequence of coefficient functionals associated to the basis $\{x_n\}_{n=1}^{\infty}$. Observe that for each $k \in \mathbb{N}$, T^k is a positive quasinilpotent operator. By Lemma 2.5.2, $T_{jj}^k = f_j(T^k x_j) = 0$ for all k. If $Tx_j = 0$, then the null ideal

$$\mathsf{N}_T = \{\, x \in \mathsf{E} \,:\, T(|x|) = 0 \,\}$$

is a nontrivial closed invariant ideal for T.

Suppose $Tx_j \neq 0$ and consider the closed ideal $\overline{\mathsf{J}_T}$, where J_T is the ideal generated by the set $\{\, T^k x_j \,:\, k \in \mathbb{N} \,\}$. Since

$$\mathsf{J}_T = \{\, x \in \mathsf{E} \,:\, \exists\, n_1, \cdots, n_m \in \mathbb{N} \text{ and } \lambda_1, \cdots, \lambda_m \in \mathbb{R}^+ \text{ with } |x| \le \sum_{i=1}^{m} \lambda_i T^{n_i} x_j \,\},$$

it is easy to check that J_T is invariant under T. Hence $\overline{\mathsf{J}_T}$ is also invariant under T.

Let $x \in \mathsf{J}_T$. Then there exists positive integers $n_1, \cdots, n_m$ and positive scalars $\lambda_i, \cdots, \lambda_m$ such that $|x| \le \sum_{i=1}^{m} \lambda_i T^{n_i} x_j$. Since f_j is positive,

$$|f_j(x)| \le f_j(|x|) \le \sum_{i=1}^{m} \lambda_i f_j(T^{n_i} x_j)\,.$$

Therefore $f_j(x) = 0$ for all $x \in \mathsf{J}_T$ and hence $f_j(x) = 0$ for all $x \in \overline{\mathsf{J}_T}$. This shows that $\overline{\mathsf{J}_T}$ is a nontrivial invariant ideal for T.

■

Theorem 2.5.4 *Let* $\{x_n\}_{n=1}^{\infty}$ *be an unconditional basis for a discrete Banach lattice with order continuous norm* E. *Suppose* $\mathcal{S}$ *is a nonzero semigroup of quasinilpotent positive operators on* E. *Then* $\mathcal{S}$ *is decomposable.*

Proof: As in the proof of the previous theorem, for each j the closed ideal $\overline{\mathsf{J}_{\mathcal{S}}}$, where $\mathsf{J}_{\mathcal{S}}$ is the ideal generated by the set $\{ Sx_j : S \in \mathcal{S} \}$, is invariant under $\mathcal{S}$ and $f_j(x) = 0$ for each $x \in \overline{\mathsf{J}_{\mathcal{S}}}$. Thus $\overline{\mathsf{J}_{\mathcal{S}}}$ is a nontrivial closed invariant ideal for $\mathcal{S}$ unless $Sx_j = 0$ for all $S \in \mathcal{S}$, in which case the null ideal

$$\mathsf{N}_{\mathcal{S}} = \{ x \in \mathsf{E} : S(|x|) = 0 \}$$

is a nontrivial closed invariant ideal for $\mathcal{S}$.

■

Chapter 3

Ideal-Triangularizability of Positive Operators on Banach Lattices

It is the purpose of this chapter to introduce a Banach lattice version of triangularizability and a few related results.

3.1 Ideal Chains

Let E be a Banach lattice and let $A(\mathsf{E})$ denote the collection of all closed ideals of E, partially ordered by inclusion. A totally ordered subset $\mathcal{C}$ of $A(\mathsf{E})$ will be called an *ideal chain*. If each element of $\mathcal{C}$ is invariant under a collection of operators $\mathfrak{C}$ on E, we shall call $\mathcal{C}$ as an *invariant ideal chain*. A trivial example of an invariant ideal chain is $\{\{0\}, \mathsf{E}\}$. The existence of nontrivial invariant ideal chains for several classes of operators on Banach lattices can be deduced from some of the theorems in Chapter 2.

A similar argument to the one given at the beginning of [41, Section 4.3], shows the following:

If $\mathfrak{G}$ denotes the class of all ideal chains that is partially ordered by the inclusion relation then

(1) every totally ordered subset $\mathfrak{G}_0$ of $\mathfrak{G}$ has a least upper bound and hence

an application of Zorn's Lemma reveals that $\mathfrak{G}$ contains maximal elements, which we call *maximal ideal chains*,

(2) every ideal chain $\mathcal{G}$ is contained in at least one maximal ideal chain $\mathcal{G}_1$.

(3) the class $\mathfrak{G}_i$ of all invariant ideal chains contains maximal elements, which we call *maximal invariant ideal chains*. Besides, each invariant ideal chain is contained in at least one maximal invariant ideal chain.

Remark: Of course it is not the case in general that maximal invariant ideal chains are necessarily maximal ideal chains. However, the following lemma gives a sufficient condition for this to happen.

The proof of the following lemma is a simple modification of the proof given in [41, Lemma 4.3.1 and Lemma 4.3.2].

Lemma 3.1.1 *Let* E *be a Banach lattice and* $\mathfrak{C} \subseteq \mathfrak{B}(\mathsf{E})$. *Suppose for each pair* $\mathsf{J}_1, \mathsf{J}_2 \in Ilat(\mathfrak{C})$ *with* $\mathsf{J}_1 \subseteq \mathsf{J}_2$ *and* $\dim(\mathsf{J}_2/\mathsf{J}_1) \geq 2$ *the compression*

$$\hat{\mathfrak{C}} = \{\, \hat{T} \,:\, T \in \mathfrak{C} \text{ and } \hat{T} \text{ is the compression of } T \text{ to } \mathsf{J}_2/\mathsf{J}_1 \,\}$$

is decomposable. Then $\mathfrak{C}$ *is ideal-triangularizable.*

Proof: By Definition 0.4.6, it suffices to show that each maximal invariant ideal chain is a maximal ideal chain.

Suppose that $\mathcal{C}$ is a maximal invariant ideal chain for $\mathfrak{C}$. Since $\mathcal{C}$ is maximal,

$$\{0\} \in \mathcal{C} \text{ and } \mathsf{E} \in \mathcal{C}. \qquad (1)$$

Let $\mathcal{C}_0$ be a subfamily of $\mathcal{C}$. It is clear that

$$\mathsf{I} = \cap\{\, \mathsf{J} \,:\, \mathsf{J} \in \mathcal{C}_0 \,\}$$

is a closed ideal. Since $\mathcal{C}_0$ is totally ordered by inclusion the set

$$\mathsf{V} = \cup\{\, \mathsf{J} \,:\, \mathsf{J} \in \mathcal{C}_0 \,\}$$

is a linear manifold of E. Now suppose $x \in \mathsf{V}$ and $y \in \mathsf{E}$ are such that $|y| \leq |x|$. Let $\mathsf{J} \in \mathcal{C}_0$ be such that $x \in \mathsf{J}$. Since J is an ideal, $y \in \mathsf{J}$ and hence $y \in \mathsf{V}$. Thus V is an ideal of E and hence its norm closure, *i.e.* $\mathsf{I}' = \overline{\mathsf{V}}$, is a closed ideal of E. It is obvious that both I and I' are invariant for $\mathfrak{C}$ and it can be easily checked that $\mathcal{C} \cup \{\mathsf{I}\}$ and $\mathcal{C} \cup \{\mathsf{I}'\}$ are invariant chains. Thus, by maximality of $\mathcal{C}$,

$$\mathsf{I} \textit{ and } \mathsf{I}' \textit{ are in } \mathcal{C} \textit{ for each subfamily } \mathcal{C}_0 \textit{ of } \mathcal{C}. \qquad (2)$$

Suppose $\mathsf{J} \in \mathcal{C}$ and define

$$\mathsf{W} = \cup\{\, \mathsf{J}' : \mathsf{J}' \in \mathcal{C} \textit{ and } \mathsf{J}' \subsetneqq \mathsf{J} \,\}.$$

Let

$$\mathsf{J}_- = \overline{\mathsf{W}} \qquad (\star)$$

and assume $\dim(\mathsf{J}/\mathsf{J}_-) \geq 2$. By hypothesis, there exists a nontrivial closed ideal $\hat{\mathsf{K}}$ of J/J_- such that $\hat{T}(\hat{\mathsf{K}}) \subseteq \hat{\mathsf{K}}$ for all $\hat{T} \in \hat{\mathfrak{C}}$.

If we define

$$\mathsf{K} = \{\, x \in \mathsf{J} : x + \mathsf{J}_- \in \hat{\mathsf{K}} \,\},$$

then, as the canonical map π is a lattice homomorphism, K is a closed ideal of E. It is also easy to verify that $T(\mathsf{K}) \subseteq \mathsf{K}$ for all $T \in \mathfrak{C}$ and $\mathsf{J}_- \subsetneqq \mathsf{K} \subsetneqq \mathsf{J}$. Given any ideal J_0 in $\mathcal{C}$, we have either (a) $\mathsf{J} \subseteq \mathsf{J}_0$, and so $\mathsf{K} \subsetneqq \mathsf{J}_0$, or (b) $\mathsf{J}_0 \subsetneqq \mathsf{J}$ and, by definition of J_-, $\mathsf{J}_0 \subseteq \mathsf{J}_- \subsetneqq \mathsf{K}$. Hence $\mathsf{K} \notin \mathcal{C}$ and $\mathcal{C} \cup \{\mathsf{K}\}$ is an invariant chain, contradicting the maximality of $\mathcal{C}$. Therefore,

$$\textit{for each } \mathsf{J} \in \mathcal{C}, \textit{ the quotient space } \mathsf{J}/\mathsf{J}_- \textit{ has dimension at most 1.} \qquad (3)$$

Now suppose $\mathcal{C}$ is not a maximal ideal chain. Then there is an ideal chain $\mathcal{G}$ which properly contains $\mathcal{C}$. Let $\mathsf{J} \in \mathcal{G} \setminus \mathcal{C}$. By (1), $\mathsf{J} \neq \{0\}$ and $\mathsf{J} \neq \mathsf{E}$. By (2), $\mathsf{I}, \mathsf{I}' \in \mathcal{C}$, where

$$\mathsf{I} = \cap\{\, \mathsf{J}' \in \mathcal{C} : \mathsf{J} \subseteq \mathsf{J}' \,\}$$

and $\mathsf{I}' = \overline{\mathsf{V}}$ with

$$\mathsf{V} = \cup\{\, \mathsf{J}' \in \mathcal{C} : \mathsf{J}' \subseteq \mathsf{J} \,\}.$$

Since $\mathsf{I}' \subseteq \mathsf{J} \subseteq \mathsf{I}$ and since $\mathsf{J} \notin \mathcal{C}$,

$$\mathsf{I}' \subsetneqq \mathsf{J} \subsetneqq \mathsf{I}. \qquad (\star\star)$$

Since $\mathsf{I}' \in \mathcal{C}$ and $\mathsf{I}' \subsetneq \mathsf{I}$, $\mathsf{I}' \subseteq \mathsf{I}_-$, where I_- is defined as $(\star)$. Let $\mathsf{J}' \in \mathcal{C}$ with $\mathsf{J}' \subsetneq \mathsf{I}$. Then $\mathsf{J} \not\subseteq \mathsf{J}'$, by definition of I. Since $\mathcal{G}$ is totally ordered and since $\mathsf{J}', \mathsf{J} \in \mathcal{G}$, $\mathsf{J}' \subseteq \mathsf{J}$, and so $\mathsf{J}' \subseteq \mathsf{I}'$ by definition of I'. This shows $\mathsf{I}_- \subseteq \mathsf{I}'$ and hence $\mathsf{I}' = \mathsf{I}_-$. Thus, by $(\star\star)$, $\dim(\mathsf{I}/\mathsf{I}_- \geq 2$, contradicting (3).

■

3.2 Ideal-Triangularizability of Positive Operators

In this and the next sections we present some examples of ideal-triangularizability results.

Theorem 3.2.1 *Suppose* $\mathsf{E} = C(\mathcal{K})$ *and* $T \in \mathfrak{B}(\mathsf{E})$ *is a quasinilpotent positive operator. Then* T *is ideal-triangularizable.*

Proof: By Theorem 2.1.1, T is decomposable. Let $\mathsf{J}_1, \mathsf{J}_2 \in$ *Ilat* (T), with $\mathsf{J}_1 \subseteq \mathsf{J}_2$ and $\dim(\mathsf{J}_2/\mathsf{J}_1) \geq 2$. Since E is an AM-space, J_2 is an AM-space, by Lemma 0.3.5. Since J_1 is a closed ideal of both J_2 and E, $\mathsf{J}_2/\mathsf{J}_1$ is a closed ideal of E/J_1, $\mathsf{J}_2/\mathsf{J}_1$ and E/J_1 are both AM-spaces and $1 + \mathsf{J}_1$ is a unit for E/J_1 by Lemma 0.3.6. Therefore $\mathsf{E}/\mathsf{J}_1 \cong C(\mathcal{K}')$ for a suitable compact Hausdorff space $\mathcal{K}'$, by Proposition 0.3.2, and hence $\mathsf{J}_2/\mathsf{J}_1 \cong C_0(\Omega')$, where $\Omega' \subseteq \mathcal{K}'$ is a locally compact Hausdorff space.

Since T is a quasinilpotent positive operator on E, T is a quasinilpotent positive operator on J_2. Hence $\hat{T}$ is a quasinilpotent positive operator on $\mathsf{J}_2/\mathsf{J}_1$ by Lemma 0.4.3 and Lemma 0.4.4(a). Therefore Theorem 2.1.1 implies that $\hat{T}$ is decomposable. Thus $\mathfrak{C} = \{T\}$ satisfies the condition of Lemma 3.1.1 and hence T is ideal-triangularizable.

■

Theorem 3.2.2 *If* E *is any Banach lattice and* $K \in \mathfrak{B}(\mathsf{E})$ *is a compact quasinilpotent positive operator, then* $\mathfrak{C} = \{K\}'_+$ *is ideal-triangularizable.*

Proof: By Theorem 2.1.3, $\mathfrak{C}$ is decomposable. Let $\mathsf{J}_1, \mathsf{J}_2 \in$ *Ilat* $(\mathfrak{C})$), with $\mathsf{J}_1 \subseteq \mathsf{J}_2$ and $\dim(\mathsf{J}_2/\mathsf{J}_1) \geq 2$. We know that $\hat{K}$ is a compact quasinilpotent positive operator on $\mathsf{J}_2/\mathsf{J}_1$ by Lemma 0.4.3, Lemma 0.4.4(a), and Lemma 0.4.4(b). Hence, by Theorem

2.1.3, the compression $\hat{\mathfrak{C}}$,of $\mathfrak{C}$ to J_2/J_1, is decomposable, as $\hat{\mathfrak{C}} \subseteq \{\hat{K}\}'$. Therefore $\mathfrak{C}$ satisfies the condition of Lemma 3.1.1 and hence $\mathfrak{C}$ is ideal-triangularizable.

■

Theorem 3.2.3 *Suppose $K \in \mathfrak{B}(\mathsf{E})$ is a weakly compact quasinilpotent positive operator such that $K^2 \neq 0$. Suppose also that E is either $L_1(\mu)$ or $C_0(\Omega)$. If $\mathfrak{C} = \{K\}'_+$, then $\mathfrak{C}$ is ideal-triangularizable.*

Proof: Since $L_1(\mu)$ is an AL-space and $C(\Omega)$ is an AM-space, K^2 is compact by Proposition 0.3.8. Since $\mathfrak{C} \subseteq \{K^2\}'_+$, the result follows from Theorem 3.2.2.

■

Remark Propositions 0.3.2 and 0.3.3 show that Theorem 3.2.1 and Theorem 3.2.3 are valid for more general Banach lattices, namely AM-spaces with unit and AL-spaces.

3.3 Discrete Banach Lattices and Ideal-Triangularizability

In this section we will assume that $\{x_n\}_{n=1}^{\infty}$ is the unconditional basis of the given discrete Banach lattice with order continuous norm.

Lemma 3.3.1 *Suppose E is a discrete Banach lattice with order continuous norm. Then, for each $x \in \mathsf{E}$, with $x = \Sigma_{n=1}^{\infty} \alpha_n x_n$, the following holds:*

$$| x | = \sum_{n=1}^{\infty} | \alpha_n | x_n .$$

Proof: We first find $x^+ = x \vee 0$. Let $x^+ = \Sigma_{n=1}^{\infty} \beta_n x_n$ and let $y = \Sigma_{n=1}^{\infty} (\alpha_n)^+ x_n$. Since $x^+ \geq x$ and $x^+ \geq 0$, we obtain

$$\beta_n \geq \alpha_n \text{ and } \beta_n \geq 0, \quad \forall n \in \mathbb{N}.$$

Hence $\beta_n \geq (\alpha_n)^+$ for all n which means $x^+ \geq y$. On the other hand, $y \geq x$ as $\alpha_n^+ \geq \alpha_n$ for all $n \in \mathbb{N}$ and $y \geq 0$ as $\alpha_n^+ \geq 0$ for all $n \in \mathbb{N}$. Thus $y \geq x^+$ and hence

$$y = x^+ = \sum_{n=1}^{\infty} (\alpha_n)^+ x_n .$$

Similarly,

$$x^- = \sum_{n=1}^{\infty} (\alpha_n)^- x_n .$$

Therefore;

$$| x | = x^+ + x^- = \sum_{n=1}^{\infty} \Big((\alpha_n)^+ + (\alpha_n)^- \Big) x_n = \sum_{n=1}^{\infty} | \alpha_n | x_n .$$

■

Definition 3.3.2 Let J be an ideal of a discrete Banach lattice with order continuous norm E. We say that x_i *participates* in J if there exists $x \in J$ with $x = \Sigma_{n=1}^{\infty} \alpha_n x_n$ such that $\alpha_i \neq 0$.

Lemma 3.3.3 *Suppose* J *is a closed ideal of a discrete Banach lattice with order continuous norm* E *and* $\mathcal{P} = \{ x_i : x_i$ *participates in* J $\}$. *Then,* $\mathcal{P} \subset J$ *and*

$$J = \{ x \in E : x = \sum_{x_i \in \mathcal{P}} \alpha_i x_i \} .$$

Proof: If $x_i \in \mathcal{P}$, then there exists $x = \Sigma_{n=1}^{\infty} \alpha_n x_n$ in J such that $\alpha_i \neq 0$. Since $| x | \in J$ and since $| x | = \Sigma_{n=1}^{\infty} | \alpha_n | x_n$ by Lemma 3.3.1, we can conclude, from

$$| \alpha_i | x_i \leq \sum_{n=1}^{\infty} | \alpha_n | x_n ,$$

that $| \alpha_i | x_i \in J$. Hence $x_i \in J$ as $\alpha_i \neq 0$. Thus $\mathcal{P} \subset J$ and hence

$$\{ x \in E : x = \sum_{x_i \in \mathcal{P}} \alpha_i x_i \} \subseteq J$$

as J is closed. On the other hand the definition of $\mathcal{P}$ implies that

$$J \subseteq \{ x \in E : x = \sum_{x_i \in \mathcal{P}} \alpha_i x_i \} .$$

and this completes the proof.

■

Lemma 3.3.4 *Suppose* E *is a discrete Banach lattice with order continuous norm. Then* J *is a closed ideal of* E *if and only if there exists a subset* S_J *of* $\mathbb{N}$ *such that*

$$J = \{ x = \sum_{n=1}^{\infty} \alpha_n x_n \in E : \ \alpha_m = 0 \quad \forall\, m \notin S_J \}.$$

Proof: Suppose J is a closed ideal of E. Let $\mathcal{P}$ be as in Lemma 3.3.3. Then with

$$S_J = \{ i \in \mathbb{N} : x_i \in \mathcal{P} \}$$

we have

$$J = \{ x = \sum_{n=1}^{\infty} \alpha_n x_n \in E : \ \alpha_m = 0 \quad \forall\, m \notin S_J \}.$$

Conversely, suppose S_J is a subset of $\mathbb{N}$ such that

$$J = \{ x = \sum_{n=1}^{\infty} \alpha_n x_n \in E : \ \alpha_m = 0 \quad \forall\, m \notin S_J \}.$$

Using the definition of an ideal it is easy to verify that J is an ideal of E. Now let $\{y_k\}_{k=1}^{\infty}$ be a sequence in J that converges to y in E. Suppose for each k, $\{\beta_{kn}\}_{n \in S_J}$ is a sequence of scalars such that $y_k = \Sigma_{n \in S_J} \beta_{kn} x_n$. Suppose also that $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of scalars such that $y = \Sigma_{n=1}^{\infty} \beta_n x_n$. Since the coefficient of linear functionals are continuous in E, we have

$$\lim_{k \to \infty} \beta_{kn} = \beta_n \qquad \textit{whenever } n \in S_J.$$

and

$$\lim_{k \to \infty} \beta_n = 0 \qquad \textit{whenever } n \notin S_J.$$

Thus $\beta_n = 0$ for all $n \notin S_J$, which means $y \in J$ and hence J is closed.

■

Corollary 3.3.5 *For each closed ideal* $J \neq \{0\}$, *of a discrete Banach lattice with order continuous norm* E, *there exists* $i \in \mathbb{N}$ *such that* $x_i \in J$.

Proof: If $\mathsf{J} = E$ there is nothing to prove. If J is a proper closed ideal of E, let S_J be as in Lemma 3.3.4. Since S_J is not empty, there exists $i \in S_\mathsf{J}$. For this i we have $x_i \in \mathsf{J}$.

■

Corollary 3.3.6 *The positive cone of each nonzero closed ideal* J *of a discrete Banach lattice with order continuous norm* E *contains an atom.*

Proof: Let S_J be as in Lemma 3.3.4. By Corollary 3.3.5, let $i \in S_\mathsf{J}$ be such that $x_i \in \mathsf{J}$. Suppose $y \in \mathsf{J}_+$ is such that $0 \leq y \leq x_i$ and suppose $\{\alpha_n\}_{n \in S_\mathsf{J}}$ is a sequence of nonnegative scalars such that

$$y = \sum_{n \in S_\mathsf{J}} \alpha_n x_n .$$

Then, by the properties of a basis we should have $\alpha_n = 0$ for all $n \in S_\mathsf{J} \backslash \{i\}$. Hence $y = \alpha_i x_i$ which means x_i is an atom for J_+.

■

Corollary 3.3.7 *Suppose* $\mathsf{J}_1, \mathsf{J}_2$, *with* $\mathsf{J}_1 \subseteq \mathsf{J}_2$ *are two closed ideals of a discrete Banach lattice with order continuous norm* E. *Then the positive cone* $(\mathsf{J}_2/\mathsf{J}_1)_+$ *of the quotient space* $\mathsf{J}_2/\mathsf{J}_1$ *contains an atom provided* $\mathsf{J}_2/\mathsf{J}_1 \neq \{0\}$.

Proof: It is clear that $\mathsf{J}_2 \neq \mathsf{J}_1$. Let S_{J_1} and S_{J_2} be as in Lemma 3.3.4. Clearly $S_{\mathsf{J}_1} \subsetneqq S_{\mathsf{J}_2}$. Let $j \in S_{\mathsf{J}_2}$ be such that $x_j \notin \mathsf{J}_1$. By Corollary 3.3.6, x_j generates an atom for $(\mathsf{J}_2)_+$. We claim that $x_j + \mathsf{J}_1$ is an atom for $(\mathsf{J}_2/\mathsf{J}_1)_+$.

Let $y \in \mathsf{J}_2$ be such that $y + \mathsf{J}_1 \in (\mathsf{J}_2/\mathsf{J}_1)_+$. (Without loss of generality we can assume that $y \in (\mathsf{J}_2)_+$.) Suppose the sequence of positive scalars $\{\alpha_n\}_{n \in S_{\mathsf{J}_2}}$ is such that

$$y = \sum_{n \in S_{\mathsf{J}_2}} \beta_n x_n .$$

If $y + \mathsf{J}_1 \leq x_j + \mathsf{J}_1$, then there exists $z \in \mathsf{J}_1$ such that $y \leq x_j + z$. Hence

$$\alpha_j x_j + \sum_{n \in S} \alpha_n x_n + \sum_{n \in S_{\mathsf{J}_1}} \alpha_n x_n \leq x_j + \sum_{n \in S_{\mathsf{J}_1}} \gamma_n x_n ,$$

where $S = S_{J_2} \setminus \left(S_{J_1} \cup \{j\}\right)$ and $\{\gamma_n\}_{n \in S_{J_1}}$ is a sequence of scalars such that

$$z = \Sigma_{n \in S_{J_1}} \gamma_n x_n .$$

Thus, by definition of positivity, $\alpha_n = 0$ for all $n \in S$, and hence

$$y + J_1 = \alpha_j x_j + \sum_{n \in S_{J_1}} \alpha_n x_n + J_1 = \alpha_j x_j + J_1 = \alpha_j (x_j + J_1),$$

which means $x_j + J_1$ is an atom for $(J_2/J_1)_+$.

■

Corollaries 3.3.6 and 3.3.7 now imply the main result of this section concerning discrete Banach lattices with order continuous norms and, in particular, the Banach lattices c_0 and l_p, where $1 \leq p < \infty$. In fact:

Theorem 3.3.8 *Suppose* E *is a discrete Banach lattice with order continuous norm and T is a quasinilpotent positive operator on* E. *Then T is ideal-triangularizable. In particular, any quasinilpotent positive operator on l_p, $1 \leq p < \infty$ or c_0 is ideal-triangularizable.*

Proof: Since E is a discrete Banach lattice with order continuous norm, for each closed ideal J of E, J_+ contains an atom, by Corollary 3.3.6. Hence, for each J $\in Ilat(T)$ with $\dim(J) \geq 2$, $T|_J$ is decomposable by Theorem 2.1.1. Let $J_1, J_2 \in Ilat(T)$, with $J_1 \subseteq J_2$ and $\dim(J_2/J_1) \geq 2$. Since $(J_2/J_1)_+$ contains an atom, by Corollary 3.3.7, and since $\hat{T}$ is a quasinilpotent positive operator on J_2/J_1, by Lemma 0.4.3 and 0.4.4(a), $\hat{T}$ is decomposable by Theorem 2.1.1. This shows that $\mathfrak{C} = \{T\}$ satisfies the condition of Lemma 3.1.1 and hence it is ideal-triangularizable.

■

In the following theorem we establish a stronger version of Theorem 3.3.8 by using the results of this section and Theorem 2.5.4.

Theorem 3.3.9 *Let* E *be a discrete Banach lattice with order continuous norm. If* $\mathcal{S}$ *is any semigroup of quasinilpotent positive operators on* E, *then* $\mathcal{S}$ *is ideal-triangularizable. In particular, any semigroup of quasinilpotent positive operators* *on l_p, $1 \leq p < \infty$ or c_0 is ideal-triangularizable.*

Proof: Since each discrete Banach lattice with order continuous norm has an unconditional basis, $\mathcal{S}$ is decomposable by Theorem 2.5.4. Let $\mathsf{J}_1, \mathsf{J}_2 \in Ilat(\mathcal{S})$ with $\mathsf{J}_1 \subset \mathsf{J}_2$ and $\dim \mathsf{J}_2/\mathsf{J}_1 \geq 2$. Use Lemma 3.3.4 to verify that both J_2 and $\mathsf{J}_2/\mathsf{J}_1$ are discrete Banach lattices with order continuous norms. Now consider the compression $\hat{\mathcal{S}}$ of $\mathcal{S}$ to $\mathsf{J}_2/\mathsf{J}_1$ and use Theorem 2.5.4 to show that $\hat{\mathcal{S}}$ is decomposable. This shows that $\mathcal{S}$ satisfies the condition of Lemma 3.1.1 and hence it is ideal-triangularizable.

■

3.4 Ideal-Triangularizability of Collections of Operators on General Banach Lattices

The following proposition and its corollary give us other examples of ideal-triangularizable collection of operators on certain Banach lattices.

Proposition 3.4.1 *Suppose* E *is any Banach lattice,* $\mathfrak{C} \subseteq \mathfrak{B}(\mathsf{E})$ *and* $T \in \mathfrak{C}$. *Suppose* $\{T\}$ *satisfies the condition of Lemma 3.1.1. Then* $\mathfrak{C}$ *is ideal-triangularizable if either* T *dominates all of the elements of* $\mathfrak{C}$, *or all of the elements of* $\mathfrak{C}$ *are positive operators and* T *majorizes* $\mathfrak{C}$.

Proof: Apply Lemma 0.4.5 and Lemma 2.3.13., and Lemma 3.1.1.

■

Corollary 3.4.2 *Suppose* $\mathfrak{C} \subseteq \mathfrak{B}(\mathsf{E})$, $T \in \mathfrak{C}$ *is a quasinilpotent positive operator, and either (a)* T *dominates all elements of* $\mathfrak{C}$ *or (b) all elements of* $\mathfrak{C}$ *are positive and* T *majorizes* $\mathfrak{C}$. *Then* $\mathfrak{C}$ *is ideal-triangularizable in each of the following cases;*

(i) E *is a closed ideal of an AM-space,*

(ii) E *is a discrete Banach lattice with order Continuous norm,*

(iii) E *is an AL-space and* T *is weakly compact,*

(iv) E *is any Banach lattice and* T *is compact.*

Proof: Apply the results obtained in Sections 3.2 and 3.3, and Proposition 3.4.1.

■

An application of Proposition 2.3.15 establishes the ideal triangularizability of a semigroup of operators which is generated by a certain pair of positive operators on Banach lattices.

Proposition 3.4.3 *Suppose* E *is any Banach lattice.* $T \in \mathfrak{B}(\mathsf{E})$ *is a nonzero quasinilpotent positive operator, and there exists a nonzero operator* $S \in \{T\}'_+$ *that dominates a nonzero compact operator* $K \in \mathfrak{B}(\mathsf{E})$*. Then the multiplicative semigroup* $\mathcal{S}$ *generated by* $\{S, T\}$ *is ideal-triangularizable.*

Proof: Since T is a quasinilpotent positive operator, there exists $x_0 > 0$ in E such that T is quasinilpotent at x_0, and hence, by Proposition 2.3.15, $\mathcal{S}$ is decomposable. Suppose $\mathsf{J}_1, \mathsf{J}_2 \in$ *Ilat* $(\mathcal{S})$ such that $\mathsf{J}_1 \subseteq \mathsf{J}_2$. It is easy to see that $\mathsf{J}_1, \mathsf{J}_2 \in$ *Ilat* (K) and, by Lemma 0.4.5, the compression $\hat{S}$ of S to $\mathsf{J}_2/\mathsf{J}_1$ dominates the compression $\hat{K}$ of K to $\mathsf{J}_2/\mathsf{J}_1$. Use Lemma 0.4.4, Lemma 0.4.5 and Proposition 2.3.15 to prove $\hat{\mathcal{S}}$ is decomposable, if $\dim(\mathsf{J}_2/\mathsf{J}_1) \geq 2$. Now apply Lemma 3.1.1.

■

Proposition 3.4.4 *Suppose* $[\mathcal{I}, \mathfrak{q}]$ *is a C-trace quasi-normed operator ideal. Suppose also that* E *is a reflexive Banach lattice. Let* $\mathcal{S}$ *be a semigroup of quasinilpotent positive operators that is generated by the union of a finite subset* $\mathfrak{C}$ *of* $\mathcal{I}(\mathsf{E})$ *and a positive operator,* $B \in \mathfrak{B}(\mathsf{E})$*. If some power of* B*, say* B^n*, is compact, then* $\mathcal{S}$ *is ideal-triangularizable.*

Proof: We show that $\mathcal{S}$ satisfies the condition of Lemma 3.1.1. Let $\mathsf{J}_1, \mathsf{J}_2 \in$*Ilat*$(\mathcal{S})$ with $\mathsf{J}_1 \subseteq \mathsf{J}_2$ and $\dim(\mathsf{J}_2/\mathsf{J}_1) \geq 2$. Since E is a reflexive Banach lattice, $\mathsf{J}_2/\mathsf{J}_1$ is a reflexive Banach lattice. Now an easy verification and an application of the appropriate lemmas in Section 0.4, show that all conditions of Theorem 2.2.1 are satisfied with $\mathsf{J}_2/\mathsf{J}_1$ in place of E, the compression $\hat{\mathcal{S}}$ of $\mathcal{S}$ to $\mathsf{J}_2/\mathsf{J}_1$ in place of $\mathcal{S}$, the compression $\hat{\mathfrak{C}}$ of $\mathfrak{C}$ to $\mathsf{J}_2/\mathsf{J}_1$ in place of $\mathfrak{C}$, and the compression $\hat{B}$ of B to $\mathsf{J}_2/\mathsf{J}_1$ in place of B. Thus $\hat{\mathcal{S}}$ is decomposable.

■

Proposition 3.4.5 *Suppose $[\mathcal{I}, \mathfrak{q}]$ is a C-trace quasi-normed operator ideal. Suppose also that E is a reflexive Banach lattice. Let $\mathcal{S}$ be a semigroup of quasinilpotent positive operators that is generated by a countable subset $\mathfrak{C}$ of $\mathcal{I}(\mathsf{E})$, then $\mathcal{S}$ is ideal-triangularizable.*

Proof: Proceed as in the proof of Proposition 3.4.4 and use Theorem 2.2.2 to derive the desired result.

■

Proposition 3.4.6 *Suppose $[\mathcal{I}, \mathfrak{q}]$ is a C-trace quasi-normed operator ideal. Suppose also that E is a reflexive Banach lattice. Let $\mathcal{S}$ be a semigroup of quasinilpotent positive operators in $\mathcal{I}(\mathsf{E})$ that is separable in the strong operator topology of $\mathfrak{B}(\mathsf{E})$, then $\mathcal{S}$ is ideal-triangularizable.*

Proof: Observe that: (a) For any subspace $M \in Lat(\mathcal{S})$ the semigroup $\mathcal{S}_M = \{ S|_M \ : \ S \in \mathcal{S} \}$ is separable in the strong operator topology of $\mathcal{B}(M)$. (b) If $M, N \in Lat(\mathcal{S})$ with $M \subseteq N$ then the compression $\hat{\mathcal{S}}$ of $\mathcal{S}$ to N/M is separable in the strong operator topology of $\mathcal{B}(N/M)$.

Now proceed as in the proof of Theorem 3.4.4 and use Corollary 2.2.3 to conclude the claim.

■

Using Proposition 2.2.5, we may find another ideal-triangularizability result, for a semigroup of quasinilpotent positive operators in C-trace ideals.

Proposition 3.4.7 *Let $[\mathcal{I}, \mathfrak{q}]$ be a C-trace quasi-normed operator ideal and let E be a reflexive Banach lattice. Suppose $\mathcal{S}$ is a semigroup of quasinilpotent positive operators which is a countably generated subset of $\mathfrak{B}(\mathsf{E})$. If there exists an integer $k \in \mathbb{N}$ such that $\mathcal{S}^k \subseteq \mathcal{I}(\mathsf{E})$, then $\mathcal{S}$ is ideal-triangularizable.*

Proof: First, use Lemma 1.1.15 and proceed as in the proof of Corollary 1.4.5. Then, use Proposition 2.2.5 to show that $\mathcal{S}$ satisfy the condition of Lemma 3.1.1.

■

3.5 Ideal-Triangularizability of a Semigroup of Integral Operators on $C(\mathcal{K})$

Under suitable conditions, we can say more about a semigroup $\mathcal{S}$ of quasinilpotent integral operators on $E = C(\mathcal{K})$, each of whose members has a non-negative lower semicontinuous kernel. To do this we need the following lemmas.

Lemma 3.5.1 *Let μ be a finite regular Borel measure on Ω, Ω_0 a nonempty compact subset of Ω, and $h_0 \in C(\Omega_0)$. Then, given $\kappa > 0$, there exist a closed subset A of $B = \Omega \setminus \Omega_0$ and a continuous extension h of h_0 to Ω such that the following hold:*

(a) $\mu(B \setminus A) \leq \kappa$.

(b) $h(x) = 0$ *for all* $x \in A$.

(c) $|h(x)| \leq \|h_0\|_\infty$ *for all* $x \in \Omega$.

Proof: If $\Omega_0 = \Omega$ there is nothing to prove. Otherwise, use Tietze Extension Theorem (cf. [43, Theorem 20.4]), to find a continuous extension g of h_0 to Ω such that $\|g\|_\infty = \|h_0\|_\infty$. Then use the regularity of μ to find a compact subset A of B with $\mu(B \setminus A) \leq \kappa$. This can be done as μ is also a finite measure. Since Ω is a Hausdorff space A is a closed subset of Ω. Now use the normality of Ω and the fact that $A \cap \Omega_0 = \emptyset$ to find a continuous function f on Ω such that $f(A) = \{0\}$, $f(\Omega_0) = \{1\}$, and $0 \leq f(x) \leq 1$ for all $x \in \Omega$. Finally define $h = fg$. Then h is a continuous function on Ω,

$$h(y) = f(y)g(y) = 1 \cdot h_0(y) = h_0(y) \qquad \text{for all } y \in \Omega_0,$$
$$h(t) = f(t)g(t) = 0 \cdot g(t) = 0 \qquad \text{for all } t \in A,$$

and

$$|h(x)| = f(x) \cdot |g(x)| \leq |g(x)| \leq \|h_0\|_\infty \qquad \text{for all } x \in \Omega.$$

■

Lemma 3.5.2 *Assume all the conditions of Lemma 3.5.1 and let K be a bounded integrable function on $\Omega \times \Omega$. Then, given $\epsilon > 0$, there exists a continuous extension* *h of h_0 to Ω such that*

$$\left| \int_\Omega K(x,t)h(t)d\mu(t) - \int_{\Omega_0} K(x,t)h_0(t)d\mu(t) \right| \le \epsilon .$$

for all $x \in \Omega$.

Proof: Put $\kappa = \epsilon/(M\|h_0\|_\infty)$, where M is a bound for K, and use Lemma 3.5.1 to find a continuous extension h of h_0 to Ω with the stated properties given in that lemma. Then

$$\begin{aligned} \int_\Omega K(x,t)h(t)d\mu(t) &= \int_{\Omega_0} K(x,t)h(t)d\mu(t) + \int_A K(x,t)h(t)d\mu(t) \\ &\quad + \int_{B\setminus A} K(x,t)h(t)d\mu(t) \\ &= \int_{\Omega_0} K(x,t)h_0(t)d\mu(t) + \int_{B\setminus A} K(x,t)h(t)d\mu(t) , \end{aligned}$$

for any $x \in \Omega$, and hence

$$\begin{aligned} \left| \int_\Omega K(x,t)h(t)d\mu(t) - \int_{\Omega_0} K(x,t)h_0(t)d\mu(t) \right| &\le \int_{B\setminus A} |K(x,t)| \cdot |h(t)| d\mu(t) \\ &\le M\|h_0\|_\infty \mu(B \setminus A) \le \kappa M \|h_0\|_\infty = \epsilon . \end{aligned}$$

for all $x \in \Omega$.

■

The following lemma is known and was implicity used in [44]. For completeness we state and prove it here.

Lemma 3.5.3 *Let J be a closed ideal in $C(\mathcal{K})$. Then the quotient $C(\mathcal{K})/\mathsf{J}$ can be canonically identified with $C(\mathcal{K}_0)$ where $\mathcal{K}_0$ is a suitable closed subset of $\mathcal{K}$.*

Proof: Since J is a closed ideal of $C(\mathcal{K})$, there exists a closed, and hence compact, subset $\mathcal{K}_0$ of $\mathcal{K}$ such that

$$\mathsf{J} = \{ f \in C(\mathcal{K}) \ : \ f(t) = 0 \ \ for \ all \ \ t \in \mathcal{K}_0 \} .$$

Define $\rho : C(\mathcal{K}_0) \longrightarrow C(\mathcal{K})/\mathsf{J}$ by $\rho(f_0) = f + \mathsf{J}$, where f is a continuous extension of f_0 to $\mathcal{K}$. Tietze's Extension Theorem and the structure of J imply that ρ is well defined, and it can be easily verified that ρ is linear, one-to-one, onto, and $\rho^{-1}(f + \mathsf{J}) = f_0$, where $f_0 = f|_{\mathcal{K}_0}$.

We show that $\|\rho(f_0)\| = \|f_0\|_\infty$. First observe that for each $f \in C(\mathcal{K})$ and $g \in \mathsf{J}$

$$\begin{aligned}&\sup\{|(f+g)(x)| \,:\, x \in \mathcal{K}\}\\ &= \sup\Big\{\{\,|(f+g)(x)| \,:\, x \in \mathcal{K}\setminus\mathcal{K}_0\} \cup \{|f(x)| \,:\, x \in \mathcal{K}_0\}\Big\},\end{aligned}$$

and hence $\|f_0\|_\infty \leq \|f + g\|_\infty$ for all $g \in \mathsf{J}$. This shows that $\|f_0\|_\infty \leq \|f + \mathsf{J}\|$. On the other hand, if we use Tietze's Extension Theorem to find a continuous extension h of f_0 to $\mathcal{K}$ with $\|h\|_\infty = \|f_0\|_\infty$, then

$$\|f + \mathsf{J}\| = \|h + \mathsf{J}\| \leq \|h\|_\infty = \|f_0\|_\infty .$$

Thus ρ is an isometric isomorphism from $C(\mathcal{K}_0)$ to $C(\mathcal{K})/\mathsf{J}$.

■

Lemma 3.5.4 *Suppose μ is a regular Borel measure on $\mathcal{K}$. Let T be an integral operator on $C(\mathcal{K})$ with a bounded kernel K_T. If $\mathsf{J} \in$ Ilat(T), then the operator $\hat{T}$: $C(\mathcal{K})/\mathsf{J} \longrightarrow C(\mathcal{K})/\mathsf{J}$ can be identified with an integral operator.*

Proof: Suppose $\mathcal{K}_0$ is a closed, and hence a compact, subset of $\mathcal{K}$ such that

$$\mathsf{J} = \{\, f \in C(\mathcal{K}) \,:\, f(t) = 0 \;\; for\ all \;\; t \in \mathcal{K}_0 \,\} .$$

Since $\mathcal{K}_0$ is a Borel subset of $\mathcal{K}$, the restriction μ_0 of μ to $\mathcal{K}_0$ is well defined. Define T_0 on $C(\mathcal{K}_0)$ by

$$T_0 f_0(y) = \int_{\mathcal{K}_0} K_T(y,t) f_0(t)\, d\mu_0(t) \quad \forall\, y \in \mathcal{K}_0 .$$

We claim that $T_0 = \rho^{-1}\hat{T}\rho$, where ρ is as in Lemma 3.5.3, and hence $\hat{T}$ can be identified with the kernel operator T_0. To prove the claim, let $f_0 \in C(\mathcal{K}_0)$. Then $\rho^{-1}\hat{T}\rho(f_0) = (Tf)|_{\mathcal{K}_0}$, where f is any continuous extension of f_0 to $\mathcal{K}$. Let $\epsilon > 0$ and use Lemma 3.5.2, with $\Omega = \mathcal{K}$, $\Omega_0 = \mathcal{K}_0$, and $h_0 = f_0$, to find an extension h of f_0 to

$\mathcal{K}$ such that

$$\left| \int_{\mathcal{K}} K_T(y,t)h(t)\, d\mu(t) - \int_{\mathcal{K}_0} K_T(y,t)f_0(t)\, d\mu(t) \right| \le \epsilon ,$$

for all $y \in \mathcal{K}_0$. Since

$$(Tf)|_{\mathcal{K}_0}(y) = (Th)|_{\mathcal{K}_0}(y) = \int_{\mathcal{K}} K_T(y,t)h(t)\, d\mu(t)$$

and

$$T_0 f_0(y) = \int_{\mathcal{K}_0} K_T(y,t)f_0(t)\, d\mu_0(t) = \int_{\mathcal{K}_0} K_T(y,t)f_0(t)\, d\mu(t) .$$

for each $y \in \mathcal{K}_0$. $\|\rho^{-1}\hat{T}\rho(f_0) - T_0(f_0)\|_\infty \le \epsilon$, and hence $\rho^{-1}\hat{T}\rho = T_0$, as desired.

■

Lemma 3.5.5 *Assume all the conditions of Lemma 3.5.4. Then $T|_{\mathrm{J}}$ can be identified with an integral operator.*

Proof: Let $\mathcal{K}_0$ be as in the Proof of Lemma 3.5.4. Put $\mathcal{U} = \mathcal{K} \setminus \mathcal{K}_0$. then $\mathcal{U}$ is locally compact and J is isomorphic to $C_0(\mathcal{U})$. In fact $\tau : \mathrm{J} \longrightarrow C_0(\mathcal{U})$ defined by $\tau(f) = f|_{\mathcal{U}}$ is an isometric isomorphism. Now for each $g \in C_0(\mathcal{U})$ we have

$$\tau T|_{\mathrm{J}}\tau^{-1}g = \tau T|_{\mathrm{J}}f = (Tf)|_{\mathcal{U}} .$$

where $f \in \mathrm{J}$ is such that $f|_{\mathcal{U}} = g$. But $Tf(x) = 0$. for all $x \in \mathcal{K}_0$, and. for each $x \in \mathcal{U}$.

$$Tf(x) = \int_{\mathcal{K}} K_T(x,t)f(t)\, d\mu(t) = \int_{\mathcal{U}} K_T(x,t)g(t)\, d\mu_{\mathcal{U}}(t) .$$

where $\mu_{\mathcal{U}}$ is the restriction of μ to $\mathcal{U}$, hence $T|_{\mathrm{J}}$ can be identified with an integral operator on $C_0(\mathcal{U})$.

■

We are now ready to state and prove the main result of this section.

Theorem 3.5.6 *Let μ be a regular Borel measure on $\mathcal{K}$ and suppose $\mathcal{S}$ is a semigroup of quasinilpotent integral operators on $C(\mathcal{K})$ by way of the measure μ, each of whose members has a non-negative bounded lower-semicontinuous kernel. Then $\mathcal{S}$ is ideal-triangularizable.*

Proof: By Theorem 2.4.4, $\mathcal{S}$ is decomposable. Let $J_1, J_2 \in$, $Ilat(\mathcal{S})$ such that $J_1 \subseteq J_2$ and $\dim(J_2/J_1) \geq 2$. Let $\hat{\mathcal{S}}$ be the compression of $\mathcal{S}$ to $C(\mathcal{K})/J_1$. By Lemma 3.5.4, each $\hat{T} \in \hat{\mathcal{S}}$ can be identified with an integral operator on $C(\mathcal{K}_0)$ by way of the regular Borel measure $\mu|_{\mathcal{K}_0}$, where $\mathcal{K}_0$ is a closed subset of $\mathcal{K}$ such that

$$J_1 = \{ f \in C(\mathcal{K}) \, : \, f(t) = 0 \; for \;\; all \; t \in \mathcal{K}_0 \} .$$

By Lemma 3.5.5, since $J_2/J_1 \in Ilat\hat{\mathcal{S}}$ for each $\hat{T} \in \hat{\mathcal{S}}$, each $\hat{T}|_{(J_2/J_1)}$ can be identified with a non-negative integral operator on $C_0(\mathcal{U}_0)$ by way of the regular Borel measure $\mu|_{\mathcal{U}_0}$, where $\mathcal{U}_0 = \mathcal{K}_0 \setminus \mathcal{K}_{00}$ and $\mathcal{K}_{00}$ is a closed subset of $\mathcal{K}_0$ such that

$$J_2/J_1 \cong \{ f_0 \in C(\mathcal{K}_0) \, : \, f_0(t) = 0 \; for \;\; all \; t \in \mathcal{K}_{00} \} .$$

Since, for each $T \in \mathcal{S}$, the compression of $\widehat{T|_{J_2}}$ of $T|_{J_2}$ to J_2/J_1 is $\hat{T}|_{(J_2/J_1)}$, and since for such T, $\hat{T}|_{(J_2/J_1)}$ is, by Lemma 0.4.4(a), a quasinilpotent operator, the semigroup

$$\mathcal{S}_{J_2} = \{ \hat{T}|_{(J_2/J_1)} \, : \, \hat{T} \in \hat{\mathcal{S}} \}$$

can be identified with a semigroup of quasinilpotent integral operators on $C_0(\mathcal{U}_0)$ each of whose members has a nonnegative lower-semicontinuous kernel. Therefore; $\mathcal{S}_{J_2}$ is decomposable by Theorem 2.4.4. This shows that $\mathcal{S}$ satisfies the condition of Lemma 3.1.1 and hence it is ideal-triangularizable.

■

Chapter 4

Examples, Remarks, Consequences, and Open Questions

In this chapter we present examples and comments on some questions concerning the results obtained in the previous sections.

4.1 Indecomposable Quasinilpotent Positive Operators

First we recall the following definitions, notations, and facts from [18]. Consider a measure space $(\mathcal{X}, \mathcal{A}, \mu)$. Redefine the concept of equality: if two sets A and B in $\mathcal{A}$ are such that $\mu(A \triangle B) = 0$, consider them equal and write $A = B \ (mod\ [\mu])$. With the altered concept of equality observe that $\mu(A) = \mu(B)$ if $A = B \ (mod\ [\mu])$. Besides, if $A_n = B_n \ (mod\ [\mu])$ for all $n \in \mathbb{N}$, then

$$A_1 \setminus B_1 = A_2 \setminus B_2 \quad and \quad \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \ (mod\ [\mu]).$$

Therefore, after the alteration of the concept of equality, $\mathcal{A}$ is a σ-algebra with respect to the familiar set operations, μ is unambiguously defined on $\mathcal{A}$, and μ is a positive measure. We shall use the symbol $\mathcal{A}(\mu)$ to denote the σ-algebra $\mathcal{A}$ with equality interpreted modulo μ and call $(\mathcal{A}(\mu), \mu)$ a *measure algebra*.

Remark 4.1.1 Fix $1 \leq p < \infty$. It is obvious that $\chi_A = \chi_B$ *a.e.* $[\mu]$, whenever $A = B$ $(mod\ [\mu])$. Thus

$$\int_A \phi\, d\mu = \int \phi\, \chi_A\, d\mu = \int \phi\, \chi_B\, d\mu = \int_B \phi\, d\mu \qquad\qquad \phi \in F_\mu$$

where F_μ is the set of all measurable simple functions ϕ on $\mathcal{X}$ with

$$\mu(\{x \in \mathcal{X} : \phi(x) \neq 0\}) < \infty .$$

Since F_μ is dense in $\mathsf{E} = L_p(\mathcal{X}, \mathcal{A}, \mu)$ (cf. [43, Theorem 3.13]), this shows that, if $f \in \mathsf{E}$, we can define

$$\int_{[A]} |f|^p\, d\mu \ = \ \int_A |f|^p\, d\mu ,$$

where $[A] = \{\, B \in \mathcal{A} \ : \ A = B \ \ (mod\ [\mu]\,\}$. Therefore, we may consider $\mathcal{A}(\mu)$ as an algebra of sets and deduce that E and $L_p(\mathcal{X}, \mathcal{A}(\mu), \mu)$ are identical.

A mapping Φ of a measure algebra $(\mathcal{A}(\mu), \mu)$ into a measure algebra $(\mathcal{B}(\nu), \nu)$ is called an *isomorphism* if Φ is a one-to-one mapping from $\mathcal{A}$ onto $\mathcal{B}$ such that

$$\Phi(A \setminus B) = \Phi(A) \setminus \Phi(B), \qquad \Phi(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \Phi(A_n) .$$

and

$$\mu(A) = \nu(\Phi(A)) ,$$

whenever A, B, and A_n are elements of $\mathcal{A}(\mu)$, $n \in \mathbb{N}$. If such an isomorphism exists between two measure algebras, we say that they are isomorphic. Two measure spaces $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ are *isomorphic* if $(\mathcal{A}(\mu), \mu)$ and $(\mathcal{B}(\nu), \nu)$ are isomorphic.

An element $A \neq 0$ in a measure algebra $\mathcal{A}$ is called an *atom* if $B \subseteq A$ can occur only for $B = A$ and $B = 0$.

Let $\mathcal{F}(\mu)$ denotes the set of elements of finite measure in $\mathcal{A}(\mu)$. Then $\mathcal{F}(\mu)$ becomes a metric space if we define

$$d_\mu(A, B) = \mu(A \triangle B) \qquad\qquad A, B \in \mathcal{F}(\mu) .$$

This metric space is always complete, and the mappings $A \longrightarrow A^C$, $(A, B) \longrightarrow A \cup B$, and $(A, B) \longrightarrow A \cap B$ are continuous. A measure algebra $(\mathcal{A}(\mu), \mu)$ is called *separable* if $\mathcal{F}(\mu)$ is separable.

Let m be (the) Lebesgue measure on $[0, 1]$ and $\mathcal{M}$ the algebra of all m-measurable subsets of $[0, 1]$. It is known that $(\mathcal{M}(m), m)$ is a separable measure algebra without atoms. The following proposition asserts that, it is (up to an isomorphism) the only such measure algebra.

Proposition 4.1.2 (Carathéodory) *Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$ and let $(\mathcal{A}(\mu)\mu)$ be its associated measure algebra. If $(\mathcal{A}(\mu), \mu)$ is separable, then there is an isomorphism Φ from $(\mathcal{A}(\mu), \mu)$ into $(\mathcal{M}(m), m)$. If Φ is onto then $\mathcal{A}(\mu)$ has no atoms: if $\mathcal{A}(\mu)$ has no atoms then Φ can be taken to be onto.*

Suppose $(X, \mathcal{A}, \mu)$ is a measure algebra, with $\mu(X) = \infty$, such that its associated measure algebra is σ-finite and non-atomic. Let $A_0 \in \mathcal{A}(\mu)$. It can be shown that for every extended real number α with $0 \leq \alpha \leq \mu(A_0)$, there exists an element A in $\mathcal{A}(\mu)$ such that $A \subseteq A_0$ and $\mu(A) = \alpha$. It follows that there exists a sequence $\{X_n\}$ of pairwise disjoint elements in $\mathcal{A}(\mu)$ such that $X = \cup_{n=1}^{\infty} X_n$ and $\mu(X_n) = 1$, $n \in \mathbb{N}$. Now if $(\mathcal{A}(\mu), \mu)$ is separable, an application of the previous proposition yields the following;

Corollary 4.1.3 *Let m be Lebesgue measure and let $\mathcal{B}$ be the algebra of all m-measurable subsets of the real numbers $\mathbb{R}$. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, with $\mu(X) = \infty$ such that $(\mathcal{A}(\mu), \mu)$ is separable, non atomic, and σ-finite. Then, the measure algebras $(\mathcal{A}(\mu), \mu)$ and $(\mathcal{B}(m), m)$ are isomorphic.*

All of the above facts can be found with more details in [18, sections 40 and 41].

The following lemma shows that every Banach lattice $L_p(X, \mathcal{A}, \mu)$, $1 \leq p < \infty$, subject to certain not too restrictive conditions, is (up to an isomorphism) the Banach lattice L_p on $[0, 1]$.

Lemma 4.1.4 *Let $(X, \mathcal{A}, \mu)$ be a measure space such that μ is finite and $(\mathcal{A}(\mu), \mu)$ is a separable measure algebra with no atoms. Let m be the Lebesgue measure on $Y = [0,1]$ and let $\mathcal{M}$ be the algebra of all m-measurable subsets of Y. Then for each $1 \le p < \infty$ the Banach lattices $L_p(X, \mathcal{A}, \mu)$ and $L_p(Y, \mathcal{M}, m)$ are isometrically isomorphic.*

Proof: According to Remark 4.1.1 it is enough to show that the Banach lattices $\mathsf{E} = L_p(X, \mathcal{A}(\mu), \mu)$ and $\mathsf{F} = L_p(Y, \mathcal{M}(m), m)$ are isometrically isomorphic.

Suppose S_μ and S_m are the respective sets of all measurable simple functions on X and Y. Let

$$F_\mu = \{\phi \in S_\mu : \mu(\{x \in X : \phi(x) \neq 0\}) < \infty\},$$

and

$$F_m = \{\psi \in S_m : m(\{y \in Y : \psi(y) \neq 0\} < \infty\}.$$

It is known (cf. [43 Theorem 3.13]) that F_μ and F_m are dense in E and F, respectively.

Pick $\phi \in F_\mu$. Suppose $\phi = \sum_{i=1}^n a_i \chi_{A_i}$ is the canonical representation of ϕ, i.e. $a_i \in \mathbb{R}$ are distinct and $A_i \in \mathcal{A}(\mu)$ are disjoint, $1 \le i \le n$. Define $\psi = \sum_{i=1}^n a_i \chi_{\Phi(A_i)}$, where Φ is the isomorphism from $(\mathcal{A}(\mu), \mu)$ onto $(\mathcal{M}(m), m)$ in Proposition 2. Then $\psi \in F_m$ and the relation $T_0\phi = \psi$ defines a mapping T_0 from F_μ into F_m. It is easy to verify that T_0 is linear. Since Φ is onto, T_0 is also onto. The fact that Φ is one to one implies that the image of each simple function, that is represented in its canonical form, is a simple function in its canonical form. Therefore if $T_0\phi_1 = T_0\phi_2$, we conclude that $\phi_1 = \phi_2$ and hence T_0 is one-to-one. We also have

$$\|T_0\phi\|_p^p = \int_Y |T_0\phi|^p dm = \sum_{i=1}^n |a_i|^p \int_Y \chi_{\Phi(A_i)} dm = \sum_{i=1}^n |a_i|^p m(\Phi(A_i))$$

$$= \sum_{i=1}^n |a_i|^p \mu(A_i) = \sum_{i=1}^n |a_i|^p \int_X \chi_{A_i} d\mu = \int_X |\phi|^p d\mu = \|\phi\|_p^p,$$

and hence $\|T_0\phi\|_p = \|\phi\|_p$ for all $\phi \in F_\mu$. Finally

$$|T_0\phi| = |\sum_{i=1}^n a_i \chi_{\Phi(A_i)}| = \sum_{i=1}^n |a_i| \chi_{\Phi(A_i)} = T_0(|\phi|).$$

Thus $|T_0\phi| = T_0(|\phi|)$ for all $\phi \in F_\mu$. These properties of T_0 show that the extension $T : \mathsf{E} \longrightarrow \mathsf{F}$ of T_0, defined by

$$Tf = \lim_{n\to\infty} T_0(\phi_n),$$

where $\{\phi_n\}_{n=1}^{\infty}$ is a sequence in F_μ that converges to f, is an isometric isomorphism that is also a lattice homomorphism from E onto F.

■

With the same procedure as the one used in the proof of the previous lemma and by using Corollary 3 we can also prove the following.

Lemma 4.1.5 *Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) = \infty$, μ is σ-finite, and $(\mathcal{A}(\mu), \mu)$ is separable with no atoms. Then, for each $1 \le p < \infty$, the Banach lattices $L_p(X, \mathcal{A}, \mu)$ and $L_p(\mathbb{R}, \mathcal{M}, m)$, where m is the Lebesgue measure and $\mathcal{M}$ is the algebra of all m-measurable subsets of $\mathbb{R}$, are isometrically isomorphic.*

We now recall the following fact from [45].

Proposition 4.1.6 *Let Γ be the circle group, Θ the Haar measure on Γ, and $\mathcal{T}$ the algebra of all Θ-measurable subsets of Γ. Then there exists an indecomposable quasinilpotent positive operator on $L_p(\Gamma, \mathcal{T}, \Theta)$, $1 \le p < \infty$.*

Using Proposition 4.1.6, the fact that $L_p(\Gamma, \mathcal{T}, \Theta)$ is isometrically isomorphic to $L_p([0,1], \mathcal{M}, m)$, and Lemma 4.1.4, we can deduce the following:

Lemma 4.1.7 *Let $(X, \mathcal{A}, \mu)$ be a measurable space such that $\mu(X) < \infty$ and $(\mathcal{A}(\mu), \mu)$ is a separable measure algebra with no atoms. Then there exists an indecomposable quasinilpotent positive operator on $L_p(X, \mathcal{A}, \mu)$, $1 \le p < \infty$.*

To extend the above lemma to the case of σ-finite measure we should first recall the following. Let $\{\mathsf{E}_i\}_{i=1}^{\infty}$ be a sequence of normed vector lattices. It is known that, for $1 \le p < \infty$, the space

$$\bigoplus_p \mathsf{E}_i = \left\{ x = (x_1, x_2, x_3, \cdots) \in \prod_{i=1}^{\infty} \mathsf{E}_i \ : \ \left[\sum_{i=1}^{n} \|x_i\|^p\right]^{1/p} < \infty \right\}$$

is a normed vector space once we define

$$\|x\| = \left[\sum_{i=1}^{\infty} \|x_i\|^p\right]^{1/p} \qquad \forall x = (x_1, x_2, x_3, \cdots) \in \prod_{i=1}^{\infty} \mathsf{E}_i .$$

Let $x = (x_1, x_2, x_3, \cdots), y = (y_1, y_2, y_3, \cdots) \in \bigoplus_p \mathsf{E}_i$. Under the ordering $x \geq y$ whenever $x_i \geq y_i$ holds for all $i \in \mathbb{N}$, the space $\bigoplus_p \mathsf{E}_i$ is a normed vector lattice. It is a Banach lattice if and only if each E_i is a Banach lattice. It can be verified that every operator $T : \bigoplus_p \mathsf{E}_i \longrightarrow \bigoplus_p \mathsf{F}_i$ between two direct sums of normed vector lattices can be represented by a matrix $T = (T_{ij})$, where $T_{ij} : \mathsf{E}_i \longrightarrow \mathsf{F}_j$ are operators defined appropriately. Finally, each closed ideal J of $\bigoplus_p \mathsf{E}_i$ is of the form $\mathsf{J} = \bigoplus_p \mathsf{J}_i$, where each J_i is a closed ideal of E_i.

Now consider the following example.

Example 4.1.8 Let $\mathsf{E} = \bigoplus_p \mathsf{E}_i$, where $\mathsf{E}_i = \mathsf{F} = L_p([0,1], \mathcal{A}, \mu)$, $1 \leq p < \infty$, for each $i \in \mathbb{N}$. Define T in E by

$$T = \begin{bmatrix} S & (\frac{1}{2})S & (\frac{1}{4})S & (\frac{1}{8})S & \cdots \\ (\frac{1}{2})S & (\frac{1}{4})S & (\frac{1}{8})S & (\frac{1}{16})S & \cdots \\ (\frac{1}{4})S & (\frac{1}{8})S & (\frac{1}{16})S & (\frac{1}{32})S & \cdots \\ (\frac{1}{8})S & (\frac{1}{16})S & (\frac{1}{32})S & (\frac{1}{64})S & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} .$$

where S is an indecomposable quasinilpotent positive operator on F that has been found in Proposition 4.1.7. Since $\mathsf{E} = \mathsf{F} \bigotimes l_p$ and $T = S \bigotimes R$, where

$$R = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

is an operator on l_p, $T^n = S^n \bigotimes R^n$ for all $n \in \mathbb{N}$. Thus $\|T^n\| = (\|S^n\|)(\|R^n\|)$ for all $n \in \mathbb{N}$ and hence $r(T) = 0$ as S is a quasinilpotent operator.

Let J be a closed ideal of E that is invariant under T. As we mentioned above, we have $\mathsf{J} = \bigoplus_p \mathsf{J}_i$ where each J_i is a closed ideal of F, $i \in \mathbb{N}$. Suppose, if possible, that J is nontrivial. Then there exists $k \in \mathbb{N}$ such that $\mathsf{J}_k \neq \{0\}$. With no loss of generality we may assume that k=1. Choose a nonzero positive element $f \in \mathsf{J}_1$ and consider the element $x = (x_1, x_2, x_3, \cdots) \in \mathsf{E}$ with $x_1 = f$ and $x_i = 0$ for all $i \geq 2$. Then $x \in \mathsf{J}$ and we must have $Tx \in \mathsf{J}$. Since

$$Tx = T\begin{bmatrix} f \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} S(f) \\ (\frac{1}{2})S(f) \\ (\frac{1}{4})S(f) \\ \vdots \end{bmatrix}, \qquad (\star)$$

and since the null ideal N_S is not $\{0\}$, as S is indecomposable, $\mathsf{J}_i \neq \{0\}$ for all $i \in \mathbb{N}$. Thus there exists at least one $k \in \mathbb{N}$ such that $\mathsf{J}_k \neq \mathsf{F}$. Once again, with no loss of any generality assume $k = 1$; $(\star)$ shows that $S(f) \in \mathsf{J}_1$ for each $f \in \mathsf{J}_1$ which means J_1 is invariant under S contradicting the fact that S is indecomposable. Therefore the only closed ideals that are invariant under T are the trivial ideals. This means T is indecomposable.

The above example, the fact that, for each $1 \leq p < \infty$, $L_p(\mathbb{R}, \mathcal{M}, m)$ is isometrically isomorphic to $\bigoplus_{n=1}^{\infty} L_p([0,1], \mathcal{M}, m)$, and Lemma 4.1.5, imply the following Lemma:

Lemma 4.1.9 *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space such that $\mu(X) = \infty$. If $(\mathcal{A}(\mu), \mu)$ is a separable measure algebra with no atoms, then, for each $1 \leq p < \infty$, there exist an indecomposable quasinilpotent positive operator on $L_p(X, \mathcal{A}, \mu)$.*

We may summarize our previous results in the following Theorem:

Theorem 4.1.10 *Suppose $(X, \mathcal{A}, \mu)$ is a σ-finite measure space. If $(\mathcal{A}(\mu), \mu)$ is a separable measure algebra with no atoms, then there exists an indecomposable quasinilpotent positive operator on $L_p(X, \mathcal{A}, \mu)$, $1 \leq p < \infty$.*

Remark 4.1.11 Recall that for each measure space $(X, \mathcal{A}, \mu)$, the Banach lattice

$\mathsf{E} = L_\infty(X, \mathcal{A}, \mu)$ is an AM-space with unit (cf. [46, Example II.7.3]). Therefore each quasinilpotent positive operator on E is decomposable by Theorem 2.1.1.

It is natural to ask what would happen if, in the Theorem 4.1.10, μ was not σ-finite or the measure algebra $(\mathcal{A}(\mu), \mu)$ was not separable or $(\mathcal{A}, \mu)$ has an atom? The following lemma is a partial answer to this questions.

Lemma 4.1.12 *Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. If μ is not σ-finite or the measure algebra $(\mathcal{A}(\mu), \mu)$ has an atom then every quasinilpotent positive operator on $\mathsf{E} = L_p(X, \mathcal{A}, \mu)$ is decomposable.*

Proof: Suppose μ is not σ-finite. Then, by [46, Example II.6.1], E does not contain any quasi-interior positive element and hence every positive operator on E is decomposable by a Corollary of [46, Proposition III.8.3].

Now let $(\mathcal{A}(\mu), \mu)$ be a measure algebra with an atom, say A. Observe that $f = \chi_A$ is an atom for E_+. Therefore every quasinilpotent positive operator on E is decomposable by Theorem 2.1.1.

■

4.2 Miscellaneous Examples

The first example, which was introduced in [32], shows that the condition of reflexivity cannot be omitted from Theorem 1.2.4.

Example 4.2.1 Let X be a non-reflexive Banach space and let $\mathfrak{A}$ be the norm-closure of $\mathfrak{F}(X)$ in $\mathfrak{B}(X)$. Then $\mathfrak{A}^*$ contains finite-rank operators and it is transitive, however $\mathfrak{F}(X^*) \not\subset \mathfrak{A}^*$.

The assumption $\overline{\mathfrak{F}(X)}^{q} = \mathcal{I}(X)$ is essential in Lemma 1.3.2, as the following example will show.

Example 4.2.2 Let X be a reflexive Banach space without the approximation property. For such X we know that $\mathfrak{F}(X)$ is not norm-dense in $\mathfrak{K}(X)$. Using the Hahn-Banach Theorem we can find a norm-continuous nonzero linear functional on

$\mathfrak{K}(X)$ such that f is zero on the norm-closure of $\mathfrak{F}(X)$ in $\mathfrak{K}(X)$, however $\mathcal{S} = \mathfrak{F}(X)$ is not a reducible semigroup.

The next example shows that the functional trace is not norm-continuous on $\mathfrak{F}(X)$ even if X is a Hilbert space. Hence $\mathfrak{K}(X)$ is not even a trace ideal.

Example 4.2.3 Let H be a separable Hilbert space and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for H. For each $n \in \mathbb{N}$ define T_n on $\{e_1, e_2, \cdots\}$ by

$T_n e_i = \frac{1}{n} e_i \qquad \text{if } i \leq n$

$T_n e_i = 0 \qquad \text{if } i > n,$

and extend T_n by linearity to H. An easy verification shows that each T_n is a finite-rank operator. $tr(T_n) = 1$ for each n, and T_n converges in norm to 0, however

$$\lim_{n \to \infty} tr(T_n) \neq 0 = tr(0).$$

As the following example shows, there are different quasi-normed operator ideals even on reflexive Banach spaces.

Example 4.2.4 Consider $X = l_p$ where $1 < p < \infty$ and $p \neq 2$. We know that X is a reflexive Banach space which is not isomorphic to a Hilbert space. Hence $\mathcal{N}(X) \neq S_1^a(X)$ and $\mathcal{N}(X) \neq \Pi_2^2(X)$ by [26, Corollary 4.b.12].

The condition of convexity imposed on a semigroup of quasinilpotent positive operators is very strong. A trivial example is the one that is generated by a single quasinilpotent positive operator. The following non-trivial examples illustrate the usefulness of the results that were found in section 2.3.

Example 4.2.5 Let $\mathsf{E} = l_p$, $1 \leq p < \infty$, and let $\mathcal{K}(\mathsf{E})$ be the set of all compact operators on E. Let

$$\mathcal{S} = \{ T \in \mathcal{K}(\mathsf{E}) : T = (a_{ij})_{i,j \in \mathbb{N}}, \text{ with } a_{ij} = 0 \text{ if } i \geq j \text{ and } a_{ij} \geq 0 \text{ if } i < j \}.$$

Then $\mathcal{S}$ is a convex semigroup of quasinilpotent positive operators.

Example 4.2.6 Suppose m is the Lebesgue measure on $\mathcal{X} = [0, 1]$ and $\mathcal{M}$ is the algebra of all m-measurable subsets of $\mathcal{X}$. Let $\mathsf{E} = L_2(\mathcal{X}, \mathcal{M}, m)$ and let $\mathcal{S}$ be the set of all positive Volterra operators on E, i.e. $\mathcal{S}$ is the set of all operators $T \in \mathcal{B}(\mathsf{E})$ for which there exists a positive function K_T in $L_2(m \times m)$, with $K_T(x, y) = 0$ when $x < y$, such that

$$(Tf)(x) = \int_0^x K_T(x, y) f(y)\, dy \qquad f \in \mathsf{E}.$$

Then $\mathcal{S}$ is a convex semigroup of quasinilpotent positive operators (cf. [19, Problems 186 and 187]).

Some other examples of convex semigroups of quasinilpotent positive operators can be found by using Lemma 2.3.8.

In the following we shall construct a concrete process of ideal-triangularization.

Example 4.2.7 Let $\mathsf{E} = L^p[0, 1]$, $1 \leq p < \infty$, or $\mathsf{E} = C[0, 1]$, and let V be the Volterra integral operator on E, *i.e*, the indefinite integral

$$Vf(t) = \int_0^t f(x)\, dx .$$

It is known that V is a compact quasinilpotent operator on E. If we define

$$\mathsf{E}_+ = \{ f \in \mathsf{E} : f \geq 0 \ a.e. \ on \ [0, 1] \},$$

where $\mathsf{E} = L^p[0, 1]$ and

$$\mathsf{E}_+ = \{ f \in \mathsf{E} : f(t) \geq 0 \quad \forall t \in [0, 1] \},$$

where $\mathsf{E} = C[0, 1]$, then it is also obvious that V is a positive operator.

For each $s \in [0, 1]$, let

$$\mathsf{I}_s = \{ f \in \mathsf{E} : f = 0 \ a.e. \ on \ [0, s] \}$$

if $\mathsf{E} = L^p[0, 1]$, and let

$$\mathsf{I}_s = \{ f \in \mathsf{E} : f = 0 \ on \ [0, s] \}$$

if $\mathsf{E} = C[0,1]$. In either case it is clear that the I_s form closed ideals of E which are invariant under V. It was shown, in [12], that in both cases, such subspaces(ideals) are the only closed invariant subspaces(ideals) of the operator V. In a similar discussion to that of [41, Example 4.3.12], it can also be shown that, in both cases, the family $\mathcal{F} = \{\, \mathsf{I}_s \,:\, 0 \le s \le 1 \,\}$ is a maximal chain, and hence $\mathcal{F}$ is a triangularizing ideal chain for V.

Example 4.2.8 Let $\mathcal{K} = [0,1]$ and let $\mathsf{E} = C(\mathcal{K})$. For each nonzero $f \in \mathsf{E}_+$, let α and β be two distinct positive numbers such that $[\alpha,\beta] \subset [0,1]$ and $f(t) > 0$ for all $t \in [\alpha,\beta]$. Let g be a continuous function on $[\alpha,\beta]$ such that $g(t) > 0$ for all $t \in [\alpha,\beta]$, $f(\alpha) = g(\alpha)$, $f(\beta) = g(\beta)$, and $f(t) > g(t)$ for all $t \in (\alpha,\beta)$. Define h on $[\alpha,\beta]$ by $h(t) = 2f(t) - g(t)$. Then h is also continuous on $[\alpha,\beta]$, $h(t) > 0$ for all $t \in [\alpha,\beta]$, $f(\alpha) = h(\alpha)$, $f(\beta) = h(\beta)$, and $h(t) > f(t)$ for all $t \in (\alpha,\beta)$.

Now if we define g, h on $[0,\alpha] \cup [\beta,1]$ by

$$g(t) = h(t) = f(t) \qquad \forall t \in [0,\alpha] \cup [\beta,1],$$

then it is easy to show that: $g, h \in \mathsf{E}_+$,

$$f(t) = \frac{g(t) + h(t)}{2} \qquad \forall t \in [0,1],$$

and g, h are not positive multiples of f which means E has no atoms. This shows that if we consider the Banach lattice $\mathsf{F} = \mathbb{R} \oplus \mathsf{E}$, the closed ideal E of F has no atoms. Therefore we cannot apply the procedures used in Chapter 3 to establish the ideal-triangularizability of a quasinilpotent positive operator on a Banach lattice with atoms.

Suppose $[\mathcal{I}, \Phi]$ is a quasi-normed operator ideal and X is a reflexive Banach space such that $\mathcal{I}(X)$ is a C-trace ideal. Using Theorem 1.4.1, we know that every semigroup $\mathcal{S}$ of quasinilpotent operators on X that contains a nonzero element of $\mathcal{I}(X)$ is reducible. However, we do not know, in general, if $\mathcal{S}$ is triangularizable. This is because the same procedure, as the one used in the proof of Corollary 1.4.2, cannot be applied to prove this triangularizability, as the following example shows.

Example 4.2.9 Consider $X = l_p$, $2 < p < \infty$. We know that X is a reflexive Banach space. We also know that X has the approximation property, as it has a basis, and hence $\mathfrak{N}(X)$ is a C-trace ideal by Corollary 1.1.8. By Theorem 2.d.6 of [28], X has a subspace M which does not have the approximation property, and hence $\mathfrak{N}(M/\{0\})$ is not a C-trace ideal. Therefore; if $\mathcal{S}$ is a semigroup of quasinilpotent operators on X that contains a nonzero element of $\mathcal{I}(X)$ and if $M \in Lat(\mathcal{S})$, then we cannot apply Theorem 1.4.1 to the compression $\hat{\mathcal{S}}$ of $\mathcal{S}$ to $M/\{0\}$.

4.3 Remarks, Consequences, and Open Questions

Remark 4.3.1 If we examine the proof of Theorem 1.3.8, we will realize that the theorem is also true for the normed operator ideal $\mathfrak{K}(X)$ whenever X is a reflexive Banach space with $A.P.$. This is because in such a case:

(1) $\overline{\mathfrak{F}(X)}^{\|\cdot\|} = \mathfrak{K}(X)$ by Proposition 1.1.16,

(2) we can certainly apply Corollary 1.1.7 of [8], and

(3) for the linear functional f we have

$$|f(T)| = |tr(BT)| \leq |tr(B)| \cdot \|T\| \quad \forall T \in \mathfrak{K}(X).$$

Therefore f is a norm-continuous linear functional on $\mathfrak{K}(X)$, and hence all the conditions of Lemma 1.3.2 are satisfied and we can state:

Theorem 4.3.2 *Let X be a reflexive Banach space with* A.P.. *Suppose $\mathcal{S} \subseteq \mathfrak{K}(X)$ is a semigroup such that $r(S) \leq 1$ for all $S \in \mathcal{S}$. If $\mathcal{S}$ contains an operator A that is not a contraction under any renorming of X, then $\mathcal{S}$ is reducible.*

Considering Example 4.2.3, Theorem 4.3.2 shows that, under suitable conditions, Theorem 1.3.8 is also valid for some operator ideals that are not C-trace ideals.

Remark 4.3.3 Even if a Hilbert lattice H is under discussion, one cannot say that in a semigroup of quasinilpotent positive operators the sum of two quasinilpotent positive operators is quasinilpotent. This was observed by P. Guniand in [17].

Remark 4.3.4 If the semigroup $\mathcal{S}$, in Theorem 1.3.9, is commutative the condition on the spectral radii is fulfilled and the result is trivial. However, there are many nontrivial examples of non-commutative semigroups satisfying the hypothesis of Theorem 1.3.9 even in finite dimensions. For example every semigroup of idempotents is triangularizable by [37], but not necessarily commutative by any means. For other examples of such semigroups see [14].

In Chapters 2 and 3 we observed that under suitable conditions a semigroup of (compact) quasinilpotent positive operators is decomposable. Some examples are Theorem 2.2.1, Theorem 2.4.4, and Theorem 3.3.9. The results of Section 4.1 show, in general, that this is not always the case. However, it seems that the following question has an affirmative answer.

Question 4.3.5 *Is a semigroup of compact quasinilpotent positive operators, on an arbitrary Banach lattice* E, *decomposable?*

Considering Lemma 2.2.4 and the results of [3], it is natural to ask the following (in some sense) strong version of Question 4.3.5.

Question 4.3.6 *If a compact operator on a Banach lattice* E *majorizes an element of a semigroup* $\mathcal{S}$ *of quasinilpotent positive operators on* E *, must* $\mathcal{S}$ *be decomposable?*

Remark 4.3.7 Suppose E is a Banach lattice such that E_+ does not contain any quasi-interior point. By Corollary to the proposition III.8.3 of [46], any semigroup of positive operators in $\mathfrak{B}(E)$ which is separable in the strong operator topology of $\mathfrak{B}(E)$ is decomposable. In particular, any positive operator on E is decomposable. Therefore Question 4.3.3 is much more interesting whenever a Banach lattice E, that possesses quasi-interior positive elements, is under discussion. This also shows that it seems to be unnecessary to weaken the hypothesis of Theorem 2.3.4 by withdrawing the condition on E to possess a positive quasi-interior point.

Theorems 2.1.1, 2.3.9, 2.3.10, 2.4.4, and, 2.5.4 suggest the following question about a semigroup of quasinilpotent positive operators on $C(\mathcal{K})$.

Question 4.3.8 *Is a semigroup of quasinilpotent positive operators on $C(\mathcal{K})$ decomposable?*

Remark 4.3.9 It is known that there are σ-finite measure spaces $(X, \mathcal{A}, \mu)$ for which $(\mathcal{A}(\mu), \mu)$ is a non-separable measure algebra with no atoms. For example, consider the family of topological spaces

$$\{\mathcal{X}_\alpha : \alpha \in \mathbb{R} \text{ and } \mathcal{X}_\alpha = [0, 1]\}$$

and let $(\mathcal{X}, \mathfrak{T})$ denotes its corresponding *Cartesian product space*. Let $\mathcal{U}$ be an open subset of $\mathcal{X}$. Then $\mathcal{U} = \prod_{\alpha\in\mathbb{R}} \mathcal{U}_\alpha$, where, for each α, $\mathcal{U}_\alpha$ is an open subset of $[0, 1]$ and $\mathcal{U}_\alpha = \mathcal{X}_\alpha$ for all but a finite number of indices. Define

$$\nu(\mathcal{U}) = \prod_{\alpha\in\mathbb{R}} m(\mathcal{U}_\alpha)\,.$$

Then it can be verified that ν induces a finite measure μ on $\mathcal{X}$ that has no atoms. With this measure μ, it can be verified that the space $L_2(\mathcal{X}, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the algebra of all μ-measurable subsets of $\mathcal{X}$, is not separable. Thus, by [18, Section 42.1], the measure algebra $(\mathcal{A}(\mu), \mu)$ is not separable.

At present, however, we do not know the answer to the following question:

Question 4.3.10 *Suppose* $(X, \mathcal{A}, \mu)$ *is a σ-finite measure space such that* $(\mathcal{A}(\mu), \mu)$ *is a non-separable measure algebra with no atoms. Does there exists an indecomposable quasinilpotent positive operator on* $L_p(X, \mathcal{A}, \mu)$, $1 \leq p < \infty$)*?*

Remark 4.3.11 Suppose $T \in \mathfrak{B}(\mathsf{E})$ is quasinilpotent at a nonzero element x_0 of E and let I be an ideal of E which belongs to *Ilat*(T). Since it is quite possible that $x_0 \notin \mathsf{I}$, the procedure given in Chapter 3 may fail to prove the ideal-triangularizability of classes of positive operators which contains compact operators that are quasinilpotent at a point. Therefore, one may ask the following question.

Question 4.3.12 *Suppose* E *is an arbitrary Banach lattice. Is every compact positive operator* $T \in \mathfrak{B}(\mathsf{E})$, *that is quasinilpotent at a non-zero positive element, ideal-triangularizable?*

Remark 4.3.13 In Section 3.3 we observed that a semigroup of quasinilpotent positive operators on a discrete Banach lattice with order continuous norm is ideal-triangularizable. In fact the existence of an atom in each closed ideal of such Banach lattices is one of the keys in finding ideal-triangularizing chains. However, as we observed in Example 4.2.8, the procedure of ideal-triangularizability of a single quasinilpotent positive operator given in Section 3.3 is not applicable when a general Banach lattice with atoms is under consideration. Therefore it is natural to ask:

Question 4.3.14 *Is a quasinilpotent positive operator on a Banach lattice with atoms ideal-triangularizable?*

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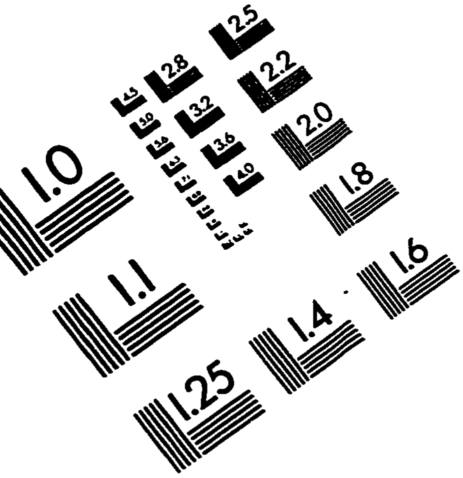
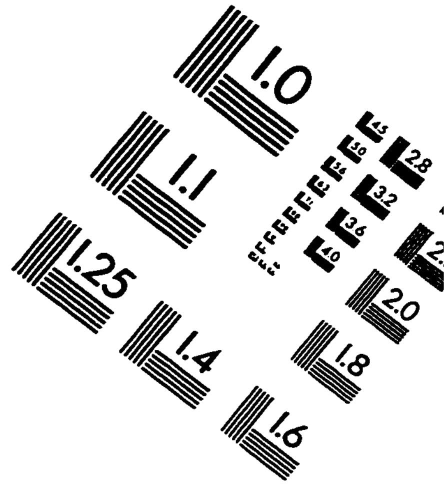
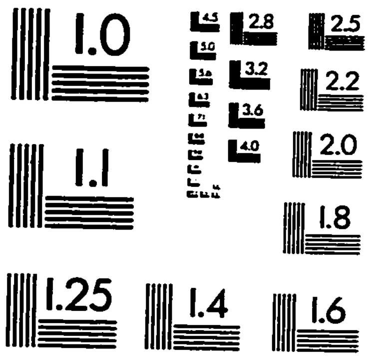
IMAGE EVALUATION TEST TARGET (QA-3)







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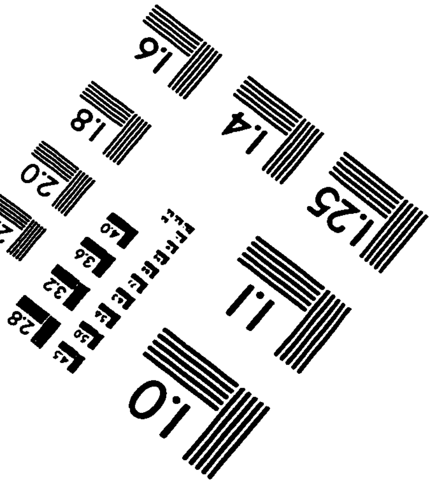

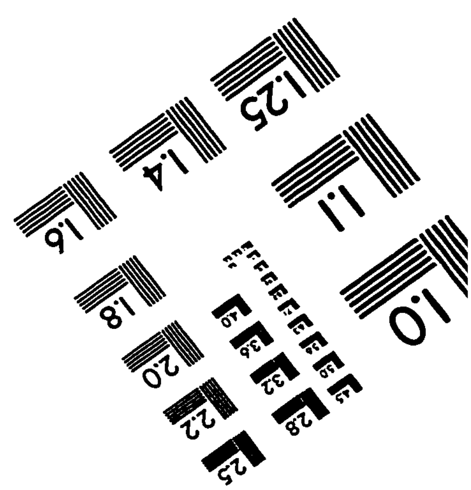

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