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New slant on tilted cosmology

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The physical properties of a class of cosmological models in which the matter is described by a perfect fluid moving relative to a shear-free, irrotational, and geodesic timelike congruence, which is assumed to be associated with the cosmic microwave background radiation field, are investigated. [S0556-2821(96)03420-0]

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I. INTRODUCTION

In a recent paper by Coley and Mc Manus [1], spacetimes admitting a shear-free, irrotational, and geodesic (SIG) timelike congruence were studied. Moreover, (single) perfect fluid spacetimes in which the fluid four-velocity is tilting relative to the SIG timelike congruence (which we shall refer to here as tilting SIG models) were investigated; in particular, it was shown that such spacetimes are not necessarily Friedmann-Robertson-Walker (FRW) models. Our aim in this paper is to study further such tilting SIG models and to investigate whether they may be of any particular physical interest.

Motivated by the existence of an isotropic cosmic microwave background (CMB) radiation field, Ehlers, Geren, and Sachs [2] studied spacetimes, satisfying the Einstein field equations, in which the gravitational field is generated by a gas with a locally isotropic (in momentum space with respect to the world-velocity field u^a of the CMB photons) one-particle distribution function obeying Liouville's equation (i.e., the model of matter was that of kinetic theory). In their classic paper, Ehlers, Geren, and Sachs [2] showed that Liouville's equation implies that the shear of the world-velocity field is necessarily zero [3], and that for an expanding world-velocity field, either in the case of massive particles or in the case of nonaccelerating particles with zero rest mass, the world-velocity field is necessarily irrotational. The field equations then imply that the resulting spacetime is FRW if the matter is assumed to be moving with the world-velocity field [2]. These results have motivated us to study further spacetimes admitting a SIG timelike congruence which, from a physical point of view, is associated with the world-velocity field of the CMB photons. However, unlike Ehlers, Geren, and Sachs, we shall not assume that the matter is comoving with the world-velocity field (although, in the case where the ''relative'' velocity is small, the resulting cosmological models can be regarded as ''generalized'' FRW models).

Recently, observations of the large-scale streaming of matter [4] and their relevance regarding theories of structure formation (see, for example, [5]) has attracted much interest, and (so-called tilting) cosmological models have been advanced to study the effects of a large-scale peculiar velocity field relative to the CMB frame [6], particularly the growth of inhomogeneities in such models and the relationship with the observed large-scale structure.

If the strong energy condition holds, the energy-momentum tensor has a unique unit timelike eigenvector u_L^a. For a general source in which the matter is described using the fluid approximation [e.g., an imperfect fluid or a multi-component fluid which is formally equivalent to a (single) imperfect fluid, of particular interest here are the two-fluid models in which the respective four-velocities are not parallel; see [7]], there exists another unique unit timelike vector u_E^a which is parallel to the particle flux; however, u_E^a is not necessarily parallel to u_L^a. Therefore, there are two different relativistic thermodynamic descriptions of the state of the fluid depending upon whether the energy-momentum tensor is decomposed relative to u_L^a (the Landau-Lifshitz [8] frame) or u_E^a (the Eckart [9] frame). Clearly, when studying spacetimes admitting a SIG timelike congruence, only one such timelike vector field is shear-free, irrotational, and geodesic. The tilting SIG models studied here bear some resemblance to those studied previously by Coley and Tupper [7,10] in that in both cases there are matter sources tilting relative to a SIG timelike congruence; however, here the source is a perfect fluid and in Coley and Tupper [10] the geometry was fixed to be Robertson-Walker but the source was an imperfect fluid.

II. THE MODELS

The stress-energy tensor can be formally decomposed with respect to the shear-free, irrotational, and geodesic timelike congruence u^a according to

$$T_{ab}=\mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \qquad (1)$$

where $q_a u^a = \pi^a_a = \pi_{ab} u^b = 0$ and the projection tensor $h_{ab} = g_{ab} + u_a u_b$ satisfies[1] $h_{ab} u^b = 0$. The quantities μ, p, q_a, and π_{ab} denote the energy density, pressure, heat conduction and anisotropic stress, respectively, as measured by an observer whose four-velocity is u^a. The existence of a

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[1]We follow the notation and conventions in Ellis [11] as utilized in Coley and Mc Manus [1]. Furthermore, Roman indices range from 0 to 3 and Greek indices range from 1 to 3.

SIG congruence implies that there exists a coordinate system [1] such that $u^a=\delta^a_t$ and such that the metric may be written as

$$ds^2=-dt^2+H^2(t,x^\gamma)h_{\alpha\beta}(x^\delta)dx^\alpha dx^\beta, \tag{2}$$

whence

$$\mu=\tfrac{1}{2}H^{-2}R-2H^{-3}\nabla^2 H+H^{-4}\nabla H\cdot\nabla H+3H^{-2}\dot{H}^2, \tag{3}$$

$$p=-2H^{-1}\ddot{H}-\tfrac{1}{3}\mu, \tag{4}$$

$$q_\alpha=2\partial_t[\nabla_\alpha(\ln H)], \tag{5}$$

$$\pi_{\alpha\beta}=R_{\alpha\beta}-\tfrac{1}{3}h_{\alpha\beta}R-\nabla_\alpha\nabla_\beta(\ln H)+(\nabla_\alpha \ln H)(\nabla_\beta \ln H)+\tfrac{1}{3}[\nabla^2(\ln H)-\nabla(\ln H)\cdot\nabla(\ln H)]h_{\alpha\beta}, \tag{6}$$

where $R_{\alpha\beta}$ is the Ricci tensor of the three-metric $h_{\alpha\beta}(x^\gamma)$, ∇_α is the covariant derivative with respect to $h_{\alpha\beta}$, $\nabla^2 H\equiv h^{\alpha\beta}\nabla_\alpha\nabla_\beta H$, and $\nabla H\cdot\nabla H\equiv h^{\alpha\beta}\nabla_\alpha H\nabla_\beta H$. The conservation laws and Bianchi identities are given by [1]

$$\dot{\mu}+3H^{-1}\dot{H}(\mu+p)+H^{-1}\nabla^\alpha q_\alpha=0, \tag{7}$$

$$\partial_t q_\alpha+3H^{-1}\dot{H}q_\alpha+\nabla_\alpha p+H^{-2}\nabla^\beta\pi_{\alpha\beta}=0, \tag{8}$$

$$H^{-1}\nabla^\beta\pi_{\alpha\beta}+H^{-3}(\nabla^\beta H)\pi_{\alpha\beta}=\nabla_\alpha\mu-H^{-1}\dot{H}q_\alpha, \tag{9}$$

$$2\partial_t\pi_{\alpha\beta}=-\nabla_\alpha q_\beta+2q_{(\alpha}\nabla_{\beta)}(\ln H)+\tfrac{1}{3}H^2\nabla^\gamma(H^{-2}q_\gamma)h_{\alpha\beta}. \tag{10}$$

We assume that the source is a single perfect fluid with stress-energy tensor

$$T_{ab}=(\overline{\mu}+\overline{p})v_a v_b+\overline{p}g_{ab}, \tag{11}$$

where $\overline{\mu}$ and $\overline{p}$ are, respectively, the energy density and pressure as measured by an observer moving with the fluid whose four-velocity v^a is tilting with respect to u^a. (If v^a is not tilting then the resulting spacetime must necessarily be FRW [1,11].) From Eq. (11), and using Eqs. (1) and (2), we deduce that

$$\pi_{\alpha\beta}=\pi\{q_\alpha q_\beta-\tfrac{1}{3}(h^{\gamma\delta}q_\gamma q_\delta)h_{\alpha\beta}\} \tag{12}$$

[where π is defined by (12)]. Equation (10), in conjunction with Eqs. (12) and (8), can now be employed to show that the unit spacelike vector $q^\gamma/\sqrt{h_{\alpha\beta}q^\alpha q^\beta}$ is both shear-free and twist-free with respect to the three-metric $h_{\alpha\beta}$. Hence, the three space admits an umbilical foliation, and there exist coordinates such that $h_{\alpha\beta}dx^\alpha dx^\beta=a^2(x^\alpha)dx^2+b^2(x^\alpha)(dy^2+dz^2)$. Furthermore, Eq. (6) then implies that the spacetime metric may be written as [1]

$$ds^2=-dt^2+H^2(t,x)\{dx^2+f^2(x)\phi^2(y,z)(dy^2+dz^2)\}, \tag{13}$$

where the term inside { } is precisely $h_{\alpha\beta}dx^\alpha dx^\beta$ and H is a nonseparable function, and where

$$q_\alpha=q\delta^x_\alpha;\quad q\equiv\sqrt{h^{\gamma\delta}q_\gamma q_\delta}=2\partial_t\partial_x(\ln H), \tag{14}$$

and the only nonzero components of $\pi_{\alpha\beta}$ are

$$\pi_{xx}=-2f^{-2}\phi^{-2}\pi_{yy}=-2f^{-2}\phi^{-2}\pi_{zz}=\tfrac{2}{3}\pi q^2. \tag{15}$$

Writing the four-velocity of the tilted fluid[2] as $v_a=(-\cosh\psi,H\sinh\psi,0,0)$, where ψ is called the tilt angle (i.e., $\cosh\psi=-v_a u^a$), we find that

$$\mu=(\overline{\mu}+\overline{p})\cosh^2\psi-\overline{p}, \tag{16}$$

$$p=\overline{p}+\tfrac{1}{3}(\overline{\mu}+\overline{p})\sinh^2\psi, \tag{17}$$

$$q=(\overline{\mu}+\overline{p})H\cosh\psi\sinh\psi=\frac{3H(\mu+p)\cosh\psi\sinh\psi}{3\cosh^2\psi+\sinh^2\psi}, \tag{18}$$

$$\pi=\frac{1}{(\overline{\mu}+\overline{p})\cosh^2\psi}=\frac{3\cosh^2\psi+\sinh^2\psi}{3(\mu+p)\cosh^2\psi}, \tag{19}$$

whence

$$\mu+p=(\overline{\mu}+\overline{p})[\cosh^2\psi+\tfrac{1}{3}\sinh^2\psi], \tag{20}$$

$$q\pi=H\tanh\psi, \tag{21}$$

$$q^2=\pi q^2[(\mu+p)-\tfrac{1}{3}\pi H^{-2}q^2]. \tag{22}$$

Now, from Eqs. (3)–(6), and using Eqs. (16)–(22), we find that

$$\phi^{-1}(y,z)=1+\frac{\kappa}{4}(y^2+z^2), \tag{23}$$

where κ is a constant. Therefore,

$$\pi_{xx}=\frac{2}{3}\left[-\frac{H_{xx}}{H}-\frac{2H_x^2}{H^2}-\frac{H_x}{H}\frac{f_x}{f}-\frac{f_{xx}}{f}+\frac{f_x^2}{f^2}-\frac{\kappa}{f^2}\right], \tag{24}$$

$$\mu+p=-\frac{2H_{tt}}{H}+\frac{2H_t^2}{H^2}+\frac{2}{3}H^{-2}\left[-\frac{2H_{xx}}{H}+\frac{H_x^2}{H^2}-\frac{4H_x}{H}\frac{f_x}{f}-\frac{2f_{xx}}{f}-\frac{f_x^2}{f^2}+\kappa f^{-2}\right]. \tag{25}$$

Finally, using Eqs. (14) and (15), Eq. (22) yields

$$\left[\frac{H_{xt}}{H}-\frac{H_x H_t}{H^2}\right]^2=\frac{3}{8}\pi_{xx}\left[(\mu+p)-\frac{1}{2}H^{-2}\pi_{xx}\right], \tag{26}$$

which becomes a differential equation for $H(t,x)$ and $f(x)$ when Eqs. (24) and (25) are used to eliminate π_{xx} and $\mu+p$ from the right-hand side of the equation. We also note that an expression for the density μ can be obtained from Eqs. (4) and (25):

[2] The inclusion of the function H in the four-velocity v_a corrects an error that appeared in our original analysis [1]. The correction has altered the original form of the Eqs. (18), (21), (22), and (26)–(35) (see [1] for a full comparison).

$$\mu=3\left(\frac{H_t}{H}\right)^2+\frac{1}{H^2}\left\{-\frac{2H_{xx}}{H}+\frac{H_x^2}{H^2}-\frac{4H_x}{H}\frac{f_x}{f}-\frac{2f_{xx}}{f}-\frac{f_x^2}{f^2}+\frac{\kappa}{f^2}\right\}. \tag{27}$$

Thus far, we have found expressions for the four quantities μ,p,q, and π in terms of the functions H,f, and $\phi(\kappa)$ and their derivatives: μ is given by Eq. (27); p is given through Eqs. (25) and (27); q by Eq. (14); and $\pi\equiv 3\pi_{xx}/2q^2$ where π_{xx} and q are given by Eqs. (24) and (14), respectively.

In addition, μ,p,q, and π are given in terms of the three (unknown) physical quantities $\bar{\mu},\bar{p}$, and ψ through Eqs. (16)–(19). Thus, Eqs. (16)–(19) can be used to yield a single algebraic identity between μ,p,q, and π, namely, Eq. (22). Hence, Eq. (26), where π_{xx} and $\mu+p$ have been eliminated using Eqs. (24) and (25), is the only differential equation that needs to be satisfied. In other words, the quantities μ,p,q, and π_{xx} are automatically specified once a solution $\{H,f,\kappa\}$ to Eq. (26) is given. Thus, Eq. (26) is the only field equation that must be satisfied (unless, of course, some additional structure is placed on the physical quantities $\bar{\mu},\bar{p}$, and ψ). Henceforth, any reference to Eq. (26) is a reference to the differential equation obtained from Eq. (26) when Eqs. (24) and (25) are used to eliminate π_{xx} and $\mu+p$ from the right-hand side of the equation.

Solutions to Eq. (26) exist. Unfortunately, the special solution given in [1] [see Eqs. (7.72) and (7.73)] is incorrect because of the error in v_a [see footnote (2)]. However, the error can be easily remedied to yield a special class of solutions. If $f=1$, $\kappa=0$, and $H=H(\tau)$, where $\tau=t+\alpha x$ ($\alpha=$ const), then Eq. (26) becomes a quadratic equation in $H''/(H')^2$ (we have introduced the notation $H'\equiv dH/d\tau$):

$$\left[\frac{H''}{(H')^2}\right]^2 3H^2(2H^2-\alpha^2)-\left[\frac{H''}{(H')^2}\right]2H(\alpha^2+15H^2)+8(\alpha^2+3H^2)=0. \tag{28}$$

Solving the above equation for $H''/(H')^2$ yields the two solutions

$$\frac{H''}{(H')^2}=\frac{\alpha^2+15H^2\pm\sqrt{25\alpha^4+54\alpha^2H^2+81H^4}}{3H(2H^2-\alpha^2)}, \tag{29}$$

which can be integrated to give H' as a function of H. If $\alpha\neq 0$, then α can always be set equal to 1 by a rescaling of both the coordinates and H. Thus, without loss of generality, we take $\alpha=1$.

We can use Eqs. (15) and (21) to obtain an expression for the tilt angle: namely,

$$\tanh\psi=\frac{3\pi_{xx}}{2qH}. \tag{30}$$

Equations (14), (24), and (29), with $\alpha=1$, then imply that

$$\tanh\psi_{\pm}=-\frac{1}{2H}\left[\frac{H''}{H(H')^2}+\frac{2}{H^2}\right]\left[\frac{H''}{H(H')^2}-\frac{1}{H^2}\right]^{-1} \tag{31}$$

$$=-\frac{1}{2H}\left[\frac{-5+27H^2\pm\sqrt{25+54H^2+81H^4}}{4+9H^2\pm\sqrt{25+54H^2+81H^4}}\right]. \tag{32}$$

We can disregard the $\tanh\psi_-$ solution since $\max(\tanh\psi_-)<-1$. The $\tanh\psi_+$ solution is valid for all values of H since $|\tanh\psi_+|\leqslant 1$.

If we now make the change of variables $(t,x,y,z)\to(w,x,y,z)$, where $w=H^2/\alpha^2$, then the metric can be written as

$$ds^2=-\left(\frac{C\,dw}{g(w)}-dx\right)^2+w(dx^2+dy^2+dz^2), \tag{33}$$

where C is an arbitrary constant and

$$g(w)=(-1+2w)^{17/2}w^{1/3}[3(1+3w)+h(w)]^{3/4}\left[\frac{5\,h(w)+25+27w}{5h(w)-25-27w}\right]^{5/12}\times\left[\frac{17h(w)+77+135w}{17h(w)-77-135w}\right]^{-17/12}, \tag{34}$$

$$h(w)\equiv\sqrt{9(3w+1)^2+16}. \tag{35}$$

The density, as calculated from Eq. (27), is given by

$$\mu=3\left(\frac{H'}{H}\right)^2\left[1-\frac{2}{3}\frac{H''}{H(H')^2}+\frac{1}{3}\frac{1}{H^2}\right] \tag{36}$$

$$=3\left(\frac{H'}{H}\right)^2\left[1+\frac{-5-24w+h(w)}{9w(2w-1)}\right]. \tag{37}$$

Thus, in the limit as w tends to infinity, that is, $H\to\infty$, Eq. (32) implies that the tilt angle tends to zero and hence the models asymptotically tend to FRW models at late times. Furthermore, the density perturbation decreases in the limit as w tends to infinity according to

$$\frac{\delta\mu}{\mu}\sim\frac{1}{w}. \tag{38}$$

The fact that the density perturbations decay in these models (which consequently tend to FRW models) can be clearly seen from the form of the ''perturbation'' terms in μ,p and $\tanh^2\psi$ [in Eqs. (25), (27), and (32)], which are of the form H^{-2}.

III. PERTURBATION ANALYSIS

Let us investigate the possible growth of inhomogeneities in these models. We write

$$H=R(t)[1+\epsilon(t,x)], \tag{39}$$

$$f(x)=1+\delta(x), \tag{40}$$

where $\epsilon(t,x)$ and $\delta(x)$ are small perturbations, and we attempt to solve Eq. (26) to successive orders in ϵ and δ. If $\kappa\neq 0$, then to ''zeroth'' order Eq. (26) yields

$$-\frac{2\ddot{R}}{R}+\frac{2\dot{R}^2}{R^2}+\frac{\kappa}{R^2}=0, \tag{41}$$

and to ''first'' order we obtain

$$2\epsilon_{tt}+\frac{1}{R^2}[\epsilon_{xx}+\delta_{xx}+2\kappa(\epsilon+\delta)]=0, \tag{42}$$

which represents a perturbation about the solution (41), which is either the Kantowski-Sachs solution if $\kappa=1$ or a Bianchi type III solution if $\kappa=-1$.

The case $\kappa=0$ is of more physical interest since the model can be interpreted as a perturbed (flat) FRW model. In this case, to ''first'' order Eq. (26) becomes (assuming $\epsilon_{xx}+\delta_{xx}\neq 0$)

$$\frac{\ddot{R}}{R}=\frac{\dot{R}^2}{R^2}, \tag{43}$$

which has the solution $R=R_0e^{\lambda t}$ (λ const); thus the ''background'' solution is given by the de Sitter metric, which is of particular relevance in early Universe cosmology (e.g., inflation). To ''second'' order we obtain

$$4\epsilon_{xt}^2=[\epsilon_{xx}+\delta_{xx}]\left[2\epsilon_{tt}+\frac{1}{R^2}(\epsilon_{xx}+\delta_{xx})\right]. \tag{44}$$

In general, further progress can only be made by making additional assumptions; we shall consider the following two assumptions separately: (i) a barotropic equation of state for the tilted fluid, and (ii) a constant tilt angle.

A. Barotropic equation of state

If we assume a barotropic equation of state of the form $\overline{p}=(\gamma-1)\overline{\mu}$, where γ is constant, then we obtain the additional equation

$$q^2(3\gamma-2)+\tfrac{9}{4}(2-\gamma)H^{-2}\pi_{xx}^2=-9\gamma H^{-1}H_{tt}\pi_{xx} \tag{45}$$

from Eqs. (4), (15)–(19), and (22); Eq. (45) holds for all values of κ.

First, we consider the case $\kappa=0$. If $\epsilon_{xx}+\delta_{xx}=0$ then Eq. (45) implies that $\epsilon_{xt}=0$ and thus to first order, $q\not\approx 0$. [If we take $\epsilon_{xx}+\delta_{xx}=0$ but assume that $q\not\approx 0$ to second order then $\epsilon(t,x)=-\delta(x)+a(t)x+b(t)$, where a,b, and δ are arbitrary functions, and the following solutions are consistent to second order: (i) $\gamma=0,a=\text{const}$, $R=R_0\exp(\lambda t)$, and (ii) $\gamma\neq 0,R=(c_0t+d_0)^{2/3\gamma}$, where R_0,λ,c_0, and d_0 are arbitrary constants.] Hence, we shall assume that $\epsilon_{xx}+\delta_{xx}\neq 0$. Equation (43) yields the solution $R=R_0\exp(\lambda t)$, where R_0 and λ are constants. Equation (45) is then trivially satisfied to zeroth order, and yields

$$\lambda\gamma=0 \tag{46}$$

to first order and

$$4(3\gamma-2)\epsilon_{xt}^2=6\gamma\epsilon_{tt}(\epsilon_{xx}+\delta_{xx})+\frac{1}{R_0^2}(\gamma-2)(\epsilon_{xx}+\delta_{xx})^2 \tag{47}$$

to second order. Comparing Eqs. (44) and (47) yields

$$2\epsilon_{tt}=\frac{\gamma}{R^2}(\epsilon_{xx}+\delta_{xx}) \tag{48}$$

and

$$\epsilon_{xt}=\nu(t)(\epsilon_{xx}+\delta_{xx}) \tag{49}$$

where

$$\nu^2(t)=\frac{\gamma+1}{4R^2(t)}. \tag{50}$$

Thus, Eq. (46), in conjunction with Eqs. (48)–(50), implies that the only consistent solutions are

(i) $\gamma=0$,

$$\epsilon(t,x)=-\delta(x)+a_0R_0x^2+(a_0x+b_0)t+c_0x+d_0, \tag{51}$$

(ii) $\gamma=1$,

$$\epsilon(t,x)=-\delta(x)+b_0(t-\sqrt{2}R_0x)+h(t+\sqrt{2}R_0x), \tag{52}$$

(iii) $\gamma\neq 0,1$,

$$\epsilon(t,x)=-\delta(x)+a_0\left(x^2+2\nu tx+\frac{\gamma}{2R_0^2}t^2\right)+b_0t+c_0x+d_0, \tag{53}$$

where $a_0(\neq 0)\cdots d_0$ are arbitrary constants, $\delta(x)$ and $h(t+\sqrt{2}R_0x)$ are arbitrary functions, and $R(t)=R_0=$ const. We note that solution (ii) exhibits wavelike behavior.

The only solution in the case $\kappa\neq 0$ is given by $\kappa=-1,\gamma=2,R(t)=t$, and

$$\epsilon(t,x)=-\delta(x)+(a_0x+b_0)t^{-1}+c_0\exp(2x)+d_0\exp(-2x). \tag{54}$$

Equation (27) gives the equation for the density:

$$\mu\cong 3R^{-2}R_t^2+\kappa R^{-2}+2[3R^{-1}R_t\epsilon_t-R^{-2}\{\kappa(\epsilon+\delta)+\epsilon_{xx}+\delta_{xx}\}]. \tag{55}$$

Setting $c_0=d_0=0$, we find that the density perturbation has the form

$$\frac{\delta\mu}{\mu}=-2(a_0x+b_0)/t. \tag{56}$$

B. Constant tilt angle

Alternatively, if we assume that the tilt angle is constant then $q\pi=\alpha H(\equiv H\tanh\psi)$ where α is constant and hence, Eq. (15) implies that

$$\pi_{xx} = \tfrac{2}{3}\,\alpha H q. \tag{57}$$

Equation (57) yields $\kappa = 0$ to zeroth order, and again Eq. (43) implies that $R = R_0 \exp(\lambda t)$, where λ and R_0 are constants (as before, we can assume that $\delta_{xx} + \epsilon_{xx} \neq 0$, else $q \approx 0$ to first order). [If $\epsilon_{xx} + \delta_{xx} = 0$, then Eqs. (26) and (57) imply that $\epsilon = -\delta(x) + a_0 x + b(t)$, where a_0 is an arbitrary constant and $b(t)$ is an arbitrary function, is a consistent solution up to first order. We note that in this case the equations up to first order give no information about the form of $R(t)$.] To first order, Eq. (57) yields

$$\epsilon_{xx} + \delta_{xx} + 2\alpha R_0 \exp(\lambda t)\epsilon_{xt} = 0. \tag{58}$$

Now, repeated use of Eq. (58), to substitute for $\epsilon_{xx} + \delta_{xx}$ in Eq. (44), yields

$$(1-\alpha^2)\epsilon_{xt} = -\alpha R_0 \exp(\lambda t)\epsilon_{tt}. \tag{59}$$

First, consider the case $\alpha^2 = 1$. Equation (59) then implies that

$$\epsilon(t,x) = a(x)t + b(x), \tag{60}$$

where $a(x)$ and $b(x)$ are arbitrary functions. Equation (58) then yields

$$a(x) = a_1 x + a_0, \tag{61}$$

where $a_1 (\neq 0)$ and a_0 are arbitrary constants, and $\lambda = 0$. Thus, integrating Eq. (58) yields the solution

$$\epsilon(t,x) = -\delta(x) + R_0 a_1 x^2 + (a_1 x + a_0)t + d_0, \tag{62}$$

where $a_1 \neq 0, a_0$ and d_0 are arbitrary constants (can always chose $\alpha = 1$ without loss of generality).

If $\alpha^2 \neq 1$ then it can be shown that the only valid solution occurs for $\alpha = 1/\sqrt{2}$, namely,

$$\epsilon(t,x) = -\delta(x) + a_0\left(x + \frac{t}{\sqrt{2}R_0}\right) + h\left(x - \frac{t}{\sqrt{2}R_0}\right), \tag{63}$$

where a_0 is an arbitrary constant and $h(x - t/\sqrt{2}R_0)$ is an arbitrary function. Again, the above solution exhibits wave-like behavior.

IV. DISCUSSION

The extreme degree of observed isotropy of the CMB plus the philosophical prejudice of the Copernican principle lead us to believe that the overall structure of the Universe is well modeled by FRW models. All scenarios advocated to explain the large-scale structure of the present observed Universe (for example, galaxies and clusters of galaxies) involve an evolution (through gravitational instability) from initially small density perturbations. Cosmological perturbation theory develops linear equations for perturbations away from spatial homogeneity and isotropy. The growth of perturbations on a given scale can be followed until they become sufficiently large (for gravitational collapse); on scales $\gtrsim 100$ Mpc cosmological perturbations can be used up until the present epoch.

In the inflationary scenario exponential expansion driven for example, by a scalar field smooths the Universe out towards a flat de Sitter-like state. Quantum-mechanical fluctuations of the (scalar) field(s) ''within the horizon'' then gives rise to new, small-scale perturbations. These de Sitter fluctuations then lead to fluctuations in the metric tensor since gravitons are the propagating modes associated with transverse, traceless metric perturbations which behave as minimally coupled scalar fields.

In the models under investigation here, there are perturbations that are due to the tilt, the evolution of the tilting modes has been studied. These models and more realistic models, including other physical perturbations, should be further analyzed, paying particular attention to the evolution of the growing modes, and the results of any such analysis should, of course, be compared to the conventional analysis [12] of density perturbations about an FRW background. However, what is really necessary is a complete *covariant* and *gauge-invariant* approach to calculate the cosmological density perturbations [12,13]. This work will be done elsewhere [14], not only for the particular type of models studied here, but for the general class of tilting models [13,15,16].

The models that have been studied here may also be of relevance in other areas of cosmology which are currently of interest. In particular, they are of relevance in the study of models which are linear perturbations of FRW models in the so-called ''longitudinal'' gauge. In this gauge the normals are both shear-free and irrotational and the matter moves relative to these normals (that is, the matter is tilted). If the matter is a barotropic perfect fluid then in this gauge the metric can be written in terms of a single scalar potential; this form of the metric has been extensively used in the study of the evolution of density perturbations and gravitational lensing calculations (for example, see [17]).

In addition, the solutions here may be of significance in the study of other, more general, classes of cosmological models. For example, some of these solutions occur as special cases (in which the shear is zero) of the so-called ''silent'' universe models, these are cosmological models with irrotational dust in which the magnetic part of the Weyl tensor vanishes [18].

In summary, we have studied tilting cosmological models admitting a shear-free, irrotational, and geodesic timelike congruence. These cosmologies may be of importance in modeling large-scale structure formation, particularly that of the observed large-scale streaming of matter relative to the CMB.

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