GENERALIZING FRÖBERG'S THEOREM ON IDEALS WITH LINEAR RESOLUTIONS

by

Emma Connon

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Abstract

In 1990, Fröberg presented a combinatorial classification of the quadratic square-free monomial ideals with linear resolutions. He showed that the edge ideal of a graph has a linear resolution if and only if the complement of the graph is chordal. Since then, a generalization of Fröberg's theorem to higher dimensions has been sought in order to classify all square-free monomial ideals with linear resolutions. Such a characterization would also give a description of all square-free monomial ideals which are Cohen-Macaulay.

In this thesis we explore one method of extending Fröberg's result. We generalize the idea of a chordal graph to simplicial complexes and use simplicial homology as a bridge between this combinatorial notion and the algebraic concept of a linear resolution. We are able to give a generalization of one direction of Fröberg's theorem and, in investigating the converse direction, find a necessary and sufficient combinatorial condition for a square-free monomial ideal to have a linear resolution over fields of characteristic 2.

List of Abbreviations and Symbols Used

$[F, G]$	closed interval from F to G in a simplicial complex, 106
$[v_0, \ldots, v_d]$	oriented d-face on vertices $v_0, \ldots, v_d$ with orientation $v_0 < \cdots < v_d$, 12
$\beta_{i,j}(M)$	the graded Betti numbers of M, 21
$\beta_i(M)$	the ith total Betti number of M, 21
$\bigsqcup$	disjoint union, 106
$\Delta(G)$	clique complex of the graph G, 18
$\Delta_d(\Gamma)$	d-closure of Γ, 61
$\dim \Gamma$	dimension of a simplicial complex Γ, 8
$\dim F$	dimension of a face F, 8
$\mathcal{F}(\Gamma)$	facet ideal of Γ, 19
$\mathcal{F}(I)$	facet complex of I, 19
$\mathrm{Facets}(\Gamma)$	facet set of Γ, 8
$\Gamma * \Delta$	join of Γ and Δ, 10
$\Gamma * v$	cone over Γ with vertex v, 11
$\Gamma^{[d]}$	pure d-skeleton of a simplicial complex Γ, 9
$\Gamma^\vee$	Alexander dual of Γ, 106
Γ_W	induced subcomplex of Γ on W, 10
$\operatorname{im} f$	the image of the map f, 14
$\ker f$	the kernel of the map f, 14
Λ_n^d	d-dimensional d-complete simplicial complex with n vertices, 10
$\langle F_1, \ldots, F_k \rangle$	simplicial complex with facets $F_1, \ldots, F_k$, 8
$\mathbb{N}$	the natural numbers $\{0, 1, 2, \ldots\}$, 19
$\mathfrak{N}(G)$	neighbourhood complex of the graph G, 105
$\mathcal{N}(\Gamma)$	Stanley-Reisner ideal of Γ, 19
$\mathcal{N}(I)$	Stanley-Reisner complex of I, 19
$\Omega(\pi_1, \ldots, \pi_p)$	vertex-partition complex whose minimal non-faces are $\pi_1, \ldots, \pi_p$, 87
$\overline{\Gamma}_d$	d-complement of a simplicial complex Γ, 9
$\overline{G}$	complement of a graph G, 18

∂_d	d-boundary map, 13
$\mathrm{pd}(I)$	projective dimension of I, 22
$\mathbb{R}$	the real numbers, 9
$\mathrm{reg}(I)$	regularity of I, 22
$\tilde{H}_d(\Gamma; A)$	dth reduced simplicial homology group of Γ over A, 15
$\mathbb{Z}$	the integers, 14
$\mathbb{Z}_2$	the integers modulo 2, 15
$\{\emptyset\}$	the empty complex, 8
$\{\}$	the void complex, 8
$C_d(\Gamma)$	the d-chains of Γ, 13
$E(G)$	edge set of the graph G, 16
$f(\Gamma)$	the f-vector of Γ, 9
G_S	induced subgraph of G on vertex set S, 16
$H_d(\Gamma; A)$	dth simplicial homology group of Γ over A, 13
$I_{[d]}$	ideal generated by the square-free monomials in I of degree d, 24
k	field, 19
$k[\Gamma]$	Stanley-Reisner ring of Γ, 19
K_n	complete graph on n vertices, 16
R	$k[x_1, \ldots, x_n]$, the polynomial ring, 19
$R(-j)$	shift of the polynomial ring by j, 21
$V(\Gamma)$	vertex set of a simplicial complex Γ, 8
$V(G)$	vertex set of a graph G, 16
x^F	monomial $x_{i_1}x_{i_2}\cdots x_{i_\ell}$ for $F = \{x_{i_1}, x_{i_2}, \ldots, x_{i_\ell}\}$, 19
$\lvert\Gamma\rvert$	geometric realization of Γ, 9

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Chapter 1

Introduction

One common method of understanding monomial ideals in a polynomial ring is to study their **minimal free resolutions**. A minimal free resolution of the monomial ideal I in $R = k[x_1, \ldots, x_n]$ is an exact sequence of maps between R-modules

$$0 \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to I \to 0$$

where F_i is a free R-module for all i and $m \leq n$. The resolution is called minimal because in each map, a basis of F_i is mapped onto a minimal generating set for the kernel of the following map. Such a resolution is unique and the ranks of the F_i's, called **Betti numbers**, provide information about the relationship between the generators of the ideal, and also about the relations between these relationships and so on. When the maps between the F_i's in these resolutions are representable with matrices whose non-zero entries are linear forms, the ideal is said to have a **linear resolution**.

To determine the monomial ideals which have linear resolutions it is enough to study square-free monomial ideals. Any monomial ideal can be turned into a square-free monomial ideal via a process called **polarization** (Herzog and Hibi [21, Section 1.6]). This process transforms a monomial ideal I which is not square-free into a square-free monomial ideal in a larger polynomial ring that has additional variables which replace those that appear more than once within a generator of I. The resulting square-free monomial ideal will have a linear resolution only when I has a linear resolution (Herzog and Hibi [21, Corollary 1.6.3]).

One advantage to studying square-free monomial ideals is that they can easily be associated to combinatorial objects and examined from a combinatorial perspective. There are two standard methods of associating a simplicial complex to a square-free monomial ideal I in the polynomial ring $R = k[x_1, \ldots, x_n]$. The **Stanley-Reisner complex** of I is the simplicial complex whose vertex set is $\{x_1, \ldots, x_n\}$ and whose faces correspond to the square-free monomials of R not belonging to I. The ideal I is the **Stanley-Reisner ideal** of this complex. We can also define the **facet complex** of I to be the simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$ whose facets correspond to the unique minimal generators of I. We say that I is the **facet ideal** of this complex.

When a square-free monomial ideal is generated in degree 2, it can be thought of as the facet ideal of a 1-dimensional simplicial complex or as the **edge ideal** of a graph. In this case the graph has vertices given by the variables in the polynomial ring and the edges of the graph correspond to the minimal monomial generators of the ideal. In 1990, Fröberg gave a combinatorial classification of the square-free monomial ideals generated in degree 2 which have linear resolutions over any field in terms of chordal graphs. A **chordal graph** is one in which every cycle of length greater than three has a chord. The **complement** of a graph G is a graph on the same set of vertices as G but whose edges correspond to the missing edges in G. In [16], Fröberg proved the following result.

Theorem 1.0.1 (Fröberg [16])**.** *The edge ideal of a graph G has a linear resolution if and only if the complement of G is chordal.*

One of the most interesting things about this theorem is that it provides a strong link between two well-known mathematical properties which originate from different mathematical disciplines and are each studied in their own right. The definition of a chordal graph is easy to understand. It is a simple geometric idea and Fröberg's theorem allows us to use it to describe an algebraic property which, at least on the surface, appears much more complicated.

The arrival of Fröberg's result inspired a new line of research. Since the introduction of his theorem there have been several attempts to generalize it in order to characterize all square-free monomial ideals having linear resolutions using a combinatorial property of an associated hypergraph or simplicial complex. Unfortunately, although possible in the 1-dimensional case, it is too much to expect that the property of having a linear resolution can be described purely through a combinatorial feature of an associated combinatorial structure. The existence of such a resolution often depends on the field over which the polynomial ring is defined and since the combinatorial structure is independent of the field, the combinatorics is not enough to determine whether or not such a linear resolution exists. The Stanley-Reisner ideal of a triangulation of the real projective plane is a typical example. This ideal has a linear resolution only when the characteristic of the field of the polynomial ring is not equal to 2. Therefore it is clear that the characteristic of the field in question also has a role to play in such a classification. However, it is not unreasonable to expect that such a complete combinatorial characterization may exist for ideals having linear resolutions over all fields.

Finding a higher-dimensional version of Fröberg's result is inherently more difficult than the 1-dimensional case for two simple reasons. First, there is the issue of orientability of the combinatorial structures which plays no role in the 1-dimensional situation (see Proposition 3.2.11). Even in the 2-dimensional case this is a problem, as illustrated by the triangulation of the real projective plane mentioned above, which is non-orientable. In addition, there are many different ways in which two higher-dimensional faces can intersect. This is in contrast to the graph case where there is essentially only one way that two edges may intersect. Thus an examination of the combinatorial objects associated to our monomial ideals is much more involved.

The general method to finding a generalization of Fröberg's theorem has been to extend the definition of a chordal graph to higher dimensions. There are several different, but equivalent, ways to define the class of chordal graphs. A **complete graph** is a graph having all possible edges. In [7], Dirac shows that all chordal graphs can be constructed by successively joining complete graphs together by identifying them along complete subgraphs. Conversely, all graphs constructed in this way are chordal. In [12], Emtander extends this idea to higher dimensions based on ideas of Hà and Van Tuyl presented in [18] and introduces the class of **generalized chordal hypergraphs**. He uses the idea of a d**-complete hypergraph**, whose hyperedges are all possible subsets of the vertex set of size d, to give a constructive definition based on Dirac's inductive framework. Emtander provides a generalization of one direction of Fröberg's theorem to higher dimensions by showing that the ideals associated to his generalized chordal hypergraphs have linear resolutions over all fields.

Another way to define a chordal graph is by requiring all induced subcomplexes of the graph to contain a **simplicial vertex**, which is a vertex with the property that all of its adjacent vertices are adjacent to each other. In [39], Woodroofe extends the notion of a simplicial vertex to hypergraphs. He calls a vertex v in a hypergraph **simplicial** if for any two hyperedges containing v there is a third hyperedge contained in their union which excludes v. Woodroofe defines a **chordal hypergraph** to be one in which all **minors**, which are subhypergraphs that result from deleting or contracting vertices, contain a simplicial vertex. Woodroofe also generalizes one direction of Fröberg's theorem by showing that the "hyperedge ideals" of the complements of his chordal hypergraphs have linear resolutions over all fields.

Unfortunately there are hyperedge ideals which have linear resolutions over all fields, but for which the associated hypergraph has a complement which is not chordal in either the sense of Emtander or Woodroofe. See Section 4.3 in Chapter 4 for an example. Therefore there do not exist complete generalizations of Fröberg's result using either of these definitions of chordal hypergraphs.

In this thesis, the approach taken to extending Fröberg's result is strongly motivated by Fr̈oberg's own method. Although Fröberg's theorem is now more commonly stated in terms of edge ideals, his theorem was originally presented using Stanley-Reisner ideals. In the following theorem, $\Delta(G)$ is the **clique complex** of the graph G. The clique complex of G is the simplicial complex that lies on the same vertex set as G and whose faces are given by the sets of vertices that correspond to complete subgraphs of G. By $\Gamma^{[1]}$ we mean the 1-skeleton of the simplicial complex Γ.

Theorem 1.0.2 (Fröberg [16])**.** *If a graph G is chordal, then the Stanley-Reisner ideal of $\Delta(G)$ has a linear resolution over any field. Conversely, if the Stanley-Reisner ideal of a simplicial complex Γ is generated in degree 2 and has a linear resolution over some field, then $\Gamma = \Delta(\Gamma^{[1]})$ and $\Gamma^{[1]}$ is chordal.*

The equivalence of Theorems 1.0.1 and 1.0.2 is straightforward to prove by showing that the edge ideal of the complement of a graph is equal to the Stanley-Reisner ideal of its clique complex. The proof of Theorem 1.0.2 relies completely on the following theorem proved by Fröberg in [15] which gives a homological classification of the square-free monomial ideals which have linear resolutions.

Theorem 1.0.3 (Fröberg [15])**.** *A square-free monomial ideal I generated in degree d has a linear resolution over a field k if and only if for every induced subcomplex Γ of the Stanley-Reisner complex of I we have $\tilde{H}_i(\Gamma; k) = 0$ for $i \neq d-2$.*

What is most interesting about this connection is that Theorem 1.0.3 is a purely homological characterization of linear resolutions whereas Theorem 1.0.2 is wholly combinatorial in nature. Furthermore, although Theorem 1.0.3 places homological conditions of all dimensions on the Stanley-Reisner complex and its induced subcomplexes, Theorem 1.0.2 requires only a simple condition on the 1-skeleton of the whole complex. The translation between these two frameworks is entirely reliant on the fact that a graph cycle, the structure at the heart of chordal graphs, is exactly the right notion to capture the idea of 1-dimensional simplicial homology. This fact and the intricate relationship between chordal graphs and their clique complexes, which forces all higher-dimensional homology to disappear also, makes Fröberg's proof possible. We discuss this in more detail in Chapter 4.

In this thesis, we attempt to generalize Fröberg's theorem by uncovering the combinatorial structures central to higher-dimensional simplicial homology. The idea is to extend the notion of chordal graphs to higher dimensions using these particular structures as higher-level "cycles". A higher-dimensional version of the clique complex is used to replicate the relationship between the Stanley-Reisner complex of the ideal and its 1-skeleton that we see in Fröberg's result.

As mentioned previously, the property of having a linear resolution is highly field-dependent. For this reason we focus much of our studies on the characteristic 2 case specifically. For these fields, we are more easily able to establish a link between the combinatorics of our complexes and their simplicial homology. This is due to the fact that we may reduce our studies to the $\mathbb{Z}_2$ case by appealing to the Universal Coefficient Theorem and that, over this field, the algebraic coefficients of homological structures play a lesser role. We also examine the ways in which our results can be generalized to the case of an arbitrary field, by including notions such as orientability.

This thesis is organized in the following way. In Chapter 2 we review definitions and notation that we will use throughout the thesis. In Chapter 3 we introduce a higher-dimensional notion of a graph cycle, the d**-dimensional cycle**. We study its combinatorial structure and discuss its homological properties over various fields. In particular, we are able to prove the following theorem which identifies the specific combinatorial properties in a simplicial complex which lead to non-zero homology over fields of characteristic 2.

Theorem 1.0.4. *Let Γ be a simplicial complex and let k be a field of characteristic 2. Then $\tilde{H}_d(\Gamma; k) \neq 0$ if and only if Γ contains a d-dimensional cycle, the sum of whose d-faces is not a d-boundary.*

In Chapter 4 we generalize the definition of chordal graphs to pure simplicial complexes in two different ways with the notions of **orientably-d-cycle-complete complexes** and **d-chorded complexes** which both rely heavily on the concept of a d-dimensional cycle. We provide a necessary condition for a square-free monomial ideal to have a linear resolution over any field, which generalizes one direction of Fröberg's theorem, and we give a stronger necessary condition in the case that the resolution is over a field of characteristic 2. In the following theorem the notation $\Gamma^{[d]}$ refers to the pure d-skeleton of the complex Γ and $\Delta_d(\Gamma^{[d]})$ is the **d-closure** of $\Gamma^{[d]}$ which is a generalization of the idea of the clique complex to higher dimensions.

Theorem 1.0.5. *Let Γ be a simplicial complex, let k be a field and let $d \geq 1$. If the Stanley-Reisner ideal of Γ is generated in degree $d+1$ and has a linear resolution over k then $\Gamma = \Delta_d(\Gamma^{[d]})$ and*

1. *$\Gamma^{[d]}$ is orientably-d-cycle-complete*
2. *$\Gamma^{[d]}$ is d-chorded if k has characteristic 2.*

After investigating the counterexamples to the converse of Theorem 1.0.5 part 2, we are able to provide a necessary and sufficient combinatorial condition for a square-free monomial ideal to have a linear resolution over fields of characteristic 2 using the notion of a **chorded** simplicial complex. This is a generalization to non-pure complexes of the idea of a d-chorded complex.

Theorem 1.0.6. *Let k be a field of characteristic 2 and let I be a square-free monomial ideal in $k[x_1, \ldots, x_n]$ that is minimally generated in a fixed degree. Then I has a linear resolution if and only if its Stanley-Reisner complex is chorded.*

In Chapter 5 we look more closely at a class of complexes arising from counterexamples to the converse of Theorem 1.0.5 part 2. These simplicial complexes have minimal non-faces which partition their vertex sets. We call these simplicial complexes **vertex-partition complexes** and we show that they have a simple homological structure.

Theorem 1.0.7. *Let Γ be a vertex-partition complex. Then Γ is a simplicial sphere and any induced subcomplex of Γ is either a simplicial sphere or is contractible.*

By exploiting the well-behaved structure of vertex-partition complexes, we are able to give simple formulas for computing the Betti numbers and other invariants of the Stanley-Reisner ideals of the entire class.

Finally, in Chapter 6 we discuss a few possible applications of the results of this thesis and related lines of research.

Chapter 2

Background

In this chapter we will review basic terminology, notation and concepts that will be used in later chapters. The approach that we will use to try and develop a generalization of Fröberg's theorem draws from many different areas of mathematics. We include here basic introductions to simplicial complexes and homology, graph theory, Stanley-Reisner theory and monomial resolutions. For more detailed expositions on these topics we refer the reader to books by Munkres for simplicial complexes and homology [28], West for graph theory [38], Bruns and Herzog for Stanley-Reisner theory [4], and Peeva for monomial resolutions [30].

2.1 Simplicial Complexes

An (abstract) **simplicial complex** Γ on the finite **vertex set** V is a set of subsets of V such that for any $F \in \Gamma$ if $G \subseteq F$ then $G \in \Gamma$. The elements of V are **vertices** of Γ and the elements of Γ are called **faces** or **simplices** of Γ. Faces of Γ that are maximal with respect to inclusion are called **facets** of Γ and we use the notation $\mathrm{Facets}(\Gamma)$ for this set of faces. We denote the vertex set of Γ by $V(\Gamma)$. If $\mathrm{Facets}(\Gamma) = \{F_1, \ldots, F_k\}$ then we write

$$\Gamma = \langle F_1, \ldots, F_k \rangle.$$

A simplicial complex with a single facet is called a **simplex**. If F is a face of Γ then the **dimension** of F, denoted by $\dim F$, is equal to $|F| - 1$ while the dimension of Γ itself is

$$\dim \Gamma = \max\{\dim F : F \in \Gamma\}.$$

By convention, the face $\emptyset$ has dimension -1. The simplicial complex $\{\emptyset\}$ is called the **empty complex** and we have $\dim\{\emptyset\} = -1$. The **void complex** $\{\}$ has no faces and by convention $\dim\{\} = -\infty$.

A face of the simplicial complex Γ of dimension d is called a d-**face** or a d-**simplex**. If all facets of Γ have the same dimension then Γ is said to be **pure**. The f-**vector** of a simplicial complex Γ is the vector $f(\Gamma) = (f_{-1}(\Gamma), f_0(\Gamma), \ldots, f_{\dim \Gamma}(\Gamma))$ where $f_i(\Gamma)$ is the number of faces of dimension i in Γ. The empty face is the only face of dimension -1 and so we have $f_{-1}(\Gamma) = 1$ unless Γ is the void complex in which case $f_{-1}(\Gamma) = 0$.

Often simplicial complexes are used as models for more complicated topological spaces. A set of points $x_1, \ldots, x_n$ in $\mathbb{R}^d$ is **affinely independent** if, for $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\sum_{i=1}^n \lambda_i x_i = 0$ and $\sum_{i=1}^n \lambda_i = 0$ imply that $\lambda_i = 0$ for all $1 \leq i \leq n$. Let Γ be a simplicial complex with $|V(\Gamma)| = n$ and choose an embedding of the n vertices of Γ in $\mathbb{R}^{n-1}$ that is affinely independent. To each face of Γ we associate the geometric simplex that is spanned by the corresponding vertices in $\mathbb{R}^{n-1}$ and denote the union of these simplices by $|\Gamma|$. We call the topological space $|\Gamma|$ the **geometric realization** of Γ. Conversely, given a topological space X, a **triangulation** of X is a simplicial complex whose geometric realization is homeomorphic to X. In Figure 2.1 we give an example of the geometric realization of a simplicial complex which is the triangulation of a 2-dimensional sphere. Throughout this thesis, we will implicity use the idea of a geometric realization to give visual representations of our simplicial complexes.

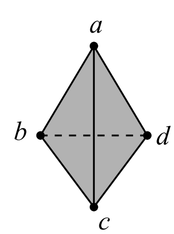


Figure 2.1: Geometric realization of $\Gamma = \langle \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \rangle$.

The d-**complement** of a pure d-dimensional simplicial complex Γ, denoted $\overline{\Gamma}_d$, is the simplicial complex on $V(\Gamma)$ whose facets are the $(d+1)$-subsets of $V(\Gamma)$ that are not faces of Γ. An example is given in Figure 2.2.

The **pure** d-**skeleton** of a simplicial complex Γ, denoted $\Gamma^{[d]}$, is the simplicial complex on the same vertex set as Γ whose facets are the d-faces of Γ. A **subcomplex** of Γ is any simplicial complex whose set of facets is a subset of the faces of Γ. Given

(a) $\Gamma = \langle\{a,b,c\},\{a,c,d\}\rangle$ (b) $\overline{\Gamma}_2 = \langle\{a,b,d\},\{b,c,d\}\rangle$

Figure 2.2: The 2-complement of a pure 2-dimensional simplicial complex.

any $W \subseteq V(\Gamma)$ the **induced subcomplex of Γ on the vertex set** W is the simplicial complex

$$\Gamma_W = \{F \in \Gamma | F \subseteq W\}.$$

See Figure 2.3 for an example of an induced subcomplex.

(a) The complex Γ (b) The complex $\Gamma_{\{a,c,f,g\}}$

Figure 2.3: The induced subcomplex of Γ on $\{a, c, f, g\}$.

A simplicial complex Γ is said to be ***d*-complete** if all possible subsets of $V(\Gamma)$ of size $d+1$ are faces of Γ. Note that if Γ is d-complete then it is d'-complete for all $d' \leq d$. The d-dimensional d-complete complex on n vertices is denoted Λ_n^d. Notice that Λ_n^d is the pure d-skeleton of the simplex on n vertices.

The **join** of the simplicial complexes Γ and Δ with $V(\Gamma) \cap V(\Delta) = \emptyset$ is the simplicial complex denoted $\Gamma * \Delta$ on the vertex set $V(\Gamma) \cup V(\Delta)$ such that

$$\Gamma * \Delta = \{F \cup G \mid F \in \Gamma, G \in \Delta\}.$$

Note that $\dim(\Gamma * \Delta) = \dim \Gamma + \dim \Delta + 1$ and that the join operation is both commutative and associative (Kozlov [26, page 12]).

In Figure 2.4 we give an example of the join of two simplicial complexes.

(a) Two 1-faces

(b) $\langle\{x_0, x_1\}\rangle * \langle\{x_2, x_3\}\rangle$

Figure 2.4: The join of two 1-faces results in a solid tetrahedron.

A special case of the join operation is when one of the complexes is a single vertex. The join of a simplicial complex Γ with a single vertex v is called the **cone over** Γ **with vertex** v and is written $\Gamma * v$. In Figure 2.5 we give an example of the cone over a simplicial complex. Notice that, in this instance, the resulting complex is actually conical in shape.

(a) The 1-dimensional complex Γ and the vertex v

(b) $\Gamma * v$

Figure 2.5: The cone over a complex Γ with the vertex v.

A facet F of a simplicial complex Γ is a **leaf** of Γ if either F is the only facet in Γ or if there exists some facet G in Γ with $G \neq F$ such that for any other facet H in Γ we have $H \cap F \subseteq G$ (Faridi [13]). In graph theory, a leaf is a vertex that is only contained in one edge, also known as a **free vertex**. In contrast, a leaf in a simplicial complex is a generalization of a graph edge that contains a leaf since, for algebraic purposes, it matters how the facet containing the free vertex connects to the rest of the simplicial complex. A nonempty simplicial complex Γ is a **simplicial cycle** if Γ has no leaves, but every subcomplex of Γ whose facets are a nonempty proper subset of Facets(Γ) has a leaf (Caboara et al. [5]). See Figure 2.6 for an example of a simplicial cycle.

To any d-face F in a simplicial complex we can assign an ordering to its vertices.

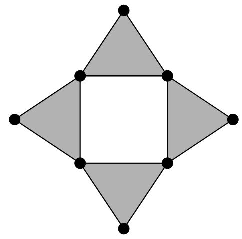


Figure 2.6: A simplicial cycle.

Two orderings are said to be **equivalent** if one is an even permutation of the other. Thus, when $\dim F > 0$, there are only two equivalence classes of orderings on F. These equivalence classes are called **orientations** of F. By an **oriented d-face** we mean a d-face with a choice of one of these two orientations. Given the vertices $v_0, \ldots, v_d$ we use the notation $[v_0, \ldots, v_d]$ to denote the oriented d-face on $v_0, \ldots, v_d$ with the choice of orientation given by the equivalence class of the ordering $v_0 < \cdots < v_d$. In other words, $[v_0, \ldots, v_d]$ represents the entire equivalence class. We will also need the concept of an **induced orientation** of a subface in the simplicial complex.

Definition 2.1.1 (induced orientation). Given an oriented d-face $[v_0, \ldots, v_d]$ in a simplicial complex, the **induced orientation** of any $(d-1)$-subface of $[v_0, \ldots, v_d]$ is given by the following procedure, where v_0 is considered to be in an even position:

- if the vertex removed to obtain the $(d-1)$-face was in an even position of the ordering then the orientation of the $(d-1)$-face is given by the ordering of its vertices in the d-face

- if the vertex removed to obtain the $(d-1)$-face was in an odd position of the ordering then the orientation of the $(d-1)$-face is given by any odd permutation of the ordering of the vertices in the d-face

Example 2.1.2. Let Γ be a simplicial complex and let $[a, b, c, d, e]$ be an oriented 4-face in Γ. The induced orientation of the 3-face $\{a, b, d, e\}$ is given by $a < b < d < e$ and the induced orientation of the 3-face $\{a, c, d, e\}$ is given by $a < c < e < d$.

Notice that a $(d-1)$-face which belongs to more than one oriented d-face in a simplicial complex will have an induced orientation corresponding to each oriented d-face to which it belongs. These induced orientations may be non-equivalent.

2.2 Simplicial Topology

An R-module F is **free** if there exists a set of elements $y_1, \ldots, y_m$ of F called a **basis** such that for any non-zero element x of F there exist unique elements $r_1, \ldots, r_n$ of R such that $x = r_1 y_1 + \cdots + r_m y_m$. When R is commutative all bases of a free module have the same cardinality (Rotman [33, Proposition 7.50]). In this case $F \cong R^m$ and F is said to have **dimension** or **rank** m.

Let A be a commutative ring with identity. We define $C_d(\Gamma)$ to be the free A-module whose basis is the oriented d-faces of Γ with the relations $[v_0, v_1, \ldots, v_d] = -[v_1, v_0, \ldots, v_d]$ for each oriented d-face $[v_0, v_1, \ldots, v_d]$. The elements of $C_d(\Gamma)$ are called d**-chains**.

There is a boundary map homomorphism ∂_d from the space of d-chains to the space of $(d-1)$-chains that is defined by setting

$$\partial_d([v_0, \ldots, v_d]) = \sum_{i=0}^{d} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d]$$

for each oriented d-face $[v_0, \ldots, v_d]$ and extending to $C_d(\Gamma)$ by linearity. Notice that the map ∂_d is consistent with Definition 2.1.1 in the sense that ∂_d maps an oriented d-face to the sum of the induced orientations of its $(d-1)$-faces. It is easy to show that $\partial_d \partial_{d+1} = 0$ for all d. Thus this sequence of homomorphisms forms the **oriented chain complex of** Γ

$$\cdots \longrightarrow C_{n+1}(\Gamma) \xrightarrow{\partial_{n+1}} C_n(\Gamma) \xrightarrow{\partial_n} C_{n-1}(\Gamma) \longrightarrow \cdots \longrightarrow C_1(\Gamma) \xrightarrow{\partial_1} C_0(\Gamma) \xrightarrow{\partial_0} 0.$$

The kernel of the map ∂_d is called the group of d**-cycles** and the image of ∂_d is called the group of $(d-1)$**-boundaries**. Since $\partial_d \partial_{d+1} = 0$ for all d, the group of d-boundaries is contained in the group of d-cycles. The d**th simplicial homology group** of Γ over A is equal to the quotient of the group of d-cycles over the group of d-boundaries and is denoted $H_d(\Gamma; A)$. Roughly speaking, a non-zero element of $H_d(\Gamma; A)$ indicates the presence of a "d-dimensional hole" in the simplicial complex. One can show that the dimension of $H_0(\Gamma; A)$ corresponds to the number of connected components of the simplicial complex.

Example 2.2.1. Consider the 2-dimensional simplicial complex Γ illustrated in Figure 2.7 where vertices with the same label are identified. This simplicial complex is a triangulation of a torus. We will compute the simplicial homology groups of Γ over $\mathbb{Z}$. We can see that

$$\dim C_2(\Gamma) = 18, \quad \dim C_1(\Gamma) = 27, \quad \text{and} \quad \dim C_0(\Gamma) = 9$$

by counting the number of 0, 1, and 2-faces. The map ∂_2 is represented by a 27×18 matrix, ∂_1 by a 9×27 matrix, and ∂_0 by a 1×9 matrix. By row-reducing these matrices we can compute that

$$\dim \ker \partial_2 = 1, \quad \dim \ker \partial_1 = 19, \quad \text{and} \quad \dim \ker \partial_0 = 9.$$

Because $\operatorname{im} \partial_i \cong C_i(\Gamma)/\ker \partial_i$ we see that

$$\dim \operatorname{im} \partial_2 = 17, \quad \dim \operatorname{im} \partial_1 = 8, \quad \text{and} \quad \dim \operatorname{im} \partial_0 = 0.$$

Therefore, since $H_i(\Gamma; \mathbb{Z}) = \ker \partial_i / \operatorname{im} \partial_{i+1}$, we have

$$H_2(\Gamma; \mathbb{Z}) = \mathbb{Z}, \quad H_1(\Gamma; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}, \quad \text{and} \quad H_0(\Gamma; \mathbb{Z}) = \mathbb{Z}\,.$$

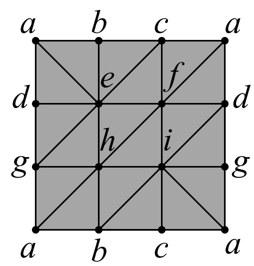


Figure 2.7: A triangulated torus.

We can also set $C_{-1}(\Gamma) = A$ since Γ has exactly one face of dimension -1, the empty face. We can then redefine the homomorphism ∂_0 by setting $\partial_0 : C_0(\Gamma) \to C_{-1}(\Gamma)$ where $\partial_0(v) = 1$ for each $v \in V(\Gamma)$. In this way we obtain the **augmented oriented chain complex of** Γ

$$\cdots \longrightarrow C_{n+1}(\Gamma) \xrightarrow{\partial_{n+1}} C_n(\Gamma) \longrightarrow \cdots \longrightarrow C_1(\Gamma) \xrightarrow{\partial_1} C_0(\Gamma) \xrightarrow{\partial_0} C_{-1}(\Gamma) \xrightarrow{\partial_{-1}} 0.$$

We denote the homology groups of this complex by $\tilde{H}_i(\Gamma; A)$ and we clearly have that $\tilde{H}_i(\Gamma; A) = H_i(\Gamma; A)$ for $i > 0$. We also have

$$\tilde{H}_0(\Gamma; A) = \ker \partial_0 / \operatorname{im} \partial_1$$

and

$$\tilde{H}_{-1}(\Gamma; A) = \ker \partial_{-1} / \operatorname{im} \partial_0.$$

It is the case that $\dim \tilde{H}_0(\Gamma; A) = \dim H_0(\Gamma; A) - 1$. The group $\tilde{H}_i(\Gamma; A)$ is called the **reduced homology group** of Γ over A in dimension i.

Remark. For the void complex we have $\tilde{H}_i(\{\}; A) = 0$ for all i whereas for the empty complex we have $\tilde{H}_{-1}(\{\emptyset\}; A) = A$ and $\tilde{H}_i(\{\emptyset\}; A) = 0$ for $i \neq -1$.

A simplicial complex Γ is called **acyclic over** A if $\tilde{H}_i(\Gamma; A) = 0$ for all i. It is well-known that a simplicial complex is acyclic over $\mathbb{Z}$ if and only if it is acyclic over all fields (Björner [3, page 1853]).

Throughout the thesis we will make use of the following fact, which is a consequence of the Universal Coefficient Theorem for homology and basic properties of the tensor product.

Lemma 2.2.2. *Let k be a field of characteristic 2 and let Γ be a simplicial complex. Then*

$$\dim_k \tilde{H}_i(\Gamma; k) = \dim_{\mathbb{Z}_2} \tilde{H}_i(\Gamma; \mathbb{Z}_2)$$

for all $0 \leq i \leq \dim \Gamma$.

Proof. By the Universal Coefficient Theorem given in [22, Theorem 2.5, page 176] we know that $\tilde{H}_i(\Gamma; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} k \cong \tilde{H}_i(\Gamma; k)$ since $\operatorname{Tor}_1^{\mathbb{Z}_2}(\tilde{H}_{i-1}(\Gamma; \mathbb{Z}_2), k) = 0$, because k is a $\mathbb{Z}_2$-vector space and hence is flat.

Furthermore, by [8, Corollary 18, page 373],

$$\dim_k(\tilde{H}_i(\Gamma; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} k) = \dim_{\mathbb{Z}_2} \tilde{H}_i(\Gamma; \mathbb{Z}_2).$$

Therefore

$$\dim_k \tilde{H}_i(\Gamma; k) = \dim_{\mathbb{Z}_2} \tilde{H}_i(\Gamma; \mathbb{Z}_2).$$

□

A simplicial complex Γ is **contractible** if its geometric realization is homotopy equivalent to the topological space having a single point. A simplex is the easiest example of a contractible simplicial complex. It is a consequence of the definition that contractible simplicial complexes are acyclic over $\mathbb{Z}$ and therefore over all fields. Also it is not difficult to show that the cone of a simplicial complex, introduced in Section 2.1, is contractible (Rotman [32, Theorem 1.11]).

A d-dimensional simplicial complex Γ is called a **simplicial sphere** if its geometric realization is homeomorphic to a d-dimensional sphere. See Figure 2.1 for an example. A simplicial sphere is pure and has non-zero reduced homology only in its top dimension (Munkres [28, page 230]). As discussed in (Hatcher [19, page 9]), the d-dimensional sphere is homeomorphic to the geometric realization of the join of $d+1$ two-point complexes. Since the join operation is associative, the following result follows immediately.

Proposition 2.2.3. *The join of an n-dimensional simplicial sphere with an m-dimensional simplicial sphere is an $(n+m+1)$-dimensional simplicial sphere.*

2.3 Graph Theory

A **finite simple graph** G consists of a finite nonempty set of **vertices** denoted $V(G)$ and a set of **edges** denoted $E(G)$ where the elements of $E(G)$ are unordered pairs from $V(G)$. In this thesis we will use the term **graph** to refer to a finite simple graph. A **subgraph** H of the graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The **induced subgraph** of G on the set $S \subseteq V(G)$ is the subgraph G_S with $V(G_S) = S$ and $E(G_S) = \{e \in E(G) \mid e \subseteq S\}$.

A 1-dimensional simplicial complex can be thought of as a graph where the edges are given by the 1-faces of the simplicial complex. The opposite is also true and we will make use of this relationship throughout the thesis.

Two vertices u and v in a graph G are **adjacent** if $\{u, v\}$ is an edge of G. The **degree** of a vertex v is the number of vertices to which v is adjacent.

The **complete graph** on n vertices, denoted K_n, is the graph whose edges are all possible pairs of these n vertices. A complete graph corresponds to a 1-complete 1-dimensional simplicial complex. See Figure 2.8 for an example of a complete graph.

A **path** in a graph G is a sequence of vertices $v_0, \ldots, v_n$ from $V(G)$ such that $\{v_{i-1}, v_i\} \in E(G)$ for $1 \leq i \leq n$. A **connected** graph is a graph in which every pair of vertices is connected by a path.

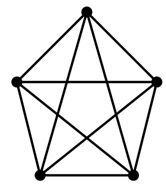


Figure 2.8: The complete graph on five vertices, K_5.

A **cycle** in a graph G is an ordered list of distinct vertices $v_0, \ldots, v_n$ from $V(G)$ where $\{v_{i-1}, v_i\} \in E(G)$ for $1 \leq i \leq n$ and $\{v_n, v_0\} \in E(G)$. The **length** of a cycle is its number of vertices. In Figure 2.9 we give an example of a graph cycle of length six.

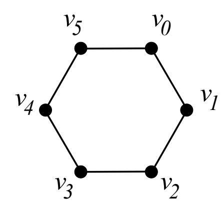


Figure 2.9: Graph cycle on six vertices.

In Chapter 3, we will make use of the following well-known lemma from graph theory. A proof can be found in (West [38, Section 1.2]).

Lemma 2.3.1. *If G is a graph in which every vertex has degree at least 2 then G contains a cycle.*

A graph G is called **chordal** if all cycles in G of length greater than three have a **chord**, which is an edge between non-adjacent vertices of the cycle. In Figure 2.10 we give an example of a graph that is chordal and one that is not.

A graph G has a **perfect elimination ordering** if there is an ordering of its vertices so that for each vertex v of G, the vertices adjacent to v that occur after v in the ordering form a complete subgraph of G. It is well-known that a graph is chordal exactly when it has a perfect elimination ordering (see West [38, Section 5.3] for a proof).

Lemma 2.3.2. *A graph is chordal if and only if it has a perfect elimination ordering.*

(a) A non-chordal graph (b) A chordal graph

Figure 2.10: Examples of a non-chordal and a chordal graph.

The **complement** of a graph G, denoted $\overline{G}$, is the graph on the same vertex set as G but whose edges are exactly those 2-sets that are not edges of G. When the graph is thought of as a 1-dimensional simplicial complex, the graph complement is equivalent to the 1-complement. See Figure 2.11 for an example of a graph and its complement.

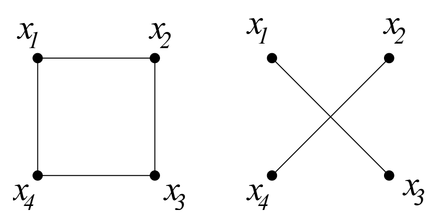
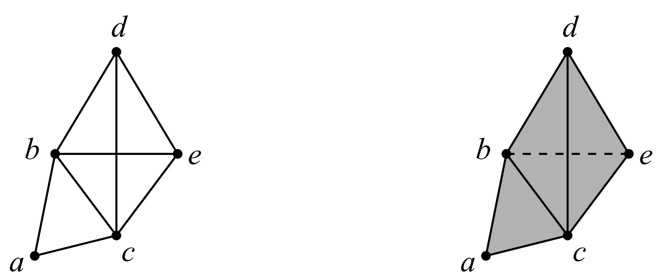


Figure 2.11: A graph and its complement.

Given a graph G we can obtain a simplicial complex $\Delta(G)$, called the **clique complex** of G, by taking the sets of vertices of complete subgraphs of G as the faces of $\Delta(G)$. Figure 2.12 gives an example of a graph and its associated clique complex.



(a) The graph G (b) $\Delta(G) = \langle\{a, b, c\}, \{b, c, d, e\}\rangle$

Figure 2.12: A graph and its clique complex.

2.4 Stanley-Reisner Ideals and Facet Ideals

Let k be a field and let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over k in n variables. Recall that a **monomial** in R is a product of variables of the form $x_1^{a_1} \cdots x_n^{a_n}$ where $a_i \in \mathbb{N}$ for all $1 \leq i \leq n$. The **support** of the monomial $x_1^{a_1} \cdots x_n^{a_n}$ is the set $\{x_i \mid a_i \neq 0\}$. The **degree** of the monomial $x_1^{a_1} \cdots x_n^{a_n}$ is equal to $\sum_{i=1}^{n} a_i$. A **monomial ideal** is an ideal of R generated by monomials. Such an ideal is always finitely generated since R is Noetherian. We say that a monomial ideal I is **generated in degree** d if the monomial generators of I which are minimal in terms of divisibility all have degree d. A **square-free monomial** is a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $0 \leq a_i \leq 1$ for $1 \leq i \leq n$. A **square-free monomial ideal** of R is an ideal of R whose unique minimal generators are all square-free monomials.

To any simplicial complex Γ on the vertex set $\{x_1, \ldots, x_n\}$ we can associate, in two different ways, a square-free monomial ideal in the polynomial ring R. Given a subset $F = \{x_{i_1}, x_{i_2}, \ldots, x_{i_\ell}\}$ of $V(\Gamma)$ we define x^F to be the square-free monomial $x_{i_1} x_{i_2} \cdots x_{i_\ell}$ in R. The **facet ideal** of Γ (or the **edge ideal** if Γ is 1-dimensional and we think of it as a graph) is the ideal

$$\mathcal{F}(\Gamma) = \left(\{x^F \mid F \in \text{Facets}(\Gamma)\}\right).$$

The **facet complex** of the square-free monomial ideal I is the complex $\mathcal{F}(I)$ whose facets are given by the minimal monomial generators of I. The **Stanley-Reisner ideal** of Γ is the ideal

$$\mathcal{N}(\Gamma) = \left(\{x^F \mid F \notin \Gamma\}\right).$$

In other words, $\mathcal{N}(\Gamma)$ is minimally generated by the **minimal non-faces** of Γ which are the minimal subsets of $V(\Gamma)$ which are not faces of Γ. The **Stanley-Reisner ring** of Γ is the ring $k[\Gamma] = R/\mathcal{N}(\Gamma)$. The **Stanley-Reisner complex** of the square-free monomial ideal I is the complex $\mathcal{N}(I)$ whose faces are given by the square-free monomials not in I. See Figure 2.13 for examples of these relationships.

Notice that, for a fixed set of vertices, $\mathcal{N}(I)$ and $\mathcal{F}(I)$ are uniquely determined by the ideal I and, similarly $\mathcal{N}(\Gamma)$ and $\mathcal{F}(\Gamma)$ are uniquely determined by Γ.

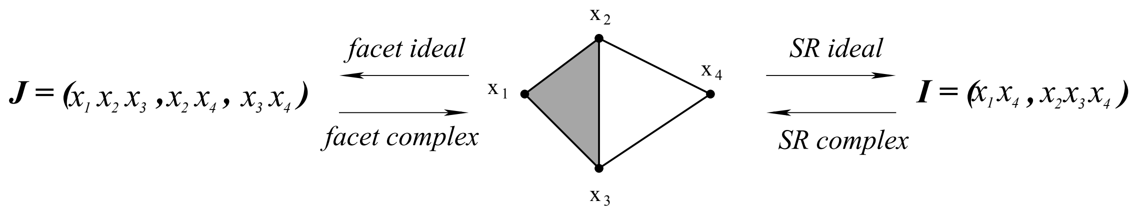


Figure 2.13: Relationship between simplicial complexes and ideals.

2.5 Graded Free Resolutions

For a commutative monoid H, an H-**graded ring** is a ring R with a direct sum decomposition as abelian groups

$$R = \bigoplus_{a \in H} R_a$$

satisfying $R_a R_b \subseteq R_{a+b}$ for all $a, b \in H$. A module M over an H-graded ring R having a direct sum decomposition

$$M = \bigoplus_{a \in H} M_a$$

that satisfies $R_a M_b \subseteq M_{a+b}$ for all $a, b \in H$ is an H-**graded module**. An element m in M is **homogeneous of degree** a if $m \in M_a$ for some $a \in H$.

For example, the polynomial ring $R = k[x_1, \ldots, x_n]$ over the field k is an $\mathbb{N}$-graded ring. We have

$$R = \bigoplus_{a \in \mathbb{N}} R_a$$

where R_a is the set of all polynomials of R whose terms are monomials of degree a with coefficients in k. This grading is referred to as the **standard grading** on R.

A module homomorphism φ between H-graded modules M and N is **homogeneous** if we have $\varphi(M_a) \subseteq N_a$ for all $a \in H$.

A sequence of homomorphisms

$$\cdots \longrightarrow X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow \cdots$$

between the R-modules X_i is said to be **exact** if $\operatorname{im} \alpha_{i-1} = \ker \alpha_i$ for all i.

Definition 2.5.1 (minimal graded free resolution). A **graded free resolution** of the graded R-module M is an exact sequence of R-modules

$$\cdots \longrightarrow F_m \xrightarrow{\delta_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

where each F_i is a graded free R-module with grading such that the maps δ_i are homogeneous. This resolution is said to be **minimal** if, for each i, the map δ_i sends a basis of F_i to a minimal homogeneous generating set for $\operatorname{im}\delta_i$.

A minimal graded free resolution of the graded R-module M provides structural information about the module. Since each F_i is a free R-module we know that $F_i \cong R^{\beta_i(M)}$ for some $\beta_i(M) \in \mathbb{N} \setminus \{0\}$. The rank, $\beta_0(M)$, of the free module F_0 tells us the size of a minimal generating set of M. The rank of F_1 gives the size of a minimal generating set for $\ker\delta_0$. This provides information about the number of relations that exist between the generators of M. Similarly the rank of F_2 tells us about the relations between these relations and so on. If R is a field then M is a vector space and we obtain the simplest resolution possible where the only non-zero map is δ_0 since no relations exist between the generators of M.

When $R = k[x_1, \ldots, x_n]$ for some field k, a minimal graded free resolution of a $\mathbb{Z}$-graded R-module M is unique up to isomorphism (Eisenbud [10, Theorem 1.6]). Each free module F_i in such a resolution can be written

$$F_i = \bigoplus_j R(-j)^{\beta_{i,j}(M)}$$

where $R(-j)$ denotes the $\mathbb{Z}$-graded module isomorphic to R but with $R(-j)_a = R_{a-j}$ so that the degrees of R are shifted by j. This shift ensures that the maps in the resolution are homogeneous. Since $R(-j)_j = R_0$ we have that $R(-j)$ is generated as a free R-module by one element of degree j. The numbers $\beta_{i,j}(M)$ are called the **graded Betti numbers** of M and they are uniquely determined by the module M due to the uniqueness of the resolution. The rank, $\beta_i(M)$, of F_i is called the ***i*th total Betti number** of M and we have the relation

$$\beta_i(M) = \sum_j \beta_{i,j}(M).$$

In this thesis we are interested in resolutions of monomial ideals in R. The monomial ideal I is $\mathbb{N}$-graded by $I_d = I \cap R_d$ and has a minimal graded free resolution of the form

$$0 \to \bigoplus_j R(-j)^{\beta_{m,j}(I)} \to \bigoplus_j R(-j)^{\beta_{m-1,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0$$

where $m \leq n$ by (Eisenbud, [10, Theorem 1.1]).

Example 2.5.2. Let $R = k[x,y]$ and let $I = (x^3, xy)$. A minimal graded free resolution of I is given by

$$0 \to R(-4) \xrightarrow{\begin{pmatrix} -y \\ x^2 \end{pmatrix}} R(-3) \oplus R(-2) \xrightarrow{\begin{pmatrix} x^3 & xy \end{pmatrix}} I \to 0.$$

The first non-zero map takes an element r of $R(-4)$ and maps it to the element $(-ry, rx^2)$ in $R(-3) \oplus R(-2)$. The second map takes the element (r_1, r_2) of $R(-3) \oplus R(-2)$ to the element $r_1x^3 + r_2xy$ in I. In this example we have $\beta_{0,2}(I) = 1$, $\beta_{0,3}(I) = 1$, and $\beta_{1,4}(I) = 1$ with the remaining graded Betti numbers all equal to zero.

There are certain invariants associated with a minimal graded free resolution of the monomial ideal I. The **projective dimension** of I is

$$\operatorname{pd}(I) := \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

The **Castelnuovo-Mumford regularity** of the ideal I is

$$\operatorname{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

In Example 2.5.2 we see that $\operatorname{pd}(I) = 1$ and $\operatorname{reg}(I) = 3$.

Remark 2.5.3. There is a bijective correspondence between minimal graded free resolutions of the R-module I and the R-module R/I. In particular

$$\cdots \to F_m \to F_{m-1} \to \cdots \to F_1 \to R \to R/I \to 0$$

is a minimal graded free resolution of R/I if and only if

$$\cdots \to F_m \to F_{m-1} \to \cdots \to F_1 \to I \to 0$$

is a minimal graded free resolution of I. Therefore we have

$$\beta_{i,j}(I) = \beta_{i+1,j}(R/I).$$

In 1977, Hochster gave a formula for the Betti numbers of the Stanley-Reisner ideal of a simplicial complex Γ in terms of the dimensions of the homology groups of Γ and its induced subcomplexes.

Theorem 2.5.4 (Hochster [23])**.** *For a simplicial complex* Γ*, the graded Betti numbers of the Stanley-Reisner ideal* $\mathcal{N}(\Gamma)$ *are given by*

$$\beta_{i,j}(\mathcal{N}(\Gamma)) = \sum_{W\subseteq V,\ |W|=j} \dim_k \tilde{H}_{j-i-2}(\Gamma_W; k)$$

where $i \geq 0$*.*

A monomial ideal I in $R = k[x_1, \dots, x_n]$ has a **d-linear resolution over k** if $\beta_{i,j}(I) = 0$ for all $j \neq i+d$. In this case, the resolution has the following simple form:

$$0 \longrightarrow R(-d-m)^{\beta_{m,d+m}(I)} \xrightarrow{\delta_m} \cdots \longrightarrow R(-d-1)^{\beta_{1,d+1}(I)} \xrightarrow{\delta_1} R(-d)^{\beta_{0,d}(I)} \xrightarrow{\delta_0} I \longrightarrow 0$$

We say that I has a **linear resolution** if it has a d-linear resolution for some d. In a linear resolution, the non-zero entries in the matrices corresponding to all maps in the resolution between the graded free R-modules are all linear forms in the variables $x_1, \dots, x_n$.

Remark 2.5.5. It is not hard to see that if a monomial ideal I has a d-linear resolution then it is generated in degree d. This is because the basis elements of $R(-d)^{\beta_{0,d}(I)}$ have degree d and the map δ_0 is homogeneous. Therefore the basis elements of $R(-d)^{\beta_{0,d}(I)}$ get mapped to elements of degree d in I. Since the resolution is minimal, these elements form a minimal generating set for $\operatorname{im}\delta_0 = I$.

Example 2.5.6. Let $R = k[x, y, z]$ and $I = (xy, yz)$. The following is a 2-linear resolution of I over k.

$$0 \to R(-3) \xrightarrow{\begin{pmatrix} z \\ -x \end{pmatrix}} R(-2)^2 \xrightarrow{\begin{pmatrix} xy & yz \end{pmatrix}} I \to 0.$$

We have $\beta_{0,2}(I) = 2$ and $\beta_{1,3}(I) = 1$ and the remaining graded Betti numbers are all equal to zero.

It is known that classifying monomial ideals with linear resolutions is equivalent to classifying monomial ideals whose quotients are Cohen-Macaulay rings (Eagon and Reiner [9]). This is a particularly nice class of commutative rings whose structure generalizes that of polynomial rings. When characterizing monomial ideals with linear resolutions it is sufficient to consider only square-free monomial ideals. This is due to the technique of **polarization** whereby a monomial ideal I is transformed into a square-free monomial ideal I' in a polynomial ring in a larger number of variables. In particular, generators of I having variables which appear more than once are replaced in I' by generators containing new variables in lieu of repeated ones. The ideal I has a linear resolution if and only if I' has a linear resolution (Herzog and Hibi [21, Corollary 1.6.3]). This transformation makes it possible to study any monomial ideal from a combinatorial perspective by examining a simplicial complex associated to its polarization.

For square-free monomial ideals whose generators are not all of the same degree, there exists an analogous property to having a linear resolution. A square-free monomial ideal I is called **componentwise linear** over the field k if $I_{[d]}$ has a linear resolution over k for all d, where $I_{[d]}$ is the ideal generated by the square-free monomials in I of degree d (Herzog and Hibi [20]).

Chapter 3

The Structure of d-Dimensional Cycles and the Vanishing of Simplicial Homology

As discussed in Chapter 1, Fröberg classifies the monomial ideals with linear resolutions through the simplicial homology of their Stanley-Reisner complexes and their induced subcomplexes in [15]. Nevertheless a simplicial complex may be thought of as a purely combinatorial object and so the question arises as to whether or not one may attribute the existence of non-zero homology in a simplicial complex to a combinatorial structure present in that complex. This is an interesting mathematical question in its own right, but at the same time its possible solution has the potential to enable easier translation between algebraic properties of monomial ideals and the combinatorial framework. In this thesis we are interested in translating Fröberg's homological characterization from [15] into a purely combinatorial one.

Despite the substantial use of simplicial homology in classifying algebraic properties and the concrete combinatorial nature of the theory, there appears to be very little literature on the explicit combinatorial structures necessary for a simplicial complex to exhibit non-zero simplicial homology. However, studies have been made into the combinatorics of acyclic simplicial complexes – those for which simplicial homology vanishes in all dimensions. In [25], Kalai gave a characterization of the f-vectors of such simplicial complexes. This was followed by work of Stanley in [34] on a combinatorial decomposition of these complexes. On the other hand, the literature relating to the combinatorial structures associated with non-zero simplicial homology is scant. In [14], Fogelsanger studied the "rigidity" of the 1-skeletons of the underlying complexes of minimal homological d-cycles and concluded that these 1-skeletons are rigid graphs when embedded in $\mathbb{R}^{d+1}$. However, the question remains as to whether or not one may describe non-zero homology in a simplicial complex with a purely combinatorial structure. In this chapter we answer this question fully over fields of characteristic 2.

In addition we describe a combinatorial structure which results in non-zero homology over all fields.

3.1 Motivation

We would like to turn the vanishing of simplicial homology into an explicitly combinatorial property and relate the concept of non-zero homology in a simplicial complex to the existence of precise combinatorial patterns in the complex which are distinct from the more algebraic notion of a d-cycle.

It is not difficult to show that this goal is achievable for non-zero simplicial homology in dimension 1. In this case, the combinatorial structure associated to non-zero homology is the graph cycle. This can be deduced from Biggs [2, Chapters 4 and 5], but in this section we provide an explicit proof of this relationship as motivation for the general case. The techniques used in this proof also illustrate our overall approach to this problem.

The **support complex** of a homological d-chain is the simplicial complex whose facets are the d-faces in the d-chain with non-zero coefficients.

Theorem 3.1.1 (Non-zero 1-dimensional homology corresponds to graph cycles). *For any simplicial complex Γ and any field k, $\tilde{H}_1(\Gamma; k) \neq 0$ if and only if Γ contains a graph cycle, which is not the support complex of a 1-boundary.*

Proof. Suppose that $\tilde{H}_1(\Gamma; k) \neq 0$. Then Γ contains a 1-cycle c that is not a 1-boundary. We may assume that the support complex Ω of c is minimal with respect to this property. In other words, no proper subset of the 1-faces of c is the support complex of a 1-cycle which is not a 1-boundary. First we would like to show that Ω is a connected graph.

The 1-cycle c is of the form

$$c = \alpha_1 F_1 + \cdots + \alpha_n F_n \tag{3.1}$$

for some oriented 1-faces $F_1, \ldots, F_n$ of Γ and where $\alpha_i \in k$. If Ω is not connected then we can partition the 1-faces $F_1, \ldots, F_n$ into two sets having no vertices in common. Without loss of generality let these two sets be $\{F_1, \ldots, F_\ell\}$ and $\{F_{\ell+1}, \ldots, F_n\}$. Since

$$\partial_1(c) = \partial_1(\alpha_1 F_1 + \cdots + \alpha_n F_n) = 0$$

and since there are no vertices shared between the two sets we must have

$$\partial_1(\alpha_1 F_1 + \cdots + \alpha_\ell F_\ell) = 0 \text{ and } \partial_1(\alpha_{\ell+1} F_{\ell+1} + \cdots + \alpha_n F_n) = 0$$

and so $\alpha_1 F_1 + \cdots + \alpha_\ell F_\ell$ and $\alpha_{\ell+1} F_{\ell+1} + \cdots + \alpha_n F_n$ are both 1-cycles. By the assumption of minimality of Ω we know that these 1-cycles must also be 1-boundaries. However, since c is the sum of these two 1-chains and they are both 1-boundaries, c must be a 1-boundary as well, which is a contradiction. Therefore Ω must be a connected graph.

Next, note that the degree of all vertices in Ω must be at least two. This follows since $\partial_1(c) = 0$ and this may only be achieved if all vertices present cancel out in this sum. Therefore each vertex must appear at least twice. Hence by Lemma 2.3.1 we know that Ω contains a graph cycle. Let $v_1, \ldots, v_m$ be the vertices in this cycle where v_i is adjacent to v_{i+1} for $1 \leq i \leq m-1$ and v_m is adjacent to v_1. By relabeling if necessary we may assume that $F_1, \ldots, F_m$ are the oriented 1-faces corresponding to the edges in this cycle with $F_i = \varepsilon_i[v_i, v_{i+1}]$ for $1 \leq i \leq m-1$ and $F_m = \varepsilon_m[v_m, v_1]$ where $\varepsilon_i = \pm 1$ depending on the orientation of F_i. Suppose that $m < n$. We have

$$\varepsilon_1 F_1 + \ldots + \varepsilon_m F_m = \sum_{i=1}^{m-1} [v_i, v_{i+1}] + [v_m, v_1]$$

is a 1-chain and it is straightforward to see that

$$\partial_1(\varepsilon_1 F_1 + \ldots + \varepsilon_m F_m) = 0.$$

Let $b = \alpha_1 \varepsilon_1(\varepsilon_1 F_1 + \cdots + \varepsilon_m F_m)$. Then

$$\partial_1(b) = \alpha_1 \varepsilon_1 \partial_1(\varepsilon_1 F_1 + \ldots + \varepsilon_m F_m) = 0$$

and so b is a 1-cycle. We also have

$$\partial_1(c - b) = \partial_1(c) - \partial_1(b) = 0 - 0 = 0$$

and so $c - b$ is also a 1-cycle. Now $m < n$ and so by our assumption of minimality of Ω we know that b is a 1-boundary. Since

$$c - b = (\alpha_2 - \alpha_1\varepsilon_1\varepsilon_2)F_2 + (\alpha_3 - \alpha_1\varepsilon_1\varepsilon_3)F_3 + \cdots (\alpha_m - \alpha_1\varepsilon_1\varepsilon_m)F_m + \alpha_{m+1}F_{m+1} + \cdots + \alpha_n F_n,$$

it is supported on a proper subset of the 1-faces of Ω and so, by the assumption of minimality, $c - b$ is a 1-boundary also. Therefore, since both b and $c - b$ are 1-boundaries and $c = b + (c - b)$ then c is a 1-boundary. This is a contradiction and so we must have $m = n$. Therefore Ω itself is a graph cycle.

Next we will show that Ω is not the support complex of a 1-boundary. First we show that for any 1-cycle of the form

$$d = \beta_1 F_1 + \cdots + \beta_n F_n$$

for non-zero $\beta_1, \ldots, \beta_n \in k$ we have $\beta_1 \varepsilon_1 = \cdots = \beta_n \varepsilon_n$ with ε_i defined as before. Recall that $F_i = \varepsilon_i[v_i, v_{i+1}]$ for $1 \leq i \leq n-1$ and $F_n = \varepsilon_n[v_n, v_1]$. Since d is a 1-cycle we have

$$0 = \partial_1(d) = \beta_1\varepsilon_1(v_2 - v_1) + \beta_2\varepsilon_2(v_3 - v_2) + \cdots + \beta_n\varepsilon_n(v_1 - v_n). \tag{3.2}$$

Therefore we have

$$\beta_1\varepsilon_1 = \beta_2\varepsilon_2 = \cdots = \beta_n\varepsilon_n$$

as each vertex appears exactly twice in (3.2). Since the β_i's are arbitrary we also have

$$\alpha_1\varepsilon_1 = \alpha_2\varepsilon_2 = \cdots = \alpha_n\varepsilon_n.$$

Suppose that we have the 1-cycle

$$e = \gamma_1 F_1 + \cdots + \gamma_n F_n$$

where $\gamma_1, \ldots, \gamma_n \in k$ and suppose that e is also a 1-boundary of a 2-chain f. For any $1 \leq j \leq n$ we have

$$\gamma_j \left(\frac{\alpha_1}{\gamma_1}\right) = \left(\frac{\varepsilon_1}{\varepsilon_j}\right) \alpha_1 = \alpha_j.$$

since $\gamma_1\varepsilon_1 = \gamma_j\varepsilon_j$ and $\alpha_1\varepsilon_1 = \alpha_j\varepsilon_j$. Hence we have

$$\left(\frac{\alpha_1}{\gamma_1}\right) e = c$$

and so since e is the 1-boundary of f, c is the 1-boundary of $\left(\frac{\alpha_1}{\gamma_1}\right) f$. This is a contradiction and so Ω is a graph cycle which is not the support complex of a 1-boundary.

Conversely, suppose that Γ contains a graph cycle Ω, which is not the support complex of a 1-boundary. Let $v_1, \dots, v_k$ be the vertices of Ω where v_i is adjacent to v_{i+1} for $1 \leq i \leq k-1$ and v_k is adjacent to v_1. Then

$$c = \sum_{i=1}^{k-1} [v_i, v_{i+1}] + [v_k, v_1]$$

is a 1-chain whose support complex is Ω and it is easy to see that $\partial_1(c) = 0$. Therefore c is a 1-cycle, and by assumption it is not a 1-boundary. Therefore $\tilde{H}_1(\Gamma; k) \neq 0$.

$\square$

3.2 The d-Dimensional Cycle and its Structure

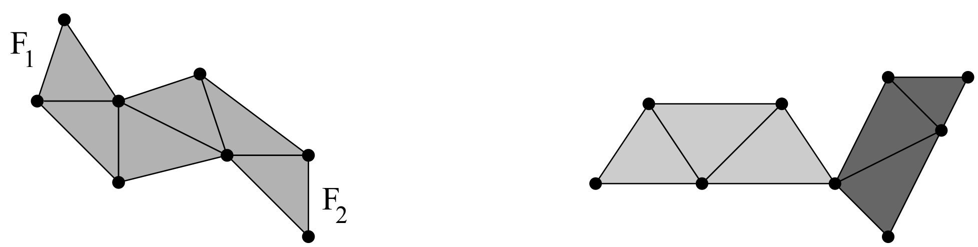
The role of the graph cycle in graph theory is substantial. The existence, frequency, and lengths of cycles in a graph play important roles in many different areas of graph theory such as connectivity, perfect graphs, graph colouring, and extremal graph theory. In recent years, with the introduction of edge ideals of graphs by Villarreal in [36], the influence of the graph cycle has extended into combinatorial commutative algebra.

As we saw in Section 3.1, the graph cycle, when thought of as a simplicial complex, is exactly the right structure to describe non-zero homology in dimension 1. It is this specific property which is at the heart of Fröberg's result. With the goal of extending Fröberg's classification to simplicial complexes we would like to capture the idea of non-zero higher-dimensional simplicial homology by generalizing the idea of a graph cycle to higher dimensions. In the past, versions of the graph cycle with higher-dimensional faces such as the Berge cycle (Berge [1]) and the simplicial cycle (Caboara et al. [5]) have been introduced to extend the usefulness of graph cycles to hypergraphs or simplicial complexes. In contrast to the higher-dimensional cycle that we will introduce in this section, these cycles in general only generate 1-dimensional simplicial homology.

When we examine the support complexes of "minimal" homological d-cycles we see that such complexes must be connected in a particularly strong way. We will use the following definitions.

Definition 3.2.1 (*d*-**path,** *d*-**path-connected,** *d*-**path-connected components).** A sequence $F_1, \ldots, F_k$ of d-dimensional faces in a simplicial complex Γ is a ***d*-path** between F_1 and F_k if either $k = 1$ or $|F_i \cap F_{i+1}| = d$ for all $1 \leq i \leq k-1$. If Γ is a pure d-dimensional simplicial complex and there exists a d-path between each pair of its d-faces or $|\operatorname{Facets}(\Gamma)| = 1$ then Γ is ***d*-path-connected**. The maximal subcomplexes of Γ which are d-path-connected are called the ***d*-path-connected components** of Γ.

In Figure 3.1a we give an example of a 2-path between the 2-faces F_1 and F_2. Figure 3.1b shows a pure 2-dimensional simplicial complex with two 2-path-connected components shown by different levels of shading. Notice that the existence of a d-path between two d-faces in a simplicial complex Γ is an equivalence relation on the d-faces of Γ. The d-path-connected components of Γ correspond to these equivalence classes. A d-path-connected simplicial complex is sometimes referred to as **strongly connected** (see, for example, [3]). Note that this is not a generalization of the idea in graph theory of a strongly connected directed graph.



(a) A 2-path between F_1 and F_2 (b) 2-path-connected components

Figure 3.1: Examples for path-connected complexes.

A graph cycle is characterized by two features. It is connected and each of its vertices is of degree two. By generalizing these two properties we arrive at our combinatorial definition of a higher-dimensional cycle.

Definition 3.2.2 (*d*-**dimensional cycle).** A pure d-dimensional simplicial complex Ω is a ***d*-dimensional cycle** if

1. Ω is d-path-connected, and
2. every $(d-1)$-face of Ω is contained in an even number of d-faces of Ω.

Notice that a graph cycle is a 1-dimensional cycle, but a 1-dimensional cycle need not be a graph cycle. See Figure 3.2 for an example. This difference is due

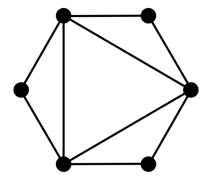


Figure 3.2: A 1-dimensional cycle which is not a graph cycle.

to the second property in Definition 3.2.2 which requires $(d-1)$-faces to belong to an even number of d-faces rather than exactly two as in the graph cycle case. This modification is motivated by the goal of describing the combinatorial structures that correspond to non-zero simplicial homology over fields of characteristic 2 as we will see in Section 3.3. A d-dimensional cycle does generalize the notion of a **circuit** from graph theory, which is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ where $e_i = \{v_{i-1}, v_i\}$ for $1 \leq i \leq n$, $v_0 = v_n$ and the edges $e_1, \ldots, e_n$ are all distinct. One can show that in a circuit each vertex belongs to an even number of edges.

Notice that a d-dimensional cycle by definition has only one d-path-connected component. We will see in Proposition 3.2.12 that a d-dimensional cycle must contain at least $d+2$ facets.

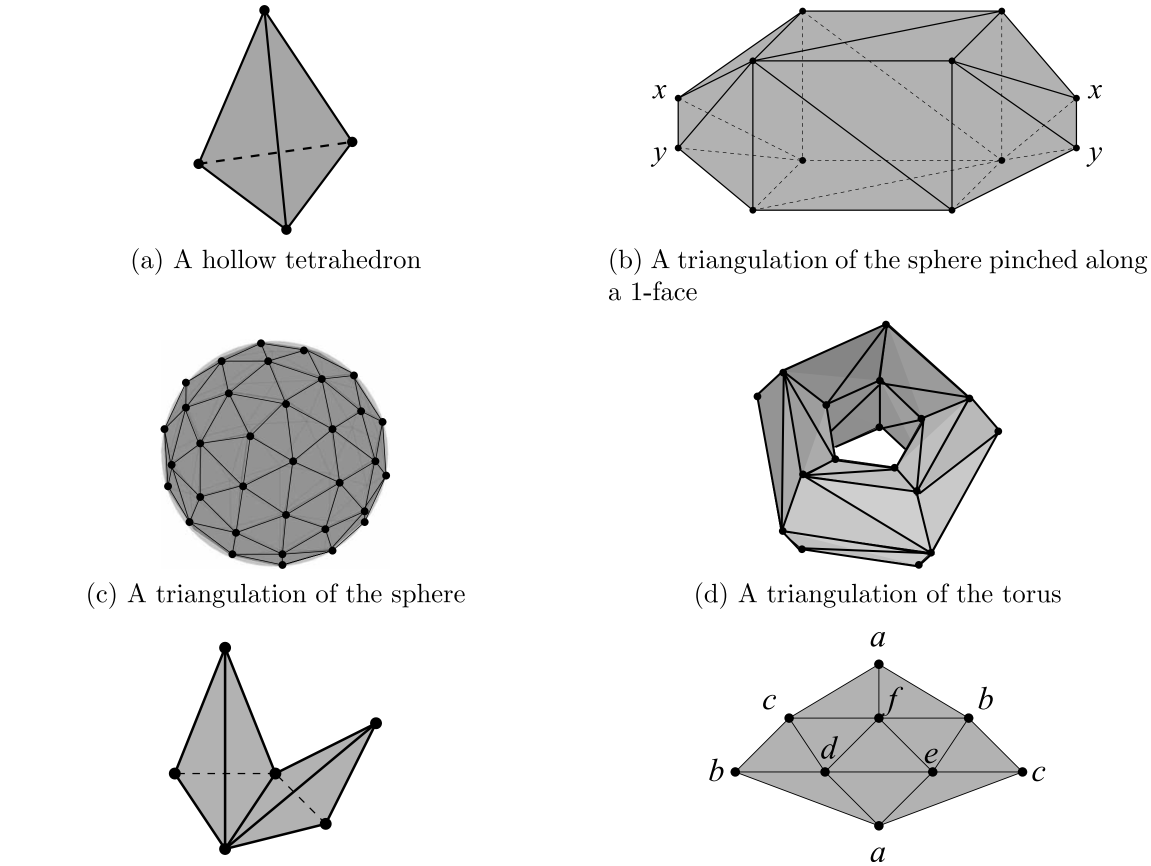
Examples of 2-dimensional cycles are given in Figure 3.3.

One of the properties of a d-dimensional cycle is that it contains $(d-1)$-dimensional cycles as subcomplexes. As an example, in Figure 3.4 we have a 2-dimensional cycle containing a 1-dimensional cycle within the 2-faces containing the vertex v. The 1-dimensional cycle is shown with dotted lines.

Proposition 3.2.3 (All d-dimensional cycles contain $(d-1)$-dimensional cycles). *Let Ω be a d-dimensional cycle and let $v \in V(\Omega)$. If $F_1, \ldots, F_k$ are the d-faces of Ω containing v then the $(d-1)$-path-connected components of the complex*

$$\langle F_1 \setminus \{v\}, \ldots, F_k \setminus \{v\} \rangle$$

are $(d-1)$-dimensional cycles.



(a) A hollow tetrahedron

(b) A triangulation of the sphere pinched along a 1-face

(c) A triangulation of the sphere

(d) A triangulation of the torus

(e) Two hollow tetrahedra glued along a 1-face

(f) A triangulation of the real projective plane

Figure 3.3: Examples of 2-dimensional cycles.

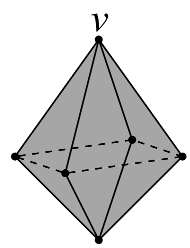


Figure 3.4: A 2-dimensional cycle containing a 1-dimensional cycle.

Proof. Let $\Omega_v = \langle F_1 \setminus \{v\}, \ldots, F_k \setminus \{v\} \rangle$ and let Ω'_v be a $(d-1)$-path-connected component of Ω_v with $(d-1)$-faces $F_{i_1} \setminus \{v\}, \ldots, F_{i_\ell} \setminus \{v\}$. To show that Ω'_v is a $(d-1)$-dimensional cycle we need only show that each of its $(d-2)$-faces is contained in an even number of the faces $F_{i_1} \setminus \{v\}, \ldots, F_{i_\ell} \setminus \{v\}$. Let f be a $(d-2)$-face of Ω'_v. Note that, for $1 \leq r \leq \ell$, f is a $(d-2)$-face of $F_{i_r} \setminus \{v\}$ if and only if $f \cup \{v\}$ is a

$(d-1)$-face of F_{i_r}. The face $f \cup \{v\}$ belongs to an even number of the d-faces in Ω since Ω is a d-dimensional cycle. Since these faces all contain v, they belong to the set $\{F_1, \ldots, F_k\}$ and thus $f \cup \{v\}$ belongs to an even number of the d-faces $F_1, \ldots, F_k$. Note that, after removing v from these d-faces, they are $(d-1)$-path-connected in Ω_v since they all contain f. Hence, with v removed, these faces all lie in Ω'_v and so f is contained in an even number of the faces $F_{i_1} \setminus \{v\}, \ldots, F_{i_\ell} \setminus \{v\}$. Therefore Ω'_v is a $(d-1)$-dimensional cycle. □

In contrast to Proposition 3.2.3 it is also possible to create higher-dimensional cycles from lower-dimensional ones under certain homological conditions over $\mathbb{Z}_2$ by using the cone operation. We will see this in Proposition 3.3.2.

The notion of a d-dimensional cycle extends the classical concept of a pseudo d-manifold that appears in algebraic topology. For example see (Munkres, [28]).

Definition 3.2.4 (pseudo d-manifold). A pure d-dimensional d-path-connected simplicial complex Γ is a **pseudo d-manifold** if every $(d-1)$-face of Γ is contained in exactly two d-faces of Γ.

It is not difficult to see that the 2-dimensional cycles in Figures 3.3a, 3.3c, 3.3d, and 3.3f are all examples of pseudo 2-manifolds. The simplicial complexes in Figures 3.3b and 3.3e are not pseudo 2-manifolds as they each have 1-faces belonging to more than two 2-faces.

We would like to be able to describe a simplicial complex which "minimally" generates homology in a particular dimension in the sense that no proper subcomplex also generates homology. One can easily show that the 2-dimensional cycle given in Figure 3.3e does not satisfy this property. With this in mind, we introduce notions of minimality into the idea of a d-dimensional cycle.

Definition 3.2.5 (face-minimal d-dimensional cycle). A d-dimensional cycle Γ is **face-minimal** when there are no d-dimensional cycles on a proper subset of the d-faces of Γ.

One can easily show that a 1-dimensional cycle is a graph cycle if and only if it is face-minimal.

Definition 3.2.6 (vertex-minimal d-dimensional cycle). A d-dimensional cycle Ω in a simplicial complex Γ is called **vertex-minimal** if there is no d-dimensional cycle in Γ on a proper subset of the vertices of Ω. In other words, for all $W \subsetneq V(\Omega)$, the induced subcomplex Γ_W does not contain a d-dimensional cycle.

The 2-dimensional cycles in Figure 3.3 are all face-minimal and vertex-minimal except for the complex in Figure 3.3e. This 2-dimensional cycle can be thought of as the boundaries of two 3-simplices glued along a 1-face. This cycle is neither face-minimal nor vertex-minimal because each of the two boundaries is also a 2-dimensional cycle which consists of a proper subset of the 2-faces and vertices from the original cycle.

Remark 3.2.7. Notice that the face-minimality of a d-dimensional cycle Ω is not affected by whether or not it sits inside a larger simplicial complex. On the other hand, a d-dimensional cycle can be vertex-minimal when considered as a stand-alone simplicial complex, but not vertex-minimal when considered as a subcomplex of another simplicial complex. As an example consider the simplicial complex in Figure 3.5. This complex consists of an "outer" triangulated sphere and an "inner" triangulated sphere that is suspended inside the first sphere from a proper subset of its vertices. The outer sphere is not vertex-minimal in this complex as there exists another 2-dimensional cycle, the inner sphere, on a proper subset of its vertices. However, when considered as a simplicial complex on its own, the outer sphere is a vertex-minimal 2-dimensional cycle.

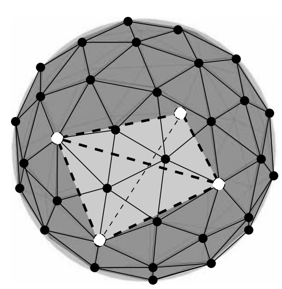


Figure 3.5: Triangulated sphere with suspended hollow tetrahedron.

Next we show that any d-dimensional cycle can be broken down into face-minimal cycles. For an illustration of the idea used in the proof of the following lemma see

Figure 3.6. The simplicial complex Ω in this figure is a 2-dimensional cycle that consists of four hollow tetrahedra that are glued in a "chain" along shared 1-faces. The 2-faces in these four face-minimal 2-dimensional cycles form a partition of the 2-faces of Ω.

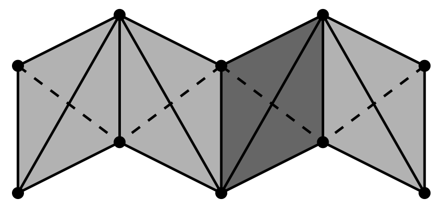


Figure 3.6: Example of a 2-dimensional cycle Ω from the proof of Lemma 3.2.8 with Φ_1 shown in dark grey and Ω_1 in light grey.

Lemma 3.2.8 (A d-dimensional cycle can be partitioned into face-minimal cycles). *Any d-dimensional cycle Ω can be written as a union of face-minimal d-dimensional cycles $\Phi_1, \ldots, \Phi_n$ such that every d-face of Ω belongs to some Φ_i and such that Φ_i and Φ_j have no d-faces in common when $i \neq j$.*

Proof. If Ω is a face-minimal d-dimensional cycle then we are done. So suppose that Ω is not face-minimal and let Φ_1 be a face-minimal d-dimensional cycle on a proper subset of the d-faces of Ω. Consider the d-path-connected components of the complex Ω_1 whose facets are the d-faces of Ω not belonging to Φ_1. We claim that each such component is a d-dimensional cycle. Since each component is d-path-connected by definition, we need only show that each $(d-1)$-face in the component is contained in an even number of its d-faces.

Let Ψ be one of the d-path-connected components of Ω_1 and let f be a $(d-1)$-face of Ψ. Suppose first that f also belongs to one of the d-faces of Φ_1. Since Φ_1 is a d-dimensional cycle f belongs to an even number of its d-faces. However Ω is also a d-dimensional cycle and f belongs to an even number of its d-faces. Since Ω_1 is the complex whose facets are the d-faces of Ω not in Φ_1, f belongs to an even number of d-faces in Ω_1. However the collection of all d-faces of Ω_1 containing f is clearly d-path-connected and so these d-faces all lie in Ψ. Hence f belongs to an even number of d-faces of Ψ.

If f does not belong to any d-faces of Φ_1 then all of the d-faces of Ω which contain f lie in Ω_1. Since there are an even number of such faces and they are d-path-connected they all lie in Ψ. Hence f belongs to an even number of d-faces of Ψ. Therefore each d-path-connected component of Ω_1 is a d-dimensional cycle.

Each of these components is either a face-minimal d-dimensional cycle, or contains a face-minimal d-dimensional cycle on a proper subset of its d-faces. We may repeat the argument above on the simplicial complex whose facets are the d-faces of Ω_1 belonging to the components that are not face-minimal cycles. Iterating this procedure we see that, since we have a finite number of d-faces, eventually the procedure must terminate. We are left with face-minimal cycles $\Phi_1, \ldots, \Phi_n$ in which every d-face of Ω belongs to some Φ_i and, by our construction, no two distinct cycles Φ_i and Φ_j share a d-face. $\square$

Proposition 3.2.9. *A pseudo d-manifold is a face-minimal d-dimensional cycle.*

Proof. Let Ω be a pseudo d-manifold. Then Ω is d-path connected and every $(d-1)$-face in Ω belongs to exactly two d-faces. Hence Ω is a d-dimensional cycle. Suppose that Ω is not face-minimal. By Lemma 3.2.8 we can partition Ω into face-minimal d-dimensional cycles $\Phi_1, \ldots, \Phi_n$ where $n \geq 2$. Since Ω is d-path connected by definition, the d-faces in the distinct d-dimensional cycles $\Phi_1, \ldots, \Phi_n$ are all joined by d-paths. In particular, there must exist $(d-1)$-faces which appear in more than one of these cycles. Therefore there exists some pair of indices i, j with $i \neq j$ with $F_1 \in \Phi_i$ and $F_2 \in \Phi_j$ where $F_1 \cap F_2 = f$ for some $(d-1)$-face f of Ω. Since Φ_i is a d-dimensional cycle, f belongs to an even number of d-faces of Φ_i and similarly f belongs to an even number of d-faces of Φ_j. By Lemma 3.2.8, these d-faces are all distinct which means that f belongs to at least four d-faces of Ω. This is a contradiction since Ω is a pseudo d-manifold. Hence Ω is a face-minimal d-dimensional cycle. $\square$

The converse of this theorem does not hold. The simplicial complex in Figure 3.3b, a triangulated sphere pinched along a 1-dimensional face, is a counterexample. It is a face-minimal 2-dimensional cycle, but it is not a pseudo 2-manifold as it has a 1-dimensional face, $\{x, y\}$, belonging to four distinct 2-dimensional faces.

A pseudo d-manifold can be classified as either **orientable** or **non-orientable** and this idea can be generalized to the case of d-dimensional cycles.

Definition 3.2.10 (orientable d-dimensional cycle). Let Ω be a d-dimensional cycle. If it is possible to choose orientations of the d-faces of Ω such that for any $(d-1)$-face of Ω its induced orientations are divided equally between the two orientation classes then we say that Ω is **orientable**. Otherwise Ω is **non-orientable**.

Note that when we talk about the oriented d-faces of an orientable d-dimensional cycle we are referring to any set of orientations that is consistent with Definition 3.2.10.

As mentioned previously, many of the combinatorial complexities that exist in higher dimensions are not present in the 1-dimensional case. Non-orientable cycles are an example of this.

Proposition 3.2.11. *Any* 1*-dimensional cycle is orientable.*

Proof. Let Ω be a 1-dimensional cycle. If Ω is face-minimal then it is a graph cycle. It is straightforward to see that any graph cycle is orientable by choosing a "direction" in which to traverse the cycle and orienting each face in a way that is consistent with this direction.

If Ω is not face-minimal then by Lemma 3.2.8 we can partition the 1-faces of Ω into face-minimal 1-dimensional cycles $\Phi_1, \dots, \Phi_n$ where $n \geq 2$. For each $1 \leq i \leq n$, Φ_i is orientable. Let v be any vertex of Ω. Then v belongs to some subset of the cycles $\Phi_1, \dots, \Phi_n$. In each such cycle there are two induced orientations of v and they are opposite to each other. Thus overall the induced orientations of v in Ω are divided equally between the two orientation classes. Therefore Ω is orientable. ☐

One can show that all 2-dimensional cycles given in Figures 3.3a to 3.3e are examples of orientable 2-dimensional cycles. In contrast, it is not possible to choose orientations of the 2-faces of the triangulation of the real projective plane given in Figure 3.3f which are consistent with Definition 3.2.10. Thus this simplicial complex is a non-orientable 2-dimensional cycle.

Within the class of d-dimensional d-complete complexes there are particularly simple examples of orientable d-dimensional cycles. In fact, these are the smallest examples of d-dimensional cycles.

Proposition 3.2.12 (The smallest d-dimensional cycle is a complete one). *The smallest number of vertices that a d-dimensional cycle can have is $d+2$ and the only d-dimensional cycle on $d+2$ vertices is the d-complete simplicial complex Λ^d_{d+2}. In addition, Λ^d_{d+2} is orientable.*

Proof. It is clear that any d-dimensional cycle must have at least $d+2$ vertices since a single d-face contains $d+1$ vertices. It is easy to see that any two d-faces of Λ^d_{d+2} have d vertices in common and so are connected by a d-path. Also, each set of d vertices in Λ^d_{d+2} belongs to exactly two d-faces. Hence Λ^d_{d+2} is a d-dimensional cycle.

Conversely, let Ω be any d-dimensional cycle on $d+2$ vertices. There are only $d+2$ possible d-faces on a set of $d+2$ vertices and so in order to show that Ω is d-complete we must show that it has $d+2$ distinct d-faces. Let F be any d-face of Ω and let f be one of its $(d-1)$-faces. We know that f must belong to at least one other d-face of Ω since it is a d-dimensional cycle. There is only one vertex v of Ω not already contained in F and so $f \cup \{v\}$ must be a d-face of Ω. Since F contains $d+1$ of these distinct $(d-1)$-faces which all must lie in another d-face of Ω this gives rise to $d+1$ distinct d-faces of Ω which all contain v. Therefore Ω contains $d+2$ distinct d-faces including F. Hence Ω is d-complete and so $\Omega = \Lambda^d_{d+2}$.

We would like to show that Λ^d_{d+2} is orientable. Each $(d-1)$-face of Λ^d_{d+2} belongs to just two d-faces and so we need to ensure that there is a way to orient the d-faces of Λ^d_{d+2} so that the orientations induced on each $(d-1)$-face are opposite to each other.

We propose assigning orientations to the d-faces in the following way. Suppose that $v_1, \ldots, v_{d+2}$ are the vertices of Λ^d_{d+2}. Let the orientation of each d-face be the induced orientation that results from thinking of the d-face as a subface of the oriented simplex $[v_1, \ldots, v_{d+2}]$.

Let f be any $(d-1)$-face of Λ^d_{d+2}. We know that f belongs to exactly two d-faces of Λ^d_{d+2} which we will call F and G. Since Λ^d_{d+2} has $d+2$ vertices then we have $V(\Lambda^d_{d+2}) \setminus f = \{v_s, v_t\}$ for some $1 \le s < t \le d+2$ where, without loss of generality, we have $v_s \in F$ and $v_t \in G$. Since $s < t$ we know that v_s appears before v_t in the ordering above. The ordering induced on f by F is achieved by first removing v_t from $[v_1, \ldots, v_{d+2}]$ to induce an ordering on F and then removing v_s to induce an ordering on f, applying odd permutations where necessary. The ordering induced on f by G is achieved by first removing v_s from $[v_1, \ldots, v_{d+2}]$ to induce an ordering on G and then removing v_t to induce an ordering on f, again applying odd permutations where necessary. Note that the removal of v_t does not change whether or not v_s is in an even or odd position in the ordering since it appears before v_t. However removing v_s before v_t causes v_t to move either from an even position to an odd one or from an odd one to an even one. Consequently, the orientation of f induced by F is necessarily an odd permutation of the orientation induced by G. Therefore the two orientations of f are opposite. Hence Λ^d_{d+2} is an orientable d-dimensional cycle. $\square$

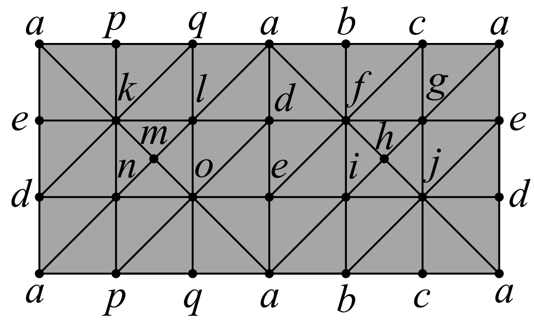
Example 3.2.13. The hollow tetrahedron Λ^2_4 is shown in Figure 3.3a. It is the boundary of a 3-simplex and by Proposition 3.2.12 it is the 2-dimensional cycle on the smallest number of vertices.

We can specialize the concepts of face-minimality and vertex-minimality to the case of orientable cycles in the following way.

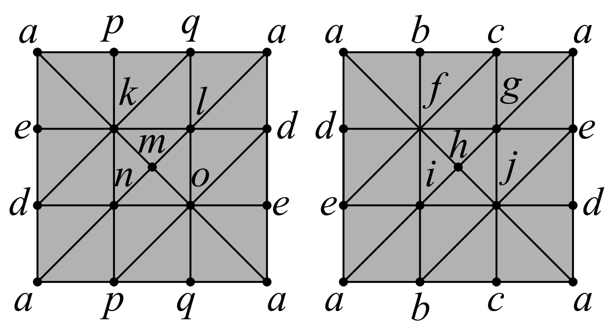
Definition 3.2.14 (orientably-face-minimal). An orientable d-dimensional cycle is called **orientably-face-minimal** if there is no orientable d-dimensional cycle on a proper subset of its d-faces.

Definition 3.2.15 (orientably-vertex-minimal). An orientable d-dimensional cycle in a simplicial complex Γ is called **orientably-vertex-minimal** if there is no orientable d-dimensional cycle in Γ on a proper subset of its vertices.

It is easy to see that an orientable d-dimensional cycle can be orientably-vertex-minimal without being vertex-minimal. It is also possible to have an orientable d-dimensional cycle which is orientably-face-minimal, but which is not face-minimal. The 2-dimensional cycle given in Figure 3.7a is an example of such a cycle. This simplicial complex is the result of gluing the two triangulated Klein bottles given in Figure 3.7b along a circle. It can be shown by careful examination that the entire complex is an orientable 2-dimensional cycle and that there are only two other 2-dimensional cycles on proper subsets of its 2-faces. It turns out that these 2-dimensional cycles, which correspond to the two triangulations of the Klein bottle, are non-orientable.



(a) Orientable face-minimal 2-dimensional cycle



(b) Non-orientable face-minimal 2-dimensional cycles

Figure 3.7: Orientably-face-minimal 2-dimensional cycle which is not face-minimal.

3.3 A Combinatorial Condition for Non-zero Homology over Fields of Characteristic 2

In this section we will show that a d-dimensional cycle is the right combinatorial structure to describe the idea of non-zero d-dimensional homology over a field of characteristic 2. We begin by investigating the relationship between the combinatorial structure of a simplicial complex and its simplicial homology over $\mathbb{Z}_2$. In this field the role played by the coefficients in a d-chain is reduced to indicating whether or not a face is present. As well, since $-1 = 1$ over $\mathbb{Z}_2$, the concept of an orientation of a face is unnecessary as all orientations of a face are equivalent. This allows us to more easily examine the connections between the combinatorics of the simplicial complex and the algebraic concepts of the d-cycle and the d-boundary. In fact, over $\mathbb{Z}_2$, we lose no information when we translate between the d-chain $\sum_{i=1}^{m} F_i$ and the support complex $\langle F_1, \ldots, F_m \rangle$. This makes the field $\mathbb{Z}_2$ an ideal setting to investigate the correspondence between complexes which generate non-zero simplicial homology and their combinatorial properties.

The following proposition demonstrates the relationship between d-dimensional cycles and homological d-cycles over $\mathbb{Z}_2$.

Proposition 3.3.1 (All d-dimensional cycles are homological d-cycles and conversely). *The sum of the d-faces of a d-dimensional cycle is a homological d-cycle over $\mathbb{Z}_2$ and the d-path-connected components of the support complex of a homological d-cycle are d-dimensional cycles.*

Proof. Let Ω be a d-dimensional cycle with d-faces $F_1, \ldots, F_m$. Setting $c = \sum_{i=1}^{m} F_i$ and applying the boundary map ∂_d over $\mathbb{Z}_2$ we have

$$\partial_d(c) = \sum_{i=1}^{m} (e_1^i + \cdots + e_{d+1}^i)$$

where $e_1^i, \ldots, e_{d+1}^i$ are the $d+1$ edges of dimension $d-1$ belonging to F_i. Since the faces $F_1, \ldots, F_m$ form a d-dimensional cycle each $(d-1)$-dimensional face appears in an even number of the faces $F_1, \ldots, F_m$. Hence, since our coefficients belong to $\mathbb{Z}_2$, we have $\partial_d(c) = 0$ and so c is a d-cycle.

Conversely, let $c = F_1 + \cdots + F_m$ be a d-cycle over $\mathbb{Z}_2$. Applying the boundary map ∂_d we have

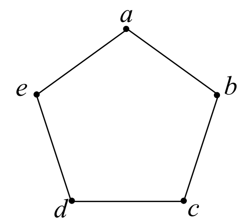
$$0 = \partial_d(F_1 + \cdots + F_m) = \sum_{i=1}^{m} (e_1^i + \cdots + e_{d+1}^i)$$

where $e_1^i, \ldots, e_{d+1}^i$ are the $d+1$ faces of dimension $d-1$ belonging to F_i. If the support complex of c is not d-path-connected then we can partition this complex into d-path-connected components $\Phi_1, \ldots, \Phi_n$ where $n \geq 2$. Let $P_i \subseteq \{1, \ldots, m\}$ be such that $F_j \in \Phi_i$ if and only if $j \in P_i$. Note that $P_1, \ldots, P_n$ form a partition of $\{1, \ldots, m\}$. Since $\Phi_1, \ldots, \Phi_n$ have no $(d-1)$-faces in common, for each $1 \leq i \leq n$ we must have

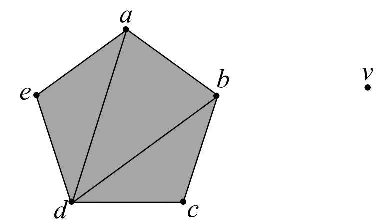
$$\sum_{j \in P_i} (e_1^j + \cdots + e_{d+1}^j) = 0$$

and so, since our sum is over $\mathbb{Z}_2$, we see that each $(d-1)$-face occurring in Φ_i belongs to an even number of the d-faces in $\{F_j | j \in P_i\}$. Therefore Φ_i is a d-dimensional cycle and hence the d-path-connected components of the support complex of c all form d-dimensional cycles. $\square$

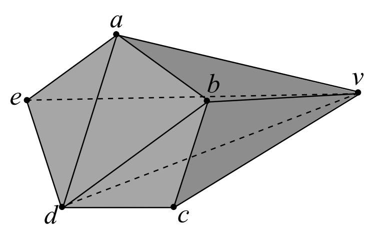
Using Proposition 3.3.1 we can construct a $(d+1)$-dimensional cycle from a d-dimensional cycle when certain homological requirements are satisfied in the complex. For an illustration of the construction used in the proof of the following proposition see Figure 3.8.



(a) The 1-dimensional cycle Ω



(b) Ω as the support complex of the 1-boundary of $[a,b,d]+[b,c,d]+[a,d,e]$ and the vertex v



(c) The 2-dimensional cycle Φ with facets $\{a,b,v\}$, $\{a,e,v\}$, $\{b,c,v\}$, $\{c,d,v\}$, $\{d,e,v\}$, $\{a,b,d\}$, $\{b,c,d\}$, $\{a,d,e\}$

Figure 3.8: Example of construction in Proposition 3.3.2.

Proposition 3.3.2 (A cone over a d-dimensional cycle which is a d-boundary is a $(d+1)$-dimensional cycle). *Let $\Omega = \langle F_1, \dots, F_k \rangle$ be a d-dimensional cycle contained in a simplicial complex Γ. Suppose that there exist $(d+1)$-faces $A_1, \dots, A_\ell$ in $\Gamma_{V(\Omega)}$ such that, over $\mathbb{Z}_2$, we have*

$$\partial_{d+1}\left(\sum_{i=1}^{\ell} A_i\right) = \sum_{j=1}^{k} F_j \tag{3.3}$$

and for no proper subset of $\{A_1, \dots, A_\ell\}$ does (3.3) hold. If $v \in V(\Gamma) \setminus V(\Omega)$ then

$$\Phi = \langle F_1 \cup \{v\}, \dots, F_k \cup \{v\}, A_1, \dots, A_\ell \rangle$$

is a $(d+1)$-dimensional cycle.

Proof. We will first show that each d-face of Φ is contained in an even number of the $(d+1)$-faces of Φ. Let f be any d-face of Φ. We have three cases to consider:

(i) First suppose that $v \in f$. In this case, f is not contained in any of the A_i's and so f is a subset of $F_j \cup \{v\}$ for some j. So we have $f \setminus \{v\} \subseteq F_j$ and so $f \setminus \{v\}$ is a $(d-1)$-face of Ω. Since Ω is a d-dimensional cycle $f \setminus \{v\}$ belongs to an even number of the d-faces $F_1, \ldots, F_k$. Therefore f belongs to an even number of the $(d+1)$-faces $F_i \cup \{v\}$ and so belongs to an even number of the $(d+1)$-faces of Φ.

(ii) Now suppose that $v \notin f$ and that f belongs to at least one $(d+1)$-face of the form $F_j \cup \{v\}$ for some j. In this case we must have $f = F_j$ and so $f \not\subseteq F_i \cup \{v\}$ for any $i \neq j$. So f belongs to exactly one $(d+1)$-face of the form $F_i \cup \{v\}$. Thus f appears exactly once on the right-hand-side of (3.3) and since this equation holds over $\mathbb{Z}_2$, f must be contained in an odd number of the A_i's. Hence overall f is contained in an even number of the $(d+1)$-faces of Φ.

(iii) Finally suppose that $v \notin f$ and that f does not belong to any $(d+1)$-faces of the form $F_j \cup \{v\}$. Then f is not a d-face of Ω and it does not appear in the right-hand-side of (3.3). Again, since (3.3) holds over $\mathbb{Z}_2$, we know that f belongs to an even number of the A_i's. Thus f is contained in an even number of the $(d+1)$-faces of Φ.

Therefore we know that the $(d+1)$-path-connected components of Φ are $(d+1)$-dimensional cycles. Note that the $(d+1)$-faces $F_i \cup \{v\}$ all lie in the same $(d+1)$-path-connected component of Φ since Ω is d-path-connected. Recall from above that, for any j, a d-face belonging to $F_j \cup \{v\}$ which does not contain v is equal to F_j and must belong to at least one of the A_i's by (3.3). Thus at least one of the A_i's belongs to the $(d+1)$-path-connected component of Φ which contains the $F_j \cup \{v\}$'s. Therefore if Φ has any other $(d+1)$-path-connected component then it consists solely of a proper subset of the A_i's. Without loss of generality let these faces be $A_1, \ldots, A_r$ where $r < \ell$.

We know that these $(d+1)$-path-connected components are all $(d+1)$-dimensional cycles by what was just shown and so by Proposition 3.3.1 we have that

$$\partial_{d+1}\left(\sum_{j=1}^{r} A_j\right) = 0$$

and so

$$\partial_{d+1}\left(\sum_{j=1}^{\ell} A_j\right) = \partial_{d+1}\left(\sum_{j=1}^{r} A_j\right) + \partial_{d+1}\left(\sum_{j=r+1}^{\ell} A_j\right) = \partial_{d+1}\left(\sum_{j=r+1}^{\ell} A_j\right).$$

Therefore by (3.3) we have

$$\partial_{d+1}\left(\sum_{j=r+1}^{\ell} A_j\right) = \sum_{i=1}^{k} F_i$$

which contradicts the minimality of our choice of $A_1, ..., A_\ell$. Hence Φ has only one $(d+1)$-path-connected component and so it is a $(d+1)$-dimensional cycle. $\square$

Due to the close association between homological d-cycles and d-dimensional cycles over the field $\mathbb{Z}_2$ we are able to obtain a necessary and sufficient combinatorial condition for non-zero homology over any field of characteristic 2.

Theorem 3.3.3 (Non-zero d-homology in characteristic 2 corresponds to d-dimensional cycles). *Let Γ be a simplicial complex and let k be a field of characteristic 2. Then $\tilde{H}_d(\Gamma; k) \neq 0$ if and only if Γ contains a d-dimensional cycle, the sum of whose d-faces is not a d-boundary.*

Proof. By Lemma 2.2.2 we know that $\tilde{H}_d(\Gamma; k) \neq 0$ if and only if $\tilde{H}_d(\Gamma; \mathbb{Z}_2) \neq 0$. Therefore we need only prove the theorem in the case that $k = \mathbb{Z}_2$.

Suppose that $\tilde{H}_d(\Gamma; \mathbb{Z}_2) \neq 0$. Then Γ contains a d-cycle c that is not a d-boundary. We may assume that the support complex of c is minimal with respect to this property. In other words no proper subset of the d-faces of c has a sum that is also a d-cycle which is not a d-boundary. First we would like to show that the support complex of c is d-path-connected.

Since we are in $\mathbb{Z}_2$, the d-cycle c is of the form

$$c = F_1 + \cdots + F_m$$

for some d-faces $F_1, \dots, F_m$ of Γ. Since c is a d-cycle then applying the boundary map ∂_d we have

$$0 = \partial_d(F_1 + \cdots + F_m) = \sum_{i=1}^{m}(e_1^i + \cdots + e_{d+1}^i)$$

where $e_1^i, \dots, e_{d+1}^i$ are the $d+1$ edges of dimension $d-1$ belonging to F_i. If the support complex of c is not d-path-connected then, without loss of generality, we can partition its set of d-faces into two sets, $\{F_1, \dots, F_\ell\}$ and $\{F_{\ell+1}, \dots, F_m\}$ such that these two sets have no $(d-1)$-faces in common. Hence we must have

$$\sum_{i=1}^{\ell}(e_1^i + \cdots + e_{d+1}^i) = 0 \qquad \text{and} \qquad \sum_{i=\ell+1}^{m}(e_1^i + \cdots + e_{d+1}^i) = 0.$$

In other words we have

$$\partial_d(F_1 + \cdots + F_\ell) = 0 \qquad \text{and} \qquad \partial_d(F_{\ell+1} + \cdots + F_m) = 0$$

and so $F_1 + \cdots + F_\ell$ and $F_{\ell+1} + \cdots + F_m$ are both d-cycles. By our assumption of minimality these d-cycles are both d-boundaries. Hence in Γ there exist $(d+1)$-faces $G_1, \dots, G_r$ and $H_1, \dots, H_t$ such that

$$\partial_{d+1}\left(\sum_{i=1}^{r} G_i\right) = F_1 + \cdots + F_\ell \qquad \text{and} \qquad \partial_{d+1}\left(\sum_{i=1}^{t} H_i\right) = F_{\ell+1} + \cdots + F_m.$$

But then we have

$$\partial_{d+1}\left(\sum_{i=1}^{r} G_i + \sum_{i=1}^{t} H_i\right) = F_1 + \cdots + F_m$$

which is a contradiction since $F_1 + \cdots + F_m$ is not a d-boundary. Therefore the support complex of c must be d-path-connected. Hence, by Proposition 3.3.1 the support complex of c is a d-dimensional cycle. Therefore Γ contains a d-dimensional cycle the sum of whose d-faces is not a d-boundary.

Conversely, suppose that Γ contains a d-dimensional cycle with d-faces $F_1, \dots, F_m$ such that $\sum_{i=1}^{m} F_i$ is not a d-boundary. By Proposition 3.3.1 we know that $\sum_{i=1}^{m} F_i$ is a d-cycle. It follows that $\tilde{H}_d(\Gamma; \mathbb{Z}_2) \neq 0$. $\square$

Remark 3.3.4. Proposition 3.3.1 gives us the most efficient algorithm that we know of for finding all of the d-dimensional cycles in a simplicial complex. The first step in such an algorithm is to row reduce a matrix representation of the map ∂_d over $\mathbb{Z}_2$ in order to obtain a basis for its kernel. This enables us to find all of the homological d-cycles in the set of d-chains. We can then enumerate the d-dimensional cycles by identifying, combinatorially, the d-path-connected components of the support complex of each homological d-cycle. This procedure may find a d-dimensional cycle more than once and so a final step would be to remove the repeated cycles.

3.4 Non-zero Homology over General Fields

When we broaden the scope of our investigations to study simplicial homology over an arbitrary field we must keep in mind examples such as the triangulation of the real projective plane given in Figure 3.3f. The simplicial homology of this complex changes significantly depending on the field under consideration. In particular, one can show by computation that this complex has non-zero 2-dimensional homology only over fields of characteristic 2. As we saw in Section 3.2, the triangulation of the real projective plane is an example of a non-orientable 2-dimensional cycle. It is this notion of orientability which leads us to a sufficient condition for a simplicial complex to have non-zero homology over any field. First we see that orientable d-dimensional cycles are homological d-cycles over any field.

Lemma 3.4.1 (Orientable d-dimensional cycles are homological d-cycles). *The sum of the oriented d-faces of an orientable d-dimensional cycle is a homological d-cycle over any field k.*

Proof. Let Ω be an orientable d-dimensional cycle with d-faces $F_1, \ldots, F_m$ and let

$$c = F_1 + \cdots + F_m$$

where $F_1, \ldots, F_m$ are given orientations consistent with Definition 3.2.10. For $1 \leq i \leq m$ let $e^i_1, \ldots, e^i_{d+1}$ be the $d+1$ faces of dimension $d-1$ belonging to F_i with vertices ordered as in F_i. Applying the boundary map to c we have, without loss of generality,

$$\partial_d(c) = \sum_{i=1}^{m} \sum_{j=1}^{d+1} (-1)^{j+1} e_j^i. \tag{3.4}$$

Notice that every $(d-1)$-face of Ω occurs an even number of times in (3.4) since Ω is a d-dimensional cycle. Furthermore, because Ω is orientable the number of times that the $(d-1)$-face appears with a positive sign is equal to the number of times that it appears with a negative sign. Hence we get $\partial_d(c) = 0$ and so c is a homological d-cycle. □

Theorem 3.4.2 (Orientable d-dimensional cycles give non-zero homology over all fields). *If a simplicial complex Γ contains an orientable d-dimensional cycle, the sum of whose oriented d-faces is not a d-boundary, then $\tilde{H}_d(\Gamma; k) \neq 0$ for any field k.*

Proof. Let Ω be the orientable d-dimensional cycle in Γ given by our assumption and with d-faces $F_1, \ldots, F_m$. By Lemma 3.4.1 we know that the d-chain $c = F_1 + \cdots + F_m$ is a homological d-cycle where $F_1, \ldots, F_m$ are oriented according to Definition 3.2.10. By assumption, c is not a d-boundary and so $\tilde{H}_d(\Gamma; k) \neq 0$. □

The converse of Theorem 3.4.2 does not hold as can be seen from the example of the triangulation of the real projective plane. As mentioned above, this simplicial complex has non-zero 2-dimensional homology over any field of characteristic 2, but this complex is a face-minimal non-orientable 2-dimensional cycle and so contains no orientable 2-dimensional cycles.

A triangulation Γ of the mod 3 Moore space shown in Figure 3.9, is another interesting counterexample to the converse of Theorem 3.4.2. First, this is an example of a simplicial complex which has non-zero 2-dimensional homology only over fields of characteristic 3. Second, and even more of note, is that Γ contains no orientable or non-orientable 2-dimensional cycles. One can show that Γ is the support complex of a minimal 2-dimensional homological cycle over fields of characteristic 3, in the sense that removing any of its facets leaves a simplicial complex with no 2-dimensional homology. Also, in some sense, Γ is quite close to being a 2-dimensional cycle. All but three of its 1-faces each lie in exactly two 2-dimensional faces. The remaining three 1-faces lie in three 2-dimensional faces each. These 1-faces are $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$. The set $\{x, y, z\}$ is not a face of Γ, but by adding this face to Γ we obtain the 2-dimensional cycle shown in Figure 3.10.

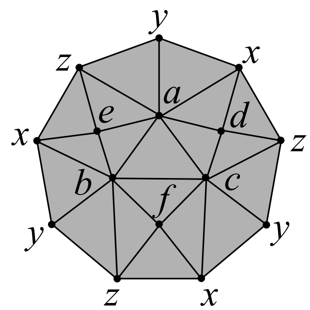


Figure 3.9: A triangulation of the mod 3 Moore space.

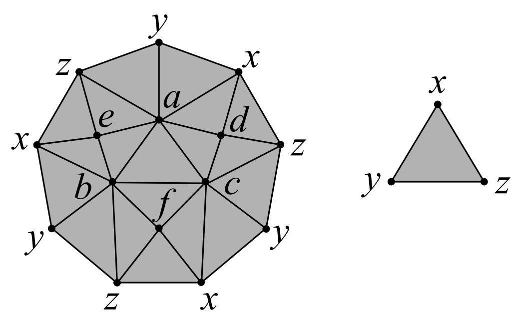


Figure 3.10: Modification of the triangulated mod 3 Moore space.

It is clear from the example of the mod 3 Moore space that one may have non-zero reduced d-dimensional homology over fields of finite characteristic without having d-dimensional cycles present. In fact, it is possible to construct a mod p Moore space, for p a prime, in any dimension $d \geq 1$ (see Hatcher [19, page 143]). A triangulation of such a space will not contain a d-dimensional cycle, but will have non-zero reduced homology only in degree d and only over fields of characteristic p. This gives us a family of counterexamples to the converse of Theorem 3.4.2.

The difficulty in trying to prove the converse of Theorem 3.4.2 for the general case is in linking the algebraic notion of a homological d-cycle with general field coefficients to the combinatorics of its underlying support complex. In particular, when the field coefficients of a d-chain are not all equal in absolute value it is difficult to find a combinatorial description of the role played by the coefficients. For this reason we had hoped to be able to prove that, over $\mathbb{Z}$, all "minimal" homological d-cycles had coefficients which were constant up to absolute value. Unfortunately, the simplicial complex in Figure 3.10 is a counterexample to this conjecture. In this case, as a homological 2-cycle over $\mathbb{Z}$, the coefficients of all facets but $\{x, y, z\}$ can be chosen to be 1 in absolute value whereas the coefficient of $\{x, y, z\}$ must be 3. Furthermore, it is not hard to show that this 2-dimensional simplicial complex is non-orientable. Therefore we have another, more interesting counterexample to the converse of Theorem 3.4.2 as we have a non-orientable 2-dimensional cycle which has non-zero homology over all fields.

Chapter 4

Generalizing Fröberg's Theorem

As mentioned in Chapter 1, in 1990 Fröberg gave a well-known combinatorial characterization of the square-free monomial ideals with 2-linear resolutions.

Theorem 4.0.1 (Fröberg [16])**.** *If a graph G is chordal, then $\mathcal{N}(\Delta(G))$ has a 2-linear resolution over any field. Conversely, given a simplicial complex Γ, if $\mathcal{N}(\Gamma)$ has a 2-linear resolution over any field, then $\Gamma = \Delta(\Gamma^{[1]})$ and $\Gamma^{[1]}$ is chordal.*

This theorem demonstrates that in order to determine whether or not a square-free monomial ideal that is generated in degree 2 has a linear resolution one need only examine the 1-skeleton of its Stanley-Reisner complex. This is an extremely strong result, especially when viewed in contrast to another classification of square-free monomial ideals with linear resolutions given by Fröberg, which is homological in nature.

Theorem 4.0.2 (Fröberg [15])**.** *A square-free monomial ideal I has a d-linear resolution over a field k if and only if for every induced subcomplex Γ of $\mathcal{N}(I)$ we have $\tilde{H}_i(\Gamma; k) = 0$ for $i \neq d-2$.*

This theorem tells us that in order to have a linear resolution, all homology groups of all induced subcomplexes of the Stanley-Reisner complex must vanish in all but one dimension, and conversely. Consequently, a simple condition on the 1-skeleton of such a complex at first glance seems inadequate for determining the existence of a 2-linear resolution. However, there are several important connections to be made between the graph condition of being chordal and the homological requirements for a linear resolution in degree 2.

In the first case, when a square-free monomial ideal is generated in degree 2 it is not hard to see that its Stanley-Reisner complex is the clique complex of its 1-skeleton. This means that all complete subgraphs of this 1-skeleton, when it is viewed as a graph, correspond to faces of the Stanley-Reisner complex. In essence, it is the 1-skeleton alone which determines which faces are present in the complex.

In addition, as we saw in Chapter 3, the graph cycle is the right combinatorial structure to capture non-zero 1-dimensional homology. In a chordal graph the manner in which a graph cycle may be present is severely restricted. In particular, all "minimal" cycles must be triangles. In other words, these cycles must be complete subgraphs and thus show up in the clique complex as 2-faces. This effectively removes all 1-dimensional homology from these complexes.

The fact that all upper-dimensional homology groups of such a Stanley-Reisner complex are also forced to vanish is impressive. In particular because such a complex may have facets of any dimension at all, since a chordal graph may have a complete subgraph on any number of vertices. It is the delicate interaction between the restrictions placed on cycles in a chordal graph and the nature of the clique complex operation which allows this vanishing of homology to occur. We elaborate on this relationship in Section 4.7.

In attempting to generalize Fröberg's theorem to an arbitrary square-free monomial ideal generated in degree $d+1$ we seek a similar relationship between the combinatorial structure of the pure d-skeleton of the Stanley-Reisner complex and the vanishing of its homology groups. Not surprisingly, such a relationship is much harder to pinpoint in higher dimensions. When we leave the 1-dimensional case, the way in which our combinatorial structures may be connected becomes far more complicated and as well we are confronted with new issues such as orientability. Recall Proposition 3.2.11 for example, which states that all 1-dimensional cycles are orientable. Hence an examination of these higher-dimensional structures from a purely combinatorial viewpoint is considerably more involved. As a simple example, a graph edge may be connected to other edges at exactly two places, its endpoints. Whereas, a 2-dimensional face may be connected to other 2-dimensional faces in six different ways.

We begin this chapter with two different generalizations of chordal graphs to higher dimensions. We use these two combinatorial notions to generalize, in two different ways, one direction of Fröberg's theorem over an arbitrary field and over fields of characteristic 2 respectively. In particular, in Theorem 4.3.1, we provide necessary conditions for a square-free monomial ideal to have a linear resolution in these two cases.

The converses to both parts of Theorem 4.3.1 do not hold and in Section 4.5 we examine the characteristic 2 case in more detail. As a result, in Section 4.6, we are able to provide a necessary and sufficient combinatorial condition for a square-free monomial ideal to have a linear resolution over any field of characteristic 2 with Theorem 4.6.1.

Much of this chapter is based on [6].

4.1 The Classes of d-Chorded and d-Cycle-Complete Complexes

In a chordal graph any cycle can be "broken down" into a set of complete cycles, or triangles, on the same vertex set, often in more than one way. The clique complex of a chordal graph introduced in Section 2.3 "fills in" these cycles by turning each such triangle into a face of the complex. See Figure 4.1 for an example. The original

(a) A chordal graph consisting of a cycle with chords shown with dotted lines

(b) The clique complex of the chordal graph

Figure 4.1: Vanishing 1-dimensional homology in the clique complex of a chordal graph.

cycle can be thought of as a sum of these triangles. Therefore, as we can see from Theorem 3.1.1, all 1-dimensional homology existing in the chordal graph when it is thought of as a simplicial complex disappears in the clique complex, as all 1-cycles are transformed into 1-boundaries.

We would like to replicate this dismantling of a cycle into smaller complete pieces in higher dimensions using the d-dimensional cycles introduced in Chapter 3. In the case of chordal graphs, cycles that are not complete must have a chord. This chord breaks the cycle into two cycles both having fewer vertices than the original. It is the inductive nature of this addition of chords which results in each cycle being dismantled into complete cycles. To achieve this same goal in d-dimensional cycles we introduce a higher-dimensional notion of a chord.

Definition 4.1.1 (chord set). Let Ω be a d-dimensional cycle in a simplicial complex Γ. A **chord set** of Ω in Γ is a nonempty set C of d-faces of $\Gamma \setminus \Omega$ contained in $V(\Omega)$ such that the simplicial complex $\langle C, \text{Facets}(\Omega)\rangle$ consists of k d-dimensional cycles, $\Omega_1, \ldots, \Omega_k$, where $k \geq 2$ and with the following conditions:

1. $\bigcup_{i=1}^{k} \text{Facets}(\Omega_i) = \text{Facets}(\Omega) \cup C$,
2. each d-face in C is contained in an even number of the cycles $\Omega_1, \ldots, \Omega_k$,
3. each d-face of Ω is contained in an odd number of the cycles $\Omega_1, \ldots, \Omega_k$,
4. $|V(\Omega_i)| < |V(\Omega)|$ for $i = 1, \ldots, k$.

A chord set of a face-minimal 1-dimensional cycle corresponds to a set of chords in a graph cycle in the traditional sense. In Figure 4.2 we have a graph cycle on six vertices with a chord set displayed with dotted lines. This cycle is broken into four smaller cycles by its chord set. A chord of a graph cycle is always a chord set of the cycle as it is a 1-face of the graph which breaks the graph cycle into two graph cycles on proper subsets of the vertices of the original graph. The edges of the original cycle appear in exactly one of the two smaller cycles and the chord appears in both.

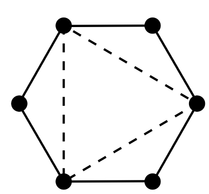
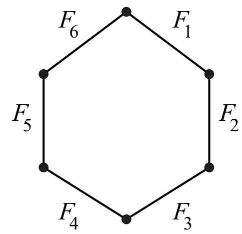
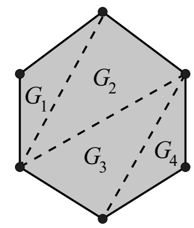


Figure 4.2: Example of a graph cycle with a chord set.

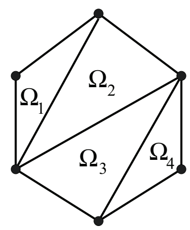
In the next lemma we see that, when a non-d-complete d-dimensional cycle is the support complex of a d-boundary over $\mathbb{Z}_2$, it is forced to have a chord set. In Figure 4.3 we illustrate the technique used in the proof of Lemma 4.1.2.



(a) The 1-dimensional cycle Ω



(b) Ω as the 1-boundary of $G_1 + \cdots + G_4$ with $E_1, \ldots, E_4$ shown by dashed lines



(c) The 1-complete 1-dimensional cycles $\Omega_1, \ldots, \Omega_4$

Figure 4.3: Example of the construction in the proof of Lemma 4.1.2.

Lemma 4.1.2 (All *d*-dimensional cycles which are boundaries have chord sets). *Let Ω be a face-minimal d-dimensional cycle that is not d-complete in a simplicial complex Γ. If, over $\mathbb{Z}_2$, Ω is the support complex of a d-boundary of faces of $\Gamma_{V(\Omega)}$ then Ω has a chord set in Γ.*

Proof. Let the d-faces of Ω be $F_1, \ldots, F_k$. Since, in $\mathbb{Z}_2$, Ω forms the support complex of a d-boundary of faces of $\Gamma_{V(\Omega)}$, there exist $(d+1)$-faces $G_1, \ldots, G_\ell$ of $\Gamma_{V(\Omega)}$ such that

$$\partial_{d+1}\left(\sum_{i=1}^{\ell} G_i\right) = \sum_{i=1}^{k} F_i.$$

Note that $\ell \geq 2$ since otherwise we have

$$\partial_{d+1}(G_1) = \sum_{i=1}^{k} F_i$$

and since $\langle G_1 \rangle^{[d]} = \Lambda^d_{d+2}$ this indicates that Ω itself must be Λ^d_{d+2} since it is face-minimal. This can't happen since Ω is not d-complete.

Let $E_1, \ldots, E_m$ be the d-faces of $\langle G_1, \ldots, G_\ell \rangle$ which are not d-faces of Ω. If the set $\{E_1, \ldots, E_m\}$ is empty then Ω contains all the d-faces belonging to the $(d+1)$-faces

$G_1, \ldots, G_\ell$. However, since $\langle G_i \rangle^{[d]} = \Lambda^d_{d+2}$ which is a d-complete d-dimensional cycle and Ω is a face-minimal non-d-complete d-dimensional cycle, we have a contradiction. Therefore the set $\{E_1, \ldots, E_m\}$ is nonempty. We claim that $\{E_1, \ldots, E_m\}$ is a chord set of Ω in Γ.

First note that the vertices of $E_1, \ldots, E_m$ are contained in $V(\Omega)$ since $G_1, \ldots, G_\ell$ are faces of $\Gamma_{V(\Omega)}$. Also, by construction we have $\{E_1, \ldots, E_m\} \cap \{F_1, \ldots, F_k\} = \emptyset$. We set $\Omega_i = \langle G_i \rangle^{[d]}$ for each $i \in \{1, \ldots, \ell\}$.

Since each d-face of the two sets $\{F_1, \ldots, F_k\}$ and $\{E_1, \ldots, E_m\}$ appears in at least one of the G_i's, we have

$$\bigcup_{i=1}^{\ell} \text{Facets}(\Omega_i) = \{F_1, \ldots, F_k, E_1, \ldots, E_m\}$$

and so property 1 of a chord set is satisfied. Over $\mathbb{Z}_2$ we have

$$\partial_{d+1}\left(\sum_{i=1}^{\ell} G_i\right) = \sum_{i=1}^{k} F_i$$

and so we know that for each $1 \leq i \leq m$ the face E_i must be contained in an even number of the faces $G_1, \ldots, G_\ell$ as E_i does not appear on the right-hand side of this equation. Therefore E_i is also contained in an even number of the cycles $\Omega_1, \ldots, \Omega_\ell$. Similarly, each d-face of Ω must be contained in an odd number of the cycles $\Omega_1, \ldots, \Omega_\ell$. Thus properties 2 and 3 of a chord set are satisfied.

Finally since Ω is not d-complete, we know that $|V(\Omega)| > d + 2$ by Proposition 3.2.12. Therefore since $|V(\Omega_i)| = d + 2$ for all $i \in \{1, \ldots, \ell\}$, we have that $|V(\Omega_i)| < |V(\Omega)|$ for all $i \in \{1, \ldots, \ell\}$ and so property 4 of a chord set is satisfied. Hence $\{E_1, \ldots, E_m\}$ is a chord set of Ω in Γ. $\square$

Next we introduce a generalization to higher dimensions of the "cycles and chords" characterization of chordal graphs.

Definition 4.1.3 (d**-chorded).** A pure d-dimensional simplicial complex Γ is d-**chorded** if all face-minimal d-dimensional cycles in Γ that are not d-complete have a chord set in Γ.

Since any d-dimensional cycle can be partitioned into face-minimal cycles by Lemma 3.2.8, the requirement in Definition 4.1.3 on face-minimal cycles is sufficient to ensure that all d-dimensional cycles in a d-chorded simplicial complex have chord sets.

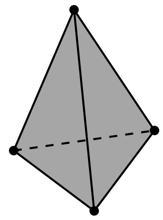
In the 1-dimensional case, Definition 4.1.3 says that a graph is 1-chorded when all face-minimal cycles that are not 1-complete have a chord set. In other words, a graph is 1-chorded when all graph cycles that are not triangles have a chord set. This agrees with the usual notion of a chordal graph.

In Figure 4.4 we give examples of simplicial complexes that are 2-chorded. The hollow tetrahedron in Figure 4.4a is 2-chorded as it is a 2-complete face-minimal 2-dimensional cycle. The pure 2-dimensional complex in Figure 4.4b is 2-chorded because it contains no 2-dimensional cycles. Notice, however, that this simplicial complex is an example of a simplicial cycle. The complex in Figure 4.4c contains three 2-dimensional cycles, the outer cycle which consists of six 2-dimensional faces and uses all vertices, and the two hollow tetrahedra which are 2-complete face-minimal cycles and which share the inner face shown in dark grey. All three cycles are face-minimal. This simplicial complex is 2-chorded because its only face-minimal cycle that is not 2-complete contains a chord set of size 1 breaking it into two 2-complete face-minimal cycles. Similarly, the simplicial complex in Figure 4.4d consists of five face-minimal 2-dimensional cycles, four of which are hollow tetrahedra breaking the remaining, non-2-complete face-minimal outer cycle into four smaller 2-dimensional cycles by a chord set of size 4 shown in dark grey.

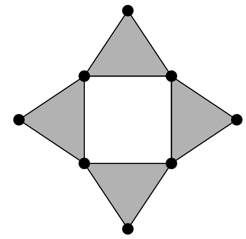
Some of the simplest examples of d-chorded simplicial complexes are the d-complete complexes.

Proposition 4.1.4 (**d-complete $\Rightarrow$ d-chorded**). *For $n \geq d+1$, the d-dimensional d-complete simplicial complex Λ_n^d is d-chorded.*

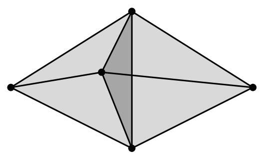
Proof. First notice that Λ_n^d is the d-skeleton of the simplex on n vertices Λ_n^{n-1}. Let Ω be a face-minimal d-dimensional cycle in Λ_n^d which is not d-complete. Then Ω is also a face-minimal d-dimensional cycle in Λ_n^{n-1}. Since any simplex is acyclic over any field, we have $\tilde{H}_d(\Lambda_n^{n-1}; \mathbb{Z}_2) = 0$. Hence Ω, which is the support complex of a d-cycle by Proposition 3.3.1, is the support complex of a d-boundary. Therefore, by Lemma 4.1.2, Ω has a chord set in Λ_n^{n-1}. Since Λ_n^d is the d-skeleton of Λ_n^{n-1}, Ω has a chord set in Λ_n^d as well. Therefore Λ_n^d is d-chorded. $\square$



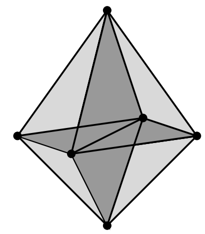
(a) 2-complete face-minimal 2-dimensional cycle



(b) Pure 2-dimensional complex with no 2-dimensional cycles



(c) Face-minimal 2-dimensional cycle with chord set of size 1



(d) Octahedron with chord set of size 4

Figure 4.4: Examples of 2-chorded simplicial complexes.

Remark 4.1.5. As a consequence of Proposition 4.1.4 we see that the pure d-skeleton of any n-simplex is d-chorded for any $d < n$.

We would like to make use of the homological characterization given in Theorem 4.0.2 to extend Fröberg's theorem to higher dimensions. For this purpose, we require the property of being d-chorded to be transferred to induced subcomplexes.

Lemma 4.1.6 (Chordedness transfers to induced subcomplexes). *The pure d-skeleton of any induced subcomplex of a d-chorded simplicial complex is d-chorded.*

Proof. Let Γ be a d-chorded simplicial complex and let $W \subseteq V(\Gamma)$. Let Ω be any face-minimal d-dimensional cycle in $(\Gamma_W)^{[d]}$ that is not d-complete. It is clear that Ω is also a d-dimensional cycle in Γ. Also, Ω must be face-minimal in Γ otherwise some proper subset of its d-faces is a d-dimensional cycle in Γ, but then also in $(\Gamma_W)^{[d]}$, which is a contradiction. Hence, since Γ is d-chorded, Ω has a chord set in Γ. Since a chord set is a set of d-faces that lie on the same vertex set as the cycle, this chord set exists in Γ_W as well. Hence Ω has a chord set in $(\Gamma_W)^{[d]}$. Thus every face-minimal d-dimensional cycle in $(\Gamma_W)^{[d]}$ that is not d-complete has a chord set. Therefore $(\Gamma_W)^{[d]}$ is d-chorded. $\square$

A chordal graph can also be defined without the notion of chords by requiring that all of its “minimal” cycles be complete. We can extend this definition to higher dimensions in the following way.

Definition 4.1.7 (**orientably-d-cycle-complete, d-cycle-complete).** A pure d-dimensional simplicial complex Γ is called (**orientably-**) **d-cycle-complete** if all of its (orientably-) vertex-minimal d-dimensional cycles are d-complete.

The examples in Figure 4.4 are all both 2-cycle-complete and orientably-2-cycle-complete. It is not hard to see that the sets of 1-cycle-complete and orientably-1-cycle-complete simplicial complexes correspond exactly to the set of chordal graphs. The fact that these classes coincide in dimension 1 follows from Proposition 3.2.11.

Remark 4.1.8. It is easy to see that the property of being either d-cycle-complete or orientably-d-cycle-complete is transferred to induced subcomplexes. This is because any (orientably-) vertex-minimal d-dimensional cycle in an induced subcomplex is also (orientably-) vertex-minimal in the original complex.

We have imposed structure on our classes of d-chorded and d-cycle-complete complexes by restricting the way in which d-dimensional cycles may exist in these complexes. A more severe restriction is to disallow d-dimensional cycles altogether.

Definition 4.1.9 (**d-dimensional forest, d-dimensional tree).** A **d-dimensional forest** is a pure d-dimensional simplicial complex containing no d-dimensional cycles. A **d-dimensional tree** Γ is a d-dimensional forest such that $\Gamma^{[1]}$ is 1-path-connected.

It is easy to see that all d-dimensional forests are d-chorded.

Notice that the notion of a graph tree agrees with that of a 1-dimensional tree. Another higher-dimensional analogue to the graph tree is the simplicial tree which is a connected simplicial complex with no simplicial cycles [13]. It is not difficult to show that a pure d-dimensional simplicial tree is a d-dimensional tree as one can show that any d-dimensional cycle contains a simplicial cycle. In particular, given any d-dimensional cycle Ω and any $(d-2)$-face f in Ω, one can show that the simplicial complex whose facets are $\{F \in \text{Facets}(\Omega) \mid f \in F\}$ contains a simplicial cycle.

Proposition 4.1.10 (*d*-**chorded** $\Rightarrow$ *d*-**cycle-complete** $\Rightarrow$ **orientably-*d*-cycle-complete).**

1. *Any d-chorded simplicial complex is d-cycle-complete.*

2. *Any d-cycle-complete simplicial complex is orientably-d-cycle-complete.*

Proof.

1. Let Γ be a d-chorded simplicial complex and let Ω be any vertex-minimal d-dimensional cycle in Γ. Suppose that Ω is not d-complete. If Ω is face-minimal then Γ contains a chord set for Ω which means that there exist d-dimensional cycles on proper subsets of the vertices of Ω. If Ω is not face-minimal then it contains a face-minimal d-dimensional cycle on its d-faces which has a chord set. This also implies that there exist d-dimensional cycles on proper subsets of the vertices of Ω. Either way we have a contradiction to vertex-minimality of Ω and so Ω must be d-complete. Hence Γ is d-cycle-complete.

2. Let Γ be a d-cycle-complete simplicial complex and let Ω be any orientably-vertex-minimal d-dimensional cycle in Γ. We know that Ω does not contain any orientable d-dimensional cycles on a proper subset of its vertices. If it also does not contain any non-orientable d-dimensional cycles on a proper subset of its vertices then it is vertex-minimal and so d-complete since Γ is d-cycle complete. Thus suppose that Ω contains a non-orientable cycle on a proper subset of its vertices. If this non-orientable cycle is vertex-minimal then it is d-complete which means that it contains a copy of Λ^d_{d+2}, an orientable d-dimensional cycle on a proper subset of the vertices of Ω by Proposition 3.2.12. This is a contradiction to the fact that Ω is orientably-vertex-minimal. If the non-orientable cycle is not vertex-minimal then it must contain a d-dimensional cycle on a proper subset of its vertices which is vertex-minimal. This cycle is d-complete since Γ is d-cycle-complete and so contains a copy of Λ^d_{d+2}. As before we have a contradiction and so Ω does not contain any non-orientable d-dimensional cycles on its vertex set. Hence Ω is vertex-minimal and so d-complete since Γ is d-cycle-complete. Thus Γ is orientably-d-cycle-complete.

□

By Proposition 4.1.10 we can see that the four classes of simplicial complexes defined in this section are “nested” with

$$\{d\text{-dimensional forests}\}$$

$$\subsetneq$$

$$\{d\text{-chorded simplicial complexes}\}$$

$$\subsetneq$$

$$\{d\text{-cycle-complete simplicial complexes}\}$$

$$\subsetneq$$

$$\{\text{orientably-}d\text{-cycle-complete simplicial complexes}\}.$$

These inclusions are strict. The simplicial complex Λ^d_{d+2} is d-chorded but not a d-dimensional forest. An example of a 2-cycle-complete complex which is not 2-chorded is given in Figure 3.5 in Section 3.2. This complex is not 2-chorded because it contains a face-minimal 2-dimensional cycle, the outer triangulated sphere, which is not 2-complete and has no chord set. The triangulation of the real projective plane given in Figure 3.3f is an example of an orientably-2-cycle-complete complex which is not 2-cycle-complete. It is orientably-2-cycle-complete since it contains no orientable 2-dimensional cycles, but it is not 2-cycle-complete because it contains a vertex-minimal 2-dimensional cycle that is not 2-complete.

4.2 The Closure of a Pure Simplicial Complex

As mentioned in Section 4.1, the clique complex of a chordal graph removes all 1-dimensional homology from its cycles by turning these cycles into sums of 2-faces. The idea of “filling in” complete subgraphs can easily be extended to simplicial complexes.

Definition 4.2.1 (*d*-**closure).** Let Γ be a simplicial complex with vertex set V where either $\Gamma = \{\emptyset\}$ or Γ is pure of dimension d. We define $\Delta_d(\Gamma)$ to be the simplicial complex with vertex set V and such that

1. $\Gamma \subseteq \Delta_d(\Gamma)$,
2. for all $S \subseteq V$ with $|S| \leq d$, we have $S \in \Delta_d(\Gamma)$, and
3. for any $S \subseteq V$ with $|S| > d+1$, if all $(d+1)$-subsets of S are faces of Γ then S is a face of $\Delta_d(\Gamma)$.

Note that, in the case that $\Gamma = \{\emptyset\}$, we have that $\Delta_d(\Gamma)$ is the pure $(d-1)$-dimensional simplicial complex whose faces are all $S \subseteq V$ such that $|S| \leq d$. In other words, $\Delta_d(\Gamma)$ is the complex $\Lambda_{|V|}^{d-1}$.

It is not difficult to see that $\Delta_d(\Gamma)$ is a simplicial complex. By property 2, all subsets of V of size less than $d+1$ are faces of $\Delta_d(\Gamma)$. Thus all proper subsets of the d-faces of $\Delta_d(\Gamma)$ belong to $\Delta_d(\Gamma)$. For any $F \in \Delta_d(\Gamma)$ with $|F| > d+1$ we know that all of its subsets of size $d+1$ are faces of Γ by definition and so are also faces of $\Delta_d(\Gamma)$ by property 1. If $G \subseteq F$ and $|G| < d+1$ then $G \in \Delta_d(\Gamma)$ by property 2. Otherwise $|G| > d+1$ and G belongs to $\Delta_d(\Gamma)$ only if all of its subsets of size $d+1$ belong to Γ. However, $G \subseteq F$ and all subsets of F of size $d+1$ are faces of Γ. Therefore $G \in \Delta_d(\Gamma)$.

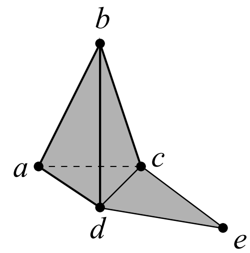
Note that the pure d-skeleton of $\Delta_d(\Gamma)$ is Γ. When $\Gamma = \{\emptyset\}$ the pure d-skeleton of $\Delta_d(\Gamma)$ is empty.

In [12], Emtander refers to $\Delta_d(\Gamma)$ as the **complex of** Γ and in [27], Morales et al. call it the **clique complex of** Γ. We will refer to $\Delta_d(\Gamma)$ as the ***d*-closure** or simply the **closure** of Γ when the dimension is clear. Note that when G is a graph, $\Delta_1(G)$ is equivalent to $\Delta(G)$, the clique complex of G.

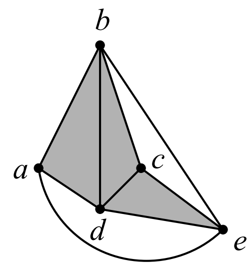
The motivation for properties 2 and 3 in Definition 4.2.1 is algebraic in nature. To generalize Theorem 4.0.1 we are interested in when the Stanley-Reisner ideals of these d-closures have $(d+1)$-linear resolutions. By Remark 2.5.5 the minimal generators of these ideals must all have size $d+1$. Thus the minimal non-faces of these complexes must all have size $d+1$. Therefore the complex must contain all faces of dimension less than d which is ensured by property 2. As well, any non-face of size larger than $d+2$ must contain a minimal non-face of size $d+1$. This follows from property 3. Therefore the minimal generators of the Stanley-Reisner ideal of the d-closure of a simplicial complex will all have degree $d+1$. This relationship is formalized in

Proposition 4.2.4.

In Figure 4.5 we give an example of a pure 2-dimensional complex Γ and its 2-closure $\Delta_2(\Gamma)$. We can see that the 2-closure transforms the hollow tetrahedron on the vertices a, b, c, d into a solid tetrahedron thus eliminating the non-zero 2-dimensional homology in Γ. The minimal non-faces of $\Delta_2(\Gamma)$ are $\{a, b, e\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, e\}$, and $\{b, d, e\}$.



(a) $\Gamma = \langle \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{c, d, e\} \rangle$



(b) $\Delta_2(\Gamma) = \langle \{a, b, c, d\}, \{c, d, e\}, \{a, e\}, \{b, e\} \rangle$

Figure 4.5: 2-closure.

The following lemma explains the results of repeated applications of the closure operation in different dimensions. First recall that the m-closure may only be applied to either an empty complex or to a pure m-dimensional simplicial complex. Therefore after taking the n-closure of a pure n-dimensional simplicial complex we must first take its pure m-skeleton before applying the m-closure operation.

Lemma 4.2.2 (The n-closure is stronger than the m-closure when $n < m$). *Let Γ be either the empty complex or a pure n-dimensional simplicial complex.*

1. *If $m < n$ then $\Delta_m(\Delta_n(\Gamma)^{[m]})$ is a simplex.*

2. *If $m = n$ then $\Delta_m(\Delta_n(\Gamma)^{[m]}) = \Delta_n(\Gamma)$.*

3. *If $m > n$ then $\Delta_m(\Delta_n(\Gamma)^{[m]})^{[t]} = \Delta_n(\Gamma)^{[t]}$ for all $t \geq m$.*

Proof.

1. If $m < n$ then $\Delta_n(\Gamma)^{[m]}$ is m-complete as the n-closure adds all faces of dimension less than n. Therefore by the definition of m-closure the set of all vertices of $\Delta_n(\Gamma)^{[m]}$ is a face of $\Delta_m(\Delta_n(\Gamma)^{[m]})$ and so $\Delta_m(\Delta_n(\Gamma)^{[m]})$ is a simplex.

2. If $m = n$ then by the nature of n-closure

$$\Delta_m(\Delta_n(\Gamma)^{[m]}) = \Delta_n(\Delta_n(\Gamma)^{[n]}) = \Delta_n(\Gamma)$$

   since the pure n-skeleton of $\Delta_n(\Gamma)$ is Γ.

3. Let $m > n$ and let $F \in \Delta_n(\Gamma)^{[t]}$ for some $t \geq m$ so that $|F| = t + 1$. Each subset A of F of size $m + 1 \leq t + 1$ is also a face of $\Delta_n(\Gamma)^{[t]}$ so $A \in \Delta_n(\Gamma)^{[m]}$. Therefore, by the definition of the m-closure, $F \in \Delta_m(\Delta_n(\Gamma)^{[m]})^{[t]}$.

   Conversely, if $F \in \Delta_m(\Delta_n(\Gamma)^{[m]})^{[t]}$ then all subsets of F of size $m + 1 \leq t + 1$ belong to $\Delta_n(\Gamma)^{[m]}$. Therefore $(\Delta_n(\Gamma)^{[m]})_F$ is m-complete. Thus all subsets of F of size $n + 1 < m + 1$ are in $\Delta_n(\Gamma)^{[m]}$ which means they are n-faces of Γ since $\Delta_n(\Gamma)^{[n]} = \Gamma$. Hence by the definition of n-closure $F \in \Delta_n(\Gamma)^{[t]}$.

□

We will see in the following lemma that the closure operation commutes with the operation of taking induced subcomplexes. Since an induced subcomplex of a nonempty pure d-dimensional simplicial complex may not be either empty or pure of dimension d, we must first take the pure d-skeleton of such a subcomplex before applying the d-closure operation.

Lemma 4.2.3 (The closure commutes with taking induced subcomplexes). *Let Γ be either the empty complex or a pure d-dimensional simplicial complex and let $W \subseteq V(\Gamma)$. Then we have $\Delta_d(\Gamma)_W = \Delta_d((\Gamma_W)^{[d]})$.*

Proof. First note that all possible faces of dimension less than d contained in W exist in both complexes, by the nature of d-closure. No other faces of dimension less than d are possible as the vertex set of both complexes is W. Next consider faces of dimension d. Any face of dimension d in $\Delta_d(\Gamma)_W$ is a face of Γ and is contained in W. Such a face is clearly a face of $(\Gamma_W)^{[d]}$ and so of $\Delta_d((\Gamma_W)^{[d]})$. Similarly any face of dimension d in $\Delta_d((\Gamma_W)^{[d]})$ is a face of Γ_W and so is a face of Γ and lies in W. So it is a face of $\Delta_d(\Gamma)$ and $\Delta_d(\Gamma)_W$ in particular. Next consider a face F of dimension greater than d that lies in $\Delta_d(\Gamma)_W$. Such a face lies in W and, by the nature of d-closure, all possible subsets of the face of size $d+1$ are d-faces of Γ. Since $F \subseteq W$, these d-faces are also faces of $(\Gamma_W)^{[d]}$ and so F lies in $\Delta_d((\Gamma_W)^{[d]})$. If F is a face of dimension larger than d in the complex $\Delta_d((\Gamma_W)^{[d]})$ then F lies in W and all possible subsets of the face of size $d+1$ are d-faces of $(\Gamma_W)^{[d]}$. Since these d-faces must be faces of Γ, F lies in $\Delta_d(\Gamma)_W$. Therefore we have $\Delta_d(\Gamma)_W = \Delta_d(\Gamma_W)$. $\square$

In the second half of Theorem 4.0.1, Fröberg states that if the Stanley-Reisner ideal of a simplicial complex has a 2-linear resolution then the complex is equal to the clique complex of its 1-skeleton. We can easily extend this idea to the higher-dimensional closure operation. In fact we only require that the generators of the ideal have the same degree. Recently, one direction of the following proposition was also independently given by Morales et al. in [27, Proposition 4.4].

Proposition 4.2.4 (ideal generated in fixed degree $\iff$ Stanley-Reisner complex is a closure). *The Stanley-Reisner ideal of a simplicial complex Γ is minimally generated in degree $d+1$ if and only if $\Gamma = \Delta_d(\Gamma^{[d]})$.*

Proof. Suppose that $\mathcal{N}(\Gamma)$ is minimally generated in degree $d+1$. Let F be any face of $\Delta_d(\Gamma^{[d]})$. We shall show that $F \in \Gamma$. We have three cases to consider.

(i) First suppose that $|F| < d+1$. If F is not a face of Γ then $x^F \in \mathcal{N}(\Gamma)$. However, $\mathcal{N}(\Gamma)$ is minimally generated by elements of degree $d+1$ and so we have a contradiction. Therefore $F \in \Gamma$.

(ii) Suppose now that $|F| = d+1$. By the definition of d-closure if $F \in \Delta_d(\Gamma^{[d]})$ then $F \in \Gamma^{[d]}$. Hence we must have $F \in \Gamma$ also.

(iii) If $|F| > d+1$ then by the definition of $\Delta_d(\Gamma^{[d]})$ all $(d+1)$-subsets of F are faces of $\Gamma^{[d]} \subseteq \Gamma$. If F is not a face of Γ then we know that $x^F \in \mathcal{N}(\Gamma)$. Hence x^F is divisible by some monomial of degree $d+1$ whose elements make up a non-face of Γ. This is not possible since all $(d+1)$-subsets of F are faces of Γ. Therefore F must be a face of Γ.

We conclude that $\Delta_d(\Gamma^{[d]}) \subseteq \Gamma$.

Now let F be a face of Γ. If $|F| < d+1$ then F is automatically a face of $\Delta_d(\Gamma^{[d]})$. If $|F| = d+1$ then F is a face of $\Gamma^{[d]}$ and so a face of $\Delta_d(\Gamma^{[d]})$. If $|F| > d+1$ then all $(d+1)$-subsets of F are clearly faces of $\Gamma^{[d]}$. By the definition of d-closure we have that F is a face of $\Delta_d(\Gamma^{[d]})$. Hence all faces of Γ are faces of $\Delta_d(\Gamma^{[d]})$. Therefore $\Gamma = \Delta_d(\Gamma^{[d]})$.

Now suppose that $\Gamma = \Delta_d(\Gamma^{[d]})$. Then Γ contains all possible faces of dimension less than d by definition. Also, any subset of vertices of size at least $d+2$, all of whose subsets are faces of Γ, must also be a face of Γ by the definition of d-closure. Hence all minimal non-faces of Γ must have size exactly $d+1$. Thus $\mathcal{N}(\Gamma)$ is minimally generated in degree $d+1$. $\square$

Remark 4.2.5. The clique complexes of graphs are sometimes referred to as **flag complexes**. These are complexes whose minimal non-faces are all of size 2. In [12], Emtander defines a d-**flag complex** to be a simplicial complex whose minimal non-faces all have size d. By Proposition 4.2.4 it is clear that the d-flag complexes are exactly the simplicial complexes which are the $(d-1)$-closure of a simplicial complex. The d-flag complexes have Stanley-Reisner ideals which are generated in degree d and conversely, any square-free monomial ideal generated in degree d has a Stanley-Reisner complex which is a d-flag complex.

By Theorem 4.0.2, in order to show that a square-free monomial ideal has a $(d+1)$-linear resolution we must show that the homology of its Stanley-Reisner complex and the homologies of all induced subcomplexes are zero in all dimensions except $d-1$. By the nature of the closure operation it is trivial to see that, for the d-closure of a complex, all homology groups in dimension less than $d-1$ are zero as all faces of dimension less than d are added by this operation. We will show next that, when the complex is d-chorded, the d-level homology of the closure is also zero.

Lemma 4.2.6 (All d-dimensional cycles in a d-chorded complex are boundaries in the closure). *Let Ω be a d-dimensional cycle in a d-chorded complex Γ. The sum of the d-faces of Ω forms a d-boundary on $V(\Omega)$ in $\Delta_d(\Gamma)$ over $\mathbb{Z}_2$.*

Proof. We will use induction on the number of vertices of Ω. By Proposition 3.2.12, the fewest number of vertices that a d-dimensional cycle can have is $d+2$ and this occurs when $\Omega = \Lambda^d_{d+2}$. In this case $\Delta_d(\Gamma)_{V(\Omega)}$ is a $(d+1)$-simplex and so the sum of the d-faces of Ω forms the d-boundary of $\Delta_d(\Gamma)_{V(\Omega)}$ on $V(\Omega)$.

Now suppose that the statement holds for all d-dimensional cycles with fewer than n vertices and let Ω have n vertices. If Ω is not face-minimal then by Lemma 3.2.8 it can be partitioned into face-minimal d-dimensional cycles. To show that the sum of the d-faces of Ω forms a d-boundary in $\Delta_d(\Gamma)_{V(\Omega)}$ we need only show that the sum of the d-faces of each such face-minimal cycle forms a d-boundary in $\Delta_d(\Gamma)_{V(\Omega)}$ since then we may add them together to show that the original sum is a d-boundary. Therefore, without loss of generality, we may assume that Ω is a face-minimal d-dimensional cycle.

If Ω is d-complete then $\Delta_d(\Gamma)_{V(\Omega)}$ is an $(n-1)$-simplex in $\Delta_d(\Gamma)$ and so the sum of the d-faces of Ω forms a d-boundary on $V(\Omega)$. If Ω is not d-complete then, since Γ is d-chorded, there exists a chord set C of Ω in Γ. Let the d-dimensional cycles associated to C be $\Omega_1, \ldots, \Omega_k$. We know that $|V(\Omega_i)| < |V(\Omega)|$ for all i and so, by induction, the sum of the d-faces of Ω_i forms a d-boundary in $\Delta_d(\Gamma)_{V(\Omega_i)}$ over $\mathbb{Z}_2$. In other words, for each $1 \le i \le k$, there exist $(d+1)$-faces $G^i_1, \ldots, G^i_{\ell_i}$ in $\Delta_d(\Gamma)_{V(\Omega_i)}$ so that, over $\mathbb{Z}_2$,

$$\partial_{d+1}\left(\sum_{j=1}^{\ell_i} G^i_j\right) = \sum (d\text{-faces of } \Omega_i).$$

By properties 2 and 3 of a chord set, over $\mathbb{Z}_2$ we have

$$\sum_{i=1}^{k}\left(\sum (d\text{-faces of } \Omega_i)\right) = \sum (d\text{-faces of } \Omega).$$

Therefore we have

$$\partial_{d+1}\left(\sum_{i=1}^{k}\left(\sum_{j=1}^{\ell_i} G^i_j\right)\right) = \sum_{i=1}^{k}\left(\sum (d\text{-faces of } \Omega_i)\right) = \sum (d\text{-faces of } \Omega)$$

and so the sum of the d-faces of Ω forms a d-boundary on $V(\Omega)$ in $\Delta_d(\Gamma)$ over $\mathbb{Z}_2$. $\square$

Proposition 4.2.7 (**Certain homology groups vanish in the d-closure of a d-chorded complex**)**.** *Let Γ be a d-chorded simplicial complex. Then for any $W \subseteq V(\Gamma)$ and any field k of characteristic 2 we have $\tilde{H}_i(\Delta_d(\Gamma)_W; k) = 0$ for $0 \leq i \leq d-2$ and $i = d$.*

Proof. By Lemma 2.2.2, it is enough to show this for the case $k = \mathbb{Z}_2$.

From the discussion preceding Lemma 4.2.6 and by Lemma 4.2.3 we know that $\tilde{H}_i(\Delta_d(\Gamma)_W; \mathbb{Z}_2) = 0$ for $0 \leq i \leq d-2$ and any $W \subseteq V(\Gamma)$. We need only show that $\tilde{H}_d(\Delta_d(\Gamma)_W; \mathbb{Z}_2) = 0$ for any $W \subseteq V(\Gamma)$. By Lemmas 4.1.6 and 4.2.3 it is enough to show that $\tilde{H}_d(\Delta_d(\Gamma); \mathbb{Z}_2) = 0$.

If $\tilde{H}_d(\Delta_d(\Gamma); \mathbb{Z}_2) \neq 0$ then by Theorem 3.3.3 we know that $\Delta_d(\Gamma)$ contains a d-dimensional cycle, the sum of whose d-faces does not form a d-boundary. This contradicts Lemma 4.2.6 as a d-dimensional cycle in $\Delta_d(\Gamma)$ is a d-dimensional cycle in Γ also. Therefore we must have $\tilde{H}_d(\Delta_d(\Gamma); \mathbb{Z}_2) = 0$. □

We can use Proposition 4.2.7 in a homological characterization of the property of being d-chorded.

Theorem 4.2.8 (d-**chorded** $\iff$ d-**homology vanishes**)**.** *The pure d-dimensional simplicial complex Γ is d-chorded if and only if $\tilde{H}_d(\Delta_d(\Gamma)_W; \mathbb{Z}_2) = 0$ for all $W \subseteq V(\Gamma)$.*

Proof. If Γ is d-chorded then $\tilde{H}_d(\Delta_d(\Gamma)_W; \mathbb{Z}_2) = 0$ for all $W \subseteq V(\Gamma)$ by Proposition 4.2.7.

Conversely, suppose that $\tilde{H}_d(\Delta_d(\Gamma)_W; \mathbb{Z}_2) = 0$ for all $W \subseteq V(\Gamma)$. Let Ω be any face-minimal d-dimensional cycle in Γ which is not d-complete. We know that $\tilde{H}_d(\Delta_d(\Gamma)_{V(\Omega)}; \mathbb{Z}_2) = 0$ and so by Proposition 3.3.1 and Theorem 3.3.3 the sum of the d-faces of Ω must form a d-boundary in $\Delta_d(\Gamma)_{V(\Omega)}$. It follows from Lemma 4.1.2 that Ω has a chord set in $\Delta_d(\Gamma)$ and so also in Γ since $\Delta_d(\Gamma)^{[d]} = \Gamma$. Therefore Γ is d-chorded. □

4.3 A Necessary Condition for a Linear Resolution

Whether or not a monomial ideal has a linear resolution over a field k depends on the characteristic of k. A typical example of this is demonstrated by the triangulation of the real projective plane in Figure 3.3f. The Stanley-Reisner ideal of this simplicial complex has a linear resolution only when the characteristic of k is not 2. This complex is an example of a non-orientable 2-dimensional cycle and it demonstrates non-zero homology in dimension 2 only over fields with characteristic 2.

It turns out, however, that when a square-free monomial ideal has a linear resolution this forces restrictions on the orientable d-dimensional cycles of the associated simplicial complex regardless of the field in question. In the case that the field has characteristic 2, the associated complex is even more restricted. The following theorem gives us a generalization of one direction of Fröberg's theorem over all fields and also a generalization over fields of characteristic 2.

Theorem 4.3.1 (Fröberg's theorem generalized). *Let Γ be a simplicial complex, let k be a field and let $d \geq 1$. If the Stanley-Reisner ideal of Γ has a $(d+1)$-linear resolution over k then $\Gamma = \Delta_d(\Gamma^{[d]})$ and*

1. *$\Gamma^{[d]}$ is orientably-d-cycle-complete*

2. *$\Gamma^{[d]}$ is d-chorded if k has characteristic 2.*

Proof. Since $\mathcal{N}(\Gamma)$ has a $(d+1)$-linear resolution, it is minimally generated in degree $d+1$ by Remark 2.5.5. Therefore by Proposition 4.2.4 we have $\Gamma = \Delta_d(\Gamma^{[d]})$.

1. For a contradiction, let Ω be any orientably-vertex-minimal d-dimensional cycle in $\Gamma^{[d]}$ which is not d-complete. Let the d-faces of Ω be $F_1, \ldots, F_m$ and let $W = \bigcup_{i=1}^m F_i$. Since Ω is not d-complete, $|W| > d+2$ by Proposition 3.2.12. We claim that $\Gamma_W = (\Delta_d(\Gamma^{[d]}))_W$ has dimension d. If $\dim \Gamma_W > d$ then Γ_W must contain a face of dimension $d+1$. By the definition of the d-closure such a face exists only when all of its subsets of size $d+1$ are d-faces of $\Gamma^{[d]}$. Therefore to show that Γ_W has dimension d we must demonstrate that every $(d+2)$-subset of W contains a $(d+1)$-subset which is not a face of $\Gamma^{[d]}$.

   Suppose that there is some $(d+2)$-subset S of W such that all $(d+1)$-subsets of S are faces of $\Gamma^{[d]}$. Then $\Gamma^{[d]}_S$ is d-complete and so, by Proposition 3.2.12, $\Gamma^{[d]}_S$ is an orientable d-dimensional cycle. This is a contradiction since Ω is orientably-vertex-minimal and $S \subsetneq W$. Therefore every $(d+2)$-subset of W must contain a

$(d+1)$-subset which is not a face of $\Gamma^{[d]}$. Therefore by the definition of d-closure $(\Delta_d(\Gamma^{[d]}))_W = \Gamma_W$ cannot contain any faces of size $d+2$ or higher. Hence Γ_W has dimension d.

Since $\dim \Gamma_W = d$, the sum of the d-faces of Ω cannot be a d-boundary. Therefore by Theorem 3.4.2 we know that $\tilde{H}_d(\Gamma_W; k) \neq 0$. This is a contradiction to Theorem 4.0.2 since $\mathcal{N}(\Gamma)$ has a $(d+1)$-linear resolution. Therefore $\Gamma^{[d]}$ has no orientably-vertex-minimal d-dimensional cycles which are not d-complete. Hence $\Gamma^{[d]}$ is orientably-d-cycle-complete.

2. Since $\mathcal{N}(\Gamma)$ has a $(d+1)$-linear resolution over k and $\Gamma = \Delta_d(\Gamma^{[d]})$ we know by Theorem 4.0.2 that for every $W \subseteq V(\Delta_d(\Gamma^{[d]}))$ we have

$$\tilde{H}_d(\Delta_d(\Gamma^{[d]})_W; k) = 0.$$

By Lemma 2.2.2 we have

$$\tilde{H}_d(\Delta_d(\Gamma^{[d]})_W; \mathbb{Z}_2) = 0.$$

Therefore by Theorem 4.2.8, $\Gamma^{[d]}$ is d-chorded. □

Unfortunately the converses to parts 1 and 2 of Theorem 4.3.1 don't hold. We will explore this in more detail in Section 4.5.

It turns out that we can strengthen part 1 of Theorem 4.3.1 and conclude that a linear resolution over an arbitrary field forces a more restrictive combinatorial condition on our simplicial complex.

Theorem 4.3.2 (A linear resolution over any field implies orientably-cycle-complete). *Let Γ be a simplicial complex, let k be a field and let $d \geq 1$. If the Stanley-Reisner ideal of Γ has a $(d+1)$-linear resolution over k then $\Gamma = \Delta_d(\Gamma^{[d]})$ and $\Gamma^{[m]}$ is orientably-m-cycle-complete for all $1 \leq m \leq \dim \Gamma$.*

Proof. By Theorem 4.3.1 we know that $\Gamma = \Delta_d(\Gamma^{[d]})$ and that $\Gamma^{[d]}$ is orientably-d-cycle-complete. By the definition of the d-closure we know that Γ is m-complete for all $m < d$. It is not hard to see, by Proposition 3.2.12, that the only orientably-vertex-minimal m-dimensional cycles in an m-complete simplicial complex are the

m-complete ones on $m+2$ vertices. Therefore $\Gamma^{[m]}$ is orientably-m-cycle-complete for all $m < d$.

Now suppose that $d < m \leq \dim \Gamma$. Let Ω be an orientably-vertex-minimal m-dimensional cycle that is not m-complete. By Lemma 3.4.1 we know that Ω is the support complex of an m-cycle over the field k. Since $\mathcal{N}(\Gamma)$ has a linear resolution over k then $\tilde{H}_m(\Gamma_{V(\Omega)}; k) = 0$ by Theorem 4.0.2. Therefore Ω must be the support complex of an m-boundary in $\Gamma_{V(\Omega)}$. In particular, $\Gamma_{V(\Omega)}$ contains $(m+1)$-faces. The m-skeleton of any such face is the m-dimensional m-complete simplicial complex Λ^m_{m+2}, which is an orientable m-dimensional cycle by Proposition 3.2.12 and which lies on a proper subset of $V(\Omega)$. Hence we have a contradiction to the fact that Ω is orientably-vertex-minimal. Thus Ω must be m-complete and so $\Gamma^{[m]}$ is orientably-m-cycle-complete for all $1 \leq m \leq \dim \Gamma$. $\square$

The following corollary to Theorem 4.3.1 part 2 gives us a necessary condition for a square-free monomial ideal to have a linear resolution over all fields.

Corollary 4.3.3 (A linear resolution over all fields implies d-chorded). *Let I be a square-free monomial ideal with Stanley-Reisner complex Γ. If I has a $(d+1)$-linear resolution over all fields then $\Gamma^{[d]}$ is d-chorded.*

As a consequence of either Theorem 4.3.1 or Corollary 4.3.3 we see that the class of d-chorded complexes contains the class of $(d+1)$-uniform chordal hypergraphs introduced by Woodroofe in [39] and the class of $(d+1)$-uniform generalized chordal hypergraphs introduced by Emtander in [12] since the hypergraphs in these classes have complements whose edge ideals have linear resolutions over all fields. This is equivalent to the simplicial complexes whose facets correspond to the hyperedges of these hypergraphs having closures whose Stanley-Reisner complex has a linear resolution over all fields. However, consider the complex in Figure 4.6 which consists of four hollow tetrahedra “glued together”. This is a 2-chorded simplicial complex, which is not chordal in the sense of Emtander or Woodroofe when considered as a hypergraph, and yet it can be shown that the Stanley-Reisner ideal of the 2-closure of this complex has a 3-linear resolution over all fields. This answers, in the positive, a question posed by Emtander in Section 5 of [12] which asks whether or not there exists a hypergraph (or, equivalently, a simplicial complex) such that the Stanley-Reisner ideal of its closure has a linear resolution over every field, but which is not a generalized chordal hypergraph.

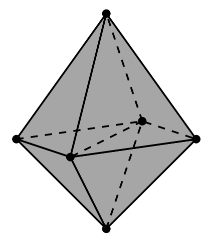


Figure 4.6: 2-chorded simplicial complex which is not "chordal".

4.4 Chorded Complexes and Componentwise Linear Ideals

Recall from Chapter 2 that the property of being componentwise linear is analogous to having a linear resolution for an ideal whose minimal generators are not all of the same degree. The Stanley-Reisner complex of such an ideal will not be the closure of a pure simplicial complex by Proposition 4.2.4. However, we may still observe some combinatorial properties of this non-pure complex itself. We introduce the notion of a **chorded** complex to restrict cycles on all dimensions of the simplicial complex.

Definition 4.4.1 (chorded). A simplicial complex Γ is **chorded** if $\Gamma^{[d]}$ is d-chorded for all $1 \leq d \leq \dim \Gamma$.

Before showing that such complexes result from componentwise linear ideals, we require the following lemma. Recall that $I_{[d]}$ is the ideal generated by the square-free monomials in I of degree d.

Lemma 4.4.2. *Given a simplicial complex* Γ *we have*

$$\Gamma^{[d-1]} = \mathcal{N}(\mathcal{N}(\Gamma)_{[d]})^{[d-1]}.$$

Proof. Let $F \in \text{Facets}(\Gamma^{[d-1]})$. Then $x^F \notin \mathcal{N}(\Gamma)$. Hence $x^F \notin \mathcal{N}(\Gamma)_{[d]}$. Therefore $F \in \mathcal{N}(\mathcal{N}(\Gamma)_{[d]})$, the Stanley-Reisner complex of $\mathcal{N}(\Gamma)_{[d]}$, and $F \in \text{Facets}(\mathcal{N}(\mathcal{N}(\Gamma)_{[d]}))^{[d-1]}$ because $|F| = d$.

Conversely, let $F \in \text{Facets}(\mathcal{N}(\mathcal{N}(\Gamma)_{[d]})^{[d-1]})$. Then $F \in \mathcal{N}(\mathcal{N}(\Gamma)_{[d]})$ and so $x^F \notin \mathcal{N}(\Gamma)_{[d]}$. Since $|F| = d$, this means that $x^F \notin \mathcal{N}(\Gamma)$. Therefore $F \in \Gamma$ and so $F \in \text{Facets}(\Gamma^{[d-1]})$. □

The following theorem is an extension of Corollary 4.3.3.

Theorem 4.4.3 (Componentwise linear implies chorded). *Let Γ be a simplicial complex. If $\mathcal{N}(\Gamma)$ is componentwise linear over every field k then Γ is chorded. In particular, if $\mathcal{N}(\Gamma)$ has a linear resolution over every field then Γ is chorded.*

Proof. Since $\mathcal{N}(\Gamma)_{[d]}$ has a linear resolution over all fields k, we have, by Theorem 4.3.1, that for all d,

$$\mathcal{N}(\mathcal{N}(\Gamma)_{[d]}) = \Delta_{d-1}\left(\mathcal{N}(\mathcal{N}(\Gamma)_{[d]})^{[d-1]}\right)$$

and $\mathcal{N}(\mathcal{N}(\Gamma)_{[d]})^{[d-1]}$ is $(d-1)$-chorded. Hence by Lemma 4.4.2 we know that $\Gamma^{[d-1]}$ is $(d-1)$-chorded for all $d \leq \dim \Gamma + 1$. Hence Γ is chorded.

The second statement follows since every ideal which has a linear resolution is componentwise linear [21, Lemma 8.2.10]. □

The converses to the statements in Theorem 4.4.3 do not hold. We will discuss this further in the next section.

4.5 Which Complexes Result in Ideals with Linear Resolution?

As mentioned in Section 4.3, the converses to parts 1 and 2 of Theorem 4.3.1 do not hold. Consider the following counterexample to the converse of part 1.

Example 4.5.1. The simplicial complex Γ in Figure 3.5 is a triangulated sphere with a hollow tetrahedron suspended within it from four pairwise non-adjacent vertices. The outer sphere is not an orientably-vertex-minimal 2-dimensional cycle as the hollow tetrahedron is an orientable 2-dimensional cycle on a proper subset of its vertices. Thus Γ is orientably-2-cycle-complete as its only orientably-vertex-minimal 2-dimensional cycle, the tetrahedron, is 2-complete. The 2-closure adds all possible 1-faces to Γ and adds the 3-face on the four vertices of the tetrahedron. It is not hard to see that the 2-faces of the outer sphere in $\Delta_2(\Gamma)$ form a homological 2-cycle which is not a 2-boundary in $\Delta_2(\Gamma)$ for any field k and therefore $\tilde{H}_2(\Delta_2(\Gamma);k) \neq 0$. Hence $\mathcal{N}(\Delta_2(\Gamma))$ does not have a linear resolution over any field k by Theorem 4.0.2.

Now consider the following counterexamples to the converse of Theorem 4.3.1 part 2. We will see in Section 5.3 that these two examples lie in a class of counterexamples that arise from a family of simplicial complexes which we call **vertex-partition complexes** and which are the subject of Chapter 5.

Example 4.5.2. Let Γ be the pure 2-dimensional simplicial complex on the vertices $x_0,\dots,x_5$ whose minimal non-faces are $\{x_0,x_1,x_2\}$ and $\{x_3,x_4,x_5\}$. By a computation using Theorem 4.2.8 one can show that the complex Γ is 2-chorded.

We claim that the 2-closure of Γ, $\Delta_2(\Gamma)$, is a 3-dimensional simplicial complex. Clearly $\dim\Delta_2(\Gamma) \geq 3$ as one can see that $\Delta_2(\Gamma)$ contains the 3-dimensional face $\{x_0,x_1,x_3,x_4\}$ whose 3-subsets are 2-dimensional faces of Γ. However, any subset of $\{x_0,\dots,x_5\}$ of size 5 contains either $\{x_0,x_1,x_2\}$ or $\{x_3,x_4,x_5\}$ and these sets are minimal non-faces of Γ and so don't belong to $\Delta_2(\Gamma)$ since $\Delta_2(\Gamma)^{[2]} = \Gamma$ by definition. Thus $\Delta_2(\Gamma)$ has no 4-faces and therefore $\dim\Delta_2(\Gamma) = 3$.

One can show that the pure 3-skeleton of $\Delta_2(\Gamma)$ is a 3-dimensional cycle, the sum of whose 3-faces cannot be a 3-boundary as $\Delta_2(\Gamma)$ is only 3-dimensional. Therefore by Theorem 3.3.3 we have $\tilde{H}_3(\Delta_2(\Gamma);\mathbb{Z}_2) \neq 0$ and so the Stanley-Reisner ideal of $\Delta_2(\Gamma)$ does not have a linear resolution over $\mathbb{Z}_2$ by Theorem 4.0.2.

Example 4.5.3. Let Γ be the pure 3-dimensional simplicial complex on the vertices $x_0,\dots,x_6$ that is obtained from Λ_7^3 by removing the following five facets:

$$\{x_0,x_1,x_5,x_6\} \quad \{x_0,x_2,x_5,x_6\} \quad \{x_0,x_3,x_5,x_6\} \quad \{x_0,x_4,x_5,x_6\} \quad \{x_1,x_2,x_3,x_4\}$$

The facets of Γ are:

$$\begin{array}{lllll}
\{x_0,x_1,x_2,x_3\} & \{x_0,x_1,x_2,x_4\} & \{x_0,x_1,x_2,x_5\} & \{x_0,x_1,x_2,x_6\} & \{x_0,x_1,x_3,x_4\} \\
\{x_0,x_1,x_3,x_5\} & \{x_0,x_1,x_3,x_6\} & \{x_0,x_1,x_4,x_5\} & \{x_0,x_1,x_4,x_6\} & \{x_0,x_2,x_3,x_4\} \\
\{x_0,x_2,x_3,x_5\} & \{x_0,x_2,x_3,x_6\} & \{x_0,x_2,x_4,x_5\} & \{x_0,x_2,x_4,x_6\} & \{x_0,x_3,x_4,x_5\} \\
\{x_0,x_3,x_4,x_6\} & \{x_1,x_2,x_3,x_5\} & \{x_1,x_2,x_3,x_6\} & \{x_1,x_2,x_4,x_5\} & \{x_1,x_2,x_4,x_6\} \\
\{x_1,x_2,x_5,x_6\} & \{x_1,x_3,x_4,x_5\} & \{x_1,x_3,x_4,x_6\} & \{x_1,x_3,x_5,x_6\} & \{x_1,x_4,x_5,x_6\} \\
\{x_2,x_3,x_4,x_5\} & \{x_2,x_3,x_4,x_6\} & \{x_2,x_3,x_5,x_6\} & \{x_2,x_4,x_5,x_6\} & \{x_3,x_4,x_5,x_6\}
\end{array}$$

By similar arguments to those given in Example 4.5.2, the complex Γ is 3-chorded but the Stanley-Reisner ideal of the 4-dimensional simplicial complex $\Delta_3(\Gamma)$ does not have a linear resolution over $\mathbb{Z}_2$. In fact, the pure 4-skeleton of $\Delta_3(\Gamma)$ is a 4-dimensional cycle and $\tilde{H}_4(\Delta_3(\Gamma); \mathbb{Z}_2) \neq 0$.

It turns out that all counterexamples to the converse of Theorem 4.3.1 part 2 share a specific property. The d-closures of these d-chorded complexes contain face-minimal non-n-complete n-dimensional cycles having complete 1-skeletons and having no chord sets, where $n > d$. It is this feature which prevents the desired linear resolution by introducing homology on a level higher than the dimension of the original complex. We will prove this in Section 4.6.

Although it is not the case that the d-closures of all d-chorded complexes have Stanley-Reisner ideals with linear resolutions over fields of characteristic 2, this does hold for the smaller class of d-dimensional forests. As we will see in the proof of the following theorem, the d-closures of these complexes have no n-dimensional cycles for $n \geq d$. It follows that, over fields having characteristic 2, all upper-level homologies in the closures of these complexes are zero.

Theorem 4.5.4 (All d-dimensional forests result in ideals with linear resolutions). *If Γ is a d-dimensional forest then $\mathcal{N}(\Delta_d(\Gamma))$ has a $(d+1)$-linear resolution over any field of characteristic 2.*

Proof. By Theorem 4.0.2 we need to show that, for any field k of characteristic 2, $\tilde{H}_i(\Delta_d(\Gamma)_W; k) = 0$ for all $i \neq d-1$ and all $W \subseteq V(\Gamma)$. However, it is not hard to see that the pure d-skeleton of any induced subcomplex of a d-dimensional forest is also a d-dimensional forest and so by Lemma 4.2.3 we need only show that $\tilde{H}_i(\Delta_d(\Gamma); k) = 0$ for all $i \neq d-1$.

Since $\Delta_d(\Gamma)$ has all possible faces of dimension less than d, by its definition, we know that $\tilde{H}_i(\Delta_d(\Gamma); k) = 0$ for all $i < d-1$. Since Γ has no d-dimensional cycles, neither does $\Delta_d(\Gamma)$ and so by Theorem 3.3.3 we must have $\tilde{H}_d(\Delta_d(\Gamma); k) = 0$.

We claim that $\Delta_d(\Gamma)$ has no faces of dimension greater than d. If $\Delta_d(\Gamma)$ contains a face of dimension greater than d then it must contain a face of dimension $d+1$. Such a face exists in $\Delta_d(\Gamma)$ only when all subsets of its vertices of size $d+1$ are faces of

Γ. But these d-faces of Γ then form a d-dimensional cycle in Γ by Proposition 3.2.12. This is a contradiction since Γ contains no d-dimensional cycles and so $\Delta_d(\Gamma)$ contains no faces of dimension greater than d. Hence it must be the case that $\tilde{H}_i(\Delta_d(\Gamma); k) = 0$ for all $i > d$.

Therefore $\tilde{H}_i(\Delta_d(\Gamma); k) = 0$ for all $i \neq d-1$ and so, by Theorem 4.0.2, $\mathcal{N}(\Delta_d(\Gamma))$ has a $(d+1)$-linear resolution over k. $\square$

The corresponding statement to Theorem 4.5.4 for a general field does not hold. For example, the triangulation of the mod 3 Moore space shown in Figure 3.9 in Chapter 3 is a 2-dimensional tree, but the Stanley-Reisner ideal of its 2-closure does not have a linear resolution over fields of characteristic 3. However, we have not found any examples of d-dimensional forests whose d-closures have Stanley-Reisner ideals without linear resolutions over fields of characteristic 0.

As mentioned earlier, the converses to both statements in Theorem 4.4.3 are also not true. The following is a counterexample to both.

Example 4.5.5. Let Γ be the triangulation of the mod 3 Moore space from Figure 3.9 in Chapter 3. The pure 1-skeleton of this simplicial complex is 1-chorded because it has a perfect elimination ordering given by $f, d, e, x, y, z, a, b, c$ and so is chordal by Lemma 2.3.2. The pure 2-skeleton of Γ is 2-chorded because it does not contain any 2-dimensional cycles. Therefore Γ is chorded, but $\mathcal{N}(\Gamma)$ is not componentwise linear over every field as one can show that $\mathcal{N}(\Gamma)_{[3]}$ does not have a 3-linear resolution over fields of characteristic 3.

Recently, in [27], Morales et al. introduced the idea of a simplicial complex that is minimal to linearity.

Definition 4.5.6 (Morales et al. [27])**.** A pure d-dimensional simplicial complex Γ is **minimal to** $(d+1)$**-linearity over the field** k if the following conditions hold:

1. $\dim \Delta_d(\Gamma) = d$
2. $\mathcal{N}(\Delta_d(\Gamma))$ does not have a $(d+1)$-linear resolution over k
3. for every proper pure d-dimensional subcomplex Γ' of Γ, $\mathcal{N}(\Delta_d(\Gamma'))$ has a linear resolution over k.

It is shown in [27] that (non-d-complete) orientable pseudo d-manifolds are minimal to $(d+1)$-linearity over all fields and that (non-d-complete) non-orientable pseudo d-manifolds are minimal to $(d+1)$-linearity only when the characteristic of the field is equal to 2. We are able to extend this idea to d-dimensional cycles.

Proposition 4.5.7 (Face-minimal d-dimensional cycles are minimal to linearity in characteristic 2). *Any face-minimal d-dimensional cycle that is not d-complete is minimal to $(d+1)$-linearity over all fields of characteristic 2.*

Proof. Let k be a field of characteristic 2 and let Ω be a face-minimal d-dimensional cycle that is not d-complete. Since Ω is face-minimal, it does not contain any d-complete subcomplexes by Proposition 3.2.12. Therefore, by the definition of the d-closure, we have $\dim \Delta_d(\Omega) = d$. By Theorem 3.3.3, $\tilde{H}_d(\Delta_d(\Omega); k) \neq 0$ and so by Theorem 4.0.2, $\mathcal{N}(\Delta_d(\Omega))$ does not have a linear resolution over k.

Let Ω' be any proper pure d-dimensional subcomplex of Ω. Then Ω' contains no d-dimensional cycles since Ω is face-minimal. Hence Ω' is a d-dimensional forest and so, by Theorem 4.5.4, $\mathcal{N}(\Delta_d(\Omega'))$ has a linear resolution over k. $\square$

We cannot conclude, however, that non-d-complete face-minimal d-dimensional cycles are minimal to $(d+1)$-linearity over an arbitrary field k. For example, the triangulation, Γ, of the real projective plane shown in Figure 3.3f is a face-minimal 2-dimensional cycle. However, if the field k has characteristic not equal to 2 then $\mathcal{N}(\Delta_2(\Gamma))$ has a $(d+1)$-linear resolution over k, as previously discussed. Thus condition 2 in Definition 4.5.6 is not satisfied and so Γ is not minimal to 3-linearity over k.

One might hope that non-d-complete, orientably-face-minimal d-dimensional cycles would be minimal to $(d+1)$-linearity over fields of characteristic 0, but this remains an open question.

4.6 A Criterion for Linear Resolution in Characteristic 2

As we have seen, for a square-free monomial ideal to have a $(d+1)$-linear resolution its Stanley-Reisner complex and its induced subcomplexes must have vanishing simplicial homology in all but dimension $d-1$. Theorem 4.3.1 shows that when this holds over fields of characteristic 2 the pure d-skeleton of the complex is d-chorded.

Conversely, in order to show that all simplicial complexes in a particular class have Stanley-Reisner ideals with linear resolutions over fields of characteristic 2, we must show that the simplicial homology of these complexes and their induced subcomplexes vanishes in the right dimensions. As we can see from Examples 4.5.2 and 4.5.3 it is not necessarily the case that the upper-level homology groups of the d-closure of a d-chorded complex vanish. This is in contrast to the 1-dimensional case where the condition of being chordal on the 1-skeleton of the complex forces all upper-level homology to disappear. In these higher-dimensional examples, the d-closure of the complex has a pure m-skeleton which is not m-chorded for some $m > d$. When we require these m-skeletons to be m-chorded we obtain a necessary and sufficient condition for linear resolution over fields of characteristic 2.

Recall from Section 2.1 that the d-complement of the pure d-dimensional simplicial complex Γ is denoted $\overline{\Gamma}_d$.

Theorem 4.6.1 (linear resolution in characteristic 2 $\iff$ chorded). *Let k be a field of characteristic 2 and let I be an ideal in $k[x_1, \ldots, x_n]$ that is generated by square-free monomials of degree $d+1$. The following are equivalent:*

a) I has a linear resolution.

b) $\mathcal{N}(I)$ is chorded.

c) $\mathcal{N}(I)^{[m]}$ is m-chorded for all $m \geq d$.

d) $\Delta_d(\overline{\mathcal{F}(I)}_d)$ is chorded.

e) $\Delta_d(\overline{\mathcal{F}(I)}_d)^{[m]}$ is m-chorded for all $m \geq d$.

Proof. Let $\Gamma = \mathcal{N}(I)$ and let $\Upsilon = \overline{\mathcal{F}(I)}_d$. Also notice that since the existence of a linear resolution is solely determined by the vanishing of certain homology groups, I has a linear resolution over any field of characteristic 2 if and only if it has a linear resolution over $\mathbb{Z}_2$, by Lemma 2.2.2.

a) $\Rightarrow$ b) If I has a linear resolution then by Theorem 4.3.1 we know that $\Gamma = \Delta_d(\Gamma^{[d]})$ and $\Gamma^{[d]}$ is d-chorded. We also know that Γ is m-complete for all $m < d$ by the definition of d-closure. Therefore it follows from Proposition 4.1.4 that $\Gamma^{[m]}$ is m-chorded for all $m < d$.

Let $m > d$ and let Ω be any face-minimal, non-m-complete m-dimensional cycle in $\Gamma^{[m]}$. By Proposition 3.3.1 we know that Ω is the support complex of a homological m-cycle over $\mathbb{Z}_2$. The ideal I has a linear resolution over $\mathbb{Z}_2$ and so we know that $\tilde{H}_m(\Gamma_{V(\Omega)}; \mathbb{Z}_2) = 0$ by Theorem 4.0.2. Thus Ω is also the support complex of an m-boundary of faces of $\Gamma_{V(\Omega)}$. Therefore by Lemma 4.1.2 we know that Ω has a chord set in $\Gamma_{V(\Omega)}$. Hence $\Gamma^{[m]}$ is m-chorded. Therefore $\Gamma^{[m]}$ is m-chorded for all $1 \leq m \leq \dim \Gamma$ and so Γ is chorded.

b) $\Rightarrow$ c) This follows from the definition of chorded.

c) $\Rightarrow$ a) Suppose that $\Gamma^{[m]}$ is m-chorded for all $m \geq d$. Since I is generated by square-free monomials of degree $d+1$ we have $\Gamma = \Delta_d(\Gamma^{[d]})$ by Proposition 4.2.4. Therefore by Proposition 4.2.7 we know that for all $W \subseteq V(\Gamma)$ we have $\tilde{H}_i(\Gamma_W; k) = 0$ for $0 \leq i \leq d-2$ and $i = d$.

Let $m > d$ and let $W \subseteq V(\Gamma)$. We would like to show that $\tilde{H}_m(\Gamma_W; k) = 0$. By assumption $\Gamma^{[m]}$ is m-chorded. Therefore by Proposition 4.2.7 we know that $\tilde{H}_m(\Delta_m(\Gamma^{[m]})_W; k) = 0$. Furthermore, by part 3 of Lemma 4.2.2 we have

$$\Delta_m(\Gamma^{[m]})^{[t]} = \Delta_m(\Delta_d(\Gamma^{[d]})^{[m]})^{[t]} = \Delta_d(\Gamma^{[d]})^{[t]}$$

for all $t \geq m$. Thus the m-faces and the $m+1$-faces of $\Delta_m(\Gamma^{[m]})_W$ and $\Gamma_W = \Delta_d(\Gamma^{[d]})_W$ are equivalent. Therefore we have

$$\tilde{H}_m(\Gamma_W; k) = \tilde{H}_m(\Delta_m(\Gamma^{[m]})_W; k) = 0$$

for all $m > d$. Consequently $\tilde{H}_m(\Gamma_W; k) = 0$ for all $m \neq d-1$. Hence I has a $(d+1)$-linear resolution by Theorem 4.0.2.

b) $\Leftrightarrow$ d) It is easy to see that the d-complement of $\mathcal{F}(I)$ is equal to the pure d-skeleton of $\mathcal{N}(I) = \Gamma$. Thus $\Upsilon = \Gamma^{[d]}$ and so $\Delta_d(\Upsilon)$ is chorded if and only $\Delta_d(\Gamma^{[d]}) = \mathcal{N}(I)$ is chorded.

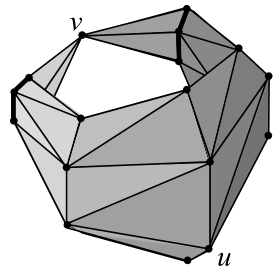
c) $\Leftrightarrow$ e) As before, $\Upsilon = \Gamma^{[d]}$ and so $\Delta_d(\Upsilon)^{[m]}$ is m-chorded for all $m \geq d$ if and only $\Delta_d(\Gamma^{[d]})^{[m]} = \Gamma^{[m]} = \mathcal{N}(I)^{[m]}$ is m-chorded for all $m \geq d$. $\square$

It is interesting to note that if the square-free monomial ideal I has a linear resolution over a field of characteristic 2 and $\dim \mathcal{N}(I) = n$ then $\mathcal{N}(I)^{[n]}$ is not only n-chorded but is, in fact, an n-dimensional forest. This follows since any n-dimensional cycle in $\mathcal{N}(I)$ would lead to non-zero n-dimensional homology in $\mathcal{N}(I)$ by Theorem 3.3.3 and this would contradict Theorem 4.0.2. By Theorem 4.6.1 we can conclude that for any square-free monomial ideal I whose minimal generators all have the same degree, if $\mathcal{N}(I)$ is chorded then $\mathcal{N}(I)^{[n]}$ is an n-dimensional forest where $n = \dim \mathcal{N}(I)$. Examining this fact independently from Theorem 4.6.1 it is not immediately obvious why this would be the case. It turns out to be due to the intricate relationship between the d-closure operation and the definition of d-chorded when I is minimally generated in degree $d+1$ as we see in the proof of the following proposition.

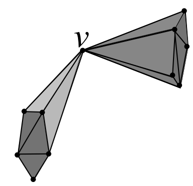
Proposition 4.6.2 (When the closure is chorded its top dimension is a forest). *Let I be a square-free monomial ideal whose minimal generators all have degree $d+1$. If $\mathcal{N}(I)$ is chorded then $\mathcal{N}(I)^{[n]}$ is an n-dimensional forest where $n =$* $\dim \mathcal{N}(I)$

Proof. Suppose that $\mathcal{N}(I)^{[n]}$ is not an n-dimensional forest and so contains an n-dimensional cycle Ω. Then, because $\mathcal{N}(I)^{[n]}$ is n-chorded, Ω is forced, by the inductive nature of the definition, to contain an n-complete cycle on some vertex set $S \subseteq V(\Omega)$. Such an n-complete cycle clearly contains all possible faces of dimension d on the set S which requires $\mathcal{N}(I)_S$ to be a simplex as $\mathcal{N}(I) = \Delta_d(\mathcal{N}(I)^{[d]})$ by Proposition 4.2.4. This simplex has dimension greater than n by Proposition 3.2.12 and so we have a contradiction. $\square$

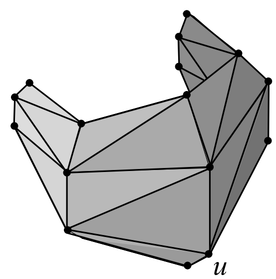
The condition for $(d+1)$-linear resolution in Theorem 4.6.1 requires ensuring that every non-m-complete, face-minimal m-dimensional cycle in $\mathcal{N}(I)^{[m]}$ has a chord set for all $m \geq d$. However, our next result shows that in most cases assuming that $\mathcal{N}(I)^{[d]}$ is d-chorded suffices. The only possible obstruction to this implication is the presence of an m-dimensional cycle of a very special and highly-connected form. In general we expect these types of cycles to occur infrequently. Thus to check for a linear resolution we need only verify that $\mathcal{N}(I)^{[d]}$ is d-chorded and that any cycles of this special nature have chord sets. For an illustration of the technique used in the proof of the following theorem see Figure 4.7.



(a) The 2-dimensional cycle Ω with 1-dimensional cycles Φ_1 and Φ_2 shown in bold



(b) The 2-dimensional cycles Ω_1 and Ω_2 joined by v



(c) The 2-dimensional cycle $\Omega_3 = \langle H_1, \ldots, H_s \rangle$

Figure 4.7: Construction used in the proof of Theorem 4.6.3.

Theorem 4.6.3 (Chordedness can be transferred upwards in the closure). *Let Γ be a d-chorded simplicial complex. Then $\Delta_d(\Gamma)$ is chorded if and only if for all $m > d$ each 1-complete, face-minimal, non-m-complete m-dimensional cycle in $\Delta_d(\Gamma)$ has a chord set in $\Delta_d(\Gamma)$.*

Proof. If $\Delta_d(\Gamma)$ is chorded then all face-minimal, non-m-complete m-dimensional cycles in $\Delta_d(\Gamma)$ have chord sets in $\Delta_d(\Gamma)$ for all m by definition.

Now suppose that for all $m > d$ each 1-complete, face-minimal, non-m-complete m-dimensional cycle in $\Delta_d(\Gamma)$ has a chord set in $\Delta_d(\Gamma)$. We would like to show that $\Delta_d(\Gamma)$ is chorded. By the nature of the d-closure we know that $\Delta_d(\Gamma)$ is t-complete for all $t < d$. Thus $\Delta_d(\Gamma)^{[t]}$ is t-chorded for all $t < d$ by Proposition 4.1.4.

For the remaining cases we will use induction on t. When $t = d$ we have $\Delta_d(\Gamma)^{[d]} = \Gamma$. Since Γ is d-chorded by assumption this proves the base case.

Now suppose that $t > d$ and that we know $\Delta_d(\Gamma)^{[n]}$ is n-chorded for all $n < t$. Let Ω be a face-minimal t-dimensional cycle that is not t-complete in $\Delta_d(\Gamma)^{[t]}$. We would like to show that Ω has a chord set in $\Delta_d(\Gamma)^{[t]}$. If Ω is 1-complete then by assumption Ω has a chord set in $\Delta_d(\Gamma)^{[t]}$, and so we may assume that Ω is not 1-complete. Then there exist $u, v \in V(\Omega)$ such that u and v are not contained in the same t-face of Ω.

Let $F_1, \ldots, F_k$ be the t-faces of Ω containing v. By Proposition 3.2.3 we know that the $(t-1)$-path-connected components of $\langle F_1 \backslash \{v\}, \ldots, F_k \backslash \{v\} \rangle$ are $(t-1)$-dimensional cycles. Call these cycles $\Phi_1, \ldots, \Phi_m$. For each $i \in \{1, \ldots, m\}$ let $P_i \subseteq \{1, \ldots, k\}$ be such that $F_j \setminus \{v\} \in \Phi_i$ if and only if $j \in P_i$. Since for each j the face $F_j \setminus \{v\}$ must belong to exactly one of $\Phi_1, \ldots, \Phi_m$, the sets $P_1, \ldots, P_m$ form a partition of $\{1, \ldots, k\}$.

The complex $\Delta_d(\Gamma)^{[t-1]}$ is $(t-1)$-chorded by assumption. Therefore, by Lemma 4.2.6, the sum of the $(t-1)$-faces of Φ_i form a $(t-1)$-boundary in $\Delta_{t-1}(\Delta_d(\Gamma)^{[t-1]})$ on $V(\Phi_i)$ over $\mathbb{Z}_2$ for each i and therefore in $\Delta_d(\Gamma)$ by Lemma 4.2.2 since $t - 1 \geq d$.

Hence for each $1 \leq i \leq m$ there exist t-faces $A^i_1, ..., A^i_{\ell_i}$ in $\Delta_d(\Gamma)_{V(\Phi_i)}$ such that

$$\partial_t \left(\sum_{j=1}^{\ell_i} A^i_j \right) = \sum_{j \in P_i} F_j \setminus \{v\}. \tag{4.1}$$

Without loss of generality we may assume that the choice of $A^i_1, ..., A^i_{\ell_i}$ is minimal in the sense that for no proper subset of $A^i_1, ..., A^i_{\ell_i}$ is (4.1) satisfied. Let Ω_i be the simplicial complex whose facets are $\{F_j | j \in P_i\} \cup \{A^i_1, ..., A^i_{\ell_i}\}$.

By Proposition 3.3.2 we know that for $1 \leq i \leq m$ each Ω_i is a t-dimensional cycle. As well we have $V(\Omega_i) \subsetneq V(\Omega)$ as $u \notin V(\Omega_i)$. Since Ω is a face-minimal t-dimensional cycle, each Ω_i must contain at least one t-face which is not in Ω. We collect all of these t-faces in the nonempty set C:

$$C = \{A^i_j \mid 1 \leq i \leq m,\ 1 \leq j \leq \ell_i,\ A^i_j \notin \Omega\}.$$

We would like to show that C is a chord set of Ω in $\Delta_d(\Gamma)^{[t]}$.

Consider the collection of t-faces in Ω and those in $\Omega_1, \ldots, \Omega_m$ with repeats. Let $H_1, \ldots, H_s$ be the t-faces in this collection which appear an odd number of times so that over $\mathbb{Z}_2$ we have

$$\sum(t\text{-faces of }\Omega) + \sum_{i=1}^{m}\sum(t\text{-faces of }\Omega_i) = \sum_{i=1}^{s} H_i. \tag{4.2}$$

Since Ω and $\Omega_1, \ldots, \Omega_m$ are all t-dimensional cycles, by Proposition 3.3.1 they correspond to homological t-cycles over $\mathbb{Z}_2$. Therefore by (4.2) over $\mathbb{Z}_2$ we have,

$$\partial_t\left(\sum_{i=1}^{s} H_i\right) = \partial_t\left(\sum(t\text{-faces of }\Omega)\right) + \sum_{i=1}^{m}\partial_t\left(\sum(t\text{-faces of }\Omega_i)\right) = 0.$$

Hence the t-path-connected components of the simplicial complex $\langle H_1, \ldots, H_s\rangle$ are t-dimensional cycles by Proposition 3.3.1. Call these cycles $\Omega_{m+1}, \ldots, \Omega_M$. We would like to show that our set C is a chord set that breaks Ω into the cycles $\Omega_1, \ldots, \Omega_M$. By (4.2), after rearranging the sums, over $\mathbb{Z}_2$ we have

$$\sum(t\text{-faces of }\Omega) = \sum_{i=1}^{M}\sum(t\text{-faces of }\Omega_i).$$

By noticing that the set C consists of exactly those t-faces on the right-hand side of this equation which do not belong to Ω we can see that properties 2 and 3 of a chord set hold for C. Also, it is clear from our construction that all t-faces of both Ω and of C appear in at least one of the Ω_i's. Therefore property 1 of a chord set holds for the set C.

Now since none of $\Omega_1, \ldots, \Omega_m$ contain u by construction we have $|V(\Omega_i)| < |V(\Omega)|$ for all $1 \leq i \leq m$. We would like to show that none of $\Omega_{m+1}, \ldots, \Omega_M$ contain v. Recall that $\Phi_1, \ldots, \Phi_m$ are the $(t-1)$-path-connected components of $\langle F_1 \setminus \{v\}, \ldots, F_k \setminus \{v\}\rangle$ and so no two such distinct components could share a face of the form $F_i \setminus \{v\}$. Thus each face F_i appears in only one of the cycles $\Omega_1, \ldots, \Omega_m$. Each such F_i is also a face of Ω and so by our choice of $H_1, \ldots, H_s$ we know that we cannot have $F_i = H_j$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, s\}$. Therefore, by the construction of the cycles $\Omega_{m+1}, \ldots, \Omega_M$ we know that none of the F_i's appear in any of these cycles. Recall that $F_1, \ldots, F_k$ are the only t-faces of Ω that contain v and none of the t-faces of C contain v since they are subsets of $\bigcup_{i=1}^{m} V(\Phi_i)$. It follows that none of $\Omega_{m+1}, \ldots, \Omega_M$ contain v. This implies that $|V(\Omega_i)| < |V(\Omega)|$ for all $m+1 \leq i \leq M$. Thus property 4 of a chord set is also satisfied by C and hence $\Delta_d(\Gamma)^{[t]}$ is t-chorded. Hence $\Delta_d(\Gamma)$ is chorded. $\square$

As a consequence of Theorems 4.6.1 and 4.6.3 we have the following theorem.

Theorem 4.6.4 (combinatorial criterion for a linear resolution). *Let I be generated by square-free monomials of degree $d+1$. Then I has a linear resolution over any field of characteristic 2 if and only if $\mathcal{N}(I)^{[d]}$ is d-chorded and for $m > d$ each 1-complete, face-minimal, non-m-complete m-dimensional cycle in $\mathcal{N}(I)$ has a chord set in $\mathcal{N}(I)$.*

□

From Theorems 4.3.1 and 4.6.4 we conclude that for any square-free monomial ideal I generated in degree $d+1$, if I has no linear resolution then either $\mathcal{N}(I)^{[d]}$ is not d-chorded or for some $m > d$ there exists a 1-complete face-minimal non-m-complete m-dimensional cycle in $\mathcal{N}(I)$ which has no chord set. Examples 4.5.2 and 4.5.3 give instances of the complex $\mathcal{N}(I)$ in the latter case. In Chapter 5 we will explore a class of examples from this case in more detail.

In the next section we prove that in the 1-dimensional case, obstructions to linear resolution of this type do not exist. In particular if $\Gamma^{[1]}$ is 1-chorded then in $\Delta_1(\Gamma^{[1]})$ all 1-complete m-dimensional cycles lie in m-complete induced subcomplexes which are m-chorded and consequently such cycles have chord sets. This leads us to a new, combinatorial proof of Theorem 4.0.1 in characteristic 2.

4.7 A New Proof of Fröberg's Theorem in Characteristic 2

As mentioned at the beginning of this chapter, in the proof of Theorem 4.0.1 in [16], Fröberg shows that the simplicial homology of the clique complex of a chordal graph vanishes on all levels greater than zero. He does so inductively by dismantling the graph at a complete subgraph and then applying the Mayer-Vietoris sequence on the resulting dismantled clique complex. This is a very clean and elegant method for demonstrating that all upper-level homologies are zero. However, this technique gives no intuitive sense as to why it should be the case that filling in complete subgraphs of a chordal graph produces a simplicial complex with no homology on higher levels. A chordal graph may contain complete subgraphs on any number of vertices and so the clique complex may have faces of any dimension. The question is why the addition of these higher-dimensional faces doesn't introduce any new homology. The following theorem, together with Proposition 4.2.7, answers this question, from a combinatorial point of view, in the case that the field of interest has characteristic 2.

Theorem 4.7.1 (chordal $\iff$ chorded clique complex). *A graph G is chordal if and only if $\Delta_1(G)$ is chorded.*

Proof. Since G is chordal it is 1-chorded and so $\Delta_1(G)^{[1]} = G$ is 1-chorded. Let $m > 1$ and let Ω be a 1-complete, face-minimal m-dimensional cycle in $\Delta_1(G)$ that is not m-complete. Then $\Delta_1(G)_{V(\Omega)}$ is a $(|V(\Omega)| - 1)$-simplex by the definition of the 1-closure and hence $\Delta_1(G)^{[m]}_{V(\Omega)}$ is m-chorded by Remark 4.1.5. Therefore Ω has a chord set in $\Delta_1(G)$ and by Theorem 4.6.3, $\Delta_1(G)$ is chorded.

Conversely, $\Delta_1(G)^{[1]}$ is 1-chorded and $\Delta_1(G)^{[1]} = G$ so G is a chordal graph. □

Comparing Theorem 4.7.1 to Theorem 4.6.3 we have another specific example of how the 1-dimensional case is more straightforward than the higher-dimensional case.

Theorem 4.7.1 gives us a new proof of Fröberg's theorem over fields of characteristic 2 using the notion of d-chorded complexes.

Theorem 4.7.2 (Fröberg's theorem in characteristic 2). *If G is chordal then $\mathcal{N}(\Delta_1(G))$ has a 2-linear resolution over any field of characteristic 2. Conversely, if $\mathcal{N}(\Gamma)$ has a 2-linear resolution over a field of characteristic 2 then $\Gamma = \Delta_1(\Gamma^{[1]})$ and $\Gamma^{[1]}$ is chordal.*

Proof. Let G be a chordal graph and let k be a field of characteristic 2. To show that $\mathcal{N}(\Delta_1(G))$ has a 2-linear resolution over k we need to show that $\tilde{H}_i(\Delta_1(G)_W; k) = 0$ for all $i \geq 1$ and all $W \subseteq V(G)$. Let $d \geq 1$ and let $W \subseteq V(G)$. We know by Theorem 4.7.1 that $\Delta_1(G)^{[d]}$ is d-chorded. Therefore by Proposition 4.2.7 we know that $\tilde{H}_d(\Delta_d(\Delta_1(G)^{[d]})_W; k) = 0$. By parts 2 and 3 of Lemma 4.2.2 we have $\Delta_d(\Delta_1(G)^{[d]})^{[t]} = \Delta_1(G)^{[t]}$ for all $t \geq d$. Therefore the complexes $\Delta_1(G)$ and $\Delta_d(\Delta_1(G)^{[d]})$ have the same d-faces and the same $(d+1)$-faces. Since these are the only faces which are taken into account when computing d-dimensional homology, we must have that $\tilde{H}_d(\Delta_d(\Delta_1(G)^{[d]})_W; k) = \tilde{H}_d(\Delta_1(G)_W; k)$. Hence $\tilde{H}_d(\Delta_1(G)_W; k) = 0$ for all $d \geq 1$ and all $W \subseteq V(G)$. Therefore $\mathcal{N}(\Delta_1(G))$ has a 2-linear resolution by Theorem 4.0.2.

The converse follows by Theorem 4.3.1 and by the equivalence of the notions of chordal and 1-chorded. $\square$

Chapter 5

Vertex-Partition Complexes

In Section 4.5 we discussed d-chorded simplicial complexes whose d-closure had a Stanley-Reisner ideal without linear resolution over fields of characteristic 2. These complexes all contain higher-dimensional cycles which have no chord sets and have complete 1-skeletons by Theorem 4.6.4. It would be an interesting point to try and characterize such counterexamples to the converse of Theorem 4.3.1 part 2 and to determine what combinatorial feature on the d-dimensional level is causing such cycles to exist in higher dimensions.

In examining such counterexamples on small sets of vertices one distinct pattern emerges in most of the cases. The salient feature in these examples is that these d-chorded complexes are the pure d-skeletons of a simplicial complex whose minimal non-faces form a partition of the vertex set and for which no part in the partition is larger than $d+1$. In this chapter we study this class of simplicial complexes to determine the barrier to linear resolution and to examine the features, both combinatorial and homological, associated to this class of complexes.

Interestingly, we discover that these complexes are all simplicial spheres. This leads us to understand why they result in ideals without linear resolution. Furthermore, these spheres are highly-connected and have relatively few vertices for their given dimension. In addition, almost all induced subcomplexes of these spheres are acyclic over all fields. This is in contrast to our general intuition about triangulated spheres which we imagine to contain many lower-dimensional spheres as induced subcomplexes. The surprisingly simple form of this class of complexes, however, allows us to easily compute the Betti numbers, projective dimension and regularity of the Stanley-Reisner ideal of any one of these simplicial complexes.

Definition 5.0.1 (vertex-partition complex). A **vertex-partition complex** is a simplicial complex whose minimal non-faces each have at least two vertices and form a partition of its vertex set. We use the notation $\Omega(\pi_1, \ldots, \pi_p)$ to indicate a vertex-partition complex whose minimal non-faces are $\pi_1, \ldots, \pi_p$.

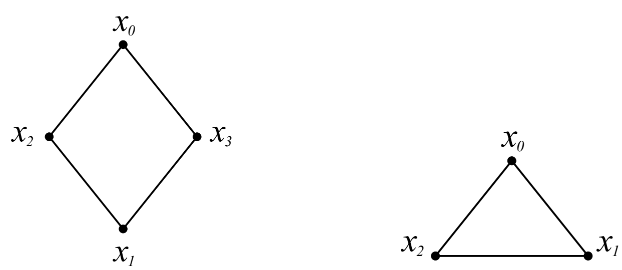
Remark. Note that the requirement that $|\pi_i| \geq 2$ for all i in the previous definition is not a real constraint as a simplicial complex with a minimal non-face of size 1 simply excludes that vertex entirely so there is no reason to include it in the underlying vertex set. In particular, if $|\pi_1| = 1$ then $\Omega(\pi_1, \ldots, \pi_p) = \Omega(\pi_2, \ldots, \pi_p)$.

Let Ω be the vertex-partition complex $\Omega(\pi_1, \ldots, \pi_p)$ on the vertex set $x_1, \ldots, x_n$. The Stanley-Reisner ideal of Ω is

$$\mathcal{N}(\Omega) = (x^{\pi_1}, \ldots, x^{\pi_p}).$$

Since $\pi_i \cap \pi_j = \emptyset$ for all $1 \leq i < j \leq p$, vertex-partition complexes correspond to Stanley-Reisner ideals whose generators have pairwise disjoint support.

In Figures 5.1 and 5.2 we give examples of all possible vertex-partition complexes in dimensions 1 and 2 up to a relabeling of the vertices. Notice that, as mentioned above, these complexes are all triangulations of spheres. In particular, these spheres have relatively few faces for their dimension and, as can be easily seen, most induced subcomplexes of these examples are acyclic.

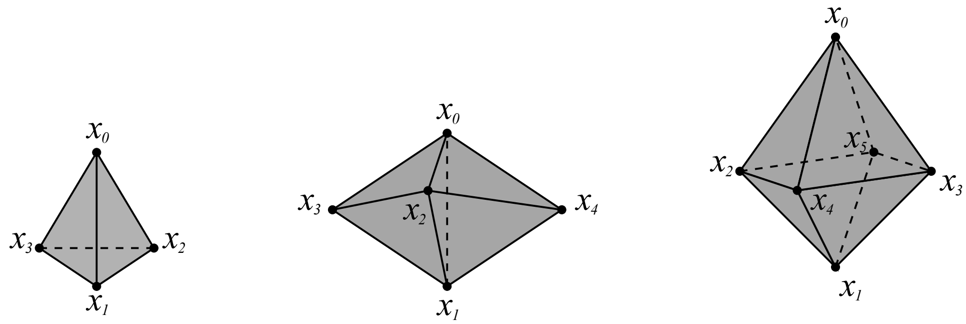


(a) $\Omega(\{x_0, x_1\}, \{x_2, x_3\})$ (b) $\Omega(\{x_0, x_1, x_2\})$

Figure 5.1: Examples of all vertex-partition complexes in dimension 1.

Next we give a more complicated and higher-dimensional example of a vertex-partition complex. We will return to this particular example again in Sections 5.2 and 5.3.

Example 5.0.2. The 48 facets of the pure 7-dimensional simplicial complex $\Omega = \Omega(\{x_0, x_1, x_2\}, \{x_3, x_4, x_5, x_6\}, \{x_7, x_8, x_9, x_{10}\})$ with vertices $x_0, x_1, \ldots, x_{10}$ are



(a) $\Omega(\{x_0, x_1, x_2, x_3\})$ (b) $\Omega(\{x_0, x_1, x_2\}, \{x_3, x_4\})$ (c) $\Omega(\{x_0, x_1\}, \{x_2, x_3\}, \{x_4, x_5\})$

Figure 5.2: Examples of all vertex-partition complexes in dimension 2.

all the maximal subsets of $\{x_0, x_1, \ldots, x_{10}\}$ which do not contain any of the minimal non-faces $\{x_0, x_1, x_2\}$, $\{x_3, x_4, x_5, x_6\}$, or $\{x_7, x_8, x_9, x_{10}\}$. The Stanley-Reisner ideal of Ω is

$$\mathcal{N}(\Omega) = (x_0x_1x_2, x_3x_4x_5x_6, x_7x_8x_9x_{10}).$$

One can show that Ω is an example of a pseudo 7-manifold. In fact, it is a simplicial sphere of dimension 7. We show that this is a general fact about vertex-partition complexes in Theorem 5.1.3.

Remark 5.0.3. It is interesting to note that a subclass of vertex-partition complexes has recently arisen in work by other researchers. In [17] and [29], Goff et al. and Nevo show a minimality property in relation to the f-vectors of the complexes in this subclass. More precisely, let $\mathcal{C}(i, d)$ be the family of d-dimensional simplicial complexes with non-zero reduced d-dimensional homology and whose minimal non-faces all have size less than or equal to i. Let $\Omega = \Omega(\pi_1, \ldots, \pi_q, \pi_{q+1})$ where $d + 1 = qi + r$ with $1 \leq r \leq i$, $|\pi_j| = i + 1$ for $1 \leq j \leq q$ and $|\pi_{q+1}| = r + 1$. In [17] and [29] it is shown that for any Γ in $\mathcal{C}(i, d)$ we have $f_j(\Gamma) \geq f_j(\Omega)$ for all $j \geq 0$ where f_j is the jth entry of the f-vector of the complex.

5.1 The Simplicial Structure of Vertex-Partition Complexes

In this section we will examine the simplicial structure of vertex-partition complexes. It turns out that this class of complexes as well as their induced subcomplexes have a particularly simple form. We will show that vertex-partition complexes are all simplicial spheres and that their induced subcomplexes are either simplicial spheres or are contractible.

Lemma 5.1.1. *The simplicial complex $\Omega(\pi)$ is a simplicial sphere with dimension $|\pi| - 2$.*

Proof. It is easy to see that, since $\Omega(\pi)$ has a single minimal non-face, it is the boundary of an $(|\pi| - 1)$-simplex. The result follows. □

In the next lemma we will see that vertex-partition complexes can be built up inductively from other vertex-partition complexes on smaller vertex sets via the join operation.

Lemma 5.1.2. $\Omega(\pi_1, \ldots, \pi_p) = \Omega(\pi_1, \ldots, \pi_{p-1}) * \Omega(\pi_p)$

Proof. Note that $F \in \Omega(\pi_1, \ldots, \pi_p)$ if and only if $F \subseteq \pi_1 \cup \cdots \cup \pi_p$ and $F \not\supseteq \pi_i$ for $1 \leq i \leq p$. Similarly $F \in \Omega(\pi_1, \ldots, \pi_{p-1}) * \Omega(\pi_p)$ if and only if $F = A \cup B$ where $A \subseteq \pi_1 \cup \cdots \cup \pi_{p-1}$ and $A \not\supseteq \pi_i$ for $1 \leq i \leq p-1$ and $B \subseteq \pi_p$ and $B \not\supseteq \pi_p$. These conditions on F are equivalent since $\pi_p \cap \pi_i = \emptyset$ for all $i < p$ and thus the two complexes are equal. □

Using these observations we are able to determine the simplicial structure of any vertex-partition complex.

Theorem 5.1.3 (Vertex-partition complexes are spheres). *The vertex-partition complex $\Omega(\pi_1, \ldots, \pi_p)$ with n vertices is a simplicial sphere of dimension $n - p - 1$.*

Proof. The first part follows from Lemmas 5.1.1 and 5.1.2 and the fact that the join of two simplicial spheres is a simplicial sphere by Proposition 2.2.3.

Since the minimal non-faces of $\Omega(\pi_1, \ldots, \pi_p)$ are $\pi_1, \ldots, \pi_p$ and these faces partition the vertex set, a face of highest dimension contains all vertices but one from each of the faces $\pi_1, \ldots, \pi_p$. Therefore such a face contains $n - p$ vertices and the dimension of $\Omega(\pi_1, \ldots, \pi_p)$ is $n - p - 1$. □

Since all simplicial spheres are pure it follows from Theorem 5.1.3 that all vertex-partition complexes are pure. It is also easy to see from the argument in the proof of Theorem 5.1.3 that all facets have $n - p$ vertices.

Remark 5.1.4. It is well-known that simplicial spheres have Stanley-Reisner rings which are Cohen-Macaulay over all fields (see Stanley [35, Corollary 4.4]). Therefore the Stanley-Reisner ring of a vertex-partition complex is Cohen-Macaulay over any field.

It turns out that vertex-partition complexes belong to an even smaller class of complexes than those that with Cohen-Macaulay Stanley-Reisner rings.

Definition 5.1.5 (shellable). A pure d-dimensional simplicial complex Γ is **shellable** if there exists some ordering of its facets $F_1, \ldots, F_m$ such that, for $i = 2, \ldots, m$, the intersection

$$F_i \cap \left(\bigcup_{j=1}^{i-1} F_j \right)$$

is a nonempty union of faces of Γ of dimension $d - 1$.

Theorem 5.1.6. *All vertex-partition complexes are shellable.*

Proof. Let $\Omega = \Omega(\pi_1, \ldots, \pi_p)$. It is easy to see that $\Omega(\pi_1)$ is the boundary of a $(|\pi_1| - 1)$-simplex by its definition. Thus $\Omega(\pi_1)$ is shellable by [31, Proposition 2.2 and Corollary 2.9]. Since $\Omega(\pi_1, \ldots, \pi_p) = \Omega(\pi_1, \ldots, \pi_{p-1}) * \Omega(\pi_p)$ by Lemma 5.1.2 and the join of two shellable complexes is shellable by [31, Corollary 2.9], by induction Ω is shellable. $\square$

Notice that there are many combinations of the parameters n and p for a vertex-partition complex which will result in a simplicial sphere of a given dimension d. However, the sensible requirement that $|\pi_i| \geq 2$ for all $1 \leq i \leq p$ means that $n \geq 2p$. This condition places constraints on the resulting pairs of n and p which will result in a given d.

Proposition 5.1.7 (Vertex-partition complexes have few vertices). *Given any vertex-partition complex Ω of dimension d with n vertices we have*

$$d + 2 \leq n \leq 2(d + 1).$$

Proof. Suppose that the vertex partition of Ω has p parts. By Theorem 5.1.3 we have $d = n - p - 1$ and so $p = n - d - 1$. Therefore $2n - 2d - 2 = 2p \leq n$ which means that $n \leq 2d + 2$.

Clearly $p \geq 1$ which means that $d = n-p-1 \leq n-2$ and therefore $d+2 \leq n$. $\square$

In particular, vertex-partition complexes are spheres of high dimension relative to the number of vertices in the complex. For example, when the dimension of the complex is equal to 10 the range is between 12 and 22 vertices at most. Thus, in some sense these spheres are very "compact" and highly-connected.

The bounds given in Proposition 5.1.7 are both achievable. The complex given in Figure 5.2a is an example of a vertex-partition complex of dimension d on $d+2$ vertices where $d = 2$ and the vertex-partition complex in Figure 5.2c has dimension d and has $2d+2$ vertices where $d = 2$.

Remarkably, in the case of vertex-partition complexes we can categorize not only their global structure, but their local structure as well. In the following theorem we see that all induced subcomplexes of a vertex-partition complex are either simplicial spheres or are contractible. Therefore all such subcomplexes either have non-zero homology only in the top dimension or have all homology groups equal to zero.

Theorem 5.1.8 (**Induced subcomplexes of vertex-partition complexes are spheres or contractible).** *Let $W \subseteq V(\Omega(\pi_1, \ldots, \pi_p))$.*

1. *If $W = \bigcup_{j=1}^{\ell} \pi_{i_j}$, $1 \leq \ell \leq p$ then $\Omega(\pi_1, \ldots, \pi_p)_W = \Omega(\pi_{i_1}, \ldots, \pi_{i_\ell})$ is a simplicial sphere of dimension $|W| - \ell - 1$.*

2. *If there exists $1 \leq i \leq p$ such that $W \cap \pi_i \neq \emptyset$ and $\pi_i \backslash W \neq \emptyset$ then $\Omega(\pi_1, \ldots, \pi_p)_W$ is contractible.*

Proof. Part 1 follows from Theorem 5.1.3.

For Part 2, we have $W = \left(\bigcup_{j=1}^{\ell} \pi_{i_j}\right) \cup S$ where $S \not\supseteq \pi_i$ for $1 \leq i \leq p$ and $S \neq \emptyset$. Let Γ be the simplex on the vertex set S. We claim that

$$\Omega(\pi_1, \ldots, \pi_p)_W = \Omega(\pi_{i_1}, \ldots, \pi_{i_\ell}) * \Gamma.$$

Notice that $F \in \Omega(\pi_1, \ldots, \pi_p)_W$ if and only if $F \subseteq W$ and $F \not\supseteq \pi_{i_j}$ for $1 \leq j \leq \ell$. On the other hand, $F \in \Omega(\pi_{i_1}, \ldots, \pi_{i_\ell}) * \Gamma$ if and only if $F = A \cup B$ such that $A \subseteq \pi_{i_1} \cup \cdots \cup \pi_{i_\ell}$ with $A \not\supseteq \pi_{i_j}$ for $1 \leq j \leq \ell$ and $B \subseteq S$. These two conditions are easily seen to be equivalent since $S \cap \pi_{i_j} = \emptyset$ for all $1 \leq j \leq \ell$ which proves the claim.

Let $S = \{x_{\alpha_1}, \dots, x_{\alpha_t}\}$. Since Γ is a simplex with vertex set S, we can write $\Gamma = \langle \{x_{\alpha_1}, \dots, x_{\alpha_{t-1}}\}\rangle * x_{\alpha_t}$. Therefore we have

$$\begin{aligned}\Omega(\pi_1, \dots, \pi_p)_W &= \Omega(\pi_{i_1}, \dots, \pi_{i_\ell}) * \Gamma \\ &= \Omega(\pi_{i_1}, \dots, \pi_{i_\ell}) * \big(\langle \{x_{\alpha_1}, \dots, x_{\alpha_{t-1}}\}\rangle * x_{\alpha_t}\big) \\ &= \big(\Omega(\pi_{i_1}, \dots, \pi_{i_\ell}) * \langle \{x_{\alpha_1}, \dots, x_{\alpha_{t-1}}\}\rangle\big) * x_{\alpha_t}\end{aligned}$$

and so $\Omega(\pi_1, \dots, \pi_p)_W$ is a cone and is therefore contractible.

□

5.2 The Betti Numbers of Vertex-Partition Complexes

As a consequence of the structural characterization given in Theorem 5.1.8 we can completely determine the Betti numbers of the Stanley-Reisner ideal of any vertex-partition complex over any field k with a simple combinatorial formula.

Theorem 5.2.1 (Betti numbers of vertex-partition complexes). *Let $\Omega = \Omega(\pi_1, \dots, \pi_p)$ be a vertex-partition complex. Then*

$$\begin{aligned}\beta_{i,j}(\mathcal{N}(\Omega)) &= |\, \{W \subseteq V(\Omega) \mid |W| = j, \\ &\qquad \Omega_W \textit{ is a simplicial sphere of dimension } j-i-2\}\,| \\ &= \left|\left\{(m_1, \dots, m_{i+1}) \;\middle|\; 1 \le m_1 < \cdots < m_{i+1} \le p \textit{ and } \sum_{\ell=1}^{i+1} |\pi_{m_\ell}| = j\right\}\right|.\end{aligned} \tag{5.1}$$

Proof. From Theorem 2.5.4, we have

$$\beta_{i,j}(\mathcal{N}(\Omega)) = \sum_{W \subseteq V,\ |W| = j} \dim_k \tilde{H}_{j-i-2}(\Omega_W; k).$$

By Theorem 5.1.8 we get a non-zero contribution to the sum on the right-hand side of the equation only when Ω_W is a simplicial sphere of dimension $j-i-2$. In this case we have

$$\dim_k \tilde{H}_{j-i-2}(\Omega_W; k) = 1.$$

This gives the first equality.

The second equality then follows from Theorem 5.1.8. □

Note that it is straightforward to compute the Betti numbers of the Stanley-Reisner ideal of a given vertex-partition complex Ω using formula (5.1). To determine the Betti number $\beta_{i,j}(\mathcal{N}(\Omega))$ we start by listing the ways of combining full parts of the partition to achieve a set of vertices of size j. The number of ways to do this with $i+1$ parts gives us the number $\beta_{i,j}(\mathcal{N}(\Omega))$. We demonstrate this technique in Example 5.2.5.

Remark. The Betti numbers of the Stanley-Reisner ideal of a vertex-partition complex can also be computed recursively by using the following formula given by Jacques and Katzman in [24],

$$\beta_{i,j}(\mathcal{N}(\Gamma_1 * \Gamma_2)) = \sum_{m+n+1=i} \sum_{r+s=j} \beta_{m,r}(\mathcal{N}(\Gamma_1))\beta_{n,s}(\mathcal{N}(\Gamma_2)).$$

However, this essentially requires first computing the reduced homology groups for all induced subcomplexes for both smaller simplicial complexes. The formula given in Theorem 5.2.1 avoids this large computation by exploiting the specific simplicial structure of vertex-partition complexes.

Recall from Section 2.5 that the projective dimension of a monomial ideal I is given by

$$\mathrm{pd}(I) = \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

As a corollary to Theorem 5.2.1 we can easily bound the projective dimension of the Stanley-Reisner ideal of any vertex-partition complex.

Corollary 5.2.2. *Let $\Omega = \Omega(\pi_1, \ldots, \pi_p)$ be a vertex-partition complex. If $i \geq p$ then $\beta_{i,j}(\mathcal{N}(\Omega)) = 0$.*

Proof. From (5.1) in Theorem 5.2.1 we see that we must have $i+1 \leq p$ in order to have $\beta_{i,j}(\mathcal{N}(\Omega)) \neq 0$. $\square$

Using Theorem 5.2.1 and Corollary 5.2.2 we can compute the projective dimension exactly.

Theorem 5.2.3. *Let $\Omega = \Omega(\pi_1, \ldots, \pi_p)$ be a vertex-partition complex. Then $\mathrm{pd}(\mathcal{N}(\Omega)) = p - 1$.*

Proof. From Corollary 5.2.2 we know that $\mathrm{pd}(\mathcal{N}(\Omega)) < p$ and so $\beta_{i,j}(\mathcal{N}(\Omega)) = 0$ for all $i \geq p$. Consider the Betti numbers $\beta_{p-1,j}(\mathcal{N}(\Omega))$. We would like to show that there exists some j such that $\beta_{p-1,j}(\mathcal{N}(\Omega))$ is non-zero. Considering the numbers $\beta_{p-1,j}(\mathcal{N}(\Omega))$ in terms of equation (5.1) in Theorem 5.2.1 we have $i + 1 = p$ and we are looking for some j such that

$$\sum_{\ell=1}^{p} |\pi_{m_\ell}| = j.$$

However, since p is the total number of parts in the vertex partition we have

$$\sum_{\ell=1}^{p} |\pi_{m_\ell}| = |V(\Omega)|$$

and so $\beta_{p-1,|V(\Omega)|}(\mathcal{N}(\Omega)) = 1$. In particular $\beta_{p-1,|V(\Omega)|}(\mathcal{N}(\Omega)) \neq 0$ and so $\mathrm{pd}(\mathcal{N}(\Omega)) = p - 1$. $\square$

Recall that the Castelnuovo-Mumford regularity of the monomial ideal I is given by

$$\mathrm{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

It is straightforward to determine the Castelnuovo-Mumford regularity of the Stanley-Reisner ideal of a vertex-partition complex.

Theorem 5.2.4. *Suppose that $\Omega = \Omega(\pi_1, \ldots, \pi_p)$ is a vertex-partition complex on n vertices. Then $\mathrm{reg}(\mathcal{N}(\Omega)) = n - p + 1$.*

Proof. We would like to find $\max\{j - i \mid \beta_{i,j}(\mathcal{N}(\Omega)) \neq 0\}$. By Theorem 5.1.8, for any $W \subseteq V(\Omega)$ with $|W| = j$ we have $\tilde{H}_{j-i-2}(\Omega_W; k) \neq 0$ if and only if $W = \bigcup_{\ell=1}^{i+1} \pi_{m_\ell}$. We claim that the maximum difference between the indices i and j occurs when $|W| = n$. To see this suppose that $|W| < n$ and $W = \bigcup_{\ell=1}^{i+1} \pi_{m_\ell}$ so necessarily we have $i+1 < p$. Choose π_r such that $r \neq m_\ell$ for any $1 \leq \ell \leq i + 1$ and let $W' = W \cup \pi_r$. Then $\tilde{H}_{|W'|-(i+1)-2}(\Omega_{W'}; k) \neq 0$ by Theorem 5.1.8 and

$$|W'| - (i + 1) = |W| + |\pi_r| - (i + 1) > |W| + 1 - (i + 1) = |W| - i$$

since $|\pi_r| \geq 2$. Therefore we can strictly increase the difference between i and j by adding additional pieces of the partition until we have exhausted all parts of the partition and thus the maximum occurs when $j = n$ and $i + 1 = p$. $\square$

Example 5.2.5. We can use (5.1) from Theorem 5.2.1 to easily compute the Betti numbers of the Stanley-Reisner ideal of the vertex-partition complex

$$\Omega = \Omega(\{x_0, x_1, x_2\}, \{x_3, x_4, x_5, x_6\}, \{x_7, x_8, x_9, x_{10}\})$$

introduced in Example 5.0.2. We have

$$\mathcal{N}(\Omega) = (x_0x_1x_2, x_3x_4x_5x_6, x_7x_8x_9x_{10})$$

and the only non-zero Betti numbers of $\mathcal{N}(\Omega)$ are

$$\beta_{2,11}(\mathcal{N}(\Omega)) = 1,\ \beta_{1,8}(\mathcal{N}(\Omega)) = 1,\ \beta_{1,7}(\mathcal{N}(\Omega)) = 2,\ \beta_{0,4}(\mathcal{N}(\Omega)) = 2,\ \beta_{0,3}(\mathcal{N}(\Omega)) = 1.$$

For instance, to compute $\beta_{i,7}(\mathcal{N}(\Omega))$ we need only notice that there are only two different induced subcomplexes of Ω on 7 vertices with non-zero homology. These have vertex sets $\{x_0, x_1, x_2\} \cup \{x_3, x_4, x_5, x_6\}$ and $\{x_0, x_1, x_2\} \cup \{x_7, x_8, x_9, x_{10}\}$ and so both of these use two parts of the partition. Thus, in equation (5.1) we have $i + 1 = 2$ and so these subcomplexes are counted by $\beta_{1,7}(\mathcal{N}(\Omega))$. All other induced subcomplexes on 7 vertices are contractible by Theorem 5.1.8.

In agreement with Theorems 5.2.3 and 5.2.4 we can see that

$$\operatorname{pd}(\mathcal{N}(\Omega)) = 2 \ \text{ and } \ \operatorname{reg}(\mathcal{N}(\Omega)) = 9.$$

5.3 Obstructions to Linear Resolution

In this section we will examine the reason that the Stanley-Reisner ideal of the d-closure of a d-chorded pure d-skeleton of a vertex-partition complex does not have a $(d+1)$-linear resolution over any field and in particular over a field of characteristic 2. In other words, we would like to know why these complexes are counterexamples to the converse of Theorem 4.3.1 part 2. Throughout this section we will assume that the complex $\Omega(\pi_1, \ldots, \pi_p)$ has $|\pi_i| \leq d + 1$ for all $1 \leq i \leq p$ since these are the specific vertex-partition complexes which show up as minimal counterexamples. Also we assume that $d \leq \dim \Omega(\pi_1, \ldots, \pi_p)$ so that we may take the pure d-skeleton of $\Omega(\pi_1, \ldots, \pi_p)$. Finally, we require $p \geq 2$ since otherwise the d-closure of the vertex-partition complex will be a simplex and thus will not give an obstruction to linear resolution.

Lemma 5.3.1 (A vertex-partition complex and its closure have the same upper-dimensional faces). *Let $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$. Then for $d \leq m \leq \dim \Delta_d(\Gamma)$ we have*

$$\Delta_d(\Gamma)^{[m]} = \langle \{F \subseteq V(\Gamma) \mid |F| = m+1 \text{ and } \forall i\ F \not\supseteq \pi_i\}\rangle.$$

Proof. Since $|\pi_i| \geq 2$ for all i, every vertex of $\Omega(\pi_1, \ldots, \pi_p)$ is contained in a facet of $\Omega(\pi_1, \ldots, \pi_p)$. In addition, since $d \leq \dim \Omega(\pi_1, \ldots, \pi_p)$ and $\Omega(\pi_1, \ldots, \pi_p)$ is pure, each vertex of $\Omega(\pi_1, \ldots, \pi_p)$ is contained in a d-face of $\Omega(\pi_1, \ldots, \pi_p)$. Therefore, $V(\Gamma) = V(\Omega(\pi_1, \ldots, \pi_p)) = \bigcup_{i=1}^{p} \pi_i$.

When $m = d$ we have $\Delta_d(\Gamma)^{[m]} = \Gamma$ by the definition of the d-closure. Since the minimal non-faces of $\Omega(\pi_1, \ldots, \pi_p)$ are $\pi_1, \ldots, \pi_p$ and $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$, the faces of Γ are those subsets of $V(\Gamma)$ of size $d+1$ which do not contain any of the subsets $\pi_1, \ldots, \pi_p$. Therefore

$$\Delta_d(\Gamma)^{[d]} = \langle \{F \subseteq V(\Gamma) \mid |F| = d+1 \text{ and } \forall i\ F \not\supseteq \pi_i\}\rangle.$$

Now let $m > d$. If $F \in \Delta_d(\Gamma)$ and $|F| = m+1$ then all subsets of F of size $d+1$ are faces of Γ by the nature of the d-closure. If $F \supseteq \pi_i$ for some i then since $|\pi_i| \leq d+1$ it must be that Γ contains some d-face f with $f \supseteq \pi_i$. This is a contradiction since π_i is a non-face of $\Omega(\pi_1, \ldots, \pi_p)$ and so is also a non-face of Γ. Thus if $F \in \Delta_d(\Gamma)$ then $F \not\supseteq \pi_i$ for all i.

Conversely, suppose that $F \subseteq V(\Gamma)$, $|F| = m+1$ and $F \not\supseteq \pi_i$ for all i. Let $f \subseteq F$ with $|f| = d+1$. Then $f \not\supseteq \pi_i$ for any i and since $\pi_1, \ldots, \pi_p$ are the only minimal non-faces of $\Omega(\pi_1, \ldots, \pi_p)$ we must have $f \in \Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$. Thus by the nature of the d-closure we know that $F \in \Delta_d(\Gamma)$. $\square$

Remark 5.3.2. Notice that Lemma 5.3.1 implies that

$$\Delta_d(\Omega(\pi_1, \ldots, \pi_p)^{[d]})^{[m]} = \Omega(\pi_1, \ldots, \pi_p)^{[m]}$$

for all $m \geq d$ by the definition of $\Omega(\pi_1, \ldots, \pi_p)$. However, unlike $\Omega(\pi_1, \ldots, \pi_p)$, $\Delta_d(\Omega(\pi_1, \ldots, \pi_p)^{[d]})$ will not be pure in general as it contains all possible faces of dimension less than d and so likely contains a facet of dimension $d-1$.

As a consequence of Lemma 5.3.1, we have the following corollary.

Corollary 5.3.3. *Let $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$. Then F is a face of $\Delta_d(\Gamma)$ of dimension $\dim \Delta_d(\Gamma)$ if and only if $F = V(\Gamma) \setminus A$ where A contains exactly one vertex from each of $\pi_1, \ldots, \pi_p$.*

From Corollary 5.3.3, it is easy to determine the dimension of the d-closure of the pure d-skeleton of a vertex-partition complex. This dimension coincides with the dimension of the vertex-partition complex itself by Remark 5.3.2.

Corollary 5.3.4. *If $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$ and $|V(\Gamma)| = n$ then $\dim \Delta_d(\Gamma) = n-p-1$.*

Proof. There are p minimal non-faces π_i and a total of n vertices and so, by Corollary 5.3.3, a largest face will have $n-p$ vertices. Thus the dimension of $\Delta_d(\Gamma)$ will be $n-p-1$. $\square$

In the next lemma we are able to show the existence of a higher-dimensional cycle without a chord set in the d-closure of these examples.

Lemma 5.3.5 (Closure of a vertex-partition complex is a pseudo-manifold). *If $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$ and $|V(\Gamma)| = n$ then $\Delta_d(\Gamma)^{[n-p-1]}$ is a pseudo $(n-p-1)$-manifold.*

Proof. We first show that each $(n-p-2)$-face of $\Delta_d(\Gamma)^{[n-p-1]}$ is contained in exactly two of its $(n-p-1)$-faces. To this end let f be any $(n-p-2)$-dimensional face. Then f has $n-p-1$ vertices. Therefore by Lemma 5.3.1 it must be the case that there exists $1 \leq j \leq p$ such that f contains all but two vertices from π_j and for all $i \neq j$ f contains all but one vertex from π_i. It is clear from Corollary 5.3.3 that there are only two $(n-p-1)$-faces of $\Delta_d(\Gamma)^{[n-p-1]}$ that contain f and each of these is obtained from f by adding exactly one of the two remaining vertices from π_j. Therefore each $(n-p-2)$-face of $\Delta_d(\Gamma)^{[n-p-1]}$ is contained in exactly two of its $(n-p-1)$-faces.

Next we would like to show that $\Delta_d(\Gamma)^{[n-p-1]}$ is $(n-p-1)$-path-connected. To do this we will provide a way to travel an $(n-p-1)$-path to get between any two $(n-p-1)$-faces. Let F and G be any two $(n-p-1)$-faces of $\Delta_d(\Gamma)$. By Corollary 5.3.3, F and G differ in at most p places since they each contain all vertices of Γ but one from each π_i. Our $(n-p-1)$-path will have faces

$$F = F_0,\ F_1,\ \ldots,\ F_p = G$$

where, for all $i \geq 1$, if $\pi_i \setminus F = \{y_i\}$ and $\pi_i \setminus G = \{z_i\}$, then

$$F_i = (F_{i-1} \cup \{y_i\}) \setminus \{z_i\}.$$

It is clear that if $y_i \neq z_i$ then $|F_i \cap F_{i-1}| = n-p-1$ and $F_i = F_{i-1}$ otherwise. Therefore, after eliminating the repeating faces, $F_0, \ldots, F_p$ form an $(n-p-1)$-path between F and G. Hence $\Delta_d(\Gamma)^{[n-p-1]}$ is $(n-p-1)$-path-connected and hence $\Delta_d(\Gamma)^{[n-p-1]}$ is a pseudo $(n-p-1)$-manifold. $\square$

Lemma 5.3.5 can also be deduced from Theorem 5.1.3 and Remark 5.3.2, but we include the combinatorial proof above for the sake of interest.

Example 5.3.6. Consider the 2-closure of the complex $\Omega(\{x_0, x_1, x_2\}, \{x_3, x_4\}) = \Omega(\{x_0, x_1, x_2\}, \{x_3, x_4\})^{[2]}$ shown in Figure 5.2b. We have

$$\Delta_2(\Omega(\{x_0, x_1, x_2\}, \{x_3, x_4\}))^{[2]} = \Omega(\{x_0, x_1, x_2\}, \{x_3, x_4\}),$$

which is easily seen to be a pseudo 2-manifold since each 1-face is contained in exactly two 2-faces and the whole complex is 2-path-connected.

Corollary 5.3.7 (Vertex-partition complexes produce non-zero homology). *If $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$ and $|V(\Gamma)| = n$ then $\tilde{H}_{n-p-1}(\Delta_d(\Gamma); k) \neq 0$ for any field k.*

Proof. By Remark 5.3.2 and Theorem 5.1.3 we know that $\Delta_d(\Gamma)^{[n-p-1]}$ is a simplicial sphere of dimension $n-p-1$ and so $\tilde{H}_{n-p-1}(\Delta_d(\Gamma)^{[n-p-1]}; k) \neq 0$. However, since $\Delta_d(\Gamma)$ has dimension $n-p-1$ by Corollary 5.3.4, it contains no faces of dimension $n-p$ and thus $\tilde{H}_{n-p-1}(\Delta_d(\Gamma); k) \neq 0$ as well.

$\square$

Corollary 5.3.7 allows us to conclude why this particular class of simplicial complexes provides us with a collection of ideals which have no linear resolution over any field.

Corollary 5.3.8 (Vertex-partition complexes are obstructions to linear resolution). *If $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$ then $\mathcal{N}(\Delta_d(\Gamma))$ does not have a $(d+1)$-linear resolution over any field.*

Proof. This follows by Theorem 4.0.2 and Corollary 5.3.7. ☐

The counterexamples to the converse of Theorem 4.3.1 part 2 are d-chorded simplicial complexes with d-closures whose Stanley-Reisner ideals do not have linear resolutions in characteristic 2. We know by Corollary 5.3.8 that the pure d-skeletons of vertex-partition complexes have d-closures whose Stanley-Reisner ideals have no linear resolution in characteristic 2. Therefore in order to obtain a class of counterexamples we would like to determine when these d-skeletons are d-chorded.

Proposition 5.3.9 (The d-skeleton of a vertex-partition complex is d-chorded iff it contains no d-spheres). *If $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$ then Γ is d-chorded if and only if for all $i_1, \ldots, i_\ell$ with $1 \leq \ell \leq p$ and $1 \leq i_1 < \cdots < i_\ell \leq p$ we have*

$$\sum_{j=1}^{\ell} |\pi_{i_j}| - \ell - 1 \neq d.$$

Proof. By Theorem 4.2.8, Γ is d-chorded if and only if $\tilde{H}_d(\Delta_d(\Gamma)_W; \mathbb{Z}_2) = 0$ for all $W \subseteq V(\Gamma)$. By Remark 5.3.2 we know that

$$(\Delta_d(\Gamma)_W)^{[m]} = (\Omega(\pi_1, \ldots, \pi_p)_W)^{[m]}$$

for all $W \subseteq V(\Gamma)$ and all $m \geq d$. In particular, the pure m-skeleton of $\Delta_d(\Gamma)_W$ is the same as the pure m-skeleton of $\Omega(\pi_1, \ldots, \pi_p)_W$ for $m = d$ and $m = d + 1$. Therefore

$$\tilde{H}_d(\Delta_d(\Gamma)_W; \mathbb{Z}_2) = \tilde{H}_d(\Omega(\pi_1, \ldots, \pi_p)_W; \mathbb{Z}_2)$$

for all $W \subseteq V(\Gamma)$. Therefore Γ is d-chorded if and only if $\tilde{H}_d(\Omega(\pi_1, \ldots, \pi_p)_W; \mathbb{Z}_2) = 0$ for all $W \subseteq V(\Gamma)$.

Suppose that Γ is d-chorded so that $\tilde{H}_d(\Omega(\pi_1, \ldots, \pi_p)_W; \mathbb{Z}_2) = 0$ for all $W \subseteq V(\Gamma)$ and suppose, for a contradiction, that there exist $i_1, \ldots, i_\ell$ with $1 \leq \ell \leq p$ and $1 \leq i_1 < \cdots < i_\ell \leq p$ such that

$$\sum_{j=1}^{\ell} |\pi_{i_j}| - \ell - 1 = d.$$

Letting $W = \bigcup_{j=1}^{\ell} \pi_{i_j}$ we have $|W| - \ell - 1 = d$ and, by Theorem 5.1.8 part 1, we know that $\Omega(\pi_1, \ldots, \pi_p)_W$ is a d-dimensional simplicial sphere. Therefore we have

$\tilde{H}_d(\Omega(\pi_1, \ldots, \pi_p)_W; \mathbb{Z}_2) \neq 0$ which is a contradiction and so the condition given in the statement of the proposition holds.

Now assume that for all $i_1, \ldots, i_\ell$ with $1 \leq \ell \leq p$ and $1 \leq i_1 < \cdots < i_\ell \leq p$ we have

$$\sum_{j=1}^{\ell} |\pi_{i_j}| - \ell - 1 \neq d.$$

Suppose that $\tilde{H}_d(\Omega(\pi_1, \ldots, \pi_p)_W; \mathbb{Z}_2) \neq 0$ for some $W \subseteq V(\Gamma)$. By Theorem 5.1.8 we must have $W = \bigcup_{j=1}^{\ell} \pi_{i_j}$ for some ℓ and $|W| - \ell - 1 = d$. However, this contradicts our assumption and so we have $\tilde{H}_d(\Omega(\pi_1, \ldots, \pi_p)_W; \mathbb{Z}_2) = 0$ for all $W \subseteq V(\Gamma)$. Therefore Γ must be d-chorded. $\square$

Example 5.3.10. Let $\Omega = \Omega(\{x_0, x_1, x_2\}, \{x_3, x_4, x_5, x_6\}, \{x_7, x_8, x_9, x_{10}\})$, the vertex-partition complex given in Example 5.0.2, and let $\Gamma = \Omega^{[3]}$. Then we have

$$\begin{aligned} \mathcal{N}(\Delta_3(\Gamma)) \quad = \quad & (x_0x_1x_2x_3, x_0x_1x_2x_4, x_0x_1x_2x_5, x_0x_1x_2x_6, \\ & x_0x_1x_2x_7, x_0x_1x_2x_8, x_0x_1x_2x_9, x_0x_1x_2x_{10}, \\ & x_3x_4x_5x_6, x_7x_8x_9x_{10}) \end{aligned}$$

and by Corollary 5.3.8 we know that $\mathcal{N}(\Delta_3(\Gamma))$ does not have a 4-linear resolution over any field of characteristic 2.

In addition, Γ has a complete 1-skeleton and is 3-chorded by an application of Proposition 5.3.9. Therefore Γ is a counterexample to the converse of Theorem 4.3.1 part 2.

Definition 5.3.11. Two pure d-dimensional simplicial complexes Γ_1 and Γ_2 lie in the same d**-closure class** if

$$\Delta_d(\Gamma_1)^{[m]} = \Delta_d(\Gamma_2)^{[m]}$$

for all $m \geq d + 1$.

See Figure 5.3 for an example of two different 2-dimensional simplicial complexes Γ_1 and Γ_2 which lie in the same 2-closure class. Both $\Delta_2(\Gamma_1)$ and $\Delta_2(\Gamma_2)$ contain the face $\{a, b, c, d\}$ and in both cases this is the only face of dimension greater than 2 in the complex. Hence $\Delta_2(\Gamma_1)^{[m]} = \Delta_2(\Gamma_2)^{[m]}$ for all $m \geq 3$.

(a) The complex Γ_1 which has $\Delta_2(\Gamma_1) = \langle\{a,b,c,d\},\{c,d,e\}\rangle$

(b) The complex Γ_2 which has $\Delta_2(\Gamma_2) = \langle\{a,b,c,d\},\{c,d,e\},\{b,c,e\}\rangle$

Figure 5.3: 2-dimensional simplicial complexes in the same 2-closure class.

It is interesting to note that, due to the definition of d-closure, we have that

$$\Delta_d(\Gamma_1)^{[m]} = \Delta_d(\Gamma_2)^{[m]}$$

for all $m \geq d+1$ if and only if

$$\Delta_d(\Gamma_1)^{[d+1]} = \Delta_d(\Gamma_2)^{[d+1]}.$$

Notice that all d-dimensional forests lie in the same d-closure class. This is easy to see since the d-closure of any d-dimensional forest contains no faces of dimension $d+1$.

The following lemma says that the pure d-skeleton of a vertex-partition complex is the minimal element in its d-closure class.

Lemma 5.3.12 (Vertex-partition complexes are minimal in their d-closure classes). *Let $\Gamma = \Omega(\pi_1,\ldots,\pi_p)^{[d]}$ where $|V(\Omega(\pi_1,\ldots,\pi_p))| = n$ and $d < n-p-1$. If Γ' lies in the same d-closure class as Γ then $\Gamma \subseteq \Gamma'$.*

Proof. First we claim that $\Gamma = \left(\Delta_d(\Gamma)^{[d+1]}\right)^{[d]}$. It is clear, from the definition of the d-closure, that $\Gamma \supseteq \left(\Delta_d(\Gamma)^{[d+1]}\right)^{[d]}$. For the reverse inclusion, let F be a facet of Γ. We need only show that there is a $(d+1)$-face of $\Delta_d(\Gamma)$ which contains F. Since $F \not\supseteq \pi_i$ for all i and $|F| = d+1 < n-p$, there must be some $1 \leq j \leq p$ such that $|\pi_j \setminus F| \geq 2$. Letting $x \in \pi_j \setminus F$ we have $F \cup \{x\} \not\supseteq \pi_i$ for all i and thus $F \cup \{x\} \in \Delta_d(\Gamma)$ by Lemma 5.3.1. Therefore $\Gamma = \left(\Delta_d(\Gamma)^{[d+1]}\right)^{[d]}$.

Furthermore, since Γ' lies in the same d-closure class as Γ we have $\Delta_d(\Gamma')^{[d+1]} = \Delta_d(\Gamma)^{[d+1]}$ and so

$$\Gamma' \supseteq \left(\Delta_d(\Gamma')^{[d+1]}\right)^{[d]} = \left(\Delta_d(\Gamma)^{[d+1]}\right)^{[d]} = \Gamma.$$

□

Corollary 5.3.13 (Vertex-partition complexes have minimal f-vectors in their d-closure classes). *The f-vector of the simplicial complex $\Gamma = \Omega(\pi_1, \ldots, \pi_p)^{[d]}$ is term-wise minimal among the f-vectors of the simplicial complexes in the d-closure class of Γ.*

Proof. Let Γ' be a pure d-dimensional simplicial complex which lies in the same d-closure class as Γ. By Lemma 5.3.12 we have $\Gamma \subseteq \Gamma'$ and so any i-dimensional face of Γ is an i-dimensional face of Γ'. Therefore $f_i(\Gamma) \leq f_i(\Gamma')$ for all $-1 \leq i \leq d$. □

All of the examples of d-chorded complexes Γ such that $\mathcal{N}(\Delta_d(\Gamma))$ has no linear resolution over fields of characteristic 2 which were originally examined contained a vertex-partition complex as an induced subcomplex. This led to the following question.

Question 5.3.14. If Γ is d-chorded but $\mathcal{N}(\Delta_d(\Gamma))$ has no linear resolution over a field of characteristic 2 does there exist $S \subseteq V(\Gamma)$ such that Γ_S lies in the same d-closure class as the pure d-skeleton of some vertex-partition complex $\Omega(\pi_1, \ldots, \pi_p)$ with $\dim \Omega(\pi_1, \ldots, \pi_p) > d$?

Unfortunately the answer to this question is "no" as can be seen from the following counterexample.

Example 5.3.15. Let Γ be the pure 3-dimensional simplicial complex on the vertices $x_0, \ldots, x_6$ whose only missing 3-faces are $\{x_0, x_1, x_5, x_6\}$, $\{x_0, x_2, x_5, x_6\}$ and $\{x_1, x_2, x_3, x_4\}$. Then $\Delta_3(\Gamma)$ is 4-dimensional and the facets of $\Delta_3(\Gamma)^{[4]}$ are:

$$\begin{array}{lll}
\{x_0, x_1, x_2, x_3, x_5\} & \{x_0, x_1, x_2, x_3, x_6\} & \{x_0, x_1, x_2, x_4, x_5\} \\
\{x_0, x_1, x_2, x_4, x_6\} & \{x_0, x_1, x_3, x_4, x_5\} & \{x_0, x_1, x_3, x_4, x_6\} \\
\{x_0, x_2, x_3, x_4, x_5\} & \{x_0, x_2, x_3, x_4, x_6\} & \{x_0, x_3, x_4, x_5, x_6\} \\
\{x_1, x_2, x_3, x_5, x_6\} & \{x_1, x_2, x_4, x_5, x_6\} & \{x_1, x_3, x_4, x_5, x_6\} \\
\{x_2, x_3, x_4, x_5, x_6\} & &
\end{array}$$

We have

$$\mathcal{N}(\Delta_3(\Gamma)) = (x_0x_1x_5x_6, x_0x_2x_5x_6, x_1x_2x_3x_4).$$

One can show that the simplicial complex Γ is 3-chorded by computation using Theorem 4.2.8. By computing reduced simplicial homology of $\Delta_3(\Gamma)$ we determine that $\tilde{H}_3(\Delta_3(\Gamma); k) = 0$ and $\tilde{H}_4(\Delta_3(\Gamma); k) \neq 0$ for k having characteristic 2. Therefore $\mathcal{N}(\Delta_3(\Gamma))$ does not have a linear resolution over fields of characteristic 2 by Theorem 4.0.2.

We claim that there is no subset $S \subseteq V(\Gamma)$ so that Γ_S lies in the same 3-closure class as the pure 3-skeleton of some vertex-partition complex. To see this, first recall from Lemma 4.2.3 that $\Delta_3((\Gamma_S)^{[3]}) = \Delta_3(\Gamma)_S$ for any $S \subseteq V(\Gamma)$. Therefore, by Lemma 5.3.5, since $\dim \Delta_3(\Gamma) = 4$, if the conditions in Question 5.3.14 are satisfied then there exists some $S \subseteq V(\Gamma)$ so that $(\Delta_3(\Gamma)_S)^{[4]}$ is a pseudo 4-manifold. It turns out that the only 4-dimensional cycle in $\Delta_3(\Gamma)$ uses all of the vertices of Γ and thus we must have $S = V(\Gamma)$. However, by examining the list of facets given above, one can check that $\Delta_3(\Gamma)^{[4]}$ is equal to a pseudo 4-manifold with the additional face $\{x_0, x_3, x_4, x_5, x_6\}$. This means that each 3-face of $\{x_0, x_3, x_4, x_5, x_6\}$ is contained in an odd number of facets of $\Delta_3(\Gamma)^{[4]}$ and thus $\Delta_3(\Gamma)^{[4]}$ does not satisfy the definition of a pseudo 4-manifold. Therefore we have shown that there does not exist $S \subseteq V(\Gamma)$ such that Γ_S lies in the same 3-closure class as the pure 3-skeleton of some vertex-partition complex $\Omega(\pi_1, \ldots, \pi_p)$ with $\dim \Omega(\pi_1, \ldots, \pi_p) > 3$.

Chapter 6

Conclusion

There are several questions that naturally arise from the results obtained in this thesis. In particular, the question of whether or not there exists a generalization of Fröberg's theorem for ideals having linear resolutions over all fields is still open and worth investigating. It seems to be the case that an ideal having a linear resolution over every field is, in some sense, better behaved than one where the existence of a resolution depends on the field in question. Thanks to Theorem 1.0.3, to examine this problem it is possible to limit investigations to studying which simplicial complexes have certain homology groups that vanish over the ring $\mathbb{Z}$. This is because homology vanishes over $\mathbb{Z}$ if and only if it vanishes over all fields. The advantage to studying this case is that the coefficients of the homological structures under scrutiny are relatively well-behaved. An important result to obtain here would be an analogue to Theorem 3.3.3, which would be an interesting result in its own right.

Another obvious question that emerges from Section 4.5, and is partially investigated in Chapter 5, is the question of specifically which d-chorded complexes have d-closures whose Stanley-Reisner ideals fail to have a linear resolution over fields of characteristic 2. These are the counterexamples to Theorem 4.3.1 part 2. We showed in Theorem 4.6.1 that these are the d-chorded complexes whose d-closures are not chorded. Specifically, by Theorem 4.6.3, they are those in which certain types of specialized cycles appear. However, it would be interesting to investigate this class further to better grasp the intricacies of the combinatorics involved. We have seen, with the vertex-partition complexes in Chapter 5, that this class contains some easy-to-define and well-structured complexes. In addition, as a follow-up to Remark 5.0.3 and Corollary 5.3.13, it would be interesting to study the f-vectors of vertex-partition complexes more closely and to determine if they are minimal in some class of complexes that is wider than their d-closure class.

There is a more general property than having a linear resolution that one can study for edge ideals. A monomial ideal I is said to satisfy property $N_{2,p}$ for some $p \geq 1$ if it is generated in degree 2 and its minimal graded free resolution is linear up to step p. This means that we have $\beta_{i,j}(I) = 0$ for all $0 \leq i < p$ and $j > i + 2$. In other words, an ideal has a 2-linear resolution if and only if it satisfies property $N_{2,p}$ for all $p \geq 1$. In [11], Eisenbud et al. give the following theorem.

Theorem 6.0.1. *For a graph G, $\mathcal{N}(\Delta(G))$ satisfies $N_{2,p}$ for some $p \geq 1$ if and only if every cycle in G with length at most $p + 2$ has a chord.*

Theorem 6.0.1 is a refinement of Fröberg's theorem as it classifies all edge ideals with "partially linear" resolutions. It would be an interesting line of study to try and generalize this theorem to classify ideals generated in any fixed degree which are linear up to the pth step in the resolution. One could attempt to use the notions of a d-dimensional cycle given in Chapter 3 and a chord set given in Chapter 4 to try and extend this theorem to higher dimensions, at least over fields of characteristic 2.

Another possible application of the results in Chapter 3 is graph colouring. A graph is m**-colourable** if its vertices can be assigned m different colours in such a way that no two adjacent vertices have the same colour. The **neighbourhood complex** of a graph G, denoted $\mathfrak{N}(G)$, is the simplicial complex whose faces are all subsets S of $V(G)$ such that the elements of S have a common neighbour in G. In [37, Theorem 11.2], Walker proves the following theorem.

Theorem 6.0.2. *If G is a graph such that $\tilde{H}_i(\mathfrak{N}(G); \mathbb{Z}_2) = 0$ for all $i \leq m - 2$ then G is not m-colourable.*

Theorem 3.3.3 given in Chapter 3 gives a purely combinatorial description of non-zero simplicial homology over $\mathbb{Z}_2$. Therefore, in combination with Theorem 6.0.2 this theorem might be useful in uncovering specific structural properties of graphs which fail to be m-colourable through examination of the combinatorics of their neighbourhood complexes.

The characterization given in Theorem 4.6.1 of the existence of a linear resolution in characteristic 2 might also shed light on a conjecture of Stanley from [35, Conjecture

2.7]. For any two faces F and G in a simplicial complex Γ with $F \subseteq G$ the **closed interval from** F **to** G is given by

$$[F, G] = \{H \mid F \subseteq H \subseteq G\}.$$

A pure simplicial complex Γ is said to be **partitionable** if we can write

$$\Gamma = \bigsqcup_{1 \leq i \leq k} [F_i, G_i]$$

where $G_1, \ldots, G_k$ are the facets of Γ and $\bigsqcup$ indicates a disjoint union. It is not hard to show that all shellable complexes are partitionable [35, page 79]. The **Alexander dual** of the simplicial complex Γ is the simplicial complex

$$\Gamma^{\vee} = \{F \subseteq V(\Gamma) \mid V(\Gamma) \setminus F \notin \Gamma\}.$$

Conjecture 6.0.3 (Stanley)**.** *If* $\mathcal{N}(\Gamma)$ *has a linear resolution then* $\Gamma^{\vee}$ *is partitionable.*

One possible approach to this conjecture would be to try and translate the property of a complex being chorded into some combinatorial property in the Alexander dual of the complex. The idea would be to show that the resulting combinatorial property implied partitionability.

Overall, we see that there are several natural directions in which the line of research begun in this thesis could progress.

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