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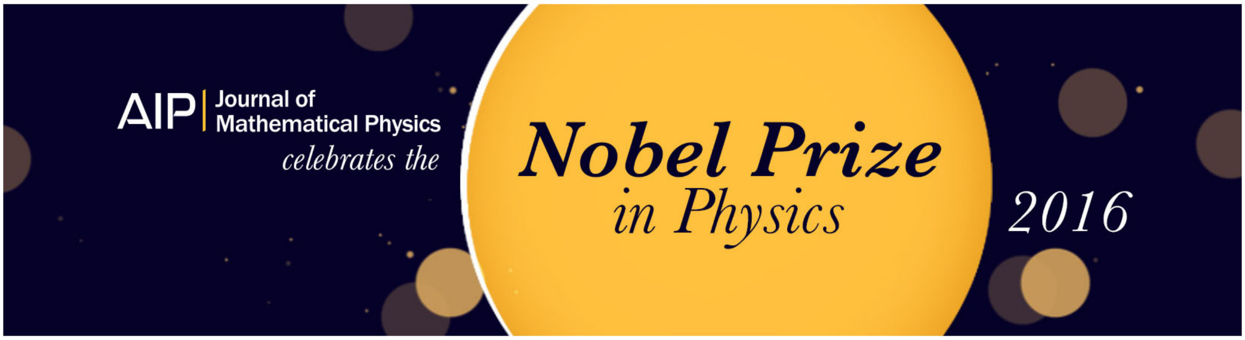

Equations of state and plane-autonomous systems in Bianchi V imperfect fluid cosmology

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A new general approach for investigating imperfect fluid cosmological models is introduced in which the equations of state are completely "dimensionless." Such equations of state are then utilized to reduce the Einstein field equations governing Bianchi V imperfect fluid cosmologies to a plane-autonomous system of equations, thus enabling the qualitative behavior of these cosmological models to be analyzed in a straightforward manner. The resulting plane-autonomous system is investigated. Finally, exact solutions of the Bianchi V imperfect fluid field equations in the case when the equations of state take on a particularly simple form are discussed.

I. INTRODUCTION

In a recent paper[1] (hereafter referred to as paper I) Bianchi V imperfect fluid cosmology was investigated. [For brevity, we will adopt the notation that an equation or reference in paper I will be referred to using a label I]. It is of interest to study cosmological models with a richer structure, both geometrically and physically, than the standard perfect fluid Friedmann–Robertson–Walker (FRW) models. Bianchi V models are of particular interest since they are sufficiently complex (e.g., the Einstein tensor has off-diagonal terms) while, at the same time, they are a simple generalization of the negative-curvature FRW models. Cosmological models that include viscosity have been investigated in an attempt to explain the currently observed highly isotropic matter distribution (I1–I3) and the high entropy per baryon in the present state of the Universe (I4, I5), and in order to further study the nature of the initial singularity (I6) and the formation of galaxies (I3). Models that include heat conduction have also been studied in spatially homogeneous cosmologies (in particular, see I7). The motivation and background for this research is discussed in more detail in Ref. I.

In MacCallum[2] a general class of Bianchi models were studied [all class A models, and the set of class B with $n_\alpha^\alpha = 0$ $(\alpha = 1,2,3)$]. In this class (that contains the Bianchi V models) the general exact (two-parameter) orthogonal perfect fluid solution is known up to quadratures[3]. Collins[4] has investigated a certain subclass of this class of models whose equations reduce to an autonomous system and are therefore susceptible to a qualitative analysis utilizing geometric techniques. More precisely, Collins studied a subclass of perfect fluid, nonrotating, spatially homogeneous cosmological models with equation of state $p = (\gamma - 1)\rho$ and zero cosmological constant. In particular, this subclass includes the (not necessarily LRS) Bianchi V models (see Fig. 3, in Ref. 4). Later, this subclass was extended to include perfect fluid LRS Bianchi models (again including type V models) with tilt.[5]

Here, we shall use the techniques and notation of Refs. 2–5 to reduce the differential equations governing the Bianchi V imperfect fluid cosmological models under consideration to a plane-autonomous system of equations.

More precisely, in this paper we will investigate a class of phenomenological equations of state (for the pressure and coefficients of bulk and shear viscosity) in imperfect fluid cosmological models. This general class of equations of state is characterized by the fact that completely dimensionless quantities are inter-related (i.e., the equations of state are "dimensionless"). It is noted that this class includes as special cases all the most commonly considered equations of state. This procedure amounts to introducing a new approach for dealing with equations of state in cosmology, an approach that is quite general, but for illustrative purposes we restrict our attention to Bianchi V cosmologies. The feature of this class of greatest interest here is that equations of state of this type are the most general under which the resulting Einstein field equations reduce to a plane-autonomous system.

The analysis will consequently enable us to write the Bianchi V imperfect fluid field equations as a plane-autonomous system. This in turn will enable us to analyze the qualitative behavior of these cosmological models in a straightforward manner. The plane-autonomous system is studied further in the case that the equations of state are of a special (power law) form; the resulting system in a particularly simple subcase is displayed in the final section for illustration.

In Sec. IV we shall look for exact solutions of the Bianchi V imperfect fluid field equations in the case when the equations of state take on the simple form $p = (\gamma - 1)\rho$, $\zeta = \zeta_0\theta$, and $\eta = \eta_0\theta$ [see Eqs. (4.1)]. Exact solutions will of course be very useful in concert with any qualitative analysis. A simple, *general* first integral of the field equations is found. Using this first integral it is then shown that the field equations reduce to a single, second-order, ordinary differential equation for a single variable. In the particular case of $\gamma = 2$ (stiff matter), a simple (albeit unphysical) solution is exhibited.

II. THE MODELS

We shall study LRS Bianchi type V spatially homogeneous cosmology, where the metric is given by

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)e^{2x}(dy^2 + dz^2), \qquad (2.1)$$

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in which the source of the gravitational field is a viscous fluid with heat conduction, so that the energy–momentum tensor is given by

$$T_{ab} = (\rho + \bar{p}) u_a u_b + \bar{p} g_{ab} - 2\eta\sigma_{ab} + q_a u_b + u_a q_b, \tag{2.2}$$

with

$$\bar{p} = p - \zeta\theta, \tag{2.3}$$

where p is the thermodynamic pressure and ζ and η are the coefficients of bulk and shear viscosity, respectively, thereby allowing dissipative processes to be included in the models.

The Einstein field equations for a comoving fluid then yield an equation defining the energy density (I.8a),

$$\rho = \frac{\dot{b}^2}{b^2} + 2\frac{\dot{a}\dot{b}}{ab} - \frac{3}{a^2}, \tag{2.4}$$

an equation that defines the only nonzero component of the heat conduction vector q_a (I.8b),

$$q_1 = 2[\dot{b}/b - \dot{a}/a], \tag{2.5}$$

and the remaining nontrivial equations (I.8c) and (I.8d),

$$\frac{1}{a^2} - \frac{\dot{b}^2}{b^2} - 2\frac{\ddot{b}}{b} = \bar{p} - \frac{4}{3}\eta\left[\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right], \tag{2.6}$$

$$\frac{1}{a^2} - \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\dot{a}\dot{b}}{ab} = \bar{p} - \frac{2}{3}\eta\left[\frac{\dot{b}}{b} - \frac{\dot{a}}{a}\right]. \tag{2.7}$$

We recall, that for the Bianchi V models under consideration,

$$\sigma^2 = \tfrac{1}{3}[\dot{a}/a - \dot{b}/b]^2, \tag{2.8}$$

$$\theta = \dot{a}/a + 2(\dot{b}/b), \tag{2.9}$$

and

$${}^3R = -6a^{-2}, \tag{2.10}$$

where ${}^3R$ denotes the Ricci curvature of the hypersurfaces of homogeneity. (Note that when $a/b = \text{const}$, the heat conduction vector and the shear are consequently both zero—this case is discussed in some detail in Ref. I.) We also recall the identity (the "generalized Friedmann equation" or "first integral"),

$$\theta^2 = 9/a^2 + 3\sigma^2 + 3\rho, \tag{2.11}$$

where use has been made of (2.10). By adding Eq. (2.6) to two times Eq. (2.7), we obtain [using (2.11)] the Raychaudhuri equation:

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - 2\sigma^2 - \tfrac{1}{2}(\rho + 3\bar{p}). \tag{2.12}$$

The second independent equation we shall write as

$$\dot{\sigma} = -2\eta\sigma - \sigma\theta, \tag{2.13}$$

which is obtained by subtracting Eq. (2.7) from Eq. (2.6). Finally, from the conservation law ($T^{ab}{}_{;b}u_a = 0$), we find that

$$\dot{\rho} = -(\rho + p)\theta + \zeta\theta^2 + 4\eta\sigma^2 + (4/\sqrt{3}) \times \sigma[\tfrac{1}{3}\theta^2 - \sigma^2 - \rho]. \tag{2.14}$$

Now, we define the new variables β and x, and the new time coordinate Ω, as follows:

$$\beta \equiv \frac{2}{\theta}\left[\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right], \quad \beta = 2\sqrt{3}\,\frac{\sigma}{\theta}, \tag{2.15a}$$

so that β measures the rate of shear in terms of the expansion,

$$x \equiv 3\rho/\theta^2, \tag{2.15b}$$

so that x measures the dynamical importance of the matter content, and

$$\ell \equiv e^{-\Omega}, \quad \frac{d\Omega}{dt} = -\frac{1}{3}\theta, \tag{2.15c}$$

where ℓ is the representative length scale with $\theta = 3\dot{\ell}/\ell$. Therefore, using Eqs. (2.15) and Eq. (2.12), we can write Eq. (2.13) as

$$\frac{d\beta}{d\Omega} = \frac{1}{2}\beta\left[4 - \beta^2 - x - \frac{9p}{\theta^2} + \frac{9\zeta}{\theta} + 12\frac{\eta}{\theta}\right], \tag{2.16}$$

and we can write Eq. (2.14) as

$$\frac{dx}{d\Omega} = x[1 - x - \beta^2] + \beta\left[2x - 2 + \frac{\beta^2}{2}\right] + 9\frac{p}{\theta^2}(1-x) - 9\frac{\zeta}{\theta}(1-x) - 3\frac{\eta}{\theta}\beta^2 \tag{2.17}$$

[where we have used Eq. (2.12) for $\bar{p}$]. Finally, we note that from (2.11) we are only interested in the region

$$\beta^2 + 4x \leqslant 4, \quad x \geqslant 0. \tag{2.18}$$

III. EQUATIONS OF STATE

In order to complete the system of equations we need to specify three equations of state for p, ζ, and η. In principle, these equations of state can be derived from kinetic theory[6–8]. For example, Collins and Stewart[9] considered a class of nonrotating Bianchi models (that included Bianchi type V's) with shear viscosity (but no bulk viscosity) in which $\eta = \frac{1}{5}pt_{\text{coll}}$, where the harmonic mean of the collision times for the various reactions, t_{coll}, is assumed to be given by $t_{\text{coll}} = 1/\sqrt{2}n\Sigma$, where n is the number density and Σ is the mean total scattering cross section (related to the temperature by a suitable approximate relationship). Subject to some additional, physically motivated assumptions, Collins and Stewart[9] concluded from a qualitative analysis that for arbitrary initial conditions the shear anisotropy could be arbitrarily large now, and that the Universe need not have been in thermal equilibrium during the early stages. These conclusions are relevant in determining whether strong dissipative mechanisms in the early Universe (such as neutrino viscosity) could produce the observed highly isotropic matter distribution.[10,11]

However, in practice, it is necessary to specify phenomenological equations of state subject to a set of thermodynamical laws.[12] Of course, specification of p, ζ, and η requires special conditions for which there may be no physical foundations. This specification should be subject to physical constraints such as p, ζ, and η should tend to zero as the density tends to zero and must be subject to the energy conditions. It goes without saying that the behavior of the fluid (e.g., it's asymptotic behavior) depends on the assumptions made on the form of these physical quantities. We also note that in writing down the energy–momentum tensor for a viscous fluid with heat conduction in the form of Eq. (2.2) we have assumed that $\pi_{ab} = -2\eta\sigma_{ab}$, where π_{ab} is the tensor of anisotropic stress. This assumption (the "viscosity assumption") is valid whenever the anisotropy is small (i.e., $|\pi_{ab}/p| \ll 1$).

There are some equations of state that are commonly used that, although not widely applicable, are obtained as a result of approximate estimates for particular fluids. The barotropic equation of state, $p = (\gamma - 1)\rho$ is often assumed. Here, $1 \leqslant \gamma \leqslant 2$ is necessary for the existence of local mechanical stability and for the speed of sound in the fluid to be no greater than the speed of light. Belinskii and Khalatnikov[13,14] consider viscous fluids in which the viscosity coefficients depend on powers of the energy density. It is argued that this approach will be valid whenever the kinetic coefficients that arise at a higher order of approximation will be proportional to the energy taken to a power greater than the one characterizing the coefficients of ζ and η. Consequently, this approach ought to be valid (at least) near the initial singularity when the energy density is very small. Moreover, it is argued that the qualitative picture ought not to change substantially from that obtained from this approach.[13]

As noted above, in order to complete the system of equations three equations of state must be given, specifying p, ζ, and η in terms of the other physical quantities. Since we are considering a viscous fluid with heat conduction, in general all physical quantities depend on two independent thermodynamical variables, one of which will be chosen as ρ and the second of which will be denoted by X (e.g., temperature or entropy density), viz.,

$$\begin{aligned} p &= p(\rho,X), \\ \zeta &= \zeta(\rho,X), \\ \eta &= \eta(\rho,X). \end{aligned} \tag{3.1}$$

As also noted earlier, in principle these equations can be obtained from kinetic theory, but in practice phenomenological equations of state need to be assumed. In addition, we also recall the variables β and x occurring in Eqs. (2.16) and (2.17), viz.,

$$\beta = 2\sqrt{3}(\sigma/\theta), \tag{3.2a}$$

$$x = 3\rho/\theta^2, \tag{3.2b}$$

and note that, firstly, β and x are dimensionless, and, secondly, in the absence of viscosity and with $p = (\gamma - 1)\rho$, Eqs. (2.16) and (2.17) form a plane-autonomous system in β and x.

Here, we are going to consider equations of state of the following form:

$$\begin{aligned} p/\theta^2 &= F(\beta,x), \\ \zeta/\theta &= G(\beta,x), \\ \eta/\theta &= H(\beta,x). \end{aligned} \tag{3.3}$$

Let us argue in favor of these equations:

(i) First, Eqs. (3.3) are completely dimensionless equations since, as noted above, β and x are dimensionless, and the ratios p/θ^2, ζ/θ, and η/θ are dimensionless. It can be argued that dimensionless equations of state are the most physically natural. In particular, it might be expected that such equations will be valid whenever the physics is scale invariant. Scale-invariant solutions in classical hydrodynamics have been a fruitful source of models for physical systems having no intrinsic units of length, mass, or time. Moreover, this situation might be especially pertinent in the qualitative analysis that we intend to carry out, where it will be of interest to study physical systems that have no intrinsic scale in an asymptotic sense.

We note that our particular "choice" of dimensionless physical quantities (3.3) is to some extent arbitrary, and the choice has been made for convenience. However, Eqs. (3.3) are independent of this choice. For example, if σ is nonzero, and if we assume that $p/\sigma^2 = f(\beta,x)$ and $\eta/\sigma = h(\beta,x)$, then

$$p/\theta^2 = [\sigma^2/\theta^2]f(\beta,x) = F(\beta,x),$$

and

$$\eta/\theta = [\sigma/\theta]h(\beta,x) = H(\beta,x).$$

In addition, $p/\rho = h(\beta,x)$ implies that $p/\theta^2 = (\rho/\theta^2)h(\beta,x) = H(\beta,x)$.

(ii) Second, Eqs. (3.3) are the *most general* equations of state such that Eqs. (2.16) and (2.17) reduce to a plane-autonomous system, enabling us to study the viscous models under consideration qualitatively in a straightforward manner. In general, it may be possible for the system of equations under investigation to reduce to an autonomous system of dimension greater than two even if Eqs. (3.3) are not assumed. However, it is strongly suggested by Eqs. (2.16) and (2.17) that equations of state (3.3) are clearly the most natural in any attempted reduction to an autonomous system, and, moreover, from the above comments Eqs. (3.3) are perhaps suggested by dimensional considerations.

(iii) Next, since we are considering spatially homogeneous models it is natural for all the physical quantities ρ, p, ζ, η (etc.) and the kinematical quantities θ and σ to depend only on t, so that p/θ^2, ζ/θ, η/θ, β, and x are function of t alone, and the equations of state can be considered in the form (3.3) in all generality.

(iv) Equations (3.3) are completely general for physical systems in which β and x can be regarded as independent thermodynamical variables.

(v) The most commonly considered equations of state are of the form (3.3). For example, the barotropic equation of state $p = (\gamma - 1)\rho$ is equivalent to $p/\theta^2 = (\gamma - 1)\rho/\theta^2 = \frac{1}{3}(\gamma - 1)x$, and is consequently of the form (3.3), where F is simply given by $F(\beta,x) = \frac{1}{3}(\gamma - 1)x$. Also, $\zeta = \zeta_0\rho^{1/2}$ and $\eta = \eta_0\rho^{1/2}$ are equivalent to $\zeta/\theta = \zeta_0[\rho/\theta^2]^{1/2}$ and $\eta/\theta = \eta_0[\rho/\theta^2]^{1/2}$, which are simple examples of Eqs. (3.3) with $G(\beta,x) = (\zeta_0/\sqrt{3})x^{1/2}$ and $H(\beta,x) = (\eta_0/\sqrt{3})x^{1/2}$. In particular, Belinskii and Khalatnikov[13] have studied viscous fluid models in which the equations of state are asymptotically of this form. In addition, since these "common" equations of state (particularly the barotropic equation of state) are derived from kinetic theory, it can be argued that there is some kinetic theoretical basis for Eqs. (3.3).

(vi) Finally, FRW models can be written in terms of a plane-autonomous system of equations in (ρ,θ) space. Murphy[15] included bulk viscosity in isotropic and spatially homogeneous cosmologies, and it can be shown that the plane-autonomous character of the resulting field equations can be retained if the bulk viscosity dissipation is modeled by means of an equation of state $\bar{p} = \bar{p}(\rho,\theta)$ (where $\zeta = -\partial\bar{p}/\partial\theta$). It is known that FRW cosmological models are structurally unstable.[5] Golda *et al.*[16] have shown that if the equation of state $\bar{p} = (\gamma - 1)\rho - \zeta\theta$ with $\zeta(\rho) = \zeta_0(\rho)^m$ is assumed, then the *only* possible solutions that are structurally stable are those with $m = \frac{1}{2}$ [that is, those in which $\zeta/\theta = \zeta_0(\rho/\theta^2)^{1/2}$, which is of the form of (3.3)].

It should be noted once again that the analysis, and, in particular, the discussion above, is quite general, and is equally applicable in all Bianchi-type models. For illustration, we are considering only Bianchi V imperfect fluid models here; the analysis will be extended elsewhere.

No qualitative analysis can be undertaken unless F, G, and H are further specified. Here we shall assume for simplicity that F, G, and H are independent of β. This, of course, still enables us to write the equations as a plane-autonomous system. In addition, if it is possible for ρ and θ to be regarded as the two independent thermodynamical variables (recall the baryon conservation law in the form $\dot{n} + n\theta = 0$, where n is the particle number density), then this assumption is the special case guaranteeing that the equations of state are dimensionless. Finally, in general, this will always be possible if all the quantities of interest are functions of t only, as is expected in the spatially homogeneous models under consideration.

Moreover, for simplicity we shall consider the case when $F(x)$, $G(x)$, and $H(x)$ are functions that depend on a power of the argument; namely,

$$p/\theta^2 = p_0 x^l, \tag{3.4a}$$

$$\zeta/\theta = \zeta_0 x^m, \tag{3.4b}$$

$$\eta/\theta = \eta_0 x^n, \tag{3.4c}$$

where l, m, and n are constants. Such equations may be valid, at least in an approximate sense, and ought to be applicable in a qualitative analysis. In addition, these equations are consistent with the "common" examples alluded to above. Using Eqs. (3.4), Eqs. (2.16) and (2.17) reduce to a plane-autonomous system. [We note that since ${}^3R<0$ [Eq. (2.10)] it follows that $\theta^2>0$ [Eq. (2.11)] and $\dot{\theta}<0$ [Eq. (2.12)] imply that if $\theta_0>0$ (at present) then $\theta>0$ for all t; hence all quantities in equations (3.4) are well defined. Care must be taken in extending this analysis to Bianchi IX models in which ${}^3R>0$ since θ is no longer always positive.]

IV. EXACT SOLUTIONS

In a series of papers cosmological models have been examined in which the condition $\sigma^2/\theta^2 = \text{const.}$ is assumed (I11, I12, and I16–21), and Bali[17] has investigated Bianchi I viscous fluid cosmology with magnetic field under the assumption $\eta = \eta_0\theta$ and has found an exact solution (in which $\lim_{t\to 0}\sigma/\theta = 0$). In this section we shall investigate exact solutions of Eqs. (2.6) and (2.7) with equations of state given by (3.4). In particular, we shall consider the simple case in which $l = 1$, $m = 0$, and $n = 0$ in Eqs. (3.4), i.e.,

$$\begin{aligned} p &= (\gamma - 1)\rho, \\ \zeta &= \zeta_0\theta, \\ \eta &= \eta_0\theta \end{aligned} \tag{4.1}$$

(where $\gamma = 3p_0 + 1$), in which first integrals of Eqs. (2.6) and (2.7) can be obtained by the method of decomposable operators of Maartens and Nel.[18] Exact solutions will be extremely useful in combination with any possible qualitative analysis.

Using equations of state (4.1) [and employing Eqs. (2.3), (2.4), and (2.9)], Eqs. (2.6) and (2.7) become, respectively,

$$\begin{aligned} -\frac{2\ddot{b}}{b} &+ \left[\zeta_0 + \frac{4}{3}\eta_0\right]\frac{\dot{a}^2}{a^2} + \left[-\gamma + 4\zeta_0 - \frac{8}{3}\eta_0\right]\frac{\dot{b}^2}{b^2} \\ &+ \left[-2(\gamma - 1) + 4\zeta_0 + \frac{4}{3}\eta_0\right]\frac{\dot{a}\dot{b}}{ab} \\ &+ \frac{1}{a^2}(3\gamma - 2) = 0, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} &+ \left[\zeta_0 - \frac{2}{3}\eta_0\right]\frac{\dot{a}^2}{a^2} \\ &+ \left[(1 - \gamma) + 4\zeta_0 + \frac{4}{3}\eta_0\right]\frac{\dot{b}^2}{b^2} \\ &+ \left[(1 - 2\gamma) + 4\zeta_0 - \frac{2}{3}\eta_0\right]\frac{\dot{a}\dot{b}}{ab} + \frac{1}{a^2}(3\gamma - 2) = 0. \end{aligned} \tag{4.3}$$

These equations constitute two independent (coupled, nonlinear, second-order, ordinary) differential equations for a and b. Multiplying Eq. (4.2) by the constant α, and Eq. (4.3) by the constant β, and adding, yields the equation

$$\begin{aligned} &[-\beta]\frac{\ddot{a}}{a} + [-2\alpha - \beta]\frac{\ddot{b}}{b} + \frac{\dot{a}^2}{a^2}[(\alpha + \beta)\zeta_0 + \tfrac{2}{3}(2\alpha - \beta)\eta_0] + \frac{\dot{b}^2}{b^2}[\beta(1 - \gamma) - \alpha\gamma + 4(\alpha + \beta)\zeta_0 + \tfrac{4}{3}(\beta - 2\alpha)\eta_0] \\ &\quad + \frac{\dot{a}\dot{b}}{ab}[(-2\alpha)(\gamma - 1) + (1 - 2\gamma)\beta + 4(\alpha + \beta)\zeta_0 + \tfrac{2}{3}(2\alpha - \beta)\eta_0] + (1/a^2)(3\gamma - 2)(\alpha + \beta) = 0. \end{aligned} \tag{4.4}$$

Using the method of decomposable differential operators[18], we can find a first integral of this equation [and hence Eqs. (4.2) and (4.3)] whenever the following algebraic equation is satisfied:

$$\begin{aligned} &\beta(\alpha + \beta)(4\alpha + \beta)(\gamma - 2) \\ &\quad + (\alpha + \beta)(2\alpha - \beta)^2[\zeta_0 + \tfrac{4}{3}\eta_0] = 0. \end{aligned} \tag{4.5}$$

The solutions of this equation are:

(i) $\alpha+\beta=0$, and (ii) $\alpha = \{2(\zeta_0+\frac{4}{3}\eta_0)^{-1}[(\xi_0+\frac{4}{3}\eta_0-(\gamma-2)) \pm\sqrt{(2-\gamma)(2+3\zeta_0+4\eta_0-\gamma)}\,]\}\beta \quad (\zeta_0+\frac{4}{3}\eta_0\neq 0)$.

(i) Taking $\alpha+\beta=0$ [i.e., subtracting Eq. (4.3) from (4.2)] yields the *general* first integral:

$$\frac{d}{dt}\left(\frac{a}{b}\right)+K\left(\frac{a}{b}\right)(ab^2)^{-(2\eta_0+1)} \tag{4.6}$$

(where K is an integration constant). Defining the new time coordinate τ by

$$\frac{d\tau}{dt}=(ab^2)^{-(2\eta_0+1)}, \tag{4.7}$$

Eq. (4.6) integrates to

$$(a/b)=C\exp(K\tau). \tag{4.8}$$

(ii) We define $\Sigma:=\frac{1}{2}[1+k\pm\sqrt{k(3+k)}\,]$, where $k=(2-\gamma)/(\zeta_0+\frac{4}{3}\eta_0)$ is non-negative (since $\gamma\leqslant 2$ and $\zeta_0>0$ and $\eta_0>0$) ensuring two real values for Σ (Σ_+ and Σ_-). Taking $\alpha=\Sigma\beta$ yields the *general* first integral(s);

$$\frac{d}{dt}\left\{a^{[-(\Sigma+1)\zeta_0-(2/3)(2\Sigma-1)\eta_0]}b^{[-4(\Sigma+1)\zeta_0-(4/3)(1-2\Sigma)\eta_0+\gamma(1+\Sigma)-(4\Sigma^2+2\Sigma+1)](2\Sigma+1)^{-1}}\frac{d}{dt}(ab^{2\Sigma+1})\right\}$$
$$=(3\gamma-2)(\Sigma+1)a^{[-(\Sigma+1)\zeta_0-(2/3)(2\Sigma-1)\eta_0-1]}b^{[-4(\Sigma+1)\zeta_0-(4/3)(1-2\Sigma)\eta_0+\gamma(1+\Sigma)+2\Sigma](2\Sigma+1)^{-1}}. \tag{4.9}$$

Defining the new variable $B:=e^{K\tau}b^{2(\Sigma+1)}$, and using the general first integral given by Eq. (4.8) in terms of the time coordinate τ, Eq. (4.9) yields the following differential equation for B:

$$\frac{B''}{B}+p\left(\frac{B'}{B}\right)^2+q\left(\frac{B'}{B}\right)=\overline{C}e^{r\tau}B^s, \tag{4.10}$$

where the constants $\overline{C}$, p, q, r, and s are given by

$$\overline{C}=(3\gamma-2)(\Sigma+1)C^{4\eta_0},$$
$$\frac{r}{K}=\frac{2(2\Sigma-1)}{(\Sigma+1)}\eta_0-\frac{2}{1+\Sigma},$$
$$s=6\eta_0/(\Sigma+1)+2/(1+\Sigma), \tag{4.11}$$
$$p=\frac{-(5+2\Sigma)(\zeta_0+\frac{4}{3}\eta_0)+(\gamma-2)}{2(2\Sigma+1)}-1,$$
$$\frac{q}{K}=\frac{-(2\Sigma+3)(2\Sigma-1)(\zeta_0+\frac{4}{3}\eta_0)+(\gamma-2)}{2(2\Sigma+1)}.$$

In the above, a prime denotes differentiation with respect to τ. Equation (4.10) is a (single) second-order, ordinary differential equation for the (single) variable B.

(iii) Let us consider the case $\gamma=2$ (corresponding to stiff matter) separately. In this case $k=0$ and $\Sigma=\frac{1}{2}$ [this case corresponds to a double root for α/β in Eq. (4.5)]. Taking $2\alpha-\beta=0$ when $\gamma=2$ [i.e., adding twice Eq. (4.3) to Eq. (4.2)] yields the first integral:

$$a^2(ab^2)^{3\zeta_0/2-1}\frac{d}{dt}\left[(ab^2)^{-3\zeta_0/2}\frac{d}{dt}(ab^2)\right]=6. \tag{4.12}$$

Again, employing the first integral $a=Ce^{K\tau}b$ and defining the new variable X by $e^X=b^3e^{K\tau}$, in terms of the time coordinate τ defined by (4.7) Eq. (4.12) reduces to a "simple" second-order differential equation for X, which we can attempt to solve in order to obtain a solution of the Bianchi V imperfect fluid field equations in the particular case of stiff matter. Alternatively, defining X by $e^X=B$, when $\gamma=2$ ($\Sigma=\frac{1}{2}$) Eq. (4.10) becomes

$$X''-[\tfrac{3}{2}\zeta_0+2\eta_0](X')^2=\overline{C}\exp\{[4\eta_0+\tfrac{4}{3}]X-\tfrac{4}{3}K\tau\}. \tag{4.13}$$

If a solution for X is found to this second-order differential equation, a and b are then obtained by

$$a=Ce^{(1/3)(X+2K\tau)},$$
$$b=e^{(1/3)(X-K\tau)}. \tag{4.14}$$

We note the simple solution

$$X=X_0+[K/(3\eta_0+1)]\tau, \tag{4.15}$$

to Eq. (4.13), where the constant X_0 satisfies

$$\overline{C}\exp[(4\eta_0+\tfrac{4}{3})X_0]$$
$$=-[(3\zeta_0+4\eta_0)/2(3\eta_0+1)^2]K^2, \tag{4.16}$$

whence

$$a=C\exp[\tfrac{1}{3}X_0+[(2\eta_0+1)/(3\eta_0+1)]K\tau],$$
$$b=\exp[\tfrac{1}{3}X_0-[\eta_0/(3\eta_0+1)]K\tau], \tag{4.17}$$

and t and τ are related by

$$(t_0-t)=C_1e^{C_2\tau}, \tag{4.18}$$

where

$$C_1=\frac{-(3\eta_0+1)}{K(2\eta_0+1)}\times\left[\frac{-(3\zeta_0+4\eta_0)K^2C^{4/3}}{12(3\eta_0+1)^2}\right]^{(2\eta_0+1)/(4\eta_0+4/3)}, \tag{4.19a}$$

and

$$C_2=(2\eta_0+1)K/(3\eta_0+1). \tag{4.19b}$$

Unfortunately, a straightforward calculation using Eq. (2.4) shows that this solution is unphysical since it leads to a negative energy density.

V. PLANE-AUTONOMOUS SYSTEMS AND DISCUSSION

Collins[4,5] was the first to use geometric techniques of standard differential equations theory, analyzing both nonrotating and tilting Bianchi-type models in the case of a perfect fluid source. Roy and Prakesh[19] derived some results for viscous fluid models of Petrov type D, under the unphysical assumption of constant shear and constant ζ. Belinskii and Khalatnikov[13,14] were the first to consider the qualitative behavior of spatially homogeneous viscous fluid cosmological models in any generality. In particular, they investigated viscous fluid Bianchi I models with barotropic equation of state $p = (\gamma - 1)\rho$ and in which the viscosity coefficients depend (only) on the powers of the energy density, in which case the field equations reduce to a plane-autonomous system.

In this article we have introduced a new approach for dealing with equations of state in Bianchi-type cosmologies and we have shown by way of illustration that the field equations in LRS Bianchi V imperfect fluid cosmologies can be written as a plane-autonomous system, facilitating a qualitative analysis of such cosmological models. This work therefore generalizes the previous results in nonrotating and (LRS) tilting perfect fluid Bianchi-type (including type V) models[4,5], and in viscous fluid Bianchi I models,[13] to the imperfect Bianchi-V case.

As noted above, Eqs. (2.16) and (2.17) reduce to a plane-autonomous system when Eqs. (3.3) or (3.4) are employed. For illustration, if we consider the equations of state in the form

$$p/\theta^2 = \tfrac{1}{3}(\gamma - 1)x, \tag{5.1a}$$

$$\zeta/\theta = \zeta_0 x^{1/2}, \tag{5.1b}$$

$$\eta/\theta = \eta_0 x^{1/2}, \tag{5.1c}$$

then Eqs. (2.16) and (2.17) reduce to

$$\frac{d\beta}{d\Omega} = \frac{1}{2}\beta\,[4 - \beta^2 - (3\gamma - 2)x + 3x^{1/2}[3\zeta_0 + 4\eta_0]], \tag{5.2}$$

and

$$\frac{dx}{d\Omega} = x[(3\gamma - 2)(1 - x) - \beta^2] + \beta\left[2x - 2 + \frac{\beta^2}{2}\right] - 3x^{1/2}[3(1 - x)\zeta_0 + \eta_0\beta^2]. \tag{5.3}$$

We shall analyze the qualitative nature of Bianchi V imperfect fluid cosmological models in a future paper.

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