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Invariant classification of vacuum pp-waves

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We solve the equivalence problem for vacuum pp-wave spacetimes by employing the Karlhede algorithm. Our main result is a suite of Cartan invariants that allows for the complete invariant classification of the vacuum pp-waves. In particular, we derive the invariant characterization of the G_2 and G_3 sub-classes in terms of these invariants. It is known [J. M. Collins, "The Karlhede classification of type N vacuum spacetimes," Class. Quantum Grav. **8**, 1859–1869 (1991)] that the invariant classification of vacuum pp-waves requires *at most* the fourth order covariant derivative of the curvature tensor, but no specific examples requiring the fourth order were known. Using our comprehensive classification, we prove that the $q \leq 4$ bound is sharp and explicitly describe all such maximal order solutions. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4791691]

I. INTRODUCTION

In general relativity, identical spacetimes are often given in different coordinate systems, thereby disguising the diffeomorphic equivalence of the underlying metrics. It is consequently of fundamental importance to have an invariant procedure for deciding the question of metric equivalence. One approach to this problem is to utilize scalar curvature invariants, obtained as full contractions of the curvature tensor and its covariant derivatives.[3] However, a particularly intriguing situation arises when we consider pp-waves, spacetimes that admit a covariantly constant null vector field (see Chapter 24 of Ref. 4 for a definition.) Some time ago it was observed that all curvature invariants of a pp-wave spacetime vanish.[5] Subsequently all spacetimes with the VSI property (vanishing scalar invariants) and the more general CSI property (constant scalar invariants) were classified.[6,7] It is now known that either a spacetime is uniquely determined by its scalar curvature invariants, or is a degenerate Kundt spacetime;[3,8] the VSI and CSI solutions belong to this more general class.

To invariantly classify the degenerate Kundt spacetimes, and pp-waves in particular, one must therefore use the Karlhede algorithm,[9] which is the Cartan equivalence method[10] adapted to the case of four-dimensional Lorentzian manifolds. The invariant classification proceeds by reducing the six-dimensional Lorentz frame freedom by normalizing the curvature tensor R and its covariant derivatives, R^q. The unnormalized components of R^q are called *Cartan invariants*. We define the *IC (invariant classification) order* of a given metric to be the highest order q required for deciding the equivalence problem for that metric. An upper bound on the IC order is often referred to as the *Karlhede bound*.

Before proceeding, let us summarize the Karlhede algorithm; for details see Chapter 9.2 of Ref. 4. Set $t_{-1} = 0$ and $d_{-1} = 6$ (the dimension of the Lorentz group). At each order $q \geq 0$, let $0 \leq t_{q-1} \leq t_q$ denote the number of functionally independent Cartan invariants, and let $6 \geq d_{q-1} \geq d_q$ denote the dimension of the joint isotropy group of the normalized $R, R^1, \ldots, R^q$. The algorithm terminates as soon as $t_{q-1} = t_q$ and $d_{q-1} = d_q$. A value of $d_q = 0$ means that there exists an invariant

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TABLE I. Type (0, 2, 2) G_2 solutions.

G_2	$f(\zeta, u)$	Invariant condition
B_{22}	$F(u^{-ik}\zeta)u^{-2}$	$B_2/B_1 = k,\ \Delta X_1 = 2X_1^2,\ \hat{\Upsilon} = 0, AA^* \neq 1, B_1 \neq 0$
C_{22}	$f(\zeta e^{iu})$	$B_1 = 0, \Delta X_2 = 0,\ \hat{\Upsilon} = 0,\ \mu \neq 0, AA^* \neq 1$
L_{22}	$g \log \zeta$	$A = 1, Y = 0$
A_{22}	$f(\zeta)$	$\mu = 0, \Delta\nu = 0$

tetrad. If $t_q < 4$, then Killing vectors are present. The dimension of the isometry group is $4 - t_q + d_q$. Henceforth, we will refer to the sequence $(t_0, t_1, \ldots, t_q)$ as the *invariant count*.

In this paper, we focus on a particularly simple class of VSI spacetimes: the vacuum pp-waves, whose metric has the simple form shown in Eq. (9) below. The class of pp-waves is perhaps the most interesting case to deal with in the context of invariant classification. In this regard, one can cite Brinkmann's theorem and its generalizations[11,12] that characterize pp-waves as the most degenerate class in the problem of algebraically recovering the metric from the curvature tensor.

The symmetry classes for pp-waves were initially classified by Kundt and Ehlers[13] (see also Table 24.2 of Ref. 4) for vacuum solutions, and subsequently extended by Sippel and Goenner[14] to the general case. The Karlhede bound for pp-waves was investigated in Refs. 1 and 15 where $q \leq 4$ was established; however, it was not known whether this bound is sharp, or if it could be lowered further. Despite the fact that these metrics have a very simple form, depending on just one parametric function $f(\zeta, u)$ (see Eq. (9) below), the present paper is the first to present a complete invariant classification for vacuum pp-waves, and to establish the sharpness of the $q \leq 4$ bound.

All vacuum pp-waves have at least one Killing vector. Kundt and Ehlers identified 3 classes of G_2 solutions (Table I), 4 classes of G_3 solutions (Table II), a universal form for the G_5 solutions, and two types of homogeneous G_6 solutions (Table III). Below, we exhibit explicit Cartan invariants that distinguish the various special sub-classes in an invariant fashion.

The G_1, G_2, G_3 solutions ($\alpha \neq 0$) and the G_5, G_6 solutions ($\alpha = 0$) form two distinct solution branches; here α is a fundamental first-order invariant which will be defined precisely in Sec. II. The classification of the $\alpha \neq 0$ class is summarized in Figure 1. The numbers in the solution labels refer to the invariant count with the initial 0 and any trailing 3 omitted. Thus, solution form AP_{123} refers to a metric with an invariant count of (0, 1, 2, 3, 3) while AP_{122} refers to a G_2 solution with an invariant count of (0, 1, 2, 2). The G_1 solutions have three independent invariants and thus their label indices end with a 3. For the same reason, the indices of the G_2 solutions end with a 2 while the indices of the G_3, G_5 solutions end with a 1.

From the point of view of invariant classification there are 4 classes of generic G_2 solutions. We label these A_{22}, B_{22}, C_{22}, L_{22} and summarize their invariant classification in Table I (the Cartan invariants in the third column will be defined in Sec. III.) Kundt-Ehlers described forms B_{22} and L_{22}. Their third G_2 form is

$$f(\zeta, u) = F(\zeta e^{iku}), \tag{1}$$

where F is a holomorphic function and k a real constant. The k parameter is not essential, and if $k \neq 0$ can be normalized to $k \to 1$ by means of a coordinate transformation. In terms of the present

TABLE II. Type (0, 1, 1) G_3 solutions.

G_3	$f(\zeta, u)$	Invariant condition
BL_{11}	$Cu^{-2} \log \zeta$	$B = 0, A = 1, \Delta\mu = \mu^2, Y = 0, \mu \neq 0$
AP_{11}	$(k_0\zeta)^{2ik_1}$	$\mu = 0, AA^* = 1, Y = 0, A^2 \neq 1$
AE_{11}	$\exp(k_0\zeta)$	$\mu = 0, A = -1, Y = 0$
AL_{11}	$e^{ik} \log \zeta$	$\mu = 0, A = 1, Y = 0$

TABLE III. The G_5 and G_6 solutions.

Label	$f(\zeta, u)$	Invariant condition
A_{11}	$g\zeta^2$	$\alpha = 0, \Delta\gamma \neq 0$
B_0	$k_1 u^{2ik_0-2}\zeta^2$	$\alpha = 0, \Delta\gamma = 0, \Re\gamma \neq 0$
C_0	$\exp(2ik_0u)\zeta^2$	$\alpha = 0, \Delta\gamma = 0, \Re\gamma = 0$

terminology, the Kundt-Ehlers solutions of type (1) belong to class C_{22} in the case of $k = 1$, and to class A_{22} if $k = 0$.

One benefit of the invariant classification is a clear description of the mechanism of specialization of the $G_1 \to G_2 \to G_3$ solutions. In order to understand the $G_1 \to G_2$ specialization one first has to understand the invariant mechanism by which the solution forms in Table I arise.

All of the above G_2 solution forms, indeed, all solutions encountered in this investigation conform to a general ansatz,

$$f(\zeta, u) = g_1 F(g_2\zeta) + g_3\zeta, \tag{2}$$

where F is a holomorphic functions and where $g_i = g_i(u), i = 1, 2, 3$ are complex valued functions of one variable. The invariant characterization of the general ansatz, namely, Eq. (68), is derived in Proposition 3.3 below.

This general form, which we name A_{23}^{**}, bifurcates into a number of more specialized forms, which we label by A, B, C, P, E, L and by numerical indices that describe the invariant count. The various possibilities are displayed in Table V. In this table, an asterisk denotes a generic precursor of a more specialized solution, while the labels P, E, L refer to, respectively, solutions of power, exponential, and logarithmic type. Roughly speaking, the Kundt-Ehlers G_2 solution forms are appropriate specializations of the A, B, C, and L solution forms.

The $G_1 \to G_2$ specialization can be understood via the notion of a "precursor solution." This is a G_1 solution that is mild generalization of a corresponding G_2 solution. For example, the precursor

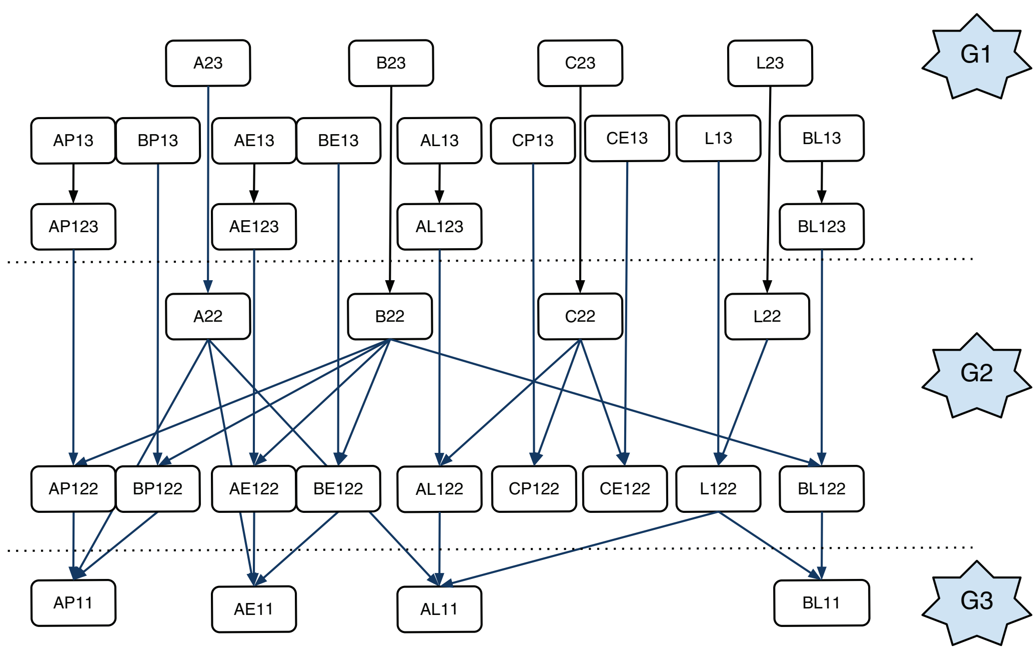

FIG. 1. Specialization of $G_1 \to G_2 \to G_3$ solutions in the $\alpha \neq 0$ class.

of the B_{22} solution

$$f(\zeta, u) = F(u^{-ik}\zeta)u^{-2}$$

is the B_{23} solution

$$f(\zeta, u) = F(u^{-ik}\zeta)u^{-2} + g\zeta, \tag{3}$$

where $g = g(u)$ is an arbitrary complex valued function of one variable. Precursors of the other G_2 solutions have an analogous form. The invariant conditions that define the various precursor classes are listed in Table VII. In each case, the specialization to a G_2 involves the loss of the $g\zeta$ term, or equivalently, the vanishing of a certain higher order invariant.

As we show below, a vacuum pp-wave has no zeroth order invariants,[4] and generically two independent first order invariants, α, α^*. In order to understand the $G_2 \to G_3$ specialization it is necessary to understand the sub-class of solutions for which $t_1 = 1$, i.e., metrics for which the invariants α and α^* are functionally dependent. We refer to such solutions as belonging to the (0,1) class and devote Sec. IV to their analysis. Thus, the specialization to the G_3 solutions follows the following path:

$$(0, 1, 3) \to (0, 1, 2, 2) \to (0, 1, 1),$$

where the middle step consists of type (0,1) G_2 solutions, summarized in Table IX.

Another consequence of our analysis is a firm determination of the Karlhede bound for vacuum pp-waves. It turns that $q \leq 4$ is the sharp bound.

Theorem 1.1. *There exist vacuum pp-wave spacetimes with an IC order $q = 4$. Every such metric belongs to one of the four classes exhibited in Table IV below.*

Note that metrics that require 4th order invariants for invariant classification necessarily have a (0,1,2,3,3) as their invariant count.

The rest of the paper is organized as follows. Section II is an introductory description of the Karlhede algorithm as it applies to the class of vacuum pp-wave metrics. In particular, this section describes the fundamental bifurcation into the generic $\alpha \neq 0$ class and the specialized $\alpha = 0$ subclass. The invariant classification of the former consists of 8 sub-class types shown in Figure 2. Section III introduces the various Cartan invariants necessary for the generic classification and derives the A, B, C, P, E, L solution forms in an invariant manner. Section IV deals with the type (0,1) solutions in the $\alpha \neq 0$ class. Section V classifies the G_2-precursor solutions. Section VII derives and classifies the G_1 metrics having maximal IC order; the proof of Theorem 1.1 is given here. Sections III, IV, V, and VII, when taken together, constitute the invariant classification of the G_1 solutions; the specialization diagram for the various G_1 sub-classes is presented in Figure 4. Sections VI and VIII deal with the invariant classification of the G_2 and G_3 solutions, respectively. The $\alpha = 0$ branch consists of G_5 and G_6 solutions. There is a generic G_5 solution that specializes into two distinct classes of homogeneous G_6 solutions, as per Figure 3. This branch of the classification is discussed in Sect. IX and summarized in Table III.

TABLE IV. Type (0, 1, 2, 3) solutions.

G_1	$f(\zeta, u)$	Invariant condition
BL_{123}	$(C\log\zeta + ke^{ih}\zeta)u^{-2}$	$B = 0, A = 1, \Delta\mu = \mu^2, \Delta\log(YY^*) = 4\mu, \mu \neq 0$
AP_{123}	$(k_0\zeta)^{2ik_1} + k_2 e^{ih(1-2ik_1)}\zeta$	$\mu = 0, AA^* = 1, A^2 \neq 1$
		$(1 - 3A)\Delta\log Y + (A - 3)\Delta\log Y^* = 0$
AE_{123}	$\exp(k_0\zeta) + e^{ik_1}e^{h}\zeta$	$\mu = 0, A = -1, \Delta(Y/Y^*) = 0$
AL_{123}	$e^{ik_0}\log\zeta + k_1 e^{ih}\zeta$	$\mu = 0, A = 1, \Delta(YY^*) = 0$

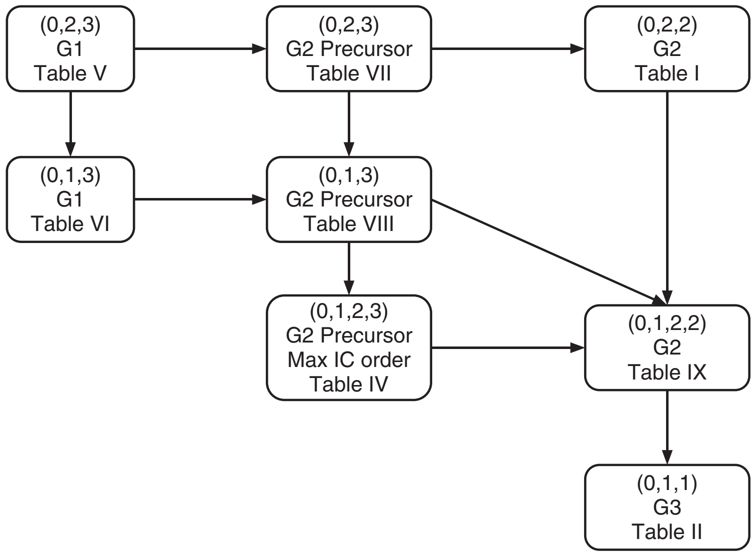

FIG. 2. The invariant classification of the $\alpha \neq 0$ class.

II. VACUUM PP-WAVE SPACETIMES

Throughout, we use the four-dimensional Newman-Penrose formalism[16] adapted to a complex, null-tetrad $(\boldsymbol{e}_a) = (m^a, m^{*a}, \ell^a, n^a) = (\delta, \delta^*, D, \Delta)$. These vectors satisfy

$$\ell_a n^a = 1, \quad m_a m^{*a} = 1,$$

with all other cross-products zero. Equivalently, letting $\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^4$ denote the dual coframe, the metric is given by

$$g = 2\boldsymbol{\theta}^1\boldsymbol{\theta}^2 - 2\boldsymbol{\theta}^3\boldsymbol{\theta}^4.$$

The connection 1-form and the curvature 2-form are defined, respectively, by

$$d\boldsymbol{\theta}^a = \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{\theta}^b, \quad \boldsymbol{\omega}_{(ab)} = 0, \tag{4}$$

$$\boldsymbol{\Omega}^a{}_b = d\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_d. \tag{5}$$

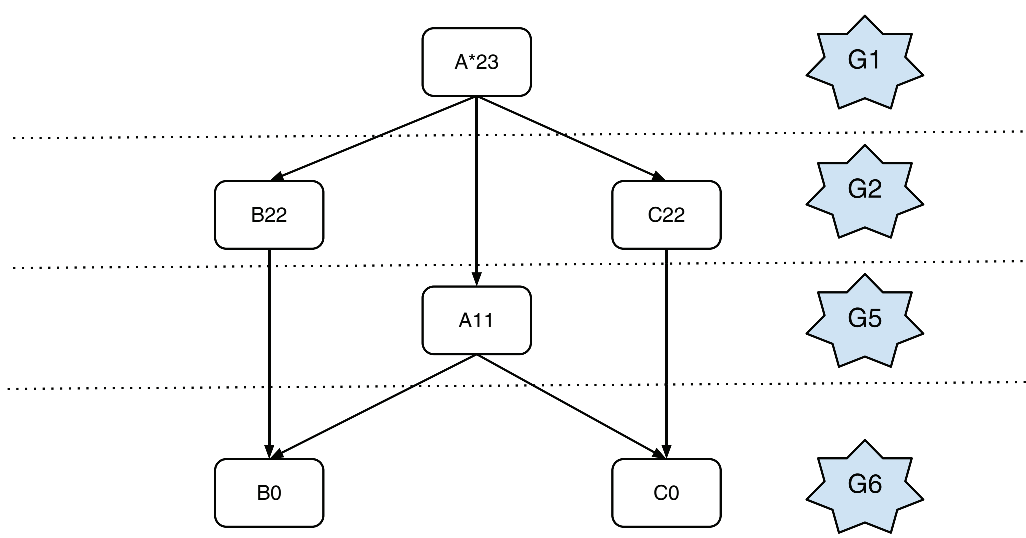

FIG. 3. Specialization diagram for the G_5, G_6 solutions.

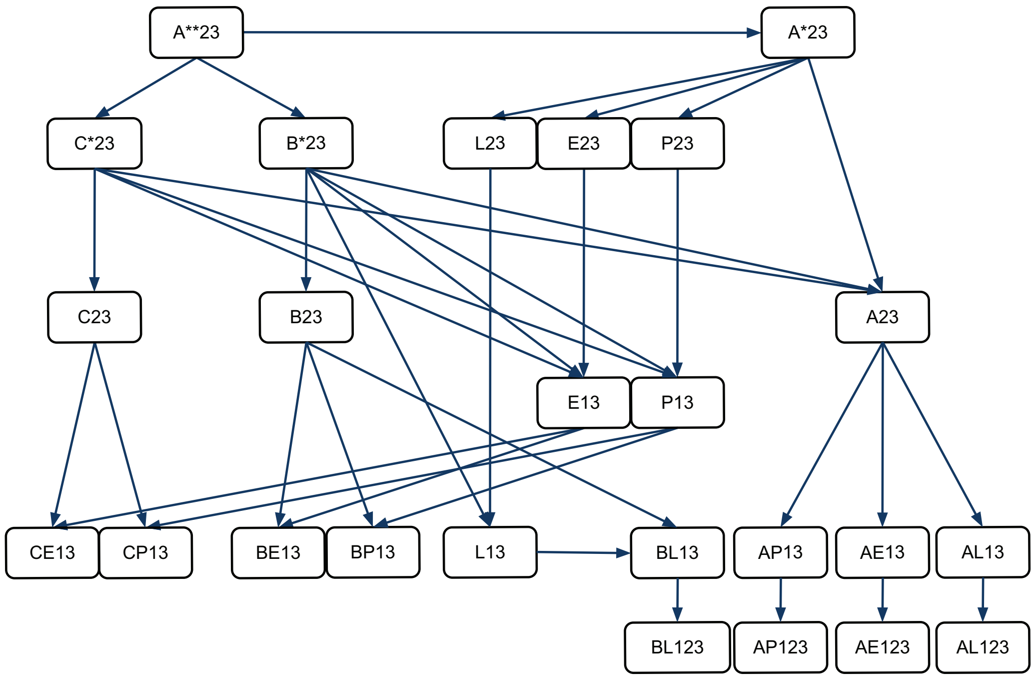

FIG. 4. G_1 solutions.

The connection components are labeled by the 12 Newman-Penrose scalars:

$$-\boldsymbol{\omega}_{14} = \sigma\,\boldsymbol{\theta}^1 + \rho\,\boldsymbol{\theta}^2 + \tau\,\boldsymbol{\theta}^3 + \kappa\,\boldsymbol{\theta}^4, \tag{6}$$

$$\boldsymbol{\omega}_{23} = \mu\,\boldsymbol{\theta}^1 + \lambda\,\boldsymbol{\theta}^2 + \nu\,\boldsymbol{\theta}^3 + \pi\,\boldsymbol{\theta}^4, \tag{7}$$

$$-(\boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{34})/2 = \beta\,\boldsymbol{\theta}^1 + \alpha\,\boldsymbol{\theta}^2 + \gamma\,\boldsymbol{\theta}^3 + \epsilon\,\boldsymbol{\theta}^4. \tag{8}$$

The curvature components are labeled by the Ricci scalar $\Lambda = \bar{\Lambda}$, traceless Ricci components $\Phi_{AB} = \bar{\Phi}_{BA},\ A, B = 0, 1, 2$, and Weyl components $\Psi_C, C = 0, \ldots, 4$:

$$\begin{aligned}
\boldsymbol{\Omega}_{14} &= \Phi_{01}(\boldsymbol{\theta}^{34} - \boldsymbol{\theta}^{12}) - \Phi_{02}\boldsymbol{\theta}^{13} + \Phi_{00}\boldsymbol{\theta}^{24} + \Psi_0\boldsymbol{\theta}^{14} - (\Psi_2 + 2\Lambda)\,\boldsymbol{\theta}^{23} + \Psi_1(\boldsymbol{\theta}^{12} + \boldsymbol{\theta}^{34}),\\
\boldsymbol{\Omega}_{23} &= \Phi_{21}(\boldsymbol{\theta}^{12} - \boldsymbol{\theta}^{34}) + \Phi_{22}\boldsymbol{\theta}^{13} - \Phi_{20}\boldsymbol{\theta}^{24} + \Psi_4\boldsymbol{\theta}^{23} - (\Psi_2 + 2\Lambda)\boldsymbol{\theta}^{14} - \Psi_3(\boldsymbol{\theta}^{12} + \boldsymbol{\theta}^{34}),\\
(\boldsymbol{\Omega}_{12} + \boldsymbol{\Omega}_{34})/2 &= -\Phi_{12}\boldsymbol{\theta}^{13} + \Phi_{10}\boldsymbol{\theta}^{24} + \Psi_1\boldsymbol{\theta}^{14} - \Psi_3\boldsymbol{\theta}^{23} +\\
&\quad + \Phi_{11}(\boldsymbol{\theta}^{34} - \boldsymbol{\theta}^{12}) + (\Psi_2 - \Lambda)(\boldsymbol{\theta}^{12} + \boldsymbol{\theta}^{34}),
\end{aligned}$$

where $\boldsymbol{\theta}^{ab} = \boldsymbol{\theta}^a \wedge \boldsymbol{\theta}^b$.

A pp-wave is a spacetime admitting a covariantly constant null vector field. This entails

$$\kappa = \sigma = \rho = \tau = 0.$$

Such spacetimes are necessarily Petrov type N or type O and belong to the Kundt class; see Sec. 24.5 of Ref. 4. A vacuum pp-wave that is not flat-space is necessarily type N:

$$\Phi_{AB'} = 0, \quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0,\ \Psi_4 \neq 0,$$

Applying a boost and a spatial rotation we normalize the tetrad by setting $\Psi_4 \to 1$. Therefore, there are no 0th order Cartan invariants. The remaining frame freedom consists of the 2-dimensional group of null rotations.

As shown in Sec. 24.5 of Ref. 4, integration of the above equations yields the following metric:

$$\mathrm{d}s^2 = 2\mathrm{d}\zeta\mathrm{d}\bar{\zeta} - 2\mathrm{d}u\mathrm{d}v - (f + \bar{f})\mathrm{d}u^2, \tag{9}$$

where $f = f(\zeta, u)$ is analytic in ζ. The above form is preserved by the following class of transformations:

$$\hat{\zeta} = \mathrm{e}^{\mathrm{i}k}(\zeta + h(u)), \tag{10}$$

$$\hat{v} = a(v + h'(u)\bar{\zeta} + \bar{h}'(u)\zeta + g(u)), \tag{11}$$

$$\hat{u} = (u + u_0)/a, \tag{12}$$

$$\hat{f} = a^2(f - \bar{h}''(u)\zeta + 1/2(h'(u)\bar{h}'(u) - g(u))). \tag{13}$$

The Bianchi identities (see Eqs. (7.32c) and (7.32d) of Ref. 4) impose:

$$\beta = \epsilon = 0. \tag{14}$$

Using the notation of Ref. 1, the non-vanishing first-order components are

$$(D\Psi)_{50'} = 4\alpha, \quad (D\Psi)_{51'} = 4\gamma.$$

The transformation law for these components under a null rotation is (see Eq. (7.7c) of Ref. 4)

$$\alpha' = \alpha, \quad \gamma' = \gamma + z\alpha, \tag{15}$$

where z is a complex valued scalar. Therefore, α is a first-order Cartan invariant and the invariant classification divides into two cases: $\alpha = 0$ and $\alpha \neq 0$. In the first case, γ is an invariant, while in the second case, we fix the tetrad by normalizing $\gamma \to 0$. These two cases correspond to class IIa and IIb of the Collins classification[1] of type N, vacuum spacetimes. We consider them below in more detail.

Proposition 2.1. Suppose that $\alpha \neq 0$. Then, $d_p = 0$ for $p \geq 1$. The possible values of the invariant count sequence are

$$(0, 2, 3, 3),\ (0, 1, 3, 3),\ (0, 1, 2, 3, 3),\ (0, 2, 2),\ (0, 1, 2, 2),\ (0, 1, 1).$$

The first 3 possibilities describe a G_1, the next 2 possibilities are a G_2, and the last possibility is a G_3. The Cartan invariants are generated by

$$\delta^{*n}\alpha, \quad \delta^j\Delta^{n-j}\mu, \quad \Delta^n\nu, \quad 0 \leq j \leq n,\ n = 0, 1, 2, \ldots$$

and their complex conjugates, where the above spin coefficients are calculated relative to the normalized $\Psi_4 \to 1$, $\gamma \to 0$ tetrad.

Proposition 2.2. Suppose that $\alpha = 0$. Then $d_p = 2$ for all p. The possible values of the invariant count sequence are

$$(0, 1, 1),\ (0, 0).$$

The first possibility describes a G_5. The second possibility describes a G_6 (homogeneous space). The Cartan invariants are generated by

$$\Delta^n\gamma, \quad n = 0, 1, 2, \ldots$$

and their complex conjugates, calculated relative to a tetrad normalized by $\Psi_4 \to 1$.

In Secs. III–IX we will show that each of these cases describes a well-defined class of solutions, and go on to derive a the canonical forms for the metric in each case. We now turn to the proof of Proposition 2.1, which concerns the $\alpha \neq 0$ case. The Newman-Penrose (NP) equations (see (7.21f) and (7.21o) of Ref. 4) imply the additional constraints:

$$\pi = \lambda = 0.$$

The non-vanishing second-order curvature components are (see Eqs. (4.2a)– (4.2t) of Ref. 1):

$$
\begin{aligned}
(D^2\Psi)_{50';00'} &= 4D\alpha,\\
(D^2\Psi)_{50';01'} &= 4\delta\alpha - 4\alpha^*\alpha,\\
(D^2\Psi)_{50';10'} &= 4\delta^*\alpha + 20\alpha^2,\\
(D^2\Psi)_{50';11'} &= 4\Delta\alpha,\\
(D^2\Psi)_{51';10'} &= -4\mu^*\alpha,\\
(D^2\Psi)_{51';11'} &= -4\nu^*\alpha.
\end{aligned}
$$

Here we are using the notation of Ref. 2 for symmetrized dyad components. No invariants will be missed, because it can be proved using the Bianchi and Ricci identities that at all orders of covariant differentiation of the Weyl spinor only the symmetrized parts are algebraically independent. The details can be found in Appendix 2 of the just cited reference.

In light of the above equations, the independent second-order Cartan invariants are μ, ν, $\delta^*\alpha$, and their complex conjugates. The commutator relations are

$$\Delta D - D\Delta = 0, \tag{16}$$

$$\delta D - D\delta = \alpha^* D, \tag{17}$$

$$\delta\Delta - \Delta\delta = -\nu^* D - \alpha^*\Delta + \mu\delta, \tag{18}$$

$$\delta^*\delta - \delta\delta^* = (\mu^* - \mu)D - \alpha^*\delta^* + \alpha\delta. \tag{19}$$

The dual structure equations are

$$\mathrm{d}\omega^1 = \alpha\omega^1 \wedge \omega^2 - \mu\omega^1 \wedge \omega^3, \tag{20}$$

$$\mathrm{d}\omega^3 = (\alpha^*\omega^1 + \alpha\omega^2) \wedge \omega^3, \tag{21}$$

$$\mathrm{d}\omega^4 = (\mu^* - \mu)\omega^1 \wedge \omega^2 + (\nu^*\omega^1 + \nu\omega^2) \wedge \omega^3 - (\alpha^*\omega^1 + \alpha\omega^2) \wedge \omega^4. \tag{22}$$

The NP-equations imply the following relations amongst the invariants:

$$D\alpha = 0, \tag{23}$$

$$\delta\alpha = \alpha\alpha^*, \tag{24}$$

$$\Delta\alpha = -\mu^*\alpha, \tag{25}$$

$$D\mu = 0, \tag{26}$$

$$\delta^*\mu = -\alpha\mu, \tag{27}$$

$$D\nu = 0, \tag{28}$$

$$\delta^*\nu = 1 - 3\alpha\nu, \tag{29}$$

$$\delta\nu = -\alpha^*\nu + \Delta\mu + \mu^2. \tag{30}$$

Higher order relations follow in a straightforward manner from these and from the commutator relations. Fixing $\Psi_4 \to 1$ reduces the isotropy to null rotation. Fixing $\gamma \to 0$ eliminates this frame

TABLE V. Type (0, 2, 3) solution classes.

G_1	$f(\zeta, u)$	Invariant condition
A_{23}^{**}	$g_1 F(g_2\zeta) + g_3\zeta$	$\delta A^* \delta^2 M = \delta M \delta^2 A^*$
A_{23}^{*}	$F(g\zeta)g^{-2} + g_1\zeta$	$\delta M = 0$
A_{23}	$f(\zeta) + g_1\zeta$	$\mu = 0$
B_{23}^{*}	$F(h^{ik}\zeta)h^2 + g\zeta$	$B_2/B_1 = k, AA^* \neq 1, B_1 \neq 0$
C_{23}^{*}	$F(e^{ih}\zeta) + g\zeta$	$B_1 = 0, A \neq 1$
P_{23}	$(e^{g_1}\zeta)^{ih} + g_2\zeta$	$AA^* = 1, A^2 \neq 1$
E_{23}	$\exp(g_1\zeta) + g_2\zeta$	$A = -1$
L_{23}	$g_1 \log\zeta + g_2\zeta$	$A = 1$

freedom. Therefore, the isotropy is trivial. Equation (24) implies that α is not constant. All invariants are annihilated by D. Therefore, there are either 3, 2, or 1 independent Cartan invariants. The conclusions of Proposition 2.1 now follow directly from the Karlhede algorithm.

Next we present the proof of Proposition 2.2, which treats the $\alpha = 0$ class. As was mentioned above, the first-order Cartan invariants are generated by

$$(D\Psi)_{51'} = 4\gamma.$$

The Newman-Penrose equations (see (7.21f), (7.21o), (7.21r) of Ref. 4) imply

$$D\gamma = 0,\ \ \delta\gamma = 0,\ \ \delta^*\gamma = 0. \tag{31}$$

There is only one non-zero second-order curvature component, namely,

$$(D^2\Psi)_{51';11'} = 4\Delta\gamma + 20\gamma^2 + 4\bar{\gamma}\gamma,$$

The operator transformation law for null rotations is (see (7.7a) of Ref. 4)

$$D' = D,\ \ \delta' = \delta + BD,\ \ \Delta' = \Delta + B\delta^* + \bar{B}\delta + B\bar{B}D.$$

Therefore, by (31), $\Delta^n\gamma$ is well-defined, despite the fact that no canonical choice of Δ exists and is invariant with respect to null rotations. By Eqs. (7.6a)– (7.6d) of Ref. 4, all commutators are spanned by δ, δ^*, D. This implies that

$$\delta\Delta^n\gamma = \delta^*\Delta^n\gamma = D\Delta^n\gamma = 0.$$

Therefore, there are two possibilities. Either γ is a constant, in which case we have a homogeneous G_6; or γ is the unique independent invariant, in which case we have a G_5. This concludes the proof of Proposition 2.2.

III. THE G_1 SOLUTIONS

In this section, we derive solutions for certain key G_1 sub-classes. We assume that $\alpha \neq 0$ for the remainder of this section. The solutions are summarized in Tables V and VI. In the tables, $F = F(z)$ is an analytic function; $g = g(u)$ is complex-valued function of u; $h = h(u)$ is a real-valued function of u; and k is a real constant. The meanings of g_1, g_2, h_1, h_2, k_1, k_2 are analogous.

TABLE VI. Type (0, 1, 3) solutions.

G_1	$f(\zeta, u)$	Invariant condition
P_{13}	$(k_0 e^{ih}\zeta)^{2ik_1} + g\zeta$	$B = 0, A^2 \neq 1, \mu \neq 0$
E_{13}	$\exp(ke^{ih}\zeta) + g\zeta$	$B = 0, A = -1$
L_{13}	$e^{ik}h \log\zeta + g\zeta$	$B = 0, A = 1$

Throughout, this section δ, δ^*, Δ, D is a tetrad defined by the following normalizations:

$$\Psi_4 = 1, \quad \gamma = 0. \tag{32}$$

Let ω^1, ω^2, ω^3, ω^4 denote the dual coframe.

We now introduce the following invariants:

$$A := \delta^*\alpha/\alpha^2, \tag{33}$$

$$B := \mu A - \mu^*, \quad B_1 = \operatorname{Re} B,\ B_2 = \operatorname{Im} B, \tag{34}$$

$$M := \alpha\mu, \tag{35}$$

$$X := B/(AA^* - 1), \quad X_1 = \operatorname{Re} X,\ X_2 = \operatorname{Im} X, \quad AA^* \neq 1, \tag{36}$$

$$Y := (3 - A)\nu - 1/\alpha + (\Delta\mu + \mu^2)/\alpha^*, \tag{37}$$

$$\hat{\nu} := \nu + X(\mu + 2X^*)/\alpha^*, \quad AA^* \neq 1, \tag{38}$$

$$\hat{\Upsilon} := \Delta(\hat{\nu}/X^*) - 2\hat{\nu} + 1/\alpha - 4\mathrm{i}XX_2/\alpha^*, \tag{39}$$

$$\hat{\Delta} := \Delta + \hat{z}^*\delta + \hat{z}\delta^* + \hat{z}\hat{z}^*D, \quad \hat{z} := X^*/\alpha, \quad AA^* \neq 1, \tag{40}$$

$$\tilde{X} := \Delta \log M^*/(1 - A^*), \quad A \neq 1, \tag{41}$$

$$\tilde{\nu} := \nu + \tilde{X}^*(\tilde{X} + 2\mu - A^*\tilde{X}^*)/\alpha^*, \quad A \neq 1, \tag{42}$$

$$\tilde{\Upsilon} := \Delta(\tilde{\nu}/\tilde{X}^*) - 2\tilde{\nu} + 1/\alpha - 4\mathrm{i}\tilde{X}\tilde{X}_2/\alpha^*, \tag{43}$$

$$\tilde{\Delta} := \Delta + \tilde{z}^*\delta + \tilde{z}\delta^* + \tilde{z}\tilde{z}^*D, \quad \tilde{z} := \tilde{X}^*/\alpha. \tag{44}$$

In the sequel we show that the above invariants suffice to invariantly classify the various subclasses of G_1, G_2, and G_3 vacuum pp-wave solutions.

The above definitions imply the following elementary identities:

$$A = (B + \mu^*)/\mu, \tag{45}$$

$$AA^* - 1 = A(B^* + \mu)/\mu^* - 1 = (AB^* + B)/\mu^*, \tag{46}$$

$$X = \frac{B\mu^*}{AB^* + B}. \tag{47}$$

In Proposition 3.1 below we show that $\hat{\Delta}\alpha = 0$ in the generic case where $AA^* \neq 1$. We therefore introduce the alternate invariant tetrad $\hat{\delta}$, $\hat{\delta}^*$, $\hat{\Delta}$, $\hat{D}$ which is defined by the normalizations

$$\Psi_4 \to 1, \quad \hat{\Delta}\alpha \to 0,$$

and which differs from the tetrad (32) by the null rotation in (40). A Killing vector V necessarily annihilates all invariants.[17] We employ the alternate tetrad because sometimes it is convenient to work in a frame where $\hat{\Delta}$ is a linear combination of Killing vectors.

For a given vector field V let us write

$$V = V^1\delta + V^2\delta^* + V^3\Delta + V^4 D,$$

where V^1, V^2 are complex conjugate and V^3, V^4 are real. The following proposition shows that if $AA^* \neq 1$, then the normalization $\hat{\Delta}\alpha \to 0$ selects a well-defined invariant tetrad.

Proposition 3.1. Suppose that $AA^* \neq 1$. Then, every vector field that satisfies

$$\mathcal{L}_V \alpha = \mathcal{L}_V \alpha^* = 0, \quad V^3 \neq 0. \tag{48}$$

has the form $V = a\hat{\Delta} + bD,\ a \neq 0$. If $AA^* = 1$, but $B \neq 0$, then (48) does not have a solution. If $AA^* = 1$ and $B = 0$, then there is a 1-parameter family of solutions to (48).

Proof. The null-rotation transformation law for Δ is (see (7.7 c) of Ref. 4),

$$\hat{\Delta} = \Delta + \hat{z}^*\delta + \hat{z}\delta^* + \hat{z}\hat{z}^* D. \tag{49}$$

Hence, by (23)–(25) and (33) we seek a scalar $\hat{z}$ such that

$$\begin{pmatrix} A\alpha & \alpha^* \\ \alpha & A^*\alpha^* \end{pmatrix} \begin{pmatrix} \hat{z} \\ \hat{z}^* \end{pmatrix} = \begin{pmatrix} \mu^* \\ \mu \end{pmatrix}. \tag{50}$$

If $AA^* \neq 1$, the solution is

$$\hat{z} = \frac{A^*\mu^* - \mu}{(AA^* - 1)\alpha} = \frac{X^*}{\alpha}. \tag{51}$$

If $AA^* = 0$, then the system has rank 1. In this case the system is consistent if and only if

$$\begin{vmatrix} A\alpha & \mu^* \\ \alpha & \mu \end{vmatrix} = \alpha B = 0.$$

□

Next, we establish some key relations for these invariants and certain other scalars that will prove useful in our calculations.

Proposition 3.2. Suppose that $\alpha \neq 0$. If the normalization (32) holds then

$$\alpha = e^{a-a^*}/(Z_a)^* = e^{a-a^*}(a_\zeta)^*, \tag{52}$$

$$\mu = e^{-a-a^*} L_u, \tag{53}$$

$$M = (e^{-2a}/Z_a)^* L_u, \tag{54}$$

$$\nu = e^{-a-3a^*}\left(Z_{uu} + (\Phi_a/Z_a)^*\right) = e^{-a-3a^*}(Z_{uu} + (f_\zeta)^*), \tag{55}$$

$$A = -1 - (L_a)^*, \tag{56}$$

$$\omega^1 = (\alpha^*)^{-1} da, \tag{57}$$

$$\omega^3 = e^{a+a^*} du, \tag{58}$$

$$\omega^4 = e^{-a-a^*}\left((f + f^* + Z_u Z_u^*)du + dv - Z_u d\zeta^* - Z_u^* d\zeta\right), \tag{59}$$

where

$$a := \frac{1}{4}\log f_{\zeta\zeta}, \quad a_\zeta \neq 0, \tag{60}$$

$$\zeta =: Z(a, u), \quad \zeta^* =: Z^*(a^*, u), \tag{61}$$

$$L := \log Z_a, \tag{62}$$

$$\Phi(a, u) := f(\zeta, u). \tag{63}$$

We also have

$$\mu = XA^* + X^*, \tag{64}$$

$$\delta A = 0. \tag{65}$$

Furthermore, if $Q = Q(a, u)$ *then*

$$\delta Q = \alpha^* Q_a, \tag{66}$$

$$\Delta Q = e^{-a-a^*} Q_u. \tag{67}$$

We begin by deriving some a key classes of G_1 solutions; *all* the various solutions discussed in this paper are subclasses of these general categories.

Proposition 3.3. Suppose that $\alpha \neq 0$. *The following conditions are equivalent:*

$$\delta A^* \, \delta^2 M = \delta M \, \delta^2 A^*, \tag{68}$$

$$f(\zeta, u) = g_1 F(g_2 \zeta) + g_3 \zeta, \tag{69}$$

where $F = F(z)$ *is an analytic function such that* $F'''(z) \neq 0$ *and where* $g_i = g_i(u), i = 1, 2, 3$ *are complex-valued such that* $g_1, g_2 \neq 0$. *Furthermore,* $\delta M = 0$ *if and only if* $g_1 = g_2^{-2}$, *i.e.,*

$$f(\zeta, u) = F(g\zeta) g^{-2} + g_3 \zeta. \tag{70}$$

In addition $M = 0$ *if and only if* $g_1 = g_2 = 1$, *i.e.,*

$$f(\zeta, u) = F(\zeta) + g\zeta. \tag{71}$$

Proof. Our first claim is that (69) is equivalent to the following chain of conditions:

$$f_{\zeta\zeta} = g_4 F_1(g_2 \zeta), \quad g_4 = g_1 g_2^2, \; F_1(z) = F''(z), \tag{72}$$

$$a = F_2(g_2 \zeta) + g_5, \quad g_5 = \frac{1}{4} \log g_4, \; F_2(z) = \frac{1}{4} \log F_1(z), \tag{73}$$

$$\zeta = Z(a, u) = F_3(a - g_5)/g_2, \quad F_3(F_2(z)) = z, \tag{74}$$

$$L = F_4(a - g_5) + g_6, \quad g_6 = -\log g_2, \; F_4(z) = \log F_3'(z), \tag{75}$$

$$L_u + g_7 L_a = g_8, \quad g_7 = g_5'(u), \; g_8 = g_6'(u). \tag{76}$$

Note that since $\alpha \neq 0$, by (52), we must have $L \neq 0$. We now consider two cases.

First, let us consider the case of $\delta M = 0$. Note that in this case (68) holds trivially. Also, in this case, $L_{ua} = 0$, and hence without loss of generality, $g_5 = 0$. The case of $M = 0$ is true if and only if $L_u = 0$. Here $g_5 = 0$ and $g_1 = g_2 = 1$.

Let us now consider the generic case where $\delta M \neq 0$. In this case, (68) can be restated as

$$\delta \left(\frac{\delta A^*}{\delta M} \right) = 0.$$

Observe that

$$\frac{\delta(A^*/M)}{\delta(1/M)} = A^* - M\frac{\delta A^*}{\delta M},$$
$$\delta\left(\frac{\delta(A^*/M)}{\delta(1/M)}\right) = -M\delta\left(\frac{\delta A^*}{\delta M}\right).$$

Hence, (68) is equivalent to

$$\delta\left(\frac{\delta(A^*/M)}{\delta(1/M)}\right) = 0.$$

Next, we observe that

$$\frac{\delta(A^*/M)}{\delta(1/M)} = -1 - L_a + \frac{L_u L_{aa}}{L_{au}}.$$

Hence,

$$\delta^*\left(\frac{\delta(A^*/M)}{\delta(1/M)}\right) = 0.$$

Hence, by (66), condition (68) is equivalent to

$$\frac{\delta(A^*/M)}{\delta(1/M)} = g, \quad g = g(u),$$
$$\delta\left(\frac{1 + A^* - g}{M}\right) = \delta\left(\frac{-L_a + g}{L_u}\right) Z^*_{a^*} e^{2a^*} = 0.$$

The latter condition is equivalent to (76). □

Proposition 3.4. Suppose that $B_1 \neq 0$ *and* $AA^* \neq 1$. *Then the following are equivalent: (i)* $B_2/B_1 = k$, *is a real constant and (ii)* $f(\zeta, u) = F(h^{\mathrm{i}k}\zeta)h^2 + g\zeta$.

Proof. Let $C = 1 + \mathrm{i}k$ so that condition (i) is equivalent to

$$\frac{B}{B^*} = \frac{B_1 + \mathrm{i}B_2}{B_1 - \mathrm{i}B_2} = \frac{C}{C^*}, \tag{77}$$

or $\mathrm{Im}(B/C) = 0$. Suppose that (i) holds. By (46),

$$AA^* - 1 = (B/\mu^*)\left(1 + (C^*/C)A\right) = (B^*/\mu)\left(1 + (C/C^*)A^*\right).$$

Hence, by (53) and (56),

$$e^{-a-a^*}(C^*/B^*)(AA^* - 1) = e^{-a-a^*}(C^* + CA^*)/\mu = (-2\mathrm{i}k - CL_a)/L_u. \tag{78}$$

By Proposition 3.2, the above is both real and holomorphic in a, and hence independent of a. Hence,

$$L_u + (1 + \mathrm{i}k)h_1 L_a = -2\mathrm{i}kh_1, \tag{79}$$

where $h_1 = h_1(u) \neq 0$ is real. Conversely, (79) with $h_1 \neq 0$ implies that C/B is real. The latter implies condition (i). Furthermore, (79) is equivalent to the following chain of conditions:

$$L = F_1(a - (1 + \mathrm{i}k)h_2) - 2\mathrm{i}kh_2, \quad h_2'(u) = h_1(u),$$
$$Z = F_2(a - (1 + \mathrm{i}k)h_2)\mathrm{e}^{-2\mathrm{i}kh_2},$$
$$a = (1 + \mathrm{i}k)h_2 + F_3(\mathrm{e}^{2\mathrm{i}kh_2}\zeta),$$
$$f_{\zeta\zeta} = h^{2+2\mathrm{i}k}F_4(h^{\mathrm{i}k}\zeta), \quad h = \mathrm{e}^{2h_2},$$
$$f = F(h^{\mathrm{i}k}\zeta)h^2 + g\zeta.$$

□

The other generic G_1 solutions are derived in the following Proposition. The proofs are appropriate modifications of the techniques utilized for the proofs of Propositions 3.3 and 3.4 above.

Proposition 3.5. Suppose that $A \neq 1$. *Then, the following are equivalent: (i)* $B_1 = 0$ *and (ii)* $f(\zeta, u) = F(\mathrm{e}^{\mathrm{i}h}\zeta) + g\zeta$.

Proposition 3.6. The following are equivalent: (i) $AA^* = 1$ *and (ii)* $L = Pa + g$, *where* $g = g(u)$ *and* $P = P(u)$ *such that* $PP^* + P + P^* = 0$.

Proposition 3.7. Suppose that $A^2 \neq 1$. *The following are equivalent: (i)* $AA^* = 1$ *and (ii)* $f(\zeta, u) = (\mathrm{e}^{g_1}\zeta)^{\mathrm{i}h} + g_2\zeta$.

Proposition 3.8. The following are equivalent: (i) $A = -1$ *and (ii)* $f(\zeta, u) = \exp(g_1\zeta) + g_2\zeta$.

Proposition 3.9. The following are equivalent: (i) $A = 1$ *and (ii)* $f(\zeta, u) = g_1 \log\zeta + g_2\zeta$.

Note that if $A = 1$, then $B = \mu - \mu^*$. Hence, if $A = 1$, then $B_1 = 0$ automatically.

IV. THE (0, 1) CLASS

Above we showed that α, α^* generate the first-order invariants. Generically, these are independent and hence, generically, the invariant count is (0, 2). However, an important subclass occurs for which $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* = 0$. We will refer to these as the (0, 1) solutions. The next two Propositions characterize the (0,1) solutions in terms of invariants.

Proposition 4.1. If $\mu \neq 0$, *then* $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* = 0$ *if and only if* $B = 0$. *In this case, the condition* $AA^* = 1$ *follows automatically. If* $\mu = 0$, *then* $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* = 0$ *if and only if* $AA^* = 1$.

Proof. By (24) and (25),

$$\delta\alpha\delta^*\alpha^* - \delta^*\alpha\delta\alpha^* = \alpha^2\alpha^{*2}(1 - AA^*),$$

$$\delta\alpha\Delta\alpha^* - \delta\alpha^*\Delta\alpha = \alpha\alpha^{*2}B^*.$$

Hence, the condition $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* = 0$ is equivalent to the conjunction of $AA^* = 1$ and $B = 0$. However, if $\mu \neq 0$ and $B = 0$, then $A = \mu^*/\mu$, and hence $AA^* = 1$ automatically. Therefore, if $\mu \neq 0$, then the condition $B = 0$ suffices for a (0,1) solution. On the other hand, if $\mu = 0$, then $B = 0$, and therefore the condition $AA^* = 1$ suffices for a (0,1) solution. □

Proposition 4.2. Suppose that $B = 0$ *and* $AA^* = 1$. *Then, necessarily A is a constant and* $\delta M = 0$.

Proposition 4.3. Suppose that $B = 0$ *and* $AA^* = 1$. *If* $A \neq 1$, *then*

$$L = -\frac{A+1}{A}\,a + \frac{A-1}{A}\,(k + \mathrm{i}h), \tag{80}$$

where k is a real constant, and $h = h(u)$ *is real. If* $A = 1$, *then*

$$L = -2a + h + k\mathrm{i}. \tag{81}$$

Proposition 4.4. A type (0, 1) *solution belongs to one of the classes shown in Table VI.*

Proof. By Proposition 4.1, $B = 0$ and $AA^* = 1$. We proceed by cases. Suppose that $A^2 \neq 1$. By Proposition 3.7,

$$f = (\mathrm{e}^{g_1}\zeta)^{2\mathrm{i}h_1} + g_2\zeta, \quad L = Pa - (1 + P/2)g_1, \quad P = -2\mathrm{i}/(h_1 + \mathrm{i}).$$

TABLE VII. Type (0, 2, 3) G_2-precursor solutions.

G_1	$f(\zeta, u)$	Invariant condition
B_{23}	$F(u^{-ik}\zeta)u^{-2} + g\zeta$	$B_2/B_1 = k,\ \Delta X_1 = 2X_1^2, AA^* \neq 1, B_1 \neq 0$
C_{23}	$f(\zeta e^{iu}) + g\zeta$	$B_1 = 0, \Delta X_2 = 0, \mu \neq 0, AA^* \neq 1$
L_{23}	$g_1 \log\zeta + g_2\zeta$	$A = 1$
A_{23}	$f(\zeta) + g_1\zeta$	$\mu = 0$

By Proposition 4.3, $h_1 = k_1$ a real constant. Since $(1 + P/2) = (A - 1)/(2A)$, we must have $g_1 = k_2 + ih_2$. This gives form P_{13}. Next, consider the case $A = -1$. Here, $L = k + ih$. By Proposition 3.8 we arrive at form E_{13}. Finally, if $A = 1$, then (81) and Proposition 3.9 give form L_{13}. □

V. THE G_2 PRECURSORS

As above, we assume that $\alpha \neq 0$ and that $\delta, \delta^*, \Delta, D$ is a tetrad normalized so that $\Psi_4 \to 1$ and $\gamma \to 0$. In this section we classify the solutions that meet the following criterion.

Definition 5.1. We say that a vacuum pp-wave metric is a G_2-precursor if there exists a real vector field $V = V^1\delta + V^2\delta^* + V^3\Delta + V^4 D$ such that

$$\mathcal{L}_V\omega^1 = \mathcal{L}_V\omega^2 = \mathcal{L}_V\omega^3 = 0, \quad V^1 \neq 0,\ \text{or}\ V^3 \neq 0. \tag{82}$$

A Killing vector annihilates all invariant scalars and invariant differential forms (see Chapters 8–10 of Ref. 17 for a proof). Thus, the "precursor" terminology reflects the fact that (82) is a necessary, but not sufficient condition, for the existence of a Killing vector independent from $D = \partial_v$. The requisite propositions and proofs are presented below. The resulting classification of precursor solutions is summarized in Tables VII and VIII.

Proposition 5.2. A vector field $V = V^1\delta + V^2\delta^* + V^3\Delta + V^4 D$ annihilates ω^1, ω^2, ω^3 if and only if $C = \alpha^* V^1$ is a constant, while V^3 satisfies

$$V^3\mu^* = C + C^* A, \tag{83}$$

$$\delta V^3 = \alpha^* V^3, \tag{84}$$

$$\Delta V^3 = -C - C^*. \tag{85}$$

TABLE VIII. Type (0, 1, 3) G_2-precursor solutions.

G_1	$f(\zeta, u)$	Invariant condition
BP_{13}	$(k_0 u^{-ik_1}\zeta)^{2ik_2} + g\zeta$	$B = 0, \Delta^2(1/\mu) = 0, \Delta\mu \neq 0, A^2 \neq 1$
CP_{13}	$(k_0 e^{iu}\zeta)^{2ik_1} + g\zeta$	$B = 0, \Delta\mu = 0, \mu \neq 0, A^2 \neq 1$
BE_{13}	$\exp(k_0 u^{-ik_1}\zeta) + g\zeta$	$B = 0, A = -1, \Delta^2(1/\mu) = 0, \Delta\mu \neq 0$
CE_{13}	$\exp(k_0 e^{iu}\zeta) + g\zeta$	$B = 0, A = -1, \Delta\mu = 0, \mu \neq 0$
L_{13}	$e^{ik}h \log\zeta + g_2\zeta$	$B = 0, A = 1$
BL_{13}	$Cu^{-2}\log\zeta + g\zeta$	$B = 0, A = 1, \Delta\mu = \mu^2, \mu \neq 0$
AP_{13}	$(k_0\zeta)^{2ik_1} + g\zeta$	$\mu = 0, AA^* = 1, A^2 \neq 1$
AE_{13}	$\exp(k\zeta) + g\zeta$	$\mu = 0, A = -1$
AL_{13}	$e^{ik}\log\zeta + g\zeta$	$\mu = 0, A = 1$

Proof. By (23)–(25) and (33),

$$\mathcal{L}_V\alpha = \alpha(C + C^*A - V^3\mu^*). \tag{86}$$

By (57) and the definition of C,

$$\mathcal{L}_V(\alpha^*\omega^1) = \mathcal{L}_V da = d(\mathcal{L}_V a) = d(V\rfloor da) = dC, \tag{87}$$

By (21),

$$\mathcal{L}_V\omega^3 = d(V\rfloor\omega^3) + V\rfloor \mathrm{d}\omega^3 \tag{88}$$

$$= d(V^3) + (\alpha^*V^1 + \alpha V^2)\omega^3 - \alpha^*V^3\omega^1 - \alpha V^3\omega^2 \tag{89}$$

$$= (\delta V^3 - \alpha^*V^3)\omega^1 + (\delta^*V^3 - \alpha V^3)\omega^2 + (\Delta V^3 + C + C^*)\omega^3. \tag{90}$$

The desired equivalence follows immediately. □

Proposition 5.3. Suppose that $B \neq 0$ and $AA^* \neq 1$. The following are equivalent:

(i) there exists a V such that

$$\mathcal{L}_V\omega^1 = \mathcal{L}_V\omega^2 = \mathcal{L}_V\omega^3 = 0, \quad V^1, V^3 \neq 0; \tag{91}$$

(ii) the invariant B/B^* is a constant, and

$$\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \wedge \mathrm{d}\mu = 0; \tag{92}$$

(iii) condition (91) holds with $V = CX^{-1}\hat{\Delta}$ for some complex constant $C \neq 0$.

Proof. Evidently, (iii) implies (i). We prove that (i) implies (ii), and then that (ii) implies (iii). Suppose that (i) holds. If V annihilates ω^1, ω^2, ω^3, then

$$\mathcal{L}_V\alpha = \mathcal{L}_V\alpha^* = \mathcal{L}_V\mu = 0,$$

because α, α^*, μ are the structure functions in (20) and (21). If 3 functions on a four-dimensional manifold are annihilated by 2 independent vector fields, then these functions are functionally dependent. Therefore, (92) holds. By Proposition 3.1,

$$V = a\hat{\Delta} + bD,$$

for some functions a, b. By Proposition 5.2 and (40),

$$C = \alpha^*V^1 = aX, \quad C^* = \alpha V^2 = aX^*$$

are constants. Hence, by (36),

$$B/B^* = X/X^* = C/C^* \tag{93}$$

is a constant. Therefore, (ii) holds.

Next, we show that (ii) implies (iii). By assumption, (93) holds for some complex constant C. Set $V = CX^{-1}\hat{\Delta}$. By construction, this is a real vector field such that $\mathcal{L}_V\alpha = \mathcal{L}_V\alpha^* = 0$. By (92), we also have $\mathcal{L}_V\mu = 0$. Also, by construction, $C = \alpha^*V^1$ is a constant, and hence by (87), $\mathcal{L}_V\omega^1 = 0$. By (20) and (90),

$$0 = \mathcal{L}_V d\omega^1 = -(\mathcal{L}_V\mu)\omega^1 \wedge \omega^3 - \mu\omega^1 \wedge \mathcal{L}_V\omega^3 = -\mu\omega^1 \wedge \mathcal{L}_V\omega^3.$$

Since $\mathcal{L}_V\omega^3$ is real and $\mu \neq 0$ by assumption, it follows that $\mathcal{L}_V\omega^3 = 0$, as was to be shown. □

Proposition 5.4. There exists a vector field V such that

$$\mathcal{L}_V\omega^1 = \mathcal{L}_V\omega^3 = 0, \quad V^1 \neq 0, V^3 = 0 \tag{94}$$

if and only if $A = 1$.

Proof. Suppose that (94) holds. By (85), $C + C^* = 0$, and hence $C = \alpha^* V^1$ is imaginary. Hence, by (83), $C + C^*A = 0$, which means that $A = 1$. Conversely, if $A = 1$, then in order for (83)–(85) to hold, it suffices to set $V^1 = \mathrm{i}/\alpha^*$, $V^3 = 0$. □

Proposition 5.5. There exists a vector field V such that

$$\mathcal{L}_V \omega^1 = \mathcal{L}_V \omega^3 = 0, \quad V^1 = 0, V^3 \neq 0 \tag{95}$$

if and only if $\mu = 0$.

Proof. Suppose that (95) holds. Hence, $C = 0$, and by (86),

$$\mathcal{L}_V \alpha = -V^3 \alpha\mu = 0.$$

Therefore, $\mu = 0$. To prove the converse, it suffices to take $V^3 = \mathrm{e}^{a+a^*}$. Relations (84) and (85) follow (66) and (67). □

We now show that type (0, 2) precursor solutions belong to the 4 classes shown in Table VII. Proposition 5.4 characterizes the precursor solutions for which $V^3 = 0$. Proposition 5.5 characterizes precursor solutions for which $V^1 = 0$. This leaves the case where both V^1, V^3 are non-zero. Since we are considering type (0, 2) solutions, we exclude the possibility that $B = 0$. The possibility that $B \neq 0$ but $AA^* = 1$ is excluded by Proposition 3.1. The remaining possibilities can be divided into the case $B_1 \neq 0$ and the case $B_1 = 0$. Proposition 5.6 deals with the former and V.7 with the latter.

Proposition 5.6. Suppose $B_1 \neq 0$, $AA^* \neq 1$. *The following are equivalent:*

(i) *there exists a* V *such that* (82) *holds*;
(ii) $B_2/B_1 = k,\ \Delta X_1 = 2X_1^2$;
(iii) $f(\zeta, u) = F(u^{-\mathrm{i}k}\zeta)u^{-2} + g\zeta$.

Proof. Observe that, if $B_1 \neq 0$, then

$$\frac{B}{B^*} = \frac{B_1 + \mathrm{i}B_2}{B_1 - \mathrm{i}B_2} = \frac{1 + \mathrm{i}B_2/B_1}{1 - \mathrm{i}B_2/B_1}.$$

Hence, the condition that B/B^* is a constant can be restated as $B_2/B_1 = k$, where k is a real constant. Furthermore, if (93) holds and if $V = CX^{-1}\hat{\Delta}$, then without loss of generality, $C = 1 + k\mathrm{i}$ and $V^3 = 1/X_1$. Therefore, if (i) holds, then (ii) follows Propositions 5.2 and 5.3.

Next, we show that (ii) implies (iii). By Proposition 3.4, $f(\zeta, u) = F(h^{\mathrm{i}k}\zeta)h^2 + g\zeta$ belongs to class B_{23}^*. In the proof of Proposition 3.4, we showed that

$$e^{-a-a^*}/X_1 = 1/h_1,$$

where $h_1 = h_1(u)$ is real. Hence, by (67)

$$\Delta(1/X_1) = \Delta(e^{a+a^*}/h_1) = (1/h_1)'(u),$$
$$(1/h_1)'(u) + 2 = 0,$$
$$h_1 = -1/(2u).$$

In the last step we can omit the constant of integration because of transformation freedom (12). Therefore,

$$L_u - \left(\frac{1 + \mathrm{i}k}{2u}\right) L_a = \frac{\mathrm{i}k}{u}.$$

Following the steps in the proof of Proposition 3.4 gives $h = u^{-1}$, which specializes solution form B_{23}^* to form B_{23}.

Finally we show that (iii) implies (i). It suffices to set $V = X_1^{-1}\hat{\Delta}$ and $C = 1 + k\mathrm{i}$. Equation (93) holds by Proposition 3.4. Hence, by construction and by (47)

$$\alpha^* V^1 = C,$$

$$V^3 \mu^* = \frac{C\mu^*}{X} = C\left(\frac{AB^*}{B} + 1\right) = AC^* + C,$$

$$V^3 = \frac{1}{X_1} = -2ue^{a+a^*}.$$

Conditions (84) and (85) follow (66) and (67). □

Proposition 5.7. Suppose that $B_1 = 0$, $\mu \neq 0$, $AA^* \neq 1$. *The following are equivalent: (i) there exists a vector field* V *such that* (82) *holds; (ii)* $\Delta X_2 = 0$;
(iii) $f(\zeta, u) = F(\mathrm{e}^{\mathrm{i}u}\zeta) + g\zeta$.

The proof is similar to that of Proposition 5.6 above.
We now classify the type (0, 1) precursor solutions.

Proposition 5.8. Suppose that $B = 0$, $A \neq 1$, $\mu \neq 0$. *Then* (82) *holds if and only if*

$$\Delta^2(1/\mu) = 0. \tag{96}$$

Proof. Suppose that (82) holds. By Proposition 4.2, A is a constant. Hence, (96) follows (83) and (85). Conversely, suppose that (96) holds.

By Proposition 5.2, we seek a constant C such that

$$V^3 = (C^* + CA^*)/\mu = (C + C^*A)/\mu^*,$$

and such that the above V^3 satisfies (84) and (85). First, observe that $A^* = 1/A$ and $\mu^* = A\mu$. Hence,

$$\frac{C + C^*A}{\mu^*} = \frac{C/A + C^*}{\mu} = \frac{C^* + CA^*}{\mu}.$$

Therefore, V^3 is well-defined for any choice of C. By Proposition 4.2, $\delta(\alpha\mu) = 0$. Hence

$$\alpha(\alpha^*\mu + \delta\mu) = 0 \quad \delta(1/\mu) = -\delta\mu/\mu^2 = \alpha^*/\mu.$$

Hence, (84) is satisfied for all choices of C. We now turn to condition (85). By (53) and (80) of Proposition 4.3

$$\frac{\mathrm{i}(A-1)}{A\mu} = \frac{\mathrm{i}(A-1)}{A}\frac{e^{a+a^*}}{L_u} = \frac{\mathrm{e}^{a+a^*}}{h'(u)},$$

where $h = h(u)$ is real. Hence, by (67),

$$\frac{\mathrm{i}(A-1)}{A}\Delta(1/\mu) = -\frac{h''(u)}{h'(u)^2} = k \tag{97}$$

is a real constant. If $k = 0$, then condition (85) can be satisfied by taking $C = \mathrm{i}$. If $k \neq 0$, (85) is satisfied by taking $C = A/(A - 1) + \mathrm{i}/k$. With this choice,

$$C^* + CA^* = \frac{1}{1-A} - \frac{\mathrm{i}}{k} + \left(\frac{A}{A-1} + \frac{\mathrm{i}}{k}\right)\frac{1}{A} = \frac{-\mathrm{i}(A-1)}{kA},$$

$$\Delta V^3 = -1,$$

$$C + C^* = \frac{A}{A-1} + \frac{1}{1-A} = 1.$$

□

Proposition 5.9. The type (0,1) *precursor solutions belong to one of the classes shown in Table VIII.*

Proof. By Proposition 5.4 the $B = 0, A = 1$ solutions are automatically precursor solutions with $V^3 = 0$, $V^1 \neq 0$. We now classify all precursor solutions that admit a vector field that satisfies (82) with $V^3 \neq 0$. We consider two cases: $\mu \neq 0$ and $\mu = 0$. Suppose the former. By Proposition 5.8, a precursor solution is characterized by the condition $\Delta^2(1/\mu) = 0$, which is equivalent to

$$h''(u) + k\, h'(u)^2 = 0. \tag{98}$$

where h is the parameter in solution forms P_{13}, E_{13}, L_{13}. This gives us four classes of solutions. Class BP_{13} corresponds to the case $A \neq -1$ and $k \neq 0$. In this case, the solution of (98), without loss of generality, is $h = \frac{1}{k}\log u$. Class CP_{13} corresponds to $A \neq -1$ and $k = 0$. Here, without loss of generality, the solution to (98) is $h = u$. Similarly, the condition $A = -1$ gives solution classes BE_{13} and CE_{13}. Finally, consider the case of $A = 1$. Here $\mu^* = \mu$. Hence, by Proposition 3.9 $L = -2a + \mathrm{i}k + h$, where h is real. By Proposition 5.2 we require that

$$V^3 = (C + C^*)/\mu \neq 0, \quad \Delta V^3 = (C + C^*)\Delta(1/\mu) = -(C + C^*).$$

Since $\delta(\alpha\mu) = 0$, we automatically have $\delta(1/\mu) = \alpha^*\mu$; condition (84) is automatically satisfied. Hence, a necessary and sufficient condition for a precursor solution is $\Delta(1/\mu) = -1$, or equivalently $\Delta\mu = \mu^2$. This is equivalent to $h''(u) = h'(u)^2$, which, by employing the freedom (12), gives us $h(u) = -\log u$. Employing the integration steps in Proposition 3.9, this gives us $f = Cu^{-2}\log\zeta + g\zeta$, which is solution form BL_{13}.

Next, suppose that $\mu = 0$, $AA^* = 1$. Here it suffice to specialize one of the Table VI solutions. For classes P_{13} and E_{13} we set $h \to 0$. For the logarithmic solution L_{13} we set $h \to k$, where the latter is a constant. □

Note that the (0,1) precursor class BL_{13} is a specialization of the (0,1) precursor class L_{13}. However, we include BL_{13} as a distinct category because it enjoys the property of having two distinct vectors V satisfying (82): one of these has $V^3 = 0$ (as a specialization of L_{13}) and the other $V^3 \neq 0$ (because Eq. (98) is satisfied). The same "double precursor" property holds for classes AP_{13}, AE_{13}, AL_{13}. As consequence, it is precisely these 4 classes that specialize to a G_3 solution.

VI. THE G_2 SOLUTIONS

In this section we characterize and classify the vacuum pp-waves with two independent Killing vectors. Since a Killing vector annihilates the invariant 1-forms ω^1, ω^2, ω^3, and ω^4, every G_2 solution is a specialization of the precursor metrics discussed in Sec. V.

We first present the invariant characterization of the generic, type (0, 2, 2) solutions, and then present the characterization of the type (0, 1, 2, 2) solutions. We then pass to a detailed classification, the results of which are displayed in Tables I and IX.

Proposition 6.1. A type (0,2,2) G_2 *solution is characterized by* (82) *and*

$$d\alpha \wedge \mathrm{d}\alpha^* \neq 0, \quad d\alpha \wedge \mathrm{d}\alpha^* \wedge \mathrm{d}\nu = 0. \tag{99}$$

Proof. By Proposition 2.1, the second-order Cartan invariants are generated by A, μ, ν. Suppose that there exists a Killing vector V independent from D. Condition (82) follows assumption. Since Killing vectors annihilate invariants, there are at most two functionally independent invariants. Hence, (99) must hold.

Conversely, suppose that (82) and (99) hold. Dependence of μ follows Proposition 5.3. Furthermore,

$$\mathcal{L}_V\alpha = \mathcal{L}_V\alpha^* = 0, \quad \mathcal{L}_V \mathrm{d}\alpha = \mathrm{d}\mathcal{L}_V\alpha = 0,$$

TABLE IX. Type (0, 1, 2, 2) G_2-solutions.

G_2	$f(\zeta, u)$	Invariant condition
BP_{122}	$((k_0 u^{-ik_1}\zeta)^{2ik_2} + k_3 u^{-ik_1}\zeta)u^{-2}$	$B=0,\ \Delta^2(1/\mu)=0,\ \tilde{\Upsilon}=0,\ \Delta\mu\neq 0,\ A^2\neq 1$
CP_{122}	$(k_0 e^{iu}\zeta)^{2ik_1} + k_2 e^{iu}\zeta$	$B=0,\ \Delta\mu=0,\ \tilde{\Upsilon}=0,\ \mu\neq 0,\ A^2\neq 1$
BE_{122}	$\exp(k_0 u^{-ik_1}\zeta) + k_2 u^{-ik_1}\zeta$	$B=0,\ A=-1,\ \Delta^2(1/\mu)=0,\ \tilde{\Upsilon}=0,\ \Delta\mu\neq 0$
CE_{122}	$\exp(k_0 e^{iu}\zeta) + k_1 e^{iu}\zeta$	$B=0,\ A=-1,\ \Delta\mu=0,\ \tilde{\Upsilon}=0,\ \mu\neq 0$
L_{122}	$e^{ik}h\log\zeta$	$B=0, A=1, Y=0$
BL_{122}	$u^{-2}(C\log\zeta + k\zeta)$	$B=0, A=1, \Delta\mu=\mu^2, \mu\neq 0$
		$\Delta\log(YY^*)=4\mu, \Delta(\alpha\Delta\log Y)=0$
AP_{122}	$(k_0\zeta)^{2ik_1} + Cu^{-2-ik_1}\zeta$	$\mu=0,\ AA^*=1,\ \Delta^2 Y^{\frac{1-A}{A-3}}=0,\ A^2\neq 1$
	$\left(k_0 u^{-i/k_1}\zeta + C\right)^{2ik_1} u^{-2}$	
AE_{122}	$\exp(k_0\zeta) + Cu^{-2}\zeta$	$\mu=0, A=-1, \Delta^2 Y^{-1/2}=0$
	$(\exp(k_0\zeta) + C\zeta)u^{-2}$	
AL_{122}	$e^{ik_0}\log\zeta + k_1 e^{iu}\zeta$	$\mu=0, A=1, \Delta^2\log Y=0$
	$e^{ik_0}\log(e^{iu}\zeta + k_1)$	

where V is the vector field in (82). By (23)–(25) and (33),

$$d\alpha = \alpha(\alpha^*\omega^1 + A\alpha\omega^2 - \mu\omega^3).$$

Hence $\mathcal{L}_V A = 0$. Therefore, the invariant count is (0, 2, 2). □

The type (0, 1, 2, 2) solutions split into two branches, depending on whether or not μ is independent of α. We consider each branch in turn.

Proposition 6.2. Suppose that $d\alpha \wedge d\alpha^* = 0$ *but that* $d\alpha \wedge d\mu \neq 0$. *Then a* G_2 *solution is characterized by the condition:*

$$d\alpha \wedge d\mu \wedge d\nu = 0. \tag{100}$$

Proof. If V is a Killing vector then $\mathcal{L}_V \nu = 0$. In a G_2 solution there are two such independent vector fields, which means that α, μ, ν must be functionally dependent. Let us prove the converse. We will show that the invariant count is (0, 1, 2, 2), which signifies a G_2 solution by the Karlhede algorithm. By Proposition 2.1, the second-order invariants are generated by μ, A, ν and their complex conjugates. Suppose that

$$d\alpha \wedge d\mu \neq 0, \quad d\alpha \wedge d\alpha^* = 0, \quad d\alpha \wedge d\mu \wedge d\nu = 0.$$

By Propositions 4.1 and 4.2,

$$B = 0, \quad AA^* = 1, \quad dA = 0, \quad \mu^* = A\mu.$$

Hence, all second order invariants depend on α, μ. The third order invariants are generated by $\delta^* A$, $\delta\mu$, $\Delta\mu$, $\Delta\nu$, and their complex conjugates. Since A is a constant and ν is a function of α, μ, and since relation (30) holds, it suffices to show that $\delta\mu$ depends on α, μ. By Proposition 4.2 and by (24),

$$\delta(\alpha\mu) = 0, \quad \delta\mu + \alpha^*\mu = 0, \tag{101}$$

as was to be shown. □

Proposition 6.3. Suppose that $d\alpha \wedge d\alpha^* = d\alpha \wedge d\mu = 0$, *but that* $d\alpha \wedge d\nu \neq 0$. *Then a* G_2 *solution is characterized by the conditions:*

$$d\alpha \wedge d\nu \wedge d\nu^* = 0, \tag{102}$$

$$d\alpha \wedge d\nu \wedge d\Delta\nu = 0. \tag{103}$$

Lemma 6.4. Suppose that $B = 0$, $\mu \neq 0$. *The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\mu = 0$ *and (ii)* $A = 1$, $\Delta\mu = \mu^2$.

Proof. By assumption, $\mu^* = A\mu$. By Proposition 4.2, relation (101) holds. Hence,

$$\begin{aligned} \mathrm{d}\mu &= -\mu\alpha^*\omega^1 - \alpha\mu\omega^2 + \Delta\mu\omega^3, \\ d\alpha &= \alpha(\alpha^*\omega^1 + A\alpha\omega^2 - \mu^*\omega^3), \\ d\alpha \wedge \mathrm{d}\mu &= (A-1)\alpha^2\alpha^*\mu\omega^1 \wedge \omega^2 - \alpha\alpha^*(A\mu^2 - \Delta\mu)\omega^1 \wedge \omega^3 \\ &\quad - A\alpha^2(\mu^2 - \Delta\mu)\omega^2 \wedge \omega^3. \end{aligned}$$

□

Proof of Proposition 6.3. By Propositions 4.1 and 4.2, A is a constant. Hence, using the reasoning in the proof of Proposition 6.2 above, ν, $\Delta\mu$, $\Delta\nu$, and their complex conjugates generate the second- and third-order invariants. If $\mu \neq 0$, then by Lemma 6.4, $\Delta\mu$ is a function of μ, which itself is a function of α. If $\mu = 0$, then *a fortiori* $\Delta\mu = 0$. Therefore, (102) and (103) suffice for a G_2 solution. □

We now classify the (0, 2, 2) solutions. Throughout, V denotes the second Killing vector independent from D. The G_2 solutions can be further subdivided according to whether $V^3 \neq 0$ or $V^3 = 0$.

By Proposition 5.4, the (0,2) precursor with $V^3 = 0$ is of class L_{22}. The remaining (0,2) precursors are B_{23}, C_{23}, A_{23}. As we show below, the specialization from the precursor class to the G_2 class is governed by the vanishing of the Y and Υ invariants, which are defined in (37) and (39), respectively.

Proposition 6.5. Suppose that $f(\zeta, u) = F(u^{-\mathrm{i}k}\zeta)u^{-2} + gu^{-2-\mathrm{i}k}\zeta, k \neq 0$ *belongs to the* B_{23} *precursor class. The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \wedge \mathrm{d}\nu = 0$, *(ii)* $\hat{\Upsilon} = 0$, *and (iii)* $g'(u) = 0$.

Proof. By Proposition 5.3, $V = X_1^{-1}\hat{\Delta}$ annihilates ω^1, ω^2, ω^3, α, μ. Above, we already noted that $\mathcal{L}_V A = 0$. By (34) and (36), $\mathcal{L}_V X = 0$ also. Let $\hat{\nu}$ be the invariant defined in (38). By Proposition 5.2, and (29) and (30),

$$\begin{gathered} \delta X = -\alpha^* X, \quad \delta^* X = -\alpha X, \quad \Delta X = 2XX_1, \\ \delta(\hat{\nu}/X^*) = -4\mathrm{i}X_2, \quad \delta^*(\hat{\nu}/X^*) = (1 - 2\hat{\nu}\alpha)/X^*, \\ \hat{\Delta}(\hat{\nu}/X^*) = \Delta(\hat{\nu}/X^*) - 4\mathrm{i}XX_2/\alpha^* + (1 - 2\hat{\nu}\alpha)/\alpha = \hat{\Upsilon}, \end{gathered}$$

where $\hat{\nu}$ is the invariant defined by (38). This proves the equivalence of (i) and (ii). A direct calculation shows that

$$\hat{\Upsilon}^* = 4u\frac{X_1^2}{X}\frac{g'(u)}{\sqrt{F''(u^{-\mathrm{i}k}\zeta)}}.$$

This proves the equivalence of (ii) and (iii). □

Remark 1. If $g'(u) = 0$, then by (10) we can absorb the $g(u)u^{-2-\mathrm{i}k}\zeta$ term into the $F(u^{-\mathrm{i}k}\zeta)u^{-2}$ term.

Remark 2. The invariant $\hat{\nu}$ can be calculated directly by employing the tetrad that respects the normalization $\hat{\Delta}\alpha = 0$. The null rotation that sends $\Delta \to \hat{\Delta}$ maps $\nu \to \hat{\nu}$.

The proof of the following 3 propositions is similar to the proof above.

Proposition 6.6. Suppose that $f(\zeta, u) = F(\mathrm{e}^{\mathrm{i}u}\zeta) + g\mathrm{e}^{\mathrm{i}u}\zeta$ *belongs to the* C_{23} *precursor class. The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \wedge \mathrm{d}\nu = 0$, *(ii)* $\hat{\Upsilon} = 0$, *and (iii)* $g'(u) = 0$.

Proposition 6.7. Suppose that $f(\zeta, u) = g_1 \log \zeta + g_2\zeta$ *belongs to the logarithmic* L_{23} *precursor class. The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \wedge \mathrm{d}\nu = 0$, *(ii)* $Y = 0$, *and (iii)* $g_2 = 0$.

Proposition 6.8. Suppose that $f(\zeta, u) = f(\zeta) + g\zeta$ *belongs to the* A_{23} *precursor class. The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \wedge \mathrm{d}\nu = 0$, *(ii)* $\Delta\nu = 0$, *and (iii)* $g'(u) = 0$.

We now classify the G_2 solutions of type (0, 1, 2, 2). By definition, these are specializations of the type (0, 1) precursors. The latter solutions fall into three groups: (i) $V^3 = 0$, (ii) $V^3 \neq 0$ and $\mathrm{d}\alpha \wedge \mathrm{d}\mu = 0$, and (iii) $V^3 \neq 0$ and $\mathrm{d}\alpha \wedge \mathrm{d}\mu \neq 0$, where V is the vector field that satisfies (82). Case (i) is class L_{23}. The specialization to a G_2 solution is described, *mutatis mutandi*, by Proposition 6.7 above. Case (ii) consists of classes L_{13}, AP_{13}, AE_{13}, and AE_{13}. The specialization to G_2 solutions is described by Propositions 7.2, 7.4, 7.6, 7.8 of Sec. VII. Case (iii) consists of classes BP_{13}, CP_{13}, BE_{13}, CE_{13}. By Proposition 6.2, the specialization to a G_2 solution is characterized by the condition $\mathrm{d}\alpha \wedge \mathrm{d}\mu \wedge \mathrm{d}\nu = 0$. The following proposition analyzes this condition. The key invariant here is $\tilde{\Upsilon}$, as defined by (43).

Lemma 6.9. Suppose that $B = 0$ *and* $AA^* = 1, A \neq 1$. *Then*

$$\{\mathrm{d}\alpha, \mathrm{d}\alpha^*, \mathrm{d}\mu, \mathrm{d}\mu^*\}^\perp = \operatorname{span}\{\tilde{\Delta}, D\}, \tag{104}$$

with $\tilde{\Delta}$ *defined as in* (44).

Proof. Since $M = \alpha\mu$, no generality is lost if replace $\mathrm{d}\mu$ with $\mathrm{d}M$. By Proposition 4.2, $\delta M = 0$. By (27)

$$\delta^* M = (A - 1)\alpha M.$$

By (34), $\mu^* = A\mu$. Hence, by (24) and (25) we seek the kernel of the following matrix:

$$\begin{pmatrix} \alpha\alpha^* & \alpha^2 A & -AM & 0 \\ \alpha^{*2}A^{-1} & \alpha\alpha^* & -M\alpha^*\alpha^{-1} & 0 \\ 0 & (A-1)\alpha M & \Delta M & 0 \end{pmatrix}. \tag{105}$$

By Proposition 4.1 $\mathrm{d}\alpha \wedge \mathrm{d}\alpha^* = 0$; hence, the above matrix has rank 2. Since $A^* = 1/A$, the kernel is invariant under complex conjugation. Therefore, since $A \neq 1$, a basis for the kernel is D and

$$\tilde{\Delta} = \tilde{X}/\alpha^*\,\delta + \tilde{X}^*/\alpha\,\delta^* + \Delta + \tilde{X}\tilde{X}^*/(\alpha\alpha^*)D, \quad \tilde{X}^* = \Delta M/(M(1-A)).$$

□

Proposition 6.10. Suppose that $f(\zeta, u) = (k_0 z)^{\mathrm{i}k_1} u^{-2} + g u^{-2} z$, *or* $f(\zeta, u) = \exp(z) + gz$, *where* $z = u^{-\mathrm{i}k}\zeta$ *or* $z = \mathrm{e}^{\mathrm{i}u}\zeta$, *i.e.,* $f(\zeta, u)$ *belongs to one of the following classes:* BP_{13}, CP_{13}, BE_{13}, *and* CE_{13}. *Then, the following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\mu \wedge \mathrm{d}\nu = 0$, *(ii)* $\tilde{\Upsilon} = 0$, *and (iii)* $g'(u) = 0$.

Proof. By assumption, $B = 0$, $AA^* = 1$, $A \neq 1$. Hence, there exists a V such that condition (82) holds. Since $\mathcal{L}_V\alpha = \mathcal{L}_V\mu = 0$, by Lemma 6.9 V is a multiple of $\tilde{\Delta}$.

Hence, $\tilde{X}/\tilde{X}^* = C/C^*$, where $C = \alpha^* V^1$, and hence $V = C/\tilde{X}\tilde{\Delta}$. In the proof of Proposition 5.8 we showed that $\Delta(1/\mu)$ is a constant. It follows that $\mathcal{L}_V\Delta\mu = 0$ and hence $\mathcal{L}_V\tilde{X} = 0$ also. Therefore, the desired condition is equivalent to $\tilde{\Delta}(\tilde{\nu}/\tilde{X}) = 0$ where $\tilde{\nu}$ is the invariant defined in (42). By (29), (30), (84), and (85):

$$\delta\tilde{X} = -\alpha^*\tilde{X}, \quad \delta^*\tilde{X} = -\alpha\tilde{X}, \quad \Delta\tilde{X} = 2\tilde{X}\tilde{X}_1,$$

$$\delta(\tilde{\nu}/\tilde{X}) = -4\mathrm{i}\tilde{X}_2, \quad \delta^*(\hat{\nu}/\tilde{X}) = (1 - 2\hat{\nu}\alpha)/\tilde{X},$$

$$\hat{\Delta}(\hat{\nu}/\tilde{X}) = \Delta(\hat{\nu}/\tilde{X}) - 4\mathrm{i}\tilde{X}\tilde{X}_2/\delta^* + (1 - 2\hat{\nu}\alpha)/\alpha = \tilde{\Upsilon}.$$

This proves the equivalence of (i) and (ii). A direct calculation shows that

$$\tilde{\Upsilon} = C\alpha\mu^2 u^{1+ik_1}\zeta^*(g_1'(u))^*,$$

where $C = C(k_0, k_1)$ is a constant. This proves the equivalence of (ii) and (iii). □

Remark 1. If $g'(u) = 0$, then by (10) we can absorb the second term in $f(\zeta, u)$ into the first term.

Remark 2. The invariant $\tilde{\nu}$ can be calculated directly by employing a null-rotated tetrad that sends $\Delta \to \tilde{\Delta}$ and $\nu \to \tilde{\nu}$.

VII. THE MAXIMAL IC ORDER CLASS

This section is devoted to the proof of Theorem 1.1; we exhibit and classify all vacuum pp-wave solutions with a (0, 1, 2, 3) invariant count. The (0, 1) class is defined by the condition $d\alpha \wedge d\alpha^* = 0$. If α, μ are independent, then the (0, 1, 2) condition requires that ν, ν^* be functions of α, μ. However, by Proposition 6.2, this forces a G_2 solution, and therefore can be excluded from the (0, 1, 2, 3) classification.

Thus, we have narrowed the search for (0, 1, 2, 3) solutions to the following class:

$$d\alpha \wedge d\alpha^* = 0, \quad d\alpha \wedge d\mu = 0, \quad d\alpha \wedge d\nu \wedge d\nu^* = 0. \tag{106}$$

The middle condition forces some restrictions.

By Lemma 6.4, the analysis divides into two cases: $B = 0$, $A = 1$, $\Delta\mu = \mu^2$, $\mu \neq 0$ and $\mu = 0$, $AA^* = 1$. The former possibility specifies class BL_{13}; the latter classes AP_{13}, AE_{13}, AL_{13}. We begin by describing the specialization from class BL_{13} to class BL_{123}. The Y invariant employed below is defined in (37).

Proposition 7.1. Suppose that $f(\zeta, u) = Cu^{-2}\log\zeta + g\zeta$ *belongs to class* BL_{13}. *The following are equivalent: (i)* $d\alpha \wedge d\nu \wedge d\nu^* = 0$, *(ii)* $\Delta\log(YY^*) = 4\mu$, *and (iii)* $g = ku^{-2}e^{ih}$, *where k is a real constant and* $h = h(u)$ *is real.*

Proof. Our assumption implies

$$A = 1, \quad B = 0,$$
$$\delta\mu = -\mu\alpha^*, \quad \Delta\mu = \mu^2,$$
$$Y = 2\nu - 1/\alpha + 2\mu^2/\alpha^*.$$

Hence, by (29) and (30)

$$\delta Y = -Y\alpha^*, \quad \delta^* Y = -3Y\alpha,$$

$$\begin{vmatrix} \delta\alpha & \delta^*\alpha & \Delta\alpha \\ \delta Y & \delta^* Y & \Delta Y \\ \delta Y^* & \delta^* Y^* & \Delta Y^* \end{vmatrix} = \begin{vmatrix} \alpha\alpha^* & \alpha^2 & -\alpha\mu \\ -Y\alpha^* & -3Y\alpha & \Delta Y \\ -3Y^*\alpha^* & -\alpha Y^* & \Delta Y^* \end{vmatrix} = 2\alpha^2\alpha^*(4YY^*\mu - \Delta(YY^*)).$$

This proves the equivalence of (i) and (ii). Writing $g = e^{h_1 + ih_2}$, a direct calculation shows that

$$\mu = -(CC^*)^{1/4}(\zeta\zeta^*)^{1/2}, \tag{107}$$

$$M = \alpha\mu = (i/2)(C^*)^{-1/2}, \tag{108}$$

$$YY^* = 4e^{2h_1}u^4\mu^4, \tag{109}$$

$$(\Delta\log YY^*)\mu = -2uh_1'(u). \tag{110}$$

Therefore, (ii) is equivalent to

$$uh_1'(u) = -2,$$

which is equivalent to (iii). □

We now prove that generically the above solution is (0,1,2,3), and in the process derive the condition for specialization to a G_2 solution.

Proposition 7.2. Suppose that $f(\zeta, u) = u^{-2}(C \log \zeta \ + \ k\mathrm{e}^{\mathrm{i}h}\zeta)$ *belongs to class* BL_{123}. *The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\nu \wedge \mathrm{d}\Delta\nu = 0$, *(ii)* $\Delta(\alpha\Delta\log Y) = 0$, *and (iii)* $\mathrm{e}^{\mathrm{i}h} = u^{\mathrm{i}k_1}$, *where* k_1 *is a real constant*.

We now consider the case of $\mu = 0$, $AA^* = 1$.

Proposition 7.3. Suppose that $f(\zeta, u) = (k_0\zeta)^{2\mathrm{i}k_1} + g\zeta$ *belongs to the* AP_{13} *class. The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\nu \wedge \mathrm{d}\nu^* = 0$, *(ii)*

$$(1 - 3A)\Delta \log Y + (A - 3)\Delta \log Y^* = 0, \tag{111}$$

and (iii) $g = k_2\mathrm{e}^{\mathrm{i}h(1-2\mathrm{i}k_1)}$, *where* k_2 *is a real constant and* $h = h(u)$ *is real*.

We now prove that generically the above solution is (0,1,2,3), and in the process derive the condition for specialization to a G_2 solution.

Proposition 7.4. Suppose that $f(\zeta, u) = (k_0\zeta)^{2\mathrm{i}k_1} + k_2\mathrm{e}^{\mathrm{i}h(1+2\mathrm{i}k_1)}\zeta$ *belongs to class* AP_{123}. *The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\nu \wedge \mathrm{d}\Delta\nu = 0$, *(ii)* $\Delta^2 Y^{\frac{1-A}{A-3}} = 0$, *and (iii)* $f(\zeta, u) = (k_0\zeta)^{2\mathrm{i}k_1} + Cu^{-2-\mathrm{i}k_1}\zeta$, *where C is a complex constant.*

Proof. All of the relations given in the proof of Proposition 7.3 hold. Furthermore, by (16)–(19),

$$\delta\Delta Y = -2\alpha^*\Delta Y, \quad \delta^*\Delta Y = -4\alpha\Delta Y.$$

From there, a direct calculation shows that

$$\begin{vmatrix} \delta\alpha & \delta^*\alpha & \Delta\alpha \\ \delta Y & \delta^* Y & \Delta Y \\ \delta\Delta Y & \delta^*\Delta Y & \Delta^2 Y \end{vmatrix} = \begin{vmatrix} \alpha\alpha^* & A\alpha^2 & 0 \\ -Y\alpha^* & -3Y\alpha & \Delta Y \\ -2\alpha^*\Delta Y & -4\alpha\Delta Y & \Delta^2 Y \end{vmatrix} =$$

$$= 2\alpha^2\alpha^*((2(2-A)(\Delta Y)^2 + (A-3)\Delta^2 Y) = 2\alpha\alpha^* Y^{\frac{3A-7}{A-3}}\frac{(A-3)^2}{1-A}\Delta^2 Y^{\frac{1-A}{A-3}}.$$

This proves the equivalence of (i) and (ii). Furthermore, a direct calculation gives

$$Y^{\frac{A-1}{A-3}}\Delta^2 Y^{\frac{1-A}{A-3}} = \tilde{C}\zeta^{1-\mathrm{i}k_1}\zeta^{1+\mathrm{i}k_1}(k_1 h_2'(u)^2 - h_2''(u)),$$

where $\tilde{C}$ is a complex constant. This proves the equivalence of (ii) and (iii). □

Finally, we consider the AE and the AL classes. Propositions 7.5 and 7.7 derive the form of the (0, 1, 2) solutions for the cases $A = -1$ and $A = 1$, respectively. Propositions 7.6 and 7.8 prove that these solutions are generically of type (0, 1, 2, 3) and derive the condition for the specialization to the corresponding (0, 1, 2, 2) G_2 solution. *Mutatis mutandi*, these Propositions are proved in the same way as Propositions 7.3 and 7.4 above.

Proposition 7.5. Suppose that $f(\zeta, u) = \exp(k\zeta) + g\zeta$ *belongs to the* AE_{13} *class. The following are equivalent: (i)* $\mathrm{d}\alpha \wedge \mathrm{d}\nu \wedge \mathrm{d}\nu^* = 0$, *(ii)* $\Delta\log(Y/Y^*) = 0$, *and (iii)* $g = \mathrm{e}^{\mathrm{i}k_1}\mathrm{e}^h$, *where* k_1 *is a real constant and* $h = h(u)$ *is real*.

Proposition 7.6. Suppose that $f(\zeta, u) = \exp(k_0\zeta) + e^{ik_1}e^h\zeta$ *belongs to the* AE_{123} *class. The following are equivalent: (i)* $d\alpha \wedge d\nu \wedge d\Delta\nu = 0$, *(ii)* $\Delta^2 Y^{-1/2} = 0$, *and (iii)* $e^h = Cu^{-2}$, *where C is a complex constant.*

Proposition 7.7. Suppose that $f(\zeta, u) = e^{ik}\log\zeta + g\zeta$ *belongs to the* AL_{13} *class. The following are equivalent: (i)* $d\alpha \wedge d\nu \wedge d\nu^* = 0$, *(ii)* $\Delta\log(YY^*) = 0$, *and (iii)* $g = k_2 e^{ih}$, *where* k_2 *is a real constant and* $h = h(u)$ *is real.*

Proposition 7.8. Suppose that $f(\zeta, u) = e^{ik_0}\log\zeta + k_1 e^{ih}\zeta$ *belongs to the* AL_{123} *class. The following are equivalent: (i)* $d\alpha \wedge d\nu \wedge d\Delta\nu = 0$, *(ii)* $\Delta^2\log Y = 0$, *and (iii)* $e^{ih} = u^{ik_2}$, *where* k_2 *is a real constant.*

Finally, we remark that a suitable change of variable (10) and (13) allows for two equivalent representation for solution classes AP_{122}, AE_{122}, AL_{122}:

$$(k_0\zeta)^{2ik_1} + Cu^{-2-ik_1}\zeta \simeq \left(k_0 u^{-i/k_1}\zeta + C\right)^{2ik_1} u^{-2}, \tag{112}$$

$$\exp(k_0\zeta) + Cu^{-2}\zeta \simeq (\exp(k_0\zeta) + C\zeta)u^{-2}, \tag{113}$$

$$e^{ik_0}\log\zeta + k_1 e^{iu}\zeta \simeq e^{ik_0}\log(e^{iu}\zeta + k_1). \tag{114}$$

It follows that classes AP_{122}, AE_{122} are specializations of the generic G_2 solution B_{22}, while AL_{122} is a specialization of C_{22}.

VIII. THE G_3 SOLUTIONS

In this section we classify the G_3 solutions. The invariant count is (0, 1, 1) and hence these solutions are characterized by $\alpha \neq 0$ and

$$d\alpha \wedge d\alpha^* = d\alpha \wedge d\mu = d\alpha \wedge dA = d\alpha \wedge d\nu = 0.$$

The condition $d\alpha \wedge dA = 0$ is redundant, because by Propositions 4.1 and 4.2, a G_3 solution satisfies $B = 0$, $AA^* = 1$, $dA = 0$. By Lemma 6.4 there are two branches: (i) $B = 0$, $A = 1$, $\Delta\mu = \mu^2$, $\mu \neq 0$; and (ii) $\mu = 0$, $AA^* = 1$. By Propositions 7.1, 7.3, 7.5, and 7.7 the condition $d\alpha \wedge d\nu \wedge d\nu^* = 0$, which is weaker than $d\alpha \wedge d\nu = 0$, specializes these two branches to (0, 1, 2, 3) solutions. Therefore, the G_3 solutions arise as the following sequence of specializations:

$$(0, 1, 3) \to (0, 1, 2, 3) \to (0, 1, 2, 2) \to (0, 1, 1).$$

Therefore, to classify the G_3 solutions it suffices to begin with the classes BL_{13}, AP_{13}, AE_{13}, AL_{13}, and impose the specialization is $d\alpha \wedge d\nu = 0$.

Proposition 8.1. Suppose that $f(\zeta, u) = Cu^{-2}\log\zeta + gu^{-2}\zeta$ *belongs to class* BL_{13}. *The following are equivalent: (i)* $d\alpha \wedge d\nu = 0$, *(ii)* $Y = 0$, *and (iii)* $g = 0$.

Proof. Using the relations from the proof of Proposition 7.1, we have

$$\delta\alpha\delta^*Y - \delta Y\delta^*\alpha = -2Y\alpha^2\alpha^*,$$

$$\delta\alpha\Delta Y - \delta Y\Delta\alpha = -2\alpha\alpha^*(Y\mu - \Delta Y),$$

$$\Delta\alpha\delta^*Y - \Delta Y\delta^*\alpha = \alpha^2(3Y\mu - \Delta Y).$$

This proves the equivalence of (i) and (ii). A direct calculation shows that

$$\alpha^*Y^* = u^2\zeta g/C.$$

This proves the equivalence of (ii) and (iii). □

Proposition 8.2. Suppose that $f(\zeta, u) = (k_0\zeta)^{2ik_1} + g\zeta$ *belongs to the* AP_{13} *class. The following are equivalent: (i)* $d\alpha \wedge d\nu = 0$*, (ii)* $Y = 0$*, and (iii)* $g = 0$.

Proof. Using the relations from the proof of Proposition 7.3, we have

$$\delta\alpha\delta^*Y - \delta Y\delta^*\alpha = (A-3)Y\alpha^2\alpha^*,$$

$$\delta\alpha\Delta Y - \delta Y\Delta\alpha = \alpha\alpha^*\Delta Y,$$

$$\Delta\alpha\delta^*Y - \Delta Y\delta^*\alpha = A\alpha^2\Delta Y.$$

This proves the equivalence of (i) and (ii). A direct calculation shows that

$$\alpha^*Y^* = C\zeta^{1-2ik_1}g,$$

where C is a constant. This proves the equivalence of (ii) and (iii). □

The proof of the following two propositions uses the same argument as above. One merely specializes $A \to -1$ and $A \to 1$, respectively.

Proposition 8.3. Suppose that $f(\zeta, u) = \exp(k\zeta) + g\zeta$ *belongs to the* AE_{13} *class. The following are equivalent: (i)* $d\alpha \wedge d\nu = 0$*, (ii)* $Y = 0$*, and (iii)* $g = 0$.

Proposition 8.4. Suppose that $f(\zeta, u) = e^{ik}\log\zeta + g\zeta$ *belongs to the* AL_{13} *class. The following are equivalent: (i)* $d\alpha \wedge d\nu = 0$*, (ii)* $Y = 0$*, and (iii)* $g = 0$.

IX. THE G_5 AND G_6 SOLUTIONS

In this section we derive and classify the metric forms in the $\alpha = 0$ class. By Proposition 2.2 the corresponding solutions are either G_5 or G_6.

Proposition 9.1. The following are equivalent: (i) $\alpha = 0$ *and (ii)* $f(\zeta, u) = g_2\zeta^2 + g_1\zeta + g_0$*, where as usual* $g_i = g_i(u), i = 0, 1, 2$ *denote complex valued functions of one variable.*

Proof. A direct calculation shows that

$$\alpha = e^{a-a^*}(a_\zeta)^*,$$

where

$$a = \frac{1}{4}f_{\zeta\zeta}.$$

□

Note that a form-preserving transformation (10)–(13) can be used to set $g_1, g_0 \to 0$. Hence, without loss of generality a solution in the $\alpha = 0$ class has the form $f(\zeta, u) = g\zeta^2$, where $g \neq 0$.

It will be convenient to set $g = e^{4A}$, where $A = A(u)$ is complex valued. A direct calculation then shows that

$$\gamma = \frac{e^{-2\Re A}A_u^*}{\sqrt{2}}, \tag{115}$$

$$\frac{\gamma}{\gamma^*} = \frac{A_u^*}{A_u}. \tag{116}$$

We are now in a position to derive and classify the homogeneous G_6 solutions. Such solutions are characterized by the condition $\Delta\gamma = 0$, which ensures that the fundamental Cartan invariant γ is a constant.

At this point the G_6 classification bifurcates, depending on the value of A_u. We consider the generic case in Proposition 9.2, and the singular case in Proposition 9.3. The classification is summarized in Table III.

Proposition 9.2. Suppose that $f(\zeta, u) = e^{4A}\zeta^2$, $\Delta\gamma = 0$, *and* $\Re\gamma \neq 0$. *Then, without loss of generality*,

$$f(\zeta, u) = k_1 u^{2\mathrm{i}k_0 - 2}\zeta^2. \tag{117}$$

Proof. If $\Delta\gamma = 0$, then γ is a constant. By assumption, $A_u \neq 0$, and so γ/γ^* is also a constant. It will therefore be convenient to write

$$1/A_u = e^{\mathrm{i}k} h, \tag{118}$$

where both k is a real constant and $h = h(u)$ is real. A direct calculation now gives

$$h_u = -\frac{1}{2}\cos k,$$

which implies

$$A_u = \frac{2\mathrm{e}^{-\mathrm{i}k}}{k_2 - u\cos k},$$

$$f = (\cos ku - k_2)^{-2+2\mathrm{i}\tan k} k_1,$$

where $k_1 \neq 0$ is a real constant. Substituting into (115) gives

$$\gamma = \frac{e^{\mathrm{i}k}}{\sqrt{8k_1}},$$

which means that k, k_1 are essential constants, while k_2 can be gauged away. Applying the change of variables (12) gives the desired solution form. □

Proposition 9.3. Suppose that $f(\zeta, u) = e^{4A}\zeta^2$, $\Delta\gamma = 0$, *and* $\Re\gamma = 0$. *Then, without loss of generality*,

$$f(\zeta, u) = \mathrm{e}^{2\mathrm{i}k_0 u}\zeta^2, \tag{119}$$

where k_0 *is a real constant*.

Proof. The super-singular case of $\gamma = 0$ corresponds to $A_u = k = 0$. From now on, we suppose that γ is a non-zero imaginary constant. It follows that

$$A_u = \mathrm{i}k,$$

where k is some real constant. The desired conclusion follows immediately. □

X. CONCLUSIONS

In our search for those vacuum pp-wave spacetimes in which the fourth-order covariant derivatives of the curvature tensor are required to classify them entirely, we have produced an approach to invariantly classifying the vacuum pp-wave spacetimes. Our approach is based on Cartan invariants and the Karlhede algorithm and is necessitated by the fact that a the class of vacuum pp-waves has vanishing scalar invariants.[3] Our classification is finer than the analysis of each spacetime's isometry group alone. The summary of this invariant approach to classification is given in Tables I and II with specialization relations summarized in Figures 1 and 4.

For any spacetime, the classification begins with the fact that the components of the curvature tensor and its covariant derivatives produce all of the invariants required. The Karlhede algorithm provides an algorithmic approach to determining the lowest order, q, of covariant differentiation needed to classify the space, canonical forms for the components of the curvature tensor and the number of functionally independent invariants (t_0, t_1, ..., t_q) arising from the collection of all components of the curvature tensor and its covariant derivatives up to order q.

For vacuum pp-waves we have demonstrated that $q \leq 4$ and have classified all solutions that attain an IC order of 4. Table IV summarizes the maximal order solutions. By characterizing the

G_2 and G_3 solutions in terms of invariant conditions, the invariant approach also sheds light on the origin of the additional Killing vectors. Another remarkable finding is the fact that the maximal order solutions of Table IV are direct precursors of the G_3 solutions first discovered by Kundt and Ehlers. In terms of the metric form, the mechanism of specialization is the disappearance of an additive term, e.g.,

$$e^{ik_0}\log\zeta + k_1 e^{ih}\zeta \to e^{ik}\log\zeta.$$

Outside of the invariant classification of spacetimes, the study of the invariant structure of the Riemann tensor and its covariant derivatives reveals the interconnection between spacetimes with less symmetry and their more symmetric counterparts and how these arise as specialization of the classifying manifold. Furthermore by imposing conditions on the Cartan invariants we produced definite examples of spacetime with little or no symmetry. This is particularly relevant for the pp-wave spacetimes as before our work little was known about those spacetimes admitting $D = \partial_v$ as the sole Killing vector.

The approach used to invariantly classify the pp-waves is not limited to this class alone. One may repeat the process for the other half of the plane-fronted waves, the Kundt waves.[18] Together these spacetimes constitute the entirety of all Petrov type N VSI spacetimes: the class of spacetimes where all scalar curvature invariants vanish. These spacetimes are a special case of the CSI spacetimes, where all scalar curvature invariants are constant, and so the Karlhede algorithm is the only approach to invariantly classifying these spaces.

Similarly, it should be possible to investigate the invariant classification of other classes of spacetimes, such as non-vacuum pp-waves[19] and conformally flat radiation solutions.[20] Furthermore, as the just cited references indicate, it is sometimes possible to utilize the classifying Cartan invariants as coordinate functions, and thereby to invariantly integrate the metrics in question. For example, vacuum pp-waves belong to class IIa of the Collins classification.[2] A complete invariant integration should then, in principle, recover a more general solution form than considered in the present article. However, the techniques would have to be appropriately generalized to account for the invariants arising from the Φ_{12}, Φ_{22} components of the Ricci tensor. In this regard, Held's approach[21] based on invariants subject to involutive constraints should prove particularly useful. It is reasonable to expect that non-vacuum exact solutions would be derived through such an approach.

Future research direction involve the extension of the invariant classification to all VSI spacetimes, and even the full class of Kundt-degenerate spacetimes. The question of the physical and phenomenological interpretation of the classifying invariants is also unresolved, although some steps in this direction are ongoing.[22]

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[1] J. M. Collins, "The Karlhede classification of type N vacuum spacetimes," Class. Quantum Grav. **8**, 1859–1869 (1991).
[2] J. M. Collins, R. A. dInverno, and J. A. Vickers, "The Karlhede classification of type D vacuum spacetimes," Class. Quantum Grav. **7**, 2005–2015 (1990).
[3] A. Coley, S. Hervik, and N. Pelavas, "Spacetimes characterized by their scalar curvature invariants," Class. Quantum Grav. **26**, 025013 (2009).
[4] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, 1980).
[5] H. J. Schmidt, "Why do all the curvature invariants of a gravitational wave vanish?" in *New Frontiers in Gravitation*, edited by G. A. Sardanashvili (Hadronic, Palm Harbor, 1994), pp. 337–344.
[6] V. Pravda, A Pravdová, A. Coley, and R. Milson, "All spacetimes with vanishing curvature invariants," Class. Quantum Grav. **19**, 6213–6236 (2002).
[7] A. Coley, S. Hervik, and N. Pelavas, "Lorentzian spacetimes with constant curvature invariants in four dimensions," Class. Quantum Grav. **26**, 125011 (2009).
[8] A. Coley, S. Hervik, G. Papadopoulos, and N. Pelavas, "Kundt spacetimes," Class. Quantum Grav. **26**, 105016 (2009).
[9] A. Karlhede, "A review of the geometrical equivalence of metrics in general relativity," Gen. Relativ. Gravit. **12**, 693 (1980).

[10] E. Cartan, *Leçons sur la Géométrie des Espaces de Riemann* (Gauthier-Villars, Paris, 1946).
[11] H. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94**, 119–145 (1925).
[12] G. S. Hall and A. D. Rendall, "Uniqueness of the metric from the Weyl and energy momentum tensors," J. Math. Phys. **28**, 1837–1839 (1987).
[13] J. Ehlers and W. Kundt, "Exact solutions of the gravitational field equations," in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 49.
[14] R. Sippel and H. Goenner, "Symmetry classes of pp-waves," Gen. Relativ. Gravit. **18**, 1229–1243 (1986).
[15] M. P. Machado Ramos and J. A. G. Vickers, "Invariant differential operators and the Karlhede classication of type N vacuum solutions," Class. Quantum Grav. **13**, 1589–1599 (1996).
[16] R. Penrose and W. Rindler, *Spinors and Spacetime* (Cambridge University Press, 1984), Vol. 1.
[17] P. J. Olver, *Equivalence, Invariants, and Symmetry* (Cambridge University Press, 1995).
[18] D. D. McNutt, R. Milson, and A. Coley, "Vacuum Kundt waves" Class. Quatum Grav. (in press); preprint arXiv:1208.5027.
[19] S. B. Edgar and M. P. Machado Ramos, "Obtaining a class of type N pure radiation metrics using invariant operators," Class. Quantum Grav. **22**, 791–802 (2005).
[20] S. B. Edgar and J. A. Vickers, "Integration using invariant operators: Conformally flat radiation metrics," Class. Quantum Grav. **16**, 589 (1999).
[21] A. Held, "A formalism for the investigation of algebraically special metrics. I & II," Commun. Math. Phys. **37**, 311–326 (1974); **44**, 211–222 (1975).
[22] A. Coley, D. D. McNutt, and R. Milson, "Vacuum plane waves: Cartan invariants and physical interpretations," Class. Quantum Grav. **29**, 235023 (2012); preprint arXiv:1210.0746 (2012).