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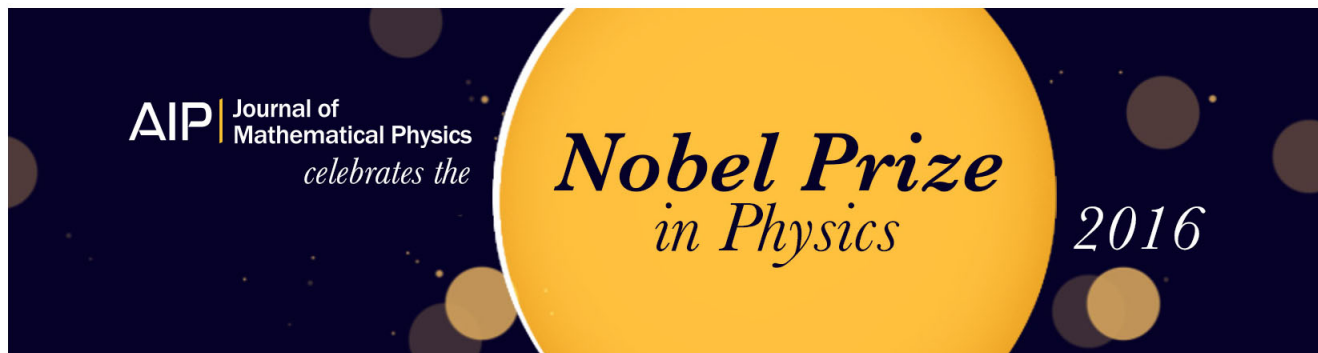
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# Algebraic exact solvability of trigonometric-type Hamiltonians associated to root systems

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In this article, we study and settle several structural questions concerning the exact solvability of the Olshanetsky–Perelomov quantum Hamiltonians corresponding to an arbitrary root system. We show that these operators can be written as linear combinations of certain basic operators admitting infinite flags of invariant subspaces, namely the Laplacian and the logarithmic gradient of invariant factors of the Weyl denominator. The coefficients of the constituent linear combination become the coupling constants of the final model. We also demonstrate the  $L^2$  completeness of the eigenfunctions obtained by this procedure, and describe a straightforward recursive procedure based on the Freudenthal multiplicity formula for constructing the eigenfunctions explicitly. © 1999 American Institute of Physics. [S0022-2488(99)01110-X]

## I. INTRODUCTION

The potentials first discovered by Calogero and Sutherland<sup>1,2</sup> and subsequently generalized to arbitrary root systems by Olshanetsky and Perelomov<sup>3</sup> play a central role in the theory of classical and quantum completely integrable systems. One of the main themes of the original work by Olshanetsky and Perelomov was to establish quantum complete integrability, that is, the existence of complete sets of commuting operators. The actual eigenfunctions of the corresponding Hamiltonians were discussed in numerous subsequent publications.<sup>4–7</sup>

Our purpose in this paper is study and settle a certain number of basic structural questions concerning the exact solvability of the Olshanetsky–Perelomov Hamiltonians. In order to outline the main results of our paper, we first need to give a precise definition of what we mean by exact solvability. We will adopt a promising approach, which has recently arisen in the framework of the theory of quasiexactly solvable potentials,<sup>8–11</sup> by defining a quantum Hamiltonian  $\mathcal{H}$  to be *algebraically exactly solvable* if one can explicitly construct an ordered basis for the underlying Hilbert space such that the corresponding flag of subspaces is  $\mathcal{H}$  invariant. In terms of this approach, the first step in the treatment of an exactly solvable operator must be the construction of an infinite flag of finite-dimensional vector spaces ordered by inclusion, the determination of a collection of basic operators that preserve this flag, and the demonstration that the operator in question is generated by the basic ones. The second step is to prove the  $L^2$  completeness in the underlying Hilbert space of this family of subspaces.

In order to fit the Olshanetsky–Perelomov Hamiltonians of trigonometric type into this framework, we first recall that these Hamiltonians are indexed by irreducible root systems, with the Calogero–Sutherland potentials corresponding to type  $A_n$  root systems. We thus consider the vector space of trigonometric functions that are invariant under the Weyl group  $W$  of the given root system  $R$ . The partial order relation on dominant weights gives rise to a natural flag of finite-dimensional subspaces of this infinite-dimensional vector space. It is quite evident that the

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flag in question is preserved by the ordinary, multidimensional Laplacian. Less evident is the fact that one can obtain other flag-preserving operators by factoring the Weyl denominator,

$$A = \prod_{\alpha \in R^+} e^{\alpha/2} - e^{-\alpha/2},$$

into factors corresponding to the various orbits of the Weyl group on  $R$ . It turns out (see Proposition 12) that the gradient of the logarithm of each of the resulting factors also preserves the flag in question. More generally, one obtains other flag-preserving second-order operators by taking linear combinations of the Laplacian and of these gradients. The Olshanetsky–Perelomov Hamiltonians are then obtained by a ground-state conjugation. This approach also sheds light on the presence of multiple coupling constants in some of the models; the number of coupling constants is precisely the number of invariant factors of  $A$ , i.e., the number of Weyl group orbits in  $R$ , or, equivalently, the number of distinct root lengths. We then show that if all the coupling constants are positive, then the action of the Hamiltonian on each subspace of the flag is diagonalizable. This is the first main result of our paper; it is given in Theorem 1. The second main result concerns the  $L^2$  completeness of the resulting eigenfunctions in the underlying Hilbert space of  $L^2$  functions on the alcove of the root system  $R$ .

It is also interesting to note that if all the coupling constants are equal to 1, then one recovers a second-order differential operator whose eigenfunctions are precisely the characters of the corresponding simple Lie algebras. For certain other values of the coupling constants, one recovers the spherical functions associated to any symmetric space  $G/K$ , where  $G$  is a semisimple real Lie group and  $K$  is a suitable compact subgroup. If the restricted root system of the symmetric space is of type  $A_{n-1}$  and  $m$  is the multiplicity of each restricted root, then the eigenfunctions corresponding to the value  $k_c = m/2$  of the deformation parameter are the zonal spherical functions on  $G/K$ , as pointed out by Macdonald.<sup>12,13</sup> Thus the coupling constants can be regarded as parameters in a deformation of the classical characters.

In the classical case, if one reexpresses the gradient of  $\log A$  in terms of a formal power series, one obtains Freudenthal’s recursion formula for the character coefficients. This trick also works for the deformed characters, and leads to a recursion formula that allows one to straightforwardly compute the eigenfunctions of the Olshanetsky–Perelomov Hamiltonians. This result is presented in Sec. IV.

We should point out that the Weyl-invariant deformed characters that appear in the expressions of the eigenfunctions of the Olshanetsky–Perelomov trigonometric Hamiltonians are related by a change of variables to the multivariate Jacobi polynomials that have been investigated by Heckman and Opdam.<sup>14</sup> In particular, the analog of the Freudenthal multiplicity formula that is at the basis of the recursion formula we give in Proposition 19 for the eigenfunctions of the Hamiltonians also appears in the context of their study. We should also mention the interesting recent contributions of Brink, Turbiner, and Wyllard<sup>15</sup> in the general effort aimed at understanding the exact solvability for multidimensional systems in an algebraic context.

## II. TRIGONOMETRIC-TYPE POTENTIALS ASSOCIATED TO ROOT SYSTEMS

We first recall the abstract definition of the trigonometric Olshanetsky–Perelomov Hamiltonians in terms of root systems. Let  $\mathbf{V}$  be a finite-dimensional real vector space endowed with a positive-definite inner product  $(u, v) \in \mathbb{R}$ ,  $u, v \in \mathbf{V}$ . We use this inner product to identify  $\mathbf{V}$  with  $\mathbf{V}^*$ . The induced positive-definite inner product on  $\mathbf{V}^*$  will also be denoted by  $(\cdot, \cdot)$ . Let  $\Delta: C^\infty(\mathbf{V}; \mathbb{R}) \rightarrow C^\infty(\mathbf{V}; \mathbb{R})$  and  $\nabla: C^\infty(\mathbf{V}; \mathbb{R}) \rightarrow \Gamma(T\mathbf{V})$  denote the corresponding Laplace–Beltrami and gradient operators.

For a nonzero  $\alpha \in \mathbf{V}^*$ , we set  $\check{\alpha} = 2\alpha/(\alpha, \alpha)$  and let  $s_\alpha$  denote the reflection across the hyperplane orthogonal to  $\alpha$ :

$$s_\alpha(\beta) = \beta - (\check{\alpha}, \beta)\alpha, \quad \beta \in \mathbf{V}^*.$$

By a root system, we mean a finite, spanning subset  $R$  of  $\mathbf{V}^*$  such that  $0 \in R$ ,  $s_\alpha(R) \subset R$  for all  $\alpha \in R$  and  $(\check{\alpha}, \beta) \in \mathbb{Z}$  for all  $\alpha, \beta \in R$ . A root system  $R$  is said to be irreducible if it cannot be partitioned into a union of root systems spanning orthogonal subspaces of  $\mathbf{V}$ .

To any root system  $R$  corresponds a root lattice  $Q = \{\sum_R m_\alpha \alpha : m_\alpha \in \mathbb{Z}\}$  and a weight lattice  $P = \{\lambda \in \mathbf{V}^* : (\check{\alpha}, \lambda) \in \mathbb{Z}, \forall \alpha \in R\}$ . The Weyl group of  $R$ , generated by  $s_\alpha, \alpha \in R$ , will be denoted by  $W$ . The subgroup of  $W$  fixing a particular  $\lambda \in \mathbf{V}^*$  will be denoted by  $W_\lambda$ .

The hyperplanes  $\{\lambda \in \mathbf{V}^* : (\alpha, \lambda) = 0\}, \alpha \in R$  define a set of open Weyl chambers in  $\mathbf{V}^*$ . We choose a Weyl chamber  $C$  and let  $R^+ = R \cap \bar{C}$  denote the corresponding subset of positive roots. Let  $B \subset R^+$  denote the set of simple roots, i.e., the positive roots that cannot be written as the sum of two positive roots. Let  $P^+ = R \cap \bar{C}$  denote the set of dominant weights.

We will say that a real number  $c > 0$  is a *root length* if there exists a  $\alpha \in R$  such that  $c = \|\alpha\|$ . Let  $c$  be a root length, and set

$$R_c = \{\alpha \in R : \|\alpha\| = c\},$$

$$R_c^+ = R_c \cap R^+,$$

$$U_c = \frac{c^2}{4} \sum_{\alpha \in R_c^+} \cos^2 \frac{\alpha}{2}.$$

Note<sup>16</sup> that if  $c$  is a root length, then,  $R_c$  is nothing but the  $W$  orbit of  $\alpha$ .

The Olshanetsky–Perelomov Hamiltonians with trigonometric potentials associated to a root system  $R$  are defined in terms of the above data by

$$\mathcal{H} = -\Delta + \sum_c a_c U_c,$$

where the sum is taken over all root lengths,  $c$ , and where the  $a_c$ 's are real coupling constants.

### III. THE ALGEBRAIC EXACT SOLVABILITY OF $\mathcal{H}$

The affine hyperplanes  $\{\lambda \in \mathbf{V}^* : (\alpha, \lambda) \in 2\pi\mathbb{Z}\}$  determine in  $\mathbf{V}^*$  a set of isometric open bounded subsets called alcoves. Let  $A$  denote the unique alcove (usually referred to as the fundamental alcove) that is contained in  $C$  and that has the origin as a boundary point. Let  $m$  denote the Lebesgue measure on  $A$ . From now on we use the inner product to identify  $A$  with the corresponding subset of  $\mathbf{V}$  and restrict the domain of functions introduced subsequently to  $A$ . Our goal is to construct a basis for the underlying Hilbert space  $L^2(A, m)$  in which the algebraic exact solvability of  $\mathcal{H}$  is manifest. The elements of this basis will be products of  $W$ -invariant trigonometric functions of certain linear forms on  $\mathbf{V}$  with a common gauge factor vanishing along the walls  $\{u \in \mathbf{V} : \alpha(u) \in 2\pi\mathbb{Z}\}, \alpha \in R$  of the potential terms  $U_c$ .

We now proceed to define this basis. Recall that a choice of positive roots naturally induces a partial order relation,  $\leq$ , on the weight lattice. For  $\lambda \in P^+$  set

$$P_\lambda = \cup_{w \in W} \{w(\mu) : \mu \in P^+ \text{ and } \mu \leq \lambda\},$$

$$P_{\lambda^-} = \cup_{w \in W} \{w(\mu) : \mu \in P^+ \text{ and } \mu \neq \lambda\}.$$

For  $S \subset \mathbf{V}^*$  let  $\text{trig}(S)$  denote the complex vector space spanned by functions of the form  $e^{i\lambda}, \lambda \in S$ . If  $S$  is a  $W$ -invariant subset of  $\mathbf{V}^*$ , then there is a well-defined action of  $W$  on  $\text{trig}(S)$ , namely

$$w \cdot e^{i\lambda} = e^{iw(\lambda)}, \quad w \in W, \quad \lambda \in S.$$

In this case, let  $\text{trig}(S)^W$  denote the subspace of  $W$ -invariant functions.

Recall that a root system  $R$  is said to be reduced if for every  $\alpha \in R$ , the only roots homothetic to  $\alpha$  are  $-\alpha$  and  $\alpha$  itself. A root  $\alpha$  will be called nondivisible if  $\alpha/2$  is not a root. Similarly,  $\alpha$  will be called nonmultiplicable if  $2\alpha$  is not a root. Of course, if  $R$  is reduced, then all roots are both nondivisible and nonmultiplicable. An irreducible nonreduced system must be isomorphic to a root system of type  $BC_n$  for some  $n$ . To describe the latter, take  $\mathbf{V} = \mathbb{R}^n$  and let  $\epsilon_1, \dots, \epsilon_n$  denote the dual basis of the standard basis of  $\mathbb{R}^n$ . The root system in question consists of three types of roots: short roots  $\pm \epsilon_i$ , medium roots  $\pm \epsilon_i \pm \epsilon_j$ ,  $i \neq j$ , and long roots  $\pm 2\epsilon_i$ .

For reasons that will become clear later, it is convenient to reexpress the coupling constants  $a_c$  appearing in  $\mathcal{H}$  as follows. We let  $a_c = k_c(k_c - 1)$  if  $c$  is the length of a nonmultiplicable root, and  $a_c = k_c(k_c + k_{2c} - 1)$  if  $R$  is nonreduced and  $c$  is the length of the short roots. Let

$$A_c = \prod_{\alpha \in R_c^+} \sin \frac{\alpha}{2}, \quad F = \prod_c |A_c|^{k_c}, \quad \rho_c = \frac{1}{2} \sum_{\alpha \in R_c^+} \alpha, \quad \rho = \sum_c k_c \rho_c.$$

The following theorems, which are the main results of our paper, shows that the Olshanetsky–Perelomov trigonometric Hamiltonians  $\mathcal{H}$  are exactly solvable in the algebraic sense, and that the corresponding eigenfunctions are physically meaningful.

**Theorem 1:** *Let  $\lambda$  be a dominant weight. If  $k_c \geq 0$  for each root length  $c$ , then there exists a unique  $\phi_\lambda \in \text{trig}(P_\lambda)^W$  such that  $F\phi_\lambda$  is an eigenfunction of  $\mathcal{H}$  with eigenvalue  $\|\lambda + \rho\|^2$ . Furthermore, if  $F\phi, \phi \in \text{trig}(P)^W$  is an eigenfunction of  $\mathcal{H}$ , then  $\phi = \phi_\lambda$  for some  $\lambda \in P^+$ .*

**Theorem 2:** *The subspace  $F \text{trig}(P)^W$  is dense in  $L^2(A, m)$ . Moreover, if  $k_c \geq 0$  for all root lengths  $c$ , then the operator  $\mathcal{H}$  is essentially self-adjoint on the domain  $F \text{trig}(P)^W \subset L^2(A, m)$ .*

We begin with the proof of Theorem 2, assuming Theorem 1 to be true. We first have the following.

*Lemma 3:* *Let  $D$  be an open, bounded subset of Euclidean space, and  $f: D \rightarrow \mathbb{R}$  a bounded continuous function that does not vanish on  $D$  (but may vanish on the boundary). With these assumptions,  $fL^2(D, m)$  is a dense subset of  $L^2(D, m)$ .*

*Proof:* Let  $D_0$ , an open subset of  $D$ , be given, and choose  $D_1$  such that  $\bar{D}_1 \subset D_0$  and such that  $m(D_0) - m(D_1)$  is smaller than a given  $\epsilon > 0$ . Note that  $h = f^{-1}\chi_{D_1}$  is a well-defined element of  $L^2(D)$  and that  $fh = \chi_{D_1}$ . Consequently,  $\chi_{D_0}$  lies in the closure of  $fL^2(D)$ . The conclusion follows from the fact that the characteristic functions form a dense subset of  $L^2(D)$ .  $\square$

*Proof of Theorem 2:* Let  $\mathbf{T}$  denote the torus  $\mathbf{V}^*/(2\pi Q)$ . We use the inner product on  $\mathbf{V}$  to identify  $\mathbf{T}$  with the identical quotient of  $\mathbf{V}$ . Recall that  $\text{trig}(P)$  is dense in  $L^2(\mathbf{T})$  by the Fourier representation theorem. Now  $W$  acts on  $\mathbf{T}$  and  $A$  serves as a fundamental region for this action (Ref. 17, Chap. VI, No. 2.1). Consequently,  $\text{trig}(P)^W$  is dense in  $L^2(T)^W$  and the latter is naturally isomorphic to  $L^2(A, m)$ . We therefore conclude that  $F \text{trig}(P)^W$  is dense in  $L^2(A, m)$  by applying the preceding Lemma with  $f = F$ .

We now prove the essential self-adjointness of  $\mathcal{H}$  on the domain  $F \text{trig}(P)^W$ . Let  $A_0 \subset A$  be an open subset with a piecewise smooth boundary. Let  $\phi_1, \phi_2 \in \text{trig}(P)^W$  be given. Setting  $\psi_i = F\phi_i$ ,  $i = 1, 2$ , we have

$$\int_{A_0} \mathcal{H}(\psi_1)\psi_2 - \psi_1\mathcal{H}(\psi_2) = \int_{A_0} \text{div}(\psi_2 \nabla \psi_1 - \psi_1 \nabla \psi_2) = \int_{\partial A_0} F^2(\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2).$$

Hence, as the boundary of  $A_0$  approaches the boundary of  $A$ , the above integrals tend to zero, so that the operator  $\mathcal{H}$  is a symmetric. By Theorem 1 and the density of  $F \text{trig}(P)^W$  in  $L^2(A, m)$ , the span of eigenfunctions of  $\mathcal{H}$  is dense in  $L^2(A)$ , and therefore  $\mathcal{H}$  must be essentially self-adjoint.  $\square$

We now proceed with the proof of Theorem 1. The strategy behind the proof of this theorem is to conjugate the Olshanetsky–Perelomov Hamiltonians  $\mathcal{H}$  by a suitable multiplication operator chosen in such a way that the resulting operator has a simple action on the space  $\text{trig}(P)^W$ . This

will give rise to an essential intertwining relation that will, in turn, imply the algebraic exact solvability. In order to determine this multiplicative factor, we need a series of facts about root lengths.

Let  $M_c : W \rightarrow \{\pm 1\}$  be the class function defined by

$$M_c(s_\alpha) = \begin{cases} -1, & \text{if } \alpha \in B \cap R_c, \\ 1 & \text{if } \alpha \in B \setminus R_c. \end{cases}$$

The following result is a straightforward consequence of the definition of  $A_c$ .

*Proposition 4:* For  $w \in W$  one has  $w(A_c) = M_c(w)A_c$ . In other words,  $A_c$  is a relative invariant of  $W$  with multiplier  $M_c$ . Moreover, we have the following.

*Proposition 5:* Let  $c$  be a root length. If  $\alpha \in B$ , then  $(\check{\alpha}, \rho_c)$  takes one of four possible values: 1 if  $\|\alpha\| = c$ , 2 if  $\|\alpha\| = c/2$ ,  $1/2$  if  $\|\alpha\| = 2c$ , 0 in all other cases.

*Proof:* Let  $\alpha \in B$  be given. The action of  $s_\alpha$  maps  $\alpha$  to  $-\alpha$  and permutes the elements of  $R^+$  not homothetic to  $\alpha$  (Ref. 17, Chap. VI, No. 1.6). Let  $\beta \in R_c^+$  be given and set  $\beta' = s_\alpha(\beta)$ . Note that if  $\beta = \beta'$ , then  $(\check{\alpha}, \beta) = 0$ ; and that if  $\beta' \neq \beta$ , then  $(\check{\alpha}, \beta + \beta') = 0$ . If  $\|\alpha\| \in \{c, 2c, c/2\}$ , then  $\alpha$  is not homothetic to any element of  $R_c$ , and hence one can break up  $\rho_c$  into subterms of length one and two such that each subterm is annihilated by  $\check{\alpha}$ . This proves the fourth assertion of the proposition. If  $\|\alpha\| = c$ , then  $\rho_c$  is the sum of  $\alpha/2$  and a remainder perpendicular to  $\check{\alpha}$ . Consequently,  $(\check{\alpha}, \rho_c) = 1$ , thereby proving the first assertion. If  $\|\alpha\| = c/2$ , then  $2\alpha$  is also a root, and, consequently,  $\rho_c$  is the sum of  $\alpha$  and a remainder perpendicular to  $\check{\alpha}$ . This implies the second assertion. The case three assertion is proven similarly.  $\square$

*Corollary 6:* If  $c$  is the length of a nonmultiplicable root, then  $\rho_c$  is a weight. If  $R$  is nonreduced, and  $c$  is the length of the short roots, then  $\rho_c$  is merely a half-weight.

*Corollary 7:* Let  $c$  be a root length. Then for all  $\alpha \in R_c$ , one has  $(\check{\alpha}, \rho_c) \in \mathbb{Z}$ .

*Proof:* If  $c$  is the length of a nonmultiplicable root, then the claim follows from the preceding corollary. Suppose then that  $2c$  is also a root length. For  $\alpha \in R_c$  note that  $2(2\alpha)^\vee = \check{\alpha}$  and that  $2\rho_c = \rho_{2c}$ . Hence

$$(\check{\alpha}, \rho_c) = ((2\alpha)^\vee, \rho_{2c}).$$

Since  $2\alpha$  is nonmultiplicable, the right-hand side is an integer by the preceding corollary.  $\square$

*Corollary 8:* Let  $c$  be a root length and  $w \in W$ . Then,  $w(\rho_c) \in Q - \rho_c$ .

*Proof:* Note that

$$w(\rho_c) = \frac{1}{2} \sum_{\alpha \in R_c^+} \sigma_\alpha(w) \alpha,$$

where  $\sigma_\alpha(w)$  is either 1 or  $-1$ . Hence,  $\rho_c + w(\rho_c)$  is the sum of all  $\alpha \in R_c^+$  such that  $\sigma_\alpha(w) = 1$ .  $\square$

We are now ready for the next step leading to the required intertwining relation, which is to show that  $\text{trig}(P_\lambda)^W$  is an invariant subspace of  $\nabla \log|A_c|$ . First, we have the following.

*Proposition 9:* Let  $c$  be a root length. If  $\phi \in \text{trig}(P - \rho_c)$  is a relative invariant of  $W$  with multiplier  $M_c$ , then  $\phi = A_c \phi_0$  for some  $\phi_0 \in \text{trig}(P)^W$ .

*Proof:* By assumption,  $\phi_1 = e^{i\rho_c} \phi$  is an element of  $\text{trig}(P)$ . Let  $\alpha \in R_c^+$  be given. The first claim is that  $\phi_1$  is divisible by  $e^{i\alpha} - 1$  in  $\text{trig}(P)$ . By assumption,  $\phi$  is a linear combination of expressions of the form  $e^{i\lambda} - e^{i\lambda'}$ , where  $\lambda + \rho_c \in P$ , and  $\lambda' = s_\alpha(\lambda)$ . Since  $\lambda$  is the difference of a weight and  $\rho_c$ , Corollary 7 shows that  $(\check{\alpha}, \lambda) \in \mathbb{Z}$ . By switching  $\lambda$  and  $\lambda'$ , if necessary, one may assume without loss of generality that  $-(\check{\alpha}, \lambda) \in \mathbb{N}$ . The claim follows by noting that

$$e^{i\lambda} - e^{i\lambda'} = e^{i\lambda}(1 - e^{-i(\check{\alpha}, \lambda)\alpha}),$$

and by factoring the right-hand side in the usual fashion.

Note that  $\text{trig}(P)$  with the natural function multiplication is a unique factorization domain (Ref. 17, Chap. VI, No. 3.1). Hence, the preceding claim implies that there exists a  $\phi_0 \in \text{trig}(P)$ , such that

$$\phi_1 = \phi_0 \prod_{\alpha \in R_c^+} (e^{i\alpha} - 1).$$

The proof is concluded by noting that up to a constant factor,  $A_c$  is equal to

$$e^{-i\rho_c} \prod_{\alpha \in R_c^+} (e^{i\alpha} - 1).$$

The  $W$  invariance of  $\phi_0$  follows from the fact that  $A_c$  and  $\phi$  are relative invariants with the same multiplier. □

We have:

*Corollary 10:* Let  $c$  be a root length. One has

$$(2i)^{\#R_c} A_c = \frac{1}{\#W_{\rho_c}} \sum_{w \in W} M_c(w) e^{iw(\rho_c)}. \tag{1}$$

*Proposition 11:* The differential operator  $\nabla \log |A_c|$  has a well-defined action on  $\text{trig}(P)^W$ .

*Proof:* Let  $\phi \in \text{trig}(P)^W$ . The claim is that  $(\nabla \log |A_c|)(\phi) \in \text{trig}(P)^W$ . By Corollaries 8 and 10,  $A_c \in \text{trig}(Q - \rho_c)$ , and hence  $\nabla A_c(\phi) \in \text{trig}(P - \rho_c)$ . Since  $\nabla$  is a  $W$ -invariant operator,  $\nabla A_c(\phi)$  is a relative invariant of  $W$  with multiplier  $M_c$ . Hence, by Proposition 9, there exists a  $\phi_0 \in \text{trig}(P)^W$  such that  $\nabla A_c(\phi) = A_c \phi_0$ . □

We now have the following.

*Proposition 12:* If  $\lambda \in P^+$ , then  $\text{trig}(P_\lambda)^W$  is an invariant subspace of  $\nabla \log |A_c|$ .

*Proof:* Let  $\phi \in \text{trig}(P_\lambda)^W$  be given. Set  $\phi_0 = (\nabla \log |A_c|)(\phi)$ . By Proposition 11,  $\phi_0 \in \text{trig}(P)^W$ . Let  $\mu$  be a maximal element of  $\text{supp}(\phi_0)$ . Consequently,  $\mu + \rho_c$  is a maximal element of  $\text{suppl}(A_c \phi_0)$ . Now

$$A_c = b_1 e^{i\rho_c} + \text{lower-order terms},$$

$$\phi = b_2 e^{i\lambda} + \text{lower-order terms},$$

where  $b_1, b_2$  are nonzero constants, and hence,

$$(\nabla A_c)(\phi) = -b_1 b_2 (\rho_c, \lambda) e^{i(\rho_c + \lambda)} + \text{lower-order terms}.$$

Since  $(\rho_c, \lambda) > 0$ , one must have  $\rho_c + \lambda = \rho_c + \mu$ . Therefore  $\mu = \lambda$ , and  $\phi_0 \in \text{trig}(P_\lambda)^W$ .

The basic identity that will give rise to the intertwining relation that we are looking for is given in the following proposition.

*Proposition 13:* Let  $f_1, \dots, f_n$  be smooth real-valued functions on  $\mathbf{V}$ ; let  $k_1, \dots, k_n$  be real constants; and let

$$X = \sum_{i=1}^n 2k_i \nabla \log |f_i|, \quad F = \prod_{i=1}^n |f_i|^{k_i}.$$

We have the identity

$$F(-\Delta - X) = (-\Delta + U)F,$$

where

$$U = \sum_i k_i(k_i - 1) \frac{\|\nabla f_i\|^2}{f_i^2} + \sum_{i \neq j} k_i k_j \frac{(\nabla f_i, \nabla f_j)}{f_i f_j} + \sum_i k_i \frac{\Delta f_i}{f_i}.$$

The application of this proposition to the Olshanetsky–Perelomov Hamiltonians  $\mathcal{H}$  requires a number of intermediate formulas.

*Proposition 14:* Let  $c$  be a root length. One has

$$\Delta A_c = -\|\rho_c\|^2 A_c, \tag{2}$$

$$\|\nabla A_c\|^2 = (U_c - \|\rho_c\|^2) A_c^2. \tag{3}$$

*Proof:* Note that for  $\lambda \in \mathbf{V}^*$  one has  $\Delta e^{i\lambda} = -\|\lambda\|^2 e^{i\lambda}$ . Formula (2) follows immediately from (1). Note that

$$\nabla A_c = \frac{A_c}{2} \sum_{\alpha \in R_c^+} \cot \frac{\alpha}{2} \nabla \alpha. \tag{4}$$

Consequently,

$$\|\nabla A_c\|^2 = \left( \frac{c^2}{4} \sum_{\alpha} \cot^2 \frac{\alpha}{2} + \frac{1}{4} \sum_{\alpha \neq \beta} (\alpha, \beta) \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \right) A_c^2. \tag{5}$$

Taking the divergence of (4), one obtains

$$\frac{\Delta A_c}{A_c} = -\frac{(\#R_c)c^2}{4} + \frac{1}{4} \sum_{\alpha \neq \beta} (\alpha, \beta) \cot \frac{\alpha}{2} \cot \frac{\beta}{2}.$$

Solving for the second term of the right-hand side of the latter equation, substituting into (5) and applying (2), we obtain (3).  $\square$

*Proposition 15:* If  $c_1, c_2$  are distinct root lengths such that the corresponding roots are not homothetic, then

$$(\nabla A_{c_1}, \nabla A_{c_2}) = -(\rho_{c_1}, \rho_{c_2}) A_{c_1} A_{c_2}. \tag{6}$$

If  $R$  is nonreduced and  $c$  is the length of the short roots, then

$$(\nabla A_c, \nabla A_{2c}) = [U_c - (\rho_c, \rho_{2c})] A_c A_{2c}. \tag{7}$$

*Proof:* Let  $c_1, c_2$  be given. A straightforward generalization of the argument in Proposition 9 yields

$$A_{c_1} A_{c_2} = \frac{1}{\#W_{\rho_{c_1} + \rho_{c_2}}} \sum_{w \in W} M_{c_1}(w) M_{c_2}(w) e^{iw(\rho_{c_1} + \rho_{c_2})}.$$

Hence,

$$\Delta(A_{c_1} A_{c_2}) = -\|\rho_{c_1} + \rho_{c_2}\|^2 A_{c_1} A_{c_2},$$

and the desired conclusion follows immediately from the usual product rule for the Laplacian.

Next, assume that the second of the proposition’s hypotheses holds. Set  $S_c = \prod_{\alpha \in R_c} \cos(\alpha/2)$ , and note that  $A_{2c} = 2A_c S_c$ . Since  $R$  is of type  $BC_n$ , a direct calculation will show that  $\Delta S_c = -\|\rho_c\|^2 S_c$ . Consequently,

$$2(\nabla A_c, \nabla S_c) = \frac{1}{2} \Delta A_{2c} - A_c \Delta S_c - S_c \Delta A_c = -\|\rho_c\|^2 A_{2c}.$$



$$(\nabla A_c, \nabla A_{2c}) = -\|\rho_c\|^2 A_{2c} + 2S_c \|\nabla A_c\|^2.$$

The formula to be proved now follows from (3). □

We can now state and prove the intertwining relation, which is fundamental to the proof of our main result.

*Proposition 16:* Let

$$\tilde{\mathcal{H}} = -\Delta - \sum_c 2k_c \nabla \log |A_c|.$$

We have

$$F\tilde{\mathcal{H}} = \mathcal{H}F - \|\rho\|^2.$$

*Proof:* Apply Propositions 13, 14, and 15. □

Finally, we are ready to give the proof of Theorem 1, that is of the algebraic exact solvability of the Olshanetsky–Perelomov Hamiltonian  $\mathcal{H}$ . We begin with the following simple result from linear algebra.

*Proposition 17:* Let  $\mathbf{V}$  a finite-dimensional vector space over  $\mathbb{C}$ , and  $\mathbf{V}_1 \subset \mathbf{V}$  a codimension 1 subspace. Let  $T$  be an endomorphism of  $\mathbf{V}$  such that  $\mathbf{V}_1$  is an invariant subspace, and let  $\kappa \in \mathbb{C}$  denote the unique eigenvalue of the corresponding endomorphism of  $\mathbf{V}/\mathbf{V}_1$ . If  $\kappa$  is not an eigenvalue of  $T|_{\mathbf{V}_1}$ , then  $\kappa$  is a multiplicity 1 eigenvalue of  $T$ .

It should be noted that the assumption  $k_c \geq 0$  in Theorem 1 is crucial. The necessity of this assumption is explained by the following proposition. Indeed, one should remark that there exist certain negative values of  $k_c$  for which the action of  $\mathcal{H}$  fails to be diagonalizable.

*Proposition 18:* Let  $\mu < \lambda$  be dominant weights. If  $k_c \geq 0$  for each root length  $c$ , then  $\|\lambda + \rho\| > \|\mu + \rho\|$ .

*Proof:* Note that

$$\|\lambda + \rho\|^2 - \|\mu + \rho\|^2 = \|\lambda\|^2 - \|\mu\|^2 + 2(\lambda - \mu, \rho).$$

Using the fact that  $\lambda - \mu \in P^+$ , one can easily show that  $\|\lambda\| > \|\mu\|$ . Furthermore, since  $\lambda - \mu$  is a linear combination of basic roots with positive coefficients, Proposition 5 implies that  $(\lambda - \mu, \rho) > 0$ . □

Finally, we have the following.

*Proof of Theorem 1:* Let  $\lambda$  be a dominant weight. By Proposition 12,  $\text{trig}(P_\lambda)^W$  is an invariant subspace of  $\tilde{\mathcal{H}}$ . Using an argument similar to the one given in the proof of Proposition 12, it is not hard to verify that if  $\phi \in \text{trig}(P_\lambda)^W$ , then

$$(\tilde{\mathcal{H}} - \|\lambda\|^2 - 2(\rho, \lambda))(\phi) \in \text{trig}(P_{\lambda-})^W. \tag{8}$$

Note that  $\text{trig}(P_{\lambda-})^W$  is a codimension 1 subspace of  $\text{trig}(P_\lambda)^W$ . Furthermore, by Proposition 18,

$$\|\lambda\|^2 + 2(\lambda, \rho) > \|\mu\|^2 + 2(\mu, \rho),$$

for all dominant weights  $\mu < \lambda$ . Hence, by Proposition 17, there exists a unique  $\phi_\lambda \in \text{trig}(P_\lambda)^W$  such that  $\tilde{\mathcal{H}}\phi_\lambda = (\|\lambda\|^2 + 2(\rho, \lambda))\phi$ . The first of the desired conclusions now follows by Proposition 16.

To prove the converse let  $F\phi$  with  $\phi \in \text{trig}(P)^W$  be an eigenfunction of  $\mathcal{H}$  with eigenvalue  $\kappa$ . Let  $\lambda \in P^+$  be a maximal element of  $\text{supp}(\phi)$ . Since  $\text{trig}(P_{\lambda-})^W$  is a codimension 1 subspace of  $\text{trig}(P_\lambda)^W$ , (8) implies that  $\kappa = \|\lambda\|^2 + 2(\lambda, \rho)$ . Consequently,  $\lambda$  is the unique maximal element of  $\text{supp}(\phi)$ . By Proposition 17,  $\kappa$  has multiplicity 1, and this gives the desired conclusion. □

**IV. A RECURSION FORMULA FOR THE EIGENFUNCTIONS OF  $\tilde{\mathcal{H}}$**

In the present section we show how to explicitly compute the eigenfunctions of the Olshanetsky–Perelomov Hamiltonian by using a  $k_c$ -parametrized analog of the Freudenthal multiplicity formula. The generalized formula actually yields the eigenfunctions  $\phi_\lambda$  of the related operator  $\tilde{\mathcal{H}}$ . One should mention that the eigenfunctions  $\phi_\lambda$  first appeared in the investigations of Heckman and Opdam,<sup>14</sup> who regard these functions as multivariable generalizations of the Jacobi polynomials. The eigenfunctions of  $\mathcal{H}$  are, of course, obtained by multiplication with the gauge factor  $F$ .

By way of motivation it will be useful to recall the context of the original Freudenthal formula. Suppose that  $R$  is reduced and let  $\chi_\lambda, \lambda \in P^+$  denote a character of the corresponding compact, simply connected Lie group. The Weyl character formula states that

$$\chi_\lambda = \frac{\sum_{w \in W} \text{sgn}(w) e^{iw(\lambda + \tilde{\rho})}}{\sum_{w \in W} \text{sgn}(w) e^{iw(\lambda)}}, \tag{9}$$

where  $\tilde{\rho}$  is the half-sum of the positive roots. Now if  $k_c = 1$  for all  $c$ , then the potential term of  $\mathcal{H}$  is zero, and the gauge factor  $F$  is nothing but the  $W$ -antisymmetric denominator of (9). Furthermore, the numerator in (9) is the unique  $W$ -antisymmetric eigenfunction of  $\Delta$  with highest-order term  $e^{i(\lambda + \tilde{\rho})}$ . Hence, by the intertwining relation described in Proposition 16, the Weyl character formula is equivalent to the statement that  $\chi_\lambda$  is an eigenfunction of  $\tilde{\mathcal{H}}$  with eigenvalue  $(\lambda, \lambda + 2\tilde{\rho})$ . This observation leads directly to the classical Freudenthal formula for the multiplicities of  $\chi_\lambda$ , and to the following generalization involving the parameters  $k_c$ . (See Ref. 18 for more details regarding the Weyl and Freudenthal formulas.)

*Proposition 19:* Let  $\phi_\lambda = e^{i\lambda + \sum_{\mu < \lambda} n_\mu e^{i\mu}}$  be the eigenfunction of  $\tilde{\mathcal{H}}$  described in the statement and proof of Theorem 1. Setting  $n_\lambda = 1$  and  $n_\nu = 0$  for  $\nu \neq \lambda$ , the remaining coefficients  $n_\mu, \mu < \lambda$ , are given by the following recursion formula:

$$(\|\lambda + \rho\|^2 - \|\mu + \rho\|^2)n_\mu = 2 \sum_{\alpha \in R^+} \sum_{j \geq 1} k_{|\alpha|}(\alpha, \mu + j\alpha)n_{\mu + j\alpha}. \tag{10}$$

*Proof:* Rewriting

$$A_c = e^{i\rho c} \prod_{\alpha \in R_c^+} (1 - e^{-i\alpha}),$$

one obtains

$$\tilde{\mathcal{H}} = -\Delta - i \nabla \rho - 2i \sum_{\alpha \in R^+} k_{|\alpha|} \frac{e^{-i\alpha}}{1 - e^{-i\alpha}} \nabla \alpha.$$

Let  $\text{trig}((P))$  denote the vector space of formal power series  $\sum_{\mu \in PC} c_\mu e^{i\mu}$ . Since elements of  $\text{trig}(P)$  are finitely supported sums, one has a well-defined multiplication operation  $\text{trig}((P)) \times \text{trig}(P) \rightarrow \text{trig}((P))$ . Thus, setting the domain of  $\tilde{\mathcal{H}}$  to be  $\text{trig}(P)$ , one can extend the operator's coefficient ring and write

$$\tilde{\mathcal{H}} = -\Delta - i \nabla \rho - 2i \sum_{\alpha \in R^+} \sum_{j \geq 1} k_{|\alpha|} e^{-ji\alpha} \nabla \alpha.$$

However, because of Proposition 11 one can take the codomain of  $\tilde{\mathcal{H}}$  to be  $\text{trig}(P)$  rather than all of  $\text{trig}((P))$ . Acting with the right-hand side of the latter equation on  $\phi_\lambda$ , collecting like terms,

and using the fact that  $\phi_\lambda$  is an eigenfunction with eigenvalue  $(\lambda, \lambda + 2\rho)$  immediately yields (10).  $\square$

It is important to remark that by Proposition 18 the coefficient of  $n_\mu$  appearing in (10) is never zero. Consequently, (10) can indeed be used as a recursive formula for the coefficients  $n_\mu$ . One should also remark that the  $W$  symmetry of  $\phi_\lambda$  means that it suffices to use formula (10) to calculate  $n_\mu$  with  $\mu \in P^+$ .

- <sup>1</sup>F. Calogero, "Solution of the one-dimensional  $N$ -body problems with quadratic and/or inversely quadratic pair potentials," J. Math. Phys. **12**, 419 (1971).
- <sup>2</sup>B. Sutherland, "Exact results for a quantum many-body problem in one dimension II," Phys. Rev. A **5**, 1372 (1972).
- <sup>3</sup>M. Olshanetsky and A. Perelomov, "Quantum integrable systems related to Lie algebras," Phys. Rep. **94**, 313 (1983).
- <sup>4</sup>A. Perelomov, E. Ragoucy, and P. Zaugg, "Explicit solution of the quantum three-body Calogero–Sutherland model," J. Phys. A **31**, L559 (1998).
- <sup>5</sup>W. Rühl and A. Turbiner, "Exact solvability of the Calogero and Sutherland models," Mod. Phys. Lett. A **10**, 2213 (1995).
- <sup>6</sup>D. Serban, "Some properties of the Calogero–Sutherland model with reflections," J. Phys. A **30**, 4215 (1997).
- <sup>7</sup>K. Sogo, "Eigenstates of Calogero–Sutherland–Moser model and generalized Schur functions," J. Math. Phys. **35**, 2282 (1994).
- <sup>8</sup>M. Shifman, "New findings in quantum mechanics (partial algebraization of the spectrum)," Int. J. Mod. Phys. A **126**, 2897 (1989).
- <sup>9</sup>A. Turbiner, "Quasi-exactly solvable problems and  $sl_2$  algebra," Commun. Math. Phys. **118**, 467 (1989).
- <sup>10</sup>A. Gonzalez-Lopez, N. Kamran, and P. J. Olver, "New quasi-exactly solvable Hamiltonians in two dimensions," Commun. Math. Phys. **159**, 503 (1994).
- <sup>11</sup>A. Gonzalez-Lopez, N. Kamran, and P. J. Olver, "Real Lie algebras of differential operators and quasi-exactly solvable potentials," Philos. Trans. R. Soc. London, Ser. A **354**, 1165 (1996).
- <sup>12</sup>I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Oxford University Press, Oxford, 1995).
- <sup>13</sup>I. G. Macdonald, "Commuting differential operators and zonal spherical functions," in *Springer Lecture Notes in Mathematics*, 1987, Vol. 1271, pp. 189.
- <sup>14</sup>G. J. Heckman and E. M. Opdam, "Root systems and hypergeometric functions I & II," *Compositio Mathematica* **64**, 329 (1987).
- <sup>15</sup>L. Brink, A. Turbiner, and N. Wyllard, "Hidden algebras of the (super) Calogero and Sutherland models," J. Math. Phys. **39**, 1285 (1998).
- <sup>16</sup>J. Humphreys, *Introduction to Lie Algebras and Representation theory* (Springer-Verlag, Berlin, 1972).
- <sup>17</sup>N. Bourbaki, *Groupes et Algèbres de Lie* (Hermann, Paris, 1968), Chaps. 4–9.
- <sup>18</sup>W. Fulton and J. Harris, *Representation Theory* (Springer-Verlag, Berlin, 1996).