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ODD-CYCLE-FREE FACET COMPLEXES AND THE KÖNIG PROPERTY

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ABSTRACT. Facet complexes and simplicial cycles were introduced to help study the interplay between graph theoretical and algebraic properties of hypergraphs. We use the definition of a simplicial cycle to define an odd-cycle-free facet complex (hypergraph). These are facet complexes that do not contain any cycles of odd length. We show that, besides one class of such facet complexes, all of them satisfy the König property. This new family of complexes includes the family of balanced hypergraphs, which are known to satisfy the König property. These odd-cycle-free facet complexes are, however, not necessarily Mengerian.

1. Introduction. Simplicial trees were introduced by the second author in [**7**] in order to generalize algebraic structures based on graph trees. More specifically, the facet ideal of a simplicial tree, which is the ideal generated by the products of the vertices of each facet of the complex in the polynomial ring whose variables are the vertices of the complex, is a normal ideal ([**7**]), is always sequentially Cohen-Macaulay ([**8**]) and one can determine exactly when the quotient of this ideal is Cohen-Macaulay based on the combinatorial structure of the tree ([**9**]). These algebraic results that generalize those associated to simple graphs, and are intimately tied to the combinatorics of the simplicial complex, have suggested that this is a promising definition of a tree in higher dimension. This fact was most recently confirmed when the authors, while searching for an efficient algorithm to determine when a given complex is a tree, produced a precise combinatorial description for a simplicial cycle that has striking resemblance to that of a graph cycle ([**4**]). The main idea here is that a complex (or a simple hypergraph) is a tree if and only if it does not contain any "holes," or any cones over

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holes. Our definition of a simplicial cycle as a "hole" is more restrictive than the classic definition of a cycle for hypergraphs due to Berge [**1, 2**]. In a way, simplicial cycles are "minimal" hypergraph cycles, in the sense that once a facet is removed, what remains is not a cycle anymore, and does not contain one.

Once the concept of a "minimal" cycle is in place, a natural question that arises is whether the length of such a cycle bears any meaning in terms of properties of the complex? In graph theory bipartite graphs are characterized as those that do not contain any odd cycles. One of their strongest features is that they satisfy the König property; namely, the minimum number of vertices required so that each edge contains at least one of the vertices is the same as the maximum number is of pairwise disjoint edges. Our purpose in this paper is to investigate whether simplicial complexes (or hypergraphs) not containing odd simplicial cycles, which we call *odd-cycle-free* complexes, also satisfy this property. The main result of the paper is the following:

Theorem. *If Δ is a facet complex that is odd-cycle-free and $L(\Delta)$ does not contain the complement of a 7-cycle as an induced subgraph, then every subset of Δ satisfies the König property.*

The proof uses tools from hypergraph theory, as well as Berge's recently proved Strong Perfect Graph Conjecture ([**5, 6**]).

A more general notion of a cycle already exists in hypergraph theory ([**1, 2**]); we call these *hyper-cycles* (Definition 5.1) to avoid confusion. It is known that hypergraphs that do not contain odd hyper-cycles are *balanced*–meaning that every one of their odd hyper-cycles has an edge containing three vertices of the hyper-cycle–and hence they satisfy the König property. The class of odd-cycle-free complexes which we study in this paper includes the class of simple hypergraphs that do not contain odd hyper-cycles, and hence our results generalize those already known for hypergraphs. We discuss these inclusions in Section 5.

Simis, Vasconcelos and Villarreal showed in [**14**] that facet ideals of bipartite graphs are normally torsion free, and hence normal. Recently Gitler, Reyes and Villarreal [**10**] have shown that facet ideals of Mengerian complexes (Definition 5.7) are normally torsion free (see also [**12**]). This includes the class of simple hypergraphs that do not contain odd hyper-cycles, and more generally, balanced hypergraphs. We point out in Section 5 that facet ideals of odd-cycle-free complexes are not necessarily normally torsion-free, although they could still be normal ideals.

While this paper refers to simplicial or facet complexes most of the time for the statements, it is important to know that these structures are identical to simple hypergraphs. The original work on higher-dimensional trees and cycles was done in the context of commutative algebra, where a rich tradition of studying ideals associated to simplicial complexes was already in place. This paper, on the other hand, uses many results from hypergraph theory. For this reason, and for the sake of consistency, in the introductory parts of the paper, we give a careful review of the structures we introduced.

2. Facet complexes, trees, and cycles. We define the basic notions related to facet complexes. More details and examples can be found in [**7, 9**].

Definition 2.1 (Simplicial complex, facet). A *simplicial complex* Δ over a finite set of vertices V is a collection of subsets of V, with the property that if $F \in \Delta$ then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ, and the maximal faces are called *facets* of Δ.

Since we are usually only interested in the facets, rather than all faces, of a simplicial complex, it will be convenient to work with the following definition:

Definition 2.2 (Facet complex). A *facet complex* over a finite set of vertices V is a set Δ of subsets of V, such that for all $F, G \in \Delta$, $F \subseteq G$ implies $F = G$. Each $F \in \Delta$ is called a *facet* of Δ.

Remark 2.3 (Equivalence of simplicial complexes and facet complexes). The set of facets of a simplicial complex forms a facet complex. Conversely, the set of subsets of the facets of a facet complex is a simplicial complex. This defines a one-to-one correspondence between simplicial complexes and facet complexes. In this paper, we will work primarily with facet complexes.

We now generalize some notions from graph theory to facet complexes. Note that a graph can be regarded as a special kind of facet complex, namely one in which each facet has cardinality 2.

Definition 2.4 (Path, connected facet complex). Let Δ be a facet complex. A sequence of facets $F_1, \dots, F_n$ is called a *path* if for all $i = 1, \dots, n-1$, $F_i \cap F_{i+1} \neq \varnothing$. We say that two facets F and G are *connected* in Δ if there exists a path $F_1, \dots, F_n$ with $F_1 = F$ and $F_n = G$. Finally, we say that Δ is *connected* if every pair of facets is connected.

In order to define a tree, we borrow the concept of *leaf* from graph theory, with a small change.

Definition 2.5 (Leaf, joint). Let F be a facet of a facet complex Δ. Then F is called a *leaf* of Δ if either F is the only facet of Δ, or else there exists some $G \in \Delta \setminus \{F\}$ such that for all $H \in \Delta \setminus \{F\}$, we have $H \cap F \subseteq G$. The facet G above is called a *joint* of the leaf F if $F \cap G \neq \varnothing$.

It follows immediately from the definition that every leaf F contains at least one *free vertex*, i.e., a vertex that belongs to no other facet.

Definition 2.6 (Forest, tree). A facet complex Δ is a *forest* if every nonempty subset of Δ has a leaf. A connected forest is called a *tree* (or sometimes a *simplicial tree* to distinguish it from a tree in the graph-theoretic sense).

It is clear that any facet complex of cardinality one or two is a forest. When Δ is a graph, the notion of a simplicial tree coincides with that of a graph-theoretic tree.

Definition 2.7 (Minimal vertex cover, vertex covering number). Let Δ be a facet complex with vertex set V and facets $F_1, \dots, F_q$. A *vertex cover* for Δ is a subset A of V, with the property that for every facet F_i there is a vertex $v \in A$ such that $v \in F_i$. A *minimal vertex cover* of Δ is a subset A of V such that A is a vertex cover, and no proper subset of A is a vertex cover for Δ. The smallest cardinality of a vertex cover of Δ is called the *vertex covering number* of Δ and is denoted by $\alpha(\Delta)$.

Definition 2.8 (Independent set, independence number). Let Δ be a facet complex. A set $\{F_1, \dots, F_u\}$ of facets of Δ is called an *independent set* if $F_i \cap F_j = \varnothing$ whenever $i \neq j$. The maximum possible cardinality of an independent set of facets in Δ, denoted by $\beta(\Delta)$, is called the *independence number* of Δ. An independent set of facets which is not a proper subset of any other independent set is called a *maximal independent set* of facets.

Of particular interest to us in this paper is the König property.

Definition 2.9 (König property). A facet complex Δ satisfies the *König property* if $\alpha(\Delta) = \beta(\Delta)$.

2.1. Cycles. In this subsection, we define a simplicial cycle as a minimal facet complex without leaf. This in turn characterizes a tree as a connected cycle-free facet complex. The main point is that higher-dimensional cycles, like graph cycles, possess a particularly simple structure: each cycle is either equivalent to a "circle" of facets with disjoint intersections, or to a cone over such a circle.

Definition 2.10 (Cycle). A nonempty facet complex Δ is called a *cycle* (or a *simplicial cycle*) if Δ has no leaf but every nonempty proper subset of Δ has a leaf.

Equivalently, Δ is a cycle if Δ is not a forest, but every proper subset of Δ is a forest. If Δ is a graph, Definition 2.10 coincides with the graph-theoretic definition of a cycle. The next remark is an immediate consequence of the definitions of cycle and forest.

Remark 2.11 (A forest is a cycle-free facet complex). A facet complex is a forest if and only if it does not contain a cycle.

We now provide a complete characterization of the structure of cycles as described in [**4**].

Definition 2.12 (Strong neighbor). Let Δ be a facet complex and $F, G \in \Delta$. We say that F and G are *strong neighbors*, written $F \sim_\Delta G$, if $F \neq G$ and for all $H \in \Delta$, $F \cap G \subseteq H$ implies $H = F$ or $H = G$.

The relation $\sim_\Delta$ is symmetric, i.e., $F \sim_\Delta G$ if and only if $G \sim_\Delta F$. Note that if Δ has more than two facets, then $F \sim_\Delta G$ implies that $F \cap G \neq \varnothing$.

A cycle can be described as a sequence of strong neighbors.

Theorem 2.13 (Structure of a cycle ([**4**])). *Let Δ be a facet complex. Then Δ is a cycle if and only if the facets of Δ can be written as a sequence of strong neighbors $F_1 \sim_\Delta F_2 \sim_\Delta \cdots \sim_\Delta F_n \sim_\Delta F_1$ such that $n \geq 3$, and for all i, j*

$$F_i \cap F_j = \bigcap_{k=1}^{n} F_k \quad \text{if } j \neq i-1, i, i+1 \pmod{n}.$$

The implication of Theorem 2.13 is that a simplicial cycle has a very intuitive structure: it is either a sequence of facets joined together to form a circle (or a *hole*) in such a way that all intersections are pairwise disjoint (this is the case where the intersection of all the facets is the empty set), or it is a *cone* over such a structure, which is the case where the intersection of all the facets is nonempty.

The following example demonstrates the impact of the second condition of being a cycle in Theorem 2.13.

Example 2.14. The facet complex Δ has no leaves but is not a cycle, as its proper subset Δ' (which is indeed a cycle) has no leaves. However, we have $F_1 \sim_\Delta F_2 \sim_\Delta G \sim_\Delta F_3 \sim_\Delta F_4 \sim_\Delta F_1$, and these are the only pairings of strong neighbors in Δ.

A property of cycles that we shall use often in this paper is the following.

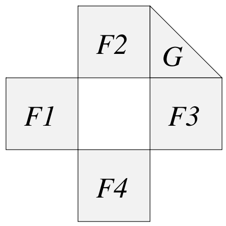

FIGURE 1a. Δ.

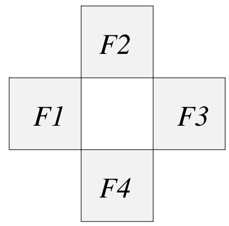

FIGURE 1b. Δ'.

Lemma 2.15. *Let F_1, F_2, F_3 be facets of a facet complex Δ, such that $F_i \cap F_j \neq \varnothing$ for $i, j \in \{1, 2, 3\}$, and $F_1 \cap F_2 \cap F_3 = \varnothing$. Then $\Gamma = \{F_1, F_2, F_3\}$ is a cycle.*

Proof. Since Γ has three facets, all its proper subsets are forests. So if Γ is not a cycle, then it must contain a leaf. Say F_1 is a leaf, and F_2 is its joint. So we have $\varnothing \neq F_1 \cap F_3 \subseteq F_2$, which implies that $F_1 \cap F_2 \cap F_3 \neq \varnothing$; a contradiction. $\square$

3. Facet complexes as simple hypergraphs.

3.1. Graph theory terminology. All graphs that are considered in this paper are simple graphs, meaning that they are undirected graphs containing no loops or multiple edges.

Definition 3.1 (Induced subgraph). Let G be a graph with vertex set V. A subgraph H of G with vertex set $W \subseteq V$ is called an *induced subgraph* of G if, for each $x, y \in W$, x and y are connected by an edge in H if and only if they are connected by an edge in G.

Definition 3.2 (Clique of a graph). A clique of a graph G is a complete subgraph of G; in other words a subgraph of G whose every two vertices are connected by an edge.

Definition 3.3 (Chromatic number). The *chromatic number* of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices (vertices that belong to the same edge) share the same color.

Definition 3.4 (Complement of a graph). The complement of a graph G, denoted by $\overline{G}$, is a graph over the same vertex set as G whose edges connect non-adjacent vertices of G.

Definition 3.5 (Perfect graph). A graph G is *perfect* if for every induced subgraph G' of G, the chromatic number of G' is equal to the size of the largest clique of G'.

We call G a *minimal imperfect* graph if it is not perfect but all proper induced subgraphs of G are perfect. There is a characterization of minimal imperfect graphs that was conjectured by Berge and known for a long time as the "Strong Perfect Graph Conjecture" and was proved recently by Chudnovsky, Robertson, Seymour and Thomas [**5**]; see also [**6**].

Theorem 3.6 (Strong Perfect Graph Theorem ([**5**])). *The only minimal imperfect graphs are odd cycles of length* ≥ 5 *and their complements.*

3.2. Hypergraphs. Hypergraphs are the higher-dimensional counterparts of graphs.

Definition 3.7 (Hypergraph, simple hypergraph ([**1**])). Let $V = \{x_1, \dots, x_n\}$ be a finite set. A *hypergraph* on V is a family $\mathcal{H} = (F_1, \dots, F_m)$ of subsets of V such that

1. $F_i \neq \varnothing$ for $i = 1, \dots, m$;
2. $V = \cup_{i=1}^{m} F_i$.

Each F_i is called an *edge* of $\mathcal{H}'$. If, additionally, we have the condition: $F_i \subseteq F_j \Rightarrow i = j$, then $\mathcal{H}$ is called a *simple hypergraph.*

A graph is a hypergraph in which an edge consists of exactly two vertices.

Definition 3.8 (Partial hypergraph). A *partial hypergraph* of a hypergraph $\mathcal{H} = \{F_1, \ldots, F_m\}$ is a subset $\mathcal{H}' = \{F_j \mid j \in J\}$, where $J \subseteq \{1, \ldots, m\}$.

It is clear that a facet complex Δ is a simple hypergraph on its set of vertices, and a partial hypergraph is just a subset of Δ. For this reason, we are able to borrow the following definitions from hypergraph theory. The main source for these concepts is Berge's book [**1**].

Definition 3.9 (Line graph of a hypergraph), Given a hypergraph $\mathcal{H} = \{F_1, \ldots, F_m\}$ on vertex set V, its *line graph* $L(\mathcal{H})$ is a graph whose vertices $e_1, \ldots, e_m$ represent the edges of $\mathcal{H}$, and two vertices e_i and e_j are connected by an edge if and only if $F_i \cap F_j \neq \varnothing$.

Definition 3.10 (Normal hypergraph ([**13**])). A hypergraph $\mathcal{H}$ with vertex set V is *normal* if every partial hypergraph $\mathcal{H}'$ satisfies the *colored edge property*, i.e., $q(\mathcal{H}') = \delta(\mathcal{H}')$, where

- $q(\mathcal{H}')$ = chromatic index of $\mathcal{H}'$, which is the minimum number of colors required to color the edges of $\mathcal{H}'$ in such a way that two intersecting edges have different colors; and
- $\delta(\mathcal{H}') = \max_{x \in V}\{$number of edges of $\mathcal{H}'$ that contain $x\}$.

Clearly, we always have $q(\mathcal{H}') \geq \delta(\mathcal{H}')$.

Definition 3.11 (Helly property). Let $\mathcal{H} = \{F_1, \ldots, F_q\}$ be a simple hypergraph, or equivalently, a facet complex. Then $\mathcal{H}$ is said to satisfy the *Helly property* if every intersecting family of $\mathcal{H}$ is a star; i.e., for every $J \subseteq \{1, \ldots, q\}$,

$$F_i \cap F_j \neq \varnothing \text{ for all } i, j \in J \Longrightarrow \bigcap_{j \in J} F_j \neq \varnothing.$$

From the above definitions, the following statement (originally due to Lovász [**13**]), which we shall rely on for the rest of this paper, makes sense.

Theorem 3.12 ([**1**, page 197]). *A simple hypergraph (or facet complex) $\mathcal{H}$ is normal if and only if $\mathcal{H}$ satisfies the Helly property and $L(\mathcal{H})$ is a perfect graph.*

4. Odd-cycle-free complexes.

As we discussed in the previous section, a facet complex is a simple hypergraph.

Definition 4.1 (Odd-cycle-free complex). We call a facet complex *odd-cycle-free* if it contains no cycles of odd length.

It is well known that odd-cycle-free graphs, which are known to be equivalent to bipartite graphs, satisfy the König property (Definition 2.9). In higher dimensions, the König property is enjoyed by simplicial trees [**9**], and complexes that do not contain odd special cycles, which are also known as *balanced hypergraphs* [**1, 2**]. For the definition of a special cycle, see Section 5. The class of odd-cycle-free complexes includes balanced hypergraphs (see Section 5).

It is therefore natural to ask if odd-cycle-free complexes satisfy the König property. The answer to this question is mostly positive: besides one specific class of odd-cycle-free complexes, all of them do satisfy the König property.

Theorem 4.2 (Odd-cycle-free complexes that satisfy König). *If Δ is a facet complex that is odd-cycle-free and $L(\Delta)$ does not contain the complement of a 7-cycle as an induced subgraph, then every subset of Δ satisfies the König property.*

A theorem of Lovász [**13**] (see also [**1**, page 195]) states that a hypergraph $\mathcal{H}$ is normal if and only if every partial hypergraph of $\mathcal{H}$ satisfies the König property. It is therefore enough to show that a facet complex Δ (and its subsets) satisfy the König property by showing that Δ is normal. By Theorem 3.12, it suffices to show that Δ satisfies the Helly property and $L(\Delta)$ is perfect. We show these two properties separately.

Proposition 4.3 (3-cycle-free complexes satisfy Helly property). *If the facet complex Δ does not contain a cycle of length 3, then it satisfies the Helly property. In particular, odd-cycle-free complexes satisfy the Helly property.*

Proof. Suppose Δ does not satisfy the Helly property, so it contains an intersecting family that is not a star. In other words, there exists a $\Gamma = \{F_1, \dots, F_m\} \subseteq \Delta$ such that

$$F_i \cap F_j \neq \varnothing \quad \text{for } i, j \in \{1, \dots, m\}, \text{ but } \bigcap_{j=1}^{m} F_j = \varnothing.$$

We use induction on m. If $m = 3$, from Lemma 2.15 it follows that Γ is a 3-cycle.

Suppose now that $m > 3$ and we know that every intersecting family of less than m facets that is not a star contains a 3-cycle. Let Γ be an intersecting family of m facets $F_1, \dots, F_m$, such that every $m-1$ facets of Γ intersect (otherwise by the induction hypothesis Γ contains a 3-cycle and we are done), but $\bigcap_{i=1}^{m} F_i = \varnothing$.

So for each $j \in \{1, \dots, m\}$, we can find a vertex x_j such that $x_j \in F_i$ if and only if $j \neq i$. Therefore we have a sequence of vertices $x_1, \dots, x_m$ such that for each i:

$$\{x_1, \dots, \widehat{x_i}, \dots, x_m\} \subseteq F_i \text{ and } x_i \notin F_i.$$

Now consider three facets F_1, F_2, F_3 of Γ. Since $\{F_1, F_2, F_3\}$ is not a cycle, it must be a tree; therefore, it has a leaf, say F_1, and a joint, say F_2. It follows that $F_1 \cap F_3 \subseteq F_2$. But then it follows that $x_2 \in F_1 \cap F_3 \subseteq F_2$, which is a contradiction. □

We now concentrate on $L(\Delta)$ and its relation to Δ.

Lemma 4.4. *If Δ is a facet complex, then for every induced subgraph G of $L(\Delta)$ there is a subset $\Gamma \subseteq \Delta$ such that $G = L(\Gamma)$.*

Proof. Let G be an induced subgraph of $L(\Delta)$. Then two vertices x and y of G are connected by an edge in G if and only if they are connected by an edge in $L(\Delta)$. This means that, if $F_1, \dots, F_m$ are the facets of Δ corresponding to the vertices of G, and $\Gamma = \{F_1, \dots, F_m\}$, then G is precisely $L(\Gamma)$. □

Lemma 4.5. *If Δ is a facet complex and $L(\Delta)$ is a cycle of length $\ell > 3$, then Δ is a cycle of length ℓ.*

Proof. Suppose $L(\Delta)$ is the cycle

$$\{w_1, w_2\} \sim_{L(\Delta)} \{w_2, w_3\} \sim_{L(\Delta)} \cdots \sim_{L(\Delta)} \{w_{\ell-1}, w_\ell\} \sim_{L(\Delta)} \{w_\ell, w_1\},$$

where each vertex w_i of $L(\Delta)$ corresponds to a facet F_i of Δ. Since w_i is only adjacent to w_{i-1} and $w_{i+1} \pmod{\ell}$, it follows that

$$F_i \cap F_j \neq \varnothing \iff j = i-1, i, i+1 \pmod{\ell},$$

which implies that $\Delta = \{F_1, \ldots, F_\ell\}$ where

$$F_1 \sim_\Delta F_2 \sim_\Delta \cdots \sim_\Delta F_\ell \sim_\Delta F_1.$$

Moreover, since $\ell > 3$, we have $\bigcap_{i=1}^{\ell} F_i = \varnothing$. Theorem 2.13 now implies that Δ is a cycle of length ℓ. □

Proposition 4.6 (The line graph of an odd-cycle-free complex). *If Δ is an odd-cycle-free facet complex, then $L(\Delta)$ is either perfect, or contains the complement of a 7-cycle as an induced subgraph.*

Proof. Suppose $L(\Delta)$ is not perfect, and let G be a minimal imperfect induced subgraph of $L(\Delta)$. By Lemma 4.4, for some subset Γ of Δ, $G = L(\Gamma)$. Any induced subgraph of G is, by Lemma 4.4, the line graph of some $\Delta' \subseteq \Gamma \subseteq \Delta$, and is hence perfect. So G is a minimal imperfect graph, and by the Strong Perfect Graph Theorem (Theorem 3.6), G is either an odd cycle of length ≥ 5, or the complement of one. If G is an odd cycle, then so is Γ by Lemma 4.5, and therefore Δ is not odd-cycle-free and we are done.

So assume that G is the complement of an odd cycle of length $\ell \geq 5$. We consider two cases.

1. $\ell = 5$. Since the complement of a 5-cycle is a 5-cycle, it immediately follows from the discussions above that Γ is a cycle of length 5, and hence Δ is not odd-cycle-free.

2. $\ell \geq 9$. We show that Γ contains a cycle of length 3.

Let $G = \overline{C_\ell}$, where C_ℓ is the ℓ-cycle

$$\{w_1, w_2\} \sim_{C_\ell} \{w_2, w_3\} \sim_{C_\ell} \cdots \sim_{C_\ell} \{w_{\ell-1}, w_\ell\} \sim_{C_\ell} \{w_\ell, w_1\},$$

and a vertex w_i of G corresponds to a facet F_i of Γ. This means that $F_i \cap F_j \neq \varnothing$ unless $j = i-1, i+1 \pmod{\ell}$. With this indexing, consider the subset $\Gamma' = \{F_1, F_4, F_7\}$ of Γ. Clearly all three facets of Γ' have nonempty pairwise intersections:

$$F_1 \cap F_4 \neq \varnothing,\ F_1 \cap F_7 \neq \varnothing,\ F_4 \cap F_7 \neq \varnothing.$$

Suppose Γ' is not a cycle. Since Γ' has only three facets it must be a tree and must therefore have a leaf, say F_1, and a joint, say F_4 (and the other cases will be similar as explained below). So

$$\varnothing \neq F_1 \cap F_7 \subseteq F_4. \tag{1}$$

Now consider the subset $\Gamma'' = \{F_1, F_3, F_7\}$ of Δ. We know that

$$F_1 \cap F_3 \neq \varnothing,\ F_1 \cap F_7 \neq \varnothing,\ F_3 \cap F_7 \neq \varnothing. \tag{2}$$

If $F_1 \cap F_3 \cap F_7 \neq \varnothing$, then from (1) we see that $F_3 \cap F_4 \neq \varnothing$, which is a contradiction. Therefore $F_1 \cap F_3 \cap F_7 = \varnothing$, which along with the properties in (2) and Lemma 2.15 implies that Γ'' is not a tree, so it must be a cycle.

We can make similar arguments if F_1 or F_7 are joints of Γ': if F_1 is a joint, then we can show that $\Gamma'' = \{F_2, F_4, F_7\}$ is a cycle, and if F_7 is a joint, then $\Gamma'' = \{F_1, F_4, F_6\}$ is a cycle. So we have shown that either Γ' is a 3-cycle, or one can form another 3-cycle Γ'' in Δ. Either way, Δ contains an odd cycle, and is therefore not odd-cycle-free. □

Proof of Theorem 4.2. Propositions 4.3 and 4.6, along with Theorem 3.12 immediately imply the statement of Theorem 4.2. □

4.1. Are these conditions necessary for satisfying the König property? A natural question is whether the conditions in Theorem 4.2 are necessary for a facet complex whose every subset satisfies the König property. The answer in general is negative. In this section, we explore various properties and examples related to this issue.

The first observation is that not all odd cycles lack the König property. Indeed, if the cycle Δ (or in fact any complex) is a cone, in the sense that all facets share a vertex, then it always satisfies the König property with $\alpha(\Delta) = \beta(\Delta) = 1$.

But if we eliminate the case of cones, all remaining odd cycles lack the König property.

Lemma 4.7 (Odd cycles that lack König). *Suppose the facet complex $\Delta = \{F_1, \dots , F_{2k+1}\}$ is a cycle of odd length such that $\bigcap_{i=1}^{2k+1} F_i = \varnothing$. Then Δ lacks the König property.*

Proof. Suppose without loss of generality that Δ can be written as

$$F_1 \sim_\Delta F_2 \sim_\Delta \cdots \sim_\Delta F_{2k+1} \sim_\Delta F_1.$$

Then a maximal independent set of facets of Δ can have at most k facets; say $B = \{F_1, F_3, \dots , F_{2k-1}\}$ is such a set, and by symmetry, all maximal independent sets will consist of alternating facets, and will have cardinality k. Hence $\beta(\Delta) = k$.

But we need at least $k + 1$ vertices to cover Δ. To see this, let us suppose that Δ has a vertex cover $A = \{x_1, \dots , x_k\}$. Since B is an independent set, we can without loss of generality assume that

$$x_1 \in F_1, x_2 \in F_3, \dots , x_i \in F_{2i-1}, \dots , x_k \in F_{2k-1}.$$

The other facets $F_2, F_4, \dots , F_{2k}, F_{2k+1}$ have to also be covered by the vertices in A. Since $F_{2k+1} \cap G = \varnothing$ for all $G \in B$ except for $G = F_1$, we must have $x_1 \in F_{2k+1}$. Working our way forward in the cycle, and using the same argument, we get

$$x_2 \in F_2, x_3 \in F_4, \dots , x_i \in F_{2i-2}, \dots , x_k \in F_{2k-2}.$$

But we have still not covered the facet F_{2k}, who is forced to share a vertex of A from one of its two neighbors: either $x_1 \in F_{2k}$ or $x_k \in F_{2k}$. Neither is possible as $F_{2k} \cap F_1 = F_{2k} \cap F_{2k-2} = \varnothing$, and so A cannot be a vertex cover.

Adding a vertex of F_{2k} solves this problem though, so $\alpha(\Delta) = k + 1$ and $\beta(\Delta) = k$, and hence Δ lacks the König property. □

The previous lemma then brings us to the question: can we replace the condition "odd-cycle-free" with "odd-hole-free" (where an *odd hole* is referring to an odd cycle that is not a cone) in the statement of Theorem 4.2? The answer is again negative, as clarified by the example below.

Example 4.8. The hollow tetrahedron $\Delta = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_4\}, \{x_3, x_4\}, \{x_2, x_4\}\}$ is odd-hole-free (but it does contain four 3-cycles). However it lacks the König property, since $\alpha(\Delta) = 2$, but $\beta(\Delta) = 1$. Similar examples in higher dimensions can be constructed, e.g., if $d \geq 3$ is odd then the facet complex $\Delta = \{F \subset \{x_1, \dots, x_{d+1}\} : |F| = d\}$ is odd-hole-free and lacks König, since we have $\alpha(\Delta) = 2$ but $\beta(\Delta) = 1$.

We next focus on the second condition in the statement of Theorem 4.2, which turns out to be inductively necessary for satisfying the König property.

Lemma 4.9. *Let Δ be a facet complex such that $L(\Delta)$ is the complement of an odd cycle of length $k > 3$. Then Δ lacks the König property.*

Proof. Suppose $L(\Delta) = \overline{C_k}$ where C_k is a k-cycle and k is an odd number. Let $\Delta = \{F_1, \dots, F_k\}$ be such that the vertices of C_k correspond to the facets $F_1, \dots, F_k$ in that order; in other words, F_1 intersects all other facets but F_2 and F_k, and so on (the case $k = 7$ is illustrated in Figure 2).

Let B be a maximal independent set of facets, and assume $F_1 \in B$. Then, since F_1 intersects all facets but F_2 and F_k, B can contain one of F_2 and F_k (but not both, since they intersect). So $|B| = 2$. The same argument holds if B contains any other facet than F_1, so we conclude that $\beta(\Delta) = 2$.

Now suppose Δ has a vertex cover of cardinality 2, say $A = \{x, y\}$. Then each facet of Δ must contain one of x and y. Without loss of generality, suppose $x \in F_1$. Since each facet does not intersect the next one in the sequence $F_1, F_2, \dots, F_k, F_1$, we have

$$x \in F_1 \Longrightarrow y \in F_2 \Longrightarrow x \in F_3 \Longrightarrow y \in F_4 \Longrightarrow \cdots \Longrightarrow y \in F_{k-1} \Longrightarrow x \in F_k.$$

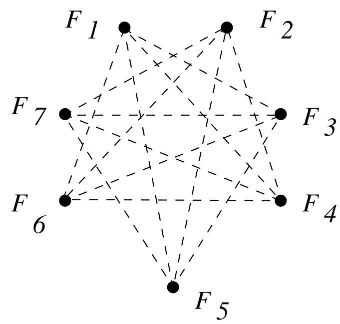

FIGURE 2. Complement of a 7-cycle.

But now $x \in F_1 \cap F_k = \varnothing$, which is a contradiction. So $\alpha(\Delta) \geq 3$, and hence Δ does not satisfy the König property. □

Corollary 4.10. *If every subset of a facet complex Δ satisfies the König property, then $L(\Delta)$ cannot contain the complement of an odd cycle of length > 3 as an induced subgraph.*

Remark 4.11 (The case of the complement of a 7-cycle). As suggested above, if $L(\Delta)$ contains the complement of a 7-cycle as an induced subgraph, Δ may lack the König property, even though it may be odd-cycle-free. For example, consider the complex Δ on seven vertices $x_1, \ldots, x_7$: $\Delta = \{F_1, \ldots, F_7\}$ where $F_1 = \{x_1, x_2, x_3\}$, $F_6 = \{x_2, x_3, x_4\}$, $F_4 = \{x_3, x_4, x_5\}$, $F_2 = \{x_4, x_5, x_6\}$, $F_7 = \{x_5, x_6, x_7\}$, $F_5 = \{x_6, x_7, x_1\}$, $F_3 = \{x_7, x_1, x_2\}$.

The graph $L(\Delta)$ is the complement of a 7-cycle (the labels of the facets correspond to those in Figure 2). One can verify that Δ contains no 3, 5, or 7-cycles, so it is odd-cycle-free. However by Lemma 4.9, the facet complex Δ lacks the König property; indeed $\alpha(\Delta) = 3$ but $\beta(\Delta) = 2$.

On the other hand, it is easy to expand Δ to get another complex Γ, such that $L(\Gamma)$ does contain the complement of a 7-cycle as an induced subgraph, and Γ satisfies the König property. For example, consider $\Gamma = \{G, F_1', F_2, \ldots, F_7\}$, where $F_2, \ldots, F_7$ are the same facets as above, and we introduce two new vertices u, v to build the new facets

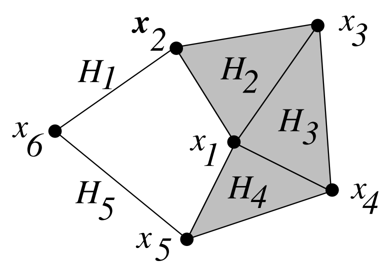

FIGURE 3. Not balanced odd-cycle-free complex.

$F_1' = \{u, x_1, x_2, x_3\}$, and $G = \{u, v\}$.

The set $B = \{G, F_2, F_3\}$ is a maximal independent set of facets, so $\beta(\Gamma) = 3$. Also, we can find a vertex covering $A = \{u, x_4, x_7\}$, which implies that $\alpha(\Delta) = 3$. Note, however, that Γ does not satisfy the König property "inductively": it contains a subset $\{F_1', F_2, \dots, F_7\}$ that lacks the König property by Lemma 4.9.

5. Balanced complexes are odd-cycle-free.

The notion of a cycle has already been defined in hypergraph theory, and is much more general than our definition of a cycle (see [**2**], or [**1**, Chapter 5]). To keep the terminologies separate, in this paper we refer to the traditional hypergraph cycles as *hyper-cycles*. In particular, hypergraphs that do not contain hyper-cycles of odd length are known to satisfy the König property. In this section, we introduce this class of hypergraphs and show that hypergraphs not containing odd hyper-cycles are odd-cycle-free, and their line graphs cannot contain the complement of a 7-cycle as an induced subgraph.

Definition 5.1 (Hyper-cycle [**1, 2**]). Let $\mathcal{H}$ be a hypergraph on vertex set V. A *hyper-cycle* of length ℓ ($\ell \geq 2$), is a sequence $(x_1, F_1, x_2, F_2, \dots, x_\ell, F_\ell, x_1)$ where the x_i are distinct vertices and the F_i are distinct edges of $\mathcal{H}$, and moreover $x_i, x_{i+1} \in F_i \pmod{\ell}$ for all i.

Definition 5.2 (Balanced hypergraph ([**1, 2**])). A hypergraph is said to be *balanced* if every odd hyper-cycle has an edge containing three vertices of the hyper-cycle.

Note that a "balanced hypergraph" is different than a "balanced simplicial complex" defined by Stanley [**15**].

A hyper-cycle has been called by hypergraph theorists a *special cycle* or a *strong cycle* if, with notation as in Definition 5.1, for all i we have $x_i \in F_j$ if and only if $j = i-1, i \pmod{\ell}$. In other words, if each vertex x_i of the hyper-cycle appears in exactly two facets, the hyper-cycle is a special cycle. So a balanced hypergraph is one that does not contain any special cycle of odd length.

In general, the following fact holds:

Lemma 5.3. *A facet complex contains a cycle if and only if it contains a hyper-cycle.*

Moreover, it is easy to see that a cycle Δ defined as

$$F_1 \sim_\Delta F_2 \sim_\Delta \cdots \sim_\Delta F_\ell \sim_\Delta F_1$$

produces a hyper-cycle; just pick any vertex $x_i \in F_i \cap F_{i+1} \pmod{\ell}$, Δ produces a hyper-cycle, or in fact a special cycle, of the same length ℓ

$$(x_1, F_1, x_2, F_2, \dots, x_\ell, F_\ell, x_1).$$

It follows that a balanced simple hypergraph is odd-cycle-free. The converse, however, is not true.

Example 5.4 (Not all odd-cycle-free complexes are balanced). Consider the complex Δ in Figure 3, which is odd-cycle-free, as the only cycle is the 4-cycle $\{H_1, H_2, H_4, H_5\}$. But Δ is not balanced, as all of Δ forms the special 5-cycle

$$(x_6, H_1, x_2, H_2, x_3, H_3, x_4, H_4, x_5, H_5, x_6).$$

The complex in Example 5.4 is an example of how our main result (Theorem 4.2) generalizes the fact that balanced complexes satisfy the König property. In fact, we can show that all balanced complexes satisfy the hypotheses of Theorem 4.2.

Proposition 5.5. *Let Δ be a balanced complex. Then Δ is odd-cycle-free and $L(\Delta)$ does not contain the complement of a 7-cycle as an induced subgraph.*

Proof. Let Δ be balanced. We have already shown that Δ is odd-cycle-free. Suppose that $L(\Delta)$ contains the complement of a 7-cycle as an induced subgraph.

By Lemmas 4.4 and 4.5, Δ contains a subset Γ whose line graph is the complement of a 7-cycle on the facets $F_1, \dots, F_7$ as in Figure 2.

We claim that we can find vertices $x_{15} \in F_1 \cap F_5$, $x_{57} \in F_5 \cap F_7$, $x_{72} \in F_2 \cap F_7$, $x_{26} \in F_2 \cap F_6$, and $x_{61} \in F_1 \cap F_6$, such that

$$(x_{61}, F_1, x_{15}, F_5, x_{57}, F_7, x_{72}, F_2, x_{26}, F_6, x_{61}) \tag{3}$$

is a special cycle of length 5, and if not, Δ contains a special cycle of length 3.

If the hyper-cycle in (3) is not a special 5-cycle, then at least one of the following statements hold:

1. $F_1 \cap F_5 \subseteq F_2 \cup F_6 \cup F_7$. This is not possible, since we know that $F_1 \cap F_2 = F_1 \cap F_7 = \varnothing$, and $F_5 \cap F_6 = \varnothing$. Since $F_1 \cap F_5 \neq \varnothing$, one can choose $x_{15} \in F_1 \cap F_5$ such that $x_{15} \notin F_2 \cup F_6 \cup F_7$.

2. $F_5 \cap F_7 \subseteq F_1 \cup F_2 \cup F_6 \Rightarrow F_5 \cap F_7 \subseteq F_2$ (since $F_7 \cap F_1 = F_7 \cap F_6 = \varnothing$). In this case, consider the facet complex $\{F_3, F_5, F_7\}$. Then, since $F_2 \cap F_3 = \varnothing$ and $F_5 \cap F_7 \subseteq F_2$, we have $F_3 \cap F_5 \cap F_7 = \varnothing$. Lemma 2.15 now implies that $\{F_3, F_5, F_7\}$ is a 3-cycle, and hence can be written as a special 3-cycle.

3. $F_2 \cap F_7 \subseteq F_1 \cup F_5 \cup F_6 \Rightarrow F_2 \cap F_7 \subseteq F_5$ (since $F_7 \cap F_1 = F_7 \cap F_6 = \varnothing$).

Similar to Case 2. it follows that $\{F_2, F_4, F_7\}$ is a (special) 3-cycle.

4. $F_2 \cap F_6 \subseteq F_1 \cup F_5 \cup F_7$. Fails with argument similar to Case 1. So one can choose $x_{26} \in F_2 \cap F_6$ such that $x_{26} \notin F_1 \cup F_5 \cup F_7$.

5. $F_1 \cap F_6 \subseteq F_2 \cup F_5 \cup F_7$. Fails with argument similar to Case 1. So one can choose $x_{61} \in F_1 \cap F_6$ such that $x_{61} \notin F_2 \cup F_5 \cup F_7$.

So we have shown that either there are vertices $x_{15}, \dots, x_{61}$ such that the sequence in (3) is a special 5-cycle, or otherwise, either cases 2 or 3 above would hold, in which case Δ would contain a (special) 3-cycle. Either way, Δ is not balanced. $\square$

As a result, we have another proof to the following known fact (see [**1, 2**]).

Corollary 5.6 (Balanced complexes satisfy König). *If Δ is a balanced facet complex, then all subsets of Δ satisfy the König property.*

In fact, a stronger version of the above statement was proved for balanced hypergraphs by Berge and Las Vergnas; see [**1**, page 178].

In closing, we would like to briefly discuss some algebraic properties of Mengerian facet complexes.

Definition 5.7. Let $\Delta = \{F_1, \dots, F_n\}$ be a facet complex and $M = (a_{ij})$ its vertex-edge incidence matrix, i.e., $a_{ij} = 0$ if $i \notin F_j$ and $a_{ij} = 1$ if $i \in F_j$. Also, for all vectors $\underline{v} \in \mathbf{N}^n$, we define $\nu(\underline{v}) = \max\{\underline{u} \cdot (1, \dots, 1) \mid \underline{u} \in \mathbf{R}^m_+, M \cdot \underline{u} \leq \underline{v}\}$ and $\tau(\underline{v}) = \min\{\underline{w} \cdot \underline{v} \mid \underline{w} \in \mathbf{N}^n,\ M^{tr} \cdot \underline{w} \geq 1\}$ where $M^{tr} = (a_{ji})$. A facet complex Δ is called *Mengerian* if $\nu(\underline{v}) = \tau(\underline{v})$ for all $\underline{v} \in \mathbf{N}^n$.

As mentioned earlier in the paper, the study of combinatorial properties of hypergraphs has been much motivated by algebraic structures; one example is the search for normal ideals. Normal ideals are ideals whose every power is integrally closed. Bipartite graphs and balanced complexes are known to have normally-torsion-free facet ideals, and therefore their facet ideals are normal (see [**10, 12, 14**]).

Odd-cycle-free facet complexes, on the other hand, are not necessarily Mengerian. Since Mengerian implies the König property, this follows from Remark 4.11. Therefore, the facet ideal of an odd-cycle-free complex may not be normally-torsion-free. However, this ideal could still be a normal ideal. We have run several examples using the computer algebra softwares *Normaliz* [**3**] and *Singular* [**11**] that confirm this statement. It would be of great interest to know whether odd-cycle-free complexes provide a new class of normal ideals; this would generalize results in [**7, 12, 14**].

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