NOTES AND CORRESPONDENCE

Time-Averaged Forms of the Nonlinear Stress Law

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ABSTRACT

On the assumption that the mean velocity and the probability distribution of the higher frequency fluctuating motions are known, an expression for the mean surface stress is given. For the case of isotropic background variations, the mean stress is shown to be a simple nonlinear function of the mean velocity and the standard deviation of the fluctuations. Results should be useful in studies concerning the stress at the bottom of either the ocean or the atmosphere. For use in the oceanic case, a constant drag coefficient is considered. For the atmospheric case, the drag coefficient is a function of wind speed. Results are compared for several previously proposed forms of this functional dependence.

1. Introduction

The determination of the exchange of momentum across both horizontal boundaries of the ocean is clearly crucial to our understanding of the ocean–atmosphere system. Although measurements of the stress on the ocean's surface and bottom are growing in numbers, they are difficult to make. For large-scale studies it is useful to parameterize the fluxes in terms of easily measured quantities. Generally, a quadratic friction law is assumed to hold, i.e.,

$$\tau = \rho c_D |\mathbf{U}|\mathbf{U}, \qquad (1.1)$$

where $\mathbf{U}$ is the mean horizontal velocity over a period of about 1 h measured at some specified height above the surface (typically 10 m for the atmospheric case and of order 1 m for the oceanic case), and c_D is the drag coefficient determined from direct measurements.

Recent measurements (e.g., Smith, 1980; Large and Pond, 1981—henceforth LP81) indicate an increase in c_D with wind speed and several functional forms of c_D have been proposed. One of the aims of this note is to clarify the relationships between the mean stresses estimated using these different forms of $c_D(|\mathbf{U}|)$.

For studies involving large spatial and long temporal scales it is useful to simplify (1.1) further by appropriate averaging. Our primary motivation for considering this problem is the desire to compute accurately surface stress from monthly-mean air pressure charts. If this can be done, it will make readily available years of historical information on the wind stress at the sea surface.

In the following section, our general approach is described. In Section 3, examples are considered. A simple formula relating the mean stress to the mean velocity and the standard deviation of the fluctuating velocity field is given in Section 4, and conclusions are summarized in Section 5.

2. General approach

Consider a velocity $\mathbf{U}(t)$. Assume that we know the mean velocity $\mathbf{U}_0$ over some time interval T and the probability distribution P corresponding to the fluctuating velocity field $\mathbf{U}_1(t)$.

Then, using (1.1), the mean stress is given by

$$\bar{\tau} = \iint_{-\infty}^{\infty} \rho c_D(|\mathbf{U}|)|\mathbf{U}|\mathbf{U} P(U_1, V_1) dU_1 dV_1, \qquad (2.1)$$

where $\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1$. The usefulness of (2.1) lies in the fact that even when the details of $\mathbf{U}_1(t)$ are unknown, one can frequently obtain a reasonable estimate of $P(U_1, V_1)$.

Without loss of generality, we shall henceforth assume that our coordinate system is chosen such that the x-axis is parallel to the direction of the mean velocity so that $\mathbf{U}_0 = (U_0, 0)$.

3. Examples

For the deep ocean, where tidal velocities are expected to be relatively small, a reasonable first approximation to $P(U_1, V_1)$ is an isotropic, bivariate Gaussian distribution with standard deviation σ_u, i.e.,

$$P(U_1, V_1) = \frac{1}{2\pi\sigma_u^2} \exp\left[-\frac{1}{2\sigma_u^2}(U_1^2 + V_1^2)\right]. \qquad (3.1)$$

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This is also appropriate for the atmospheric boundary layer (the surface layer) at periods greater than one month (Thompson *et al.*, 1983). With this form for P, we now consider the mean stress resulting from various forms of $c_D(|\mathbf{U}|)$.

(i) c_D = constant (bottom stress on the ocean)

For c_D constant and P given by (3.1), a change of variables in (2.1) yields

$$\bar{\tau}_x = \frac{\rho c_D U_0^2 (U_0/\sigma_u)^2}{2\pi} \iint_{-\infty}^{\infty} [(1+x)^2 + y^2]^{1/2}(1+x) \times \exp\left[-\frac{U_0^2}{2\sigma_u^2}(x^2+y^2)\right] dx dy, \quad (3.2)$$

and

$$\bar{\tau}_y = 0. \quad (3.3)$$

For $U_0 \ll \sigma_u$, (3.2) can be approximated by

$$\bar{\tau}_x = 1.5(\pi/2)^{1/2}\rho c_D \sigma_u U_0. \quad (3.4)$$

Indeed, for $U_0 \ll \sigma_u$ it is straightforward to show that if P in (2.1) is any isotropic probability density function, then

$$\bar{\tau}_x = 1.5\rho c_D \overline{|\mathbf{U}_1|} U_0, \quad (3.5)$$

$$\bar{\tau}_y = 0, \quad (3.6)$$

consistent with results of previous investigators (e.g., Rooth, 1972; Hunter, 1975; Heaps, 1978).

Although (3.4) is a useful result, it gives us no idea of the value of $\bar{\tau}_x$ for $U_0/\sigma_u \geq 1$. However, using (3.2) we can clearly consider arbitrary values of U_0/σ_u. Noting from (3.2) that the quantity $\bar{\tau}_x/\rho c_D \sigma_u U_0$ is a function of U_0/σ_u only, Fig. 1 is easily generated by numerical integration. This figure can be reproduced with a relative error of less than 2% by

$$\bar{\tau}_x = \rho c_D \sigma_u U_0 \{[1.5(\pi/2)^{1/2}]^2 + (U_0/\sigma_u)^2\}^{1/2}. \quad (3.7)$$

Note that $\bar{\tau}_x \rightarrow 1.5(\pi/2)^{1/2}\rho c_D \sigma_u U_0$ as $U_0/\sigma_u \rightarrow 0$ and $\bar{\tau}_x \rightarrow \rho c_D U_0^2$ (the broken line in Fig. 1) as $U_0/\sigma_u \rightarrow \infty$. This equation allows one easily to determine $\bar{\tau}_x$ for arbitrary U_0/σ_u. Further, for $U_0/\sigma_u \leq 1$,

$$\bar{\tau}_x \approx 2.0\rho c_D \sigma_u U_0 \quad (3.8)$$

with a relative error of less than 6%. Hence for the isotropic probability distribution considered here, the stress law can be linearized over a much wider range than one might have expected (past linear formulas relating U_0 to $\bar{\tau}$ have always been restricted to the range $U_0/\sigma_u \ll 1$).

(ii) $c_D = c_D(|\mathbf{U}|)$ (bottom stress on the atmosphere)

In this section we consider the relationships between the stress estimates determined from different forms of c_D as a function of wind speed for the atmospheric case (e.g., Hellerman, 1967; Smith and Banke, 1975; Smith, 1980; LP81). A generalization of (3.7) is given in Section 4.

Consider

$$c_D = c'_D F(|\mathbf{U}|), \quad (3.9)$$

where c'_D is a constant. Then the equation corresponding to (3.2) is

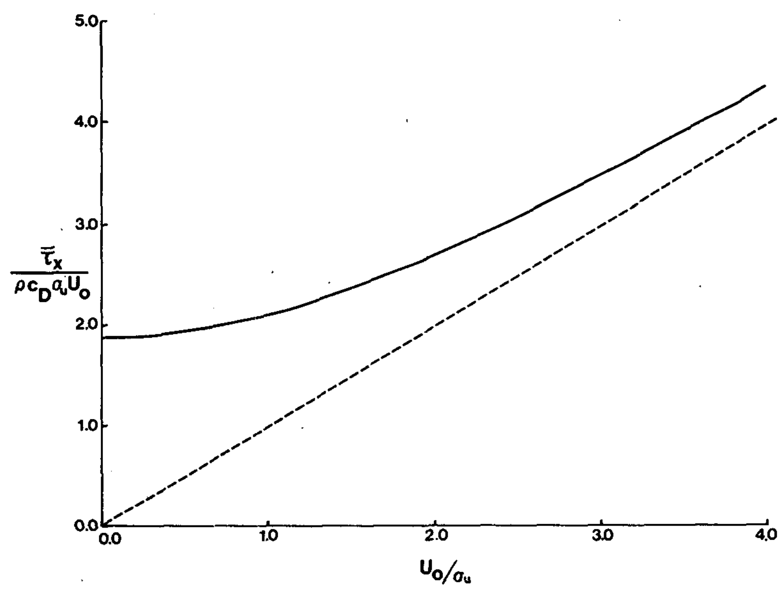

FIG. 1. The normalized stress plotted against U_0/σ_u for the case c_D = constant. Eq. (3.7) fits this curve within 2% everywhere.

$$\frac{\bar{\tau}_x}{\rho c'_D \sigma_u U_0} = \frac{(U_0/\sigma_u)^3}{2\pi} \int\int_{-\infty}^{\infty} F\{[(1+x)^2+y^2]^{1/2}U_0\} \times [(1+x)^2+y^2]^{1/2}(1+x) \times \exp\left[-\frac{U_0^2}{2\sigma_u^2}(x^2+y^2)\right]dxdy. \quad (3.10)$$

The right side of this equation is clearly a function of both U_0 and σ_u (not just of U_0/σ_u).

To compare mean stresses derived from the various forms of c_D (Fig. 2), the recent results of LP81 have been chosen as a reference case. Although the form of c_D given by Smith (1980) is not explicitly included in this comparison, his results are not significantly different from those determined using Large and Pond's formula.

Fig. 3 illustrates the results of this comparison where Fig. 3a gives $\bar{\tau}_x$ as a function of U_0 for several values of σ_u. Using this form one can readily check the range over which $\bar{\tau}_x$ varies linearly with U_0. As expected this range increases with σ_u. Further, Fig. 3a illustrates the increase in stress for a given mean velocity as σ_u increases. This is the effect which Saunders (1977) refers to when he attributes much of the offshore increase in mean stress in the Mid-Atlantic Bight to "the intensity (and frequency) of cyclonic activity." It is also this effect which necessitates the introduction of "correction factors" in the bulk aerodynamic formula (1.1) for averaging periods longer than two days (Fissel *et al.* 1977). With the present results, the estimation of correction factors is replaced by the estimation of σ_u, an easier task.

Fig. 3b shows the variation of $\bar{\tau}_x$ over a wide range of values of U_0 and σ_u. The broken line is the curve $U_0 = \sigma_u$. From the results of the previous section we expect $\bar{\tau}_x$ to vary approximately linearly with U_0 to the left of this line. Though this is difficult to see from Fig. 3b, it is readily seen in Fig. 3a.

Figs. 3c–3e show the ratios of the results derived from the other c_D forms to the results derived from the LP81 form. Except for small values of U_0 and σ_u, the results derived using Smith and Banke's (1975) formula are in good agreement with those from LP81. The case $c_D = 1.5 \times 10^{-3}$ gives reasonable results over a limited range of values of U_0 and σ_u. For large wind speeds it significantly underestimates the stress, as expected. Similarly, Hellerman's (1967) results underestimate the stress at large wind speeds. However, the most significant errors in this case occur in the region near the curve $U_0^2 + (2\sigma_u)^2 = 12^2$.

Figs. 3d and 3e have been included here primarily to point out the inadequacies of using the forms of c_D proposed by Hellerman (1967) and Pond (1975). Indeed, Hellerman's form severely overestimates the stress in the region of primary interest for a one month averaging period. The constant value of c_D proposed by Pond (1975) does appear to be an optimum choice if one is restricted to using c_D = constant. However, there seems to be little justification for this restriction in the light of recent measurements.

4. Approximate forms

The format of Fig. 3 is useful for the purpose of comparisons. However, for the purpose of compu-

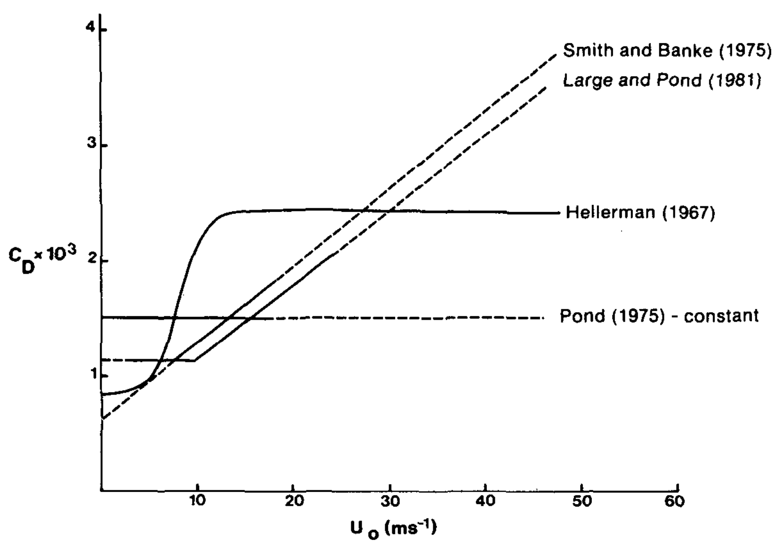

FIG. 2. The various forms of $c_D(|\mathbf{U}|)$ considered. Broken lines correspond to regions where we have arbitrarily extrapolated the author's original results. The curve labeled Hellerman (1967) is the continuous representation of Hellerman's results used by Saunders (1976).

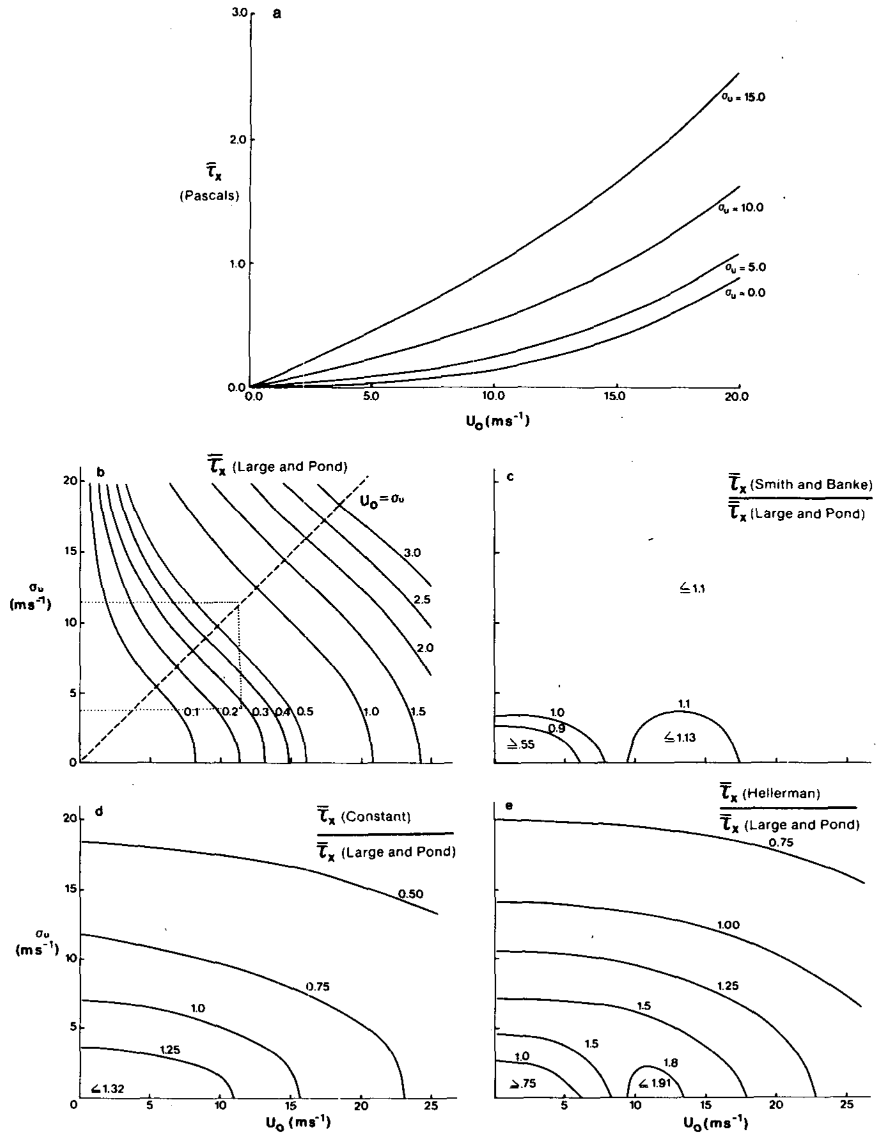

FIG. 3. (a) $\bar{\tau}_x$ as a function of U_0 and σ_u derived from the form of c_D proposed by Large and Pond (1981); (b) contours of $\bar{\tau}_x$ derived from Large and Pond's formula for a wide range of values of U_0 and σ_u (for a 1 month averaging period, U_0 and σ_u generally lie within the region enclosed by dots); (c), (d), and (e) the ratio of $\bar{\tau}_x$ derived using various forms of $c_D(|U|)$ to $\bar{\tau}_x$ derived using Large and Pond's formulation.

tation, we have found that the mean stress can be accurately determined using the simple formula (see Appendix)

$$\bar{\tau} = \rho c_D(a) a \mathbf{U}_0, \qquad (4.1)$$

where $a = [U_0^2 + (2\sigma_u)^2]^{1/2}$.

Comparing (4.1) with (1.1), we see that a is the effective wind speed for the purpose of determining the mean stress. Note that $a > U_0$ for $\sigma_u \neq 0$. Because of the nonlinearity of the stress law, the larger wind speeds tend to dominate in the determination of the mean stress.

The relative error in writing (4.1) is largest for U_0, σ_u and U_0/σ_u all small. For the forms of c_D given by Smith and Banke (1975) or LP81, it is never larger than 6%. The region of small U_0 and σ_u is not of particular interest as the stress is very small there. Further, the form of c_D is not well defined in this region. The error decreases as U_0 and/or σ_u increase and at $U_0 = \sigma_u = 5$ m s^{-1} the relative error is less than 3%.

5. Conclusions

The use of probability–density functions has provided a unified approach to the derivation of time-averaged forms of the nonlinear stress law. Though the method is general, we have concentrated our attention on the case of an isotropic Gaussian fluctuating velocity field. However, any systematic deviations from this form could be incorporated through the use of a Gram–Charlier expansion (Kendall and Stuart, 1958).

The central result of this paper is a simple formula for the mean stress in terms of the mean velocity U_0 and the standard deviation σ_u of the fluctuating velocity field (4.1). This form holds for arbitrary U_0 and σ_u, and clearly quantifies the effect of σ_u on the mean stress. Further, a linearized version of (4.1) can be obtained simply by replacing a by $2\sigma_u$. For c_D a constant or as given by Smith and Banke (1975) or LP81 this linear formula gives reasonable results (relative error $\leqslant 6\%$) for the entire range $U_0 \leqslant \sigma_u$. This should be particularly useful for analytical models where a linear form is desirable.

Finally we note that the approach taken here does not require the determination of "correction factors" as a function of space, time and averaging period as required in the approach taken by Fissel *et al.* (1977). The use of Eq. (4.1) replaces this problem with the task of determining appropriate values of σ_u (Thompson *et al.*, 1983).

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APPENDIX

Derivation of Approximate Formulas

Eq. (4.1) is an approximation to a slightly more complicated formula derived below (A3).

We consider the general form of $c_D(|\mathbf{U}|)$ given by Smith and Banke (1975) and by Smith (1980), i.e.,

$$c_D(|\mathbf{U}|) = c'_D + b|\mathbf{U}|. \qquad \text{(A1)}$$

The mean stress derived using (A1) can be determined in the following manner:

$$\begin{aligned}\bar{\tau}_x &= \rho\overline{[c'_D + b|\mathbf{U}|]|\mathbf{U}|(U_0 + U_1)} \\ &= \rho c'_D\overline{|\mathbf{U}|(U_0 + U_1)} + \rho b\overline{|\mathbf{U}|^2(U_0 + U_1)} \\ &\approx \rho c'_D U_0[U_0^2 + (1.88\sigma_u)^2]^{1/2} \\ &\qquad + \rho b U_0[U_0^2 + (2\sigma_u)^2]. \qquad \text{(A2)}\end{aligned}$$

The last approximation is obtained by using (3.7) in the first term. The second term is exact for isotropic background variations with $\overline{U_1^2} = \overline{V_1^2} = \sigma_u^2$, and is easily obtained by expanding $|\mathbf{U}|^2(U_0 + U_1)$ and averaging. Hence, use of (A2) introduces a relative error of less than 2%.

Eq. (A2) can now be re-written in the form

TABLE 1. Bounds on the relative errors introduced by using Eqs. (A3) or (4.1) for various forms of $c_D(|U|)$.

	Eq. (A3)	Eq. (4.1)
Smith and Banke (1975)	<2%	<6%
Large and Pond (1981)	<10%	<6%
c_D constant (1.5×10^{-3})	<2%	<6%
Hellerman (1967)	<17%	<15%

$$\bar{\tau}_x = \rho c_D(a) a U_0 + \rho c'_D U_0[a' - a], \qquad \text{(A3)}$$

where

$$a = [U_0^2 + (2\sigma_u)^2]^{1/2} \text{ and } a' = [U_0^2 + (1.88\sigma_u)^2]^{1/2}.$$

Eq. (4.1) is obtained from (A3) simply by neglecting the second term. The relative error introduced by dropping this term is clearly largest for $U_0 \ll \sigma_u$, and is never greater than 6%. This bound on the relative error is, of course, only valid for $c_D(|\mathbf{U}|)$ given by (A1). Eq. (A3) can, however, be used for other forms of c_D, if c'_D is interpreted as the value of c_D for $|\mathbf{U}| = 0$. For c_D constant, (A3) is then equivalent to (3.7). For this case (appropriate to the bottom boundary of the ocean), Eq. (3.7), which involves a relative error of less than 2%, should be used rather than (4.1). The use of (A3) with Large and Pond's (1981) formula results in relative errors as large as 9% and hence (4.1) (for which the relative error is not greater than 6%) should be used in this case. Relative errors involved in the use of (A3) and (4.1) are summarized in Table 1.

REFERENCES

Fissel, D. B., S. Pond and M. Miyake, 1977: Computation of surface fluxes from climatological and synoptic data. *Mon. Wea. Rev.*, **105**, 26–36.

Heaps, N. S., 1978: Linearized vertically-integrated equations for residual circulation in coastal seas. *Dtsch. Hydrogr. Z.*, **31**, 147–169.

Hellerman, S., 1967: An updated estimate of the wind stress on the world ocean. *Mon. Wea. Rev.*, **95**, 607–626. [Corrigendum (1968), **96**, 63–74].

Hunter, J. R., 1975: A note on quadratic friction in the presence of tides. *Estuarine Coastal Mar. Sci.*, **3**, 473–475.

Kendall, M. G., and Stuart, A., 1958: *The Advanced Theory of Statistics*, Vol. I. Griffin & Co., 439 pp.

Large, W. G., and S. Pond, 1981: Open ocean momentum flux measurements in moderate to strong winds. *J. Phys. Oceanogr.*, **11**, 324–336.

Pond, S., 1975: The exchange of momentum, heat and moisture at the ocean–atmosphere interface. *Numerical Models of Ocean Circulation*, Nat. Acad. Sci., Washington, DC, 26–38.

Rooth, C., 1972: A linearized bottom friction law for large scale oceanic motions. *J. Phys. Oceanogr.*, **2**, 509–510.

Saunders, P. M., 1976: On the uncertainty of wind stress curl calculations. *J. Mar. Res.*, **34**, 155–160.

——, 1977: Wind stress on the ocean over the eastern continental shelf of North America. *J. Phys. Oceanogr.*, **7**, 555–566.

Smith, S. D., 1980: Wind stress and heat flux over the ocean in gale force winds. *J. Phys. Oceanogr.*, **10**, 709–726.

——, and E. G. Banke, 1975: Variation of the sea-surface drag coefficient with wind speed. *Quart. J. Roy. Meteor. Soc.*, **101**, 665–673.

Thompson, K. R., R. F. Marsden, and D. G. Wright, 1983: Estimation of low-frequency wind stress fluctuations over the open ocean. In preparation.